Ruhr- Universität Bochum

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Technical Inefficiencies and Profit-Maximization

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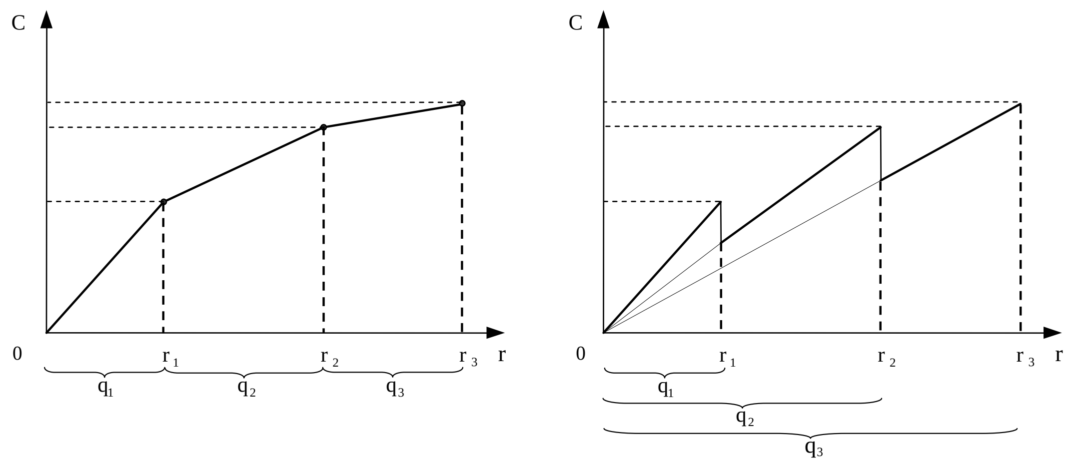
1 Introduction

The theory of production focuses on the quantitative analysis of input/ output-relations. Based on the economic principle, technical efficiency is considered as a matter of rationality and as a necessary condition for profit-optimization. Due to this assumption, technically inefficient production alternatives are generally excluded from traditional considerations. In contrast, the objective of this paper is to demonstrate that it is not reasonable to focus only on technically efficient production alternatives in order to maximize profit.

2 Effects of Quantity Discounts

From a traditional perspective, technical efficiency is based on rare goods and perfect markets and therefore implies strictly positive and constant input- and output-prices. In reality, however, on the one hand there exist non-rare goods and on the other hand there are also imperfect markets with variable prices. Therefore, the assumption of strictly positive and constant prices is dropped. The further analysis concentrates on quantity discounts as a tool of nonlinear price discrimination. For this, two types of quantity discount are distinguished. In the case of all-unit quantity discount the lower price applies to all units purchased, while in the case of incremental quantity discount the new price refers only to those units within the corresponding interval [1][4]. Figure 1 illustrates the slope of a cost function depending on the two different types of quantity discount.[1] This results in a non-monotonic profit function, so that technically inefficient production alternatives may also maximize profit [3], which is analysed in the subsequent chapter.



(a) incremental quantity discount

(b) all-unit quantity discount

Figure 1: Cost function depending on quantity discounts

[1]Cf. e.g. [4], pp. 63, 66.

3 Analysis of technically inefficient Profit-Maxima

Based on the approach [3] and in consideration of the respective types of quantity discount, different price situations are analysed in order to demonstrate that the optimal combination of input/output-relations might be located in the interior of the technology set.[2]

3.1 The Optimization Approach

Assume that an enterprise uses only one kind of input $r \in \Re_+$ to produce one kind of output $x \in \Re_+$ with the exclusive objective of profit maximization.[3] Additionally, stock-keeping is excluded from the analysis because any produced output quantity is sold immediately. Furthermore, all technically feasible input/output-combinations are denoted by the technology set T and the efficient boundary is specified by a (explicit) production function:

$$T_{eff} = \{(r, x) \in T \,|x = f(r)\}\,. \tag{1}$$

Thus, the technology set is defined as

$$T = \{(r, x) \in \Re_+ \times \Re_+ \,|x \leq f(r)\}\,. \tag{2}$$

In case of quantity discounts, the selling price $p\,(x)$ varies with the output-quantity and the input price $q\,(r)$ depends on the quantity of input. Pursuant, the enterprise obtains the following revenue and cost function

$$R\,(x) = p\,(x) \cdot x \quad \text{and} \quad C\,(r) = q\,(r) \cdot r \tag{3}$$

and hence the profit function

$$\Pi\,[r, x] = p\,(x) \cdot x - q\,(r) \cdot r. \tag{4}$$

To identify technical (in)efficiency with regard to the profit maximum (r^*, x^*) the non-linear optimization problem has to be solved:

$$\max_{(r,x)\in\, T} \Pi\,(r, x) = p\,(x) \cdot x - q\,(r) \cdot r \tag{5}$$

$$\text{subject to} \qquad x \leq f\,(r)$$

$$r, x \geq 0.^4$$

Based on the Lagrangian

$$L\,(\lambda, r, x) = p\,(x) \cdot x - q\,(r) \cdot r - \lambda\,[x - f\,(r)] \tag{6}$$

[2]The considerations based on the Linear Activity Analysis [5].
[3]Cf. in the following [3], pp. 3-7.
[4]The profit and production function are continuously differentiable.

the solution to this problem has to satisfy the Kuhn-Tucker (KT-) conditions [6][2]:

$$\text{(a)} \quad r \geq 0, \quad \frac{\partial L}{\partial r} = -\frac{dq(r)}{dr} \cdot r - q(r) + \lambda \cdot \frac{df(r)}{dr} \leq 0, \quad r \cdot \frac{\partial L}{\partial r} = 0,$$

$$\text{(b)} \quad x \geq 0, \quad \frac{\partial L}{\partial x} = \frac{dp(x)}{dx} \cdot x + p(x) - \lambda \leq 0, \quad x \cdot \frac{\partial L}{\partial x} = 0,$$

$$\text{(c)} \quad \lambda \geq 0, \quad \frac{\partial L}{\partial \lambda} = -\left[x - f(r)\right] \geq 0, \quad \lambda \cdot \frac{\partial L}{\partial \lambda} = 0.$$ [5]

In consideration of these conditions, two economically relevant solutions can be identified:[6] (1) technically feasible production alternatives with marginal revenue > 0, marginal costs > 0 and marginal productivity > 0, which imply a positive Lagrangian multiplier λ. Because of that, the technology constraint is binding ($x = f(r)$) and technical efficiency is indispensable. (2) Technically feasible production alternatives with marginal revenue $= 0$, marginal costs $= 0$ and marginal productivity > 0, which require a Lagrangian multiplier λ of zero. As a consequence, the technology constraint is non-binding ($x \leq f(r)$) and the solution may be technically efficient but do not need to be.[7]

3.2 Consideration of different Price Situations

Assume that the dependency of output price and output quantity is described by a linear price-response-function with non-negative output price:[8]

$$p(x) = a - b \cdot x, \quad a, b > 0. \tag{7}$$

Due to that, the revenue function is quadratic:[9]

$$R(x) = a \cdot x - b \cdot x^2. \tag{8}$$

Let three different, constant input prices vary with the quantity of input ($0 < \tilde{r} < \tilde{\tilde{r}}$), the cost function regarding the incremental quantity discount then is piecewise linear

$$C(r) = \begin{cases} q_1 \cdot r & , 0 \leq r < \tilde{r} \\ q_1 \cdot \tilde{r} + q_2 \cdot (r - \tilde{r}) & , \tilde{r} \leq r < \tilde{\tilde{r}} \\ q_1 \cdot \tilde{r} + q_2 \cdot \left(\tilde{\tilde{r}} - \tilde{r}\right) + q_3 \cdot \left(r - \tilde{\tilde{r}}\right) & , \tilde{\tilde{r}} \leq r. \end{cases} \tag{9}$$

[5]If the quasi-concave programming conditions hold, the problem has an unique solution (see for further information [3], p. 5).

[6]For a detailled representation cf. [3], pp. 5-7.

[7]Corner solutions ($r = 0$, $x = 0$) are economically less interesting; solutions with marginal revenue/costs/productivity < 0 are excluded from the considerations.

[8]Cf. in the following [3], pp. 9-11.

[9]This function is strictly concave, continuously differentiable and has an unique maximum with marginal revenue $= 0$ for $\hat{x}$. For sales volume $< \hat{x}$ the marginal revenue is positive and for sales volume $> \hat{x}$ it is negative (cf. [3], p. 10).

The profit function is specified as:

$$\Pi\left[r,x\right]=\begin{cases} a\cdot x-b\cdot x^2-q_1\cdot r & ,0\leq r<\tilde{r}\\ a\cdot x-b\cdot x^2-q_1\cdot\tilde{r}-q_2\cdot(r-\tilde{r}) & ,\tilde{r}\leq r<\tilde{\tilde{r}}\\ a\cdot x-b\cdot x^2-q_1\cdot\tilde{r}-q_2\cdot\left(\tilde{\tilde{r}}-\tilde{r}\right)-q_3\cdot\left(r-\tilde{\tilde{r}}\right) & ,\tilde{\tilde{r}}\leq r. \end{cases} \tag{10}$$

With regard to the production function the profit maximum is determined by the iso-profit curves in consideration of an arbitrarily chosen, constant profit level $\bar{\Pi}$:

$$\begin{aligned} 0\leq r<\tilde{r}: &\qquad x_{1,2}=\tfrac{a}{2b}\pm\sqrt{\left(-\tfrac{a}{2b}\right)^2-\tfrac{q_1\cdot r+\bar{\Pi}}{b}}\\ \tilde{r}\leq r<\tilde{\tilde{r}}: &\qquad x_{1,2}=\tfrac{a}{2b}\pm\sqrt{\left(-\tfrac{a}{2b}\right)^2-\tfrac{q_1\cdot r+q_2\cdot(r-\tilde{r})+\bar{\Pi}}{b}}\\ \tilde{\tilde{r}}\leq r: &\qquad x_{1,2}=\tfrac{a}{2b}\pm\sqrt{\left(-\tfrac{a}{2b}\right)^2-\tfrac{q_1\cdot r+q_2\cdot\left(\tilde{\tilde{r}}-\tilde{r}\right)+q_3\cdot\left(r-\tilde{\tilde{r}}\right)+\bar{\Pi}}{b}}. \end{aligned} \tag{11}$$

As a simplification the linear production function

$$x=m\cdot r,\qquad m>0 \tag{12}$$

is assumed. Figure 2(a) depicts the cost situation for $q_1>q_2>q_3$, in which the cost function increases monotonically, possesses kink points and positive marginal costs. In consideration of the KT-conditions and positive marginal revenue, the positive

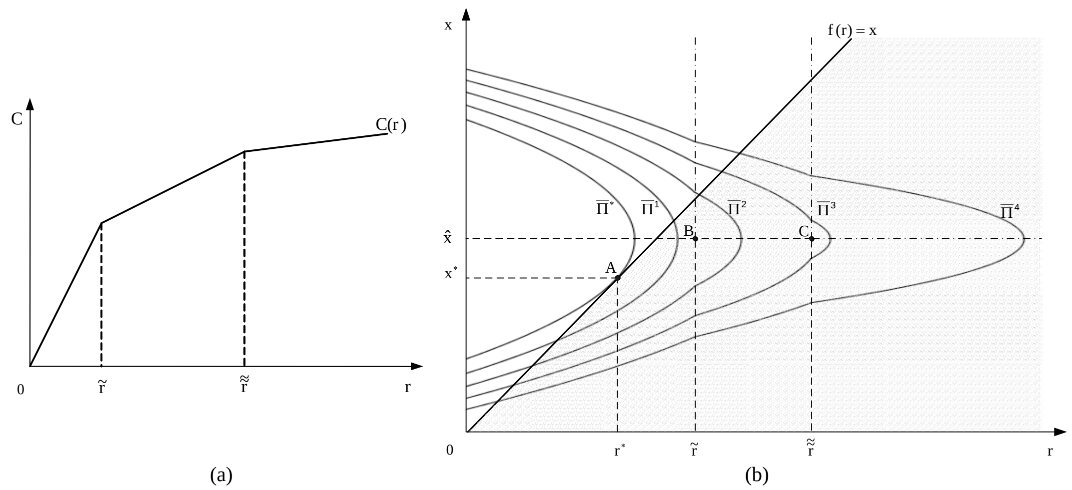


Figure 2: Profit-Maximization and monotonic cost function with kink points

marginal costs implicate a Lagrangian multiplier greater than zero and due to that, a technically efficient profit maximum ($A=(r^*,x^*)\in T_{eff}$ on the iso-profit curve $\bar{\Pi}^*$ in figure 2(b)). If there exists a technically inefficient maximum it can only be located at the kinks of the cost function, because no information about the marginal costs is obtainable. A technical inefficient production possesses zero marginal revenue and due to this a non-binding technology constraint. Such production points (B and C), which fulfil these conditions and are located in the interior of the technology set are depicted in figure 2(b). In contrast, point A yields the highest profit regarding

$\bar{\Pi}^* > \bar{\Pi}^1 > \bar{\Pi}^2 > \bar{\Pi}^3 > \bar{\Pi}^4$. As a consequence, monotonically increasing costs cause a technically efficient profit maximum.

In the case of all-unit quantity discounts assume the following piecewise linear cost function:

$$C(r) = \begin{cases} q_1 \cdot r & , 0 \leq r < \tilde{r} \\ q_2 \cdot r & , \tilde{r} \leq r. \end{cases} \tag{13}$$

In conjunction with the revenue function (8) the profit function is defined as:

$$\Pi[r, x] = \begin{cases} a \cdot x - b \cdot x^2 - q_1 \cdot r & , 0 \leq r < \tilde{r} \\ a \cdot x - b \cdot x^2 - q_2 \cdot \tilde{r} & , \tilde{r} \leq r \end{cases} \tag{14}$$

and the iso-profit curves are deduced as:

$$\begin{aligned} 0 \leq r < \tilde{r}: \qquad & x_{1,2} = \tfrac{a}{2b} \pm \sqrt{\left(-\tfrac{a}{2b}\right)^2 - \tfrac{q_1 \cdot r + \bar{\Pi}}{b}} \\ \tilde{r} \leq r: \qquad & x_{1,2} = \tfrac{a}{2b} \pm \sqrt{\left(-\tfrac{a}{2b}\right)^2 - \tfrac{q_2 \cdot r + \bar{\Pi}}{b}}. \end{aligned} \tag{15}$$

Assuming $q_1 > q_2$, the cost function has a saltus and therefore is non-monotonically increasing as shown in figure 3(a). For $r < \tilde{r}$ and $r > \tilde{r}$ the marginal costs are positive

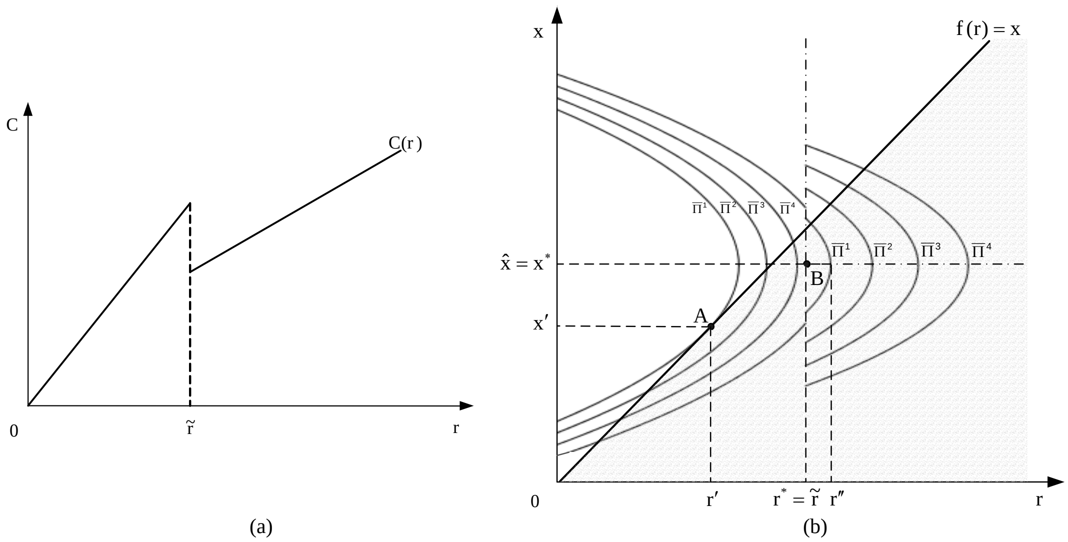


Figure 3: Profit-Maximization and non-monotonic cost function with saltus

and in consideration of the KT-conditions, this implies a technical efficient profit maximum. Therefore, a technically inefficient production point, which maximizes profit exists at the saltus of the cost function ($r = \tilde{r}$). At the same time such a production point has to satisfy the KT-conditions and hence possesses zero marginal revenue. As shown in figure 3(b) the technically inefficient point B maximizes profit ($\bar{\Pi}^*$) while point $A \in T_{eff}$ (a technically efficient production point) yields the highest profit in comparison to all other technically efficient input/output-combinations. However, point B dominates point A because $\bar{\Pi}^* > \bar{\Pi}^1 > \bar{\Pi}^2 > \bar{\Pi}^3 > \bar{\Pi}^4$.

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