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Sebastian Siebertz

Nowhere Dense Classes of Graphs

Characterisations and Algorithmic Meta-Theorems

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Kurzfassung

Wir zeigen, dass jede Eigenschaft von Graphen aus einer nowhere dense Klasse von Graphen, die in der Prädikatenlogik formuliert werden kann, in fast linearer Zeit $O(n^{1+\epsilon})$ entschieden werden kann. Dieses Ergebnis ist optimal für Klassen von Graphen, die unter Subgraphen abgeschlossen sind (unter einer Standardannahme aus der Komplexitätstheorie).

Um den obigen Satz zu beweisen, führen wir zwei neue Charakterisierungen von nowhere dense Klassen von Graphen ein. Zunächst charakterisieren wir solche Klassen durch ein Spiel, das die lokalen Eigenschaften von Graphen beschreibt. Weiter zeigen wir, dass eine Klasse, die unter Subgraphen abgeschlossen ist, genau dann nowhere dense ist, wenn alle lokalen Nachbarschaften von Graphen der Klasse dünn überdeckt werden können. Weiterhin beweisen wir eine erweiterte Version von Gaifman's Lokalitätssatz für die Prädikatenlogik, der eine Übersetzung von Formeln in lokale Formeln des gleichen Ranges erlaubt. In Kombination erlauben diese neuen Charakterisierungen einen effizienten, rekursiven Lösungsansatz für das Model-Checking Problem der Prädikatenlogik.

Die Charakterisierung der nowhere dense Graphklassen durch die oben beschriebenen Überdeckungen basiert auf einer bekannten Charakterisierung durch verallgemeinerte Färbungszahlen. Unser Studium dieser Zahlen führt zu neuen, verbesserten Schranken für die verallgemeinerten Färbungszahlen von nowhere dense Klassen von Graphen, insbesondere für einige wichtige Subklassen, z. B. für Klassen mit ausgeschlossenen Minoren und für planare Graphen.

Zuletzt untersuchen wir, welche Auswirkungen eine Erweiterung der Logik durch Ordnungs- bzw. Nachfolgerrelationen auf die Komplexität des Model-Checking Problems hat. Wir zeigen, dass das Problem auf fast allen interessanten Klassen nicht effizient gelöst werden kann, wenn eine beliebige Ordnungs- oder Nachfolgerrelation zum Graphen hinzugefügt wird. Andererseits zeigen wir, dass das Problem für ordnungsinvariante monadische Logik zweiter Stufe auf allen Klassen, für die bekannt ist, dass es für monadische Logik zweiter Stufe effizient gelöst werden kann, auch effizient gelöst werden kann. Wir zeigen, dass das Problem für nachfolgerinvariante Prädikatenlogik auf planaren Graphen effizient gelöst werden kann.

Abstract

We show that every first-order property of graphs can be decided in almost linear time $O(n^{1+\epsilon})$ on every nowhere dense class of graphs. For graph classes closed under taking subgraphs, our result is optimal (under a standard complexity theoretic assumption): it was known before that for all classes $\mathscr{C}$ of graphs closed under taking subgraphs, if deciding first-order properties of graphs in $\mathscr{C}$ is fixed-parameter tractable, parameterized by the length of the input formula, then $\mathscr{C}$ must be nowhere dense.

Nowhere dense graph classes form a large variety of classes of sparse graphs including the class of planar graphs, actually all classes with excluded minors, and also bounded degree graphs and graph classes of bounded expansion. For our proof, we provide two new characterisations of nowhere dense classes of graphs. The first characterisation is in terms of a game, which explains the local structure of graphs from nowhere dense classes. The second characterisation is by the existence of sparse neighbourhood covers. On the logical side, we prove a rank-preserving version of Gaifman's locality theorem.

The characterisation by neighbourhood covers is based on a characterisation of nowhere dense classes by generalised colouring numbers. We show several new bounds for the generalised colouring numbers on restricted graph classes, such as for proper minor closed classes and for planar graphs.

Finally, we study the parameterized complexity of the first-order model-checking problem on structures where an ordering is available to be used in formulas. We show that first-order logic on ordered structures as well as on structures with a successor relation is essentially intractable on nearly all interesting classes. On the other hand, we show that the model-checking problem of order-invariant monadic second-order logic is tractable essentially on the same classes as plain monadic second-order logic and that the model-checking problem for successor-invariant first-order logic is tractable on planar graphs.

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Contents

Introduction

A *graph* consists of *vertices* and *edges* connecting the vertices. The concept of modelling relations between objects as a graph is one of the most important and fundamental concepts in mathematics with many applications in other disciplines. Its most important applications are probably in computer science, where graphs are used to represent networks of communication or transportation, computational devices, flows of computation and many more. Once a graph theoretical formulation of a real world situation has been established, it is a challenging task to efficiently solve various optimization problems on the resulting graph instances. For instance, a telecommunication network may be modelled by a graph in which terminals are represented by vertices and transmission links between the terminals by edges of the graph. An example of an optimization problem in a communication network is to place a minimum number of communication antennas to cover all important locations with wireless network. In its graph theoretic formulation the problem corresponds to the problem of finding a small set of vertices in the network graph which dominate all other vertices, that is, which are connected to all other vertices by an edge.

It has long been realized that a large class of important algorithmic problems seems to evade all attempts of solving them efficiently in general. The meaning of efficiency and tractability varies, for example, it may be polynomial time solvability, fixed-parameter tractability, or polynomial time approximability to some ratio. For example, the dominating set problem on the class of all graphs is NP-hard, W[2]-hard and Log-APX-hard, hence considered intractable with respect to all of the above notions of efficiency. However, on restricted graph classes, the complexity of a problem may be quite different from the general worst-case complexity. In particular, instances of graphs arising in applications often have more structure than general graphs. For instance, road or railway maps correspond to nearly planar graphs and telecommunication networks are modelled by sparse graphs, that is, by graphs with a moderate number of edges.

Researchers have studied many structural properties of graphs which can be used to design efficient algorithms for otherwise hard problems. Among the most important ones are properties of planar graphs or, much more generally, properties of graph classes that exclude a fixed minor. Robertson and Seymour [119] developed a celebrated structure theory of graphs with excluded minors which had an immense influence on the design of efficient algorithms [32, 33]. In particular the concept of tree-width, which they introduced as part of their graph minors project, proved extremely valuable in the algorithmic context [14].

The classification of (infinite) classes of finite graphs into nowhere dense and somewhere dense classes was introduced by Nešetřil and Ossona de Mendez in [104, 105]. Many familiar sparse graph classes, like proper minor closed classes, classes of bounded degree and, most generally, classes of locally bounded expansion are nowhere dense. Nowhere density turns out to be a very robust concept with several seemingly unrelated natural characterisations. This fact has very nice algorithmic consequences, as each characterisation yields different algorithmic techniques. For instance, Dawar and Kreutzer [31] showed that problems such as network centres and dominating sets can be solved by fixed-parameter algorithms on nowhere dense classes of graphs using uniformly quasi-wideness. Using low tree-depth colourings, Nešetřil and Ossona de Mendez [104] showed that the subgraph isomorphism or homomorphism problem is fixed-parameter tractable on nowhere dense classes. Using the same characterisation, Gajarsky et al. [55] extended the meta-kernelisation framework of Bodlaender et al. [15] to nowhere dense classes of graphs providing polynomial kernels for a large number of algorithmic problems. On the important subclass of bounded expansion graphs, even more algorithmic applications are known, for instance in database query answering and enumeration [73], which relies on the concept of augmentations, or in approximating dominating sets [41], which is based on generalised colouring numbers.

Algorithmic meta theorems attempt to explain and unify algorithmic results by proving tractability not only for individual problems, but for whole classes of problems and are therefore some of the most sought-after theorems in algorithmic research. The prototypical example of an algorithmic meta theorem is Courcelle's Theorem [22], stating that all properties of graphs of bounded tree-width that are definable in monadic second-order logic are decidable in linear time. Another well-known example is Papadimitriou and Yannakakis's [113] result that all optimisation problems in

the class MAXSNP, which is defined in terms of a fragment of existential second-order logic, admit constant-ratio polynomial time approximation algorithms or the result of Dawar et al. [29] that every first-order definable optimisation problem admits a PTAS on any proper minor closed class of graphs. By now, there is a rich literature on algorithmic meta theorems (see for example [15, 24, 25, 28, 29, 42, 53, 83, 84, 125] and the surveys [61, 63, 62, 82]). In general, the main goals of this whole line of research, sometimes referred to as *algorithmic graph structure theory*, are to understand for natural and important classes of graphs what kind of problems can be solved efficiently on these graphs and develop the corresponding graph structural and algorithmic techniques. In particular, for natural classes of problems we want to understand their general tractability frontier, i.e. the most general classes of graphs on which these problems are tractable.

The quest of finding the largest classes of graphs where problems definable in first-order logic are tractable has been pursued intensively in research on algorithmic meta-theorems. First-order definable problems define a natural class of problems including dominating sets, vertex covers, network centres and many others. Deciding first-order properties of graphs in general is known to be complete for the parameterized complexity class AW[$\star$] and thus widely believed to be not fixed-parameter tractable [38].

There is a long list of meta theorems for first-order logic. The starting point is Seese's [125] result that first-order properties of bounded degree graphs can be decided in linear time. Frick and Grohe [53] gave linear time algorithms for planar graphs and all apex-minor-free graph classes and $O(n^{1+\epsilon})$ algorithms for graphs of bounded local tree-width. Flum and Grohe [50] proved that deciding first-order properties is fixed-parameter tractable on graph classes with excluded minors, and Dawar, Grohe and Kreutzer [28] extended this to classes of graphs locally excluding a minor. Finally, Dvořák, Král and Thomas [42] proved that first-order properties can be decided in linear time on graph classes of bounded expansion and in time $O(n^{1+\epsilon})$ on classes of locally bounded expansion.

The main contribution of this thesis is the presentation of a new meta theorem for first-order logic on nowhere dense classes of graphs. We proved in [65] that every first-order property can be decided in time $O(n^{1+\epsilon})$ on every nowhere dense class of graphs.
What makes our theorem particularly interesting is not primarily that it is yet another extension of the previous results, but that it is optimal for

classes $\mathscr{C}$ closed under taking subgraphs: under the standard complexity theoretic assumption FPT $\neq$ W[1], Kreutzer [82] and Dvořák et al. [42] proved that if a class $\mathscr{C}$ closed under taking subgraphs is somewhere dense, then deciding first-order properties of graphs in $\mathscr{C}$ is not fixed-parameter tractable (parameterized by the length of the input formula). Note that all classes considered in the previous results are closed under taking subgraphs. Hence our result supports the intuition that nowhere dense classes are the natural limit for many algorithmic techniques for sparse graph classes.

For the proof of our main theorem we extend the theory of nowhere dense graphs in several ways. Our starting point is Gaifman's Theorem [54] which states that every first-order sentence φ is equivalent to a Boolean combination of *basic local sentences*. Such a sentence states the existence of a fixed number (depending on the sentence φ) of wide-spread elements of a certain type which depends only on their local neighbourhood. Gaifman's Theorem gives rise to a model-checking method which was first employed by Frick and Grohe in [53]. We can first compute the local type of each element and then solve the generalised independent set problems described in the basic local sentences. Hence, to use this approach, it is a key requirement to understand the structure of local neighbourhoods in a graph (and to be able to exploit this structure algorithmically).

We provide a new characterisation of nowhere dense classes of graphs which describes the local neighbourhoods of their elements in terms of a game, called the *splitter game*. From this game we can derive a structural decomposition of local neighbourhoods which, on nowhere dense classes, leads to computations of bounded recursive depth.

However, we cannot simply start the recursion for the local neighbourhood of every element, as this yields a running time of $O(n^\ell)$ if the neighbourhoods overlap in many elements, where ℓ is the recursion depth. Instead, we group close-by elements into clusters and compute the local type of all elements of the cluster in one recursive call. Such families of clusters are known as *r-neighbourhood covers*, where r is the radius of the neighbourhoods that should be covered in some cluster. We show that every nowhere dense class admits r-neighbourhood covers with clusters of radius at most $2r$ such that every vertex appears in $n^{o(1)}$ clusters. We show that nowhere dense classes, which are closed under subgraphs, can in fact be characterised by the existence of such covers. The combination of the two new characterisations yields a powerful algorithmic technique on nowhere dense classes of graphs.

Our construction of sparse neighbourhood covers is based on the concept of generalised colouring numbers introduced by Kierstead and Yang [77]. Based on a result of Zhu [134], Nešetřil and Ossona de Mendez [105] proved that nowhere dense classes can be characterised by the generalised colouring numbers. We greatly improve on previous bounds for the generalised colouring numbers for nowhere dense classes, especially for several restricted classes such as proper minor closed classes and classes of bounded genus. We provide tight bounds for classes of bounded tree width. For classes that exclude a fixed minor, these results yield r-neighbourhood covers of radius $2r$ and degree $c(r)$ for some constant depending only on r and the class under consideration. In the literature [4, 18] one finds constructions for these classes which have much larger radius, but their degree depends on the class only.

Finally, we prove a strengthened version of Gaifman's locality theorem. We introduce a new, discounted rank measure for first-order formulas, which allows a translation into local formulas such that the rank is preserved. This makes a recursive evaluation of formulas by the method described above possible.

In the last part of the thesis we study the complexity of first-order model-checking on structures where an ordering is available to be used in formulas. The methods based on Gaifman's locality theorem do not readily extend to ordered structures. We do so in two different settings. The first is that the input structures are equipped with a fixed order or successor relation. We show that first-order logic on ordered structures as well as on structures with a successor relation is essentially intractable on nearly all interesting classes.

The other case we consider is order-invariant monadic second-order logic and successor-invariant first-order logic. In order-invariant logics, we are allowed to use an order relation in the formulas but whether a formula is true in a given structure must not depend on the particular choice of order. Order-invariant logics have been studied in database theory and finite model theory in the past.

We show that model-checking for order-invariant MSO on graphclasses of bounded clique-width is fixed-parameter tractable. Furthermore, combining the result of Courcelle [22] and a result in [21, 95] we find that model-checking for order-invariant MSO_2 on graphs of bounded tree-width is fixed-parameter tractable. For successor-invariant FO we are able to show that the model-checking problem is fixed-parameter tractable on planar

graphs. Using the result of Seese [125] we get the same result for FO on any class of graphs of bounded degree equipped with an arbitrary fixed number of successor relations.

This thesis is structured as follows.

- In Part I, we provide background from logic, graph theory, computability and complexity theory. We will provide more specific background about the respective topic at the beginning of each chapter.

In the following parts we present our own contributions.

- In Part II, we develop the theory of nowhere dense graphs required for our model-checking algorithm.
  - We introduce the generalised colouring numbers in Section 4.1 of Chapter 4. We prove a tight bound on classes of bounded tree-width in Section 4.2 and improve on previous bounds for the generalised colouring numbers for nowhere dense classes in Section 4.3. We further investigate the generalised colouring numbers of proper minor closed classes in Section 4.4 and classes of bounded genus in Section 4.5.
  - In Chapter 5, we introduce neighbourhood covers, show that sparse neighbourhood covers exist for nowhere dense classes and show that in fact nowhere dense classes closed under taking subgraphs can be characterised by the existence of sparse neighbourhood covers.
  - We provide the game characterisation of nowhere dense classes in Chapter 6.
- In Part III, we develop the logical theory required for our model-checking algorithm and present the model-checking algorithm itself.
  - In Chapter 7 we strengthen Gaifman's locality theorem for first-order logic such that ranks are preserved. The key innovation is a new discounted rank measure for first-order formulas.
  - Our model-checking algorithm is presented in Chapter 8.
  - Finally, in Chapter 9, we consider the model-checking problem of order-invariant first-order formulas.
- We conclude in Chapter 10.

The splitter game, introduced in [65] and presented in Chapter 6, was invented by Martin Grohe and Stephan Kreutzer. The author of this thesis joined the research group shortly after its invention. The results of Chapter 7 and Chapter 8, which are published in [65], were obtained in close cooperation with Martin Grohe and Stephan Kreutzer.

The results of Chapter 4, which were published in [64] and [109], were obtained in collaboration with Martin Grohe, Stephan Kreutzer, Patrice Ossona de Mendez, Roman Rabinovich, Konstantinos Stavropoulos and Jan van den Heuvel.

The results of Chapter 5 were published in [65] and [64] and are the result of a collaboration with Martin Grohe, Stephan Kreutzer and Roman Rabinovich.

The results of Chapter 9 were published in [46] and were found by Viktor Engelmann, Stephan Kreutzer and the author of this thesis.

Part I.

Background

1. General Background

1.1. Numbers, sets and functions

Throughout this work, we use lower case letters $a,\ldots,t$ to name integers, lower case Greek symbols $\alpha,\beta,\ldots$ to name reals and capital letters $A,B,\ldots$ to name sets. Logarithms written as 'log' are at base 2, the natural logarithm will be denoted by 'ln'. For a non-negative real number x, we denote by $\lfloor x \rfloor$ the greatest integer $\leq x$ and by $\lceil x \rceil$ the least integer $\geq x$. The cardinality of a set A is written as $|A|$ and its power set is denoted $\mathscr{P}(A)$. We will often call a set of sets a *family*.

For a positive integer $s > 0$ and a non-empty set A non-empty, a *partition* of A into s sets is a family $\{A_1,\ldots,A_s\}$ of non-empty subsets of A such that $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq s$ and $\bigcup_{1\leq i\leq s} A_i = A$. For sets $A_1,\ldots,A_k$,

$$A_1 \times \cdots \times A_k := \{(a_1,\ldots,a_k) : a_i \in A_i, 1 \leq i \leq k\}$$

denotes the *Cartesian product* of $A_1,\ldots,A_k$. For a non-negative integer k, we write A^k for $A \times \cdots \times A$ and $A^\star$ for $\bigcup_{k\geq 0} A^k$. We often abbreviate a k-tuple $(a_1,\ldots,a_k)$ as $\overline{a}$ and we write $a \in \overline{a}$ instead of $a \in \{a_1,\ldots,a_k\}$. We write $\overline{a}a$ for the tuple $(a_1,\ldots,a_k,a)$.

A *k-ary relation* R over $A_1,\ldots,A_k$ is a subset of $A_1 \times \cdots \times A_k$. A k-ary relation *on* a set A is a subset of A^k. The set A^0 contains only the empty tuple (), and hence there are exactly two 0-ary, or *Boolean relations* on A, namely $\emptyset$ and $\{()\}$. 1-ary relations are *unary* relations and 2-ary relations are *binary* relations.

For sets A,B, a binary relation $f \subseteq A \times B$ is *functional* if $(a,b) \in f$ and $(a,c) \in f$ implies $b = c$ for all $a \in A$ and $b,c \in B$. If f is functional, we use the standard notation $f(a) = b$ for $(a,b) \in f$. We also call f a *partial function*. Then the *domain* of f is the set

$$\mathrm{dom}(f) = \{a \in A : \text{ there exists } b \in B \text{ with } f(a) = b\}$$

and the *range* of f is the set

$$\mathrm{ran}(f) = \{b \in B : \text{ there exists } a \in A \text{ with } f(a) = b\}.$$

A functional relation f is a *function* from A to B, denoted $f : A \to B$, if $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$. The set of all functions from A to B is denoted B^A. In the following, let f be a function.

If $\mathrm{dom}(f) = A^k$, then f is a k-ary function on A. If f is a function on A and $A' \subseteq A$, then the *image* of A' by f is

$$f(A') = \{b \in B : \text{ there exists } a \in A' \text{ with } f(a) = b\}.$$

If $\mathrm{ran}(f) = B$, then f is called *surjective* or a function *onto* B. If $f(a) = f(b)$ implies $a = b$ for all $a, b \in \mathrm{dom}(f)$, then f is called *injective* or *one-to-one*. If f is surjective and injective, then it is called *bijective*. In this case, the *inverse* of f is

$$f^{-1} = \{(b, a) \in B \times A : (a, b) \in f\}.$$

The *restriction* of f to A' is the function

$$f \restriction A' = \{(a, b) \in f : a \in A'\}.$$

A function g is an *extension* of f if $f \subseteq g$, that is, $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g(a) = f(a)$ for all $a \in \mathrm{dom}(f)$. We write $f[a \mapsto b]$ for the function $\{(a, b)\} \cup \{(c, f(c)) : c \in \mathrm{dom}(f), c \neq a\}$. If $\overline{a} = (a_1, \ldots, a_k)$ and $\overline{b} = (b_1, \ldots, b_k)$ are such that $b_i = b_j$ if $a_i = a_j$ for $1 \leq i, j \leq k$, then we write $\overline{a} \mapsto \overline{b}$ for the function $\{(a_1, b_1), \ldots, (a_k, b_k)\}$. If f is a unary function, we write $f(\overline{a})$ for the tuple $(f(a_1), \ldots, f(a_k))$ and for a k-ary relation R, we write $f(R)$ for the relation $\{f(\overline{a}) : \overline{a} \in R\}$. We sometimes call a function a *mapping*.

1.2. Computability and complexity

A *finite alphabet* Σ is any finite and non-empty set whose elements are called *symbols*. We call an element of $\Sigma^\star$ a *word* or *string* over Σ. A *decision problem* over $\Sigma^\star$ is a subset $L \subseteq \Sigma^\star$. In the following, let Σ and Γ be finite alphabets.

When studying questions of *computability* or *decidability* we assume the standard multi-tape Turing machine model as in [69, 127]. A decision problem $L \subseteq \Sigma^\star$ is *recursively enumerable* if there is a Turing machine which

accepts every word $w \in L$. L is *decidable* if there is a Turing machine which accepts every word $w \in L$ and rejects every word $w \notin L$. A function $f : \Sigma^\star \to \Gamma^\star$ is *computable* if there is a deterministic Turing machine which on input $w \in \Sigma^\star$ computes the output $f(w)$. We always assume that abstract objects, such as numbers, graphs or formulas are encoded in an appropriate way as words over Σ. We may then write e.g. $f : \mathbb{N} \to \mathbb{N}$, even though $f : \Sigma^\star \to \Sigma^\star$, but we interprete the input and output of the function according to the fixed encoding of natural numbers.

For the analysis of running times of specific algorithms we assume the *random access machine (RAM)* model. However, we will never provide an algorithm as a Turing machine or as a RAM program, but always provide a high level description only. For more background on complexity theory we refer to [112]. We use the usual Landau notation to describe the limit behaviour of a function. For a function $f : \mathbb{N} \to \mathbb{N}$,

$$O(f) = \{g : \mathbb{N} \to \mathbb{N} : \limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty\}$$

and

$$o(f) = \{g : \mathbb{N} \to \mathbb{N} : \limsup_{n\to\infty} \frac{f(n)}{g(n)} = 0\}.$$

We write $g \in \Theta(f)$ if asymptotically the bounds are sharp, i.e. if $f \in O(g)$ and $g \in O(f)$, and $g \in \Omega(f)$ if $f \in O(g)$.

We write PTIME for the class of all problems that can be decided in polynomial time by a deterministic Turing machine, NP for the class of all problems that can be decided in polynomial time by a non-deterministic Turing machine and PSPACE for the class of all languages that can be decided with polynomial space by a deterministic Turing machine. It is conjectured and widely believed that

$$\text{PTIME} \subsetneq \text{NP} \subsetneq \text{PSPACE}.$$

An *oracle machine* with oracle $\mathcal{Q} \in \Sigma^\star$ is a machine with a distinguished *oracle tape*. The machine can write a word $w \in \Sigma^\star$ to the oracle tape and then query in unit time whether $w \in \mathcal{Q}$.

A *polynomial time (many-one) reduction* from a problem $L \subseteq \Sigma^\star$ to a problem $M \in \Gamma^\star$ is a mapping $R : \Sigma^\star \to \Gamma^\star$ such that for all $w \in \Sigma^\star$ we have $w \in L \iff R(w) \in M$ and $R(w)$ is computable in polynomial time by a deterministic Turing machine.

A *polynomial time Turing reduction* from a problem $L \in \Sigma^\star$ to a problem $M \in \Gamma^\star$ is a deterministic oracle machine that decides L with oracle M in polynomial time.

We write $L \leq^p M$ if there is a polynomial time many-one reduction from L to M and $L \leq^T M$ if there is a polynomial time Turing reduction from L to Q. Often it is easier (and more intuitive) to establish that $L \leq^p Q$, which implies $L \leq^T M$.

A problem L is *hard* for a complexity class C, or *C-hard*, if $M \leq^T L$ for every $M \in C$. L is *complete* for C, or *C-complete*, if $L \in C$ and L is hard for C. In order to show that a problem L is hard for C it suffices to show $M \leq^T L$ for some problem M which is hard for C.

In *classical complexity theory* one says that a problem can be solved *efficiently* if the problem lies in PTIME. In order to show that a problem can in general not be solved efficiently (under the assumption that PTIME $\neq$ NP), one shows that the problem is NP-hard.

The first-order model-checking problem is complete for PSPACE [133], even on structures with only two elements. Therefore, classical complexity theory renders the problem intractable.

While it is interesting to know whether a problem is NP-hard, we often want to apply a finer analysis, that is, we do not want to measure the complexity only in terms of the size of the input, but rather measure it with respect to one or more parameters. This is the subject of *parameterized complexity theory* [37, 51].

A *parameterized problem* is a pair (L, κ), where $L \subseteq \Sigma^\star$ is a problem and $\kappa : \Sigma^\star \to \mathbb{N}$ is a polynomial time computable function. We call $\kappa(w)$ the *parameter* of the instance w.

A parameterized problem is *(uniformly) fixed-parameter tractable* (*fpt*) if there exists an algorithm (called a *fixed-parameter algorithm*), a (computable) function $f : \mathbb{N} \to \mathbb{N}$, and a constant c such that, given $w \in \Sigma^\star$, the algorithm decides whether $w \in L$ in time bounded by $f(\kappa(w)) \cdot |w|^c$. The complexity class containing all fixed-parameter tractable problems is called FPT.

Let (L, κ) and (Q, γ) be parameterized problems over the alphabets Σ and Γ, respectively. An *fpt-reduction* from (L, κ) to (Q, κ) is a mapping R from $\Sigma^\star$ to $\Gamma^\star$ such that there are computable functions f, g and a constant c such that for all $w \in \Sigma^\star$ we have $w \in L \iff R(w) \in M$, $R(w)$ is computable in time $f(\kappa(w)) \cdot |w|^c$ and $\gamma(R(w)) \leq g(\kappa(w))$.

We write $(L,\kappa) \leq^{fpt} (M,\gamma)$ if there exists a fixed-parameter reduction from (L,κ) to (M,γ). If (L,κ) is fixed-parameter tractable and $(M,\gamma) \leq^{fpt} (L,\kappa)$, then also (M,γ) is fixed-parameter tractable. A parameterized problem (L,κ) is *hard* for a class C of parameterized problems if every parameterized problem in C is fpt-reducible to (L,κ) and it is *complete* for C if it is hard for C and lies in C. For a parameterized problem (L,κ) let

$$[(L,\kappa)]^{fpt} := \{(M,\gamma) : (M,\gamma) \leq^{fpt} (L,\kappa)\}.$$

1.3. Structures and isomorphisms

A (finite and purely relational) *vocabulary* or *signature* τ is a finite set of *relation symbols* $R_1,\ldots,R_s$ for some non-negative integer $s \geq 0$, each with an associated *arity*. In the following, let τ be a finite and purely relational vocabulary.

A τ*-structure* $\mathfrak{A}$ consists of a non-empty set $V(\mathfrak{A})$, called the *universe* of $\mathfrak{A}$, and for each non-negative integer $k \geq 0$ an interpretation

- $R(\mathfrak{A}) \subseteq V(\mathfrak{A})^k$ of each k-ary relation symbol R from τ.

A τ-structure $\mathfrak{A}$ is called *finite* if its universe $V(\mathfrak{A})$ is finite. We write $n(\mathfrak{A})$ for $|V(\mathfrak{A})|$, $m(R(\mathfrak{A}))$ for the number of tuples in $R(\mathfrak{A})$ and $m(\mathfrak{A})$ for $\sum_{R\in\tau} m(R(\mathfrak{A}))$. If $\tau = \{R_1,\ldots,R_s\}$, we write $\mathfrak{A} = (V(\mathfrak{A}),R_1(\mathfrak{A}),\ldots,R_s(\mathfrak{A}))$.

In the following, let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures. $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if

- $V(\mathfrak{A}) \subseteq V(\mathfrak{B})$, and
- for each non-negative integer $k \geq 0$ and each k-ary symbol R from τ we have $R(\mathfrak{A}) \subseteq R(\mathfrak{B}) \cap V(\mathfrak{A})^k$.

$\mathfrak{A}$ is an *induced substructure* of $\mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and

- for each non-negative integer $k \geq 0$ and each k-ary symbol R from τ we have $R(\mathfrak{A}) = R(\mathfrak{B}) \cap V(\mathfrak{A})^k$.

If $A \subseteq V(\mathfrak{B})$ is non-empty, the *substructure of* $\mathfrak{B}$ *induced* or *spanned* by A is the τ-structure $\mathfrak{B}[A]$ with universe A and $R(\mathfrak{B}[A]) = R(\mathfrak{B}) \cap A^k$ for each k-ary relation symbol R from τ, $k \geq 0$. Note that in model theory one usually

writes $\mathfrak{A} \subseteq \mathfrak{B}$ to indicate that $\mathfrak{A}$ is an induced substructure of $\mathfrak{B}$ whereas we use it as in graph theory for substructures.

Let $\sigma \subseteq \tau$ be a vocabulary. The *σ-reduct* of $\mathfrak{A}$ is the σ-structure $\mathfrak{A} \restriction \sigma$ with universe $V(\mathfrak{A})$ in which all relation symbols in σ are interpreted as in $\mathfrak{A}$. If $\mathfrak{A}$ is a σ-reduct of $\mathfrak{B}$, then $\mathfrak{B}$ is a *τ-expansion* of $\mathfrak{A}$.

A *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a mapping $\pi : V(\mathfrak{A}) \to V(\mathfrak{B})$ such that

- if $\overline{a} \in R(\mathfrak{A})$, then $\pi(\overline{a}) \in R(\mathfrak{B})$ for all non-negative integers $k \geq 0$, all k-ary relation symbols R from τ and all $\overline{a} \in V(\mathfrak{A})^k$.

An injective homomorphism is called an *embedding*. A homomorphism π is *strong* if it satisfies the stronger condition

- $\overline{a} \in R(\mathfrak{A})$ if and only if $\pi(\overline{a}) \in R(\mathfrak{B})$ for all non-negative integers $k \geq 0$, all k-ary relation symbols R from τ and all $\overline{a} \in V(\mathfrak{A})^k$.

A bijective, strong homomorphism is an *isomorphism*. We call $\mathfrak{A}$ and $\mathfrak{B}$ *isomorphic* and write $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. For a non-negative integer $k \geq 0$ and $\overline{a} \in V(\mathfrak{A})^k$ and $\overline{b} \in V(\mathfrak{B})^k$, we write $(\mathfrak{A}, \overline{a}) \cong (\mathfrak{B}, \overline{b})$ if there exists an isomorphism π between $\mathfrak{A}$ and $\mathfrak{B}$ with $\pi(\overline{a}) = \overline{b}$. The *isomorphism type* of $(\mathfrak{A}, \overline{a})$ is the set

$$\mathrm{tp}(\mathfrak{A}, \overline{a}) := \{(\mathfrak{B}, \overline{b}) : (\mathfrak{B}, \overline{b}) \cong (\mathfrak{A}, \overline{a})\}.$$

If $\mathrm{tp}(\mathfrak{A}, \overline{a}) = \gamma$, we say that $\overline{a}$ *realises* the type γ in $\mathfrak{A}$.

A *partial* isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ is an isomorphism from $\mathfrak{A}[A']$ to $\mathfrak{B}[B']$ for non-empty and finite subsets $A' \subseteq V(\mathfrak{A}), B' \subseteq V(\mathfrak{B})$. For convenience, we also call the empty mapping $\emptyset$ a partial isomorphism (recall that we demand that structures have a non-empty universe).

For a non-negative integer $m \geq 0$, an m-ary *query* on τ-structures is a mapping Q that associates with each τ-structure $\mathfrak{A}$ an m-ary relation on $V(\mathfrak{A})$ which is closed under isomorphism, i.e. if $\mathfrak{A} \cong \mathfrak{B}$ via isomorphism $\pi : V(\mathfrak{A}) \to V(\mathfrak{B})$, then $Q(\mathfrak{B}) = \pi(Q(\mathfrak{A}))$.

An important special case is that of $m = 0$. As there are exactly two Boolean relations on $V(\mathfrak{A})$, a 0-ary query, also called *Boolean query*, is a mapping from τ-structures to a set with two elements, which we name *true* and *false*. We can hence associate a Boolean query with a set $\mathscr{C}$ of τ-structures closed under isomorphism:

$$\mathfrak{A} \in \mathscr{C} \text{ if and only if } Q(\mathfrak{A}) = \mathit{true}.$$

2. First-order and monadic second order logic

2.1. Syntax and Semantics

Let V be a countably infinite set of *variables* and let τ be a finite relational vocabulary. The set FO[τ] of *first-order τ-formulas* is defined inductively as follows.

- $\top$ and $\bot$ are (atomic) formulas.
- If $x_1, x_2 \in V$, then $x_1 = x_2$ is an (atomic) formula.
- If $x_1, \ldots, x_k \in V$ and R is a k-ary relation symbol from τ, $k \geq 0$, then $R(x_1, \ldots, x_k)$ is an (atomic) formula.
- If φ, χ are τ-formulas, then $\varphi \wedge \chi$, $\varphi \vee \chi, \varphi \rightarrow \chi, \varphi \leftrightarrow \chi$ and $\neg\varphi$ are τ-formulas.
- If φ is a τ-formula and $x \in V$, then $\exists x\varphi$ and $\forall x\varphi$ are τ-formulas.

The class of first-order formulas FO is the union of FO[τ] for all vocabularies τ. In the following, let φ, ψ, χ be first-order τ-formulas.

Variables can occur *free* or *bound* in a formula.

- The variables of atomic formulas are free.
- The free variables of a formula $\varphi \circ \chi$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ are the free variables of φ and χ and the free variables of a formula $\neg\varphi$ are the free variables of φ.
- The free variables of a formula $\exists x\varphi$ and $\forall x\varphi$ are the free variables of φ except x.

Variables that are not free are bound. A formula without free variables is a *sentence*. For a non-negative integer $m \geq 0$, we write $\varphi(x_1, \ldots, x_m)$ to indicate that the free variables of φ are among $x_1, \ldots, x_m$.

If $\mathscr{F}$ is a set of formulas, then any formula constructed from formulas in $\mathscr{F}$ using only the Boolean connectives $\vee, \wedge$ and $\neg$ is called a *Boolean combination* of formulas from $\mathscr{F}$.

The *quantifier rank* $\mathrm{qr}(\varphi)$ of φ is its depth of quantifier nesting, i.e.

- if φ is atomic, then $\mathrm{qr}(\varphi) = 0$,
- if $\varphi = \psi \circ \chi$, then $\mathrm{qr}(\varphi) = \max\{\mathrm{qr}(\psi), \mathrm{qr}(\chi)\}$ for $\circ \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$,
- if $\varphi = \neg\psi$, then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi)$ and
- if $\varphi = \exists x \psi$ or $\varphi = \forall x \psi$, then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi) + 1$.

A formula of quantifier rank 0 is also called *quantifier free*. Let Σ_0 and Π_0 denote the class of all quantifier free first-order formulas. For a non-negative integer $t \geq 0$, let Σ_{t+1} be the class of all first-order formulas

$$\exists x_1 \ldots \exists x_k \, \varphi,$$

where $\varphi \in \Pi_t$ and $k \geq 0$ is a non-negative integer and let Π_{t+1} be the class of all first-order formulas

$$\forall x_1 \ldots \forall x_k \, \varphi,$$

where $\varphi \in \Sigma_t$ and $k \geq 0$ is a non-negative integer.

A τ-*interpretation* is a tuple $\mathfrak{I} = (\mathfrak{A}, \beta)$, consisting of a τ-structure $\mathfrak{A}$ and a mapping $\beta : V \to V(\mathfrak{A})$. We write $\mathfrak{I}[x \mapsto a]$ for $(\mathfrak{A}, \beta[x \mapsto a])$. A τ-interpretation $\mathfrak{I}$ evaluates a variable x to a value in $V(\mathfrak{A})$, namely $\mathfrak{I}(x) = \beta(x)$. We define the *satisfaction relation*, $\mathfrak{I} \models \varphi$, between τ-interpretations and formulas from $\mathrm{FO}[\tau]$ as follows.

- $\mathfrak{I} \models \top$ and $\mathfrak{I} \not\models \bot$,
- $\mathfrak{I} \models x_1 = x_2$ if and only if $\mathfrak{I}(x_1) = \mathfrak{I}(x_2)$,
- for a non-negative integer $k \geq 0$, a k-ary relation symbol $R \in \tau$ and $x_1, \ldots, x_k \in V$, $\mathfrak{I} \models R(x_1, \ldots, x_k)$ if and only if $(\mathfrak{I}(x_1), \ldots, \mathfrak{I}(x_k)) \in R(\mathfrak{A})$,
- $\mathfrak{I} \models \neg\psi$ if and only if $\mathfrak{I} \not\models \psi$,

- $\mathfrak{I} \models \psi \vee \chi$ if and only if $\mathfrak{I} \models \psi$ or $\mathfrak{I} \models \chi$,
- $\mathfrak{I} \models \psi \wedge \chi$ if and only if $\mathfrak{I} \models \psi$ and $\mathfrak{I} \models \chi$,
- $\mathfrak{I} \models \psi \rightarrow \chi$ if and only if ($\mathfrak{I} \models \chi$ if $\mathfrak{I} \models \psi$),
- $\mathfrak{I} \models \psi \leftrightarrow \chi$ if and only if ($\mathfrak{I} \models \chi$ if and only if $\mathfrak{I} \models \psi$),
- $\mathfrak{I} \models \exists x \psi$ if and only if there exists $a \in V(\mathfrak{A})$ such that $\mathfrak{I}[x \mapsto a] \models \psi$ and
- $\mathfrak{I} \models \forall x \psi$ if and only if for all $a \in V(\mathfrak{A})$ we have $\mathfrak{I}[x \mapsto a] \models \psi$.

Monadic second-order logic is the following extension of first-order logic. Let W be a fixed countably infinite set of unary predicates which are called *second-order set variables* and let W_2 be a fixed countably infinite set of binary predicates which are called *guarded second-order variables*.

The set of *monadic second-order τ-formulas*, denoted $\mathrm{MSO}[\tau]$, is defined by extending the rules for first-order logic with the following additional rules.

- If X is a second-order set variable and $\varphi \in \mathrm{MSO}[\tau \cup X]$, then $\exists X \varphi \in \mathrm{MSO}[\tau]$ and $\forall X \varphi \in \mathrm{MSO}[\tau]$.

Finally, the set of $\mathrm{MSO}_2[\tau]$-formulas, is defined by extending the rules for $\mathrm{MSO}[\tau]$ with the following additional rules.

- If Y is a guarded second-order set variable and $\varphi \in \mathrm{MSO}_2[\tau \cup Y]$, then also $\exists Y \varphi \in \mathrm{MSO}_2[\tau]$ and $\forall Y \varphi \in \mathrm{MSO}_2[\tau]$.

We define free and bound variables and the quantifier-rank of a monadic second-order formula as for first-order formulas. A τ-interpretation for a monadic second-order formula is a tuple $\mathfrak{I} = (\mathfrak{A}, \alpha, \beta)$, for a τ-structure $\mathfrak{A}$, a mapping $\alpha : V \to V(\mathfrak{A})$ and $\beta : W \to \mathscr{P}(V(\mathfrak{A}))$. We write $\mathfrak{I}[X \mapsto A]$ for the interpretation $(\mathfrak{A}, \alpha, \beta[X \mapsto A])$. We extend the satisfaction relation $\mathfrak{I} \models \varphi$ for $\mathrm{MSO}[\tau]$-formulas by the following rules. For a set variable X, we have

- $\mathfrak{I} \models \exists X \psi$ if and only if there exists $A \subseteq V(\mathfrak{A})$ such that $\mathfrak{I}[X \mapsto A] \models \psi$ and
- $\mathfrak{I} \models \forall X \psi$ if and only if for all $A \subseteq V(\mathfrak{A})$ we have $\mathfrak{I}[X \mapsto A] \models \psi$.

We define the semantics for $\mathrm{MSO}_2[\tau]$ only for restricted vocabularies $\tau = \{E, C_1, \ldots, C_k\}$, where E is a binary relation symbol and the C_i are unary relation symbols.

A τ-interpretation for a $\mathrm{MSO}_2[\tau]$-formula is a tuple $\mathfrak{I} = (\mathfrak{A}, \alpha, \beta, \gamma)$, where $(\mathfrak{A}, \alpha, \beta)$ is a τ-interpretation as above and $\gamma : W_2 \to \mathscr{P}(E(\mathfrak{A}))$. We write $\mathfrak{I}[Y \mapsto A]$ for the interpretation $(\mathfrak{A}, \alpha, \beta, \gamma[Y \mapsto A])$. We extend the satisfaction relation for guarded variables Y as follows.

- $\mathfrak{I} \models \exists Y \psi$ if and only if there exists $A \subseteq E(\mathfrak{A})$ such that $\mathfrak{I}[X \mapsto A] \models \psi$ and
- $\mathfrak{I} \models \forall Y \psi$ if and only if for all $A \subseteq E(\mathfrak{A})$ we have $\mathfrak{I}[Y \mapsto A] \models \psi$.

If $\mathfrak{I} \models \varphi$, we call $\mathfrak{I}$ a *model* of φ or say that the interpretation $\mathfrak{I}$ *satisfies* φ. A formula is *satisfiable* if it has a model. Without loss of generality we may assume that the free variables of φ are named $x_1, \ldots, x_m$ for some non-negative integer $m \geq 0$. Then, if $\beta(x_1) = a_1, \ldots, \beta(x_m) = a_m$, we write $(\mathfrak{A}, a_1, \ldots, a_m) \models \varphi$ or $\mathfrak{A} \models \varphi(a_1, \ldots, a_m)$ instead of $(\mathfrak{A}, \beta) \models \varphi$.

For a set Φ of τ-formulas, $\mathfrak{I} \models \Phi$ means that $\mathfrak{I} \models \varphi$ for all $\varphi \in \Phi$. A τ-formula ψ is a *consequence* of Φ, written $\Phi \models \psi$, if $\mathfrak{I} \models \Phi$ implies $\mathfrak{I} \models \psi$ for all τ-interpretations $\mathfrak{I}$. A τ-formula ψ is *valid* if $\emptyset \models \psi$, i.e. if ψ is true in all τ-interpretations. Formulas φ and ψ are *equivalent* if $\varphi \leftrightarrow \psi$ is valid.

A class $\mathscr{C}$ of τ-structures is *first-order definable* if there is an $\mathrm{FO}[\tau]$-formula φ such that for all τ-structures $\mathfrak{A}$

$$\mathfrak{A} \in \mathscr{C} \iff \mathfrak{A} \models \varphi.$$

2.2. Satisfiability and model-checking

A *formal proof system* consists of a decidable set of *axioms* and a decidable set of *inference rules*. A *formal proof* (in the proof system) for the validity of a formula is a finite sequence of first-order formulas. The formulas which may occur in a proof are either axioms or they must be the product of applying an inference rule on previous formulas in the proof sequence.

Let Φ be a set of formulas and let ψ be a formula. We write $\Phi \vdash \psi$ if ψ can be derived from Φ in a formal proof system. A proof system is *sound* if $\Phi \vdash \psi$ implies $\Phi \models \psi$ and it is *complete* if $\Phi \models \psi$ implies $\Phi \vdash \psi$. Gödel's Completeness Theorem states that there exists a sound and complete

proof system for first-order logic. As a consequence, the set of valid FO-sentences is recursively enumerable. By the famous result of Church and (independently) Turing, the set of valid first-order sentences is not decidable (and hence not co-recursively enumerable). Consequently, equivalence of two formulas is recursively enumerable but not decidable. As ψ is not valid if and only if $\neg\psi$ is satisfiable, the set of satisfiable sentences is not recursively enumerable.

On the other hand, the set of sentences which have a *finite* model is recursively enumerable. By Trakhtenbrot's Theorem, the set of first-order sentences valid in all *finite* structures is *not* recursively enumerable and hence finite satisfiability is not decidable. Proofs of the above mentioned results can be found for example in [11].

Our main concern is the study of the parameterized complexity of the *model-checking problem* for a logic $\mathscr{L} \in \{\mathrm{FO}, \mathrm{MSO}, \mathrm{MSO}_2\}$ on a class $\mathscr{C}$ of finite structures.

$\mathrm{MC}(\mathscr{L},\mathscr{C})$	
Input:	$\mathfrak{A} \in \mathscr{C}$, $\varphi \in \mathscr{L}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide whether $\mathfrak{A} \models \varphi$

Throughout this thesis we will use the above way of presenting parameterized problems. Formally, we fix an alphabet Σ with a designated separating symbol #, an encoding $\langle\cdot\rangle$ which maps τ-structures $\mathfrak{A}$ to words $\langle\mathfrak{A}\rangle \in \Sigma^\star$ and formulas $\varphi \in \mathscr{L}[\tau]$ to words $\langle\langle\varphi\rangle\rangle \in \Sigma^\star$ and a parametrization $\kappa(\langle\mathfrak{A}\rangle \# \langle\langle\varphi\rangle\rangle) \mapsto |\varphi|$. Then

$$\mathrm{MC}(\mathscr{L},\mathscr{C}) = \big(\{\langle\mathfrak{A}\rangle \# \langle\langle\varphi\rangle\rangle : \mathfrak{A} \in \mathscr{C}, \varphi \in \mathscr{L}[\tau], \mathfrak{A} \models \varphi\}, \kappa\big).$$

The W-hierarchy is a collection of parameterized complexity classes. Our definition follows [51] which is much more natural in the model-checking context than the original definition (see e.g. [38]) in terms of the *weft of a circuit*. Let τ be a finite relational vocabulary and for some positive integer $s > 0$, let R be an s-ary relation symbol which does not occur in τ. Let Φ be a set of $\mathrm{FO}[\tau \cup R]$-formulas. We define the *weighted satisfiability problem* for Φ as the problem

WS-Φ	
Input:	Finite τ-structure $\mathfrak{A}$, $\varphi \in \Phi$, $k \in \mathbb{N}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide whether there exists $R \subseteq V(\mathfrak{A})^s$ with $\lvert R\rvert = k$ such that $(\mathfrak{A}, R) \models \varphi$

For every $t \geq 1$, we let

$$\mathrm{W}[t] := [\text{WS-}\Pi_t]^{fpt}$$

and

$$\mathrm{AW}[\star] := [\mathrm{MC(FO)}]^{fpt}.$$

For example, the parameterized independent set problem (parameterized by the size of the independent set) is in W[1]. To see this, take a unary relation symbol R and state with first-order logic that every two elements from R are not connected by an edge, which is a Π_1-formula. It was shown that the parameterized independent set problem is complete for W[1] [36].

The parameterized dominating set problem (parameterized by the size of the dominating set) is in W[2]. To see this, take a unary relation symbol R and state with first-order logic that for every element v it either holds that $v \in R$ or that there exists $u \in R$ such that $\{u, v\}$ is an edge of the input graph. This is a Π_2-formula. In fact, the parameterized dominating set problem is complete for W[2] [35].

It is conjectured an widely believed that

$$\mathrm{FPT} \subsetneq \mathrm{W}[1] \subsetneq \mathrm{W}[2] \subsetneq \cdots \subsetneq \mathrm{AW}[\star]$$

By our definition, the first-order model-checking problem (parameterized by the length of the formula or by its number of quantifiers) is complete for the parameterized class AW[$\star$] and hence it is widely believed that no efficient first-order model-checking algorithm on the class of all finite structures exists.

2.3. First-Order Types

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures and let $k \geq 0$ be a non-negative integer. $\mathfrak{A}$ and $\mathfrak{B}$ are *k-equivalent*, written $\mathfrak{A} \equiv_k \mathfrak{B}$, if they agree on all τ-sentences of quantifier rank at most k, i.e. if for all $\varphi \in \mathrm{FO}[\tau]$ with $\mathrm{qr}(\varphi) \leq k$ we have

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathfrak{B} \models \varphi.$$

We say that $\mathfrak{A}$ and $\mathfrak{B}$ have the same *rank-k type*. More generally, for an integer $m \geq 0$, tuples $\overline{a} = (a_1, \ldots, a_m) \in V(\mathfrak{A})^m$ and $\overline{b} = (b_1, \ldots, b_m) \in V(\mathfrak{B})^m$ and $\overline{x} = (x_1, \ldots, x_m)$. The *rank-$k$ m-type* of $\overline{a}$ over $\mathfrak{A}$ is

$$\mathrm{tp}_k(\mathfrak{A}, \overline{a}) := \{\varphi(\overline{x}) \in \mathrm{FO}[\tau] : \mathrm{qr}(\varphi) \leq k, \mathfrak{A} \models \varphi(\overline{a})\}.$$

A *rank-k m-type* is any set of formulas of the form $\mathrm{tp}_k(\mathfrak{A}, \overline{a})$ for a τ-structure $\mathfrak{A}$ and $\overline{a} \in V(\mathfrak{A})^m$. When m is clear from the context, we simply speak of *rank-k types*. Note that a rank-k type T is *maximally consistent*, i.e. satisfiable and for each formula $\varphi(\overline{x})$ of quantifier rank at most k, either $\varphi \in T$ or $\neg\varphi \in T$. We write $(\mathfrak{A}, \overline{a}) \equiv_k (\mathfrak{B}, \overline{b})$ if $\mathrm{tp}_k(\mathfrak{A}, \overline{a}) = \mathrm{tp}_k(\mathfrak{B}, \overline{b})$.

Types as defined above are infinite. However, we may assume that formulas $\varphi(x_1, \ldots, x_m)$ of quantifier rank k use only the variables $x_1, \ldots, x_{k+m}$, that all Boolean combinations are in disjunctive normal form and that there are no duplicate entries in disjunctions and conjunctions. We call a formula with these properties *normalised* and we write $\Phi(\tau, k, m)$ for the set of all normalised τ-formulas $\varphi(x_1, \ldots, x_m)$ of quantifier rank at most k. It is not hard to see that $\Phi(\tau, k, m)$ is finite and computable from τ, k and m. By systematically renaming bound variables, bringing Boolean combinations into disjunctive normal form and deleting duplicate entries from the disjunctions and conjunctions, we can effectively normalise any τ-formula φ.

Hence each rank-k type is uniquely defined by a subset $\Theta \subseteq \Phi(\tau, k, m)$. We will often not distinguish between a type and the formula which is the conjunction of the type. Every τ-formula φ of quantifier rank k is either unsatisfiable or equivalent to a disjunction of rank-k types, namely of those types which contain φ.

This implies that it is not decidable whether a set $\Gamma \subseteq \Phi(\tau, k, m)$ is a type (otherwise we could simply test whether a τ-formula φ of quantifier rank k with m free variables occurs in one of the finitely many types with parameters τ, k and m and thereby decide whether φ is satisfiable). We can transform any formula φ into a disjunction $\bigvee_{i \in I} \bigwedge \Gamma_i$ though, where $\Gamma_i \subseteq \Phi(\tau, k, m)$ is a set with the property that for all $\psi \in \Phi(\tau, k, m)$ either ψ or the normalised version of $\neg\psi$ is contained in Γ_i. Each such Γ_i is a either equivalent to a type or unsatisfiable. Let $\Gamma_1, \ldots, \Gamma_s$ be an enumeration of the finitely many subsets of $\Phi(\tau, k, m)$ with the above property. We know that for some $I \subseteq \{1, \ldots, s\}$ the formula φ is equivalent to $\psi_I := \bigvee_{i \in I} \bigwedge \Gamma_i$ and hence the formula $\varphi \leftrightarrow \psi_I$ is valid. We can test this for all I in parallel and eventually find the desired disjunction. We can generalise this principle as follows.

Theorem 2.3.1 *Let Φ be a finite set of first-order τ-formulas, let $\mathscr{C}$ be a first-order definable class of τ-structures and let $k, m \geq 0$ be non-negative integers. If for all τ-structures $\mathfrak{A}, \mathfrak{B}$ from $\mathscr{C}$ and all $\overline{a} \in V(\mathfrak{A})^m$, $\overline{b} \in V(\mathfrak{B})^m$,*

$$((\mathfrak{A}, \overline{a}) \models \chi \Leftrightarrow (\mathfrak{B}, \overline{b}) \models \chi \text{ for all } \chi \in \Phi) \quad \text{implies} \quad (\mathfrak{A}, \overline{a}) \equiv_k (\mathfrak{B}, \overline{b}),$$

then every first-order formula $\varphi(x_1, \ldots, x_m)$ of quantifier-rank at most k can effectively be translated to a Boolean combination of formulas from Φ which is equivalent to φ on all structures from $\mathscr{C}$.

PROOF. The Φ-*type* of a tuple $\overline{a}$ in a τ-structure $\mathfrak{A}$ is defined as

$$\Phi(\mathfrak{A}, \overline{a}) := \{\psi \in \Phi : (\mathfrak{A}, \overline{a}) \models \psi\} \cup \{\neg\psi : \psi \in \Phi, (\mathfrak{A}, \overline{a}) \models \neg\psi\}.$$

As Φ is finite, $\Phi(\mathfrak{A}, \overline{a})$ for any τ-structure $\mathfrak{A}$ and tuple $\overline{a} \in V(\mathfrak{A})^m$ is also finite. Let $\varphi(x_1, \ldots, x_m)$ be any first-order τ-formula of quantifier-rank at most k. We claim that for all $\mathfrak{B} \in \mathscr{C}$ and $\overline{b} \in V(\mathfrak{B})^m$ it holds that

$$(\mathfrak{B}, \overline{b}) \models \varphi \quad \text{if and only if} \quad (\mathfrak{B}, \overline{b}) \models \bigvee_{\substack{\mathfrak{A} \in \mathscr{C}, \overline{a} \in V(\mathfrak{A})^m \\ (\mathfrak{A}, \overline{a}) \models \varphi}} \bigwedge \Phi(\mathfrak{A}, \overline{a}),$$

where we remove duplicate entries from the big disjunctions. For readability, we will simply write $\bigvee_{(\mathfrak{A}, \overline{a}) \models \varphi}$ for $\bigvee_{\mathfrak{A} \in \mathscr{C}, \overline{a} \in V(\mathfrak{A})^m, (\mathfrak{A}, \overline{a}) \models \varphi}$.

Let $\mathfrak{B} \in \mathscr{C}$ and $\overline{b} \in V(\mathfrak{B})^m$. If $(\mathfrak{B}, \overline{b}) \models \varphi$, then $(\mathfrak{B}, \overline{b}) \models \bigvee_{(\mathfrak{A}, \overline{a}) \models \varphi} \bigwedge \Phi(\mathfrak{A}, \overline{a})$, since $\mathfrak{B} \models \Phi(\mathfrak{B}, \overline{b})$.

Conversely, assume that $(\mathfrak{B}, \overline{b}) \models \bigvee_{(\mathfrak{A}, \overline{a}) \models \varphi} \bigwedge \Phi(\mathfrak{A}, \overline{a})$. Then for some model $(\mathfrak{A}, \overline{a})$ of φ, $(\mathfrak{B}, \overline{b}) \models \Phi(\mathfrak{A}, \overline{a})$. By definition of $\Phi(\mathfrak{A}, \overline{a})$, $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{B}, \overline{b})$ satisfy the same formulas of Φ and satisfy, by assumption, $(\mathfrak{A}, \overline{a}) \equiv_k (\mathfrak{B}, \overline{b})$. Hence $(\mathfrak{B}, \overline{b}) \models \varphi$.

It remains to show that the transformation is effective. Let ψ be a formula such that $\mathfrak{A} \in \mathscr{C}$ if and only if $\mathfrak{A} \models \psi$. Let $\Gamma_1, \ldots, \Gamma_s$ be an enumeration of the finitely many sets Γ with the property that for each $\gamma \in \Phi$, either $\gamma \in \Gamma$ or $\neg\gamma \in \Gamma$. As shown above, for some $I \subseteq \{1, \ldots, s\}$ the formula φ is equivalent to $\psi_I := \bigvee_{i \in I} \bigwedge \Gamma_i$ on structures from $\mathscr{C}$ and hence the formula $\psi \to (\varphi \leftrightarrow \psi_I)$ is valid. We can test this for all I in parallel and eventually find the desired Boolean combination. □

Note that again we cannot decide whether a set Γ_i is a Φ-type and that the above method does not provide a way to decide whether φ is satisfiable.

The equivalence relation $\equiv_k$ can be characterised in an algebraic way which greatly helps to understand the expressive power of first-order logic.

$(\mathfrak{A},\overline{a})$ and $(\mathfrak{B},\overline{b})$ are *k-isomorphic*, written $(\mathfrak{A},\overline{a}) \cong_k (\mathfrak{B},\overline{b})$, if there is a sequence $(I_j)_{0\leq j\leq k}$ with the following properties.

- For every non-negative integer j, $0 \leq j \leq k$, I_j is a non-empty set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and $\overline{a} \mapsto \overline{b} \in I_k$.
- (*Forth property*) For every non-negative integer j, $0 \leq j < k$, $q \in I_{j+1}$ and $a \in V(\mathfrak{A})$ there is $p \in I_j$ such that p extends q and $a \in \mathrm{dom}(p)$.
- (*Back property*) For every non-negative integer j, $0 \leq j < k$, $q \in I_{j+1}$ and $b \in V(\mathfrak{B})$ there is $p \in I_j$ such that p extends q and $b \in \mathrm{ran}(p)$.

Hence a partial isomorphism from I_j can be extended for j times while preserving the back and forth property (the back-and-forth property is not required to hold for I_0). The sequence $(I_j)_{0\leq j\leq k}$ is called a *back-and-forth system*.

Back-and-forth systems can be understood in a very intuitive, game theoretic way. The *Ehrenfeucht-Fraïssé game* $G_k(\mathfrak{A},\overline{a},\mathfrak{B},\overline{b})$ is played by two players, called *Spoiler* and *Duplicator*. Each play of the game has k rounds, where in the i-th round Spoiler selects one of the structures $\mathfrak{A}$ or $\mathfrak{B}$ and an element of the chosen structure. If he chooses a_{m+i} in $\mathfrak{A}$, then Duplicator answers with an element b_{m+i} in $\mathfrak{B}$ and if he chooses b_{m+i} in $\mathfrak{B}$, then Duplicators answers with a_{m+i} in $\mathfrak{A}$. Duplicator *wins* the play after k rounds if $\{(a_i,b_i) : 1 \leq i \leq m+k\}$ is a partial isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$. Otherwise, Spoiler wins. Equivalently, Spoiler wins the play if after some $\ell \leq k$, $\{(a_i,b_i) : 1 \leq i \leq m+\ell\}$ is not a partial isomorphism. We say that a player has a *winning strategy* for $G_k(\mathfrak{A},\overline{a},\mathfrak{B},\overline{b})$, or shortly that he *wins* $G_k(\mathfrak{A},\overline{a},\mathfrak{B},\overline{b})$, if he can win each play of the game whatever choices are made by the opponent. We omit a formal definition of games and winning strategies because we will work with back-and-forth systems and simply note that Duplicator wins $G_k(\mathfrak{A},\overline{a},\mathfrak{B},\overline{b})$ if and only if there is a back-and-forth system $(I_j)_{0\leq j\leq k}$ for $\mathfrak{A}$ and $\mathfrak{B}$ with $\overline{a} \mapsto \overline{b} \in I_k$.

For a non-negative integer $n \geq 0$, a τ-structure $\mathfrak{A}$ and $\overline{a}_n \in V(\mathfrak{A})^n$, let $\overline{x}_n := (x_1,\ldots,x_n)$ and let $\Phi^0_{\mathfrak{A},\overline{a}_n}$ be the set of atomic or negated atomic formulas $\varphi(\overline{x}_n)$ such that $\mathfrak{A} \models \varphi(\overline{a}_n)$. Let

$$\varphi^0_{\mathfrak{A},\overline{a}_n}(\overline{x}_n) := \bigwedge \Phi^0_{\mathfrak{A},\overline{a}_n}.$$

$\Phi^0_{\mathfrak{A},\overline{a}_n} \subseteq \Phi(\tau,0,n)$ is finite, and hence the above conjunction is a well defined formula. For an integer $i > 0$, assume that

$$\Phi^{i-1}_{n+1} := \{\varphi^{i-1}_{\mathfrak{A},\overline{a}_{n+1}}(\overline{x}_{n+1}) : \mathfrak{A} \ \tau\text{-structure}, \overline{a}_{n+1} \in V(\mathfrak{A})^{n+1}\} \subseteq \Phi(\tau, i-1, n+1)$$

has been defined and is finite. Then

$$\varphi^i_{\mathfrak{A},\overline{a}_n}(\overline{x}_n) := \bigwedge_{a \in V(\mathfrak{A})} \exists x_{n+1} \varphi^{i-1}_{\mathfrak{A},\overline{a}_n a}(\overline{x}_n, x_{n+1}) \wedge \forall x_{n+1} \bigvee_{a \in V(\mathfrak{A})} \varphi^{i-1}_{\mathfrak{A},\overline{a}_n a}(\overline{x}_n, x_{n+1}),$$

where we remove duplicate entries from the conjunctions and disjunctions is a well defined formula. We call $\varphi^k_{\mathfrak{A},\overline{a}_n}(\overline{x}_n)$ the *rank-k Hintikka type* of $\overline{a}_n \in V(\mathfrak{A})^n$. Note that $\varphi^k_{\mathfrak{A},\overline{a}}(\overline{x})$ for $\overline{x} = (x_1, \ldots, x_m)$ uses only the variables $x_1, \ldots, x_{k+m}$ and that all Boolean combinations are in disjunctive normal form (the outermost disjunction contains exactly one disjunct), hence it is a normalised formula.

The following theorem which links rank-k types, rank-k Hintikka types and back-and-forth systems, is known as the Theorem of Ehrenfeucht and Fraïssé [44, 52].

Theorem 2.3.2 *Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures, let $k, m \geq 0$ be non-negative integers and let $\overline{a} \in V(\mathfrak{A})^m$, $\overline{b} \in V(\mathfrak{B})^m$. The following are equivalent.*

1. $\mathrm{tp}_k(\mathfrak{A}, \overline{a}) = \mathrm{tp}_k(\mathfrak{B}, \overline{b})$.
2. $\mathfrak{B} \models \varphi^k_{\mathfrak{A},\overline{a}}(\overline{b})$.
3. *There is a back-and-forth system $(I_j)_{0 \leq j \leq k}$ for $\mathfrak{A}$ and $\mathfrak{B}$ and $\overline{a} \mapsto \overline{b} \in I_k$.*

As rank-k Hintikka types are normalised formulas from $\Phi(\tau, k, m)$ and by Theorem 2.3.2 each rank-k type is uniquely determined by some rank-k Hintikka type, namely by the rank-k Hintikka type it contains, we can replace rank-k types by rank-k Hintikka types. Hence, every formula of quantifier rank k is either unsatisfiable or equivalent to a disjunction of rank-k Hintikka types.

2.4. Definable Interpretations

Interpretations concern the logical definition of one structure within another. Let $\mathfrak{A}$ be a τ-structure and let $m, \ell \geq 0$ be non-negative integers. A formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_\ell) \in \mathrm{FO}[\tau]$ naturally defines the query

$$\varphi[\mathfrak{A}, \overline{b}] := \{\overline{a} \in V(\mathfrak{A})^m : \mathfrak{A} \models \varphi(\overline{a}, \overline{b})\}$$

on $\mathfrak{A}$ with parameters $\overline{b} \in V(\mathfrak{A})^\ell$. Alternatively, φ may be seen as defining a structure of vocabulary $\{R\}$, where R is an m-ary relation symbol, namely the structure $(V(\mathfrak{A}), \varphi[\mathfrak{A}, \overline{b}])$.

This concept is generalised by the notion of *first-order definable interpretations*. Let σ be a vocabulary and let $k > 0, \ell \geq 0$ be integers. A *first-order definable k-ary (σ, τ)-interpretation with ℓ parameters* $\mathscr{I} = \left(\varphi_u, (\varphi_R)_{R \in \sigma}\right)$ consists of

- a formula $\varphi_u(x_1, \ldots, x_k, y_1, \ldots, y_\ell) \in \mathrm{FO}[\tau]$ defining the universe of the interpreted structure (which is a subset of $V(\mathfrak{A})^k$), and
- formulas $\varphi_{R_i}(\overline{x}, y_1, \ldots, y_\ell) \in \mathrm{FO}[\tau]$ for $R_i \in \sigma$ of arity m_i, where

  $$\overline{x} = (x_{(1,1)}, \ldots, x_{(1,k)}, \ldots, x_{(m_i,1)}, \ldots, x_{(m_i,k)}),$$

  defining the relation R_i of the interpreted structure.

If $\varphi_u[\mathfrak{A}, \overline{b}] \neq \emptyset$, then the σ-structure defined by $\mathscr{I}$ over $\mathfrak{A}$ and $\overline{b}$ is

$$\mathscr{I}(\mathfrak{A}, \overline{b}) = \left(\varphi_u[\mathfrak{A}, \overline{b}], \left(\varphi_{R_i}[\mathfrak{A}, \overline{b}] \cap (\varphi_u[\mathfrak{A}, \overline{b}])^{m_i}\right)\right)_{R_i \in \sigma}.$$

Every (σ, τ)-interpretation also induces a translation from σ-formulas to τ-formulas.

Lemma 2.4.1 (Interpretation Lemma) *Let $k > 0, \ell \geq 0$ be integers. Let $\mathscr{I}$ be a k-ary (σ, τ)-interpretation with ℓ parameters and let $\varphi(x_1, \ldots, x_m)$ be a σ-formula. There is a formula $\psi = \mathscr{I}(\varphi) \in \mathrm{FO}[\tau]$ with free variables $x_{(1,1)}, \ldots, x_{(1,k)}, \ldots, x_{(m,1)}, \ldots, x_{(m,k)}, y_1, \ldots, y_\ell$ such that for all τ-structures $\mathfrak{A}$ and all $\overline{b} \in V(\mathfrak{A})^\ell$, if $\varphi_u[\mathfrak{A}, \overline{b}] \neq \emptyset$, then for all*

$$\overline{c} = \left((c_{(1,1)}, \ldots, c_{(1,k)}), \ldots, (c_{(m,1)}, \ldots, c_{(m,k)})\right) \in (\varphi_u[\mathfrak{A}, \overline{b}])^k$$

we have

$$\mathscr{I}(\mathfrak{A}, \overline{b}) \models \varphi(\overline{c}) \iff \mathfrak{A} \models \psi(\overline{c}, \overline{b}).$$

Here we interprete $\overline{c}$ on the left side of the equivalence as an m-tuple over $V(\mathfrak{A})^k$ and on the right side of the equivalence as a km-tuple over $V(\mathfrak{A})$. The quantifier rank of ψ depends only on $\mathscr{I}$.

PROOF. Assume that for some integer s, φ uses only the variables $x_1, \dots, x_s$. Then $(x_{(i,1)}, \dots, x_{(i,k)})$ in ψ will take the role of x_i in φ for all $1 \leq i \leq s$. An easy induction shows that the following interpretation of formulas satisfies the claimed properties.

- $\mathscr{I}(\top) = \top$ and $\mathscr{I}(\bot) = \bot$,
- $\mathscr{I}(x_i = x_j) = (x_{(i,1)} = x_{(j,1)}) \wedge \ldots \wedge (x_{(i,k)} = x_{(j,k)})$,
- $\mathscr{I}(R(x_{i_1}, \dots, x_{i_t})) = \varphi_R(x_{(i_1,1)}, \dots, x_{(i_1,k)}, \dots, x_{(i_t,1)}, \dots, x_{(i_t,k)}, y_1, \dots, y_\ell)$ for each t-ary relation symbol $R \in \sigma$,
- $\mathscr{I}(\chi_1 \circ \chi_2) = \mathscr{I}(\chi_1) \circ \mathscr{I}(\chi_2)$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ and $\mathscr{I}(\neg\chi) = \neg(\mathscr{I}(\chi))$,
- $\mathscr{I}(\exists x_i \chi) = \exists x_{(i,1)} \dots \exists x_{(i,k)}(\varphi_u(x_{(i,1)}, \dots, x_{(i,k)}, y_1, \dots, y_\ell) \wedge \mathscr{I}(\chi))$ and
- $\mathscr{I}(\forall x_i \chi) = \forall x_{(i,1)} \dots \forall x_{(i,k)}(\varphi_u(x_{(i,1)}, \dots, x_{(i,k)}, y_1, \dots, y_\ell) \rightarrow \mathscr{I}(\chi))$ □

Similarly, we can define MSO-definable interpretations. However, we must restrict ourselves to one-dimensional interpretations, as MSO can quantify only over sets and not over k-ary relations.

Lemma 2.4.1 gives rise to an algorithmic method which allows to transfer complexity results from one class of structures to other classes of structures. Let $\mathscr{D}$ be a class of σ-structures and let $\mathscr{C}$ be a class of τ-structures such that the first-order (monadic second-order) model-checking problem can be solved efficiently on $\mathscr{C}$. If there is a (σ, τ)-interpretation $\mathscr{I}$ such that for each $\mathfrak{B} \in \mathscr{D}$ we can efficiently compute a structure $\mathfrak{A} \in \mathscr{C}$ such that $\mathscr{I}(\mathfrak{A}) \cong \mathfrak{B}$, then we can also efficiently solve the model-checking problem on $\mathscr{D}$. On input $\mathfrak{B}$ and $\varphi \in \mathrm{FO}[\sigma]$, in order to decide whether $\mathfrak{B} \models \varphi$, we compute $\mathfrak{A} \in \mathscr{C}$ such that $\mathscr{I}(\mathfrak{A}) \cong \mathfrak{B}$ and then decide whether $\mathfrak{A} \models \mathscr{I}(\varphi)$.

2.4.1. Defining distances – the Gaifman graph of a structure

For a τ-structure $\mathfrak{A}$, we define a binary relation $\sim$ on $V(\mathfrak{A})$, where two elements $a, b \in V(\mathfrak{A})$ stand in relation if $a \neq b \in V(\mathfrak{A})$ and a and b occur

together in some tuple of a relation of $\mathfrak{A}$. We call $G(\mathfrak{A}) := (V(\mathfrak{A}), \sim)$ the *Gaifman graph* of $\mathfrak{A}$. It is common in model theory to consider the Gaifman graph of a structure in order to transfer concepts, such as distances or structural measures for graphs, from graph theory to model theory.

The *distance* between a and b in $\mathfrak{A}$ is the minimum number r such that there are $a_2, \ldots, a_{r-1}$ such that $a \sim a_2, a_2 \sim a_3, \ldots, a_{r-1} \sim b$ or ∞ if no such r exists and is denoted by $\mathrm{dist}_{\mathfrak{A}}(a,b)$, or simply by $\mathrm{dist}(a,b)$, if there is no danger of confusion. For an integer $r \geq 0$, the *r-neighbourhood* $N_r^{\mathfrak{A}}(a)$, or simply $N_r(a)$, of a is the set of elements at distance at most r from a in $\mathfrak{A}$,

$$N_r^{\mathfrak{A}}(a) = \{b \in V(\mathfrak{A}) : \mathrm{dist}(a,b) \leq r\}.$$

For an integer $k > 0$ and a tuple $\overline{a} = (a_1, \ldots, a_k) \in V(\mathfrak{A})^k$, we let

$$N_r^{\mathfrak{A}}(\overline{a}) := \bigcup_{1 \leq i \leq k} N_r(a_i).$$

We can find the r-neighbourhood of a tuple $\overline{a} \in V(\mathfrak{A})^m$ in $\mathfrak{A}$ via a one-dimensional (τ, τ)-interpretation with m parameters.

Lemma 2.4.2 *For every integer $r \geq 0$ there is a τ-formula $\delta_{\leq r}(x,y)$ such that for all τ-structures $\mathfrak{A}$ and $a, b \in V(\mathfrak{A})$,*

$$\mathfrak{A} \models \delta_{\leq r}(a,b) \Longleftrightarrow \mathrm{dist}_{\mathfrak{A}}(a,b) \leq r.$$

PROOF. Assume that $\tau = \{R_1, \ldots, R_s\}$, where R_i is of arity m_i for $1 \leq i \leq s$. Then

$$\delta_{\leq 0}(x,y) := (x = y)$$

and for $r > 0$

$$\delta_{\leq r+1}(x,y) := \delta_{\leq r}(x,y) \vee \exists z \Big(\delta_{\leq r}(x,z) \wedge \bigvee_{1 \leq i \leq s} \exists u_1 \ldots \exists u_{m_i} \big(R_i(u_1, \ldots, u_{m_i}) \wedge \bigvee_{1 \leq j, \ell \leq m_i} (u_j = z \wedge u_\ell = y) \big) \Big). \quad \square$$

We write $\delta_{>r}$ for $\neg \delta_{\leq r}$. For a tuple $\overline{x} = (x_1, \ldots, x_m)$ let

$$\delta_{\leq r}(\overline{x}, y) := \bigvee_{1 \leq i \leq m} \delta_{\leq r}(x_i, y).$$

Now we can define the r-neighbourhood of $\overline{a}$ by by $\mathscr{I}_r = (\varphi_u, (\varphi_R)_{r \in \tau})$, where $\varphi_u(x, \overline{y}) = \delta_{\leq r}(\overline{y}, x)$ and $\varphi_R(\overline{x}, \overline{y}) = R\overline{x}$ for all $R \in \tau$. For a formula ψ, we write $\psi^{(r)}$ for $\mathscr{I}_r(\psi)$.

Corollary 2.4.3 *Let $\mathfrak{A}$ be a τ-structure, let $m > 0, r \geq 0$ be integers and let $\overline{a} \in V(\mathfrak{A})^m$. Then*

$$\mathfrak{A}[N_r(\overline{a})] \models \psi(\overline{a}) \Longleftrightarrow \mathfrak{A} \models \psi^{(r)}(\overline{a}) \quad .$$

The quantifier rank of $\psi^{(r)}$depends on the quantifier rank of ψ and on τ, r and s only.

A formula $\psi(\overline{x})$ is called *r-local around* $\overline{x} \in V^m$ if for all $\mathfrak{A}$ and $\overline{a}$ in $V(\mathfrak{A})^m$,

$$\mathfrak{A}[N_r(\overline{a})] \models \psi(\overline{a}) \Longleftrightarrow \mathfrak{A} \models \psi(\overline{a}).$$

2.5. Locality of first-order logic

A fundamental property of first-order logic is its *locality*. In this section we are going to review the main ingredients of Gaifman's Theorem [54], which states that the truth of a formula $\varphi(\overline{a})$ depends only on the local neighbourhood of $\overline{a}$ and on whether there exist scattered sets of elements with a certain local type. The proof of our extended locality theorem in Section 7.2 parallels the proof of Gaifman's theorem.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures and let $m \geq 0$ be a non-negative integer. Let $\overline{a} \in V(\mathfrak{A})^m$ and $\overline{b} \in V(\mathfrak{B})^m$. We are looking for a combinatorial condition that guarantees the existence of a back-and-forth system $(I_j)_{0 \leq j \leq k}$ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $\overline{a} \mapsto \overline{b} \in I_k$. Let us assume $\overline{c} \mapsto \overline{d} \in I_j$ and consider the forth-property: For any $c \in V(\mathfrak{A})$ there must be $d \in V(\mathfrak{B})$ such that $\overline{c}c \mapsto \overline{d}d \in I_{j-1}$. If c is chosen close to an element of $\overline{c}$, then the existence of d will follow from a local property of $\mathfrak{A}$ around $\overline{c}$, i.e. a property of $\mathfrak{A}[N_{r(j)}(\overline{c})]$ for some function $r(j)$ to be defined. If on the other hand c is chosen far from all elements of $\overline{c}$, then some global property of the structure must guarantee the existence of d. The global property in Gaifman's Theorem is based on the following combinatorial idea.

Let $r > 0$ be a positive integer. A set $S \subseteq V(\mathfrak{A})$ is *r-scattered*, if $N_r(a) \cap N_r(b) = \emptyset$ for all $a, b \in S$.

Recall that $\psi^{(r)}(\overline{x})$ relativises a formula ψ to the r-neighbourhood of $\overline{x}$. For a formula $\psi(x)$, the following lemma provides a necessary and sufficient condition for the existence of an element a which satisfies $\psi^{(r)}$ at distance greater than $2r+1$ from $\overline{a}$.

Lemma 2.5.1 *Let $\mathfrak{A}$ be a τ-structure, let $m > 0, r \geq 0$ be integers, $\overline{a} \in V(\mathfrak{A})^m$ and let $\psi(x) \in \mathrm{FO}[\tau]$. There is an element $a \in V(\mathfrak{A})$ with $\mathfrak{A} \models \psi^{(r)}(a)$ at distance greater than $2r+1$ from $\overline{a}$ if and only if one of the following holds:*

- *there is an element $a \in V(\mathfrak{A})$ with $\mathfrak{A} \models \psi^{(r)}(a)$ with*

$$2r+1 < \mathrm{dist}(\overline{a}, a) \leq 6r+3,$$

- *there exists a $(2r+1)$-scattered set of elements which all satisfy $\psi^{(r)}$ of size $s \leq m+1$ in $\mathfrak{A}$ but no such set of size s in the $(2r+1)$-neighbourhood of $\overline{a}$.*

PROOF. Observe that the $2r+1$-neighbourhood of each element $a_i \in \overline{a}$ does not contain two elements of mutual distance greater than $4r+2$, hence $N_{2r+1}(a_i)$ for each $a_i \in \overline{a}$ can contain at most one element of a $2r+1$-scattered set. Hence, every element which satisfies $\psi^{(r)}$ lies either at distance at most $6r+3$ to $\overline{a}$ or can be added to any $2r+1$-scattered set (of size at most m) in $N_{2r+1}(\overline{a})$. □

Our motivation to find an element at distance greater than $2r+1$ from $\overline{a}$ comes from the following observation.

Let us write $\mathfrak{A} \dot\cup \mathfrak{B}$ for the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$, where $V(\mathfrak{A} \dot\cup \mathfrak{B}) = V(\mathfrak{A}) \dot\cup V(\mathfrak{B})$ and $R(\mathfrak{A} \dot\cup \mathfrak{B}) = R(\mathfrak{A}) \cup R(\mathfrak{B})$ for all $R \in \tau$. For $\overline{a} \in V(\mathfrak{A})^m$ and $a \in V(\mathfrak{A})$, if $\mathrm{dist}(\overline{a}, a) > 2r+1$, then $N_r(\overline{a}) \cap N_r(a) = \emptyset$ and no relation involves a tuple with elements from both $N_r(\overline{a})$ and $N_r(a)$. Hence $\mathfrak{A}[N_r(\overline{a}) \cup N_r(a)]$ can be seen as the disjoint union of $\mathfrak{A}[N_r(\overline{a})]$ and $\mathfrak{A}[N_r(a)]$. This allows to obtain the rank-k type of $\mathfrak{A}[N_r(\overline{a}) \cup N_r(a)]$ from the rank-k types of $\mathfrak{A}[N_r(\overline{a})]$ and $\mathfrak{A}[N_r(a)]$, as proposed in the following lemma. The lemma, which goes back to Feferman and Vaught, can easily be proved by using the characterisation of k-equivalence by the existence of back-and-forth systems (see e.g. [94] for a proof of how to effectively compute the type of the combined structure from the type of the isolated structures).

Lemma 2.5.2 *Let $k \geq 0, m_1, m_2 > 0$ be integers and let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$ be τ-structures. If $\overline{a}_1 \in V(\mathfrak{A}_1)^{m_1}$, $\overline{a}_2 \in V(\mathfrak{A}_2)^{m_2}$, $\overline{b}_1 \in V(\mathfrak{B}_1)^{m_1}$ and $\overline{b}_2 \in V(\mathfrak{B}_2)^{m_2}$ satisfy*

$$\mathrm{tp}_k(\mathfrak{A}_1, \overline{a}_1) = \mathrm{tp}_k(\mathfrak{B}_1, \overline{b}_1) \quad \text{and} \quad \mathrm{tp}_k(\mathfrak{A}_2, \overline{a}_2) = \mathrm{tp}_k(\mathfrak{B}_2, \overline{b}_2),$$

then

$$\mathrm{tp}_k(\mathfrak{A}_1 \dot\cup \mathfrak{A}_2, \overline{a}_1 \overline{a}_2) = \mathrm{tp}_k(\mathfrak{B}_1 \dot\cup \mathfrak{B}_2, \overline{b}_1 \overline{b}_2).$$

By Corollary 2.4.3, the conditions in Lemma 2.5.1 are first-order definable and the quantifier rank of the respective formulas depend on the quantifier rank of ψ and τ, m and r only.

A sentence is *basic local with parameters r and s* if it has the form

$$\exists x_1 \dots \exists x_s \Big(\bigwedge_{1 \le i < j \le s} \delta_{>2r}(x_i, x_j) \wedge \bigwedge_{1 \le i \le s} \psi^{(r)}(x_i) \Big),$$

where $\psi(x)$ is a first-order formula. It states that there is an r-scattered set R of size s such that all elements $a \in R$ satisfy $\mathfrak{A}[N_r(a)] \models \psi(a)$.

Theorem 2.5.3 (Gaifman [54]) *Every* FO$[\tau]$*-formula $\varphi(\overline{x})$ can effectively be translated into an equivalent formula ψ which is a Boolean combination of r-local formulas around $\overline{x}$ and basic local sentences with parameters r and s. If* $\operatorname{qr}(\varphi) = k$ *and* $\overline{x} = (x_1, \dots, x_m)$*, then* $s \le k + m$ *and* $r \le 7^k$.

For the proof one shows that there is a function $f(\tau, m, k)$ such that for all τ-structures $\mathfrak{A}, \mathfrak{B}$ and $\overline{a} \in V(\mathfrak{A})^m, \overline{b} \in V(\mathfrak{B})^m$, if $((\mathfrak{A}, \overline{a}) \models \chi \Leftrightarrow (\mathfrak{B}, \overline{b}) \models \chi)$ for all $r(k)$-local formulas of quantifier rank $f(\tau, m, k)$ and $(\mathfrak{A} \models \chi \Leftrightarrow \mathfrak{B} \models \chi)$ for all basic local sentences with parameters $r(k)$ and $s(k)$ of quantifier rank $f(\tau, m, k)$), then

$$(\mathfrak{A}, \overline{a}) \equiv_k (\mathfrak{B}, \overline{b}).$$

Then the claim follows from Theorem 2.3.1. The function f is defined inductively in the course of the proof.

For more background on first-order logic (and especially first-order logic in the finite) we refer to [43, 59, 89].

3. Graphs

3.1. Graphs

A *graph* G is a pair consisting of a finite and non-empty *vertex* set $V(G)$ and an *edge* set $E(G) \subseteq [V(G)]^2$, the set of all 2-element subsets of $V(G)$. We always assume $V(G) \cap E(G) = \emptyset$. In the following, let G and H be graphs, let u, v, w be vertices of G and let e be an edge of G. Let k be a non-negative integer. For convenience, we usually write $uv \in E(G)$ (or $vu \in E(G)$) instead of $\{u, v\} \in E(G)$. The number of vertices of G is its *order*, denoted by $n(G)$, and its number of edges is its *size*, denoted by $m(G)$. If $uv \in E(G)$ then u and v are *adjacent* or *neighbours* and u and v are *incident with* or the *ends of* the edge uv. Two edges $e \neq f$ are *adjacent* if they have a vertex in common.

We denote by $N_G(v) := \{u : uv \in E(G)\}$, or simply by $N(v)$ if there is no danger of confusion, the set of *neighbours* of v in G. Their number is called the *degree* of v, denoted by $d_G(v)$, or simply by $d(v)$. We have

$$\sum_{v \in V(G)} d(v) = 2 \cdot m(G).$$

An important measure is the ratio

$$\epsilon(G) := \frac{m(G)}{n(G)}$$

which is called the *edge density* of G. It is closely related to the *average degree* of G

$$d(G) := \frac{1}{n(G)} \sum_{v \in V(G)} d(v) = 2 \cdot \epsilon(G).$$

The number

$$\delta(G) := \min\{d(v) : v \in V(G)\}$$

is the *minimum degree* of G and the number

$$\Delta(G) := \max\{d(v) : v \in V(G)\}$$

is its *maximum degree*. If all vertices of G have the same degree k, then G is *k-regular*, or simply *regular*. Clearly,

$$\delta(G) \le d(G) \le \Delta(G)$$

and for every integer $k > 0$, at most $(d(G)/k) \cdot n(G)$ vertices of G can have degree at least k.

A set of vertices is *independent* (or *stable*) if no two of its elements are adjacent. Conversely, a graph is *complete* if all of its vertices are adjacent. The complete graph of order n is denoted K_n.

A set $D \subseteq V(G)$ *dominates* G if every vertex of G either lies in D or is adjacent to a vertex of D.

If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, written as $H \subseteq G$. If $\mathscr{C}$ is a class of graphs which is closed under subgraphs, i.e. $H \subseteq G$ for some graph $G \in \mathscr{C}$ implies $H \in \mathscr{C}$, then $\mathscr{C}$ is called *monotone*. If $H \subseteq G$ and $H \neq G$, then H is a *proper subgraph* of G. If $H \subseteq G$ and $E(H) = E(G) \cap [V(H)]^2$, then H is an *induced subgraph* of G. If $\mathscr{C}$ is a class of graphs which is closed under induced subgraphs, then $\mathscr{C}$ is called *hereditary*. For a subset $U \subseteq V(G)$ we write $G[U]$ for the induced subgraph of G with vertex set U and say that U *induces* or *spans* $G[U]$ in G. If $H \subseteq G$ we abbreviate $G[V(H)]$ to $G[H]$. If $H \subseteq G$ with $V(H) = V(G)$, we call H a *spanning subgraph* of G. We write $G - U$ for $G[V(G) \setminus U]$. For a subset $F \subseteq [V(G)]^2$ we write $G - F$ for the graph $(V(G), E(G) \setminus F)$ and $G + F$ for the graph $(V(G), E(G) \cup F)$.

G is *k-degenerate* if every subgraph $H \subseteq G$ has a vertex of degree at most k. If G is k-degenerate then $m(G) < k \cdot n(G)$.

G is *isomorphic* to H, written $G \cong H$, if there is a bijection $\pi : V(G) \to V(H)$ with $uv \in E(G)$ if and only if $\pi(u)\pi(v) \in E(H)$ for all $u, v \in V(G)$. We do not distinguish between isomorphic graphs, e.g. when we write G contains H as a subgraph we mean that G has a subgraph isomorphic to H. A mapping $h : V(G) \to V(H)$ is a *homomorphism* from G to H if $uv \in E(H)$ implies $h(u)h(v) \in E(H)$ for all $u, v \in V(H)$.

G is *maximal* (*minimal*) with some property if there is no graph $H \supsetneq G$ ($H \subsetneq G$) with the property. G is *maximum* (*minimum*) with the property if there is no graph $H \supsetneq G$ with $n(H) > n(G)$ ($H \supsetneq G$ with $n(H) < n(G)$) satisfying the property.

A *walk* W of length k from $v_0 \in V(G)$ to $v_k \in V(G)$ in G is a sequence $v_0 v_1 \dots v_k$ with $v_i v_{i+1} \in E(G)$ for $i < k$. The vertex set of W is $V(W) =$

$\{v_0,\dots,v_k\}$ and the edge set of W is $E(W) = \{v_iv_{i+1} : i < k\}$. W is *closed* if $v_0 = v_k$. It is a *path* if all its vertices are distinct, a *trail* if all its edges are distinct and a *cycle* if $k \geq 3$, $v_0 = v_k$ and $v_i \neq v_j$ for $i < j < k$. We usually identify a path P and a cycle C with the graphs $(V(P),E(P))$ and $(V(C),E(C))$, respectively. We denote the path of length k by P_k and the cycle of length k by C_k. A path *links* or *connects* its *end-vertices* v_0 and v_k. The vertices $v_1,\dots,v_{k-1}$ are its *inner vertices*. A shortest walk between two vertices is always a path and every walk between two vertices contains a path between these vertices.

Two or more paths are *independent* or *internally vertex disjoint* if none of them contains an inner vertex of another. They are *strongly independent* if they have no vertex in common.

G is *connected* if any two of its vertices are connected by a path. If $U \subseteq V(G)$ and $G[U]$ is connected, we also call U itself connected (in G). A maximal connected subgraph of G is a *component* of G. G is *k-connected* if it has more than k vertices and remains connected whenever fewer than k vertices are removed.

The *distance* between u and v in G is the minimum length of a path linking u and v (or ∞ if no such path exists) and is denoted by $\mathrm{dist}_G(u,v)$, or simply by $\mathrm{dist}(u,v)$. The *d-neighbourhood* $N_d^G(u)$, or simply $N_d(u)$, of u is the set of vertices at distance at most d from u in G,

$$N_d^G(u) = \{v \in V(G) : \mathrm{dist}(u,v) \leq d\}.$$

The greatest distance between any two vertices in G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. A vertex u is *central* in G if

$$\max_{w\in V(G)} d(u,w) = \min_{v\in V(G)} \max_{w\in V(G)} d(v,w)$$

and the number

$$\mathrm{rad}(G) = \min_{v\in V(G)} \max_{w\in V(G)} d(v,w)$$

is the *radius* of G. We have

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2 \cdot \mathrm{rad}(G).$$

If G does not contain a cycle, it is called an *acyclic graph* or a *forest*. A connected forest is called a *tree*, hence a forest if a graph whose components are trees. The vertices of degree 1 in a tree are its *leaves*. It is often

convenient to consider one vertex of a tree as a special vertex, called the *root* of the tree. We call a tree T with a fixed root $r \in V(T)$ a *rooted tree*. The *standard tree-order* associated with T and r is the partial order $\leq$ on $V(T)$ with $u \leq v$ for $u, v \in V(T)$ if u lies on the unique path in T between v and r.

G is *k-partite* with *vertex classes* $V_1, \dots V_k$ if $V(G)$ is the disjoint union of the V_i and adjacent vertices come from different vertex classes. Instead of 2-partite we say *bipartite*. For $n_1, \dots, n_k \geq 1$, we denote by $K_{n_1,\dots n_k}$ the complete k-partite graph whose ith class contains exactly n_i vertices. Its number of edges is

$$\sum_{1 \leq i < j \leq k} n_i n_j.$$

A *k-colouring* of a set A is a mapping $c : A \to \{c_1, \dots, c_k\}$ to colours $c_1, \dots c_k$. A *proper colouring* of the vertices or edges of G is a colouring of $V(G)$ or $E(G)$, respectively, such that adjacent elements are assigned different colours. If G has a proper k-colouring of its vertices then it is called *k-colourable*. The *chromatic number* of G is

$$\chi(G) = \min\{k : G \text{ is } k\text{-colourable}\}.$$

If $\chi(G) = k$ we say that G is *k-chromatic*. In the following, when we speak of a colouring, unless explicitly stated differently, we mean a proper vertex colouring.

It is easy to see that if G is minimal k-chromatic, then $\delta(G) \geq k-1$. For if there is $v \in V(G)$ with $d(v) \leq k-2$, then a $(k-1)$-colouring of $G - v$ can be extended to a $(k-1)$-colouring of G. In particular, if $\chi(G) \geq k$, then G has a subgraph H with $\delta(H) \geq k-1$. Conversely, if G is k-degenerate then it is $k+1$-colourable and if G is k-colourable then it contains an independent set of size at least $n(G)/k$.

If G has large average degree, then it has a subgraph of large minimum degree. This is made precise in the following lemma.

Lemma 3.1.1 *Let G be a graph. If $m(G) > 0$, then G has a subgraph H with $\delta(H) > \epsilon(H) \geq \epsilon(G)$.*

The following lemma helps to gain more control over the size of the subgraph H.

Lemma 3.1.2 *Let G be a graph and let μ be a real with $0 < \mu < 1$. Then G has a subgraph H with*

$$\delta(H) \geq (1-\mu) \cdot \epsilon(G) \qquad \text{and} \qquad m(H) \geq \mu \cdot m(G).$$

In the previous lemmas, we eliminated vertices of small degree. As shown by Erdős and Simonovits [47], for sufficiently dense graphs, we can also eliminate high degrees. G is *d-almost-regular* if $\Delta(G) < d \cdot \delta(G)$.

Theorem 3.1.3 (Δ-almost-regularisation [47]) *Let G be a graph, let ϵ be a real with $0 < \epsilon < 1$ and let $d := 20 \cdot 2^{\epsilon^{-2}}$. If G is large and $m(G) > n(G)^{1+\epsilon}$, then G has a d-almost-regular subgraph H with*

$$m(H) > \frac{2}{5} n(H)^{1+\epsilon} \quad \text{and} \quad n(H) > n(G)^{\epsilon \frac{1-\epsilon}{1+\epsilon}}.$$

Note that G is k-colourable if and only if it is k-partite. This is why a vertex class in a k-partite graph is often referred to as a colour class. The difference between k-partite graphs and k-colourable graphs is that when we speak of k-partite graphs we usually fix the vertex classes but the colour classes of a k-colourable graph are almost never supposed to be given a priori.

We can make any graph k-partite with partitions of balanced size by dropping only few edges.

Lemma 3.1.4 *Let G be a graph. If $m(G) > 0$, then G contains a k-partite subgraph $H \subseteq K_{n_1,\dots,n_k}$ with $\sum_{1 \le i \le k} n_i = n(G)$, $|n_i - n_j| \le 1$ for $1 \le i, j \le k$ and $m(H) > (1 - \frac{1}{k}) \cdot m(G)$.*

A *directed graph* D is a pair consisting of a finite and non-empty vertex set $V(D)$ and an *arc* set $A(D) \subseteq V(D)^2$. We always assume $V(D) \cap A(D) = \emptyset$. An *orientation* of a graph G is a directed graph D with $V(D) = V(G)$ and which contains for each edge $uv \in E(G)$ exactly one of the arcs (u,v) or (v,u). We let

$$N_D^+(v) := \{w \in V(D) : (v,w) \in A(D)\}$$

be the *out-neighbours* of a vertex v and call

$$d_D^+(v) := |N_D^+(v)|$$

the *out-degree* of v. Similarly, $N_D^-(v) := \{u \in V(D) : (u,v) \in A(D)\}$ are the *in-neighbours* of v and $d_D^-(v) := |N_D^-(v)|$ is its *in-degree*. The *maximum out- and in-degree* of D are defined as

$$\Delta^+(D) := \max_{v \in V(D)} d_D^+(v) \quad \text{and} \quad \Delta^-(D) := \max_{v \in V(D)} d_D^-(v).$$

The following observation is crucial for the running times of our algorithms. If G is k-degenerate then it has an acyclic orientation D with $\Delta^-(D) \le k$.

Our representation of graphs as input for an algorithm is based on this observation.

Remark 3.1.5 We assume that vertices the vertices of G are labelled as $\{1,\dots,n(G)\}$ and are represented by these numbers in binary. For a k-degenerate graph G we store an (acyclic) orientation D of G such that $|N^+(v)| \leq k$ for all $v \in V(D)$. Even though it would suffice to store $N^+(v)$ for each vertex, we store both $N^+(v)$ and $N^-(v)$ (in some ordered way). This allows us to check adjacency for two vertices $u,v \in V(G)$ in time $O\big(k \cdot \log n(G)\big)$, by simply checking whether $u \in N^+(v)$ or $v \in N^+(u)$. It is easy to see that if G is given as a list of adjacency lists, we can compute k such that G is k-degenerate and the above representation in time $O\big(n(G) + m(G) \cdot \log n(G)\big)$. On the other hand, by storing both $N^+(v)$ and $N^-(v)$ for all vertices v, if $H = G[N_r(w)]$ is k-degenerate for some vertex w, then we can perform the first r levels of a breadth-first search starting at w in time $O\big(k \cdot n(H) \cdot \log n(G)\big)$. Along with the search we can compute a spanning breadth-first search tree of H and new vertex identifiers, such that the size the representation of a vertex is $\log n(H)$ in the subgraph H. Also note that $n \log n \leq n^{1+\epsilon}$ for all $\epsilon > 0$ and n sufficiently large.

A directed graph can be understood as a relational structure G over signature $\tau = \{E\}$, where E is a binary relation symbol. The structure G is an undirected graph if $E(G)$ is irreflexive, i.e. $(v,v) \notin E(G)$ for all $v \in V(G)$ and symmetric, i.e. if $(u,v) \in E(G)$ implies $(v,u) \in E(G)$ for all $u,v \in V(G)$. More generally, a *coloured graph* is a relational structure G over signature $\{C_1,\dots,C_k,E_1,\dots,E_\ell\}$, where the C_i are unary relation symbols and the E_j are binary relation symbols, $1 \leq i \leq k, 1 \leq j \leq \ell$. The C_i are called the *vertex colours* and the E_j are the *edge colours*. The structure G is an undirected coloured graphs if each E_j is irreflexive and symmetric.

3.2. Tree width and forbidden minors

Tree width (see e.g. [13, 34]) is a fundamental structural measure with many applications in graph structure theory and graph algorithms.

A *tree decomposition* of a graph G is a pair (T,X), where T is a tree and $X = \{X_t : t \in V(T)\}$ is a family of subsets of $V(G)$ (called bags) such that

- $\bigcup_{t\in V(T)} X_t = V(G)$,
- for every edge $\{u,v\}$ of G there exists $t \in V(T)$ with $u,v \in X_t$ and
- if $r,s,t \in V(T)$ and s is on the path of T between r and t, then $X_r \cap X_t \subseteq X_s$.

A graph G has *tree width* at most k if it admits a tree decomposition (T,X) such that $|X_t| \leq k+1$ for each $t \in V(T)$ and we write $\mathrm{tw}(G)$ for the tree width of G.

We may assume that every tree decomposition (T,X) has the property that for every $\{s,t\} \in E(T)$ we have $|X_s \setminus X_t| \leq 1$. We call such decompositions *smooth*.

A graph of small tree width has small *separators*, as described in the following lemma.

Lemma 3.2.1 *Let G be a graph and let (T,X) be a tree decomposition of G. If $r,s,t \in V(T)$, $u \in X_r$ and $v \in X_t$ and s lies on the path of T between r and t, then every path from u to v in G uses a vertex contained in X_s.*

Many problems which are intractable in general can be solved efficiently on graphs of bounded tree width. Most general is Courcelle's theorem [22] which states that every property of graphs of bounded tree width that are definable in monadic second-order logic can be decided in linear time.

An equivalent definition of tree width is in terms of *elimination orders*. We represent an order of $V(G)$ as an injective mapping $L : V(G) \to \mathbb{N}$. Let G be a graph and let L be a linear order on $V(G)$. Without loss of generality we have $V(G) = \{1,\dots,n\}$ and $L(i) = i$. With L we associate a sequence $G_0,\dots,G_{n-1}$ of graphs as follows. Let $G_0 := G$ and for $0 < i \leq n$, let $V(G_i) := V(G_{i-1}) \setminus \{i\}$ and

$$E(G_i) := \Big(E(G_{i-1}) \setminus \big\{\{i,j\} : j \leq n\big\}\Big) \cup \big\{\{\ell,j\} : \{\ell,i\},\{i,j\} \in E(G_{i-1})\big\},$$

i.e. we eliminate vertex i and make a clique out of the neighbours of i in G_{i-1}. We call the maximum size of a clique over all G_i the *elimination width* of the order. The *elimination width* of G is the minimum width over all possible widths of elimination orders of G. It is well known that the tree width of G is equal to its elimination width, see e.g. [13].

Another definition of tree width which is easily seen to be equivalent is the following. A graph G has *tree width* at most k if and only if its vertices can be ordered such that each vertex v is separated from all smaller vertices by at most k smaller vertices, that is, for every $v \in V(G)$ there are $w_1,\ldots,w_k$ with $L(w_1) \leq \ldots \leq L(w_k) < L(v)$ such that every path between v and w with $L(w) < L(v)$ contains one of $w_1,\ldots,w_k$.

A graph H with vertex set $\{v_1,\ldots,v_n\}$ is a *minor* of a graph G, denoted $H \preccurlyeq G$, if there are pairwise vertex disjoint connected subgraphs $G_1,\ldots,G_n$ of G such that for every edge v_iv_j of H there is at least one edge $e_{ij} \in E(G)$ joining G_i and G_j in G.

Robertson and Seymour, in a series of 23 papers [119], developed a deep theory of graph minors. One of the best known results from their theory is their proof of Wagner's conjecture, stating that graphs are well-quasi ordered with respect to the graph minor relation.

Theorem 3.2.2 ([122]) *Let $G_1,G_2,\ldots$ be an infinite sequence of graphs. Then there are two integers $i < j$ such that $G_i \preccurlyeq G_j$.*

A class $\mathscr{C}$ of graphs is *minor closed* if for each $G \in \mathscr{C}$, every minor $H \preccurlyeq G$ belongs to $\mathscr{C}$ as well. A minor closed class $\mathscr{C}$ is *proper* if it is not the class of all graphs. Each minor closed class $\mathscr{C}$ is characterised by a set $\mathscr{F}$ of *forbidden minors*, that is, the set of graphs that do not belong to $\mathscr{C}$ but all their subgraphs do. The result of Robertson and Seymour implies that every minor closed class is characterised by a finite set of forbidden minors.

The concept of tree width is central in Robertson and Seymour's analysis. Maybe the most important part of their theory is a decomposition theorem describing the structure of all graphs that exclude a fixed minor. At a high level, every graph which excludes a fixed minor can be decomposed into a collection of graphs, each of which can be "almost" embedded into a surface of bounded genus, and which are combined in a tree-like structure.

The *local tree width* of a graph G is the function $\mathrm{ltw}^G : \mathbb{N} \to \mathbb{N}$ which associates with every $r \geq 0$ the number

$$\mathrm{ltw}^G(r) := \max\{\mathrm{tw}(G[N_r(v)]) : v \in V(G)\}.$$

A class $\mathscr{C}$ of graphs is said to have *bounded local tree width* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathscr{C}$ and all integers $r \geq 0$ we have $\mathrm{ltw}^G(r) \leq f(r)$.

Based on Robertson and Seymour's results, Grohe [60] obtained another decomposition of graphs that exclude a fixed minor. Informally, every such graph can be decomposed into a tree of graphs that, after removing a bounded number of vertices, have bounded local tree-width. This decomposition can be used to find a PTAS for many optimization problems [60] and is also the basis for the first-order model-checking algorithm on classes that exclude a fixed minor [50].

By Gaifman's Theorem, first-order formulas can express only local properties of graphs. It was shown that for efficient first-order model-checking it suffices in many cases that a class has locally nice properties, e.g. classes of bounded local tree width [53], classes locally excluding a minor [28] and classes of locally bounded expansion [42], which will be defined later.

However, it turns out that there is a much more general way to guarantee that a class has good local properties. Instead of locally excluding a minor, we may exclude *bounded-depth minors*. This leads to the definition of *nowhere dense classes* as introduced by Nešetřil and Ossona de Mendez [107], which will be introduced in the next section. Surprisingly, the notion of *generalised colouring numbers* as introduced by Kierstead and Yang [77], which can be understood as a local variant of tree width, also provide a characterisation of nowhere dense graphs.

3.3. Nowhere dense classes of graphs

For a non-negative integer $r \geq 0$, a graph H with vertex set $\{v_1, \ldots, v_n\}$ is a *depth-r minor* of a graph G, denoted $H \preccurlyeq_r G$, if there are pairwise vertex disjoint connected subgraphs $G_1, \ldots, G_n$ of G, each of *radius* at most r, such that for every edge $v_i v_j$ of H there is at least one edge $e_{ij} \in E(G)$ joining G_i and G_j in G. The set of all depth-r minors of G is denoted by $G \triangledown r$.

Let $t \geq 3$ be an integer. It is a classical result of Mader that graphs with $K_t \not\preccurlyeq G$ are sparse. In [91], Mader showed that every graph of average degree at least $c(t) = 2^{t-2}$ contains K_t as a minor, before showing that $c(t) \in O(t \log t)$ [92]. Kostochka [81] and Thomason [128] independently found the correct order, showing $c(t) \in \Theta(t\sqrt{\log t})$. Thomason [129] then showed that $c(t) = (\alpha + o(1))t\sqrt{\log t}$ for a very small constant α. It was shown by Fiorini et al. [48] that for every $\epsilon > 0$, if G has average degree at least $2^{t-1} + \epsilon$ then there is a constant $C(\epsilon, t)$ such G contains a K_t minor which uses only $C(\epsilon, t) \cdot \log n(G)$ vertices. Shapira and Sudakov [126] showed that

average degree $c(t)+\epsilon$ suffices to find a K_t minor which uses $C(\epsilon,t)(\log n(G))^3$ vertices and finally, Montgomery [97] showed that this degree suffices to find the K_t minor on $C(\epsilon,t)\log n(G)$ vertices. Hence, a linear edge density forces a low depth minor model of K_t.

Surprisingly, the question of what edge density forces a constant depth minor model of K_t was studied only much later. But let us first turn to subdivisions or topological minors of a graph.

A graph H is a *subdivision* of a graph G if H is obtained from G by replacing each edge by a path. H is a *topological subgraph* or *topological minor* of G, denoted $H \preccurlyeq^t G$, if a subdivision of H is isomorphic to a subgraph of G. For an integer $r \geq 0$, a graph H is an r-subdivision of G if H can be obtained from G by replacing each edge by a path of length at most $r+1$. Observe that if H is an r-subdivision of G then $H \preccurlyeq_{\lceil r/2 \rceil} G$, hence we define the topological depth-r minor relation as follows. A graph H is a *topological depth-r minor of G*, written $H \preccurlyeq^t_r G$, if a $2r$-subdivision of H is isomorphic to a subgraph of G. We write $G \,\widetilde{\triangledown}\, r$ for the set of topological depth-r minors of G.

Again it was Mader [91, 93] who showed first that graphs which exclude a fixed graph as a topological minor are sparse, i.e. that there is a constant $s(t)$ such that every graph G with average degree at least $s(t)$ satisfies $K_t \preccurlyeq^t G$. Bollobás and Thomason [16] and independently Komlós and Szemerédi [79] found the correct order, showing that $s(t) \in \Theta(t^2)$. The best known bound at this point of time is by Kühn and Osthus [86], showing that $s(t) \leq (1+o(1))10t^2/23$. The question of how many vertices must be used by a topological minor was considered by Kostochka and Pyber [80], who showed that if G has edge density $4^{t^2} n(G)^\epsilon$ then $K_t \preccurlyeq^t G$ and the model which witnesses this uses at most $7t^2 \log t/\epsilon$ many vertices. Fiorini et al. [48] remark that with the same methods one can show that for every integer $t \geq 2$ and $\epsilon > 0$ there is a constant $C(t,\epsilon)$ such that every graph with average degree at least $4^{t^2} + \epsilon$ contains K_t as a subdivision which uses $C(t,\epsilon)\log n$ vertices.

Dvořák [40] was the first to show that for every real number $\epsilon > 0$ there is a constant c depending only on ϵ and not on t, such that all sufficiently large graphs G (only the order of G depends on t and ϵ) of edge density at least $n(G)^\epsilon$ contain a c-subdivision of K_t. Jiang [70] independently found this result and provides the best bound for the constant $c(\epsilon)$ known today, $c(\epsilon) \leq \lfloor 10/\epsilon \rfloor$. Note that this bound is almost best possible. Any c-subdivision

of K_t, where $t \geq 3$, must contain a cycle of length at most $3c$, that is, the girth of G can be at most $3c$. For every g, there are well known classes of graphs of girth at least g and $\Omega((n^{1+1/g})$ edges. Hence there are n-vertex graphs with edge density $\Omega(n^{1/(3c+1)})$ and no c-subdivision of K_t.

These results imply the following theorem by Nešetřil and Ossona de Mendez [105].

Theorem 3.3.1 (Nešetřil and Ossona de Mendez [105]) *If $\mathscr{C}$ is an infinite class of graphs, then either for all integers $r \geq 0$*

$$\lim_{n\to\infty} \sup \left\{ \frac{\log m(H)}{\log n(H)} \;\middle|\; H \preceq_r^t G \text{ with } n(H) \geq n, G \in \mathscr{C} \right\} \leq 1 \tag{3.1}$$

or there exists an integer $r \geq 0$ with

$$\lim_{n\to\infty} \sup \left\{ \frac{\log m(H)}{\log n(H)} \;\middle|\; H \preceq_r^t G \text{ with } n(H) \geq n, G \in \mathscr{C} \right\} = 2. \tag{3.2}$$

Here we take $\frac{\log m(H)}{\log n(H)}$ to be 0 if $m(H) = 0$,

Nešetřil and Ossona de Mendez in fact showed that the limits defined there form a trichotomy, i.e. that for all classes $\mathscr{C}$ of graphs the lim sup can only take the values $\{0, 1, 2\}$. They called those classes for which the limit is ≤ 1 *nowhere dense*. With algorithmic applications in mind, we rephrase the definition in the following way.

Corollary 3.3.2 *An infinite class $\mathscr{C}$ of graphs is nowhere dense if, and only if, for all integers $r \geq 0$ and all reals $\epsilon > 0$ there is an integer $n_0(r, \epsilon)$ such that all n-vertex graphs $H \preccurlyeq_r G \in \mathscr{C}$ with $n \geq n_0$ vertices satisfy $m(H) \leq n^{1+\epsilon}$.*

Nowhere dense classes of graphs can also be characterised by excluded bounded-depth minors.

Theorem 3.3.3 (Nešetřil and Ossona de Mendez [105]) *A graphclass $\mathscr{C}$ is nowhere dense if and only if there is a function f such that for all integers $r \geq 0$ we have $K_{f(r)} \not\preccurlyeq_r G$ for all $G \in \mathscr{C}$.*

The density of (topological) depth-r minors is central to Nešetřil and Ossona de Mendez's theory of nowhere dense graphs. The *greatest reduced average density* (shortly *grad*) with rank r of G [100] is

$$\nabla_r(G) := \max \left\{ \frac{|E(H)|}{|V(H)|} : H \preccurlyeq_r G \right\}$$

and we write $\nabla(G)$ for $\nabla_{n(G)}(G)$. The *topological greatest reduced average density* (shortly *top-grad*) with rank r of G is

$$\widetilde{\nabla}_r(G) := \max\left\{\frac{|E(H)|}{|V(H)|} : H \preccurlyeq_r^t G\right\}$$

and we write $\widetilde{\nabla}(G)$ for $\widetilde{\nabla}_{n(G)}(G)$.

The original interest of Nešetřil and Ossona de Mendez when studying sparse graph classes was the following subclass of nowhere dense classes [100, 101, 102].

Definition 3.3.4 A class $\mathscr{C}$ of graphs has *bounded expansion* if for every integer $r \geq 0$ there exists $c(r)$ such that all graphs $H \preccurlyeq_r G \in \mathscr{C}$ satisfy $m(H) \leq c(r) \cdot n(H)$. □

Not surprisingly, densities of bounded depth minors and bounded depth topological subgraphs are strongly related and we can give the above definitions with respect to either notation.

Theorem 3.3.5 (Dvořák [40]) *For every graph G and every integer $r \geq 0$,*

$$\widetilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(4\widetilde{\nabla}_r(G))^{(r+1)^2}.$$

Let us conclude this section with a short remark on the methods used to prove Dvořák's and Jiang's results on the edge density of graphs that exclude K_t as a depth-r minor.

First of all, using Lemma 3.1.4 and Theorem 3.1.3, when we want to find a subgraph in a graph G with $n(G)^{1+\epsilon}$ many edges, we may assume that G is bipartite and that the minimum and maximum degree of G differ only by a constant factor. These assumptions simplify many arguments.

The second tool which is used in almost all proofs for finding minors or topological minors, in one or the other way, is graph expansion. Classically, a graph is said to be a λ-expander for some constant λ, if for every vertex set $A \subseteq V(G)$ of small size, the neighbourhood of A has size at least $\lambda \cdot |A|$. In the situation when we have $n^{1+\epsilon}$ many edges, the definition is slightly changed. The key idea is then to show that every sufficiently dense graph contains an expander subgraph of almost the same degree as G. The expansion properties then allow us to blow up the degree of bounded depth subdivisions or minors. Dvořák applies this concept iteratively, until a too

dense subdivision is obtained, Jiang applies an even finer analysis to obtain the subdivision without iterations.

To get a feeling how expansion properties may help to find dense minors, consider the following theorem of Thomassen [130] (our proof follows [34]). We state a slight modification such that the theorem is about depth-r minors.

Theorem 3.3.6 *Let $d \geq 3, r \geq 0$ be integers and let G be a graph of minimum degree at least d and girth at least $8r+3$. Then G has a depth $2r$-minor H of minimum degree at least $d(d-1)^r$.*

PROOF. Let $X \subseteq V(G)$ be maximal with $\text{dist}(u,v) > 2r$ for all $u,v \in X$ with $u \neq v$. For each $u \in X$, choose a set M_u such that $\bigcup_{u \in X} M_u = V(G)$, $M_u \cap M_v = \emptyset$ for $u \neq v$ and such that $G[M_u]$ has radius at most $2r$. Such choice is possible as each element has distance at most $2r$ to some $u \in X$ (we associate each element w not in $N_r(u)$ for all u to some M_u such that the distance to u is minimum among all such possibilities). We obtain a depth-$2r$ minor H of G by contracting all sets M_u. To conclude that H has minimum degree at least $d(d-1)^r$, observe that M_u induces a tree for all $u \in X$ with at least $d(d-1)^{r-1}$ leaves which each send out at least $d-1$ edges and that M_u and M_v for distinct $u,v \in X$ are connected by at most one edge (the girth is high). □

In fact, the class of all graphs G with $\Delta(G) \leq \mathit{girth}(G)$ serves as a good example for a nowhere dense class with well understood properties.

3.4. Everything is a (coloured) graph

Tree decompositions can also be defined for general relational structures. Just as for graphs we want each tuple in a relation to be covered by a bag of the decomposition. Let τ be a relational signature.

A *tree decomposition* of a τ-structure $\mathfrak{A}$ is a pair (T,X), where T is a tree and $X = \{X_t : t \in V(T)\}$ is a family of subsets of $V(\mathfrak{A})$ such that

- $\bigcup_{t \in V(T)} X_t = V(\mathfrak{A})$,
- for every m-ary relation symbol $R \in \tau$ and every tuple $(a_1,\ldots,a_m) \in R(\mathfrak{A})$ there exists $t \in V(T)$ with $a_1,\ldots,a_m \in X_t$ and
- if $r,s,t \in V(T)$ and s lies on the path of T between r and t, then $X_r \cap X_t \subseteq X_s$.

It is easy to see that the tree decompositions of $\mathfrak{A}$ coincide with the tree decompositions of its Gaifman graph $G(\mathfrak{A})$. Similarly, we can define a class $\mathscr{C}$ of τ-structures to be nowhere dense if the class $\mathscr{D} = \{G(\mathfrak{A}) : \mathfrak{A} \in \mathscr{C}\}$ of its Gaifman graphs is nowhere dense. Note however, that the class $\mathscr{D}$ does not necessarily carry all information required to recover the class $\mathscr{C}$. We can instead construct for every relational structure $\mathfrak{A}$ a the coloured incidence graph $H(\mathfrak{A})$ such that in $H(\mathfrak{A})$ we find $\mathfrak{A}$ via a first-order interpretation. The class $\mathscr{H} = \{H(\mathfrak{A}) : \mathfrak{A} \in \mathscr{C}\}$ will have bounded tree width if $\mathscr{D}$ has bounded tree width and it will be nowhere dense if $\mathscr{D}$ is nowhere dense. Hence, with the interpretation method at hand, for our purpose it suffices to solve the model-checking problem for (coloured) graphs.

Let $\tau = \{R_1, \dots, R_s\}$ and let k be maximal such that τ contains a k-ary relation symbol. We let $\tau' := \{C_1, \dots, C_s, E_1, \dots, E_k\}$, where the C_i are unary and the E_j are binary relation symbols, $1 \leq i \leq s, 1 \leq j \leq k$. For a τ-structure $\mathfrak{A}$, let $H(\mathfrak{A})$ be the τ'-structure whose universe contains $V(\mathfrak{A})$ and an element $v_{\overline{a}}$ for each tuple $\overline{a} \in R_i(\mathfrak{A})$, for $R_i \in \tau$. We add the element $v_{\overline{a}}$ to C_i and if $\overline{a} = (a_1, \dots, a_m)$, then we connect a_j with $v_{\overline{a}}$ via an edge of E_j. It is easy to recover $\mathfrak{A}$ from $H(\mathfrak{A})$ via a one-dimensional (τ, τ')-interpretation.

Let us show that $\mathscr{H}$ has bounded tree width if $\mathscr{D}$ has bounded tree width and that $\mathscr{H}$ is nowhere dense if $\mathscr{D}$ is nowhere dense.

Lemma 3.4.1 *Let $\mathfrak{A}$ be a τ-structure. If the Gaifman graph $G(\mathfrak{A})$ of $\mathfrak{A}$ has tree width k, then $H(\mathfrak{A})$ has tree width at most $k+1$.*

PROOF. Let (T, X) be a tree decomposition of $G(\mathfrak{A})$ of width k. We define a tree decomposition (T', X') of $H_{\mathfrak{A}}$ of width $k+1$. We let T' be such that $T \subseteq T'$ and for each $t \in V(T)$, we let $X'_t = X_t$. For every m-ary relation symbol $R \in \tau$ and every tuple $\overline{a} = (a_1, \dots, a_m) \in R(\mathfrak{A})$ there exists $t \in V(T)$ with $a_1, \dots, a_m \in X_t$. Fix one such t and attach a leaf t' to t in T' with bag $X_{t'} = X_t \cup \{v_{\overline{a}}\}$. Clearly, (T', X') is a tree decomposition of $H(\mathfrak{A})$ of width $k+1$. □

Lemma 3.4.2 *Let $\mathfrak{A}$ be a τ-structure and let k be the maximal arity of a relation symbol in τ. For all integers r and $t > k$, if the Gaifman graph $G(\mathfrak{A})$ of $\mathfrak{A}$ does not contain an r-subdivision of a complete graph K_t, then $H_{\mathfrak{A}}$ does not contain a $2r$-subdivision of K_t.*

PROOF. If $H(\mathfrak{A})$ contains a $2r$-subdivision of K_t, then all principal vertices of the model must be elements of $V(\mathfrak{A})$, as degrees cannot increase in

topological minors. Furthermore, every path in $H(\mathfrak{A})$ of length $2s$ induces a unique path of length s in $G(\mathfrak{A})$ and two paths in $H(\mathfrak{A})$ are vertex disjoint if and only if the paths induced in $G(\mathfrak{A})$ are vertex disjoint in $G(\mathfrak{A})$. □

Corollary 3.4.3 *Let $\mathscr{C}$ be a nowhere dense class of τ-structures. Then the class $\mathscr{H} = \{H(\mathfrak{A}) : \mathfrak{A} \in \mathscr{C}\}$ is nowhere dense. Furthermore, for all τ-structures $\mathfrak{A}$ we have $n(H(\mathfrak{A})) + m(H(\mathfrak{A})) \in O(n(\mathfrak{A}) + m(\mathfrak{A}))$.*

Part II.

Colouring, covering and playing games

4. Generalised colouring numbers

The colouring number $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there exists a linear ordering L of the vertices of G such that each vertex v has back-degree at most k, i.e. v has at most k neighbours u with $L(u) < L(v)$. It is easily seen that the colouring number of G is equal to its degeneracy. Recall that a graph G is k-degenerate if every subgraph of G has a vertex of degree at most k. Hence the colouring number is a structural measure that measures the edge density of subgraphs of G. The colouring number gets its name from the fact that it bounds the chromatic number – we can simply colour the vertices in the order L such that every uncoloured vertex gets a colour not used by its at most $\mathrm{col}(G)$ smaller neighbours. This bound is very useful, as computing the chromatic number of G is NP-complete (even for planar graphs of degree at most 4 [26]), whereas the colouring number can be computed by a greedy algorithm in time $O\big(n(G) + m(G)\big)$.

At the other extreme there are orderings which measure some global structural property of graphs. As described in the previous chapter, a graph G has *tree width* at most k if its vertices can be ordered such that each vertex v can be separated from all smaller vertices by at most k smaller vertices, that is, for every $v \in V(G)$ there are $w_1, \ldots, w_k \in V(G)$ with $L(w_1), \ldots, L(w_k) < L(v)$ such that every path between v and w with $L(w) < L(v)$ contains one of $w_1, \ldots, w_k$.

In this section we consider orderings with properties which lie between these extremes. Early examples of such properties are *arrangeability* [20], *admissibility* [76], and *rank* [75] of a graph. Each of these orderings can be used to attack different optimisation problems. For example the arrangeability was introduced by Chen and Schelp [20] in their study of the Burr-Erdős conjecture [17]. Nešetřil and Ossona de Mendez [103] showed that the arrangeability of G is bounded in terms of $\nabla_1(G)$. In their study of colouring games and marking games on graphs, Kierstead and Yang [77]

introduced three families of generalised colouring numbers called *r-admissibility*, $\mathrm{adm}_r(G)$, *r-colouring number*, $\mathrm{col}_r(G)$ and *weak r-colouring number*, $\mathrm{wcol}_r(G)$, for $r \geq 1$. All of these measures are more restrictive than degeneracy and we have $\mathrm{col}(G)+1 = \mathrm{col}_1(G) \leq \ldots \leq \mathrm{col}_n(G) = \mathrm{tw}(G)$ and $\mathrm{col}(G)+1 = \mathrm{wcol}_1(G) \leq \ldots \leq \mathrm{wcol}_n(G) = \mathrm{td}(G)$, where $\mathrm{td}(G)$ denotes the tree depth of G. Kierstead and Yang showed that $\mathrm{col}_r(G)$ is bounded by a function of the average degree of topological subgraphs of G. Zhu [134] observed that the bound in [77] can actually be made in terms of $\nabla_r(G)$. From this, Nešetřil and Ossona de Mendez derived characterisations of bounded expansion and nowhere dense classes in terms of the generalised colouring numbers [105], which makes them an extremely valuable tool in the study of these classes.

This chapter is organised as follows.

- We first introduce the generalised colouring numbers and prove some of their basic properties in Section 4.1.
- In Section 4.2 we prove a tight bound for the weak r-colouring number for classes of bounded tree width.
- In Section 4.3 we present a new proof of the relation between generalised colouring numbers and the edge density of bounded depth minors. This greatly improves the bounds obtained by Zhu [134].
- We consider the generalised colouring number of proper minor closed classes in Section 4.4. The special cases of bounded genus graphs is considered in Section 4.5. Formerly, no specific analysis for these restricted classes was done and the bounds we obtain are much better than those obtained from the general bounds in terms of $\widetilde{\nabla}_r$.

4.1. Definition and basic properties

Let G be a graph. We write $\Pi(G)$ for the set of linear orders on $V(G)$. Let $u, v \in V(G)$ and let $r \geq 0$ be a non-negative integer. Vertex u is *weakly r-reachable* from v with respect to an order $L \in \Pi(G)$, if there exists a path P of length $0 \leq \ell \leq r$ between u and v such that $L(u)$ is minimum in $\{L(w) : w \in V(P)\}$. Let $\mathrm{WReach}_r[G, L, v]$ be the set of vertices that are weakly r-reachable from v with respect to L.

Vertex u is *strongly r-reachable* from v with respect to an order $L \in \Pi(G)$, if there is a path P of length $0 \leq \ell \leq r$ connecting u and v such that $L(u) \leq L(v)$ and such that all inner vertices w of P satisfy $L(w) > L(v)$. Let $\mathrm{SReach}_r[G,L,v]$ be the set of vertices that are strongly r-reachable from v with respect to L.

The *r-admissibility* $\mathrm{adm}_r[G,L,v]$ of v with respect to L, is the maximum size k of a family $\{P_1,\ldots,P_k\}$ of paths of length at most r in G that start in v, end at a vertex with $L(w) \leq L(v)$ and satisfy $V(P_i) \cap V(P_j) = \{v\}$ for $1 \leq i \neq j \leq k$. As we can always let the paths end in the first vertex smaller than v (for $r > 0$), we can assume that the internal vertices of the paths are larger than v. Note that $\mathrm{adm}_r[G,L,v]$ is an integer, whereas $\mathrm{WReach}_r[G,L,v]$ and $\mathrm{SReach}_r[G,L,v]$ are sets of vertices.

The *weak r-colouring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |\mathrm{WReach}_r[G,L,v]|,$$

the *r-colouring number* $\mathrm{col}_r(G)$ of G is defined as

$$\mathrm{col}_r(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |\mathrm{SReach}_r[G,L,v]|$$

and the *r-admissibility* $\mathrm{adm}_r(G)$ of G is

$$\mathrm{adm}_r(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} \mathrm{adm}_r[G,L,v].$$

For all integers $r \geq 0$ it holds that $\mathrm{adm}_r(G) \leq \mathrm{col}_r(G) \leq \mathrm{wcol}_r(G)$. We have increasing sequences, that is, for an n-vertex graph G we have $\mathrm{col}(G)+1 = \mathrm{adm}_1(G) \leq \mathrm{adm}_2(G) \leq \ldots \leq \mathrm{adm}_n(G)$, $\mathrm{col}(G)+1 = \mathrm{col}_1(G) \leq \mathrm{col}_2(G) \leq \ldots \leq \mathrm{col}_n(G)$ and $\mathrm{col}(G)+1 = \mathrm{wcol}_1(G) \leq \mathrm{wcol}_2(G) \leq \ldots \leq \mathrm{wcol}_n(G)$.

It was shown that the generalised colouring numbers are strongly related ([41, 77]). To get a better intuition for the generalised colouring numbers, let us prove these relations.

Let $T[G,L,v]$ be a tree which results from a breadth-first search of G starting at v which stops in every search-branch after a vertex with $L(w) < L(v)$ is reached. Let $T_r[G,L,v]$ be the maximal subtree of $T[G,L,v]$ of depth r such that all leaves w of $T_r[G,L,v]$ satisfy $L(w) < L(v)$. Then

$$\mathrm{SReach}_r[G,L,v] = \{v\} \cup \{u : u \text{ is a leaf of } T_r[G,L,v]\}.$$

Observation 4.1.1 *Let G be a graph and let $c, r > 0$ be positive integers. If $\mathrm{adm}_r(G) \leq c$, then $\Delta(T_r[G,L,v]) \leq c-1$ for all $v \in V(G)$, where $T_r[G,L,v]$ is defined as above.*

PROOF. Let $v \in V(G)$ and let w be an inner vertex of $T_r[G,L,v]$ of degree $d \geq 1$ (possibly $w = v$). Then in $T_r[G,L,v]$ there are d paths of length $1 \leq \ell \leq r$ which start in w and end either in a leaf of $T_r[G,L,v]$ or in v (in any case they end in a vertex smaller than w) and which have only w in common. Then the initial segments of the paths from w to the first vertex which is smaller than w together with the path of length 0 consisting of w itself show that $\mathrm{adm}_r[G,L,w] \geq d+1$. □

Corollary 4.1.2 *Let G be a graph and let $r > 0$ be a positive integer. Then $\mathrm{col}_r(G) \leq (\mathrm{adm}_r(G)-1) \cdot (\mathrm{adm}_r(G)-2)^{r-1} + 1$.*

A similar relation between $\mathrm{wcol}_r(G)$ and $\mathrm{adm}_r(G)$ can be established.

Theorem 4.1.3 *Let G be a graph and let $r > 0$ be a positive integer. Then $\mathrm{wcol}_r(G) \leq \mathrm{adm}_r(G)^r$.*

PROOF. Let $L \in \Pi(G)$ be such that $\mathrm{adm}[G,L,v] \leq c$ for all $v \in V(G)$. We show by induction on r that $|\mathrm{WReach}_r[G,L,v]| \leq \sum_{i=0}^{r} (c-1)^i$ for all $v \in V(G)$. We have $\mathrm{wcol}_0(G) = \mathrm{adm}_0(G) = 1$ and $\mathrm{adm}_1(G) = \mathrm{col}(G) + 1 = \mathrm{wcol}_1(G)$. For $r \geq 2$ we have $\sum_{i=0}^{r} (c-1)^i < \sum_{i=0}^{r} c^i - c = (c^{r+1}-1)/(c-1) - c \leq c^r$.

Hence let $r \geq 2$ and assume that the claim holds for $r-1$. Fix some $v \in V(G)$. For $u \in \mathrm{WReach}_r[G,L,v]$, let P_{vu} be a shortest $v-u$-path such that every vertex $w \in V(P)$ satisfies $L(u) \leq L(w)$. We may assume that the path P_{vu} uses as its first edge an edge of $T_r[G,L,v]$ to reach a vertex w for which $u \in \mathrm{WReach}_{r-1}[G,L,w]$ holds. By Observation 4.1.1, there are at most $c-1$ choices for this first edge. We conclude that

$$\begin{aligned} |\mathrm{WReach}_r[G,L,v]| &\leq 1 + (c-1) \max_{w \in V(G)} |\mathrm{WReach}_{r-1}[G,L,w]| \\ &\leq 1 + (c-1) \sum_{i=0}^{r-1} (c-1)^i \\ &= \sum_{i=0}^{r} (c-1)^i \end{aligned}$$

□

As $\mathrm{adm}_r(G) \leq \mathrm{col}_r(G)$ we have the following corollary.

Corollary 4.1.4 *Let G be a graph and let $r > 0$ be a positive integer. Then $\mathrm{wcol}_r(G) \leq \mathrm{col}_r(G)^r$.*

4.2. A tight bound on bounded tree width graphs

Surprisingly, it was long unnoticed that for an n-vertex graph G, $\mathrm{col}_n(G)$ is equal to the tree width plus one. We remarked this in [64]. To see this, consider an elimination order L such that the elimination width of L is minimum. Let $G_0,\dots,G_{n-1}$ be the sequence of graphs associated with L and let L' be the reverse to L. An easy induction shows that the neighbours of a vertex i in G_{i-1} are exactly the vertices of $\mathrm{SReach}_n[G,L',i]\setminus\{i\}$.

Corollary 4.2.1 *For an n-vertex graph G we have* $\mathrm{col}_n(G)=\mathrm{tw}(G)+1$.

On the other hand it was well known that $\mathrm{wcol}_n(G)=\mathrm{td}(G)$,where $\mathrm{td}(G)$ is the tree depth of G, see e.g. [106].

It is hence immediate that $\mathrm{wcol}_r(G)\le \mathrm{tw}(G)^r$. The following tight bounds for classes of bounded tree width were shown by Konstantinos Stavropoulos. The presented proof is based on Konstantinos Stavropoulos original ideas and is only included in this thesis for completeness on the subject. Roman Rabinovich and the author of this thesis provided only technical details to the proof.

Theorem 4.2.2 ([64]) *Let G be a graph and let $r,k>0$ be positive integers. If* $\mathrm{tw}(G)\le k$, *then* $\mathrm{wcol}_r(G)\le\binom{r+k}{k}$.

PROOF. Let G be of tree width k and let r be a positive integer. Recall that a tree decomposition of a graph G is a pair (T,X), where T is a tree and $X=\{X_t : t\in V(T)\}$ is a family of subsets of $V(G)$ (called bags) such that

- $\bigcup_{t\in V(T)} X_t = V(G)$,
- for every edge $\{u,v\}$ of G there exists $t\in V(T)$ with $u,v\in X_t$ and
- if $r,s,t\in V(T)$ and s lies on the path of T between r and t, then $X_r\cap X_t\subseteq X_s$.

The width of the decomposition is $\max_{t\in V(T)}|X_t|+1$. We also fix a tree decomposition (T,X) of G of width at most k. Since the weak colouring number can only increase when we add edges to G, we may assume that G is edge maximal of tree width k, i.e. that each bag induces a clique in G. Furthermore, we may assume that a tree decomposition (T,X) is

smooth, that is, that it has the property that for every $\{s,t\} \in E(T)$ it holds that $|X_s \setminus X_t| \leq 1$. Finally, we may assume that there is one bag X_s which contains only a single vertex. In the following, we consider (T,s) as a rooted tree, where s is the root of T.

The vertices of every rooted tree (T,s) are partially ordered by the following *tree order* $L^{T,s}$ of $V(T)$. The root s is the minimal vertex of the order and we define $t \in V(T)$ to be smaller than $u \in V(T)$ with respect to $L^{T,s}$ if t lies on the unique path from s to u in T. As we assume the tree decomposition (T,X) to be smooth, we can establish a bijection from $V(G)$ to $V(T)$ such that $v \in V(G)$ is mapped to the unique node t_v of T such that t_v is minimal with respect to $L^{T,s}$ such that $v \in X_{t_v}$. Let L be any linear extension of $L^{T,s}$. Using the above bijection, we also interprete L as a linear order on $V(G)$.

Now fix some $v \in V(G)$ and let $w \in \mathrm{WReach}_r[G,L,v]$. By Lemma 3.2.1 and the definition of L, it is immediate that t_w lies on the path from t_v to s in T. Let $u \in X_{t_v}$ be such that $L(u) \leq L(u')$ for all $u' \in X_{t_v}$. If $t_v = s$, then $|\mathrm{WReach}_r[G,L,v]| = 1$ and we are done. Otherwise, as the decomposition is smooth, $L'(t_u) < L'(t_v)$.

We define two subgraphs G_1 and G_2 of G as follows. The graph G_1 is induced by the vertices from the bags between s and t_u, i.e. by the set $\bigcup\{X_t \in V(T) : L^{T,s}(t) \leq L^{T,s}(t_u)\}$. The graph G_2 is induced by $\bigcup\{X_t \in V(T) \mid L^{T,s}(t_v) \leq L^{T,s}(t) < L^{T,s}(t_u)\} \setminus V(G_1)$. Note that $V(G_1) \cap V(G_2) = \emptyset$. Let L_i be the restriction of L to $V(G_i)$, for $i = 1,2$, respectively. Using the assumption that any two vertices of a bag are connected by an edge and Lemma 3.2.1, it is easy to see that for $w \in \mathrm{WReach}_r[G,L,v]$, we have $w \in \mathrm{WReach}_{r-1}[G_1,L_1,u] \cup \mathrm{WReach}_r[G_2,L_2,v]$.

Hence $|\mathrm{WReach}_r[G,L,v]| \leq |\mathrm{WReach}_{r-1}[G_1,L_1,u]| + |\mathrm{WReach}_r[G_2,L_2,v]|$. Note that the restrictions of (T,X) to G_1 and G_2 (in the obvious way), respectively, proves that the tree width of G_1 is at most k and the tree width of G_2 is at most $k-1$, as we removed u from every bag. We can now conclude by induction.

For the induction base, recall that $\mathrm{wcol}_1(G)$ equals the degeneracy of G plus one and that every graph of tree width $\leq k$ is k-degenerate. Furthermore, wcol_r of a tree (with tree width 1) is at most $r+1$ (using any linearisation of the standard tree order).

We recursively define the following numbers $w(k,r)$ such that $w(k,1) = k+1$ for $k \geq 1$, $w(1,r) = r+1$ for $r \geq 1$ and $w(k,r) = w(k,r-1) + w(k-1,r)$ for $k,r > 1$.

By our above argumentation, $|\mathrm{WReach}_r[G,L,v]| \leq w(k,r)$. We observe that this is the recursive definition of the binomial coefficients and conclude that $|\mathrm{WReach}_r[G,L,v]| \leq \binom{r+k}{k}$. □

The proof of Theorem 4.2.2 gives rise to a construction of a class of graphs that matches the upper bound. We construct a graph of tree width k and weak r-colouring number $\binom{k+r}{k}$ whose tree decomposition has a highly branching host tree. This enforces a path in the tree from the root to a leaf that realises the recursion from the proof of Theorem 4.2.2.

Theorem 4.2.3 ([64]) *Let $k,r > 0$ be positive integers. There is a family of graphs G_r^k with $\mathrm{tw}(G_r^k) = k$, such that $\mathrm{wcol}_r(G_r^k) = \binom{r+k}{k}$.*

Proof. Fix $r,k > 0$ and let $c = \binom{r+k}{k}$. We define graphs $G(k',r')$ for all $r' \leq r$, $k' \leq k$ and corresponding tree decompositions $\mathcal{T}(k',r') = (T(k',r'),X(k',r'))$ of $G(k',r')$ of width k' with a distinguished root $s(T(k',r'))$ by induction on k' and r'. We guarantee several invariants for all values of k' and r' which will give us control over a sufficiently large part of any order that witnesses $\mathrm{wcol}_{r'}(G(k',r')) = \binom{r'+k'}{k'}$.

1. There is a bijective function $f : V(T(k',r')) \to V(G(k',r'))$ such that $f(s(T(k',r')))$ is the unique vertex contained in $X_{s(T(k',r'))}$ and if t is a child of t' in $T(k',r')$, then $f(t)$ is the unique vertex of $X_t \setminus X_{t'}$. Hence any order defined on $V(T)$ directly translates to an order of $V(G)$ and vice versa.

2. In any order L of $V(G(k',r'))$ which satisfies $\mathrm{wcol}_r(G(k',r')) \leq c$, there is some root-leaf path $P = t_1,\ldots,t_m$ such that $L(f(t_1)) < \ldots < L(f(t_m))$.

3. Every bag of $T(k',r')$ contains at most $k'+1$ vertices.

It will be convenient to define the tree decompositions first and to define the corresponding graphs as the unique graphs induced by the decomposition in the following sense. For a tree T and a family of finite and non-empty sets $(X_t)_{t\in V(T)}$ such that if $z,s,t \in V(T)$ and s is on the path of T between z and t, then $X_z \cap X_t \subseteq X_s$, we define the graph *induced* by $(T,(X_t)_{t\in V(T)})$ as the graph G with $V(G) = \bigcup_{t\in V(T)} X_t$ and $\{u,v\} \in E(G)$ if and only if $u,v \in X_t$ for some $t \in V(T)$. Then $(T,(X_t)_{t\in V(T)})$ is a tree decomposition of G.

For $k' \geq 1$ and $r' = 1$, let $T(k',r') := T$ be a tree of depth $k'+1$ and branching degree c with root s. Let $L^{T,s}$ be the natural partial tree order. Let f :

$V(T) \to V$ be a bijection to some new set V. We define $X_t := \{f(t) : L^{T,s}(t') \leq L^{T,s}(t)\}$. Let $G(k',r')$ be the graph induced by the decomposition. The first and the third invariants clearly hold. For the second invariant, consider a simple pigeon-hole argument. For every non-leaf node t, the vertex $f(t)$ has c neighbours $f(t')$ in the child bags $X_{t'}$ of t. Hence some $f(t')$ must be larger in the order. This guarantees the existence of a path as required.

For $k' = 1, r' \geq 1$, let $T(k',r') =: T$ be a tree depth $r'+1$ and branching degree c with root s and let f be as before. Let $X_s := \{f(s)\}$ and for each $t' \in V(T)$ with parent $t \in V(T)$ let $X_{t'} := \{f(t), f(t')\}$. Let $G(k',r')$ be the graph induced by the decomposition. All invariants hold by the same arguments as above. Note that G_1^1 is the same graph in both constructions and is hence well defined.

Now assume that $G(k',r'-1)$ and $G(k'-1,r')$ and their respective tree decompositions have been defined. Let $T(k',r')$ be the tree which is obtained by attaching c copies of $T(k'-1,r')$ as children to each leaf of $T(k',r'-1)$. We define the bags that belong to the copy of $T(k',r'-1)$, exactly as those of $T(k',r'-1)$. To every bag of a copy of $T(k'-1,r')$ which is attached to a leaf z, we add $f'(z)$ (where f' is the bijection from $T(k',r'-1)$). Let $G(k',r')$ be the graph induced by the decomposition.

It is easy to see how to obtain the new bijection f on the whole graph such that it satisfies the invariant. It is also not hard to see that each bag contains at most $k'+1$ vertices. For the second invariant, let $P_1 = t_1, \ldots, t_m$ be some root-leaf path in $T(k',r'-1)$ which is ordered such that $L(f(t_1)) < \ldots < L(f(t_m))$. Let $v = f(t_m)$ be the unique vertex in the leaf bag in which P_1 ends. By the same argument as above, this vertex has many neighbours s' such that $f^{-1}(s')$ is a root of a copy of $T(k'-1,r')$. One of them must be larger than v. In appropriate copy we find a path P_2 with the above property by assumption. We attach the paths to find the path $P = t_1 \ldots t_\ell$ in $T(k',r')$.

We finally show that $\mathrm{WReach}_r[G(k,r), L, f(t_\ell)] = c$. This is again shown by an easy induction. For the graph $G(k',r')$, using the notation of the proof of Theorem 4.2.2, we observe that the graph G_1 is isomorphic to $G(k',r'-1)$ and G_2 is isomorphic to the graph $G(k'-1,r')$. Furthermore we observe that the number of vertices reached in these graphs are exactly $w(k',r'-1)$ and $w(k'-1,r')$, so that the upper bound is matched. The theorem follows by letting $G_r^k := G(k,r)$. □

4.3. Bounding the numbers in terms of low depth minors

Kierstead and Yang [77] and Zhu [134] showed that the generalised colouring numbers and the edge density of bounded depth minors are strongly related. From this, Nešetřil and Ossona de Mendez [105, 106] concluded that bounded expansion and nowhere dense classes can be characterised by the generalised colouring numbers. We significantly simplify the proofs of [77, 134] and improve the bounds from their papers. Zhu [134] defines $q_1 := 2\nabla_r(G)$ and for $i \geq 1$, $q_{i+1} := q_1 q_i^{2i^2}$ and shows that $\mathrm{col}_r(G) \leq 1 + q_r$. We are going to prove the following theorem. The results of this section were published in [64].

Theorem 4.3.1 *Let G be a graph and let $r > 0$ be a positive integer. Then* $\mathrm{adm}_r(G) \leq 12r\widetilde{\nabla}_{r-1}(G)^3$.

For the proof of Theorem 4.3.1 we need a lemma which is a variation of a result of Dvořák [41]. For a set $S \subseteq V(G)$ and $v \in S$, let $b_r(S,v)$ be the maximum number k of paths $P_1,\dots,P_k$ of length at most r from v to S with internal vertices in $V(G) \setminus S$ and with $V(P_i) \cap V(P_j) = \{v\}$ for $1 \leq i \neq j \leq k$.

Lemma 4.3.2 ([41]) *For all graphs G and integers $r \geq 0$, there exists a set $S \subseteq V(G)$ such that $b_r(S,v) \geq \mathrm{adm}_r(G)$ for all $v \in S$.*

Proof. Assume that all $S \subseteq V(G)$ contain a vertex v such that $b_r(S,v) < \mathrm{adm}_r(G)$. We order the vertices $L(v_1) < L(v_2) < \ldots < L(v_n)$ of $V(G)$ as follows. If $v_{i+1},\dots,v_n$ have already been ordered, choose v_i such that if $S_i = \{v_1,\dots,v_i\}$, then $b_r(S_i,v_i)$ is minimal. Clearly, the r-admissibility of the resulting order is one of the values $b_r(S_i,v_i)$ occurring in its construction. This implies $\mathrm{adm}_r(G) < \mathrm{adm}_r(G)$, a contradiction. □

Proof. (of Theorem 4.3.1) Let G be a graph with $\widetilde{\nabla}_{r-1}(G) \leq c$, and let $\ell := 12rc^3$. Suppose towards a contradiction that $\mathrm{adm}_r(G) > \ell$. According to Lemma 4.3.2, there exists a set S such that $b_r(S,v) > \ell$ for all $v \in S$. For $v \in S$, let $\mathscr{P}_v$ a set of paths from v to S witnessing this, and let $s := |S|$. Without loss of generality we assume that $s \geq 2$.

Choose a maximal set $\mathscr{P}$ of pairwise internally vertex-disjoint paths of length at most $2r-1$ connecting pairs of vertices from S whose internal vertices belong to $V(G) \setminus S$ such that each pair of vertices is connected by

at most one path. Let H be the graph with vertex set S and edges between all vertices $v, w \in S$ connected by a path in $\mathscr{P}$. Then $H \preccurlyeq^t_{r-1} G$ and hence $|\mathscr{P}| = |E(H)| \leq s \cdot c$. Let M be the set of all internal vertices of the paths in $\mathscr{P}$, and let $m := |M|$. Then $m \leq s \cdot c \cdot (2r-2)$.

Note that we not only have $H \preccurlyeq^t_{r-1} G$, but also $H' \preccurlyeq^t_{r-1} G$ for all $H' \subseteq H$. Thus every subgraph H' of H has edge density at most c and in particular, a vertex of degree at most $2c$. Then H contains an independent set R of size $\lfloor s/(2c+1) \rfloor$. We can iteratively build the set as follows. Choose a vertex v of minimum degree and add it to R. Delete v and all its neighbours from $V(H)$ and continue inductively. Clearly the resulting set R is independent in H. As all subgraphs of H have a vertex of degree at most $2c$, we delete at most $2c+1$ vertices from S in each step. Hence R has size at least $\lfloor s/(2c+1) \rfloor$.

For every $v \in S$, we let $\mathscr{Q}_v$ be the set of initial segments of paths in $\mathscr{P}_v$ from v to a vertex in $(M \cup S) \setminus \{v\}$ with all internal vertices in $V(G) \setminus (M \cup S)$. Observe that for distinct $u, v \in R$ the paths in $\mathscr{Q}_v$ and $\mathscr{Q}_u$ are internally disjoint, because if $Q \in \mathscr{Q}_u$ and $Q' \in \mathscr{Q}_v$ had an internal vertex in common, then $Q \cup Q'$ would contain a path of length at most $2r-2$ that is internally disjoint from all paths in $\mathscr{P}$, contradicting the maximality of $\mathscr{P}$.
Let G' be the union of all paths in $\mathscr{P}$ and all paths in $\mathscr{Q}_v$ for $v \in R$, and let H' be obtained from G' by contracting all paths in $\bigcup_{v \in R} \mathscr{Q}_v$ to single edges. Then $H' \preccurlyeq^t_{r-1} G$.

We have $|V(H')| \leq s \cdot c \cdot 2r$ and $|E(H')| \geq \lfloor s/(2c+1) \rfloor \cdot \ell \geq (s-1) \cdot \ell/(2c+1)$. Thus

$$\epsilon(H') \geq \frac{(s-1)}{s} \cdot \frac{\ell}{2rc(2c+1)} \geq \frac{1}{2} \cdot \frac{\ell}{6rc^2} = \frac{\ell}{12rc^2} > c.$$

A contradiction. □

Conversely, if G has a topological depth-r minor H of edge density c, then $\mathrm{adm}_{2r}(G) \geq c$. To see this, let $H' \subseteq H$ be of minimum degree c. Such H' exists by Lemma 3.1.1. Consider an arbitrary ordering of $V(G)$ and consider the largest vertex v which is a principal vertex of H. Then there are at least c internally vertex disjoint paths of length at most $2r$ leading to the c neighbours of v in H.

Combining Corollary 3.3.2, Theorem 4.3.1, Theorem 4.1.3 and the above observation, we obtain Nešetřil and Ossona de Mendez's result that the generalised colouring numbers characterise nowhere dense graph classes.

Theorem 4.3.3 ([105]) *A monotone class $\mathscr{C}$ of graphs is nowhere dense if and only if for all integers $r \geq 0$ and reals $\epsilon > 0$ there is a positive integer n_0 such that for all n-vertex graphs $G \in \mathscr{C}$ with $n > n_0$, $\operatorname{wcol}_r(G) \leq n^\epsilon$.*

4.4. A bound on proper minor closed classes

Every class that excludes a topological minor has constant expansion. More precisely, there is a small constant $c \in \mathbb{R}$ such that for all integers $t > 0$, if G has average degree at least ct^2, then G contains K_t as a topological minor [16, 79] (currently, the best bound on c is 10/23 [86]). Hence $\widetilde{\nabla}_r(G) \leq ct^2$ for all positive integers r if G excludes K_t as a topological minor. We obtain the following corollary of Theorem 4.3.1.

Corollary 4.4.1 *Let G be a graph and let $k, r > 0$ be integers. If $K_t \not\preccurlyeq^t G$, then $\operatorname{adm}_r(G) \leq \frac{3}{2}t^6 \cdot r$ and $\operatorname{wcol}_r(G) \leq \left(\frac{3}{2}t^6\right)^r \cdot r^r$.*

In this section we will show that for classes that exclude a fixed graph as a minor we can do significantly better. The results of this section were published in [109].

Let G be a graph. We call a path P in G a *shortest path* if there is no shorter path between its endpoints. A *shortest paths decomposition* of G is a sequence $P_0, \ldots, P_\ell$ of paths such that $\bigcup_{i=0}^{\ell} V(P_i) = V(G)$, defined inductively as follows. Let P_0 be an arbitrary shortest path in G and let $G_0 := P_0$. For $i > 0$, let $P_i = v_0, \ldots, v_n$ be a shortest path in $G - E(G_{i-1})$ such that $V(G_{i-1}) \cap V(P_i) \subseteq \{v_0, v_n\}$ and let $G_i := G_{i-1} + P_i$ (the graph induced by $V(G_{i-1}) \cup P_i$). Let $\mathscr{C}_i$ be the set of components of $G - G_i$. The separating number of a component $C \in \mathscr{C}_i$ is the minimum number s of paths $Q_1, \ldots, Q_s \in \{P_0, \ldots, P_\ell\}$ such that $\bigcup_{1 \leq j \leq s} V(Q_s)$ separates C from $G - G_i$. The *width* of $P_0, \ldots, P_\ell$ is the maximum separating number over all i and all $C \in \mathscr{C}_i$.

This definition is inspired by work of Abraham and Gavoille [2] and Abraham et al. [3]. The authors of [2] define the notion of *k-path-separability*, where k corresponds to the width of our shortest paths decompositions. They furthermore require that the components that are created by the separators are of balanced size. This allows a recursive applications of their separator theorem which terminates after a logarithmic number of steps. Based on Robertson and Seymour's structure theorem [120, 121] they show that graphs which exclude a fixed minor are k-path separable. In [3], the condition on balanced component sizes was dropped which lead

to a much simpler proof that shortest paths decompositions of small width exist for graphs which exclude a fixed minor. The proof is based on results about cops and robber games [5, 7, 118] and shortest paths decompositions are therefore called *cop-decompositions* in [3]. We therefore attribute the following results to Abraham et al. as well as to the authors of [5, 7, 118].

Theorem 4.4.2 *Let G be a graph and let $g \geq 0$ be an integer. If G has genus g, then G has a shortest paths decomposition of width $2g+2$.*

Theorem 4.4.3 *Let G be a graph and let $t \geq 3$ be an integer. If G excludes K_t as a minor, then G has a shortest paths decomposition of width $\binom{t-1}{2} - 1$.*

From a shortest paths decomposition $P_0, \dots, P_\ell$, we define a linear order L on $V(G)$ as follows. For $v, w \in V(G)$, set $L(v) < L(w)$ if $v \in V(P_i) = v_0, \dots, v_n$, $w \in V(P_j) \setminus V(P_i)$ and $i < j$, or $i = j$, $v = v_x$, $w = v_y$ and $x < y$. We write $P(v)$ for the path P_m with minimum index m such that $v \in V(P_m)$. In the following, let $v \in V(G)$ and let m be such that $P(v) = P_m$. The following observations are easy consequences of the above definitions.

Lemma 4.4.4 *Let P be a shortest path in a graph G and let $r \geq 0$ be an integer. Then $|N_r(v) \cap V(P)| \leq 2r+1$ for all $v \in V(G)$.*

PROOF. Assume $P = v_0, \dots, v_n$ and $|N_r(v) \cap V(P)| > 2r+1$. Let i be minimal such that $v_i \in N_r(v)$ and let j be maximal such that $v_j \in N_r(v)$. P is not only a shortest path between its endpoints but also the path $v_i, \dots, v_j$ is a shortest path in G between v_i and v_j. As $|N_r(v) \cap V(P)| > 2r+1$, the distance between v_i and v_j is greater than $2r$ in G (a path with $2r+2$ vertices has length $2r+1$). As $v_i, v_j \in N_r(v)$, there are paths P_i, P_j of length at most r between v_i and v and between v_j and v, respectively. Concatenating these paths gives us a walk of length at most $2r$ between v_i and v_j, which contains a path of length at most $2r$. Then the distance between v_i and v_j is at most $2r$, a contradiction. □

The paths P_i for $i > 1$ of a shortest paths decomposition are not necessarily shortest paths in G, only in $G' = G - \bigcup_{1 \leq j < i} V(P_j)$. We argue that for the reachability properties we consider for the generalised colouring numbers, we may consider the graph G' instead of G when arguing about the path P_i.

Lemma 4.4.5 *Let G be a graph and let $k, r \geq 0$ be integers. Let $P_1, \dots, P_\ell$ be a shortest paths decomposition of G of width k and let L be an order induced*

by the decomposition. Let let $i > 1$ be an integer and let $G' := G - \bigcup_{1 \le j < i} V(P_j)$. Then

$$\mathrm{SReach}_r[G,L,v] \cap P_i \subseteq N_r^{G'}(v) \cap P_i$$

and

$$\mathrm{WReach}_r[G,L,v] \cap P_i \subseteq N_r^{G'}(v) \cap P_i.$$

PROOF. If a path P with one endpoint v visits a vertex that is smaller than a vertex of P_i, then the path cannot be continued to weakly or strongly visit a vertex of P_i. □

Now it is easy to prove a good bound on $\mathrm{col}_r(G)$ in terms of the width of a shortest paths decomposition.

Theorem 4.4.6 *Let G be a graph and let $k, r \ge 0$ be integers. If G has a shortest paths decomposition of width k, then $\mathrm{col}_r(G) \le (k+1) \cdot (2r+1)$.*

PROOF. Let $P_1, \ldots, P_\ell$ be a shortest paths decomposition of G of width k and let L be an order induced by the decomposition. Let $v \in V(G)$ be an arbitrary vertex and let q be minimum such that $v \in V(P_q)$. If $q = 0$, then $|\mathrm{SReach}_r[G,L,v]| \le 2r+1$ by Lemma 4.4.4. Hence assume $q > 0$. Let C be the component of $G - \bigcup_{1 \le p \le q-1} V(P_p)$ which contains v and let $Q_1, \ldots, Q_m$, $1 \le m \le k$, be the paths among $P_1, \ldots, P_{q-1}$ which separate the component C from $G - G_{q-1}$. By definition of L the vertices of $\mathrm{SReach}_r[G,L,v]$ can only lie on $Q_1, \ldots, Q_m$ and on P_q. For $1 \le j \le q$, assume that $Q_j = P_i$ and let $G' := G - \bigcup_{1 \le p < i} V(P_p)$. Then by Lemma 4.4.5, $\mathrm{SReach}_r[G,L,v] \cap Q_j = N_r^{G'}(v) \cap Q_j$. As P_i is a shortest path in G', by Lemma 4.4.4 applied to G', $|\mathrm{SReach}_r[G,L,v] \cap Q_j| \le 2r+1$. □

Similarly, we can bound $\mathrm{wcol}_r(G)$.

Theorem 4.4.7 *Let G be a graph and let $k, r \ge 0$ be integers. If G has a shortest paths decomposition of width k, then $\mathrm{wcol}_r(G) \le ((3k)^r + 1)(2r+1)$.*

PROOF. Let $P_1, \ldots, P_\ell$ be a shortest paths decomposition of G of width k and let L be an order induced by the decomposition and let $v \in V(G)$. As in the proof of Theorem 4.4.6, we first count the number of paths that contain weakly reachable vertices. Observe that

$$\mathrm{WReach}_r[G,L,v] \subseteq \{v\} \cup \bigcup_{w \in \mathrm{SReach}_r[G,L,v]} \mathrm{WReach}_{r-1}[G,L,w]. \tag{4.1}$$

Observe that for every pair of inner vertices u, w of a path P_i, the component C of $G - \bigcup_{1 \leq p \leq i-1} V(P_p)$ is the same and so are the paths $Q_1, \ldots, Q_m$ that separate C from $G - G_{i-1}$. The endpoints of P_i however are anchored at paths P_s, P_t with smaller indices (unless $i = 0$ and possibly $s = t$). We may assume that P_i is anchored at inner vertices of P_s and P_t (unless $s = 0$ or $t = 0$), because if P_i anchors at an end-vertex of P_s, then this vertex of P_s lies on a smaller path P_x (unless $s = 0$) and no vertex of P_s is weakly or strongly reachable that does not lie already on P_x. When counting paths, we will hence count the paths P_i, P_s and P_t and continue inductively for all their inner vertices with one inductive call.

Let q be minimum such that $v \in V(P_q)$. If $q = 0$, then $|\mathrm{SReach}_r[G, L, v]| \leq 2r + 1$ by Lemma 4.4.4. Hence assume $q > 0$. Let C be the component of $G - \bigcup_{1 \leq p \leq q-1} V(P_p)$ which contains v and let $Q_1, \ldots, Q_m$, $1 \leq m \leq k$, be the paths among $P_1, \ldots, P_{q-1}$ which separate C from $G - G_{q-1}$. We conclude that $\mathrm{SReach}_r[G, L, v]$ contains an inner vertex of at most $3k$ paths. As argued above we can treat all inner vertices of a paths as one vertex. By induction, we there are at most $(3k)^r$ paths that contain weakly r-reachable vertices of v. Just as in the proof of Theorem 4.4.6, we conclude with Lemma 4.4.5 and Lemma 4.4.4. □

Combining the above theorems, we obtain the following corollary.

Corollary 4.4.8 *Let G be a graph and let $r > 0, t \geq 3$. If G excludes K_t as a minor, then*

1. $\mathrm{col}_r(G) \leq \binom{t-1}{2} \cdot (2r + 1)$, *and*
2. $\mathrm{wcol}_r(G) \leq \left(3\binom{t-1}{2}^r + 1\right) \cdot (2r + 1)$.

4.5. A bound in terms of genus

We first consider the planar case and then generalise it to bounded genus graphs. A graph is *planar* if it can be drawn in the plane such that no edges intersect, except at a common end-vertex. We refrain from providing the topological background required for the study of planar graphs because our construction is based on very simple topological arguments which follow directly from the Jordan Curve Theorem. For background on graphs on surfaces we refer the reader to [96]. The results of this section were published in [109]. We state the Jordan Curve Theorem for the sake of completeness.

Theorem 4.5.1 *A simple closed curve C in the plane divides the plane into exactly two arc-wise connected components. Both of these regions have C as the boundary.*

We need the following properties of planar graphs.

Theorem 4.5.2 *Let G be a graph that is maximally planar (and satisfies $n(G) \geq 4$). Then G is 3-connected.*

For the rest of this section, we fix a planar graph G of order at least 4 and, as adding edges to a graph can only increase its weak r-colouring number, we may assume without loss of generality that G is maximally planar and hence 3-connected. As $\mathrm{wcol}_1(G)$ is equal to the degeneracy of G plus one, we always assume that $r \geq 2$.
We inductively define a shortest paths decomposition of G. Along with the construction we guarantee that for all i, if C is a component of $G - G_i$, then there are at most two paths P_j and P_ℓ with $j \leq \ell \leq i$ such that C is separated from $V(G_i)$ in G by $V(P_j) \cup V(P_\ell)$. We write $S_1(C) = P_j$ and $S_2(C) = P_\ell$ for the least possible j and ℓ with that property and call S_1, S_2 the *separating paths* of the component C. Note that if S_1 alone separates C, then $S_1 = S_2$. As G is 3-connected, C has at least three neighbours in $V(S_1) \cup V(S_2)$. Hence some $P \in \{S_1, S_2\}$ has at least two C-*neighbours*, i.e. vertices which are adjacent to a vertex of C.

Our construction. The path P_0 is an arbitrary shortest path in G. Now let $i > 0$ and assume $P_0, \ldots, P_{i-1}$ have been defined such that for each component C of $G - V(G_{i-1})$ there are at most two separating paths $S_1(C)$ and $S_2(C)$. Let C be a component of $G - V(G_{i-1})$. Then some $P = w_0, \ldots, w_\ell \in \{S_1, S_2\}$ has two C-neighbours. Let $w_{\min}$ ($w_{\max}$) be the C-neighbours of P with the least (greatest) index. We define P_i as a shortest path between $w_{\min}$ and $w_{\max}$ in $G - E(G_{i-1})$ with internal vertices from C (note that P_i has an internal vertex as P is a shortest path in $G - G_{i-1}$). We say that P_i is *anchored* at P. The procedure stops when no $v \in V(G) \setminus V(G_{i-1})$ can be found, hence when $V(G_i) = V(G)$, i.e. when a shortest paths decomposition of G was found.

The next lemma follows easily by the Jordan Curve Theorem and our choice of anchoring new paths at minimal and maximal C-neighbours.

Lemma 4.5.3 *For $i > 0$, if C is a component of $G - G_i$, then there are two paths P_j and P_ℓ with $j \leq \ell \leq i$ such that C is separated from $V(G_i)$ in G by $V(P_j) \cup V(P_\ell)$.*

Lemma 4.5.4 *Let C be a component of $G - G_i$. Then $P \in \{S_1(C), S_2(C)\}$ (for $P \neq P_0$) has an inner vertex which is a C-neighbour.*

PROOF. S_1 and S_2 are paths with minimal indexes with the separator property. Their endpoints lie on paths with smaller indices. □

Let P be a path from the shortest paths decomposition. The *chain* $\chi(P)$ of P is the sequence $Q_0, \ldots, Q_n$ of paths from the shortest paths decomposition where $Q_0 = P$, $Q_n = P_0$ and for $0 < j < n$, $Q_j = P'$ if and only Q_{j-1} is anchored at P'. For $w \in V(G)$, $\chi(w)$ is defined as $\chi(P(w))$. Note that any two chains $\chi_1 = U_1, \ldots, U_m$ and $\chi_2 = U'_1, \ldots, U'_n$ coincide from some path on. The *meeting path* of χ_1 and χ_2 is the path P_i such that $P_i = U_i = U'_j$ for the least i (and j).

Lemma 4.5.5 *In the subgraph induced by the vertices of $\chi(v)$, there are at most r^3 weakly r-reachable paths from v.*

PROOF. Let $0 \leq i \leq r$ and let $P_{j(i)}$ be the path of the chain with the minimum index such that $P_{j(i)}$ is weakly reachable from v in i steps. Let χ_i be the chain that contains only the paths with index at least as large as $j(i)$ (in the chain order). We show by induction on i that there are at most $i \cdot r$ pairs of endpoints of paths from χ_i which are weakly r-reachable from v. Clearly, we reach only $P(v)$ in 0 steps. Let $i > 0$ and assume that the claim holds for all $\ell < i$. We can reach only an inner vertex on $P_{j(i-1)}$ in $i - 1$ steps (if we could reach an endpoint, then $j(i-1)$ would not be the minimal index).

We count the tuples of endpoints of paths which lie in χ_i and which are weakly reachable in $r - i$ steps from some inner vertex v' of $P_{j(i-1)}$. As $P_{j(i-1)}$ separates $\chi_{i-1} - P_{j(i-1)}$ from $P_{J(i)}$, the path $P_{j(i)}$ is reached in one step from $P_{j(i-1)}$ and gives us one additional endpoint tuple (or $P_{j(i-1)} = P_{j(i)}$ and we are done in this step).

Now one endpoint, say x, of $P_{j(i-1)}$ is an endpoint of $P_{j(i)}$. Otherwise let P be the path at which $P_{j(i-1)}$ is anchored. Then P separates $P_{j(i-1)}$ from $P_{j(i)}$ and $P_{j(i)}$ is not reachable from $P_{j(i-1)}$ in one step.

All paths P from χ that are weakly reachable from $P_{j(i-1)}$ in χ_i have x as an endpoint, otherwise P separates $P_{j(i-1)}$ from $P_{j(i)}$. Thus we reach at most $r - i$ additional paths with a different second endpoint in $r - i$ steps from $P_{j(i-1)}$.

To conclude the proof, note that χ_r contains all weakly r-reachable paths. For every pair (x,y) of endpoints, there are at most r weakly reachable paths with those endpoints (x,y). This is because every such path P separates the chain and x and y are smaller than every inner vertex of P with respect to $\sqsubseteq$.

Hence in χ_i there are at most $i \cdot r^2$ weakly r-reachable paths. □

Lemma 4.5.6 *There are at most $2r^4$ weakly reachable paths.*

PROOF. For a chain $\chi = Q_1, \ldots, Q_m$, let ${}_\sim\chi$ be the chain $Q_2, \ldots, Q_m$. For $i > 0$ and a path P_i from the decomposition let $C(P_i)$ be the component of $G - G_{i-1}$ which contains an inner vertex of P_i (this is well defined). For $j = 1,2$, let $\chi_j(P_i) = \chi(S_j(C))$, i.e. the chains of the separating paths.

Then for every $S \in \{S_1(C), S_2(C)\}$, $\chi_1(S) \in \{{}_\sim\chi_1(P_i), {}_\sim\chi_2(P_i)\}$ or $\chi_2(S) \in \{{}_\sim\chi_1(P_i), {}_\sim\chi_2(P_i)\}$. As one needs at least one step to change a chain, we can reach at most $2r$ chains in r steps. The result follows by Lemma 4.5.5. □

Theorem 4.5.7 *If G is planar and $r \geq 2$, then* $\mathrm{wcol}_r(G) \leq 2r^4 \cdot (2r+1) \leq 5r^5$.

A cycle C in G is *non separating* if $G - C$ is connected. It is well known (see e.g. [96], Lemma 4.2.4, or [118]) that for a graph of genus $g > 0$, there exists a non-separating cycle C which consists of two shortest paths such that $G - C$ has genus $g - 1$. We can iteratively eliminate those cycles first (that is, define the vertices of the cycles to be the smallest ones of the order) and end up in the planar case after g iterations.

Theorem 4.5.8 *Let G be a graph and let $g \geq 0, r \geq 2$ be integers. If G is of genus g, then* $\mathrm{wcol}_r(G) \leq (2g + 2r^4)(2r+1)$.

4.6. Computing the numbers

In this section we consider the complexity of computing $\mathrm{wcol}_r(G)$ for a given input graph G. It is conjectured in [41] that this problem is NP-complete for $r \geq 2$. We were able to show that it is indeed NP-complete for $r \geq 3$ [64]. The case $r = 2$ remains an open question.

Clearly, the problem is in NP, hence it remains to show NP-hardness. Our proof is a straightforward modification of a proof of Pothen [116], showing that computing a minimum elimination tree height problem is NP-complete.

It is based on a reduction from the NP-complete problem BALANCED COMPLETE BIPARTITE SUBGRAPH (BCBS, problem GT24 of [58]): given a bipartite graph G and a positive integer k, decide whether there are two disjoint subsets $W_1, W_2 \subseteq V(G)$ such that $|W_1| = |W_2| = k$ and such that $u \in W_1, v \in W_2$ implies $\{u,v\} \in E(G)$. For a graph G, let $\bar{G}$ be its complement graph.

Lemma 4.6.1 *Let $G = (V_1 \cup V_2, E)$ be a bipartite n-vertex graph and let $k \geq 0$. Then G has a balanced complete bipartite subgraph with partitions W_1, W_2 of size k if and only if $\mathrm{wcol}_r(\bar{G}) = \mathrm{wcol}_3(\bar{G}) \leq n - k$ for all $r \geq 3$.*

PROOF. $\bar{G}$ is the complement of a bipartite graph, i.e. V_1 and V_2 induce complete subgraphs in $\bar{G}$ and there are possibly further edges between vertices of V_1 and V_2. Thus, for any two vertices u, v which are connected in $\bar{G}$ by a path P, there is a subpath of P between u and v of length at most 3. Hence $\mathrm{wcol}_r(\bar{G}) = \mathrm{wcol}_3(\bar{G})$ for any $r \geq 3$ and it suffices to show that G has a balanced complete bipartite subgraph with partitions W_1, W_2 of size k if and only $\mathrm{wcol}_3(\bar{G}) = n - k$.
First assume that there are sets $W_1 \subseteq V_1, W_2 \subseteq V_2$ with $|W_1| = |W_2| = k$ and such that for all $u \in W_1, v \in W_2$ there is an edge $\{u,v\} \in E(G)$. Let L be some order which satisfies $L(u) < L(v)$ if $u \in V(\bar{G}) \setminus (W_1 \cup W_2)$ and $v \in W_1 \cup W_2$ and $L(v) < L(w)$ if $v \in W_1$ and $w \in W_2$. Then any vertex from $V(\bar{G}) \setminus (W_1 \cup W_2)$ weakly reaches at most $n - 2k$ vertices and any vertex from W_i for $1 \leq i \leq 2$ weakly reaches at most $n - k$ vertices.

Now let L be an order with $\mathrm{WReach}_3[\bar{G}, L, v] \leq n - k$ for all $v \in V(G)$. Assume without loss of generality that $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $L(v_i) < L(v_{i+1})$ for all $i < n$. Denote by $\bar{G}_i$ the subgraph $\bar{G}[\{v_i, \ldots, v_n\}]$ and let $V_1^i := V(\bar{G}_i) \cap V_1$ and $V_2^i := V(\bar{G}_i) \cap V_2$. Let $\ell \geq 1$ be minimal such that there is no edge between V_1^ℓ and V_2^ℓ in $\bar{G}$. It exists because one of V_1^n or V_2^n is empty. Clearly, V_1^ℓ and V_2^ℓ induce a complete bipartite graph in G. Let $j_1 := |V_1^\ell|$ and $j_2 := |V_2^\ell|$. We show that $j_1, j_2 \geq k$. It is easy to see that $\mathrm{WReach}_3[\bar{G}, L, w_1] \leq \ell + j_1$ for the maximal element $w_1 \in V_1^\ell$ and $\mathrm{WReach}_3[\bar{G}, L, w_2] \leq \ell + j_2$ for the maximal element $w_2 \in V_2^\ell$. We have $j_1 + j_2 = n - \ell$ and, without loss of generality, $\ell + j_1 \leq \ell + j_2 \leq n - k$. Hence $j_1 \leq j_2 \leq n - \ell - k = j_1 + j_2 - k$, which implies both $j_1 \geq k$ and $j_2 \geq k$. □

The above reduction is polynomial time computable, so we obtain the following theorem.

Theorem 4.6.2 *Given a graph G and $k,r \in \mathbb{N}, r \geq 3$, it is NP-complete to decide whether* $\mathrm{wcol}_r(G) = k$.

For our purpose, we need an efficient algorithm to compute the weak r-colouring numbers. For every nowhere dense class $\mathscr{C}$, Dvořák [41] provides an approximation algorithm for computing for each sufficiently large n-vertex graph $G \in \mathscr{C}$ an ordering which witnesses that $\mathrm{wcol}_r(G) \leq n^\epsilon$, but its running time is $O(r \cdot n^3)$ which is too expensive for our purpose. In [65] we therefore used another approximation algorithm which is based on Nešetřil and Ossona de Mendez's augmentation technique [100].

Let $\vec{G}$ be an orientation of G. A 1-*transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with $V(\vec{H}) = V(\vec{G})$, including all arcs of $\vec{G}$ and such that for all $u,v,w \in V(\vec{G})$

- if $(u,v) \in E(\vec{G})$ and $(v,w) \in E(\vec{G})$ then $(u,w) \in E(\vec{H})$ (*transitivity*) and
- if $(u,v) \in E(\vec{G})$ and $(w,v) \in E(\vec{G})$ then exactly one of (u,w) or (w,u) is in $E(\vec{H})$ (*fraternity*).

A *transitive fraternal augmentation* of $\vec{G}$ is a sequence $\vec{G}_1 \subseteq \vec{G}_2 \subseteq \dots$ with $\vec{G}_1 = \vec{G}$ and such that for all $i \geq 1$, $\vec{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\vec{G}_i$.

Theorem 4.6.3 ([100]) *For every positive integer $r > 0$ there is a polynomial $p_r(x,y)$ such that for every graph G and every orientation $\vec{G}$ of G with maximum in-degree $\Delta^-(\vec{G})$ there is a transitive fraternal augmentation $\vec{H}$ of $\vec{G}$ such that the underlying undirected graph H of $\vec{H}$ satisfies*

$$\nabla_r(H) \leq p_r\big(\Delta^-(\vec{G}), \nabla_{2r+1}(G)\big).$$

Note that this implies that every class of bounded expansion is stable under transitive fraternal augmentations, however, this does not hold for nowhere dense classes. Indeed, if a vertex of $\vec{G}$ has in-degree $n(G)^\epsilon$ then H contains a complete subgraph of this size.

By greedily orienting G and then iteratively inserting transitive fraternal edges and greedily reorienting the newly inserted fraternal edges, one obtains an efficient way to compute transitive fraternal orientations. This is stated for classes of bounded expansion in [101], however, the result clearly generalises to nowhere dense classes.

Theorem 4.6.4 ([101]) *For every integer $r > 0$ there is a polynomial $q_r(x)$ such that for every graph G one can compute in time $q_r\big(\nabla_{2^{r+1}}(G)\big)$ an orientation $\vec{G}$ of G and a transitive fraternal augmentation $\vec{G}_1 \subseteq \ldots \subseteq \vec{G}_r$ with $\vec{G}_1 = \vec{G}$ such that $\Delta^-(\vec{G}_r) \leq q_r\big(\nabla_{2^{r+1}}(G)\big)$.*

The following property of transitive fraternal augmentations is noted as Theorem 5.1 in [99]. Note that the theorem does not appear in the conference version of the paper.

Lemma 4.6.5 ([99]) *Let G be a graph and let $r > 0$ be a positive integer. Let $\vec{G}_1 \subseteq \ldots \subseteq \vec{G}_r$ be a transitive fraternal augmentation of an orientation $\vec{G}_1$ of G. Let $v_1 \ldots v_k$ be a path of length at most r in G. Then either $(v_1, v_k) \in E(\vec{G}_r)$ or $(v_k, v_1) \in E(\vec{G}_r)$ or there is some v_i, $1 < i < k$ such that $(v_i, v_1), (v_i, v_k) \in E(\vec{G}_r)$.*

Theorem 4.6.6 *Let G be a graph and let $d, r > 0$ be positive integers. Let $\vec{G}_1 \subseteq \ldots \subseteq \vec{G}_r$ be a transitive fraternal augmentation of an orientation $\vec{G}_1$ of G such that $\Delta^-(\vec{G}_r) \leq d$. Then $\operatorname{wcol}_r(G) \leq 4(d+1)^2$.*

PROOF. As $\Delta^-(\vec{G}_r) = d$, the underlying undirected graph G_r of $\vec{G}_r$ is $2d$-degenerate and we can order the vertices of G_r such that each vertex has at most $2d$ smaller neighbours. Denote the order which witnesses this by L. We claim that $|\operatorname{WReach}_r[G, L, v]| \leq 4(d+1)^2$ for all $v \in V(G)$. Let $w \in \operatorname{WReach}_r[G, L, v]$. Then there is a path of length at most r from v to w such that w is the smallest vertex of the path. By Lemma 4.6.5, we either have an edge (w, v) or an edge (v, w) or there is u on the path and we have edges $(u, v), (u, w)$ in $E(\vec{G}_r)$.

Let us count how many choices we have for w. There are at most $2d$ edges (w, v), as v has small in-degree in $\vec{G}_r$. There are at most $2d$ edges (v, w) such that $L(w) < L(v)$ by construction of the order. Furthermore, we have at most $2d$ edges (u, v), as v has small in-degree and each u has at most $2d$ edges (u, w) such that $L(w) < L(u)$ by construction of the order. These are exactly the pairs of edges we have to consider, as no vertex on the path from v to w may be smaller than w. Hence in total we have $|\operatorname{WReach}_r[G, L, v]| \leq 4d + 4d^2 + 1 \leq 4(d+1)^2$. □

Corollary 4.6.7 *Let $\mathscr{C}$ be a nowhere dense class of graphs. Let $r \geq 1$ be an integer and $\epsilon > 0$ a real. There are constants $c = c(r, \epsilon)$ and $n_0 = n_0(r, \epsilon)$ such that for every $G \in \mathscr{C}$ with $n(G) > n_0$ we can construct an order L of $V(G)$ such that $|\operatorname{WReach}_r(G, L, v)| \leq n(G)^\epsilon$ for all $v \in V(G)$ in time $c \cdot n(G)^{1+\epsilon}$.*

PROOF. Choose n_0 and $\delta < 1$ such that $n^\epsilon \geq 4\big(q_r(n^\delta)+1\big)^2$ for $n > n_0$, where q_r is the polynomial from Theorem 4.6.4. We now combine Theorem 4.6.4 and Theorem 4.6.6 and choose $c = c(r,\epsilon)$ appropriately (where we also ensure that all graphs $G \in \mathscr{C}$ with $n(G) \leq n_0$ can be handled in constant time c). □

5. Neighbourhood covers

Let $r > 0$ be a positive integer. An *r-neighbourhood cover* $\mathscr{X}$ of a graph G is a set of connected subgraphs of G called *clusters*, such that for every vertex $v \in V(G)$ there is some $H \in \mathscr{X}$ with $N_r(v) \subseteq V(H)$. The *radius* $\mathrm{rad}(\mathscr{X})$ of a cover $\mathscr{X}$ is the maximum radius of any of its clusters. The *degree* $d^{\mathscr{X}}(v)$ of v in $\mathscr{X}$ is the number of clusters that contain v and the degree of $\mathscr{X}$ is $\max_{v \in V(G)} d^{\mathscr{X}}(v)$.

Neighbourhood covers play a key role in the design of many data structures for distributed systems. In the design of these data structures, the radius of the cover often translates to some notion of *latency* and the degree of a vertex to some notion of *load* imposed on the vertex by the data structure. It is hence desirable to have a *sparse cover* whose degree is small and whose radius is small compared to r. We refer to the textbook [114] for an extensive survey of properties and applications of sparse neighbourhood covers.

We say that a class $\mathscr{C}$ of graphs *admits sparse neighbourhood covers* if for every integer $r > 0$, there exists an integer $c > 0$ such that for all reals $\epsilon > 0$, there is an integer $n_0 \geq 1$ such that for all $G \in \mathscr{C}$ of order $n \geq n_0$, there exists an r-neighbourhood cover of radius at most $c \cdot r$ and degree at most n^{ϵ}.

Awerbuch and Peleg [9] present an algorithm for constructing for all integers $k, r > 0$ an r-neighbourhood cover of radius $(2k-1) \cdot r$ and degree at most $2k \cdot n^{1/k}$. It was shown in [132] that asymptotically, these bounds cannot be improved simultaneously, i.e. the class of all graphs does not admit sparse neighbourhood covers.

Theorem 5.1.1 (Theorem 16.2.4 of [114], [132]) *Let $k \geq 3, r > 0$ be integers. There exist infinitely many graphs G for which every r-neighbourhood cover of radius at most k has degree $\Omega(n(G)^{1/k})$.*

In this chapter we show that every nowhere dense class admits sparse neighbourhood covers and in fact that monotone nowhere dense classes can be characterised by the existence of sparse neighbourhood covers. The results of this chapter were published in [65] and [64].

Theorem 5.1.2 *Let $r > 0$ be a positive integer. Every graph G admits an r-neighbourhood cover of radius $2r$ and degree at most $\mathrm{wcol}_{2r}(G)$.*

PROOF. Let L be a linear order of $V(G)$ with $|\mathrm{WReach}_{2r}[G,L,v]| \leq s := \mathrm{wcol}_{2r}(G)$ for all $v \in V(G)$. For every $x \in V(G)$, let

$$X_x := \{w \in V(G) : x \in \mathrm{WReach}_{2r}[G,L,w]\}.$$

We claim that $\mathscr{X} := \{X_x : x \in V(G)\}$ is an r-neighbourhood cover of radius $2r$ and degree at most s.

Clearly the radius of each cluster is at most $2r$, because if x is weakly $2r$-reachable from w, then $w \in N_{2r}(x)$.

Furthermore, the r-neighbourhood of every $v \in V(G)$ lies in some cluster. To see this, let x be minimal in $N_r(v)$ with respect to L. Then x is weakly $2r$-reachable from every $w \in N_r(v)$ as there is a path from w to x which uses only vertices of $N_r(v)$ and has length at most $2r$ and x is the minimal element of $N_r(v)$. Thus $N_r(v) \subseteq X_x$.

Finally observe that for every $v \in V(G)$,

$$\begin{aligned} d^{\mathscr{X}}(v) &= |\{x \in V(G) : v \in X_x\}| \\ &= |\{x \in V(G) : x \in \mathrm{WReach}_{2r}[G,L,v]\}| \\ &= |\mathrm{WReach}_{2r}[G,L,v]| \leq s. \end{aligned}$$

□

Corollary 5.1.3 *If $\mathscr{C}$ is nowhere dense, then $\mathscr{C}$ admits sparse neighbourhood covers.*

Using our results of Chapter 4, we find neighbourhood covers of radius $2r$ and degree bounded by some function $f(r)$ for classes that exclude K_t as a (topological) minor. Busch et al. [18] present r-neighbourhood covers of radius $24r-8$ and degree at most 18 for planar graphs and Abraham et al. [4] present r-neighbourhood covers of radius $O(t^2 \cdot r)$ and degree at most $2^{O(t)}t!$ for graphs that exclude K_t as a minor. Hence by allowing a larger radius one finds a cover with degree independent of r (depending only on the excluded minor). It is an interesting open question how large the radius must be chosen such that this is possible. It is easily seen that in general for bounded degree graphs (and hence for classes that exclude a topological minor) such covers cannot exist.

We now show that monotone nowhere dense classes are characterised by the existence of sparse neighbourhood covers. We first observe that the lower bounds in Theorem 5.1.1 come from a well known somewhere dense class.

Lemma 5.1.4 *Let $d \geq 1, k \geq 2$ be integers and let G be a graph of girth at least $k+1$ and edge density at least d. Then every 1-neighbourhood cover of radius at most k has degree at least d.*

PROOF. Let $\mathscr{X}$ be a 1-neighbourhood cover of G of radius at most k. We may assume that $\mathscr{X}$ contains at most $n(G)$ clusters. As $\mathscr{X}$ is a neighbourhood cover, every edge lies in some cluster. Because G has girth greater than k, $G[X]$ is isomorphic to a tree T_X for every cluster $X \in \mathscr{X}$. Assume that every vertex lies in at most c clusters. Then

$$m(G) \leq \sum_{X \in \mathscr{X}} m(G[X]) \leq \sum_{X \in \mathscr{X}} n(G[X]) \leq c \cdot n(G).$$

Hence $c \geq m(G)/n(G) = d$. □

Lemma 5.1.5 ([87, 88]) *Let $r \geq 5$ be an integer. There are infinitely many graphs G of girth at least $4r$ with edge density at least $c_0 \cdot n(G)^{1/(3(r-1))}$ for some constant $c_0 > 0$.*

Furthermore, subdivisions of a graph H can be covered almost like H itself by simply taking the subdivided vertices into the clusters.

Lemma 5.1.6 *Let $r, r' > 0, s \geq 0$ be integers. If an s-subdivision of H admits an $r \cdot s$-neighbourhood cover of radius r' and degree d, then H admits an r-neighbourhood cover of radius r' and degree d.*

PROOF. Let G be an s-subdivision of H and let $\mathscr{X}$ be an $r \cdot s$-neighbourhood cover of G. Let $\mathscr{Y}$ be the *projected cover* which for every $X \in \mathscr{X}$ has a cluster $Y(X) := G[X \cap V(H)]$.

Then $\mathscr{Y}$ is an r-neighbourhood cover of radius r' and degree d: clearly, every $Y(X)$ is connected and has radius at most r'. Let $v \in V(G)$. There is a cluster $X \in \mathscr{X}$ such that $N^G_{rs}(v) \subseteq X$. Then $N^H_r(v) = N^G_{rs}(v) \cap V(H) \subseteq X \cap V(H) = Y(X)$. Finally, the degree of $\mathscr{Y}$ is at most d, as every vertex v of H is exactly in those clusters $Y(X)$ with $v \in X$. This proves the lemma. ⊣

Theorem 5.1.7 *If $\mathscr{C}$ is somewhere dense and monotone, then $\mathscr{C}$ does not admit sparse neighbourhood covers.*

PROOF. Let $r \geq 5$ be an integer and assume towards a contradiction that $\mathscr{C}$ admits a sparse r-neighbourhood cover. Then for every real $\epsilon > 0$, every integer $s \geq 0$, every sufficiently large $G \in \mathscr{C}$ admits an $r \cdot s$-neighbourhood cover of radius $c \cdot s \cdot r$ (for some constant c) which has degree at most $n(G)^\epsilon$.

As $\mathscr{C}$ is somewhere dense, for some integer s, every graph H is a topological depth-s minor of some graph $G \in \mathscr{C}$.

Let H be a graph of girth greater than $c \cdot r \cdot s$ with edge density $d = c_0 \cdot n(H)^{1/(crs)}$ for some constant c_0. Such H exists by Lemma 5.1.5 and H does not admit an $r \cdot s$-neighbourhood cover of radius $c \cdot r \cdot s$ and degree d by Lemma 5.1.4. As $\mathscr{C}$ is monotone, an s-subdivision of H is a graph $G \in \mathscr{C}$ with $n(G) \leq n(H) + s \cdot m(H) \leq 2c_0 s n(H)^{1+1/(crs)}$.

By assumption, G admits an $r \cdot s$-neighbourhood cover of radius at most $c \cdot r \cdot s$ and degree at most $n(G)^\epsilon$ for $\epsilon = 1/(2crs)$ if G is large enough. It follows from Lemma 5.1.6 that H has a cover of radius $c \cdot r \cdot s$ and degree at most

$$\begin{aligned} n(G)^\epsilon \leq \left(2c_0 s n(H)^{1+1/(crs)}\right)^\epsilon &= (2c_0 s)^\epsilon \cdot n(H)^{\epsilon+\epsilon/(crs)} \\ &< c_0 n(H)^{2\epsilon} = c_0 n(H)^{1/(crs)} \end{aligned}$$

for sufficiently large $n(H)$. A contradiction. □

Based on a result of Kühn and Osthus [85], Nešetřil and Ossona de Mendez in [98] showed that a monotone class of graphs with unbounded expansion does not admit r-neighbourhood covers of radius $2r$ and degree at most $c(r)$ for any fixed function c and a somewhere dense class does not admit r-neighbourhood covers of radius $2r$ and degree at most n^ϵ for all its n-vertex graphs. In combination with our Theorem 5.1.2, this yields a similar characterisation of monotone classes of bounded expansion and of monotone nowhere dense classes.

Let us show how to efficiently compute sparse neighbourhood covers.

Theorem 5.1.8 *Let $\mathscr{C}$ be a nowhere dense class of graphs. Let $r \geq 0$ be an integer and $\epsilon > 0$ a real. Then there are integers $n_0 > 0$ and $c > 0$ such that for every $G \in \mathscr{C}$ with $n = n(G) \geq n_0$ an r-neighbourhood cover of G of radius $2r$ and maximum degree n^ϵ can be computed in time $c \cdot n^{1+\epsilon}$.*

PROOF. Choose $\delta > 0$ such that $2n^{1+2\delta} \leq n^\epsilon$ for large n. Let $G \in \mathscr{C}$ be sufficiently large such that we can order the vertices of G as in Corollary 4.6.7

such that $|\mathrm{WReach}_{2r}[G,L,v]| \leq n(G)^\delta$ for all $v \in V(G)$ in time $c' \cdot n(G)^{1+\delta}$, where c' is the constant from the corollary. We want to construct the clusters

$$X_x := \{w \in V(G) : x \in \mathrm{WReach}_{2r}[G,L,w]\}.$$

for all $x \in V(G)$, as described in the proof of Theorem 5.1.2. For $x \in V(G)$ let G_x be the subgraph of G induced by all vertices y with $L(y) \geq L(x)$. Observe that by definition of WReach_{2r} we have

$$X_x = G_x[N_{2r}^{G_x}(x)].$$

We can hence compute in ascending order for each $x \in V(G)$ the graph G_x and the set X_x as the $2r$-neighbourhood of x in G_x. From G_x, we can compute G_y, where y is the successor of x with respect to L, by simply deleting x from G_x. In order to delete x from G_x, when performing the breadth-first search to find $N_{2r}^{G_x}(x)$, we delete x from the adjacency lists of its neighbours. By our representation of G (see Remark 3.1.5), for each neighbour w, the adjacency list of w which contains x has size at most n^δ and x can hence be deleted in this time from the list. As G_x is a subgraph of G, if G_x is sufficiently large, it also has at most $n(G_x)^{1+\delta}$ edges and the extended breadth-first search described above can be performed in time $O(n(G_x)^{1+\delta})$. For convenience we drop all constants from the following estimation and multiply by a sufficiently large constant factor c in the end.

We hence get a running time of

$$\sum_{x\in V(G)} \left(n(G_x)^{1+\delta} + \sum_{y\in N^{G_x}(x)} n(G_x)^\delta\right) \leq \sum_{x\in V(G)} n(G_x)\cdot n^\delta + \sum_{x\in V(G)} \sum_{y\in N^{G_x}(x)} n^\delta$$

$$\leq 2n^\delta \sum_{x\in V(G)} n(G_x) \leq 2n^{1+2\delta} \leq n^{1+\epsilon}.$$

□

6. The splitter game

There are several structural graph measures which can equivalently be defined in terms of games, the most prominent examples are the cops-and-robber games for tree width. Also the existence of shortest paths decompositions of small width for graphs that exclude a fixed minor, as defined in Chapter 4, were originally proved in game theoretic terms and the generalised colouring numbers were introduced in the context of graph colouring games. Often one can derive a structural decomposition from the winning strategy of one of the players and the game provides a good intuition about this decomposition. In this chapter we are going to provide a game characterisation of nowhere dense classes. The game describes the local structure of a graph and the corresponding decomposition has a nice algorithmic application. The splitter game was invented by Martin Grohe and Stephan Kreutzer, Section 6.1 is based on ideas that are not originated by the author of this thesis. The results of this chapter were published in [65].

- We will define the game and show that nowhere dense classes can be characterised by it in Section 6.1.
- As an application of the splitter game we will show how to efficiently solve the distance-r independent set problem in Section 6.2.

6.1. The splitter game

Let G be a graph and let $\ell, m, r \geq 1$ be integers. The (r, ℓ, m)-*splitter game* on G is played by two players, *Connector* and *Splitter*, as follows. We let $G_0 := G$. In round $i+1$ of the game, Connector chooses a vertex $v_{i+1} \in V(G_i)$. Then Splitter picks a subset $W_{i+1} \subseteq N_r^{G_i}(v_{i+1})$ of size at most m. We let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus W_{i+1}]$. Splitter wins if $G_{i+1} = \emptyset$. Otherwise the game continues at G_{i+1}. If Splitter has not won after ℓ rounds, then Connector wins.

A *strategy* for Splitter is a function f that associates to every partial play $(v_1, W_1, \ldots, v_s, W_s)$ with associated sequence $G_0, \ldots, G_s$ of graphs and the next move $v_{s+1} \in V(G_s)$ by Connector a set $W_{s+1} \subseteq N_r^{G_s}(v_{s+1})$ of size at most m. A strategy f is a *winning strategy* for Splitter in the (r, ℓ, m)-splitter game on G if Splitter wins every play in which he follows the strategy f. We say that Splitter *wins* the (r, ℓ, m)-splitter game on G if he has a winning strategy.

In the rest of this section we are going to prove that nowhere dense classes can be characterised by the splitter game. To be precise: a class $\mathscr{C}$ of graphs is nowhere dense if and only if there are functions ℓ and m such that for every $r \geq 1$, splitter wins the $(r, \ell(r), m(r))$-splitter game on every graph $G \in \mathscr{C}$.

The proof is based on yet another characterisation of nowhere dense classes which describes the wideness of the class. The notions of almost-wideness and quasi-wideness were introduced in model theory when studying preservation theorems for first-order logic [8, 27].

A class $\mathscr{C}$ of graphs is called *uniformly almost-wide* if there is a function $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and a number $s \in \mathbb{N}$ such that for all $r, m \in \mathbb{N}$, $G \in \mathscr{C}$ and $W \subseteq V(G)$ of size at least $N(r, m)$ there is a set S of size smaller than s such that there is a set $A \subseteq W$ of size at least m which is r-scattered in $G - S$.

A class $\mathscr{C}$ of graphs is called *uniformly quasi-wide* if there is a function $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that for all $r, m \in \mathbb{N}$, $G \in \mathscr{C}$ and $W \subseteq V(G)$ of size at least $N(r, m)$ there is a set S of size smaller than $s(r)$ such that there is a set $A \subseteq W$ of size at least m which is r-scattered in $G - S$.

Atserias et al. [8] have shown that every proper minor closed class is uniformly almost-wide. Nešetřil and Ossona de Mendez [104] have shown that a hereditary class of graphs is uniformly almost-wide if and only if there is an integer $s > 0$ and a function $t : \mathbb{N} \to \mathbb{N}$ such that for all integers $r > 0$, the bipartite graph$K_{s,t(r)}$ is not a depth-r minor of any graph $G \in \mathscr{C}$. Nešetřil and Ossona de Mendez also showed that every class that excludes a topological minor is almost-wide. They observed that uniformly almost-wide classes need not be topologically closed and need not even have (locally) bounded expansion. In fact, the class of graphs G with $\Delta(G) \leq girth(G)$ is nowhere dense but has unbounded expansion (even unbounded average degree) and is obviously uniformly almost-wide. Adding to every graph a vertex which is connected to every other vertex yields a class which does not have locally bounded expansion which is still uniformly almost-wide. Most

useful for our purpose is the following characterisation of nowhere dense classes.

Theorem 6.1.1 ([104]) *A hereditary class of graphs is uniformly quasi-wide if and only if it is nowhere dense.*

We are now ready to prove the forward direction of our characterisation theorem.

Theorem 6.1.2 ([65]) *Let $\mathscr{C}$ be a nowhere dense class of graphs. Then for every integer $r > 0$ there are integers $\ell, m > 0$, such that for every $G \in \mathscr{C}$, Splitter wins the (r, ℓ, m)-splitter game on G.*

PROOF. We may assume that $\mathscr{C}$ is a monotone (and hence hereditary) class. As $\mathscr{C}$ is nowhere dense, by Theorem 6.1.1, it is also uniformly quasi-wide. Let $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ be such that for all $r, m \in \mathbb{N}$, $G \in \mathscr{C}$ and $W \subseteq V(G)$ of size at least $N(r,m)$ there is a set S of size smaller than $s(r)$ such that there is a set $A \subseteq W$ of size at least m which is r-scattered in $G - S$. Let $r \geq 1$ and let $\ell := N(r, 2s(r))$ and $m := \ell \cdot (r+1)$. Note that both ℓ and m only depend on $\mathscr{C}$ and r. We claim that for any $G \in \mathscr{C}$, Splitter wins the (r, ℓ, m)-splitter game on G.

Let $G \in \mathscr{C}$ be a graph. In the (r, ℓ, m)-splitter game on G, Splitter uses the following strategy. In the first round, if Connector chooses $v_1 \in V(G_0)$, where $G_0 := G$, then Splitter chooses $W_1 := \{v_1\}$. Now let $i > 1$ and suppose that $v_1, \ldots, v_i, G_1, \ldots, G_i, W_1, \ldots, W_i$ have already been defined. Suppose Connector chooses $v_{i+1} \in V(G_i)$. We define W_{i+1} as follows. For each integer j, $1 \leq j \leq i$, choose a path $P_{j,i+1}$ in $G_{j-1}[N_r^{G_{j-1}}(v_j)]$ of length at most r connecting v_j and v_{i+1}. Such a path exists as $v_{i+1} \in V(G_i) \subseteq V(G_j) \subseteq N_r^{G_{j-1}}(v_j)$. We let $W_{i+1} := \bigcup_{1 \leq j \leq i} V(P_{j,i+1}) \cap N_r^{G_i}(v_{i+1})$. Note that $|W_{i+1}| \leq i \cdot (r+1)$ (the paths have length at most r and hence consist of $r+1$ vertices). It remains to be shown that the length of any such play is bounded by ℓ.

Assume towards a contradiction that Connector can play on G for $\ell' = \ell + 1$ rounds. Let $(v_1, W_1, \ldots, v_{\ell'}, W_{\ell'})$ be the play with associated sequence $G_0, \ldots, G_{\ell'}$ of graphs. As $\ell' > N(r, 2s(r))$, there is a set $S \subseteq V(G)$ with $|S| < s(r)$, such that W contains an r-scattered set I of size $t := 2s(r)$ in $G \setminus S$. Assume $I = \{u_1, \ldots, u_t\}$, where $u_j = v_{i_j}$ for indices $1 \leq i_1 < i_2 < \ldots < i_t \leq \ell'$.

We now consider the pairs (u_{2j-1}, u_{2j}) for $1 \leq j \leq s(r)$. By construction, $P_j := P_{i_{2j-1}, i_{2j}}$ is a path of length at most r from u_{2j-1} to u_{2j} in $G_{i_{2j-1}-1}$

and $V(P_j) \cap V(G_{i_{2j-1}+1}) = \emptyset$, that is, the path P_j is completely deleted by Splitter in round $i_{2j-1}+1$ of the game. Because P_{j+1} starts at a vertex chosen at least 2 rounds later, we conclude that for $i \neq j$, $V(P_i) \cap V(P_j) = \emptyset$, hence there are s vertex disjoint paths of length at most r between vertices from I.

However, any path P_j must necessarily contain a vertex $s_j \in S$, as otherwise the path would exist in $G \setminus S$, contradicting the fact that I is r-scattered in $G \setminus S$. But this is not possible as there are strictly less than $s(r)$ vertices in S. A contradiction. □

In order to efficiently compute Splitter's winning strategy, for every (partial) play $(v_1, W_1, \ldots, v_i, W_i)$ with associated sequence $G_0, \ldots, G_i$, we store the sequence $G_0, \ldots, G_i$ together with a sequence $T_1, \ldots, T_{i-1}$, where T_j for $1 \leq j < i$ is a breadth-first search tree of depth r in G_{j-1} with root v_j.

Then, given G_i, $T_1, \ldots, T_i$ and Connector's move v_{i+1} in round $(i+1)$ of the game, we can compute Splitter's answer W_{i+1} as follows. For $1 \leq j \leq i$, we compute the paths $P_{j,i+1}$ as the shortest path from v_{i+1} to v_j in T_j. We can compute this path for each j in time $O(r)$ and thus all paths in time $O(ri)$. We let $W = \bigcup_j V(P_{j,i+1})$ and compute $G_{i+1} = G[N_r^{G_i}(v_{i+1}) \setminus W]$ and the spanning tree T_i by a breadth-first search which leaves out any vertex of W in time $O(ri \cdot n(G_i) + m(G_i))$.

We call the $(r, \ell, 1)$-splitter game on a graph the *simple* (r, ℓ)*-splitter game* on G. It is straight forward to verify that for all $\ell, m, r \geq 0$, if Splitter has a winning strategy in the (r, ℓ, m)-splitter game on G then Splitter has a winning strategy in the simple $(r, \ell \cdot m)$-splitter game.

We close the section by observing the converse of Theorem 6.1.2 and hence show that the splitter game provides another characterisation of nowhere dense classes of graphs.

Theorem 6.1.3 *Let $\mathscr{C}$ be a class of graphs. If for every positive integer $r > 0$ there are integers $\ell, m > 0$ such that for every graph $G \in \mathscr{C}$, Splitter wins the (r, ℓ, m)-splitter game, then $\mathscr{C}$ is nowhere dense.*

PROOF. We show that if $\mathscr{C}$ is somewhere dense, i.e., $\mathscr{C}$ contains all graphs as depth-r minors at some depth r, then for all integers $\ell, m > 0$ there is a graph $G \in \mathscr{C}$ such that Connector wins the $(4r+1, \ell, m)$-splitter game on G.

Let $\ell, m > 0$ be integers. We choose $G \in \mathscr{C}$ such that G contains the complete graph $K := K_{\ell m+1}$ as a depth-r minor. Connector uses the following strategy to win the $(4r+1, \ell, m)$-splitter game. Connector chooses any vertex

from the branch set of a vertex of K. The $4r+1$-neighbourhood of this vertex contains the branch sets of all vertices of K. Splitter removes any m vertices. We actually allow him to remove the complete branch sets of all m vertices he chose. In round 2 we may thus assume to find the complete graph $K_{(\ell-1)m+1}$ as a depth-r minor and continue to play in this way until in round ℓ at least the branch set of a single vertex remains. □

6.2. Solving the independent set problem

In this section we use the splitter game to show that the DISTANCE INDEPENDENT SET problem, which is W[1]-hard in general, is fixed-parameter tractable on nowhere dense classes of graphs. This will be used later in the proof of our main theorem but may be of independent interest. Recall that an r-independent set $W \subseteq V(G)$ is a set such that $\mathrm{dist}(u,v) > r$ for all $u,v \in W$.

Theorem 6.2.1 *Let $\mathscr{C}$ be a nowhere dense class of graphs. There is a function f such that for every real $\epsilon > 0$, every $G \in \mathscr{C}$ and all integers $r,k \geq 0$ we can decide in time $f(\epsilon,r,k) \cdot n(G)^{1+\epsilon}$ whether G contains and r-independent set of size k.*

In fact we will solve a more general problem, called the DISTANCE INDEPENDENT RAINBOW SET problem. A *coloured graph* $(G,C_1,\dots,C_t)$ is a graph G together with relations $C_1,\dots C_t \subseteq V(G)$, called *colours*, such that $C_i \cap C_j = \emptyset$ for all $i \neq j$. A vertex $v \notin \bigcup_{1\leq i\leq t} C_i$ is called *uncoloured*. A set $X \subseteq V(G)$ is a *rainbow set* if all of its elements have distinct colours (and no vertex is uncoloured).

The DISTANCE INDEPENDENT RAINBOW SET problem is the problem to decide for a given coloured graph $(G,C_1,\dots,C_t)$ and $r,k \geq 0$ whether $(G,C_1,\dots,C_t)$ contains an r-independent rainbow set of size k. We show that the DISTANCE INDEPENDENT SET problem on a nowhere dense class $\mathscr{C}$ of graphs reduces to the DISTANCE INDEPENDENT RAINBOW SET problem on a nowhere dense class $\mathscr{D}$ of coloured graphs via a fixed-parameter reduction.

The *lexicographic product* $G \bullet H$ of two graphs G and H is defined by $V(G \bullet H) = V(G) \times V(H)$ and $E(G \bullet H) = \big\{\{(x,y),(x',y')\} : \{x,x'\} \in E(G) \text{ or } \big(x = x' \text{ and } \{y,y'\} \in E(H)\big)\big\}$. The graph $G \bullet H$ has a natural coloured version $G \circ H$: we associate a colour with every vertex of H and colour every vertex of $G \bullet H$ by its projection on H. That is, the colour of (x,y) is y (or the colour

associated with y). It is easy to see that a graph G has an r-independent set of size k if and only if $G \circ K_k$ has a rainbow r-independent of size k. This gives us the reduction from distance independent sets to their rainbow variant. Furthermore, observe that if Splitter wins the (l,m,r)-splitter game on a graph G, for some $r,l,m \geq 0$, then he also wins the $(l,k \cdot m,r)$-splitter game on $G \bullet K_k$, for all k. As a consequence, together with Theorem 6.1.2 and Theorem 6.1.3 this implies a different and very simple proof of the following result by Nešetřil and Ossona de Mendez (Theorem 13.1 of [106]) that nowhere dense classes of graphs are preserved by taking lexicographic products in the following sense.

Corollary 6.2.2 *If $\mathscr{C}$ is a nowhere dense class of graphs then for every integer $k > 0$, $\{G \bullet K_k : G \in \mathscr{C}\}$ is also nowhere dense.*

We are now ready to use this reduction to complete the proof of Theorem 6.2.1. Let $\epsilon > 0$ and let ℓ, m be chosen according to Theorem 6.1.2 such that Splitter has a winning strategy for the $(4k^2r, \ell, m)$-splitter game on every graph in $\mathscr{C}$. Choose $n_0 = n_0(k,r,\epsilon)$ according to Theorem 3.3.2 such that every graph $G \in \mathscr{C}$ of order $n \geq n_0$ has at most $n^{1+\epsilon}$ many edges.

Suppose we are given an instance G,k,r,W of DISTANCE INDEPENDENT SET, where $G \in \mathscr{C}$. We first compute the coloured graph $G' := G \circ K_k$. Let $C_1,\dots,C_t$, where $t := k$, be the colours of G'. As explained above, Splitter wins the $(4k^2r, \ell, mk)$-splitter game on G' and his winning strategy can easily be computed from any winning strategy for the $(4k^2e, \ell, m)$-splitter game on G. Also his winning strategy for the simple $(4k^2r, \ell km)$-splitter game can be computed from his original strategy.

We need to decide if $(G', C_1, \dots, C_t)$ has a rainbow r-independent set of size k. If $n(G) \leq n_0$, we test whether this set exists by brute force. In this case the running time is bounded by a function of r,k and ϵ. So let us assume $n(G) \geq n_0$.

Let $G_1 := G'$. We compute an inclusion-wise maximal rainbow r-independent set $I_1 = \{x_1^1, \dots, x_1^{k_1}\}$ of size $k_1 \leq k$ by a greedy algorithm. If $k_1 = k$, we are done and return the independent set. Otherwise, we may assume without loss of generality that x_1^j has colour j. Let $X_1 := N_r(I_1)$. Then all elements with colours $k_1+1,\dots,t$ are contained in X_1. Let $Y_1 := N_r(X_1)$. Then all paths of length at most r between elements of colour $k_1+1,\dots,k$ lie inside Y_1. Let $G_2 := G_1 \setminus Y_1$.

We continue by computing an inclusion-wise maximal rainbow r-independent set in G_2. Denote this set by $I_2 = \{x_2^1, \dots, x_2^{k_2}\}$. Note that all occurring

colours are among $1,\ldots,k_1$ and in particular we have $k_2 \leq k_1$ because no other colours occur in $G_1 \setminus Y_1$. Again we may assume without loss of generality that x_2^i has colour i. Let $X_2 := N_r(I_2)$. Then we find all elements with colours $k_2+1,\ldots,t$ in $X_1 \cup X_2$. We let $Y_2 := N_r(X_2)$. Let $G_3 := G_2 \setminus Y_2$.

We repeat this construction until $k_s = k_{s+1}$ or until $G_{s+1} = \emptyset$. Note that $s \leq k$, because $k_1 < k$. In the first case we have constructed $s+1$ sets $I_i = \{x_i^1,\ldots,x_i^{k_i}\}$, X_i and Y_i such that x_i^j has colour j for $1 \leq i \leq s+1$, $1 \leq j \leq k_i$. Furthermore, the colours $k_s+1,\ldots,t$ occur only in $X_1 \cup \ldots \cup X_s$ and all paths of length at most r between vertices of these colours lie in $Y_1 \cup \ldots \cup Y_s$. By construction, no vertex of colour $k_s+1,\ldots,t$ has distance at most r to any vertex of I_{s+1}. Hence we may assume that any rainbow r-independent set includes the vertices $x_{s+1}^1,\ldots,x_{s+1}^{k_s}$ of colour $1,\ldots,k_s$. It remains to solve the rainbow r-independent set problem with parameter $k' := k - k_s$ and colours $k_s+1,\ldots,t$ on $G'' := G[Y_1 \cup \ldots \cup Y_s]$.

In the other case ($G_{s+1} = \emptyset$) we also let $G'' := G[Y_1 \cup \ldots Y_s]$. The only difference is that we have to solve the original problem with parameter $k' = k$.

If G'' is not connected, let $U_1,\ldots,U_c \subseteq G''$ be the components of G''. For all possible partitions of the set $C_1,\ldots,C_t$ of colours into parts $V_1,\ldots,V_c$ we proceed as follows. For all $1 \leq i \leq c$ we delete all colours from U_i not in V_i, i.e. work in the coloured graph (U_i,V_i). We then solve the problem separately for all components (U_i,V_i) and for each component determine the maximal value $k'' \leq k'$ so that (U_i,V_i) contains a rainbow r-independent set. We then simply check whether for some partition $(V_1,\ldots,V_c)$ of the colours the maximal values for the individual components sum up to at least k'.

Hence, we may assume that G'' is connected. Then G'' has diameter at most $4k^2 \cdot r$ (there are at most $\sum_{i=1}^{k} i \leq k^2$ many vertices in the independent sets surrounded by their $2r$-neighbourhoods of diameter at most $4r$). Hence the radius (and even the diameter) of G'' is at most $4k^2 \cdot r$.

Let v be a centre vertex of G''. We let v be Connector's choice in the simple $(4k^2r,\ell km)$-splitter game and let w be Splitter's answer. We let $G''' := G'' - w$ and continue recursively with several colourings of G''' as follows. In the first case, (which we consider only if w has a colour), we test whether w can be added to the rainbow set, that is, we remove the colour of w completely from the graph and remove the colour of all vertices from $N_r^{G''}(w)$ and call the algorithm with parameter $k'' = k' - 1$.

In the second case we test whether we can avoid adding w to the rainbow set by changing the colours of of G''' as follows. For every colour C_i, with

$1 \le i \le t$, and every number $d \in \{1, \dots, r, \infty\}$, we add a new colour $C_{i,d}$ and set $C_{i,d}$ to be the set of all vertices $x \in C_i$ such that $\mathrm{dist}_{G''}(x,w) = d$, for all $1 \le i \le t$, where we define $\mathrm{dist}_{G''}(x,w) = \infty$ if the distance is bigger then r. Note that the number of colours added in this way is only $t \cdot (r+1)$ and hence only depends on the number of original colours and r. We call a subset $C_{i_1,d_1}, \dots, C_{i_{t''},d_{t''}}$ of the colours a *valid sub-colouring* if the colours satisfy the following constraints:

1. If $C_{i_j,d_j} \neq \emptyset$ for a colour which states that the distance to w is $r' < r$, then $C_{i_{j'},d_{j'}} = \emptyset$ for all colours which state that the distance to w is at most $r - r'$.

2. If C_{i_j,d_j} and $C_{i_{j'},d_{j'}}$ are colours with $i_j = i_{j'}$ and $d_j \neq d_{j'}$ then $C_{i_j,d_j} = \emptyset$ or $C_{i_{j'},d_{j'}} = \emptyset$.

We now test for all possible sub-colourings $C_{i_1,d_{i_1}}, \dots, C_{i_{t''}d_{t''}}$ of G''' whether they are valid and for each valid sub-colouring we recursively call the algorithm on G''' with colouring $C_{i_1,d_{i_1}}, \dots, C_{i_{t''},d_{t''}}$ and parameter $k'' := k'$. The number of valid sub-colourings only depends on the original number of colours and on r.

We claim that this procedure correctly decides whether G'' contains an r-independent rainbow set of size k'.

If there exists such a set Z, assume first that $w \in Z$. Then w has a colour and no vertex from $Z \setminus \{w\}$ may have this colour hence we may remove the colours completely from the graph. Furthermore, $Z \cap N_r(w) = w$, hence we may remove the colours from $N_r(w)$. Now assume that there is $u \in Z$ with $\mathrm{dist}_{G''}(u,w) = r' < r$. Then $v \notin Z$ for all v with $\mathrm{dist}_{G''}(v,m) \le r - r'$. Hence we will find Z in the graph where all colours which state that the distance to m is at most $r - r'$ are removed.

Conversely assume that the algorithm has computed an r-independent rainbow set I in G''' of size k''. If $k'' = k' - 1$, then clearly we can add v to obtain an r-independent rainbow set in G'. If $k'' = k'$ and the rainbow set uses colours from some valid sub-colouring then by Condition (1) of valid sub-colourings, I is also an r-independent set in G''. By Condition (2) of valid sub-colourings, I is also rainbow in G''.

We now analyse the running time of the algorithm. First observe that in a recursive call the parameter r is left unchanged and k can only decrease. Moreover it follows from the definition of G''' that Splitter has a winning strategy for the simple $(4k^2r), \ell km - 1$-splitter game on G'''. Thus in each

recursive call we can reduce the first parameter. Once we have reached $\ell = 0$, the graph G''' will be empty and the algorithm terminates. Hence the complete recursion tree has depth at most ℓkm and branching degree at most $2^{k(r+1)^{\ell km}}$ (in each level the number of colours increases by a factor of $(r+1)$ and there are only so many subsets of $k(r+1)^{\ell km}$ colours,). Let us describe the running time of the algorithm on level j of the recursion. The time for computing maximal r-independent sets of size at most k and their $2r$-neighbourhoods can be bounded by time $c_0 \cdot n^{1+\epsilon}$. The factor $n^{1+\epsilon}$ stems from the breadth-first searches we have to perform in order to find the sets $Y(i)$ and Splitter's strategy and c_0 is a constant depending only on r, k, ϵ and $\mathscr{C}$. This completes the proof of Theorem 6.2.1.

Part III.

Model-checking

Introduction

Part III of this work deals with the complexity of the model-checking problem for first-order logic, successor-invariant first-order logic and order-invariant monadic second-order logic on restricted classes of structures.

We show that every first-order property can be decided in time $O(n^{1+\epsilon})$ on a nowhere dense class of graphs.

Our starting point is the locality based technique introduced in [53]. In a nutshell, this technique works as follows. Using Gaifman's theorem, the problem to decide whether a general first-order formula φ is true in a graph can be reduced to testing whether a formula is true in r-neighbourhoods in the graph, where the radius r only depends on φ, and solving a variant of the (distance d) independent set problem. Hence, if $\mathscr{C}$ is a class of graphs where r-neighbourhoods have a simple structure, such as the class of planar graphs or classes of bounded local tree-width, this method gives an easy way for deciding properties definable in first-order logic.

As shown in the first part of this work, nowhere dense classes have nice local properties. We are going to use the splitter game introduced in Chapter 6 to recursively decompose the graph into smaller parts. After each round of the game we will apply the locality based approach to compute local information which can finally be combined to obtain the answer to the original query.

A key requirement for this approach to work is that ranks are preserved during the recursion (the parameter for the splitter game must not change during the game). This however cannot be guaranteed when using Gaifman's original locality theorem. Our solution to solve this problem is to extend first-order logic by distance atoms and to define an appropriate rank measure for this extended logic FO(dist). We then use a combinatorial argument similar to that in Gaifman's original proof to show that every FO(dist)-formula can effectively be translated to a local FO(dist)-formula of the same rank.

The final step is to give the extended logic FO(dist) access to neighbourhood covers as introduced in Chapter 5. By this we can ensure that the constructed recursion tree remains small and hence obtain efficient running times.

A very important way to make first-order logic more expressive is to include additional relations on the universe, most prominently, an order relation

and arithmetic predicates. In most practical applications of first-order logic, the given structures are (at least implicitely) ordered. We are going to show that the model-checking problem for first-order logic is hard for the parameterized class AW[$\star$] when an arbitrary order is added. We will show however, that the model-checking problem for order-invariant MSO on structures of bounded clique- or tree-width ist fixed-parameter tractable.

Finally, we will show that the model-checking problem for sucessor-invariant first-order logic on planar graphs is fixed-parameter tractable.

- Chapter 7 is devoted to the proof of our rank-preserving locality theorem. We will introduce the logic FO(dist) in Section 7.1 and in Section 7.2 we are going to prove the extended locality theorem for this logic.
- In Chapter 8 we are going to present our model-checking algorithm.
- We are going to present our results about order- and successor-invariant logics in Chapter 9.

The results of Chapter 7 and Chapter 8 were published in [65]. The results of Chapter 9 were published in [46].

7. A rank preserving locality theorem

7.1. First-Order Logic with Distance Atoms

In Corollary 2.4.3, we invested many quantifiers to relativise a formula to the r-neighbourhood around a tuple $\overline{a}$. This relativisation is a very clear algorithmic instruction for which we want to avoid introducing new quantifiers. The aim of this section is to extend first-order logic to first-order logic with distance atoms.

We define an extension FO(dist) of FO by adding for all variables x, y and all non-negative integers $d \geq 0$ an atomic formula $\mathrm{dist}(x, y) \leq d$. We call these formulas *distance atoms*. The meaning of an atom $\mathrm{dist}(x, y) \leq d$ is that of $\delta_{\leq d}(x, y)$. Hence, every FO(dist)-formula φ is equivalent to an FO-formula ψ obtained from φ by replacing each distance atom $\mathrm{dist}(x, y) \leq d$ by the FO-formula $\delta_{\leq d}(x, y)$. Thus FO(dist) is only a syntactic extension of FO. However, the quantifier rank of $\delta_{\leq d}(x, y) \in \mathrm{FO}$ as we defined it is $c(\tau) \cdot d$ where $c(\tau)$ is a constant depending on τ, whereas the quantifier rank of the atomic FO(dist)-formula $\mathrm{dist}(x, y) \leq d$ is 0. With this definition as one of the base steps, we define the quantifier rank $\mathrm{qr}(\varphi)$ for FO(dist)-formulas φ recursively as for FO-formulas.

We define a refined rank measure for FO(dist) formulas as follows. For a *distance function* $dst : \mathbb{N} \to \mathbb{N}$ and a non-negative integer $k \geq 0$, we say that a formula φ has *dst-distance rank* k if φ has quantifier rank at most k and if each distance atom $\mathrm{dist}(x, y) \leq d$ in the scope of $i \leq k$ quantifiers satisfies $d \leq dst(k - i)$. We write $\mathrm{drk}_{dst}(\varphi)$, or simply $\mathrm{drk}(\varphi)$ if there is no danger of confusion, for the dst-distance rank of φ. We will only consider monotonically increasing distance functions which will make it cheaper to define distances than in FO-formulas. On the other hand, just as for plain FO, defining distances becomes more expensive in the scope of quantifiers.

For our extended locality theorem we will use the following distance function. Let $q \in \mathbb{N}$ and define $dst_q(i) := (4q)^{q+i}$. We call the distance rank of a formula φ for this distance function the *q-rank* of φ. For example, for $q = 3$, the sentence

$$\exists x \exists y \Big(\mathrm{dist}(x,y) \leq 12^5 \wedge \exists z \big(\mathrm{dist}(x,z) \leq 12^6 \wedge \forall w (\neg \mathrm{dist}(z,w) \leq 12^4 \vee \mathrm{dist}(w,y) \leq 12^4) \big) \Big)$$

has 3-rank 6, because its quantifier rank is $4 \leq 6$ and for the distance atom $\mathrm{dist}(x,z) \leq 12^6$ in the scope of 3 quantifiers we have $12^6 = (4 \cdot 3)^{3+(6-3)}$.

It is the aim of this section to extend the theory of types and back-and-forth systems for FO(dist). In the following, let *dst* be a fixed (computable) distance function. Let $\mathfrak{A}$ be a τ-structure and let $\overline{a} = (a_1, \ldots, a_m) \in V(\mathfrak{A})^m$ and $\overline{x} = (x_1, \ldots, x_m)$. The *distance rank-k m-type* of $\overline{a}$ over $\mathfrak{A}$ is

$$\mathrm{dtp}_k(\mathfrak{A}, \overline{a}) := \{\varphi(\overline{x}) \in \mathrm{FO(dist)} : \mathrm{drk}(\varphi) \leq k, \mathfrak{A} \models \varphi(\overline{a})\}.$$

Again, up to logical equivalence, for all integers $k, m \geq 0$, there are only finitely many formulas $\varphi(x_1, \ldots, x_m) \in \mathrm{FO(dist)}$ of distance rank at most k and we can effectively normalise formulas of distance rank k, i.e. rename variables such that only the variables $x_1, \ldots, x_{m+k}$ occur in φ, such that all Boolean combinations are in disjunctive normal form and such that the Boolean combinations contain no duplicate entries. We write $\Phi(\tau, dst, k, m)$ for the set of normalised FO(dist)-formulas $\varphi(x_1, \ldots, x_m)$ over vocabulary τ of *dst*-distance rank at most k.

A *partial d-isomorphism* between $\mathfrak{A}$ and a τ-structure $\mathfrak{B}$ is an isomorphism p from $\mathfrak{A}[A']$ to $\mathfrak{B}[B']$ for non-empty and finite subsets $A' \subseteq V(\mathfrak{A})$ and $B' \subseteq V(\mathfrak{B})$ which preserves distances up to d, that is, for all elements $a, a' \in dom(p)$ either distances are equal,$\mathrm{dist}(a,a') = \mathrm{dist}(p(a), p(a'))$, or large, $\mathrm{dist}(a,a') > d$ and $\mathrm{dist}(p(a), p(a')) > d$. For convenience, we also call the empty mapping $\emptyset$ a partial d-isomorphism.

Let $\mathfrak{B}$ be a τ-structure and let $\overline{b} = (b_1, \ldots, b_m) \in V(\mathfrak{B})^m$. $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{B}, \overline{b})$ are *k-isomorphic with respect to dst*, written $(\mathfrak{A}, \overline{a}) \cong_k^{dst} (\mathfrak{B}, \overline{b})$, if there is a sequence $(I_j)_{0 \leq j \leq k}$ with the following properties.

- For every integer j, $0 \leq j \leq k$, I_j is a non-empty set of partial *dst(j)*-isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and $\overline{a} \mapsto \overline{b} \in I_k$.
- (*Forth property*) For every $0 \leq j < k$, $q \in I_{j+1}$ and $a \in V(\mathfrak{A})$ there is $p \in I_j$ such that p extends q and $a \in \mathrm{dom}(p)$.

- (*Back property*) For every $0 \leq j < k$, $q \in I_{j+1}$ and $b \in V(\mathfrak{B})$ there is $p \in I_j$ such that p extends q and $b \in \mathrm{ran}(p)$.

The sequence $(I_j)_{0\leq j\leq k}$ is called a *distance preserving back-and-forth system*.

For $a, b \in V(\mathfrak{A})$, we write $\mathrm{dist}(a,b) \equiv_i d \in \{0,\dots,i\} \cup \{\infty\}$ if $\mathrm{dist}(a,b) = d \leq i$ or $\mathrm{dist}(a,b) > i$ and $d = \infty$. For $\overline{a} \in V(\mathfrak{A})^m$ we can write down a quantifier free FO(dist)-formula $\vartheta^i_{\mathfrak{A},\overline{a}}$ of distance rank i that describes the mutual distances between elements of $\overline{a}$ up to $\mathrm{dist}(i)$. Let $\overline{x} := (x_1,\dots,x_m)$.

$$\vartheta^i_{\mathfrak{A},\overline{a}}(\overline{x}) := \bigwedge_{\substack{a_i,a_j\in\overline{a}\\ \mathrm{dist}(a_i,a_j)=d\leq dst(i)}} \mathrm{dist}(x_i,x_j) = d \quad \wedge \bigwedge_{\substack{a_i,a_j\in\overline{a}\\ \mathrm{dist}(a_i,a_j)>dst(i)}} \mathrm{dist}(x_i,x_j) > dst(i).$$

For an integer $n \geq 0$, a τ-structure $\mathfrak{A}$ and $\overline{a}_n \in V(\mathfrak{A})^n$ let $\overline{x}_n := (x_1,\dots,x_n)$ and let $\Psi^0_{\mathfrak{A},\overline{a}_n}$ be the set of formulas $\varphi(\overline{x}_n) \in \mathrm{FO(dist)}$ with $\mathrm{drk}(\varphi) = 0$ such that $\mathfrak{A} \models \varphi(\overline{a}_n)$. Let

$$\psi^0_{\mathfrak{A},\overline{a}}(\overline{x}_n) := \bigwedge \Psi^0_{\mathfrak{A},\overline{a}_n}.$$

$\Psi^0_{\mathfrak{A},\overline{a}} \subseteq \Phi(\tau, dst, 0, n)$ is finite, and hence the above conjunction is a well defined formula. Note that $\vartheta^0_{\mathfrak{A},\overline{a}_n}$, which fixes distances up to $dst(0)$, is a sub-formula of $\psi^0_{\mathfrak{A},\overline{a}_n}$. For $i > 0$, assume that $\Psi^{i-1}_{n+1} := \{\psi^{i-1}_{\mathfrak{A},\overline{a}_{n+1}}(\overline{x}_{n+1}) : \mathfrak{A}\ \tau\text{-structure}, \overline{a}_{n+1} \in V(\mathfrak{A})^{n+1}\} \subseteq \Phi(\tau, dst, i-1, n+1)$ has been defined and is finite. Then

$$\psi^i_{\mathfrak{A},\overline{a}}(\overline{x}_n) := \vartheta^i_{\mathfrak{A},\overline{a}}(\overline{x}_n) \wedge \bigwedge_{a\in V(\mathfrak{A})} \exists x_{n+1}\psi^{i-1}_{\mathfrak{A},\overline{a}_n a}(\overline{x}_n, x_{n+1}) \wedge \forall x_{n+1} \bigvee_{a\in V(\mathfrak{A})} \psi^{i-1}_{\mathfrak{A},\overline{a}_n a}(\overline{x}_n, x_{n+1}),$$

where we remove duplicate entries from the conjunctions and disjunctions is a well defined formula. We call $\psi^k_{\mathfrak{A},\overline{a}_n}$ the *dst-distance rank-k Hintikka type* of $\overline{a}_n \in V(\mathfrak{A})^n$ in $\mathfrak{A}$. Note that the distance rank-k Hintikka type of an n-tuple is a normalised formula from $\Phi(\tau, dst, k, n)$ of distance rank k (the outermost disjunction contains exactly one disjunct). The following is the analogue of Theorem 2.3.2.

Theorem 7.1.1 *Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures, let $m, k \geq 0$ be integers and let $\overline{a} \in V(\mathfrak{A})^m$, $\overline{b} \in V(\mathfrak{B})^m$. The following are equivalent.*

1) $\mathrm{dtp}_k(\mathfrak{A}, \overline{a}) = \mathrm{dtp}_k(\mathfrak{B}, \overline{b})$.
2) $\mathfrak{B} \models \psi^k_{\mathfrak{A},\overline{a}}(\overline{b})$.
3) *There is a distance preserving back-and-forth system $(I_j)_{0 \leq j \leq k}$ for $\mathfrak{A}$ and $\mathfrak{B}$ with $\overline{a} \mapsto \overline{b} \in I_k$.*

PROOF. As already noted, the distance rank-k Hintikka type of $\overline{a}$ is a formula of distance rank k. Hence 1) implies 2).

We show that 2) implies 3) by induction on k. For $k = 0$, $\mathfrak{B} \models \psi^k_{\mathfrak{A},\overline{a}}(\overline{b})$ if and only if $\overline{a}$ and $\overline{b}$ satisfy the same FO(dist)-formulas of distance rank 0. This is the case if and only if the mapping $\overline{a} \mapsto \overline{b}$ is a partial $dst(k)$-isomorphism. Hence we can define I_k as $\{\overline{a} \mapsto \overline{b}\}$.

For $k > 0$, $\mathfrak{B} \models \psi^k_{\mathfrak{A},\overline{a}}(\overline{b})$ if and only if

- $\overline{a}$ and $\overline{b}$ satisfy the same distance formulas up to dist(k),
- for all $a \in V(\mathfrak{A})$ there exists $b \in V(\mathfrak{B})$ such that $\mathfrak{B} \models \psi^{k-1}_{\mathfrak{A},\overline{a}a}(\overline{b}, b)$ and
- for all $b \in V(\mathfrak{B})$ there exists $a \in V(\mathfrak{A})$ such that $\mathfrak{B} \models \psi^{k-1}_{\mathfrak{A},\overline{a}a}(\overline{b}, b)$

by definition of $\psi^k_{\mathfrak{A},\overline{a}}(\overline{b})$. By induction hypothesis this is equivalent to

- $\overline{a}$ and $\overline{b}$ satisfy the same distance formulas up to $dst(k)$,
- for all $a \in V(\mathfrak{A})$ there exists $b \in V(\mathfrak{B})$ and a distance preserving back-and-forth system $(J^a_j)_{0 \leq j \leq k-1}$ for $\mathfrak{A}$ and $\mathfrak{B}$ with $\overline{a}a \mapsto \overline{b}b \in J^a_{k-1}$ and
- for all $b \in V(\mathfrak{B})$ there exists $a \in V(\mathfrak{A})$ and a distance preserving back-and-forth system $(L^b_j)_{0 \leq j \leq k-1}$ for $\mathfrak{A}$ and $\mathfrak{B}$ with $\overline{a}a \mapsto \overline{b}b \in L^b_{k-1}$.

We let $I_k := \{\overline{a} \mapsto \overline{b}\}$ and $I_j := \bigcup_{a \in V(\mathfrak{A})} J^a_j \cup \bigcup_{b \in V(\mathfrak{B})} L^b_j$ for $0 \leq j < k$. Then

- for $q \in I_k$ and $a \in V(\mathfrak{A})$ there is $p \in J^a_{k-1} \subseteq I_{k-1}$ such that p extends q and $a \in \mathrm{dom}(p)$ and
- for $q \in I_k$ and $b \in V(\mathfrak{B})$ there is $p \in L^b_{k-1} \subseteq I_{k-1}$ such that p extends q and $b \in \mathrm{ran}(p)$.

For $j < k$ the distance preserving back-and-forth property of I_j is inherited from that of the $(J^a_j)_{0\leq j\leq k-1}$ and $(L^b_j)_{0\leq j\leq k-1}$. As $\overline{a}$ and $\overline{b}$ satisfy the same distance formulas up to $dst(k)$, it remains to show that $\overline{a} \mapsto \overline{b}$ is a partial isomorphism (it then is a partial $dst(k)$-isomorphism). But if $\overline{a}a \mapsto \overline{b}b$ for any $a \in V(\mathfrak{A}), b \in V(\mathfrak{B})$ is a partial $dst(k-1)$-isomorphism (as witnessed by $p \in I_{k-1}$), then also $\overline{a} \mapsto \overline{b}$ is a partial isomorphism.

We show that 3) implies 1) again by induction on k. The case $k = 0$ is handled exactly as above. Let $k > 0$ and suppose that there is a distance preserving back-and-forth system $(I_j)_{1\leq j\leq k}$ for $\mathfrak{A}$ and $\mathfrak{B}$ with $\overline{a} \mapsto \overline{b} \in I_k$. Clearly, the set of formulas $\varphi(x_1,\dots,x_m)$ that satisfy

$$\mathfrak{A} \models \varphi(a_1,\dots,a_m) \iff \mathfrak{B} \models \varphi(b_1,\dots,b_m)$$

is closed under the Boolean connectives. Hence it suffices to consider formulas of the form $\varphi(\overline{x}) = \exists y\psi(\overline{x},y)$ with $\mathrm{drk}(\psi) \leq k-1$. Assume, for instance, $\mathfrak{A} \models \varphi(\overline{a})$. Then there is $a \in V(\mathfrak{A})$ such that $\mathfrak{A} \models \psi(\overline{a},a)$. By assumption, there is $b \in V(\mathfrak{B})$ such that $\overline{a}a \mapsto \overline{b}b \in I_{k-1}$. Hence $(I_j)_{1\leq j\leq k-1}$ is a distance preserving back-and-forth system with $\overline{a}a \mapsto \overline{b}b \in I_{k-1}$. By induction hypothesis, $\mathrm{dtp}_{k-1}(\mathfrak{A},\overline{a},a) = \mathrm{dtp}_{k-1}(\mathfrak{B},\overline{b},b)$ and hence $\mathfrak{B} \models \psi(\overline{b},b)$. Then $\mathfrak{B} \models \varphi(\overline{b})$. □

7.2. Rank-preserving locality

In this section we are going to prove the rank-preserving locality theorem.

The following observation allows us to avoid introducing many quantifiers in the neighbourhood of tuples. Let $\mathfrak{A}$ be a τ-structure and let $\overline{a} \in V(\mathfrak{A})^m$ be $2r$-scattered. Then there exists an r-scattered set of elements which satisfy $\psi(x)$ of size s in the r-neighbourhood of $\overline{a}$ if and only if there is a sub-tuple of $\overline{a}$ of size s such that the r-neighbourhood of each element of the sub-tuple contains (exactly) one element which satisfies ψ. We can express this as a disjunction over the sub-tuples of $\overline{a}$ without investing additional quantifiers for the scattered set formula.

Furthermore, observe that any model-checking algorithm which directly implements Lemma 2.5.1 has to solve scattered set problems in the $(2r+1)$-neighbourhoods of many tuples $\overline{a}$. We want to avoid this and rather deal with many elements at once. For this purpose, we incorporate neighbourhood covers, as introduced in Chapter 5, into the logic FO(dist). Note that

this is not necessary to obtain a rank-preserving locality theorem and we will present a version of the theorem which does not refer to neighbourhood covers.

Let $\mathfrak{A}$ be a τ-structure. For an integer $r \geq 1$, an *r-neighbourhood cover $\mathscr{X}$* of is an r-neighbourhood cover of the Gaifman graph of $\mathfrak{A}$. For every $a \in V(\mathfrak{A})$, we fix some cluster $\mathscr{X}(a) \in \mathscr{X}$ such that $N_r(a) \subseteq \mathscr{X}(a)$. We write $\mathscr{X}(\overline{a})$ for $\bigcup_{a \in \overline{a}} \mathscr{X}(a)$. We obtain the following analogue of Lemma 2.5.1.

Lemma 7.2.1 *Let $\mathfrak{A}$ be a τ-structure, let $m, r \geq 1$ be integers, let $\overline{a} \in V(\mathfrak{A})^m$, let $\psi(x)$ be a first-order formula and let $\mathscr{X}$ be a $(4m+2) \cdot r$-neighbourhood cover of $\mathfrak{A}$. There is an element a at distance greater than r from $\overline{a}$ which satisfies ψ if and only if one of the following holds:*

- *there exists an element $a \in \mathscr{X}(\overline{a})$ which satisfies ψ with $r < \operatorname{dist}(\overline{a}, a)$, or*
- *there is an r-scattered set of size $m+1$ of elements that satisfy ψ, or*
- *for all $d \geq 2r, D \geq d + 4r$ with $D - r \leq (4m+1) \cdot r$ and $0 \leq \ell \leq m$, if*
  - *there are exactly ℓ elements of mutual distance larger than d that satisfy ψ and*
  - *there are exactly ℓ elements of mutual distance larger than D that satisfy ψ,*

  then no set of ℓ elements $\{a_1, \ldots, a_\ell\} \subseteq \overline{a}$, such that each a_i contains an element which satisfies ψ in its r-neighbourhood, is $(D/2 - r)$-scattered.

PROOF. First assume that there is an element a at distance greater than r from $\overline{a}$ which satisfies ψ. If a does not lie in $\mathscr{X}(\overline{a})$ and if there is no r-scattered set of size $m+1$ of elements that satisfy ψ, fix any d and D with $d \geq 2r, d + 4r \leq D \leq (4m+2) \cdot r$ and let ℓ be maximal such that there are both exactly ℓ elements of mutual distance larger than d and exactly ℓ elements of mutual distance larger than D that satisfy ψ. Note that in this case we have $1 \leq \ell \leq m$ (otherwise we are in case two). Let p be maximal such that there are $a_1, \ldots, a_p \in \overline{a}$ with mutual distance greater than $D - 2r$ and such that the r-neighbourhood of each a_i contains an element b_i satisfying ψ. Then $\operatorname{dist}(b_i, b_j) > D - 4r \geq d$ for $1 \leq i < j \leq p$. As $a \notin \mathscr{X}(\overline{a})$, $\operatorname{dist}(a, b_i) > D - r > d$ for all $1 \leq i \leq p$. This implies $p < \ell$, as otherwise the set $\{a, b_1, \ldots, b_p\}$ contains more than ℓ elements of mutual distance greater than d, all satisfying ψ.

Conversely, assume that there is no element a which satisfy ψ and lies outside the r-neighbourhood of $\overline{a}$. Clearly, the first two items do not hold. If there is no element a at all which satisfies ψ, then also the third item does not hold (the empty set is r-scattered). Assume that there is an element a which satisfies ψ but all such elements lie inside the r-neighbourhood of $\overline{a}$. Let us show that there are integers d, D, ℓ such that $d \geq 2r$, $D \geq d + 4r$, $D \leq (4m+2) \cdot r$ and $\ell \leq m$ such that there are exactly ℓ elements of mutual distance larger than d and exactly ℓ elements of mutual distance larger than D satisfying ψ, and there are ℓ elements $a_1, \ldots, a_\ell \in \overline{a}$ of mutual distance at least $D - 2r$ such that each element a_i contains an element satisfying ψ in its r-neighbourhood, $1 \leq i \leq \ell$.

Let $d_0 := 2r$ and let ℓ_0 be maximal such that there are $a_0^1, \ldots, a_0^{\ell_0}$ which satisfy ψ with $\mathrm{dist}(a_0^i, a_0^j) > d_0$ for all $i \neq j \in \{1, \ldots, \ell_0\}$. As $d_0 > 2r$, no two of the a_0^j can belong to the same r-neighbourhood of an element of $\overline{a}$. Hence $\ell_0 \leq m$.

If d_h, ℓ_h are defined for some $h \geq 0$, let $d_{h+1} := d_h + 4r$, and let ℓ_{h+1} be maximal such that there are $a_{h+1}^1, \ldots, a_{h+1}^{\ell_{h+1}}$ which all satisfy ψ with $\mathrm{dist}(a_{h+1}^i, a_{h+1}^j) > d_{h+1}$ for all $i \neq j \in \{1, \ldots, \ell_{h+1}\}$. Then $\ell_{h+1} \leq \ell_h$. If we have $\ell_{h+1} = \ell_h$ for the first time, we stop the construction. Then $h \leq m$ and thus $d_{h+1} = (4h+2)r \leq (4m+2)r$. Let $d := d_h$, $D := d_{h+1}$ and $\ell := \ell_h = \ell_{h+1}$.

By assumption, for each $i \in \{1, \ldots, \ell\}$ there is $j(i) \in \{1, \ldots, m\}$ such that the $j(i)$ are mutually distinct and such that $\mathrm{dist}(a_h^i, a_{j(i)}) \leq r$. As $\mathrm{dist}(a_h^i, a_h^j) > D$, we have $\mathrm{dist}(a_{j(i)}, a_{j(j)}) > D - 2r$ for $i \neq j$, as desired. □

We define yet another extension of first-order logic, $\mathrm{FO}(\mathrm{dist} + \mathscr{X})$, in order to make neighbourhood covers accessible to the logic and directly implement Lemma 7.2.1. For variables x, y, we define as a new atom the formula $y \in \mathscr{X}(x)$ with the obvious semantics. We write $(\mathfrak{A}, \mathscr{X}) \models \varphi$ for an $\mathrm{FO}(\mathrm{dist} + \mathscr{X})$ formula φ to indicate that φ refers to the neighbourhood cover $\mathscr{X}$. Let us prove the analogue of Theorem 2.3.1 for $\mathrm{FO}(\mathrm{dist} + \mathscr{X})$ formulas.

Theorem 7.2.2 *Let Φ be a finite set of $\mathrm{FO}(\mathrm{dist} + \mathscr{X})$ formulas and let $m, r \geq 0$ be integers. If for all τ-structures $\mathfrak{A}, \mathfrak{B}$, $\overline{a} \in V(\mathfrak{A})^m$, $\overline{b} \in V(\mathfrak{B})^m$ and all r-neighbourhood covers $\mathscr{X}$ of $\mathfrak{A}$ and $\mathscr{Y}$ of $\mathfrak{B}$,*

$$((\mathfrak{A}, \mathscr{X}, \overline{a}) \models \chi \Leftrightarrow (\mathfrak{B}, \mathscr{Y}, \overline{b}) \models \chi \text{ for all } \chi \in \Phi) \text{ implies } \mathrm{dtp}_k(\mathfrak{A}, \overline{a}) = \mathrm{dtp}_k(\mathfrak{B}, \overline{b}),$$

then every FO(dist) *formula* $\varphi(x_1,\dots,x_m)$ *of distance-rank at most* k *can effectively be translated to a Boolean combination of formulas from* Φ *which is equivalent to* φ.

PROOF. The proof which shows the existence of the equivalent Boolean combination is analogous to that of Theorem 2.3.1. Let us show that the transformation is effective.

We make the r-neighbourhood covers part of the structures and show that we obtain a first-order definable class of structures.

Let R be a fresh binary relation symbol and $\tau_R := \tau \cup \{R\}$. For every τ-structure $\mathfrak{A}$ and every r-neighbourhood cover $\mathscr{X}$ of $\mathfrak{A}$, we let $\mathfrak{A}_{\mathscr{X}}$ be the τ_R-expansion of $\mathfrak{A}$ with

$$R(\mathfrak{A}_{\mathscr{X}}) = \{(a,b) : b \in \mathscr{X}(a)\}.$$

We define the distance function on a τ_R-structure with respect to the Gaifman graph of $\mathfrak{A}$ (we do not let the new relation mess with distances). Clearly, the class $\{\mathfrak{A}_{\mathscr{X}} : \mathfrak{A}\ \tau\text{-structure}, \mathscr{X}\ r\text{-neighbourhood cover of } \mathfrak{A}\}$ is first-order definable. We let

$$\gamma := \forall x \forall y (\mathrm{dist}(x,y) \le r \longrightarrow R(x,y)).$$

Then $\mathscr{X}$ is an r-neighbourhood cover of $\mathfrak{A}$ if, and only if, $\mathfrak{A}_{\mathscr{X}} \models \gamma$. We already noted that FO(dist) is only a syntactic extension of FO. Hence, as r is fixed, any finite set Φ of FO(dist$+\mathscr{X}$) formulas corresponds to a finite set Φ' of FO formulas about a first-order definable class of τ_R-structures. We conclude with Theorem 2.3.1. □

For $n \ge 0$, a τ-structure $\mathfrak{A}$ and $\overline{a}_n \in V(\mathfrak{A})^n$ let $\overline{x}_n := (x_1,\dots,x_n)$. Recall that $\Psi^0_{\mathfrak{A},\overline{a}_n}$ is the set of formulas $\varphi(\overline{x}_n) \in \mathrm{FO(dist)}$ with $\mathrm{drk}(\varphi) = 0$ such that $\mathfrak{A} \models \varphi(\overline{a}_n)$. We let $\Lambda^0_{\mathfrak{A},\overline{a}} := \Psi^0_{\mathfrak{A},\overline{a}}$. Let

$$\lambda^0_{\mathfrak{A},\mathscr{X},\overline{a}}(\overline{x}_m) := \bigwedge \Lambda^0_{\mathfrak{A},\overline{a}}.$$

For $i > 0$, assume that $\Lambda^{i-1}_{n+1} := \{\lambda^{i-1}_{\mathfrak{A},\mathscr{X},\overline{a}_{n+1}}(\overline{x}_{n+1}) : \mathfrak{A}\ \tau\text{-structure}, \mathscr{X}\ \text{neighbourhood cover of } \mathfrak{A}, \overline{a}_{n+1} \in V(\mathfrak{A})^{n+1}\}$ has been defined and is finite. Recall that

$$\vartheta^i_{\mathfrak{A},\overline{a}}(\overline{x}) := \bigwedge_{\substack{a_i,a_j \in \overline{a} \\ \mathrm{dist}(a_i,a_j) = d \le \mathrm{dist}(i)}} \mathrm{dist}(x_i,x_j) = d \quad \wedge \bigwedge_{\substack{a_i,a_j \in \overline{a} \\ \mathrm{dist}(a_i,a_j) > \mathrm{dist}(i)}} \mathrm{dist}(x_i,x_j) > \mathrm{dist}(i)$$

describes distances relative to the dist-function.

Let $a_{n+1} \in V(\mathfrak{A})$ and $\overline{a}_{n+1} := \overline{a}_n a_{n+1}$. We define the dist($i$)-*connectivity graph* $H_{i-1}(\overline{a}_{n+1})$ of $\overline{a}_{n+1}$ as the graph with vertex set $\{1,\dots,n+1\}$ and an edge uv if and only if $\mathrm{dist}(a_u,a_v) \leq \mathrm{dist}(i-1)$. We write $J(j)$, for the connected component of $H_{i-1}(\overline{a}_{n+1})$ which contains vertex j and we let $\overline{x}_{J(j)}$ denote the sub-tuple of $\overline{x}_{n+1}$ with entries x_u for $u \in V(J(j))$, and $\overline{a}_{J(j)}$ denote the corresponding sub-tuples of $\overline{a}_{n+1}$. Then

$$\begin{aligned}
\lambda^i_{\mathfrak{A},\mathscr{X},\overline{a}_n}(\overline{x}_n) := \vartheta^i_{\mathfrak{A},\overline{a}_n}(\overline{x}_n) \wedge \\
\bigwedge_{1\leq j\leq n} \Bigg(\bigwedge_{\substack{a_{n+1}\in\mathscr{X}(a_j)\\ \mathrm{dist}(\overline{a}_n,a_{n+1})\leq \mathrm{dist}(i-1)}} \exists x_{n+1}\Big(x_{n+1} \in \mathscr{X}(x_j) \\
\wedge \mathrm{dist}(\overline{x}_n,x_{n+1}) \leq \mathrm{dist}(i-1) \wedge \lambda^{i-1}_{\mathfrak{A},\mathscr{X},\overline{a}_{J(j)}}(\overline{x}_{J(j)}) \\
\bigwedge_{\substack{a_{n+1}\in\mathscr{X}(a_j)\\ \mathrm{dist}(\overline{a}_n,a_{n+1})> \mathrm{dist}(i-1)}} \exists x_{n+1}\Big(x_{n+1} \in \mathscr{X}(x_j) \\
\wedge \mathrm{dist}(\overline{x}_n,x_{n+1}) > \mathrm{dist}(i-1) \wedge \lambda^{i-1}_{\mathfrak{A},\mathscr{X},a_{n+1}}(x_{n+1})\Big)\Big) \\
\wedge \forall x_{n+1}\Big(\bigvee_{1\leq j\leq n} x_{n+1} \in \mathscr{X}(x_j) \wedge \\
\bigvee_{\substack{a_{n+1}\in\mathscr{X}(a_j)\\ \mathrm{dist}(\overline{a}_n,a_{n+1})\leq \mathrm{dist}(i-1)}} \Big((\mathrm{dist}(\overline{x}_n,x_{n+1}) \leq \mathrm{dist}(i-1) \wedge \lambda^{i-1}_{\mathfrak{A},\mathscr{X},\overline{a}_{J(j)}}(\overline{x}_{J(j)})\Big) \vee \\
\bigvee_{\substack{a_{n+1}\in\mathscr{X}(a_j)\\ \mathrm{dist}(\overline{a}_n,a_{n+1})> \mathrm{dist}(i-1)}} (\mathrm{dist}(\overline{x}_n,x_{n+1}) > \mathrm{dist}(i-1) \wedge \lambda^{i-1}_{\mathfrak{A},\mathscr{X},a_{n+1}}(x_{n+1}))\Big),
\end{aligned}$$

where we remove duplicate entries from the conjunctions and disjunctions is a well defined formula. It is easy to see that Λ^k_n is finite for all $n,k \in \mathbb{N}$. We call $\lambda^k_{\mathfrak{A},\mathscr{X},\overline{a}}$ the dist(k)-*cluster type* of $\overline{a} \in V(\mathfrak{A})^m$ in $\mathfrak{A}$ with respect to a neighbourhood cover $\mathscr{X}$.

Note that in the sub-formula $\mathrm{dist}(\overline{a}_n,a_{n+1}) \leq \mathrm{dist}(i-1) \wedge \lambda^{i-1}_{\mathfrak{A},\mathscr{X},\overline{a}_{J(j)}}$, the cluster type fixes distances of a_{n+1} to every element of $\overline{a}_n$ up to distance $\mathrm{dist}(i-1)$ exactly (in the sub-formula $\lambda^{i-1}_{\mathfrak{A},\mathscr{X},\overline{a}_{J(j)}}(\overline{x}_{J(j)})$ the first conjunct is the sub-formula $\vartheta^{i-1}_{\mathfrak{A},\overline{a}_{J(j)}}(\overline{x}_{J(j)})$).

An (s,k)-*independence sentence* is a sentence of the form

$$\exists x_1 \dots \exists x_{s'} \Big(\bigwedge_{1 \le i < j \le s'} \operatorname{dist}(x_i, x_j) > 2 \cdot \ell \wedge \bigwedge_{1 \le i \le s'} \lambda^k_{\mathfrak{A},\mathscr{X},a}(x_i) \Big)$$

for any $s' \le s, \ell \le \operatorname{dist}(k)$ and a $\operatorname{dist}(k)$-cluster type $\lambda^k_{\mathfrak{A},\mathscr{X},a}(x_i)$.

We now fix the distance function $\operatorname{dist}_q(i) = (4q)^{q+i}$ which was introduced in the example of the last section as the *q-rank* of a formula. For readability, we drop the index q and always when we are dealing with a formula $\varphi(x_1,\dots,x_m)$ of quantifier rank k, we let $q := k+m$. Note that if $\varphi(x_1,\dots,x_m) = \exists x_{m+1}\psi(x_1,\dots,x_{m+1})$ is of q-rank k, then ψ is of q-rank $k-1$, i.e. the q-rank is compatible with sub-formulas.

Theorem 7.2.3 *Let $k,m,r \ge 0$ be non-negative integers and let $q = k+m$. Let $\varphi(x_1,\dots,x_m) \in \mathrm{FO}(\mathrm{dist})$ be of q-rank k. Then φ is equivalent to an $\mathrm{FO}(\mathrm{dist}+\mathscr{X})$-formula $\widehat{\varphi}(x_1,\dots,x_m)$, which is a Boolean combination of (q,k)-independence sentences and $\operatorname{dist}(k)$-cluster types $\lambda^k_{\mathfrak{A},\mathscr{X},\overline{a}}(\overline{x})$, such that for every τ-structure $\mathfrak{A}$, every $\operatorname{dist}(q)$-neighbourhood cover $\mathscr{X}$ of $\mathfrak{A}$ and every $\overline{a} \in V(\mathfrak{A})^m$,*

$$\mathfrak{A} \models \varphi(\overline{a}) \iff (\mathfrak{A},\mathscr{X}) \models \widehat{\varphi}(\overline{a}).$$

Furthermore, $\widehat{\varphi}$ is computable from φ.

As there are only finitely many non-equivalent (q,k)-independence sentences and finitely many non-equivalent $\operatorname{dist}(k)$-cluster types and as the radius of the neighbourhood covers is fixed, by Theorem 7.2.2, it suffices to prove the following lemma.

Lemma 7.2.4 *Let $k,m \ge 0$ be integers and let $q : k+m$. Let $\mathfrak{A},\mathfrak{B}$ be τ-structures and $\mathscr{X}$, $\mathscr{Y}$ $\operatorname{dist}(q)$-neighbourhood covers of $\mathfrak{A},\mathfrak{B}$, respectively. Let $\overline{a} \in V(\mathfrak{A})^m, \overline{b} \in V(\mathfrak{B})^m$ such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same (q,k)-independence sentences and such that $\overline{a}$ and $\overline{b}$ have the same $\operatorname{dist}(k)$-cluster type in $\mathfrak{A}$ with respect to $\mathscr{X}$ and $\mathfrak{B}$ with respect to $\mathscr{Y}$, respectively. Then*

$$\operatorname{dtp}_k(\mathfrak{A},\overline{a}) = \operatorname{dtp}_k(\mathfrak{B},\overline{b}).$$

PROOF. Throughout the proof, for $0 \le j \le k$ we let $\overline{a}_j := (a_1,\dots,a_{m+k-j})$, $\overline{b}_j := (b_1,\dots,b_{m+k-j})$ and $\overline{x}_j := (x_1,\dots,x_{m+k-j})$. We define the $\operatorname{dist}(j)$-*connectivity graph* $H_j(\overline{a}_j)$ of $\overline{a}_j$ as the graph with vertex set $\{1,\dots,m+k-j\}$ and an edge uv if and only if $\operatorname{dist}(a_u,a_v) \le \operatorname{dist}(j)$. We write $J \sqsubseteq H_j$ to denote that J

is a connected component of H_j and we let $\overline{x}_J$ denote the sub-tuple of $\overline{x}_j$ with entries x_u for $u \in V(J)$, and $\overline{a}_J, \overline{b}_J$ denote the corresponding sub-tuples of $\overline{a}_j, \overline{b}_j$. Throughout the proof, we write $\mathfrak{A}_J$ and $\mathfrak{B}_J$, respectively, for the structures $\mathfrak{A}[\mathscr{X}(\overline{a}_J)]$ and $\mathfrak{B}[\mathscr{X}(\overline{b}_J)]$ for $J \sqsubseteq H_j$.

For $0 \leq j \leq k$, let I_j be the set of partial dist(j)-isomorphisms $\overline{a}_j \mapsto \overline{b}_j$ such that $H_j(\overline{a}_j) = H_j(\overline{b}_j) =: H_j$ and for every component $J \sqsubseteq H_j$ we have

$$\lambda^j_{\mathfrak{A},\mathscr{X},\overline{a}_J}(\overline{a}_J) = \lambda^j_{\mathfrak{B},\mathscr{Y},\overline{b}_J}(\overline{b}_J).$$

We show that $(I_j)_{0 \leq j \leq k}$ is a distance preserving back-and-forth system.

Note that the statement of the lemma is trivially true if $k = 0$. We hence assume $k > 0$. The proof is by (backward) induction on $j \leq k$.

For the base step $j = k$, first observe that $H_j(\overline{a}_j) = H_j(\overline{b}_j)$, as $\overline{a}_k = \overline{a}$ and $\overline{b}_k = \overline{b}$ and $\overline{a}$ and $\overline{b}$ have the same dist(k)-cluster type, and in particular, satisfy the same atomic dist(k)-formulas. Hence let $H_j := H_j(\overline{a}_j)$ and let $J \sqsubseteq H_j$. Then $\lambda^j_{\mathfrak{A},\mathscr{X},\overline{a}_J}(\overline{a}_J) = \lambda^j_{\mathfrak{B},\mathscr{Y},\overline{b}_J}(\overline{b}_J)$ because $\overline{a}_j$ and $\overline{b}_j$ have the same cluster type in $\mathfrak{A}$ and $\mathfrak{B}$ with respect to $\mathscr{X}$ and $\mathscr{Y}$, respectively.

Now let $0 \leq j < k$ and let $\overline{a}_{j+1} \mapsto \overline{b}_{j+1} \in I_{j+1}$. By symmetry, it suffices to consider the forth-property. Let $a \in V(\mathfrak{A})$. Note that the cluster type of $\overline{a}$ for any tuple also determines the cluster type of $\overline{a}'$ for any sub-tuple of $\overline{a}$.

If $\mathrm{dist}(\overline{a}_{j+1}, a) \leq \mathrm{dist}(j)$, say $\mathrm{dist}(a_i, a) \leq \mathrm{dist}(j)$, let $J \sqsubseteq H_{j+1}$ be the connected component of i. Then $a \in V(\mathfrak{A}_J)$ and by definition of $\lambda^{j+1}_{\mathfrak{A},\mathscr{X},\overline{a}_J}$,

$$\mathfrak{A}_J \models \exists x (x \in \mathscr{X}(x_i) \wedge \mathrm{dist}(x_i, x) \leq \mathrm{dist}(j) \wedge \lambda^j_{\mathfrak{A},\mathscr{X},\overline{a}_J a}(\overline{x}_J x)).$$

As by assumption, $\lambda^{j+1}_{\mathfrak{A},\mathscr{X},\overline{a}_J}(\overline{a}_J) = \lambda^{j+1}_{\mathfrak{B},\mathscr{Y},\overline{b}_J}(\overline{b}_J)$, also

$$\mathfrak{B}_J \models \exists x (x \in \mathscr{X}(x_i) \wedge \mathrm{dist}(x_i, x) \leq \mathrm{dist}(j) \wedge \lambda^j_{\mathfrak{A},\mathscr{X},\overline{a}_J a}(\overline{x}_J x)).$$

Let $b \in \mathscr{X}(b_i)$ be a witness for this. Let $\overline{a}_j := \overline{a}_{j+1} a$ and $\overline{b}_j := \overline{b}_{j+1} b$. Let $J' \sqsubseteq H_j(\overline{a}_j)$. Then there is $I \sqsubseteq H_{j+1}$ such that $V(J') \cap \{1, \ldots, m+k-j\} \subseteq V(I)$. To see this, note that if $u(m+k-j) \in E(H_j(\overline{a}_j))$ and $(m+k-j)v \in E(H_j(\overline{a}_j))$ then $uv \in E(H_{j+1})$, because $\mathrm{dist}(j+1) > 2 \cdot \mathrm{dist}(j)$. Thus, whenever there is a path between two vertices $u, v \in \{1, \ldots, m+k+j\}$ in $H_j(\overline{a}_j)$ there also is a path in H_{j+1}. The same holds for $H_j(\overline{b}_j)$ and in particular, $H_j(\overline{a}_j) = H_j(\overline{b}_j)$. If J' is a subcomponent of an old component, the condition on cluster types

is inherited from the old component. If J' is a new component which contains a, then the condition holds as shown above. As distances up to $\mathrm{dist}(j)$ are preserved, $\overline{a}_j \mapsto \overline{b}_j$ is a partial $\mathrm{dist}(j)$-isomorphism in I_j.

If $\mathrm{dist}(\overline{a}_{j+1}, a) > \mathrm{dist}(j)$, let $r := \mathrm{dist}(j)$. By Lemma 7.2.1,

- there is an element $c \in \mathscr{X}(\overline{a}_{j+1})$ with $r < \mathrm{dist}(\overline{a}_{j+1}, c)$ which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$, or
- there is a r-scattered sets of elements that satisfy $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$ of size $m+k-j$ $(= m+k-(j+1)-1)$, or
- for all $d \geq 2r, D \geq d+4r$ with $D - r \leq (4(m+k-(j+1))+1)r$ and $\ell \leq m+k-(j+1)$, if
  - there are exactly ℓ elements of mutual distance larger than d that satisfy $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$ and
  - there are exactly ℓ elements of mutual distance larger than D that satisfy $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$,

  then no set of ℓ elements $\{a_1, \ldots, a_\ell\} \subseteq \overline{a}$ such that each a_i contains an element which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$ in its r-neighbourhood is is $(D/2-r)$-scattered.

Note that $D \leq 4qr$, and hence the q-rank is defined exactly to match our needs.

If $a \in \mathscr{X}(a_i)$ for some i let $c := a$, and if $a \notin \mathscr{X}(\overline{a}_{j+1})$ but there is $c \in \mathscr{X}(a_i)$ for some i with $\mathrm{dist}(\overline{a}_{j+1}, c) > r$ which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$, fix this c. Let $J \sqsubseteq H_{j+1}$ be the connected component of i. Then $c \in V(\mathfrak{A}_J)$ and by definition of $\lambda^{j+1}_{\mathfrak{A},\mathscr{X},\overline{a}_J}$,

$$\mathfrak{A}_J \models \exists x(x \in \mathscr{X}(x_J) \wedge \mathrm{dist}(x_J, x) > \mathrm{dist}(j) \wedge \lambda^j_{\mathfrak{A},\mathscr{X},a}(x)).$$

As by assumption, $\lambda^{j+1}_{\mathfrak{A},\mathscr{X},\overline{a}_J}(\overline{a}_J) = \lambda^{j+1}_{\mathfrak{B},\mathscr{Y},\overline{b}_J}(\overline{b}_J)$, also

$$\mathfrak{B}_J \models \exists x(x \in \mathscr{X}(x_J) \wedge \mathrm{dist}(x_J, x) > \mathrm{dist}(j) \wedge \lambda^j_{\mathfrak{A},\mathscr{X},a}(x)).$$

Let $b \in \mathscr{X}(b_J)$ be a witness for this. Let $\overline{a}_j := \overline{a}_{j+1}a$ and $\overline{b}_j := \overline{b}_{j+1}b$.

If there is no element $c \in \mathscr{X}(\overline{a}_{j+1})$ with $\mathrm{dist}(\overline{a}_{j+1}, c) > r$ which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$, then the same holds for $\mathfrak{B}$, as $\overline{a}_J$ and $\overline{b}_J$ have the same $\mathrm{dist}(j)$-cluster type for each component $J \sqsubseteq H_{j+1}$. Then the second or third condition of the lemma holds in $\mathfrak{A}$.

If the second condition holds in $\mathfrak{A}$ then, as $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same (q,k)-independence sentences, the second condition also holds in $\mathfrak{B}$.

If the third condition holds in $\mathfrak{A}$ then if for all $d \geq 2r, D \geq d + 4r$ with $D - r \leq (4(m + k - (j+1)) + 1)r$ and $\ell \leq m + k - (j+1)$ and

- there are exactly ℓ elements of mutual distance larger than d that satisfy $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$ and
- there are exactly ℓ elements of mutual distance larger than D that satisfy $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$,

the same holds in $\mathfrak{B}$, as $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same (q,k)-independence sentences. We consider each component $J \sqsubseteq H_{j+1}$ and the corresponding cluster types $\lambda^{j+1}_{\mathfrak{A},\mathscr{X},\overline{a}_J}$ and $\lambda^{j+1}_{\mathfrak{B},\mathscr{X},\overline{b}_J}$ which are equal by assumption. In $\mathfrak{A}$, no set of ℓ elements $\{a_1, \ldots, a_\ell\} \subseteq \overline{a}_{j+1}$ such that each a_i contains an element which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(x)$ in its r-neighbourhood is is $(D/2 - r)$-scattered. By definition of the connectivity graph, vertices from different components are at distance at least $\mathrm{dist}(j+1)$ and hence we can combine the scattered subsets of each component to one big scattered set. Hence, also in $\mathfrak{B}$ no ℓ elements with the above property exist.

Using the other direction of Lemma 7.2.1, there exists b which satisfies $\lambda^j_{\mathfrak{A},\mathscr{X},a}(b)$ with $\mathrm{dist}(\overline{b}, b) > r$. Let $\overline{a}_j := \overline{a}_{j+1}a$ and $\overline{b}_j := \overline{b}_{j+1}b$.

In any case, if $J' \sqsubseteq H_j(\overline{a}_j)$, then either $V(J') = \{m + k - j\}$, or there is $J \sqsubseteq H_{j+1}$ such that $V(J') \subseteq V(J)$. Again, it is easy to see that $H_j(\overline{a}_j) = H_j(\overline{b}_j)$ and the cluster types of each component are equal (either by inheritance from the old component or because the component consists of the single element a or b, respectively, and a and b have the same cluster type as shown above). As the distance of a to all elements of $\overline{a}_{j+1}$ is larger than $\mathrm{dist}(j)$ and the same holds for b and $\overline{b}_{j+1}$, $\overline{a}_j \mapsto \overline{b}_j$ is a partial $\mathrm{dist}(j)$-isomorphism in I_j. □

Our model-checking algorithm will deal with formulas with one free variable only. We will iteratively compute the cluster types of single elements and make them available as colours. Thereby, sub-formulas $y \in \mathscr{X}(x)$ will only

implicitly occur in our formulas. In other words, in the model-checking algorithm, we can restrict ourselves to the logic FO(dist) if we expand the signature appropriately.

Let $\tau \star k$ be the vocabulary obtained from τ by adding a fresh unary relation symbol P_φ for each $\varphi = \varphi(x) \in \Phi(\tau, dst, k, 1)$. For a τ-structure $\mathfrak{A}$, let $\mathfrak{A} \star_{\mathscr{X}} k$ be the $\tau \star k$-expansion of $\mathfrak{A}$ in which P_φ is interpreted by the set of all $a \in V(\mathfrak{A})$ such that $\mathfrak{A}\big[\mathscr{X}(a)\big] \models \varphi(a)$. We let $\tau \star^0 k := \tau$ and $\mathfrak{A} \star^0_{\mathscr{X}} k := \mathfrak{A}$. For $i \geq 0$, we let $\sigma \star^{i+1} k := (\tau \star^i k) \star k$ and $\mathfrak{A} \star^{i+1}_{\mathscr{X}} k := \big(\mathfrak{A} \star^i_{\mathscr{X}} k\big) \star_{\mathscr{X}} k$.

Recall that $\psi^k_{\mathfrak{A},a}(x)$ denotes the distance rank k Hintikka type of a in $\mathfrak{A}$. Also recall, that the rank-k Hintikka types are normalised formulas from $\Phi(\tau, dst, k, 1)$. The following is proved by an easy induction on k.

Lemma 7.2.5 *Let $i \geq 0, r > 0$ be integers. Let $\mathscr{X}$ be an r-neighbourhood cover of $\mathfrak{A}$, let $a \in V(\mathfrak{A})$ and let $\psi^i := \psi^i_{\mathfrak{A}[\mathscr{X}(a)] \star^i_{\mathscr{X}} k, a}(x)$, i.e. the rank i Hintikka type of a in the coloured cluster of a. Then*

$$\mathfrak{A} \models \lambda^i_{\mathfrak{A},\mathscr{X},a}(a) \iff \mathfrak{A} \star^i_{\mathscr{X}} k \models P_{\psi^i}(a).$$

An *atomic (s,k)-independence sentence* is a sentence of the form

$$\exists x_1 \ldots \exists x_{s'} \Big(\bigwedge_{1 \leq i < j \leq s'} \operatorname{dist}(x_i, x_j) > 2 \cdot \ell \wedge \bigwedge_{1 \leq i \leq s'} P_\psi(x_i) \Big)$$

for any $s' \leq s, \ell \leq \operatorname{dist}(k)$ and $\psi \in \Phi(\tau, dst, k, 1)$.

The following theorem is proved along the lines of proof of the previous theorem with the help of Lemma 7.2.5.

Theorem 7.2.6 *Let $k \geq 0$ be an integer and let $q \geq k+1$. For every* FO(dist)*-formula $\varphi(x)$ of q-rank k there is an* FO(dist)*-formula $\widehat{\varphi}(x)$ over signature $\tau \star^k k$, which is a Boolean combination of atomic $(k+1,k)$ independence sentences and atomic formulas, such that for every τ-structure $\mathfrak{A}$, every* $\operatorname{dist}(k+1)$*-neighbourhood cover $\mathscr{X}$ of $\mathfrak{A}$, and every $a \in V(\mathfrak{A})$,*

$$\mathfrak{A} \models \varphi(a) \iff \mathfrak{A} \star^k_{\mathscr{X}} k \models \widehat{\varphi}(a).$$

Furthermore, $\widehat{\varphi}$ is computable from φ.

As already mentioned in the introduction, we do not need neighbourhood covers to obtain a rank-preserving locality theorem. We state, as a corollary,

a version that does not refer to any neighbourhood cover. It is obtained by applying the theorem to the generic r-neighbourhood cover $\mathscr{X} = \{N_r(v) \mid v \in V(G)\}$. We omit the index $\mathscr{X}$ in the $\star$-notation when we refer to this neighbourhood cover.

Corollary 7.2.7 *Let $k \geq 0$ be an integer and let $q \geq k+1$. For every* FO(dist)*-formula $\varphi(x)$ of q-rank k there is an* FO(dist)*-formula $\widehat{\varphi}(x)$ over signature $\tau \star^k k$, which is a Boolean combination of atomic $(k+1,k)$ independence sentences and atomic formulas, such that for every τ-structure $\mathfrak{A}$ and every $a \in V(\mathfrak{A})$,*

$$\mathfrak{A} \models \varphi(a) \iff \mathfrak{A} \star^k k \models \widehat{\varphi}(a).$$

Furthermore, $\widehat{\varphi}$ is computable from φ.

As a final step, we observe that if we restrict ourselves to coloured graphs then the rank of a formula is stable under deletions of elements if we expand the signature.

Lemma 7.2.8 *Let τ be a coloured graph vocabulary, let $\ell, q \geq 0$ be integers with $0 \leq \ell \leq q$ and let $\varphi(x) \in$ FO(dist) of q-rank ℓ. Then there are*

1. *a coloured graph vocabulary $\tau' \supseteq \tau$,*
2. *for every τ-coloured graph G and every $a \in V(G)$ a τ'-expansion G' of $G - a$, and*
3. *for every τ-coloured graph G and every $a \in V(G)$ an* FO(dist)*-formula $\varphi_{G,a}(x)$ and an* FO(dist)*-sentence $\psi_{G,a}$ over signature τ' of q-rank at most ℓ*

such that for all $b \in V(G) \setminus \{a\}$

$$G \models \varphi(b)) \iff G' \models \varphi_{G,a}(b) \quad \text{and} \quad G \models \varphi(a) \iff G' \models \psi_{G,a}.$$

Furthermore, $\varphi_{G,a}$ and $\psi_{G,a}$ are computable from φ, G and a, and G' is computable from G and a in time $f(\ell,q) \cdot (n(G) + m(G))$.

PROOF. For $1 \leq i \leq \operatorname{dist}(\ell)$, let Q_i be a fresh relation symbol and let τ' be the union of τ with all these Q_i. For every τ-coloured graph G and $a \in V(G)$, let G' be the τ'-expansion of $G - a$ with $Q_i^{G'} := \{b \in V(G') : \operatorname{dist}^G(a,b) = i\}$, that is, we delete a from $V(G)$ and colour all elements that were at distance i in G with the colour Q_i.

We show how to obtain the formula $\varphi_{G,a}$, the case $\psi_{G,a}$ is similar. As a first step, we expand the signature τ to $\tau_w := \tau \cup \{w\}$, where w is a constant symbol. We recursively (starting with $\varphi(x)$) transform all sub-formulas $\psi(\overline{x})$ of φ to formulas $\psi^*(\overline{x})$ as follows. If $\psi(\overline{x})$ is a Boolean combination, we translate each formula $\chi(\overline{x})$ of the combination to $\chi^*(\overline{x})$ and form the same combination from the χ^*. If $\psi(x_1,\dots,x_k) = \exists x_{k+1}\chi(x_1,\dots,x_{k+1})$, we let $\psi^*(x_1,\dots,x_k) := \exists x_{k+1}\chi^*(x_1,\dots,x_{k+1}) \vee \chi^*(x_1,\dots,x_k,w)$. We may assume that φ contains no universal quantifiers. Now from $\varphi^*(x)$, we compute $\varphi_{\mathfrak{A},a}$ by replacing each atom $E(u,w)$ by $Q_1(u)$, each atom $\mathrm{dist}(u,w) \leq d$ by $Q_d(u)$ and each atom $\mathrm{dist}(x,y) \leq d$ for variables x,y by $\mathrm{dist}(x,y) \leq d \vee \bigvee_{i+j\leq d}(Q_i(x) \wedge Q_j(y))$. Observe that $\varphi_{\mathfrak{A},a}$ is over signature τ'. An easy induction shows that $\varphi_{\mathfrak{A},a}$ has the claimed properties. □

Note that even though the distance-rank is preserved by the transformation of Theorem 7.2.6, the length of the translated formula may be huge in comparison to the input formula. It was shown in [30] that there is no elementary bound on the length of the local sentences occurring in Gaifman's theorem in terms of the original sentence. The proof in [30] can easily be modified to show that the same holds for our extended locality theorem.

8. First-order model-checking

We are now ready to combine all results and prove one of the main theorem of this part. We fix the q-rank as our distance function *dst*.

Theorem 8.1.1 *For every nowhere dense class $\mathscr{C}$ of τ-structures, every real number $\epsilon > 0$ and every formula $\varphi(x) \in \mathrm{FO}(\mathrm{dist})[\tau]$, there is an algorithm that, given a τ-structure $\mathfrak{A} \in \mathscr{C}$ with n elements, computes the set of all $a \in V(\mathfrak{A})$ such that $\mathfrak{A} \models \varphi(a)$ in time $O(n^{1+\epsilon})$.*

By Lemma 2.4.1 and Corollary 3.4.3, it suffices to prove the following theorem.

Theorem 8.1.2 *Let τ be a coloured graph vocabulary. For every nowhere dense class $\mathscr{C}$ of τ-coloured graphs, every real $\epsilon > 0$, and every formula $\varphi(x) \in \mathrm{FO}(\mathrm{dist})$ of vocabulary τ, there is an algorithm that, given a τ-coloured graph G from $\mathscr{C}$ with n vertices, computes the set of all $v \in V(G)$ such that $G \models \varphi(v)$ in time $O(n^{1+\epsilon})$.*

PROOF. Let $\mathscr{C}$ be a nowhere dense class of τ-coloured graphs and let $\epsilon > 0$. Without loss of generality we may assume that $\epsilon \leq 1/2$, which implies $\epsilon^2 \leq \epsilon/2$, and that $\mathscr{C}$ is closed under taking subgraphs.

Assume that φ is of q-rank k, $(k+1 \leq q)$ and let $r = \mathrm{dist}(k+1)$. By Theorem 7.2.6, we can find an FO(dist)-formula $\widehat{\varphi}(x)$ over vocabulary $\tau \star^k k$, which is a Boolean combination of $(k+1,k)$-independence sentences and atomic formulas, such that for all τ-coloured graphs G, all r-neighbourhood covers $\mathscr{X}$ of G, and all $v \in V(G)$ we have

$$G \models \varphi(v) \iff G \star^k_{\mathscr{X}} k \models \widehat{\varphi}(v).$$

We choose ℓ according to Theorem 6.1.2 such that Splitter has a winning strategy for the simple $(\ell, 2r)$-splitter game on every graph in $\mathscr{C}$. Note that q, r, ℓ and $\widehat{\varphi}$ only depend on φ and the class $\mathscr{C}$, but not on ϵ or the input graph G.

Let $\delta = \epsilon/(2\ell)$. Choose $n_0 = n_0(\delta, r)$ according to Theorem 5.1.8 such that every graph $G \in \mathscr{C}$ of order $n \geq n_0$ has an r-neighbourhood cover of radius at most $2r$ and maximum degree at most n^δ. Choose $n_1 \geq n_0$ such that $n_1^{\delta/2} \geq 2$ and that every graph $G \in \mathscr{C}$ of order $n \geq n_1$ has at most $n^{1+\delta}$ edges. The existence of such n_1 follows from Corollary 3.3.2.

Now fix the τ-coloured input graph G. If $n := n(G) < n_1$, we compute the set of all $v \in V(G)$ such that $G \models \varphi(v)$ by brute force; in this case the running time can be bounded in terms of φ, ϵ, and $\mathscr{C}$. So let us assume that $n \geq n_1$. We compute an r-neighbourhood cover $\mathscr{X}$ of G of radius $2r$ and maximum degree n^δ. The main task of our algorithm will be to compute $G \star_{\mathscr{X}}^{q+1} q$. Before we describe how to do this, let us assume that we have computed $G \star_{\mathscr{X}}^{q+1} q$ and describe how the algorithm proceeds from there. The next step is to evaluate all $(k+1,k)$-independence sentences in the Boolean combination $\widehat{\varphi}(x)$ in $G \star_{\mathscr{X}}^{k} k$. Consider such a sentence

$$\chi = \exists x_1 \dots \exists x_{k'} \Big(\bigwedge_{1 \leq i < j \leq k'} \operatorname{dist}(x_i, x_j) > 2r' \wedge \bigwedge_{1 \leq i \leq k'} P_\psi(x_i) \Big)$$

for some $k' \leq k+1, r' \leq \operatorname{dist}(k+1) = r$. Remember that $\psi(x_i)$ is a formula from $\Phi(\tau, dst, k, 1)$ and $P_\psi(G)$ contains all elements of G which satisfy ψ. We can easily compute the set U of all $v \in V(G)$ such that $G \star_{\mathscr{X}}^{q+1} q \models P_\psi(v)$. We use the algorithm of Theorem 6.2.1 to decide whether U has k' elements of pairwise distance greater than $2r'$. This is the case if and only if $G \star_{\mathscr{X}}^{k} k \models \chi$. It remains to evaluate the atomic formulas in $\widehat{\varphi}(x)$ and combine the results to evaluate the Boolean combination. Both tasks can be solved in linear time.

Let us now turn to computing $G \star_{\mathscr{X}}^{k} k$. We inductively compute $G \star_{\mathscr{X}}^{i} k$ for $0 \leq i \leq k$. The base step $i = 0$ is trivial, because $G \star_{\mathscr{X}}^{0} k = G$. As each $G \star_{\mathscr{X}}^{i} k$ is a τ' coloured graph for some τ' (to be precise, $\tau' = \tau \star^i k$), it suffices to show how to compute $G \star_{\mathscr{X}} k$ from G. To do this, for each formula $\psi(x) \in \Phi(\tau, dst, k, 1)$ we need to compute the set $P_\psi(G \star_{\mathscr{X}} k)$ of all $v \in V(G)$ such that $G[\mathscr{X}(v)] \models \psi(v)$. Let us fix a formula $\psi(x) \in \Phi(\tau, dst, k, 1)$.

For every $X \in \mathscr{X}$, let $v_X \in X$ be a centre of $G[X]$, that is, a vertex with $X \subseteq N_{2r}(v_X)$. Such v_X exists because the radius of $G[X]$ is at most $2r$. Let $w_X \in N_{2r}^G(v_X)$ be Splitter's response if Connector chooses v_X in the first round of the simple $(\ell, 2r)$-splitter game on G. We apply Lemma 7.2.8 to $G[X]$ and ψ, that is, we delete w_X from $G[X]$ and translate ψ appropriately. Let τ' be the vocabulary obtained by Lemma 7.2.8 (1), and let G_X be the graph

obtained from $G[X]$ and w_X by Lemma 7.2.8 (2). (Neither τ' nor G_X depend on the formula.) Let $\psi'(x)$ and ψ'' be the formulas obtained from ψ by Lemma 7.2.8 (3). We recursively evaluate the formulas ψ' and ψ'' in G_X. This gives us the set Ξ_X of all $v \in V(G)$ such that $G[X] \models \psi(v)$. Doing this for all $X \in \mathscr{X}$, we can compute the set

$$\begin{aligned} P_\psi(G \star_{\mathscr{X}} k) &= \{v \in V(G) \mid G[\mathscr{X}(v)] \models \psi(v)\} \\ &= \bigcup_{X \in \mathscr{X}} \big(\Xi_x \cap \{v \in V(G) \mid \mathscr{X}(v) = X\}\big). \end{aligned}$$

Observe that in a recursive call with input G_X, ψ' and G_X, ψ'' the parameters q and r can be left unchanged. Moreover, it follows from the definition of G_X that Splitter has a winning strategy for the $(\ell - 1, 2r)$-splitter game on G_X. Thus we can reduce the parameter ℓ by 1. Once we have reached $\ell = 0$, the graph G_X will be empty and the algorithm terminates. This completes the description of the algorithm.

Let us analyse the running time. The crucial parameters are the order n of the input graph and the level j of the recursion. As argued above, we have $j \leq \ell$. We write the running time as a function T of j and n. We first observe that the time used by the algorithm without the recursive calls can be bounded by $c_1 n^{1+\delta}$ for a suitable constant c_1 depending on the input sentence φ, the parameter ϵ, and the class $\mathscr{C}$, but not on n or j. Furthermore, for $n < n_1$ the running time can be bounded by a constant c_2 that again only depends on φ, ϵ, and $\mathscr{C}$, and for $j = 0$ the running time can be bounded by c_3. Furthermore, for each $X \in \mathscr{X}$ only two recursive calls are made to the graph G_X. Let $n_X = n(V(G_X)) \leq |X|$ and $c = \max\{c_1, c_2, c_3\}$. We obtain the following recurrence for T:

$$\begin{aligned} &T(0,n) \leq c, \\ &T(j,n) \leq c && \text{for all } n < n_1, \\ &T(j,n) \leq \sum_{X \in \mathscr{X}} cT(j-1, n_X) + cn^{1+\delta} && \text{for all } j \geq 1, n \geq n_1 \end{aligned}$$

We claim that for all $n \geq 1$ and $0 \leq j \leq \ell$ we have $T(j,n) \leq c^j n^{1+2j\delta} \leq c^\ell n^{1+\epsilon}$. As c and ℓ are bounded in terms of $\varphi, \epsilon, \mathscr{C}$, this proves the theorem.

$T(j,n) \leq c^j n^{1+2j\delta}$ can be proved by a straightforward induction. The crucial observation is

$$\sum_{X \in \mathscr{X}} n_X = \sum_{v \in V(G)} |\{X \in \mathscr{X} \mid v \in X\}| \leq n n^\delta = n^{1+\delta}. \tag{8.1}$$

The base steps $j = 0$ and $n < n_1$ are trivial. In the inductive step, we have

$$\begin{aligned}
&T(j,n)\\
&\leq \sum_{X\in\mathscr{X}} cT(j-1,n_X) + cn^{1+\delta}\\
&\leq \sum_{X\in\mathscr{X}} cc^{j-1}n_X^{1+2(j-1)\delta} + cn^{1+\delta} && \text{(Induction Hypothesis)}\\
&\leq c^j\Big(\sum_{X\in\mathscr{X}} n_X\Big)^{1+2(j-1)\delta} + cn^{1+\delta}\\
&\leq c^j n^{(1+\delta)(1+2(j-1)\delta)} + cn^{1+\delta} && \text{(by (8.1))}\\
&\leq c^j\big(n^{1+(2j-1)\delta+2(j-1)\delta^2} + n^{1+\delta}\big)\\
&\leq c^j\left(\frac{n^{1+2j\delta}+n^{1+(3/2)\delta}}{n^{\delta/2}}\right) && \text{(as } 2(j-1)\delta^2 \leq \frac{\epsilon^2}{2\ell} \leq \delta/2)\\
&\leq c^j n^{2j\delta} && \text{(because } n^{\delta/2} \geq 2). \qquad \square
\end{aligned}$$

Let us conclude with a remark on the uniformity of this result.

The way Theorem 8.1.2 is stated asserts that deciding first-order properties of nowhere dense graphs is *non-uniformly* fixed-parameter tractable. That is, for every real $\epsilon > 0$ and every sentence φ of first-order logic there is an algorithm deciding the property defined by φ in time $O(n^{1+\epsilon})$. This allows for the algorithms for different sentences to be unrelated. As our proof shows, there is an algorithm which handles all inputs in a uniform way, however, we cannot provide a uniform time bound for all inputs.

We call a class $\mathscr{C}$ *effectively nowhere dense* if there is a computable function f such that $K_{f(r)} \not\preceq_r G$ for all $G \in \mathscr{C}$. All natural nowhere dense classes are effectively nowhere dense, but it is possible to construct artificial classes that are nowhere dense, but not effectively so. For effectively nowhere dense classes $\mathscr{C}$, we obtain uniform fixed-parameter tractability, that is, a single algorithm that, given a graph $G \in \mathscr{C}$, $\epsilon > 0$ and a sentence φ of first-order logic, decides whether φ holds in G in time $f(|\varphi|,\epsilon)\cdot n^{1+\epsilon}$, for some computable function f.

9. Order-invariant model-checking

In this last part of the thesis we study the complexity of first-order model-checking on structures where an ordering is available to be used in formulas. The methods based on locality theorems for first-order logic such as Gaifman's locality theorem do not readily extend to ordered structures. We do so in two different settings. The first is that the input structures are equipped with a fixed order or successor relation. We show that first-order logic on ordered structures as well as on structures with a successor relation is essentially intractable on nearly all interesting classes.

The other case we consider is order- or successor-invariant first-order or monadic second-order logic. In order-invariant first-order logic, we are allowed to use an order relation in the formulas but whether the formula is true in a given structure must not depend on the particular choice of order. Order-invariant logics have been studied in database- and finite model-theory in the past.

It is easily seen that the expressive power of order-invariant MSO is greater than that of plain MSO, as with an order we can formalise in MSO that a structure has an even number of elements, a property not definable without an order. In fact, the expressive power of order-invariant MSO is even greater than the expressive power of the extension of MSO with counting quantifiers CMSO [57]. Over restricted classes of structures, order-invariant MSO and CMSO have the same expressive power (see e.g. [23]). This holds true for successor-invariant MSO as well, as an order is definable from a successor relation via MSO. An unpublished result of Gurevich (which then appeared as an exercise in [1] states that the expressive power of order-invariant FO is stronger than that of plain FO. For a proof see e.g. [89, 124] and for more examples showing that the expressive power of order-invariant FO is stronger than that of plain FO see e.g. [110, 117]. It is known that order-invariant FO collapses to FO on trees [12, 108], and that order-invariant FO is a subset of MSO on graphs of bounded degree and

on graphs of bounded tree-width [12]. It is also known that order-invariant FO can express only local queries [66], however, the proof does translate formulas into local formulas which could be evaluated algorithmically.

Weaker than order-invariance is successor-invariance, where the formulas are allowed to use a successor relation but must be invariant under the particular choice of successor relation. It was shown by Rossman [123] that successor-invariant FO is more expressive than FO without access to a successor relation.

As already the model-checking problem for plain FO is not tractable on the class of all graphs, order-invariant FO is not tractable on the class of all graphs either. We therefore follow the approach taken in the study of algorithmic meta-theorems and analyse the complexity of order- or successor-invariant FO and MSO on specific classes of structures or graphs. The outline for this chapter is as follows.

- We formally define the setting of our work in Section 9.1.
- In Section 9.2 we study the case of ordered structures, i.e. structures equipped with a fixed order or successor relation.
- In Section 9.3, we show that order-invariant MSO is tractable on essentially the same classes of graphs as plain MSO, i.e. we can increase the expressive power without restricting the tractable cases. To be precise, we show that the model-checking problem for order-invariant MSO on graphs of bounded clique-width is fixed-parameter tractable. Furthermore, combining the result of Courcelle [22] and a result in [21, 95] we find that model-checking for order-invariant MSO_2 on graphs of bounded tree-width is fixed-parameter tractable.
- For successor-invariant FO we show that the model-checking problem is fixed-parameter tractable on planar graphs. Using the result of Seese [125] we get the same result for FO on any class of graphs of bounded degree equipped with an arbitrary fixed number of successor relations. This result is presented in Section 9.4.

9.1. Successor- and Order-Invariance

Let τ be a vocabulary and let $<$ be a binary relation symbol which does not occur in τ. Let $\mathscr{C}$ be a class of τ-structures. We call a formula $\varphi(\overline{x})$ over

vocabulary $\tau \cup \{<\}$ *order-invariant* on $\mathscr{C}$ if for every τ-structure $\mathfrak{A} \in \mathscr{C}$, all tuples $\overline{a} \in V(\mathfrak{A})^m$ and all linear orders $<_1, <_2$ of $V(\mathfrak{A})$ we have

$$(\mathfrak{A}, <_1) \models \varphi(\overline{a}) \Leftrightarrow (\mathfrak{A}, <_2) \models \varphi(\overline{a}).$$

A successor relation S on $V(\mathfrak{A})$ is a binary relation such that $(V(\mathfrak{A}), S)$ is a directed path of length $n(\mathfrak{A}) - 1$. Let S be a binary relation symbol which does not occur in τ. As above, we call a formula $\varphi(\overline{x})$ over vocabulary $\tau \cup \{S\}$ *successor-invariant* on $\mathscr{C}$ if for every τ-structure $\mathfrak{A} \in \mathscr{C}$, all tuples $\overline{a} \in V(\mathfrak{A})^m$ and all successor relations S_1, S_2 on $V(\mathfrak{A})$ we have

$$(\mathfrak{A}, S_1) \models \varphi(\overline{a}) \Leftrightarrow (\mathfrak{A}, S_2) \models \varphi(\overline{a}).$$

We write FO[<-*inv*] and MSO[<-*inv*] for the set of all order-invariant FO and MSO formulas, respectively, and FO[+1-*inv*] and MSO[+1-*inv*] for the set of all successor invariant FO and MSO formulas, respectively. We write FO[<] and MSO[<] for the set of all FO and MSO formulas, respectively, over a signature which contains at least the binary relation symbol $<$ and similarly for FO[+1] and MSO[+1]. Note that it is not decidable whether a formula over vocabulary $\tau \cup \{<\}$ or $\tau \cup \{S\}$ is order- or successor-invariant, respectively, see e.g. [89]. Hence we do not speak of order- or successor-invariant logics, as it is a requirement of a logic to have a decidable syntax [67].

Also note that we will always consider formulas which are order- or successor-invariant on restricted classes of structures. Craig's interpolation theorem (see e.g. [19]) implies that a formula which is order- or successor-invariant on the class of all structures is equivalent to a plain FO formula.

9.2. Model-Checking on Ordered Structures

9.2.1. Successor Structures

In this section we analyse the complexity of first-order logic on graphs with an additional successor relation. We show that model-checking becomes intractable even on very restricted classes of graphs. More precisely, we show that on the class of forests with a successor relation first-order model-checking becomes as hard as the general first-order model-checking problem. The results of Section 9.2.1 and Section 9.2.2 were found by Stephan Kreutzer and the author of this thesis.

We write $\mathrm{MC}(\mathscr{L}[<],\mathscr{C})$ for the problem to decide for a given graph $G \in \mathscr{C}$, $<$ a linear order on $V(G)$ and $\varphi \in \mathscr{L}(\{E,<\})$ whether $(G,<) \models \varphi$. We write $\mathrm{MC}(\mathscr{L}[+1],\mathscr{C})$ for the problem to decide for a given graph $G \in \mathscr{C}$, $<$ a successor relation on $V(G)$ and $\varphi \in \mathscr{L}(\{E,S\})$ whether $(G,S) \models \varphi$. The parameter in both cases is $|\varphi|$.

Theorem 9.2.1 *Let $\mathscr{F}$ be the class of finite forests. Then* $\mathrm{MC}(\mathrm{FO}[+1],\mathscr{F})$ *is hard for the parameterized class* $\mathrm{AW}[\star]$.

PROOF. We show how to construct for every graph G a forest G' equipped with a successor relation S such that $n(G') + m(G') \leq p(n(G))$ for some polynomial p and for every sentence $\varphi \in \mathrm{FO}(\{E\})$ a sentence $\varphi' \in \mathrm{FO}(\{E,S\})$ with $|\varphi'| \in q(|\varphi|)$ for some polynomial q such that $G \models \varphi \Leftrightarrow G' \models \varphi'$. Both transformations will be polynomial time computable with makes the above reduction a fixed-parameter reduction, and hence the claim follows from the fact that the model-checking problem for FO on graphs is complete.

Let $G = (V,E)$, where $V = \{v_1,\ldots,v_n\}$. We will construct a forest $G' = (V',E',S)$ equipped with a successor relation as follows. G' will consist of n trees $T_1,\ldots,T_n$. The root of tree T_i will represent the vertex v_i of G. Denote the root of tree T_i by v'_i. To ensure that the roots are uniquely definable via first-order logic, we will guarantee that each root has at least three children and no other vertices of degree three exist. We define the successor relation on the v'_i in the natural way, i.e. $S(v'_i,v'_{i+1})$ holds true for all $i < n$. We will then add a child l^i_j to a root v'_i if there is an edge $\{v_i,v_j\}$ in $E(G)$. Thus, for each edge $\{v_i,v_j\}$ we have two vertices l^i_j and l^j_i, l^i_j being the child of v'_i and l^j_i being the child of v'_j. To associate those vertices with an edge, we define l^i_j as a direct predecessor of l^j_i if $i < j$. As the linear order is not definable from the successor relation with first-order logic, we have to add more information to ensure that exactly the described pairs of direct successors are interpreted as edges in the original graph. As an intermediate step we color l^i_j blue if $i < j$. We complete the definition of the successor relation in an arbitrary way. Observe that the successor of every blue vertex is already defined and thus by completing the successor relation we only define successors of colorless vertices. In particular, vertex v_i is connected to v_j in G if v'_i has a child c_i and v'_j has a child c_j such that c_i and c_j are direct successors and the smaller one with respect to the successor relation is blue. As the final step, we remove the color of every blue vertex

and attach a vertex instead. Then the vertices which were originally blue are the only ones of degree two and are thus uniquely definable. Again, we may complete the successor relation in an arbitrary way. We now give the formal details. Let

$$\begin{aligned} R &:= \{v'_1, \ldots, v'_n\}, \\ D_i &:= \{d^i_1, d^i_2, d^i_3\}, \\ L_i &:= \{l^i_j : \{v_i, v_j\} \in E(G)\} \quad \text{and} \\ B_i &:= \{b^i_j : \{v_i, v_j\} \in E(G), i < j\} \end{aligned}$$

for each $1 \leq i \leq n$. Let

$$V' := R \cup \bigcup_{1 \leq i \leq n} D_i \cup \bigcup_{1 \leq i \leq n} L_i \cup \bigcup_{1 \leq i \leq n} B_i.$$

Define

$$E' := \{\{v'_i, d^i_s\} : 1 \leq i \leq n, d^i_s \in D_i\} \cup \{\{v'_i, l^i_j\} : 1 \leq i \leq n, l^i_j \in L_i\} \cup \{\{l^i_j, b^i_j\} : 1 \leq i \leq n, l^i_j \in L_i, b^i_j \in B_i\}.$$

Define $S(v'_i, v'_{i+1})$ to hold if $i < n$ and $S(l^i_j, l^j_i)$ for all i, j with $i < j$. Complete the definition of the successor relation in an arbitrary way. By construction, for every edge $\{v_i, v_j\} \in E(G)$ there are exactly two vertices c_i and c_j with $\{v'_i, c_i\} \in E(G')$ and $\{v'_j, c_j\} \in E(G')$ such that c_i and c_j are direct neighbours in the successor relation and the smaller one of c_i and c_j has degree two and the larger one has degree one. Obviously, $n(G') + m(G') \leq p(n(G))$ for some polynomial p.

We now rewrite any sentence $\varphi \in \mathrm{FO}(\{E\})$ to a sentence $\varphi' \in \mathrm{FO}(\{E, S\})$. Let $\varphi \in \mathrm{FO}(\{E\})$. Replace quantifiers by quantifiers restricted to vertices of degree at least three. Further, replace an atom Exy by a formula $\psi(x, y)$ which states that either x is connected to a vertex x' of degree two and y is connected to a vertex y' of degree one and $S(x', y')$ holds true or vice versa. Then $|\varphi'| \leq q(|\varphi|)$ for some polynomial q and $G \models \varphi \Leftrightarrow G' \models \varphi'$ as required.□

As a corollary of the previous lemma we get that $\mathrm{MC}(\mathrm{FO}[+1], \mathscr{C})$ is AW[⋆]-hard for the class $\mathscr{C}$ of planar graphs and the class of graphs of tree-width one. However, for the proof to work it is essential that the trees are allowed to have unbounded degree. And indeed, on graph classes of bounded degree, successor-invariant FO model-checking is tractable.

Lemma 9.2.2 *For every integer $d \geq 0$ let $\mathscr{C}_d$ be the class of graphs of maximum degree at most d. Then for all $d \geq 0$,* MC(FO[+1], $\mathscr{C}_d$) *is fixed-parameter tractable. In fact, we can allow any (fixed) number of successor relations on top of $\mathscr{C}_d$ and still have tractable first-order model-checking.*

PROOF. By a result of Seese [125] the model-checking problem for FO on graphs of bounded degree and also on all structures with Gaifman-graph of bounded degree is fixed-parameter tractable. Adding a successor relation increases the degree of the Gaifman-graph of a structure by at most two.□

9.2.2. Order Relation

In the previous section we have shown that first-order logic with successor becomes intractable on very simple classes of graphs but is still tractable on classes of bounded degree. As the following lemma implies, first-order logic plus order is intractable even on degree two graphs.

Theorem 9.2.3 *Let $\mathscr{C}$ be the class of finite linear orders with one additional unary predicate. Then* MC(FO[+1], $\mathscr{C}$) *is* AW[⋆]*-hard. Consequently, if $\mathscr{S}$ is the class of finite successor structures with one unary predicate, then* MC(FO[<], $\mathscr{S}$) *is* AW[⋆]*-hard.*

PROOF. Let $\tau = \{<, S, P\}$, where S is a binary and P is a unary relation symbol. We show how to construct for every graph G a τ-structure $\mathfrak{A}$ of $m(\mathfrak{A}) \leq p(n(G))$ for some polynomial p and for every sentence $\varphi \in \mathrm{FO}[\{E\}]$ a sentence $\varphi' \in \mathrm{FO}[\tau]$ with $|\varphi'| \leq q(|\varphi|)$ for some polynomial q such that $G \models \varphi \Leftrightarrow \mathfrak{A} \models \varphi'$. Both transformations will be polynomial time computable which makes the above reduction a fixed-parameter reduction, and hence the claim follows from the fact that the model-checking problem for FO on graphs is complete.

Let $G = (V, E)$ be a graph, where $V = \{v_1, \ldots, v_n\}$. We mimic the construction of Theorem 9.2.1. We construct a structure $\mathfrak{A}$ of signature $\{<, S, P\}$ as follows. For each $v_i \in V$ we will have an element v'_i. In the construction of Theorem 9.2.1 the elements v'_i were uniquely definable as the roots of trees. Here we make the v'_i uniquely definable by assigning them the predicate P. We order the v'_i in their natural order, i.e. we let $v'_i < v'_j$ if $i < j$. In the construction of Theorem 9.2.1 we had elements l^i_j and l^j_i for each edge $\{v_i, v_j\}$, where l^i_j was the child of v'_i and l^j_i was the child of v'_j. We

have those elements again and if l^i_j was a child of v'_i in the construction of Theorem 9.2.1 we define l^i_j to lie between v'_i and v'_{i+1} (or simply after v'_i if v'_{i+1} does not exist). We furthermore order the l^i_j such that $l^i_j < l^i_k$ if $j < k$. Again, we encode an edge by defining l^i_j as the direct predecessor of l^j_i if there is an edge $\{v_i, v_j\} \in E(G)$ and $i < j$. We do not have to mark the smaller vertex by a color in this case because we can directly query whether $v'_i < v'_j$. We complete the definition of the successor relation such that no successors are added from any l^i_j to an l^k_l if $l^i_j < l^k_l$. This is done as follows. At this point of time, all l^i_j with $i < j$ have a successor and all l^j_i with $i < j$ have a predecessor. We define $S(l^j_i, l^i_k)$ to hold true if $i < j$ for the smallest $k > j$ such that l^i_k exists or $S(l^j_i, v'_{i+1})$ to hold true if no such l^i_k exists. After this step, all l^i_j have a successor and for each i only the smallest l^i_j with $i < j$ does not have a predecessor. All v'_i do not have a successor yet and v'_i does not have a predecessor if and only if $i = 1$ or $i > 1$ and between v'_{i-1} and v'_i there is no element with a successor larger than v'_i. We can thus complete the definition of the successor relation by defining $S(v'_i, l^i_j)$ to hold true if j is minimal with $i < j$ such that l^i_j exists and $S(v'_i, v'_{i+1})$ to hold true if no such l^i_j exists.

Then there is an edge $\{v_i, v_j\} \in E$ for $i < j$ if and only if there are elements a, b with $v'_i < a < v'_{i+1}$ if $i < n$ and $v'_j < b$ and $b < v'_{j+1}$ if $j < n$ that are direct successors with respect to S. We now give the formal details. Let

$$\begin{aligned} R &:= \{v'_i, \ldots, v'_n\}, \\ L_i &:= \{l^i_j : \{v_i, v_j\} \in E(G)\} \quad \text{and} \\ A &:= R \cup \bigcup_{1 \le i \le n} L_i. \end{aligned}$$

Let $P := R$. We define $v'_i < v'_j$ if $i < j$ and $v'_i < l^i_j < l^i_k$ for all i and for all $j < k$. We define $l^i_j < v'_{i+1}$ if $i < n$ and we define $S(v'_i, l^i_j)$ to hold true for the smallest possible j such that l^i_j with $i < j$ exists and $S(v'_i, v'_{i+1})$ to hold true if no such l^i_j exists. Finally, we define $S(l^i_j, l^j_i)$ to hold true if $i < j$ and $S(l^j_i, l^i_k)$ to hold true if $i < j$ for the smallest $k > j$ such that l^i_k exists or $S(l^j_i, v'_{i+1})$ if no such l^i_k exists. This completes the definition of $\mathfrak{A}$. Obviously, $n(strA) + m(\mathfrak{A}) \le p(n(G))$ for some polynomial p.

We now rewrite any sentence $\varphi \in FO[\{E\}]$ to a sentence $\varphi' \in \mathrm{FO}[\tau]$ with the required properties. Let $\varphi \in \mathrm{FO}[\{E\}]$. Replace quantifiers by quantifiers restricted to elements of P. Further, replace an atom Exy by a formula $\psi(x,y)$ which states that either $x < y$ and there are a and b such that $x < a < x'$ for the unique $x' > x$ with Px' and $y < b < y'$ for the unique $y' > y$ with Py' or no such y' exists and such that b is a direct successor of a, or vice versa. Then $|\varphi'| \leq q(|\varphi|)$ for some polynomial q and $G \models \varphi \Leftrightarrow \mathfrak{A} \models \varphi'$ as required. □

9.3. Model-checking for order-invariant MSO-formulas

In this section we consider the model-checking problem for order-invariant MSO-formulas. The results of this section were found by Stephan Kreutzer.

It was shown by Courcelle [22], that the model-checking problem for MSO_2 is fixed-parameter tractable on every class of graphs of bounded tree-width. Later, Courcelle et al. [24] showed that the model-checking problem for MSO is fixed-parameter tractable on every class of graphs of bounded clique-width, a concept more general than bounded tree-width. In this section, we show that for both logics we can allow order-invariance without increase in complexity.

Instead of designing new model-checking algorithms, we reduce the verification of order-invariant MSO on classes of small tree- or clique-width to the standard model-checking algorithms for MSO on classes of (slightly larger) tree- and clique-width, respectively. The advantage of this approach is that we can reuse existing results on MSO on such classes of graphs. For instance, in [78] the authors report on a practical implementation of Courcelle's theorem, i.e. on the implementation of a model-checker for MSO_2 on graph classes of bounded tree-width, and obtain astonishing performance results in practical tests. Our technique allows us to reuse this implementation so that with minimal effort it is possible to implement our algorithm on top of the work in [78].

Furthermore, in [49] it is shown that on graph classes $\mathscr{C}$ of bounded tree-width, the set of all satisfying assignments of a given MSO formula $\varphi(X)$ with free variables in a graph $G \in \mathscr{C}$ can be computed in time linear

in the size of the output and the size of G. Again we can use the same algorithm to obtain the same result for order-invariant MSO.

We first consider the case of MSO_2. As stated in [95] (see also the exposition in [21]), for every graph G of tree-width k there is a successor relation S on $V(G)$ such that the graph obtained from G by adding the edges in S has tree-width at most $k+5$. From the proof one can derive an algorithm running in time $f(k)\cdot p(G)$, where f is an exponential function and p a fixed polynomial, which, given a graph G of tree-width k as input, computes this successor relation. In combination with Courcelle's theorem, this implies the following result.

Theorem 9.3.1 ([46]) $\mathrm{MC}(\mathrm{MSO}[<\text{-}inv],\mathscr{C})$ *is fixed-parameter tractable on any class $\mathscr{C}$ of bounded tree-width.*

In fact, $\mathrm{MC}(\mathrm{MSO}[<\text{-}inv])$ is fixed-parameter tractable with parameter $|\varphi| + \mathrm{tw}(G)$, where $\mathrm{tw}(G)$ is the tree-width of a graph G. We prove next that also for MSO we can allow order-invariance without loss of tractability.

Theorem 9.3.2 ([46]) $\mathrm{MC}(\mathrm{MSO}[<\text{-}inv],\mathscr{C})$ *is fixed-parameter tractable on every class $\mathscr{C}$ of graphs of bounded clique-width.*

We quickly review the definition of clique-width. For the rest of this section we fix a relational signature τ in which every relation symbol has arity at most 2.

Let $k \geq 0$ be an integer. A *τ-clique-expression of width k* is a pair (T,λ), where T is a directed tree and

$$\lambda : V(T) \to \{\mathbf{1},\dots,\mathbf{k},\oplus,\mathrm{edge}_{R,i\to j},\mathrm{rename}_{i\to j} : 1 \leq i,j \leq k, R \in \tau\},$$

such that for all $t \in V(T)$: if $\lambda(t) \in \{\mathbf{1},\dots,\mathbf{k}\}$ then t is a leaf of T, if $\lambda(t) = \oplus$ then t has exactly two successors and in all other cases t has exactly one successor.

Let (T,λ) be a τ-clique-expression of width k. With every $t \in V(T)$ we associate a τ-structure $G(t)$ in which vertices are coloured by colours $\mathbf{1},\dots,\mathbf{k}$ as follows.

- If t is a leaf, then $G(t)$ consists of one element coloured by $\lambda(t)$.
- If $\lambda(t) = \oplus$ and t has successors t_1, t_2 then $G(t)$ is the disjoint union $G(t_1) \dot\cup G(t_2)$.

- If $\lambda(t) = \mathrm{edge}_{R,i\to j}$ and t_1 is the successor of t, then $G(t)$ is the structure obtained from $G(t_1)$ by adding to the relation $R(G(t))$ all pairs (u,v) such that u has colour i and v has colour j.
- If $\lambda(t) = \mathrm{rename}_{i\to j}$ and t_1 is the successor of t then $G(t)$ is the structure obtained from $G(t_1)$ by changing the colour of all vertices v which have colour i in $G(t_1)$ to colour j in $G(t)$.

The *τ-structure generated by* (T,λ) is the structure $\mathfrak{A}(r)$, where r is the root of T, from which we remove all colours $\{\mathbf{1},\dots,\mathbf{k}\}$. Finally, the *clique-width* of a τ-structure $\mathfrak{A}$ is the minimal width of a clique-expression generating $\mathfrak{A}$.

Combining results from [68] and [111] yields the following well-known result. In the following, we view graphs as $\{E\}$-structures in the obvious way.

Theorem 9.3.3 *There are computable functions $f,g:\mathbb{N}\to\mathbb{N}$ and an algorithm which, given a graph G of clique-width at most k as input, computes a clique-expression of width at most $g(k)$ in time $f(k)\cdot n(G)^3$.*

Here, the function $g(k)$ can be taken as $g(k) = 2^{k+1}$. The following result is due to Courcelle et al. [24].

Theorem 9.3.4 $\mathrm{MC}(\mathrm{MSO},\mathscr{C})$ *is fixed-parameter tractable on any class $\mathscr{C}$ of graphs of bounded clique-width.*

In fact, the result applies to any τ-structure of bounded clique-width provided that the clique-expression generating the structure is given. The next lemma is the main technical ingredient for the theorem above.

Lemma 9.3.5 *There is an algorithm which, on input a graph G of clique-width at most k, computes a linear order $<$ on $V(G)$ and a clique-expression of width at most $2g(k)$ generating the structure $(G,<)$, where g is the function defined in Theorem 9.3.3.*

PROOF. Let G and k be given. Using Theorem 9.3.3, we first compute an $\{E\}$-clique-expression (T,λ) of width at most $g(k)$ generating G. Let r be the root of T. For every node $t\in V(T)$ we fix an ordering of its successors. Let $\prec$ be the partial order on $V(T)$ induced by this.

Let $t\in V(T)$ be a node and let $s\neq t$ be the first node on the path P from t to r with $\lambda(s)=\oplus$, if it exists. Let t_1,t_2 be the successors of s with $t_1\prec t_2$.

We call t a *left node* if $t_1 \in V(P)$ and a *right node* otherwise. If there is no node labelled $\oplus$ strictly above t then we call t a left node as well.

For every $t \in V(T)$ let T_t be the subtree of T with root t and let $\lambda_{|T_t}$ be the restriction of λ to the subtree T_t. We recursively define a transformation $\rho(T_t, \lambda_{|T_t})$ on the sub-trees of T defined as follows. Intuitively, we will produce a new clique-expression (T', λ') over the signature $\{E, <\}$ using colours $\{(i,l),(i,r) : 1 \leq i \leq k\}$. Essentially, the new clique-expression will generate the same graph as (T, λ) but so that if t is a node in T and T_t generates the graph G_t, then T' contains a node t' generating an ordered version $G'_t := (G_t, <)$ of G_t so that if $v \in V(G_t)$ has colour i then, in G'_t, v has colour (i,l) if t is a left node and (i,r) if t is a right node. Hence, whenever in T we take the disjoint union of G_t and G_s and $t \prec s$ then we can define the ordering on $G'_t \dot\cup G'_s$ by adding all edges from nodes in G'_t to G'_s, i.e. all edges from vertices coloured (i,l) to (j,r) for all pairs i, j. Formally, the transformation is defined as follows.

- If $t \in V(T)$ is a leaf, then $\rho(t) := (T', \lambda')$, where T' consists only of t and $\lambda'(t) := (\lambda(t), l)$ if t is a left node and $\lambda'(t) := (\lambda(t), r)$ if t is a right node.
- If $\lambda(t) = \text{rename}_{i \to j}$ and s is the successor of t, then

  $$\rho(T_t, \lambda_{|T_t}) := (T', \lambda'),$$

  where T' is a tree defined as follows. Let $(T'', \lambda'') := \rho(T_s, \lambda_{|T_s})$ and let r'' be the root of T''. Then T' is obtained from T'' by adding a new root r' with successor r''. We define $\lambda'(r') := \text{rename}_{(i,l) \to (j,l)}$ and $\lambda'(r'') := \text{rename}_{(i,r) \to (j,r)}$ and $\lambda'(u) = \lambda''(u)$ for all other $u \in V(T'')$.
- If $\lambda(t) = \text{edge}_{E, i \to j}$ and s is the successor of t, then

  $$\rho(T_t, \lambda_{|T_t}) := (T', \lambda'),$$

  where T' is a tree defined as follows. Let $(T'', \lambda'') := \rho(T_s, \lambda_{|T_s})$ and let r'' be the root of T''. Then T' is obtained from T'' by adding a path (v_1, v_2, v_3) of length 2 and making r'' a successor of v_3. We let $\lambda'(v_1) := \text{edge}_{E,(i,l) \to (j,l)}$, $\lambda'(v_2) := \text{edge}_{E,(i,r) \to (j,l)}$, $\lambda'(v_3) := \text{edge}_{E,(i,l) \to (j,r)}$ and $\lambda'(r'') := \text{edge}_{E,(i,r) \to (j,r)}$ and $\lambda'(u) = \lambda''(u)$ for all other $u \in V(T'')$.
- Finally, suppose $\lambda(t) = \oplus$ and let t_1, t_2 be the successors of t such that $t_1 \prec t_2$. Then $\rho(T_t, \lambda_{|T_t}) := (T', \lambda')$ where T' is a tree defined as follows.

For $i = 1,2$ let $(T_i, \lambda_i) = \rho(T_{t_i}, \lambda_{|T_{t_i}})$ and let r_i be the root of T_i. T' consists of the union of T_1, T_2 and additional vertices $v_1, \ldots, v_k, v_e, v_o$, edges (v_i, v_{i+1}) for all $1 \leq i < k$, (v_k, v_e), (v_e, v_o) and (v_o, r_i), for $i = 1,2$. For every node $s \in V(T_i)$ we define $\lambda'(s) := \lambda_i(s)$, $i = 1,2$. Furthermore, we define $\lambda(v_o) := \oplus$ and $\lambda'(v_e) := \mathrm{edge}_{<,l\to r}$. Finally, if t is a left node then we define $\lambda(v_i) := \mathrm{rename}_{(i,r)\to(i,l)}$, for all $i \leq k$, and if t is a right node then we define $\lambda(v_i) := \mathrm{rename}_{(i,l)\to(i,r)}$.

Now, it is easily seen that (T', λ') generates an $\{E, <\}$-structure $(V, E, <)$ where (V, E) is the graph generated by (T, λ) and $<$ is a linear order on V. The width of (T', λ') is twice the width of (T, λ) and hence at most $2g(k)$. □

We are now ready to prove Theorem 9.3.2.

PROOF. (PROOF OF THEOREM 9.3.2) Let $k \geq 0$ be an integer and let $\mathscr{C}$ be a class of tree-width at most k. On input $G \in \mathscr{C}$ and $\varphi \in \mathrm{MSO}(\{E, <\})$, we apply Lemma 9.3.5 to obtain a clique-expression (T, λ) of width $2g(k)$ generating an ordered copy $(G, <)$ of G, where g is the function from Theorem 9.3.3. We can now apply Theorem 9.3.4 to decide whether $(G, <) \models \varphi$ in time $f(2g(k)) \cdot p(n(G))$, where f is a computable function and p a polynomial. As φ is order-invariant, if $(G, <) \models \varphi$ then $(G, <') \models \varphi$ for any linear order $<'$ on G. Hence, if $(G, <) \models \varphi$ we accept and otherwise reject the input. This concludes the proof. □

9.4. Model-checking for successor-invariant FO-formulas

In this section we study the model-checking problem first-order formulas that are successor-invariant. Recall that the input to the successor-invariant model-checking problem is a graph G (from a class $\mathscr{C}$) and a successor-invariant formula φ which refers to a successor relation but its truth in G must be invariant under the exact choice of the successor relation. Hence, to verify whether φ is true in G it suffices to compute one particular successor relation S and decide whether $(G, S) \models \varphi$. The main result of this section was published in [46] and is due to Viktor Engelmann, Stephan Kreutzer and the author of this thesis.

In graph theoretical terms, a successor relation corresponds to a Hamiltonian path in a graph G. Hence, if $\tau := \{E, F\}$ is a signature with two binary relation symbols and $\mathscr{C}$ is a class of graphs such that

1. first-order model-checking is fixed-parameter tractable on the class of all τ-structures whose Gaifman-graphs are in $\mathscr{C}$, and
2. all graphs in $G \in \mathscr{C}$ are Hamiltonian and, furthermore, a Hamiltonian path in G can be computed in polynomial time,

then we can conclude that $MC(\mathrm{FO}[+1\text{-}inv], \mathscr{C})$ is fixed-parameter tractable as follows: on input $G \in \mathscr{C}$ we first compute a Hamiltonian path P and colour its edges by the colour F to obtain a τ-structure with Gaifman-graph G. We can then use the model-checking algorithm on $\mathscr{C}$ to decide whether $(G, F) \models \varphi$.

One example of this is the class of 4-connected planar graphs. Tutte and later Thomassen (see [131] and references therein) showed that any 4-connected planar graph contains a Hamiltonian path which can be computed in polynomial time. Together with the result in [53] that the model-checking problem for first-order logic on planar graphs is fixed-parameter tractable, we immediately get the following result.

Theorem 9.4.1 *Denote by $\mathscr{C}$ the class of 4-connected planar graphs. Then* $\mathrm{MC}(\mathrm{FO}[+1\text{-}inv], \mathscr{C})$ *is fixed-parameter tractable.*

There are some other classes of graphs with similar properties, but not every planar graph is Hamiltonian and it is not always possible to obtain a Hamiltonian planar graph from an arbitrary planar graph by adding edges or vertices in a way that would be useful for our purposes. See e.g. [71] and references therein. Therefore, to show that successor-invariant FO is fixed-parameter tractable on planar graphs, the main result of this section, we have to use different techniques.

Theorem 9.4.2 $\mathrm{MC}(\mathrm{FO}[+1\text{-}inv], \textsc{Planar})$ *is fixed-parameter tractable.*

The approach we take to prove the theorem is based on the interpretation method presented in Section 2.4. We state our exact requirements as a lemma.

Lemma 9.4.3 *Let $\mathscr{C}$ be a class of graphs, τ a signature and $\mathscr{D}$ a class of τ-structures with the following properties.*

1. *There is a polynomial-time algorithm which, on input $G \in \mathscr{C}$, computes a τ-structure $H \in \mathscr{D}$ and τ-formulas $\varphi_V(x), \varphi_E(x,y), \varphi_S(x,y)$ such that $G' := (\varphi_V(H), \varphi_E(H))$ is isomorphic to G and $\varphi_S(H)$ defines the edge set of a successor relation on $V(G')$.*

2. $\mathrm{MC}(\mathrm{FO}, \mathscr{D})$ *is fixed-parameter tractable.*

Then $\mathrm{MC}(\mathrm{FO}[+1\text{-}inv], \mathscr{C})$ *is fixed-parameter tractable.*

PROOF. Given $G \in \mathscr{C}$ and a successor-invariant sentence $\varphi \in \mathrm{FO}(E,S)$, we can decide $G \models \varphi$ as follows. We apply our algorithm to obtain the structure H and the formulas $\varphi_V, \varphi_E, \varphi_S$ as above. Using Lemma 2.4.1, we transform the formula φ into a formula $\varphi' \in \mathrm{FO}[\tau]$ such that $H \models \varphi'$ if and only if for some, and hence every, successor relation S of G, $(G,S) \models \varphi$. Essentially, we relativise all quantifiers in φ to φ_V, replace atoms $E(x,y)$ by $\varphi_E(x,y)$ and atoms $S(x,y)$ by $\varphi_S(x,y)$. We can then apply the model-checking algorithm for first-order logic on $\mathscr{D}$ to decide whether $H \models \varphi'$. □

In the rest of this section we show how this lemma can be used to prove Theorem 9.4.2. For simplicity, we will aim for a class $\mathscr{D}$ which is nowhere dense. In [46] we showed that the class even has bounded local tree-width. Essentially, we will add to a planar graph G some copies of G with slight modifications, called *circular extensions* below, and show how to define Hamiltonian paths in these copies. This path will then be projected to G by first-order formulas.

Recall that $G \bullet H$ denotes the lexicographic product of G and H. Also recall that by Corollary 6.2.2, if $\mathscr{C}$ is nowhere dense and $t \geq 0$, then $\mathscr{C} \bullet K_t = \{G \bullet K_t : G \in \mathscr{C}\}$ is nowhere dense.

For the following exposition we fix a planar graph G and a plane embedding Γ of G. For a vertex $v \in V(G)$, a *cyclic ordering of v respecting* Γ is an ordering (bijection) $\rho : \{0, \dots, l-1\} \to N(v)$, where $l := |N(v)|$, of the neighbours of v obtained by listing $N(v)$ in clockwise order starting with some $v_0 \in N(v)$. For each $v \in V(G)$ we fix a cyclic ordering ρ_v of v respecting Γ. We now define a graph $\mathfrak{C}(G)$ obtained from G by the following operations. We first subdivide every edge twice. We then connect the new neighbours of a vertex v in the order specified by ρ_v to form a cycle around v respecting the plane embedding of G.

Formally, the *circular extension* $\mathfrak{C}(G)$ of G is the graph with vertex set

$$V(G) \cup \{u_{v,e} : e = \{v, v'\} \in E(G)\}$$

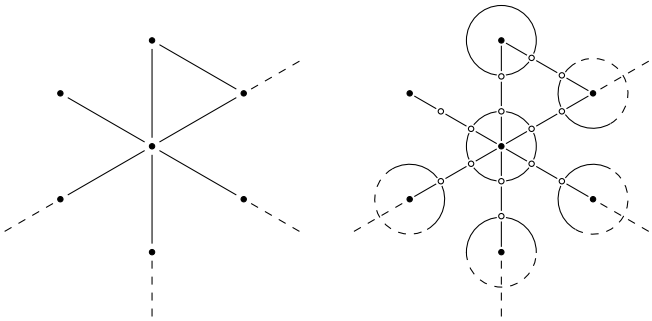

Figure 9.1.: A planar graph G and its circular extension $\mathfrak{C}(G)$. The original vertices are black and the circular vertices are white.

and edge set

$$\big\{\{v,u_{v,e}\} : e=\{v,v'\}\in E(G)\big\} \cup \big\{\{u_{v,e},u_{v',e}\} : e=\{v,v'\}\in E(G)\big\} \cup E'$$

where E' is defined as follows. Let $v \in V(G)$ with $l := |N(v)| >$ and let

$$e_0 = \{v,\rho_v(0)\},\dots,e_{l-1} = \{v,\rho_v(l-1)\}$$

be the edges of G incident to v. Let E' contain all edges $\{u_{v,e_i},u_{v,e_{i+1 \bmod l}}\}$. We call the vertices of the set $V(G)$ *original vertices* and the vertices of the set $\{u_{v,e} : e=\{v,v'\}\in E(G)\}$ *circular vertices*.

The concept of circular extensions is illustrated in Figure 9.1. The figure shows a (part of a) planar graph and its circular extension, where the original vertices are black and the circular vertices are white. It is easily seen that if G is planar, then so is $\mathfrak{C}(G)$. In particular, the class $\mathscr{C} := \{\mathfrak{C}(G) : G$ is a planar graph$\}$ is nowhere dense.

Let G be a connected planar graph and let T be a spanning tree of G with root t and consider the graph $\mathfrak{C}(T)\bullet K_2$. Recall that $\mathfrak{C}(T)\bullet K_2$ consists of two copies of $\mathfrak{C}(T)$ and some edges between them. We will refer to these copies as $\mathfrak{C}(T)$ and $\mathfrak{C}(T)'$ respectively and call $\mathfrak{C}(T)$ the *principle copy* of $\mathfrak{C}(T)\bullet K_2$. For a vertex $v \in V(\mathfrak{C}(T))$ we denote by v' the corresponding vertex

of $\mathfrak{C}(T)'$. We show that $\mathfrak{C}(T)\bullet K_2$ contains a Hamiltonian path H such that the original vertices of the principal copy $\mathfrak{C}(T)$ have short distance along H. This will allow us to define a successor relation on G in first-order logic. To find such a Hamiltonian path, we follow a depth-first, left-to-right traversal of T, walking alternatingly on vertices of $\mathfrak{C}(T)$ and $\mathfrak{C}(T)'$. Instead of using a vertex v in T multiple times, we use the circular edges between the successors of v, which exist in $\mathfrak{C}(T)$ (this is why we use a left-to-right traversal of T – the additional edges exist only between neighbouring successors).

Lemma 9.4.4 *Let T be a tree with root t. Then $\mathfrak{C}(T)\bullet K_2$ contains a Hamiltonian path $H := (t = h_1, h_2, \ldots, h_n = t')$ such that every subpath of H of length at least 7 contains an original vertex of the principle copy $\mathfrak{C}(T)$. Furthermore, such a path can be computed in polynomial time.*

PROOF. We show the following property by induction on the height h of T. $\mathfrak{C}(T)\bullet K_2$ contains a Hamiltonian path $H := (h_1, h_2, \ldots, h_n)$ with $h_1 = t$ and $h_n = t'$ such that $\{h_i : \max\{1, n-3\} \leq i \leq n\}$ contains an original vertex of the principle copy $\mathfrak{C}(T)$ and every subpath of H of length at least 7 contains an original vertex of the principle copy $\mathfrak{C}(T)$.

For $h = 0$, T contains only an isolated vertex t and $H = (t, t')$ is a Hamiltonian path in $\mathfrak{C}(T)\bullet K_2 = (\{t, t'\}, \{\{t, t'\}\})$ with the above properties.

For $h > 0$, let $t_1, \ldots, t_d$ be the successors of t and let $T_1, \ldots, T_d$ be the sub-trees of T rooted at $t_1, \ldots, t_d$. Each T_i has height $< h$ and by induction hypothesis, $\mathfrak{C}(T_i)\bullet K_2$ has a Hamiltonian path $H_i = (t_i = h_{i_1}, \ldots, h_{i_{n_i}} = t_i')$ with the desired properties. For a path P write P^{rev} for the path P in reverse direction. Then

$$\begin{aligned} H := (t', & u_{t,\{t,t_1\}}, (u_{t_1,\{t_1,t\}})', H_1, u_{t_1,\{t_1,t\}}, (u_{t,\{t,t_1\}})', \\ & u_{t,\{t,t_2\}}, (u_{t_2,\{t_2,t\}})', H_2, u_{t_2,\{t_2,t\}}, (u_{t,\{t,t_2\}})', \ldots, \\ & u_{t,\{t,t_d\}}, (u_{t_d,\{t_d,t\}})', H_d, u_{t_d,\{t_d,t\}}, (u_{t,\{t,t_d\}})', \\ & t)^{rev} \end{aligned}$$

is a Hamiltonian path in $\mathfrak{C}(T)\bullet K_2$ with the desired properties. It is clear that H can be computed in polynomial time. □

We are now ready to show that for any planar graph G we can construct a τ-structure H which satisfies the requirements of lemma 9.4.3.

Let G be a connected graph. Let $\sigma := \{E, F, R, B\}$, where E, F are binary and R, B are unary relation symbols. The *Hamiltonian extension* $\mathscr{H}(G)$ of G is a τ-structure $H := \mathscr{H}(G)$ defined as follows. Let T be a spanning tree of G. Then H consists of the disjoint union of G and $T' := \mathfrak{C}(T) \bullet K_2$. Furthermore, if C is the principle copy of $\mathfrak{C}(T)$ in T', then we add an edge between every $v \in V(G)$ and its copy in C. All edges defined so far together define the relation $E(H)$. Let P be the Hamiltonian path in T' defined in the proof of Lemma 9.4.4. We define $F(H)$ as the directed edges of an orientation P so that F is a directed Hamiltonian path. Finally, we colour the components of H by defining $R(H) := V(G)$ and $B(H) := V(C)$.

Note that $\mathscr{H}(G)$ depends on T. However, our results below do not depend on the particular choice of T and we therefore write $\mathscr{H}(G)$ for any choice of T.

We now show how to find the successor of a vertex $v \in V(G)$. Let $v' \in V(C)$ be the copy of v in C. Let u' be the next vertex on the directed Hamiltonian path $F(\mathscr{H}(G))$ in $V(C)$ such that u' is connected to a vertex u in $V(G)$, i.e. u' corresponds to an original vertex of G. We then define $u \in V(G)$ as the successor of v. As F is a Hamiltonian path of T', this construction defines indeed a successor relation, on $V(G)$. By construction, u' has distance at most 7 from v' in F and thus it is easily seen that S is first-order definable by a formula $\varphi_S(x, y) \in \mathrm{FO}(\tau)$, as the colors R, B can be used to distinguish between the copies of $V(G)$.

Lemma 9.4.5 *The class $\{\mathscr{H}(G) : G$ is a planar connected graph$\}$ is nowhere dense.*

PROOF. As observed above, the class $\{\mathfrak{C}(G) \bullet K_3 : G$ is a planar connected graph$\}$ is nowhere dense and hence there is a function ℓ such that for every integer $r \geq 0$, splitter wins the simple $(r, \ell(r))$-splitter game on any $\mathfrak{C}(G) \bullet K_3$ for G planar and connected. We define $\ell'(r) := 3\ell(3r)$. Let us transform splitter's strategy from the $(3r, \ell(3r))$-game to a strategy for the $(r, \ell'(r))$-game.

Note that in the first copy of $\mathscr{H}(G)$, edges are not subdivided, hence in order to preserve neighbourhoods, we consider the $3r$ game on $\mathfrak{C}(G) \bullet K_3$.

We only have to take care of one complication concerning circular edges of $\mathfrak{C}(T)$ which are not present in the respective copy $\mathfrak{C}(G)$ of $\mathfrak{C}(G) \bullet K_3$. Let $v \in V(G)$ be a vertex and let $N_v := (v_0, \ldots, v_l) \subseteq N^T(v)$ be the set of neighbours of v in T listed in the order given by the cyclic ordering ρ_v fixed in the

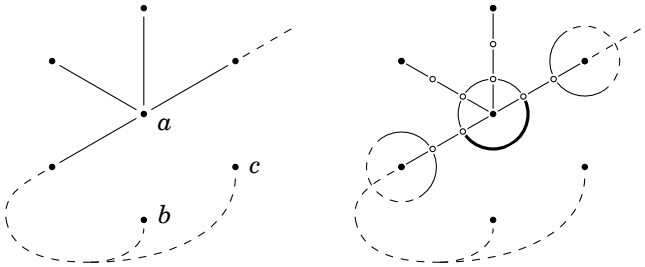

Figure 9.2.: A spanning tree T of the graph G of Figure 9.1 and its circular extension $\mathfrak{C}(T)$. Vertices b and c are not connected to T via a and thus $\mathfrak{C}(T)$ contains edges that are not edges of $\mathfrak{C}(G)$.

construction of $\mathfrak{C}(G)$. If for some $i \leq l$, v_i and $v_{i+1 \bmod (l+1)}$ are not also neighbours in the ordering ρ_v, then in $\mathfrak{C}(G)$ there is a path between any v_i and $v_{i+1 \bmod l+1}$ consisting of vertices used to subdivide original edges and their connecting new edges. This is illustrated in Figure 9.2. The figure shows a spanning tree of the graph displayed in Figure 9.1 and the corresponding circular extension. Note that the vertices b and c are neighbours of a in the graph G but they are no longer neighbours of a in the spanning tree T. The edge in the circular extension marked by a thick line in Figure 9.2 therefore corresponds to a path between neighbours v_i and $v_{i+1 \bmod l+1}$ of a in the circular extension of the graph G.

If both v_i and $v_{i+1 \bmod l+1}$ lie in the r-neighbourhood of some vertex u in $\mathscr{H}(G)$ then every vertex on the path between v_i and $v_{i+1 \bmod l+1}$ lies in the $3r+2$-neighbourhood of u in $\mathfrak{C}(G)\bullet K_3$. This follows from the fact that every circular edge of $\mathfrak{C}(T)$ can be replaced by a path length 2 in $\mathfrak{C}(G)$. We simply adapt splitter's strategy such that whenever he deletes v or any of its subdivided neighbours in $\mathfrak{C}(G)\bullet K_3$, we use the two following rounds to delete also v_i and $v_{i+1 \bmod l+1}$.
Now it is easy to see that for all i, the graph G_i reached in round i in the (r, ℓ') game on $\mathscr{H}(G)$ is a subgraph of the graph H_{3i} reached in round $3i$ in the $(3r, \ell(3r))$ game on $\mathfrak{C}(G)\bullet K_3$. Hence we reach the empty graph after at most $3\ell(3r) = \ell'(r)$ rounds. This concludes the proof. □

Clearly, Lemma 9.4.5 does not only hold for connected planar graphs but for planar graphs in general.

Lemma 9.4.6 *Let $\tau := \{E,F,R,B\}$, where E,F are binary and R and B are unary relation symbols. There is a polynomial time algorithm which, given a planar graph G as input, computes a τ-structure $\mathscr{H}(G)$ and a formula $\varphi_S(x,y) \in \mathrm{FO}[\tau]$ such that the class $\{\mathscr{H}(G) : G \text{ a planar graph}\}$ is nowhere dense and $\varphi_S(\mathscr{H}(G))$ defines a successor relation on G.*

PROOF. Construct the Hamiltonian extension $\mathscr{H}(C)$ for every component C of G and connect the resulting Hamiltonian paths in some order. After connector's first choice v in the (r,ℓ) game on the resulting graph, at most r components lie in the r-neighbourhood of v. Splitter invests $2r$ additional moves to disconnect the components again by deleting the connection points of the introduced Hamilton-edges. The following game is played as the original game on some component.

The formula defining the successor relation is modified accordingly. □

Combining Lemma 9.4.6 and Theorem 8.1.2 we can now apply the method established in Lemma 9.4.3 to obtain the main theorem of the section, Theorem 9.4.2.

Eickmeyer et al. [45] have extended these results to show that the model-checking problem for successor-invariant first-order formulas on any class of graphs that excludes a fixed minor is fixed-parameter tractable. Their result is also based on the interpretation method, however, their way of adding definable successor relations is much more straight forward.

For $k \geq 1$, a *k-walk* P of length ℓ through a graph G is a surjective mapping $w : \{1,\dots,\ell\} \to V(G)$ such that $w(i)w(i+1) \in E(G)$ for all $1 \leq i < \ell$ and such that $|\{i : w(i) = v, 1 \leq 1 < \ell\}| \leq k$ for all $v \in V(G)$. The authors of [45] showed that if G excludes K_t as a minor, then there is k such that one can find a k-walk in a super-graph H of G which excludes K_k as a minor. Furthermore, H and the k-walk can be found in polynomial time. They further showed that for every graph with a k-walk, one can expand the signature appropriately such that from the k-walk one can define a successor relation by first-order logic and then apply the interpretation method. It is an interesting open question to which classes of graphs this method may be extended.

10. Conclusion and open problems

We have shown that any first-order definable property of graphs can be decided in almost linear time $O(n^{1+\epsilon})$ on any nowhere dense class of graphs. First-order definable problems define a natural and very important class of problems including dominating sets, vertex covers, network centres and many others. Our result allows to quickly tell whether a problem is fixed-parameter tractable on a nowhere dense class of graphs, simply by formulating the problem in first-order logic. A problem specific analysis, such as the analysis of the r-independent set problem in Section 6.2, may then lead to faster algorithms.

For the proof of our theorem we have extended the theory of nowhere dense graphs in several ways. We have provided a new characterisation of nowhere dense classes of graphs which describes the local neighbourhoods of their elements in terms of the splitter game. From the winning strategy of splitter on nowhere dense classes one can derive a very simple structural decomposition of local neighbourhoods which allows to solve many local problems recursively with a recursion tree of bounded depth.

We have shown that every nowhere dense class of graphs admits sparse neighbourhood covers and that in fact every nowhere dense class which is closed under subgraphs can be characterised by the existence of such covers. This characterisation is based on a known characterisation of nowhere dense classes in terms of generalised colouring numbers. We have improved previously known bounds for the generalised colouring numbers on nowhere dense classes, proper minor closed classes, classes of bounded genus and classes of bounded tree-width. These improved bounds lead to neighbourhood covers of radius $2r$ and degree bounded by some function $f(r)$ for classes that exclude K_t as a (topological) minor. Busch et al. [18] present r-neighbourhood covers of radius $24r-8$ and degree at most 18 for planar graphs and Abraham et al. [4] present r-neighbourhood covers of radius $O(t^2 \cdot r)$ and degree at most $2^{O(t)}t!$ for graphs that exclude K_t as a minor.

Hence by allowing a larger radius one finds a cover with degree independent of r (depending only on the excluded minor). It is an interesting open question how large the radius must be chosen such that this is possible.

We have shown that in general, computing the exact value of $\mathrm{wcol}_3(G)$ of an input graph G is NP-complete. The graphs occurring in the reduction are very dense. It remains an open question whether computing $\mathrm{wcol}_2(G)$ is NP-complete in general and whether computing $\mathrm{wcol}_r(G)$ is NP-complete for sparse classes of graphs. The same questions are open for the strong colouring numbers col_r.

The combination of the two new characterisations has led to a powerful algorithmic technique on nowhere dense classes of graphs. Our algorithmic techniques are quite elementary and very easy to implement. The algorithmic construction of sparse neighbourhood covers is based on a simple greedy algorithm which is based on Nešetřil and Ossona de Mendez's augmentation technique and also the structural decomposition for local neighbourhoods resulting from the splitter game is easily computable. This is in sharp contrast to most algorithms for proper minor closed classes which are based on Robertson and Seymour's structure theorem for such classes.

On the logical side, we have proved a strengthened version of Gaifman's locality theorem. We introduced a new, discounted rank measure for first-order formulas which allows a translation into local formulas such that the rank is preserved. This makes a recursive evaluation of formulas possible. However, in general, a non-elementary growth of formula length cannot be avoided when translating formulas into their equivalent local form. This is likely to be a serious problem in a practical implementation of our algorithm for general formulas. Again, for many natural problems a careful analysis of the problem may enable us to avoid this problem.

The first-order model-checking problem is hard for AW[$\star$] on any somewhere dense class $\mathscr{C}$ which is closed under subgraphs and hence by our result we capture all classes of graphs which are closed under subgraphs on which the first-order model-checking problem is fixed-parameter tractable (under the assumption FPT $\neq$ AW[$\star$]).

Much less is known about the problem for classes which are not closed under subgraphs. Clearly, the theorem extends to all classes which can be found via first-order interpretations in a nowhere dense class. However, there are not many known useful interpretations. By Courcelle et al. [24], the problem is fixed-parameter tractable even for MSO on classes of bounded clique width. Already in 2007, Martin Grohe [61] raised the question, for

which algebraic structures such as groups, rings, fields, et cetera the model-checking problem is tractable. A first result in this direction was recently obtained by Gajarský et al. [56], who showed that the problem is fixed-parameter tractable on partially ordered sets of bounded width. However, there remain a lot of interesting classes for which the problem is open.

Another generalisation of the problem asks for the enumeration of tuples $\overline{a} \in V(\mathfrak{A})^m$ such that $\mathfrak{A} \models \varphi(\overline{a})$ for some formula $\varphi(\overline{x})$. Kazana and Segoufin [73] have shown that after a linear time preprocessing, one can enumerate all answers to the query $\varphi(\overline{x})$ with constant delay, on any class of bounded expansion. Earlier results showed that this is possible on classes of bounded degree [39, 72] and for MSO on classes of bounded tree-width [74, 10]. It is an interesting question whether Kazana and Segoufin's result can be extended to nowhere dense classes.

Finally, we have shown that first-order logic on ordered structures as well as on structures with a successor relation is intractable on nearly all interesting classes of graphs. However, we have shown that the model-checking problem for order-invariant MSO on graphs of bounded clique-width and on graphs of bounded tree-width is fixed-parameter tractable.

For successor-invariant FO we were able to show that the model-checking problem is fixed-parameter tractable on planar graphs. These results were extended by Eickmeyer et al. [45] to all proper minor closed classes. It is an interesting open question whether the result extends further to classes of bounded expansion or nowhere dense classes. In fact, not much is known about the expressive power of successor-invariant first-order logic on sparse structures. All structures which are used to show that such formulas can express more than plain first-order formulas are very dense. It is an intriguing task to find a successor-invariant sentence on a nowhere dense class which cannot be expressed by a plain first-order sentence.

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Nowhere Dense Classes of Graphs

In this thesis we study graph theoretic and algorithmic properties of nowhere dense classes of graphs. Nowhere dense graph classes form a large variety of classes of sparse graphs including all classes with excluded minors, bounded degree graphs and classes of bounded expansion.
We show that every first-order property of graphs can be decided in almost linear time on every nowhere dense class of graphs. For graph classes closed under taking subgraphs, this result is optimal (under a standard complexity theoretic assumption).
For our proof, we provide two new characterisations of nowhere dense classes in terms of a game and by the existence of sparse neighbourhood covers. The characterisation by neighbourhood covers is based on generalised colouring numbers, for which we provide several new bounds on restricted graph classes. On the logical side, we prove a rank-preserving version of Gaifman's locality theorem.
Finally, we show that first-order logic on ordered structures is essentially intractable on nearly all interesting classes. On the other hand, we show that the model-checking problem for order-invariant MSO is tractable essentially on the same classes as plain MSO and that the model-checking problem for successor-invariant first-order logic is tractable on planar graphs.

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