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Christian Proaño, Reiner Franke and Peter Flaschel

Hans-Böckler-Straße 39
D-40476 Düsseldorf
Germany
Phone: +49-211-7778-331
IMK@boeckler.de
http://www.imk-boeckler.de

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On the Determinacy of New Keynesian Models with Staggered Wage and Price Setting

Peter Flaschel
Bielefeld University, Germany

Reiner Franke
Kiel University, Germany

Christian Proaño
Macroeconomic Policy Institute (IMK), Germany

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Abstract

This paper shows that an analytical determinacy analysis of the baseline New Keynesian model with both staggered wages and prices developed by Erceg, Henderson and Levin (2000) is possible despite the high dimensional nature of this model. It is possible if the formulation of the model is translated from discrete to continuous time. Our findings corroborates in an analytical manner Galí's (2008) numerical findings regarding the determinacy frontier and the Taylor principle for this model type, where a generalized Taylor rule that employs a weighted combination of wage and price inflation is used as a measure for the inflation gap.

Keywords: Period models, continuous time, (in)determinacy.

JEL CLASSIFICATION SYSTEM: E24, **E31**, **E32**.

1 Introduction

In this paper we focus on the determinacy of the baseline New Keynesian model with both staggered wage and price setting. Starting from the fact that the dynamics and determinacy (stability) properties of a (macroeconomic) model should not depend on whether it is formulated in continuous- or discrete time, we use a (deterministic) continuous time representation of the New Keynesian model featuring staggered wage and price setting to investigate its (in-)determinacy properties in an analytical manner.

Despite the high dimensionality of this model, we can corroborate analytically the numerical results of Galí (2008) concerning the model's determinacy frontier determined by the validity of the Taylor principle in a generalized Taylor rule which employs a weighted combination of the wage and the price inflation gaps.

The remainder of this paper is organized as follows: In section 2 we deliver some considerations concerning the equivalence of discrete- and continuous time models and provide some motivation for the continuous time approach we use here to analyze determinacy. In Section 3 describe the continuous time formulation of the baseline New Keynesian model with staggered wages and prices. In section 4 we corroborate analytically the numerical results by Galí concerning the determinacy frontier of the analyzed New Keynesian model. Section 5 draws some concluding remarks from this study.

2 Period Models and Continuous Time Analysis: Some Preliminary Considerations

Continuous vs. discrete time (period) modeling in macroeconomics was discussed extensively in the 1970s and 1980s, often by means of a highly sophisticated mathematical apparatus. There are however some statements in the literature, old an new, which have already suggested in a clear and intuitive manner that period analysis in macroeconomics, i.e. discrete-time analysis where all economic agents are forced to act in a synchronized manner (with a time unit that is usually left unspecified) can be misleading from the formal as well as from the economic point of view. Foley (1975, p.310) in particular formulates a methodological precept concerning the theoretical specification of macroeconomic models according to which *No substantive prediction or explanation in a well-defined macroeconomic period model should* *depend on the real time length of the period.* Such a statement has however been completely ignored in the numerous analytical and numerical investigations of complex or chaotic macro-dynamics. Furthermore, from the view point of economic modeling, Sims (1998, p.318) analyzes the behavior of a variety of models with real and nominal rigidities in a continuous time formulation *to avoid the need to use the uninterpretable 'one period' delays that plague the discrete time models in this literature.*

Our view concerning these issues is that if a uniform and synchronized decision making by the economic agents in the real and financial markets is assumed, then a very short time-unit (say, one day instead of a quarter) should also be assumed, should both discrete- and continuous time formulations provide the same qualitative results. In the linear case this can be motivated further by the following type of argument.

Consider the mathematically equivalent discrete and continuous-time models[1]

$$x_{t+1} = Ax_t \quad \text{and} \quad \dot{x} = (A - I)x = Jx$$

which follow the literature by assuming an unspecified time unit 1.

Our above arguments suggest that we should generalize such an approach and rewrite it with a variable period length as follows:

$$x_{t+h} - x_t = hJx_t \quad \text{and} \quad \dot{x} = Jx.$$

This gives for their system matrices the relationships

$$A = hJ + I.$$

According to Foley's postulate both J and A for example should be stable matrices if period as well as continuous-time analysis is used for macroeconomic analysis in such a linear framework, i.e., all eigenvalues of J should have negative real parts, while the eigenvalues of A should all be within the unit circle. Graphically this implies the situation shown in figure 1 (which shows that, if J's eigenvalues do not yet lie inside the unit circle shown, that they have to be moved into it by a proper choice of the time unit and thus the matrix hJ.)

If the eigenvalues of the matrix J of the continuous time case are such that they lie outside the solid circle shown, but for example within a circle of radius 2, the

[1] I the identity matrix.

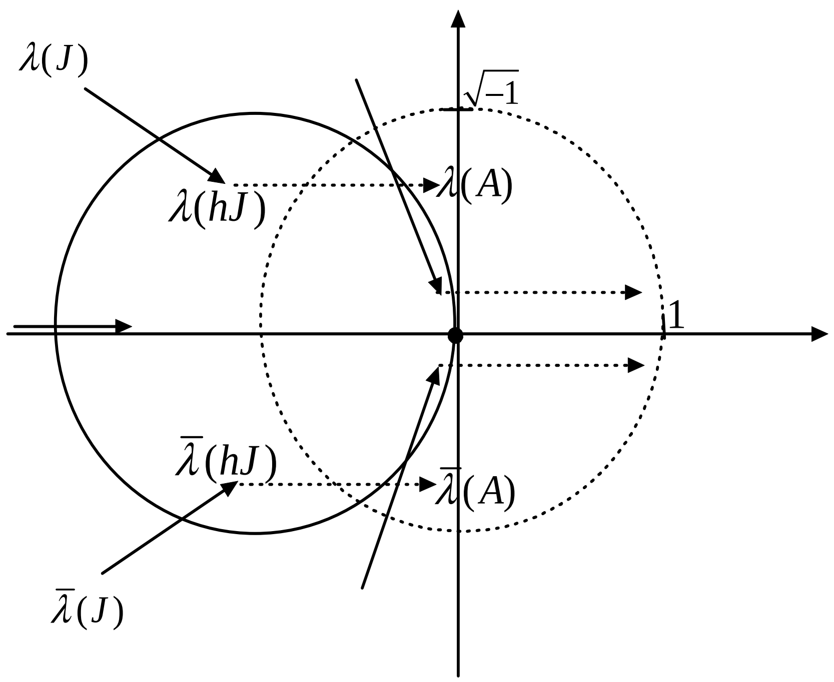

Figure 1: A choice of the period length that guarantees equivalence of continuous and discrete time analysis ($A = hJ + I$).

discrete time matrix $J + I$ would – in contrast to the continuous time case – have unstable roots (on the basis of a period length $h = 1$ that generally is left implicit in such approaches). The system $x_{t+1} = Ax_t, A = J + I$ then has eigenvalues outside the unit circle (which is obtained by shifting the shown solid unit circle by 1 to the right (into the dotted one). Choosing $h = 1/2$ would however already be sufficient to move all eigenvalues $\lambda(A) = h\lambda(J) + 1$ of $A = hJ + I$ into the unit circle, since all eigenvalues of hJ are moved by this change in period length into the solid unit circle shown in figure 1, and since $J's$ eigenvalues have all been assumed to have negative real parts and are thus moved towards the origin of the space of complex numbers when the period length h is reduced.

In view of this we claim that sensible macro-dynamic period models $x_{t+h} = (hJ + I)x_t = Ax_t$ have all to be based on a choice of the period length h such that $||\lambda(A)|| < 1$ can be achieved (if the matrix J is stable).[2] Since models of the real financial interaction suggest very small periods length and since the macroeconomy

[2]Considering in particular negative eigenvalues that are smaller than -2 it would be strange from a macroeconomic point of view to obtain from such a situation period model instability as compared to the very strong asymptotic stability they imply in the continuous time case.

is updated at the least on a daily basis in reality, such a choice should always be available for the model builder. In this way it is guaranteed that linear period and continuous-time models give qualitatively the same answer.

We also note here already (in view of the New Keynesian approach to be considered later on) that matrices J with eigenvalues with only positive real parts will always give rise to totally unstable matrices $A = hJ + I$, since the real parts are augmented by '1' in such a situation. We will however show in later sections that the here considered simple h-dependence of the eigenvalues of the matrix $A: \ \lambda(A) = h\lambda(J)+1$, in this linear comparison does not apply to baseline New Keynesian models, since they – though linear – depend nonlinearly on their period length h and are therefore only directly comparable to the above situation in the special case $h = 1$. Comparisons for larger period lengths h are therefore not so easy and demand other means in order to compare determinacy problems in both continuous and discrete time models.

As a general statement and conclusion, related to Foley's (1975) observation, we however would assert that New Keynesian period models with eigenvalue structures that differ from their continuous-time analogue should be questioned with respect to their relevance from the theoretical and – even more – from the empirical point of view. Period models, if meaningful, thus depend on their continuous-time analogues in the validity of their results.

3 New Keynesian Wage-Price Dynamics: Two Alternative Formulations

3.1 NK wage-price dynamics with forward-dated real wage dynamics

We reconsider in this section the New Keynesian model with both staggered prices and wages as described in Galí (2008, 6.2), which appropriately reformulated gives rise to a 4D loglinearized dynamical system in discrete time. This 4D NK model, the Keynesian case of the New Neoclassical Synthesis, see also Erceg et al. (2000), Woodford (2003, pp.225ff.), now with an explicitly shown period length h, reads as follows:

$$\pi_t^w \stackrel{\text{WPC}}{=} \pi_{t+h}^w + h\beta_{wy}\tilde{y}_t - h\beta_{w\omega}\tilde{\omega}_t, \quad \pi_t^w = (w_t - w_{t-h})/h \tag{1}$$

$$\pi_t^p \stackrel{\text{PPC}}{=} \pi_{t+h}^p + h\beta_{py}\tilde{y}_t + h\beta_{p\omega}\tilde{\omega}_t, \quad \pi_t^p = (p_t - p_{t-h})/h \tag{2}$$

$$\tilde{y}_t \stackrel{\text{IS}}{=} \tilde{y}_{t+h} - h\alpha_{yi}(i_t - \pi_{t+h}^p - r^n) \tag{3}$$

$$i_t \stackrel{\text{TR}}{=} r^n + \phi_{ip}\pi_t^p + \phi_{iw}\pi_t^w + \phi_{iy}\tilde{y}_t \tag{4}$$

In contrast to Galí (2008, footnote 6) we are here starting from annualized data throughout and indicate this by dividing rates of change through the period length h (usually 1/4 year in the literature). We show therewith which parameters change with the data period length or data frequency or just the iteration step-size h. We thus use the usual empirical magnitudes for the rates here under consideration, but allow for changes in the data collection frequency and thus in the iteration frequency of the considered discrete time dynamics (which in principle can also differ from the data collection frequency). We consequently consider the equations (1) – (4) from an applied perspective, i.e., we take them as starting point for an empirically motivated study of the influence of the data frequency (quarterly, monthly or weekly) on the size of the parameter values to be estimated.

We follow Galí's (2008, 6.2) presentation, but use for ease of comparison as parameter characterizations indices which indicate the variables that are related through this parameter. All variables with index $t + h$ are expected variables or should be interpreted as representing perfect foresight in the deterministic skeleton of the considered dynamics. The discount factor $\beta \approx 1$ in the Phillips curves is set equal to 1 for reasons of simplicity. This can be done without loss of generality, since we are only investigating the model from the mathematical point of view. We also assume as in Galí (2008, p.128) that the conditions stated there for the existence of a zero steady state solution are fulfilled.

The state variables of the model are the backward dated (annualized) wage and price inflation rates, the output gap and the real wage gap. We have a standard NK Wage Phillips Curve (WPC), a NK Price Phillips Curve (PPC), a dynamic IS-curve and a generalized type of Taylor interest rate policy rule, see Galí (2008, 6.2) for details. The model is more balanced in its Keynesian formulation of the New Neoclassical synthesis, since it assumes both gradual wage and price adjustments as opposed to the Classical form of the New Neoclassical Synthesis, that class of models with both perfectly flexible wages and prices.[3]

For the annualized change of the variable $\tilde{\omega}_t = \omega_t - \omega^n, \omega_t = w_t - p_t$ we get from the definition of this variable (under the assumptions that guarantee a zero steady state, see Galí (2008, p.128)) the following expression:

$$\begin{aligned}(\tilde{\omega}_{t+h} - \tilde{\omega}_t)/h &= (\omega_{t+h} - \omega_t)/h = (w_{t+h} - w_t)/h - (p_{t+h} - p_t)/h \\ &= \pi^w_{t+h} - \pi^p_{t+h}\end{aligned}$$

as relationship between the change in this real variable and the annualized rates of wage and price inflation. The fact that the model is given in loglinearized form suggests that the period length h by which the model is to be iterated should not be chosen too large, in line with what we shall do in the following. This is also suggested by the frequency of the actual data generating process which is in many respects a daily one.

The above model represents – in a simple way – an implicitly formulated system of difference equations. Making use of the TR and the PPC, it can be transformed into a complete system of difference equations as follows:[4]

$$\pi^w_{t+h} = \pi^w_t - h\beta_{wy}\tilde{y}_t + h\beta_{w\omega}\tilde{\omega}_t \tag{5}$$

$$\pi^p_{t+h} = \pi^p_t - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t \tag{6}$$

$$\tilde{y}_{t+h} = \tilde{y}_t + h\alpha_{yi}(\phi_{iw}\pi^w_t + \phi_{ip}\pi^p_t + \phi_{iy}\tilde{y}_t - (\pi^p_t - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t)) \tag{7}$$

$$\tilde{\omega}_{t+h} = \tilde{\omega}_t + h(\pi^w_{t+h} - \pi^p_{t+h}) \tag{8}$$

Since our hypothesis in this paper is that a discrete time (period) model should not depend in its fundamental qualitative properties on the length of the period h we may assume here that it must reflect the properties of its continuous time analogue, which reads:

[3] with respect to hybrid cases (one variable gradually, one instantaneously adjusting) it is interesting to note that the core cases of the old and the new neoclassical synthesis are just the opposite of each other.

[4] Note here that the above system of difference equations is no longer of the linear type we have considered in section 2 (as far as the parameter h is concerned).

$$\begin{aligned}
\dot{\pi}^w &= -\beta_{wy}\tilde{y} + \beta_{w\omega}\tilde{\omega} &(9)\\
\dot{\pi}^p &= -\beta_{py}\tilde{y} - \beta_{p\omega}\tilde{\omega} &(10)\\
\dot{\tilde{y}} &= \alpha_{yi}((\phi_{ip}-1)\pi^p + \phi_{iw}\pi^w + \phi_{iy}\tilde{y}) &(11)\\
\dot{\tilde{\omega}} &= \pi^w - \pi^p &(12)
\end{aligned}$$

We stress here that the obtained system of differential equations is to be interpreted as a mathematical approximation of the period version, not as a continuous time economic representation of it (in particular since the latter is also only a loglinear approximation of the true model). Since the steady state values of the inflation gaps, the output gap and the real wage gap are all zero, we would get in this forward dated system a unique bounded response to shocks (the steady state) if determinacy is given, i.e., if the roots of the considered continuous-time dynamics have all positive real parts. In this case, we get as response of wages and prices no change at all if the system has been in the steady state position initially (just as in the 2D Wicksellian baseline case). We thus get that those agents that have the option to adjust their prices are in fact forced (under rational expectations) by the reaction of the whole economy to not using this option, but adjust their prices to the given (inactive) prices, which in fact were not part of their optimization routine (since there holds in discrete time $x_t - x_{t-1} = (1-\theta_x)(x_t^* x_{t-1}),\ x = w, p$ in the notation of Galí (2008)).

If the assumption on the steady state history of the economy does not hold, the forwardly dated real wage dynamics, if interpreted in the above way, may however run into a consistency problem, since the equations:

$$\pi_t^w = w_t - w_{t-1} = (1-\theta_w)(w_t^* - w_{t-1}) = 0,\ \pi^p = p_t - p_{t-1} = (1-\theta_p)(p_t^* - p_{t-1}) = 0$$

imply $w_t^* = w_{t-1}$ and $p_t^* = p_{t-1}$ with respect to the new optimally chosen wages and prices. We thus get $\omega_t = \omega_{t-1}$ in contradiction to the above RE-solution $\omega_t = \omega^n$.

3.2 NK wage-price dynamics with backward-dated real wage dynamics

It is asserted in Galí (2008, p.128) – and illustrated numerically – that the considered NK model is determinate for all policy parameters ϕ_{iw}, ϕ_{ip} when the following form of the Taylor principle holds: $\phi_{iw} + \phi_{ip} > 1$. We will show in section 5 that this indeed holds for all nonnegative values of the parameter ϕ_{iy} in front of the output gap. Galí (2008, 6.2) does not investigate the analytical foundations of his (numerical) determinacy analysis in the way we have approached it in the preceding section. Instead he uses for the handling of the term $\tilde{\omega}_t$ a backward oriented definitional expression in place of our forward dated definition. For the definitional change in the variable $\tilde{\omega}_t$ he therefore makes use of the alternative representation:

$$\tilde{\omega}_t/h - \tilde{\omega}_{t-h}/h \quad = \quad \omega_t/h - \omega_{t-h}/h = (w_t - w_{t-h})/h - (p_t - p_{t-h})/h = \pi_t^w - \pi_t^p$$

as relationship between this annualized real change and the rates of wage and price inflation.

The NK model completed in this way represents an implicitly formulated system of difference equations. Making use again of the TR and the PPC, and using the above revised form for the representation of $\tilde{\omega}_t$, it can be made a complete system of difference equations as in Galí (2008, 6.2) as follows:

$$\begin{aligned}
\pi_{t+h}^w &= \pi_t^w - h\beta_{wy}\tilde{y}_t + h\beta_{w\omega}\tilde{\omega}_t && (13)\\
\pi_{t+h}^p &= \pi_t^p - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t && (14)\\
\tilde{y}_{t+h} &= \tilde{y}_t + h\alpha_{yi}(\phi_{iw}\pi_t^w + \phi_{ip}\pi_t^p + \phi_{iy}\tilde{y}_t - (\pi_t^p - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t)) && (15)\\
\tilde{\omega}_t &= \tilde{\omega}_{t-h} + h(\pi_t^w - \pi_t^p) && (16)
\end{aligned}$$

In Galí's (2008, 6.2) matrix representation of this system it is based on its following formulation (with the state variables $\tilde{y}_t, \pi_t^p, \pi_t^w, \tilde{\omega}_{t-1}$):

$$\begin{aligned}
\beta\pi_{t+1}^w - \beta_{w\omega}\tilde{\omega}_t &= \pi_t^w - \beta_{wy}\tilde{y}_t\\
\beta\pi_{t+1}^p + \beta_{p\omega}\tilde{\omega}_t &= \pi_t^p - \beta_{py}\tilde{y}_t\\
\tilde{y}_{t+1}/\alpha_{yi} + \pi_{t+1}^p &= \phi_{iw}\pi_t^w + \phi_{ip}\pi_t^p + (1/\alpha_{yi} + \phi_{iy})\tilde{y}_t\\
\tilde{\omega}_t &= \pi_t^w - \pi_t^p + \tilde{\omega}_{t-1}
\end{aligned}$$

Galí's formulation of the system leads to a two matrices representation of the considered dynamics in the form of an implicit system of difference equations where the matrix corresponding to the forward-dated expression has to be inverted in order to get an explicit system of difference equations. As this system is formulated it suggest that the state variable ω_t is partly related to the past, and it is proposed in Galí (2008) to treat it as predetermined and to update it in every iteration by the nonpredetermined wage and price inflation rates of the current point in time.

But how can it happen in a model with completely forward-looking agents (as far as the active part of the model is concerned) that one derived state variable gets the status of being backward-looking simply by making use a definitional expression that is relating it to the past? Moreover: Since $\pi_t^p = (p_t - p_{t-1})/h$ and $\pi_t^w = (w_t - w_{t-1})/h$ and thus p_t, w_t are non-predetermined: Can this fact not be used to treat the difference between these two magnitudes, i.e., the real wage as a function of its future development as well and thus as a non-predetermined variable? The observations at the end of the preceding section seem to suggest that this cannot be done in a consistent manner if the real wage has not been in its steady state position initially.

Again, our hypothesis in this paper is that a discrete time (period) model should not depend in its fundamental qualitative properties on the length of the period h. i.e., we assume again that it reflects the properties of its continuous time analogue. We therefore finally get as continuous-time analogue to the above two matrix approach of Galí (2008) the form:

$$\begin{aligned}
\dot{\pi}^w &= -\beta_{wy}\tilde{y} + \beta_{w\omega}\tilde{\omega} \\
\dot{\pi}^p &= -\beta_{py}\tilde{y} - \beta_{p\omega}\tilde{\omega} \\
\dot{\tilde{y}} &= \alpha_{yi}((\phi_{ip} - 1)\pi^p + \phi_{iw}\pi^w + \phi_{iy}\tilde{y}) \\
\dot{\tilde{\omega}} &= \pi^w - \pi^p
\end{aligned}$$

which (of course) is the same system as the one considered in the preceding section. It therefore seems that in the limit and thus also for small periods h both discrete time models give the same answer when their dynamics is investigated from the perspective of the rational expectations school. There is however one important difference between the model of this and the preceding section. Galí (2008, 6.2) considers his period model from the perspective that it exhibits 3 non-predetermined variables $\pi_t^w, \pi_t^p, \tilde{y}_t$ and 1 predetermined one, $\tilde{\omega}_{t-h}$. The preceding section however was suggesting that all of these state variables are non-predetermined ones and this on the background that the mathematical model to be used for determinacy analysis is the same in the continuous time limit. We therefore have to investigate now the eigenvalue structure of the derived continuous time analogue in order to see what results we can get for the number of stable versus unstable roots.

4 Determinacy Analysis

4.1 Indeterminacy in the Forward-Oriented Case?

In order to show from the perspective of section 3.1 that there is always a pair of one unstable and one stable real root among the 4 eigenvalues of the 4D dynamics and thus to prove the result that the version of the 4D NK dynamics of section 3.1 is always indeterminate if four forward looking variables are assumed we proceed as follows:

A useful criterion

First, let us consider a simple but useful criterion for the total instability of the equilibrium point of the dynamical system in continuous time.

Lemma 1.

Let λ_j $(j = 1, 2, \cdots, n)$ be the eigenvalues of the $(n \times n)$ matrix J. Then, $\gamma_j = -\lambda_j$ $(j = 1, 2, \cdots, n)$ are eigenvalues of the matrix $Q = -J$.

Proof:

By assumption, we have $|\lambda_j I - J| = 0$. Then, we have

$$|\gamma_j I - Q| = |-\lambda_j I + J| = (-1)^n \, |\lambda_j I - J| = 0.$$

This proves the assertion.

Corollary of Lemma 1.

All the real parts of the eigenvalues of the $(n \times n)$ matrix J are positive if and only if all the real parts of the eigenvalues of the matrix $Q = -J$ are negative.

Proof:

Let $\lambda_j = \alpha_j + \beta_j i$ $(i = \sqrt{-1}, j = 1, 2, \cdots, n)$ be the eigenvalues of the matrix J. Then, it follows from **Lemma 1** that $\gamma_j = -\lambda_j = -\alpha_j - \beta_j i$ $(i = \sqrt{-1}, j = 1, 2, \cdots, n)$ are the eigenvalues of the matrix $Q = -J$. This proves the assertion.

Analysis of the 4D forward-dated NK model

Next, let us consider the model of the purely forward-looking dynamic NK model in section 3.1. This model consists of the following four linear differential equations.

(i) $\dot{\pi}^w = -\beta_{wy}\tilde{y} + \beta_{w\omega}\tilde{\omega}$

(ii) $\dot{\pi}^p = -\beta_{py}\tilde{y} - \beta_{p\omega}\tilde{\omega}$

(iii) $\dot{\tilde{y}} = \alpha_{yi}((\phi_{ip} - 1)\pi^p + \phi_{iy}\tilde{y} + \phi_{iw}\pi^w)$

(iv) $\dot{\tilde{\omega}} = \pi^w - \pi^p$ (S_1)

where β_{wy}, $\beta_{w\omega}$, β_{py}, $\beta_{p\omega}$, ϕ_{ip}, ϕ_{iw}, ϕ_{iy}, and α_{yi} are positive parameters.

The equilibrium solution of this system is such that

$$\dot{\pi}^w = \dot{\pi}^p = \dot{\tilde{y}} = \dot{\tilde{\omega}} = 0$$

is determined by the following system of equations.

$$J \begin{pmatrix} \pi^w \\ \pi^p \\ \tilde{y} \\ \tilde{\omega} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

where

$$J = \begin{pmatrix} 0 & 0 & -\beta_{wy} & \beta_{w\omega} \\ 0 & 0 & -\beta_{py} & -\beta_{p\omega} \\ \alpha_{yi}\phi_{iw} & \alpha_{yi}(\phi_{ip} - 1) & \alpha_{yi}\phi_{iy} & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{18}$$

is the Jacobian matrix of the system, and we have

$$\det J = \alpha_{yi}(1 - \phi_{ip} - \phi_{iw})(\beta_{wy}\beta_{p\omega} + \beta_{w\omega}\beta_{py}). \tag{19}$$

Therefore, we always have an equilibrium solution such that $\begin{pmatrix} \pi_0^w, & \pi_0^p, & y_0, & \tilde{\omega}_0 \end{pmatrix} = \begin{pmatrix} 0, & 0, & 0, & 0 \end{pmatrix}$, and it is unique if $\phi_{ip} + \phi_{iw} \neq 1$.

Now, let us define

$$Q = -J = \begin{pmatrix} 0 & 0 & \beta_{wy} & -\beta_{w\omega} \\ 0 & 0 & \beta_{py} & \beta_{p\omega} \\ -\alpha_{yi}\phi_{iw} & \alpha_{yi}(1 - \phi_{ip}) & -\alpha_{yi}\phi_{iy} & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \tag{20}$$

The characteristic equation of this matrix becomes as follows.

$$\Delta(\gamma) = |\gamma I - Q| = \gamma^4 + a_1\gamma^3 + a_2\gamma^2 + a_3\gamma + a_4 = 0 \tag{21}$$

Let γ_j $(j = 1,2,3,4)$ be the characteristic roots of Eq.(21), Then, we have (see Murata, 1977)

$$\begin{aligned} a_1 &= -\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 = -\text{trace } Q = \alpha_{yi}\phi_{iy} > 0 && (22) \\ a_2 &= \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_1\gamma_4 + \gamma_2\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_4 \\ &= \text{sum of all principal second-order minors of } Q \\ &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & \beta_{wy} \\ 0 & -\alpha_{yi}\phi_{iy} \end{vmatrix} + \begin{vmatrix} 0 & -\beta_{w\omega} \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & \beta_{py} \\ \alpha_{yi}(1-\phi_{ip}) & -\alpha_{yi}\phi_{iy} \end{vmatrix} \\ &\quad + \begin{vmatrix} 0 & \beta_{p\omega} \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -\alpha_{yi}\phi_{iy} & -\alpha_{yi}\beta_{p\omega} \\ 0 & 0 \end{vmatrix} \\ &= -\beta_{w\omega} - \alpha_{yi}\beta_{py}(1-\phi_{ip}) - \beta_{p\omega} + \alpha_{yi}\phi_{iw}\beta_{wy} && (23) \end{aligned}$$

$$\begin{aligned} a_3 &= -\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_4 - \gamma_1\gamma_3\gamma_4 - \gamma_2\gamma_3\gamma_4 \\ &= -(\text{sum of all principal third-order minors of } Q) \\ &= -\alpha_{yi}\begin{vmatrix} 0 & \beta_{py} & \beta_{p\omega} \\ (1-\phi_{ip}) & -\phi_{iy} & 0 \\ 1 & 0 & 0 \end{vmatrix} - \alpha_{yi}\begin{vmatrix} 0 & \beta_{wy} & -\beta_{w\omega} \\ 0 & -\phi_{iy} & 0 \\ -1 & 0 & 0 \end{vmatrix} \\ &\quad - \begin{vmatrix} 0 & 0 & -\beta_{w\omega} \\ 0 & 0 & \beta_{p\omega} \\ -1 & 1 & 0 \end{vmatrix} - \alpha_{yi}\begin{vmatrix} 0 & 0 & \beta_{wy} \\ 0 & 0 & \beta_{py} \\ 0 & 1-\phi_{ip} & -(\phi_{iy}+\beta_{py}) \end{vmatrix} \\ &= -\alpha_{yi}\phi_{iy}(\beta_{p\omega} + \beta_{w\omega}) < 0 && (24) \\ a_4 &= \gamma_1\gamma_2\gamma_3\gamma_4 = \det Q = \det J = \alpha_{yi}(1-\phi_{ip}-\phi_{iw})(\beta_{wy}\beta_{p\omega} + \beta_{w\omega}\beta_{py}) && (25) \end{aligned}$$

It follows from **Corollary of Lemma 1** that the equilibrium point of the system (S_1) is totally unstable if and only if the following Routh-Hurwitz conditions for the matrix $Q = -J$ are satisfied (cf. Asada, Chiarella, Flaschel and Franke 2003, p.519) and also Murata (1977)).

$$a_j > 0 \quad (j = 1,2,3,4), \quad a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0 \tag{26}$$

But, we have $a_3 < 0$ from eq. (24). Therefore, the equilibrium point of the system (S_1) cannot be totally unstable. In fact, we can prove the following result.

Proposition 1

> *The characteristic equation $|\lambda I - J| = 0$ has at least one root with positive real part and at least one negative real root. In other words, the equilibrium point of the system (S_1) becomes a proper saddle point. The steady state is therefore not the only bounded solution in this purely forward-looking approach.*

Proof:

From **Lemma 1** we have $\lambda_j = -\gamma_j$, where λ_j $(j = 1, 2, 3, 4)$ are the roots of the characteristic equation $|\lambda I - J| = 0$. Therefore, equations (22) and (24) mean that

$$\begin{aligned} a_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0 && (27) \\ a_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 < 0 && (28) \end{aligned}$$

This set of inequalities proves the assertion.

We thus get the result that the purely forward looking interpretation of the NK model with both staggered wages and prices is not well defined for monetary policy rules of Taylor type which are considered in Galí (2008, ch.6).

4.2 The Determinacy Case: Galí's Formulation

Galí (2008), on the basis of two numerical examples (see his section 6.2), asserts determinacy if there holds $\beta_{w\omega} + \beta_{p\omega} > 1$, i.e., there is to be shown then that there are always 3 unstable roots and 1 stable one of the matrix J and just the opposite for the matrix $Q = -J$ we investigated in the preceding section. We shall show in the following that this holds in his formulation of the 4D NK model for all cases $\phi_{iy} \geq 0$, which in sum provides a direct generalization of the determinacy analysis of the Wicksellian case where only price inflation is staggered.

We consider for this purpose again the eigenvalues of the matrix Q, given as before by

$$Q = -J = \begin{pmatrix} 0 & 0 & \beta_{wy} & -\beta_{w\omega} \\ 0 & 0 & \beta_{py} & \beta_{p\omega} \\ -\alpha_{yi}\phi_{iw} & \alpha_{yi}(1-\phi_{ip}) & -\alpha_{yi}\phi_{iy} & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \qquad (29)$$

Let γ_j $(j = 1, 2, 3, 4)$ be again the characteristic roots of the matrix Q. Then, we have

$$\begin{aligned}
a_1 &= -\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 = -\text{trace}\, Q = \alpha_{yi}\phi_{iy} > 0 && (30)\\
a_2 &= \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_1\gamma_4 + \gamma_2\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_4 \\
&= -\beta_{w\omega} - \alpha_{yi}\beta_{py}(1 - \phi_{ip}) - \beta_{p\omega} + \alpha_{yi}\phi_{iw}\beta_{wy} && (31)\\
a_3 &= -\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_4 - \gamma_1\gamma_3\gamma_4 - \gamma_2\gamma_3\gamma_4 \\
&= -\alpha_{yi}\phi_{iy}(\beta_{p\omega} + \beta_{w\omega}) < 0 && (32)\\
a_4 &= \gamma_1\gamma_2\gamma_3\gamma_4 = \det Q = \det J = \alpha_{yi}(1 - \phi_{ip} - \phi_{iw})(\beta_{wy}\beta_{p\omega} + \beta_{w\omega}\beta_{py}) && (33)
\end{aligned}$$

On the basis of these expressions for the eigenvalues of the matrix Q we get:

Proposition 2

> *The characteristic equation $|\lambda I - J| = 0$ has 3 roots with positive real parts and 1 negative root for all positive parameter values of the model if the generalized Taylor principle $\phi_{ip} + \phi_{iw} > 1$ holds true.*

Proof: We consider first Galí's (2008, 6.2) case $\phi_{iy} = 0$ and assume for the time being in addition that $\phi_{ip}+\phi_{iw} = 1$ holds. In this case we have $a_1 = a_3 = a_4 = 0$ and get from this that two roots of the matrix Q must be zero and the other two real and of opposite sign. Let us now move away from Galí's special case and consider $\phi_{iy} > 0$ (assumed to be sufficiently small). In this case we have $a_1 > 0, a_3 < 0, a_4 = 0$. There is then still one zero root, but the other zero root must now be negative (due to $a_3 < 0$). Assume now moreover that $\phi_{ip} + \phi_{iw} > 1$ holds (sufficiently close to 1). Since $a_4 < 0$ holds in this case we have that the remaining zero eigenvalue must have become negative. The considered case therefore implies for the matrix J the existence of 3 unstable roots and 1 stable one, as was claimed by Galí (2008, 6.2). In order to show that this result can be extended to arbitrarily large parameter variations and not only holds for the small variations so far considered we have to show that a Hopf bifurcation (whereby 2 complex eigenvalues of Q pass the imaginary from the right to the left) is not possible (since $a_4 < 0$ this is already excluded as possibility with respect to a single eigenvalue). In the case of such a Hopf bifurcation we would however have as eigenvalue structure the situation $\gamma_1 = a\sqrt{-1}, \gamma_2 = -a\sqrt{-1}, \gamma_3 = b, \gamma_4 = -c$ with a, b, c being positive numbers. For the Routh Hurwitz coefficients a_1, a_3 we therefrom get:

$$a_1 = -[b - c], a_3 = -a^2[b - c].$$

This however is impossible since these coefficients are of opposite sign. This proves proposition 2.[5]

Remark: By continuity this result also holds for Galí's special case $\phi_{iy} = 0$, since a negative determinant prevents a sign switch of the real parts of eigenvalues when this limit case is approached.

The proposition 2 implies that there is a uniquely determined 'eigendirection' $x_o = (\pi_o^w, \pi_o^p, \tilde{y}_o, \tilde{\omega}_o)'$ of the matrix $J : Jx_o = \lambda x_o$, with $\lambda < 0$ for the real eigenvalue of this eigenvector. We get from this fact for Galí's version of the 4 difference equations

$$\begin{aligned}
\pi_{t+h}^w &= \pi_t^w - h\beta_{wy}\tilde{y}_t + h\beta_{w\omega}\tilde{\omega}_t & (34)\\
\pi_{t+h}^p &= \pi_t^p - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t & (35)\\
\tilde{y}_{t+h} &= \tilde{y}_t + h\alpha_{yi}(\phi_{iw}\pi_t^w + \phi_{ip}\pi_t^p + \phi_{iy}\tilde{y}_t - (\pi_t^p - h\beta_{py}\tilde{y}_t - h\beta_{p\omega}\tilde{\omega}_t)) & (36)\\
\tilde{\omega}_t &= \tilde{\omega}_{t-h} + h(\pi_t^w - \pi_t^p) & (37)
\end{aligned}$$

the relationship

$$A = hJ(h),\ J(h) \to J \text{ for } h \to 0 \text{ with } x_{t+h} = (A+I)x_t, x_t = (\pi_t^w, \pi_t^p, \tilde{y}_t, \tilde{\omega}_{t-h})']$$

where A + I denotes the system matrix of the above four difference equations. For period lengths h chosen sufficiently small we can ensure by continuity of the roots of the characteristics polynomial with respect to the parameters of the characteristic equation that the real parts of the roots of the matrix $hJ(h)$ have the same sign distribution as the ones in the continuous time limit case, and that in addition the negative real root λ lies in the interval $(-1, 0)$. There is therefore a uniquely determined root of the matrix $A + I$ within the unit circle, while the other ones are to the right of 1, since their real parts are positive in the continuous time limit case.

We denote the eigenvector of this root by $z_o = (\pi_o^w, \pi_o^p, \tilde{y}_o, \tilde{\omega}_o) \neq 0$. From the existence of this root we get that the equation

$$\alpha\tilde{\omega}_o = \tilde{\omega}_{t-h} + \alpha h\pi_o^w - \alpha h\pi_o^p$$

exhibits a unique solution for $\alpha \in \Re$ for any given $\tilde{\omega}_{t-h}$, since $\tilde{\omega}_o - h\pi_o^w + h\pi_o^p \neq 0$ must hold true on the stable manifold.

In the case $\tilde{\omega}_{t-h} = 0$, i.e., the case where the steady state is a solution of the Galí model $A+I$. we have $\alpha = 0$ so that we get that the identity $\tilde{\omega}_t = \tilde{\omega}_{t-h} + h(\pi_t^w - \pi_t^p)$ of

[5] This proposition of course also implies the validity of proposition 1, at least for the case where $\phi_{ip} + \phi_{iw} > 1$ holds true.

the Gali approach can only be valid in the steady state of the corresponding dynamics (if the solutions to this equation are restricted to the above stable manifold and its attracting point).

Moreover, in the case where $\tilde{\omega}_{t-h} \neq 0$ is given as past of the considered economy, the economy jumps to the unique point on the stable manifold where

$$\alpha\tilde{\omega}_o - \alpha h\pi_o^w + \alpha h\pi_o^p = \tilde{\omega}_{t-h} \neq 0$$

holds true (if the rational expectations methodology is applied). The economy therefore jumps to the point on the stable manifold, fulfilling

$$(\pi_t^w, \pi_t^p, \tilde{y}_t, \tilde{\omega}_{t-h}) = (\alpha\pi_o^w, \alpha\pi_o^p, \alpha\tilde{y}_o, \tilde{\omega}_{t-h}) \text{ with } \tilde{\omega}_t = \alpha\tilde{\omega}_o = \tilde{\omega}_{t-h} + h(\pi_t^w - \pi_t^p)$$

and converges from there to its steady state (the origin of the phase space) with speed λ.

In the continuous time case ($h = 0$) this means that the condition $\alpha\tilde{\omega}_o = \tilde{\omega}(0)$ must hold true, i.e., the economy is then simply started from the initial jump situation

$$(\pi^w, \pi^p, \tilde{y}, \tilde{\omega}(0)) = (\alpha\pi_o^w, \alpha\pi_o^p, \alpha\tilde{y}_o, \tilde{\omega}(0)).$$

We thus get that the Galí model always implies a unique jump of its state variables $\pi_t^w, \pi_t^p, \tilde{y}_t$ such that its real wage identity is fulfilled, i.e., it is determinate in its deterministic core. The implication of this result is that we get an (economically trivial) 1-dimensional adjustment process for the real wage in the deterministic core if the historically given real wage is not in its steady state position. The 1D deterministic core dynamics thus separates the model with both staggered wages and prices from its predecessor where only prices were reacting in a staggered way. Though the price readjusting firms and the wage readjusting households are purely forward looking in their behavior, there is nevertheless real wage inertia in the rational expectations solution of the considered dynamical system in its deterministic core, not generated by an actual interest smoothing anchor through the behavior of the central bank, but simply due to a definitional identity that relates the current real wage gap to the one of the preceding period.

5 Conclusions

We have considered in this note the NK baseline model with both staggered wages and prices and shown that there exist two ways to reduce it to a complete system of difference equations. One way sticks to the fact that the behavioral part of the model is a purely forward-looking one and thus transforms it in a way such that 4 forward-looking variables are associated with its 4 difference equations. The other approach introduced a backward-oriented state variable into the 4D dynamics and concludes on this basis that there are only three instead of four forward-looking variables to be considered. On balance, the backward-dated version appears to be the more coherent one, due to what was established in sections 4.1 and 4.2. In our view however this result needs further discussion in order to clarify in more detail the working of this version not only in its deterministic core, but also in a stochastic environment where the role of the definitionally motivated real wage rigidity (its movement along a one-dimensional stable sub-manifold) should be studied both from the theoretical as well as from the numerical perspective.

The fundamental result of this paper is however that there is an easy way to conduct determinacy analysis also in 4D situation where period analysis is based on more complex types of matrices and – more importantly – on Routh-Hurwitz type stability conditions that are very hard to apply, see the mathematical appendices in Woodford (2003) for the difficulties that exist already in the 3D case.

As an example, consider the explicit backward-based dynamics of Galí's version of the NK dynamics (with the state variables $\tilde{y}_t, \pi_t^p, \pi_t^w, \omega_{t-1}$). These dynamics read (with $\beta < 1$ and with $h = 1$ as is customary in the New Keynesian framework):

$$\begin{aligned}
\pi_{t+1}^w &= [\pi_t^w - \beta_{wy}\tilde{y}_t + \beta_{w\omega}[\pi_t^w - \pi_t^p + \tilde{\omega}_{t-1}]]/\beta \\
\pi_{t+1}^p &= [\pi_t^p - \beta_{py}\tilde{y}_t - \beta_{p\omega}[\pi_t^w - \pi_t^p + \tilde{\omega}_{t-1}]]/\beta \\
\tilde{y}_{t+1} &= \alpha_{yi}\left[(1/\alpha_{yi} + \phi_{iy})\tilde{y}_t + \phi_{ip}\pi_t^p + \phi_{iw}\pi_t^w\right. \\
&\quad \left.+\{\beta_{py}\tilde{y}_t - \pi_t^p + \beta_{p\omega}[\pi_t^w - \pi_t^p + \tilde{\omega}_{t-1}]\}/\beta\right] \\
\tilde{\omega}_t &= \pi_t^w - \pi_t^p + \tilde{\omega}_{t-1}
\end{aligned}$$

We could show the general validity of Galí's determinacy assertions by simplifying this matrix to the form

$$\begin{aligned}
\dot{\pi}^w &= -\beta_{wy}\tilde{y} + \beta_{w\omega}\tilde{\omega} \\
\dot{\pi}^p &= -\beta_{py}\tilde{y} - \beta_{p\omega}\tilde{\omega} \\
\dot{\tilde{y}} &= \alpha_{yi}((\phi_{ip} - 1)\pi^p + \phi_{iw}\pi^w + \phi_{iy}\tilde{y}) \\
\dot{\tilde{\omega}} &= \pi^w - \pi^p
\end{aligned}$$

which not only is a simpler structured matrix for determinacy analysis, but also a version of the model where for example the conditions for complete instability simply read

$$a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0, \quad a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0$$

If this approach is valid for the period length considered in the NK framework, it makes determinacy analysis much easier then in the 2D and 3D cases that are studied in Woodford (2003).

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