Universität Ulm

Institut für Stochastik

Adding Randomness to Nonlinear Semigroups: Existence, Uniqueness and Asymptotic Results

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat. der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

vorgelegt von

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aus

Luckenwalde

im Jahr

2018

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Tag der Promotion:	12.10.2018

Acknowledgment

First of all, I would like to thank my advisor Evgeny Spodarev for his support in the past three years. He not only helped me with the mathematical challenges I faced while writing this dissertation, but also to get the funding for my thesis and provided me with a friendly, supportive and encouraging work environment. In addition, I am grateful to my co-advisor Wolfgang Arendt, who helped me to overcome some of the functional analytic obstacles I faced in the past years. Last but not least, I would like to thank Alexei Kulik who took the time to introduce me to the beautiful theory of Markov processes during one of my most fruitful research stays at TU Berlin.

Abstract

The origins of this thesis lie in the present author's study of the asymptotic behavior of the weighted p-Laplacian evolution equation, which is one of the bench marks of nonlinear evolution equations and used to model many physical phenomena, such as the evolution of fluvial landscapes. Due to these modeling aspects, it is of great interest to add randomness to such an equation. In this thesis, we investigate two different ways of doing that: We literally add a pure-jump noise to a nonlinear semigroup and derive an existence, uniqueness and asymptotic theory for the resulting process; and secondly, we replace the weight function occurring in the p-Laplacian evolution equation by a random quantity. Hereby, the first approach will be set up for arbitrary nonlinear semigroups, and the applicability of our results will be demonstrated at hand of the p-Laplacian evolution equation; and the latter approach is written specifically for the p-Laplacian case.

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Chapter 1

Introduction

The purpose of this thesis is to present new insights into the existence/uniqueness and asymptotic theory of some deterministic and random nonlinear partial differential equations. Even though the different results obtained here are to some extend connected, they are not as intertwined as it seems on a first glance. Particularly, we may consider the same objects in many chapters of this thesis, but in each chapter we shoot at them from different angles; and by doing so we cover a variety of mathematical areas, including: Classical functional analysis, the theory of nonlinear semigroups and abstract Cauchy problems, random variables taking values in (separable) Banach spaces, and Markov processes taking values in infinite-dimensional spaces.

In this introduction, we do not intend to give a precise and rigorous outline of the results which we will obtain, but we would rather like to: Explain why we consider these objects as well as their connection to one another, and roughly explain the type of results we shall prove.

In general three different topics are covered by this thesis:

- Asymptotic results for the strong solution of a weighted p-Laplacian evolution equation with Neumann boundary conditions acting on an L^1-space. (Chapter 3)

- Existence, Uniqueness and asymptotic results for a class of stochastic processes which arise by exposing a nonlinear semigroup to time-discrete big-jump noise. For this class of processes, we coin the term "ACPRM-process", where ACPRM is an acronym for abstract Cauchy problem driven by a random measure. (Chapters 4, 5 and 6)

- Existence, uniqueness and asymptotic results for the strong solution of a randomized weighted p-Laplacian evolution equation with Neumann boundary conditions acting on an L^1-space. (Chapter 7)

The above list covered all chapters, except for Chapter 2 and Chapter 8. The former is a general introduction to the existence/uniqueness theory of abstract Cauchy problems (Section 2.1) and to ACPRM-processes (Section 2.2), and in the latter chapter, we briefly summarize our results and outline how they might be extended.

As mentioned, in this thesis we employ techniques from many mathematical fields. Therefore, to make the reader's life as pleasant as possible, the present author tried to write the respective chapters as self-contained as possible. More accurately, the chapters/sections depend upon each other in the following way:

- Section 2.1 is needed in every chapter of this thesis. And it is of extreme relevance to Chapters 3, 4 and 7. In particular, we use some notations/definitions from nonlinear semigroup theory in this introduction. One finds precise definitions of all these quantities in Section 2.1.
- Section 2.2 builds the foundation for Chapters 4, 5 and 6; and it is irrelevant to Chapters 3 and 7.
- Chapter 3 is of extreme importance to Chapter 7. Moreover, the p-Laplacian semigroup introduced in Chapter 3, serves as an example in the chapters dealing with ACPRM-processes, that means it is relevant to Sections 4.4, 5.3 and 6.4; but it is irrelevant to any other section in Chapters 4, 5 and 6.
- Chapters 4, 5, 6 and 7 are entirely independent of each other.

In conclusion, reading Chapter 2 suffices to understand most parts of this thesis, and additionally reading Chapter 3 suffices to understand all of them.

In the next three sections of this introduction, we will have a closer look at each of the central topics covered by this thesis. These sections are about conveying why we consider the objects we consider, and briefly outlining our results. Particularly, we will try to keep the amount of formulas to a necessary minimum. Note that each chapter (except for Chapters 2 and 8) starts with a section called "Outline & Highlights", where one finds a rigorous description of the proven results.

1.1 The weighted p-Laplacian evolution Equation

The weighted p-Laplacian evolution equation with Neumann boundary conditions is the central object of Chapter 3. It is given by

$$\begin{cases} u'(t) = \operatorname{div}\left(\gamma|\nabla u(t)|^{p-2}\nabla u(t)\right) & \text{on } S, \\ \gamma|\nabla u(t)|^{p-2}\nabla u(t)\cdot\Upsilon = 0 & \text{on } \partial S, \\ u(0) = v, \end{cases} \tag{1.1}$$

for a.e. $t \in (0,\infty)$; where $p \in (1,\infty)\backslash\{2\}$, $S \subseteq \mathbb{R}^n$ is a sufficiently regular set, $n \in \mathbb{N}\backslash\{1\}$, $\gamma : S \to (0,\infty)$ is a weight function fulfilling some technical conditions, Υ is the unit outer normal on ∂S, and $v : S \to \mathbb{R}$ is an integrable initial.

From the applied point of view, the solution u can be used to model diffusion processes: One has some initially given quantity $v = u(0)$ which changes over time due to an external force γ and the resulting quantity at time t is $u(t)$. For example, as B. Birnir and J. Rowlett demonstrated in [BR], the solution u of (1.1) can be used to describe the evolution of a fluvial landscape v (for example a hill) which changes over time due to rain that determines the water depth γ.

Moreover, F. Andreu, J. Mazón, J. Rossi and J. Toledo employed the following technique in [3] to show that (1.1) has in some sense a unique solution: They introduced a certain p-Laplace operator A_p, and then use nonlinear semigroup theory to prove that the initial value problem

$$0 \in u'(t) + \mathcal{A}_p u(t), \text{ for a.e. } t \in (0, \infty), \ u(0) = v, \tag{1.2}$$

has a unique strong solution, where $\mathcal{A}_p$ is the closure of A_p. In fact, the general results from nonlinear semigroup theory used by them, are also stated in Section 2.1.

Now, let us give a brief outline of Chapter 3: To this end, let $\mathfrak{B}(S)$ denote the Borel σ-Algebra on S and let λ denote the n-dimensional Lebesgue measure. Moreover, for any $q \in [1, \infty]$, we denote by $L^q(S) := L^q(S; \mathfrak{B}(S), \lambda; \mathbb{R})$ the usual Lebesgue spaces. In addition, for any $v \in L^1(S)$, let $T_{\mathcal{A}_p}(\cdot)v : [0, \infty) \to L^1(S)$, be the unique strong solution of (1.2). In the sequel we refer to the family of mappings $(T_{\mathcal{A}_p}(t))_{t \geq 0}$ as p-Laplacian semigroup, or the semigroup associated to $\mathcal{A}_p$.
In Chapter 3, we will prove that $\lim\limits_{t \to \infty} T_{\mathcal{A}_p}(t)v = \overline{(v)}_S$ in $L^q(S)$, for all $v \in L^q(S)$ and $q \in [1, \infty)$, where $\overline{(v)}_S := \frac{1}{\lambda(S)} \int\limits_S v d\lambda$. Moreover, we will strengthen these results in many ways:

- We will prove an L^∞-L^p-contraction principle for "large" p and sufficiently integrable initials.
- We will derive that the solution extincts in finite time for "small" p and sufficiently integrable initials.
- We will derive further decay estimates of polynomial order.

Hereby, "large" means that p is in a sub-interval of $(2, \infty)$ and "small" that p is in a sub-interval of $(1, 2)$. The concrete shape of the sub-interval, might vary from one result to another and usually depends on the dimension n and (in some sense) on how close γ is to zero. Of course, throughout an actual theorem, the precise shape of the sub-interval is given.

Chapter 3 is the only part of this thesis which is not concerned with probability theory at all, but solely deals with the aforementioned asymptotic results. In all chapters subsequent to Chapter 3 we deal with two different ways of adding randomness to either a general (time-continuous, contractive) semigroup or specifically to the p-Laplacian semigroup. The former approach leads to ACPRM-Processes, where the p-Laplacian evolution equation is the most important example we consider, and the latter leads to the randomized p-Laplacian evolution equation. Consequently, this equation is the glue that holds the chapters of this thesis together and connects them to one another.

1.2 Abstract Cauchy Problems driven by a random Measure (ACPRM)

ACPRM-processes are the central object of Chapters 4, 5 and 6. Before rigorously defining them, let us give an intuitive explanation of what an ACPRM-Process is.
Imagine one has some physical phenomena that starts at a state v and is in the state $T(t)v$ after $t \in [0, \infty)$ units of time. Hereby, we will assume that v is an element of some (separable) Banach space $(V, ||\cdot||_V)$, and that $(T(t))_{t\geq 0}$, where $T(t) : V \to V$ for all $t \geq 0$, is a (time-continuous, contractive) semigroup on V. Now, one gets an ACPRM-process by disturbing this semigroup in the following way: After a random amount of time α_1 the semigroup is exposed to the random shock η_1, then after α_2 random units of time the semigroup is exposed to the random shock η_2, and so on. In fact, we also allow that the initial state $v \in V$ is no longer deterministic but might be random as well.
For instance, if the considered semigroup is our p-Laplacian semigroup, then the shocks $\eta_1, \eta_2, ...$ could model changes in vegetation (caused due to rain showers, or drought periods), occurring at the random times $\alpha_1, \alpha_2, ...$. This is an important generalization, since the weight function γ considered in (1.1) is neither time-dependent, nor random. Consequently, (1.1) does not allow for any changes in vegetation; particularly, it does not allow that these changes occur at random times and have random intensities.

To be able to better describe what we intend to do with ACPRM-Processes, let us formalize the above procedure: To this end, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(V, ||\cdot||_V)$ be a separable Banach space. In addition, let $(\beta_m)_{m\in\mathbb{N}}$, where $\beta_m : \Omega \to (0, \infty)$, be a sequence of real-valued random variables, and introduce $\alpha_m := \sum_{k=1}^{m} \beta_k$ for all $m \in \mathbb{N}$ as well as $\alpha_0 := 0$. Moreover, let $(\eta_m)_{m\in\mathbb{N}}$, where $\eta_m : \Omega \to V$, be a sequence of V-valued random variables. Furthermore, let $x : \Omega \to V$ be another V-valued random variable and let $(T(t))_{t\geq 0}$, where $T(t) : V \to V$ for all $t \geq 0$, be a time-continuous, contractive semigroup on V. Now, introduce the sequence $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ by $\mathbb{x}_{x,0} := x$ and

$$\mathbb{x}_{x,m} := T(\alpha_m - \alpha_{m-1})\mathbb{x}_{x,m-1} + \eta_m = T(\beta_m)\mathbb{x}_{x,m-1} + \eta_m, \ \forall m \in \mathbb{N}.$$

Finally, introduce the stochastic process $\mathbb{X}_x : [0, \infty) \times \Omega \to V$ by

$$\mathbb{X}_x(t) := \sum_{m=0}^{\infty} T((t - \alpha_m)_+)(\mathbb{x}_{x,m})\mathbb{1}_{[\alpha_m, \alpha_{m+1})}(t), \ \forall t \geq 0, \tag{1.3}$$

where $(\cdot)_+ := \max(\cdot, 0)$.
Particularly throughout running text, we may call a stochastic process which is given by (1.3) for some not closer specified $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ an ACPRM-process. Throughout mathematical results and/or when it is of importance what $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ we consider, we call $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ the sequence and $\mathbb{X}_x : [0, \infty) \times \Omega \to V$ the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V.

Note that the process $\mathbb{X}_x$ indeed models the aforementioned scenario: For a fixed $\omega \in \Omega$, we have

$\mathbb{X}_x(t,\omega) = T(t)x(\omega)$ for all $t \in [0, \alpha_1(\omega))$, then at $\alpha_1(\omega)$, we have $\mathbb{X}_x(\alpha_1(\omega),\omega) = T(\alpha_1(\omega))x(\omega) + \eta_1(\omega)$, and until time $\alpha_2(\omega)$, we get $\mathbb{X}_x(t,\omega) = T(t - \alpha_1(\omega))\Big(T(\alpha_1(\omega))x(\omega) + \eta_1(\omega)\Big)$ for all $t \in [\alpha_1(\omega), \alpha_2(\omega))$, and so on.

In this thesis, we intend to do two different things with ACPRM-Processes: We demonstrate that these processes may arise as a solution of a random differential equation (Chapter 4), and we study the asymptotic properties of such processes (Chapters 5 and 6).

Before outlining this in more detail, the following is worth emphasizing: The assumptions regarding $(T(t))_{t\geq 0}$, $(\beta_m)_{m\in\mathbb{N}}$, $(\eta_m)_{m\in\mathbb{N}}$ and x, differ from one of the Chapters 4, 5, 6 to another one. Particularly, in Chapter 4, the considered semigroup must be (in some sense) connected to a multi-valued operator, whereas the considered semigroup in Chapters 5 and 6 does not need to be connected to a multi-valued operator at all. Moreover, the results in Chapter 4 are of a functional analytic nature, whereas the results in Chapters 5 and 6 are more of a probability theoretic one.

Now, let us briefly address the results of Chapter 4: Roughly speaking, Chapter 4 deals with extending the classical existence/uniqueness theory for deterministic abstract Cauchy problems to those driven by a random measure; more precisely: An abstract Cauchy problem is an equation (or in fact an inclusion) of the form

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0,\infty), \tag{1.4}$$

where $\mathcal{A} : V \to 2^V$ is a multi-valued operator and 2^V is the power-set of V. We are going to extend this theory to random equations of the form

$$\eta(t,z)N_\Theta(dt \otimes z) \in dX(t) + \mathcal{A}X(t)dt, \tag{1.5}$$

where N_Θ is the random counting measure induced by a finite point process Θ, $\eta : (0,\infty) \times Z \times \Omega \to V$ is a (jointly measurable) drift function, $(Z, \mathcal{Z})$ is a measurable space and X is a V-valued stochastic process supposed to fulfill (1.5) in some sense. We call an equation of the form (1.5) abstract Cauchy problem driven by a random measure.

We will set up the notions of mild and strong solution for such an equation, will demonstrate that it has, under appropriate assumptions on the involved quantities, a mild/strong solution, and that such solutions are unique, in the sense that two processes which are solutions and are almost surely equal at $t = 0$, are indistinguishable of each other. Moreover, we will see that if there is a mild/strong solution of (1.5), then this solution is the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V, where: The sequence $(\beta_m)_{m\in\mathbb{N}}$ will depend on the point process Θ, the sequence $(\eta_m)_{m\in\mathbb{N}}$ will depend on Θ and the drift η, and: The semigroup $(T(t))_{t\geq 0}$ must be such that $T(\cdot)v$ is, for any $v \in V$, a mild solution of the (deterministic) abstract Cauchy problem (1.4). Rigorously stating all other assumptions on the involved quantities and outlining the results in any more detail is fairly technical, requires to introduce much more notations, and is therefore postponed to Chapter4.

In Chapters 5 and 6 we will study conditions guaranteeing that an ACPRM-process fulfills two of the most classical results in probability theory: The strong law of large numbers (SLLN) and the central limit theorem (CLT). That means we will derive conditions such that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F(\mathbb{X}_x(\tau))d\tau = \nu$$

with probability one and

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t F(\mathbb{X}_x(\tau))d\tau - t\nu \right) = Y,$$

in distribution, where: F belongs to some class of functionals, ν is an element of the image space of F and Y is a centered Gaussian random variable taking values in the image space of F.
In Chapter 5, we will obtain these results, by assuming (among some technical conditions) that the underlying semigroup $(T(t))_{t\geq 0}$ extincts in finite time. Of course, this will also require some assumptions regarding the underlying random variables. Moreover, we will be able to prove these results for vector-valued functionals F, which are in a certain sense sub-linear; in particular, we obtain the SLLN and the CLT for $\mathbb{X}_x$ itself. These results will be proven by exploiting classical results from the theory of random variables taking values in separable Banach spaces.
In Chapter 6, we will derive distributional conditions on the involved random variables ensuring that $\mathbb{X}_x$ is a time-homogeneous Markov process. Particularly, this does not require any asymptotic assumptions on the semigroup. Moreover, we exploit Markov process techniques to prove an SLLN and a CLT for real-valued, Lipschitz continuous functionals, if the underlying semigroup fulfills a polynomial decay assumption.

Moreover, each of the Chapters 4, 5 and 6 concludes with a section called "Examples", where we illustrate the results in each of these sections at hand of the p-Laplacian semigroup and at least one real-valued semigroup.

Finally, let us point out that there are some related works, for example [28] (and the references therein) which also deals with evolution inclusions exposed to noise, but there the considered noise has a more complicated structure, but the assumptions regarding Banach spaces are more restrictive and rely on the classical variational framework formulated in Gelfand-Triplets, whereas this thesis treats arbitrary separable Banach spaces and does not rely on a variational approach, but a classical semigroup approach.

1.3 The randomized weighted p-Laplacian evolution Equation

The purpose of Chapter 7 is two-fold: Firstly, we will extend the existence/uniqueness theory for the (deterministic) weighted p-Laplacian evolution equation considered in Chapter 3 to weight functions which are no longer deterministic, but random. Secondly, we will derive asymptotic results for the solution of this equation.

That means instead of the deterministic PDE (1.1), we would like to study the randomized PDE

$$\begin{cases} U'(t)(\omega) = \operatorname{div}\left(g(\omega)|\nabla U(t)(\omega)|^{p-2}\nabla U(t)(\omega)\right) \text{ on } S, \\ g(\omega)|\nabla U(t)(\omega)|^{p-2}\nabla U(t)(\omega)\cdot\Upsilon = 0 \text{ on } \partial S, \\ U(0)(\omega) = v(\omega), \end{cases} \tag{1.6}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and a.e. $t \in (0,\infty)$, where: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $p \in (1,\infty)\setminus\{2\}$, $S \subseteq \mathbb{R}^n$ is a sufficiently regular set, $n \in \mathbb{N}\setminus\{1\}$, $g : \Omega \to L^1(S)$ is a random weight function fulfilling some technical conditions, Υ is the unit outer normal on ∂S, and $v : \Omega \to L^1(S)$ is an integrable initial.

As mentioned, the deterministic problem (1.1) can be used to model the evolution of a fluvial landscape, in which case the weight function γ considered there, models a water depth occurring due to rain. In this case, it is reasonable to assume that this water depth is not precisely known, which motivates why we would like to replace it by a random quantity. This naturally leads to the randomized PDE (1.6), which does not just enable us to consider a random weight function, but as a side effect also a random initial.

The technique employed to demonstrate that (1.6) has a unique solution is as follows: We introduce a random p-Laplace operator A_p^r acting on the space of Bochner integrable functions $L^1(\Omega, L^1(S)) := L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(S))$, and then show that its closure, which will be denoted by $\mathcal{A}_p^r$, is densely defined and m-accretive. By the results form nonlinear semigroup theory introduced in Section 2.1, this will yield that

$$0 \in U'(t) + \mathcal{A}_p^r U(t), \text{ for a.e. } t \in (0,\infty), \ U(0) = v, \tag{1.7}$$

has a unique mild solution for any initial $v \in L^1(\Omega, ; L^1(S))$. So let, for any $v \in L^1(\Omega; L^1(S))$, $T_{\mathrm{ra}}(\cdot)v : [0,\infty) \to L^1(\Omega; L^1(S))$ denote the unique mild solution of (1.7). After the existence/uniqueness of mild solutions has been established, we proceed by deriving a useful relation between the deterministic p-Laplacian semigroup and $(T_{\mathrm{ra}}(t))_{t\geq 0}$. Then we exploit this relation to demonstrate that $T_{\mathrm{ra}}(\cdot)v$ is not only a mild solution of (1.7) but also a strong one.
Moreover, this connection also enables us to derive some basic asymptotic properties of this randomized semigroup. Finally, we will use differential inequality techniques to establish bounds for the tail function of $||T_{\mathrm{ra}}(t)v - \overline{(v)}||^2_{L^2(S)}$, where $v : \Omega \to L^1(S)$, with $v \in L^2(S)$ a.s., has to be (in some sense) sufficiently

integrable.

Chapter 2

Nonlinear Semigroup Theory and Preliminaries

2.1 Nonlinear Semigroup Theory

In this section, we give a brief introduction to nonlinear semigroup theory, with a focus on their connection to abstract Cauchy problems. Hereby, we do not intend to give a complete survey of this topic, but focus solely on the results needed in this thesis. In particular, we do not prove any new, so far unknown, results here.
The inquisitive reader is referred to [8] for a very comprehensive introduction to nonlinear semigroup theory, and to the appendix of [2] for a more concise one.

This section is structured as follows: At first, we are going to define what a semigroup is, then we will introduce the concepts of mild and strong solutions of abstract Cauchy problems, and discuss when such a problem has a unique mild/strong solution. Afterwards, we are going to dive into the concept of complete accretivity. This concept was originally introduced in [7], and is also treated in the appendix of [2].
Finally, at this section's very end, we give a general lemma, which we did not find stated and proven anywhere in the literature, even though it seems to be in common use.

Throughout this section $(V, ||\cdot||_V)$ denotes an arbitrary, real Banach space.

Definition 2.1.1. *A family of mappings $(T(t))_{t\geq 0}$, where $T(t) : V \to V$ is called a semigroup on V, if $T(0)v = v$ and $T(t+h)v = T(t)T(h)v$ for all $t, h \in [0,\infty)$ and $v \in V$. A semigroup $(T(t))_{t\geq 0}$ on V is called*

i) time-continuous, if $[0,\infty) \ni t \mapsto T(t)v$ is a continuous map for all $v \in V$;

ii) contractive, if $||T(t)v_1 - T(t)v_2||_V \leq ||v_1 - v_2||_V$ for all $t \in [0,\infty)$ and $v_1,\ v_2 \in V$; and *iii) jointly continuous, if* $[0,\infty) \times V \ni (t,v) \mapsto T(t)v$ *is a continuous map.*

Time-continuous, contractive semigroups arise naturally as solutions of abstract Cauchy problems. An abstract Cauchy problem is an equation of the form

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0,\infty), \tag{ACR}$$

where $\mathcal{A} : V \to 2^V$ is a mapping, 2^V is the power set of V and $u : [0,\infty) \to V$ is supposed to fulfill (ACR) in some sense. Before defining the different notions of solutions of (ACR), let us spend some words on multi-valued operators: A mapping $\mathcal{A} : V \to 2^V$ is called a multi-valued operator, or simply operator. Moreover, $D(\mathcal{A}) := \{v \in V : \mathcal{A}v \neq \emptyset\}$ is called its domain and we may write $\mathcal{A} : D(\mathcal{A}) \to 2^V$. In addition, $G(\mathcal{A}) := \{(v,\hat{v}) : \hat{v} \in \mathcal{A}v\}$ is the graph of v. Obviously an operator is uniquely determined by its graph. In the sequel, we identify an operator with its graph, and may simply write $(v,\hat{v}) \in \mathcal{A}$, instead of $v \in D(\mathcal{A})$ and $\hat{v} \in \mathcal{A}v$. Moreover, if $\mathcal{A} : D(\mathcal{A}) \to 2^V$ is an operator with graph $G(\mathcal{A})$, then the operator whose graph is $\overline{G(\mathcal{A})}$, is called the closure of $\mathcal{A}$. In addition, for any operator $\mathcal{A} : D(\mathcal{A}) \to 2^V$, $v \in D(\mathcal{A})$, $w \in V$ and $\alpha > 0$, we set $w + \alpha\mathcal{A}v := \{w + \alpha\hat{v} : \ \hat{v} \in \mathcal{A}v\}$ and introduce

$$R(Id + \alpha A) := \bigcup_{v \in D(\mathcal{A})} v + \alpha\mathcal{A}v.$$

Finally, we call $\mathcal{A}$ single-valued if $\mathcal{A}v$ contains precisely one element for each $v \in D(\mathcal{A})$ - in this case we identify $\mathcal{A}v$ with the only element it contains and write $\mathcal{A} : D(\mathcal{A}) \to V$.
Now, let us continue with the notions of mild and strong solution of (ACR).

Definition 2.1.2. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be an operator. A continuous function* $u : [0,\infty) \to V$ *is called a strong solution of (ACR), if:* $u|_{(0,\infty)} \in W^{1,1}_{Loc}((0,\infty);V)$,[1] $u(t) \in D(\mathcal{A})$ *for almost every* $t \in (0,\infty)$ *and* $0 \in u'(t) + \mathcal{A}u(t)$, *for a.e.* $t \in (0,\infty)$.

The term strong solution is an intuitive way to set up a solution of (ACR). The following definition of mild solution is more technical. In fact, we will never directly work with this definition, but all we need is that there is this notion, that mild solutions have certain properties and are under certain conditions even strong ones. Of course, the proofs of the general results which we use to achieve this heavily rely directly on the definition of mild solution; therefore, for the sake of completeness, this definition is now given.

Definition 2.1.3. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be an operator. A continuous function* $u : [0,\infty) \to V$ *is called a mild solution of (ACR), if the following holds: For every interval* $[a,b] \subseteq [0,\infty)$ *and every* $\varepsilon > 0$, *there is an* $n \in \mathbb{N}$, *real numbers* $t_0, ..., t_n \in [0,\infty)$, *vectors* $f_1, ..., f_n \in V$ *and a function* $\tilde{u} : [t_0, t_n] \to V$ *such that*

i) $0 \le t_0 - a < \varepsilon$, $0 \le b - t_n < \varepsilon$ *and* $0 < t_i - t_{i-1} < \varepsilon$ *for all* $i = 1, ..., n$,

ii) $\sum_{i=1}^{n} (t_i - t_{i-1})||f_i||_V < \varepsilon$,

[1] As usually, $W^{1,1}_{\mathrm{Loc}}((0,\infty);V) := \{f : (0,\infty) \to V : \ f \text{ is loc. absolutely continuous and differentiable a.e.}\}$

iii) $||u(t) - \tilde{u}(t)||_V < \varepsilon$ *for all* $t \in [t_0, t_n]$,

iv) $\tilde{u}$ *is constant on* $(t_{i-1}, t_i]$ *for all* $i = 1, ..., n$, *and*

v) $\tilde{u}(t_i) \in D(\mathcal{A})$ *and* $f_i \in \frac{\tilde{u}(t_i) - \tilde{u}(t_{i-1})}{t_i - t_{i-1}} + \mathcal{A}\tilde{u}(t_i)$ *for all* $i = 1, ..., n$.

Definition 2.1.4. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be an operator and* $v \in V$. *A continuous function* $u : [0, \infty) \to V$ *is called a mild (strong, resp.) solution of the initial value problem*

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0, \infty), \ u(0) = v,$$

if u *is a mild (strong, resp.) solution of (ACR) and* $u(0) = v$.

Remark 2.1.5. *Strong solutions are mild ones, i.e.: Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be an operator and let* $u : [0, \infty) \to V$ *be continuous. Then, if* u *is a strong solution of (ACR) it is also a mild one, see [8, Theorem 1.4].*

Definition 2.1.6. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be an operator. Then* $\mathcal{A}$ *is called*

i) accretive, if $||v_1 - v_2 + \alpha(\hat{v}_1 - \hat{v}_2)||_V \geq ||v_1 - v_2||_V$ *for all* $(v_1, \hat{v}_1), \ (v_2, \hat{v}_2) \in \mathcal{A}$, *and* $\alpha > 0$;

ii) m-accretive, if it is accretive and $R(Id + \alpha\mathcal{A}) = V$, *for all* $\alpha > 0$; *and*

iii) densely defined, if $\overline{D(\mathcal{A})} = V$.

Theorem 2.1.7. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be m-accretive and densely defined; moreover, let* $v \in V$. *Then the initial initial value problem*

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0, \infty), \ u(0) = v, \tag{2.1}$$

has precisely one mild solution. Now, denote for each $v \in V$ *by* $T_{\mathcal{A}}(\cdot)v : [0, \infty) \to V$ *the uniquely determined mild solution of (2.1). Then,* $(T_{\mathcal{A}}(t))_{t \geq 0}$ *is a jointly continuous, contractive semigroup on* V.

Proof. See [8, Proposition 3.7] for the existence and uniqueness; [8, Theorem 3.10] for the contractivity and joint continuity of $(T_{\mathcal{A}}(t))_{t \geq 0}$, and [8, Theorem 1.10] for the fact that $(T_{\mathcal{A}}(t))_{t \geq 0}$ is a semigroup on V. □

Definition 2.1.8. *Let the assumptions and notations of Theorem 2.1.7 prevail. Then,* $(T_{\mathcal{A}}(t))_{t \geq 0}$ *is called the semigroup associated to* $\mathcal{A}$.

Remark 2.1.9. *Let* $\mathcal{A} : D(\mathcal{A}) \to 2^V$ *be accretive and* $\alpha > 0$. *Then it is clear that there is for every* $w \in R(Id + \alpha\mathcal{A})$ *precisely one pair* $(v, \hat{v}) \in \mathcal{A}$ *such that* $w = v + \alpha\hat{v}$. *Consequently, one can introduce* $(Id + \alpha\mathcal{A})^{-1} : R(Id + \alpha\mathcal{A}) \to V$, *where* $(Id + \alpha\mathcal{A})^{-1}w$ *is precisely the element* $v \in D(\mathcal{A})$, *such that there is an* $\hat{v} \in \mathcal{A}v$ *with* $w = v + \alpha\hat{v}$.
The mapping $(Id + \alpha\mathcal{A})^{-1}$ *is called the resolvent of* $\mathcal{A}$.

Theorem 2.1.10. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^V$ be m-accretive and densely defined. Moreover, introduce $(Id+\alpha\mathcal{A})^{-m} : V \to V$ recursively by $(Id+\alpha\mathcal{A})^{-m} := (Id+\alpha\mathcal{A})^{-1}(Id+\alpha\mathcal{A})^{-(m-1)}$ for all $m \in \mathbb{N} \setminus \{1\}$ and $\alpha > 0$. Then we have*

$$\lim_{m\to\infty} \left(Id + \frac{t}{m}\mathcal{A}\right)^{-m} v = T_{\mathcal{A}}(t)v, \ \forall t > 0, \ v \in V,$$

where $(T_{\mathcal{A}}(t))_{t\geq 0}$ is the semigroup associated to $\mathcal{A}$.

Proof. See [8, Theorem 4.2] □

Thanks to Theorem 2.1.7, proving the existence/uniqueness of mild solutions boils down to verifying that the operator at hand is m-accretive and densely defined. Moreover, under the same assumptions as in Theorem 2.1.7, we get the representation formula stated in Theorem 2.1.10.
Even though it might be a challenging task to verify that a given operator is m-accretive and densely defined, Theorem 2.1.7 is frequently used to show the existence/uniqueness of mild solutions.
Now, let us turn our focus to criteria guaranteeing that mild solutions are strong ones.

Definition 2.1.11. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^V$ be densely defined and m-accretive. Moreover, let $(T_{\mathcal{A}}(t))_{t\geq 0}$ denote the semigroup associated to $\mathcal{A}$. $(T_{\mathcal{A}}(t))_{t\geq 0}$ is called domain invariant, if $T_{\mathcal{A}}(t)v \in D(\mathcal{A})$ for all $t > 0$ and $v \in V$. Moreover, $(T_{\mathcal{A}}(t))_{t\geq 0}$ is said to admit an infinitesimal generator, if there is an operator $\mathcal{A}^\circ : V \to V$ such that*

$$-\lim_{h\searrow 0} \frac{T_{\mathcal{A}}(h)v - v}{h} = \mathcal{A}^\circ v \in \mathcal{A}v, \tag{2.2}$$

for all $v \in D(\mathcal{A})$ and $\mathcal{A}^\circ v = 0$ for all $v \in V \setminus D(\mathcal{A})$. In this case, we call $\mathcal{A}^\circ$ the infinitesimal generator of $T_{\mathcal{A}}$.[2]

Theorem 2.1.12. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^V$ be densely defined and m-accretive, and let $(T_{\mathcal{A}}(t))_{t\geq 0}$ be the semigroup associated to $\mathcal{A}$. Moreover, introduce $v \in V$, and assume that $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)}$ is locally Lipschitz continuous and almost everywhere right differentiable. Then, $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)} \in W^{1,1}_{Loc}((0,\infty);V)$ and $T_{\mathcal{A}}(\cdot)v$ is the uniquely determined strong solution of*

$$0 \in u'(t) + \mathcal{A}u(t), \ \textit{for a.e. } t \in (0,\infty), \ u(0) = v.$$

Proof. Lemma 2.1.19[3] yields that $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)}$ is differentiable almost every. Thus, as locally Lipschitz continuous functions are also locally absolutely continuous, we get $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)} \in W^{1,1}_{\mathrm{Loc}}((0,\infty);V)$. Consequently, [8, Theorem 7.1] yields the existence of a strong solution, and the uniqueness follows by combining Theorem 2.1.7 and Remark 2.1.5. □

[2] In the nonlinear setting, the existence of the limit in (2.2) is indeed an assumption and not necessarily true. Moreover, it is more common to set $\mathcal{A}^\circ v := \emptyset$ for $v \in V \setminus D(\mathcal{A})$, which clearly yields $D(\mathcal{A}) = D(\mathcal{A}^\circ)$; but in our case, setting it to zero on $V \setminus D(\mathcal{A})$ is more convenient.

[3] This is a general result which is probably available in the literature. To not disturb the flow of reading, this result is stated at this section's end. It is independent of any other result in this section.

Lemma 2.1.13. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^V$ be densely defined and m-accretive. Moreover, introduce $v \in D(\mathcal{A})$, $b > 0$ and let $(T_{\mathcal{A}}(t))_{t\geq 0}$ denote the semigroup associated to $\mathcal{A}$. Then $(T_{\mathcal{A}}(\cdot)v)|_{[0,b]}$ is Lipschitz continuous.*

Proof. See [8, Lemma 7.8]. $\square$

Theorem 2.1.14. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^V$ be densely defined and m-accretive, and let $(T_{\mathcal{A}}(t))_{t\geq 0}$ denote the semigroup associated to $\mathcal{A}$. Moreover, assume that $T_{\mathcal{A}}$ is domain invariant and admits an infinitesimal generator $\mathcal{A}^\circ : V \to V$. Finally, introduce $v \in V$. Then, $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)}$ is locally Lipschitz continuous and everywhere right differentiable, with*

$$\lim_{h\searrow 0} \frac{T_{\mathcal{A}}(t+h)v - T_{\mathcal{A}}(t)v}{h} = -\mathcal{A}^\circ T_{\mathcal{A}}(t)v,$$

for all $t > 0$. Consequently, $(T_{\mathcal{A}}(\cdot)v)|_{(0,\infty)} \in W^{1,1}_{Loc}((0,\infty); V)$, $T'_{\mathcal{A}}(t)v = -\mathcal{A}^\circ T_{\mathcal{A}}(t)v$ for a.e. $t \in (0,\infty)$, and $T_{\mathcal{A}}(\cdot)v$ is the uniquely determined strong solution of

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0,\infty), \; u(0) = v.$$

Proof. Let $w \in D(\mathcal{A})$. Then Lemma 2.1.13 yields that $[a,b] \ni t \mapsto T_{\mathcal{A}}(t-a)w$ is Lipschitz continuous for any $[a,b] \subseteq (0,\infty)$. Thus, choosing $w = T_{\mathcal{A}}(a)v$, and employing the semigroup property as well as the domain invariance, gives that $[a,b] \ni t \mapsto T_{\mathcal{A}}(t)v$ is Lipschitz continuous. Moreover, using the domain invariance again, yields

$$\lim_{h\searrow 0} \frac{T_{\mathcal{A}}(t+h)v - T_{\mathcal{A}}(t)v}{h} = \lim_{h\searrow 0} \frac{T_{\mathcal{A}}(h)T_{\mathcal{A}}(t)v - T_{\mathcal{A}}(t)v}{h} = -\mathcal{A}^\circ T_{\mathcal{A}}(t)v.$$

Now, the remaining claims follow from Theorem 2.1.12. $\square$

Definition 2.1.15. *Let $(T(t))_{t\geq 0}$ be a semigroup on V. Moreover, assume that $\hat{V} \subseteq V$ is a subspace. Then, $\hat{V}$ is called an invariant space w.r.t. $(T(t))_{t\geq 0}$, if $T(t)\hat{v} \in \hat{V}$ for all $t \geq 0$, and $\hat{v} \in \hat{V}$.*

Now, let us turn to the concept of complete accretivity, which is a powerful tool to establish that the semigroup associated to an operator is domain invariant and admits an infinitesimal generator.

Throughout the remainder of this section (K, Σ, μ) denotes a finite measure space and $\mathfrak{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Moreover, for any $q \in [1,\infty]$, $L^q(K,\Sigma,\mu)$ denotes the usual Lebesgue space of (μ-equivalence classes of) functions $v : K \to \mathbb{R}$, which are Σ-$\mathfrak{B}(\mathbb{R})$-measurable and fulfill $\int\limits_K |v|^q d\mu < \infty$, if $q \neq \infty$, and are μ-essentially bounded, if $q = \infty$. As usually, $||\cdot||_{L^q(K,\Sigma,\mu)}$ denotes the standard L^q-norm on $L^q(K,\Sigma,\mu)$.
Moreover, we introduce $J_0 := \{j : \mathbb{R} \to [0,\infty] : \; j \text{ is lower semicontinuous and convex}, \; j(0) = 0\}$. Furthermore, for any $v_1, v_2 \in L^1(K,\Sigma,\mu)$, we write $v_1 << v_2$ whenever

$$\int\limits_K j \circ v_1 d\mu \leq \int\limits_K j \circ v_2 d\mu, \; \forall j \in J_0.$$

Remark 2.1.16. *Let $q \in [1,\infty]$, $v_1, v_2 \in L^q(K,\Sigma,\mu)$ and assume that $v_1 << v_2$. Then, we have $||v_1||_{L^q(K,\Sigma,\mu)} \leq ||v_2||_{L^q(K,\Sigma,\mu)}$, since: If $q \neq \infty$ this is an immediate consequence of $|\cdot|^q \in J_0$; and if $q = \infty$, one verifies this by exploiting that $\mathbb{R} \ni x \mapsto \max(|x| - ||v_2||_{L^\infty(K,\Sigma,\mu)}, 0)$ is an element of J_0.*

Definition 2.1.17. *An operator $\mathcal{A} : D(\mathcal{A}) \to 2^{L^1(K,\Sigma,\mu)}$ is called completely accretive, if*

$$v_1 - v_2 << v_1 - v_2 + \alpha(\hat{v}_1 - \hat{v}_2)$$

for all $(v_1, \hat{v}_1), (v_2, \hat{v}_2) \in \mathcal{A}$ and $\alpha \in (0,\infty)$.

As the absolute value function is obviously an element of J_0, a completely accretive operator is also accretive. Moreover, we have the following striking result:

Theorem 2.1.18. *Let $\mathcal{A} : D(\mathcal{A}) \to 2^{L^1(K,\Sigma,\mu)}$ be completely-accretive, m-accretive and densely defined. In addition, assume that $\mathcal{A}$ is positively homogeneous of degree $m \in (0,\infty)\backslash\{1\}$, that is $\mathcal{A}(\alpha v) = \alpha^m \mathcal{A}(v)$ for all $v \in D(\mathcal{A})$ and $\alpha \geq 0$. Finally, let $(T_{\mathcal{A}}(t))_{t\geq 0}$ denote the semigroup associated to $\mathcal{A}$. Then $T_{\mathcal{A}}$ is domain invariant and admits an infinitesimal generator $\mathcal{A}^\circ : L^1(K,\Sigma,\mu) \to L^1(K,\Sigma,\mu)$. Consequently, for any $v \in L^1(K,\Sigma,\mu)$, $T_{\mathcal{A}}(\cdot)v$ is the uniquely determined strong solution of*

$$0 \in u'(t) + \mathcal{A}u(t), \text{ for a.e. } t \in (0,\infty), \ u(0) = v. \tag{2.3}$$

Moreover, the following assertions hold.

i) $T_{\mathcal{A}}(t)v_1 - T_{\mathcal{A}}(t)v_2 << v_1 - v_2$ for all $v_1, v_2 \in L^1(K,\Sigma,\mu)$ and $t \geq 0$.

ii) $T_{\mathcal{A}}(t)v << v$ for all $v \in L^1(K,\Sigma,\mu)$ and $t \geq 0$.

iii) For any $q \in [1,\infty]$, $L^q(K,\Sigma,\mu)$ is an invariant subspace w.r.t. $T_{\mathcal{A}}$.

iv) $|\mathcal{A}^\circ T_{\mathcal{A}}(t)v| \leq 2\frac{|v|}{|m-1|}\frac{1}{t}$, μ-a.e. on K, for all $v \in L^1(K,\Sigma,\mu)$ and $t > 0$.

v) $\frac{1}{h}(T_{\mathcal{A}}(t+h)v - T_{\mathcal{A}}(t)v) << \mathcal{A}^\circ T_{\mathcal{A}}(t)v$, for all $t, h > 0$ and $v \in V$.

Proof. The existence of an infinitesimal generator follows from [7, Theorem 4.2], and the domain invariance as well as iv) follow from [7, Theorem 4.4]. Moreover, v) follows from the domain invariance and [7, Theorem 4.2]. Furthermore, Theorem 2.1.14 yields that $T_{\mathcal{A}}(\cdot)v$ is the unique strong solution of (2.3). In addition, i) follows from [7, Proposition 4.1]. Moreover, the homogeneity of $\mathcal{A}$ yields $\mathcal{A}(0) = 0$, and thus $T_{\mathcal{A}}(t)(0) = 0$ for all $t \geq 0$. In light of this, it is clear that i) implies ii). Finally, combining ii) and Remark 2.1.16 yields iii). $\square$

The preceding result is a powerful tool to verify the existence of unique strong solutions. The price we had to pay, is that we had to restrict ourselves to $L^1(K,\Sigma,\mu)$. In fact, the above results can be generalized to so-called "normal Banach spaces", but are in this case more technical to formulate - The reader is referred to [7] for a comprehensive treatment of the general case.

Now, this section concludes with the following general result which we could not find in the literature. The result is independent of any other result in this section, and was used in the proof of Theorem 2.1.12.

Lemma 2.1.19. *Let $[a,b] \subseteq \mathbb{R}$ be an interval and let $f:[a,b] \to V$ be Lipschitz continuous and right differentiable almost everywhere. Then f is differentiable almost everywhere.*

Proof. Firstly, introduce $L^1([a,b];V)$ as the space of functions $\varphi:[a,b] \to V$ which are strongly measurable and fulfill $\int_{[a,b]} ||\varphi(t)||_V dt < \infty$. Note that $(V, ||\cdot||_V)$ is not necessarily separable, thus the notions of Borel measurable, and strongly measurable do not necessarily agree. This proof is the only time in this thesis, where we have to deal with integrals of functions which take values in Banach spaces that are not necessarily separable. For any $\varphi \in L^1([a,b];V)$, $\int_{[a,b]} \varphi(t)dt$ denotes its Bochner integral, see [5] for an introduction to strong measurability, and Bochner integrals in the non-separable setting.
Now, let $f:[a,b] \to V$ be Lipschitz continuous and right differentiable almost everywhere. Then f is a fortiori continuous and thus $f \in L^1([a,b];V)$.
Now introduce $f_r' : [a,b] \to V$, as the almost everywhere existing right derivative of f. Moreover, let L denote the Lipschitz constant of f.
Then we have by construction that

$$\lim_{h \searrow 0} \left|\left| \frac{f(t+h)-f(t)}{h} - f_r'(t) \right|\right|_V = 0,$$

for almost every $t \in [a,b]$. Thus f_r' is strongly measurable. Moreover, thanks to the Lipschitz continuity of f and Fatou's Lemma, we obtain

$$\int_{[a,b]} ||f_r'(t)||_V dt \leq \liminf_{h \searrow 0} \int_{[a,b]} \frac{1}{h} ||f(t+h)-f(t)||_V dt \leq L(b-a) < \infty.$$

Ergo, we get $f_r' \in L^1([a,b];V)$.
Now introduce $f_* : [a,b] \to V$, by

$$f_*(t) := \int_{[a,t]} f_r'(z)dz + f(a), \ \forall t \in [a,b].$$

Then the fundamental theorem of calculus (for Bochner integrals) yields that f_* is differentiable almost everywhere and that $f_*'(t) = f_r'(t)$ for a.e. $t \in [a,b]$, see [5, Prop. 1.2.2].
Consequently, the claim follows if $f(t) = f_*(t)$ for every $t \in [a,b]$.
To prove this, introduce $\Gamma : [a,b] \to \mathbb{R}$ by

$$\Gamma(t) := ||f(t) - f_*(t)||_V, \ \forall t \in [a,b].$$

Firstly, note that obviously $\Gamma(a) = 0$. Moreover, we have

$$\begin{aligned} & \lim_{h \searrow 0} \left| \frac{\Gamma(t+h)-\Gamma(t)}{h} \right| \\ \leq \quad & \lim_{h \searrow 0} \left(\left|\left| \frac{f(t+h)-f(t)}{h} - f_r'(t) \right|\right|_V + \left|\left| \frac{-f_*(t+h)+f_*(t)}{h} + f_r'(t) \right|\right|_V \right) \end{aligned}$$

$$= \quad 0,$$

for almost every $t \in [a,b]$, i.e. Γ is almost everywhere right differentiable and the right derivative is equal to zero.
In addition, it is also easily verified that Γ is Lipschitz continuous, which implies, as $\mathbb{R}$ has the Radon-Nikodym property, that it is differentiable almost everywhere. Since the right derivate of Γ is zero almost everywhere, the almost everywhere derivative is also zero a.e. Finally, the Lipschitz continuity of Γ yields that Γ is constant, and hence $\Gamma(t) = 0$ for all $t \in [a,b]$. □

2.2 General Results about ACPRM-Processes

In this section, we introduce ACPRM-processes and establish some of their basic properties needed in Chapters 4, 5 and 6. Moreover, we set up some general notations needed there and define two real-valued semigroups which serve as examples in the chapters dealing with ACPRM-processes.

Throughout everything which follows $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. Moreover, throughout this section $(V, ||\cdot||_V)$ denotes a separable (real) Banach space.

Remark 2.2.1. *Whenever (K, Σ) is a measurable space and (M, τ) is a topological space, we set*

$$\mathcal{M}(K,\Sigma;M) := \{f : K \to M : \ f \text{ is } \Sigma - \mathfrak{B}(M) - \text{measurable}\},$$

where $\mathfrak{B}(M)$ denotes the Borel σ-Algebra of M. Furthermore, we introduce the shortcut notation $\mathcal{M}(\Omega;V) := \mathcal{M}(\Omega,\mathcal{F};V)$. As usually, we may refer to the elements of $\mathcal{M}(\Omega;V)$ as V-valued random variables, and if $V = \mathbb{R}$ we refer to them as real-valued random variables.

Definition 2.2.2. *Let $(\beta_m)_{m\in\mathbb{N}}$, where $\beta_m : \Omega \to (0,\infty)$, be a sequence of real-valued random variables. Moreover, let $(\eta_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega;V)$, introduce $\alpha_m := \sum\limits_{k=1}^{m} \beta_k$ for all $m \in \mathbb{N}$ and set $\alpha_0 := 0$. Finally, let $x \in \mathcal{M}(\Omega;V)$ and let $(T(t))_{t\geq 0}$ be a time-continuous, contractive semigroup on V. Then the sequence $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ defined by $\mathbb{x}_{x,0} := x$ and*

$$\mathbb{x}_{x,m} := T(\alpha_m - \alpha_{m-1})\mathbb{x}_{x,m-1} + \eta_m = T(\beta_m)\mathbb{x}_{x,m-1} + \eta_m, \ \forall m \in \mathbb{N},$$

is called the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V. Moreover, the stochastic process $\mathbb{X}_x : [0,\infty) \times \Omega \to V$ defined by

$$\mathbb{X}_x(t) := \sum_{m=0}^{\infty} T((t-\alpha_m)_+)(\mathbb{x}_{x,m})\mathbb{1}_{[\alpha_m,\alpha_{m+1})}(t), \ \forall t \geq 0, \tag{2.4}$$

is called the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V, where $(\cdot)_+ := \max(\cdot, 0)$.

Remark 2.2.3. *Note that if $(\alpha_m)_{m\in\mathbb{N}_0}$ is as in the previous definition, then $(\alpha_m(\omega))_{m\in\mathbb{N}_0}$ is, for every*

$\omega \in \Omega$, a strictly increasing sequence. Thus, the right-hand-side series in (2.4) consists at most out of one non-zero summand which ensures that $\mathbb{X}_x$ is well-defined.

Remark 2.2.4. *Particularly throughout running text, we may simply call a stochastic process which is given by (2.4) for some not closer specified $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ an ACPRM-process*

Remark 2.2.5. *Let $(T(t))_{t\geq 0}$ be a time-continuous and contractive semigroup on V. Then it is easily verified that T is also jointly continuous. Consequently, this map is a fortiori $\mathfrak{B}([0,\infty)\times V)$-$\mathfrak{B}(V)$-measurable. Moreover, by separability we have $\mathfrak{B}([0,\infty)\times V) = \mathfrak{B}([0,\infty))\otimes\mathfrak{B}(V)$, see [10, page 244]; which gives that this map is $\mathfrak{B}([0,\infty))\otimes\mathfrak{B}(V)$-$\mathfrak{B}(V)$-measurable.*

Lemma 2.2.6. *Let $(\beta_m)_{m\in\mathbb{N}}$, where $\beta_m : \Omega \to (0,\infty)$, be a sequence of real-valued random variables. Moreover, let $(\eta_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ and $x \in \mathcal{M}(\Omega; V)$. In addition, introduce $\alpha_m := \sum_{k=1}^{m} \beta_k$ for all $m \in \mathbb{N}$ and set $\alpha_0 := 0$. Finally, let $\mathbb{X}_x$ be the process and $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ be the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V; where $(T(t))_{t\geq 0}$ is a time-continuous, contractive semigroup on V. Then the following assertions hold.*

i) $\mathbb{X}_x(0) = x$.

ii) $\mathbb{x}_{x,m}$ *is $\mathcal{F}$-$\mathfrak{B}(V)$-measurable for each $m \in \mathbb{N}_0$.*

iii) $(\mathbb{X}_x(t))_{t\geq 0}$ *is a stochastic process, i.e. each $\mathbb{X}_x(t)$ is $\mathcal{F}$-$\mathfrak{B}(V)$-measurable.*

iv) The mapping $[0,\infty) \ni t \mapsto \mathbb{X}_x(t,\omega)$ is right continuous for each $\omega \in \Omega$.

v) $\mathbb{X}_x$ *is $\mathfrak{B}([0,\infty))\otimes\mathcal{F}$-$\mathfrak{B}(V)$-measurable.*

vi) If in addition $\mathbb{P}(\sup_{m\in\mathbb{N}} \alpha_m = \infty) = 1$, then the stochastic process $(\mathbb{X}_x(t))_{t\geq 0}$ has almost surely càdlàg paths.

vii) If $\hat{V} \subseteq V$ is a subspace which is invariant w.r.t. $(T(t))_{t\geq 0}$, and $x, \eta_m \in \hat{V}$ for all $m \in \mathbb{N}$ a.s., then $\mathbb{x}_{x,m} \in \hat{V}$ for all $m \in \mathbb{N}_0$ a.s. and $\mathbb{X}_x(t) \in \hat{V}$ for all $t \geq 0$ with probability one.

Proof. The first assertion is trivial. Moreover, ii) is easily verified inductively: As $\mathbb{x}_{x,0} = x$, ii) holds if $m = 0$, and if it holds for an $m \in \mathbb{N}$, Remark 2.2.5 enables us to conclude that $T(\beta_{m+1})\mathbb{x}_{x,m}$ is $\mathcal{F}$-$\mathfrak{B}(V)$-measurable, which yields that $\mathbb{x}_{x,m+1} = T(\beta_{m+1})\mathbb{x}_{x,m} + \eta_{m+1}$ is $\mathcal{F}$-$\mathfrak{B}(V)$-measurable.
One now easily deduces the third assertion from the second and Remark 2.2.5.
Now, let us verify the fourth: Let $\omega \in \Omega$ and $t \in [0,\infty)$ be given. If, $t \geq \sup_{m\in\mathbb{N}} \alpha_m(\omega)$, then the same holds true for $t+h$, for any $h \geq 0$. Thus, we get $\mathbb{X}_v(t+h,\omega) = \mathbb{X}_v(t,\omega) = 0$. Moreover, if $t < \sup_{m\in\mathbb{N}} \alpha_m(\omega)$, then there is precisely one $m \in \mathbb{N}_0$ such that $t \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$. Now, for each $h \geq 0$ sufficiently small we also have $t + h \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$ which yields by the time continuity of T that

$$\lim_{h\searrow 0} \mathbb{X}_x(t+h,\omega) - \mathbb{X}_x(t,\omega) = T(t+h-\alpha_m(\omega))\mathbb{x}_{x,m} - T(t-\alpha_m(\omega))\mathbb{x}_{x,m} = 0$$

and iv) follows.
Furthermore, v) follows from iii), iv) and [31, Prop. 2.2.3.2].
Proof of vi). Invoking iv) yields that it remains to prove that $(0,\infty) \ni t \mapsto \mathbb{X}_x(t,\omega)$ has left limits for each $\omega \in \tilde{\Omega} := \{\omega \in \Omega : \sup\limits_{m\in\mathbb{N}} \alpha_m(\omega) = \infty\}$, which is by assumption a set of full $\mathbb{P}$-measure.
So fix $t > 0$ and $\omega \in \tilde{\Omega}$ and note that there is precisely one $m \in \mathbb{N}_0$ such that $t \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$.
If $t \in (\alpha_m(\omega), \alpha_{m+1}(\omega))$, it follows analogously to the proof of iv) that $\lim\limits_{h\searrow 0} \mathbb{X}_x(t-h,\omega) - \mathbb{X}_x(t,\omega) = 0$, and if $t = \alpha_m(\omega)$, we get by the time-continuity of T that

$$\lim_{h\searrow 0} \mathbb{X}_x(t-h,\omega) = T(\alpha_m(\omega) - h - \alpha_{m-1}(\omega))\mathbb{x}_{x,m-1}(\omega) = T(\beta_m(\omega))\mathbb{x}_{x,m-1}(\omega),$$

which completes the proof of vi).
Finally, let us prove vii). That $\mathbb{x}_{x,m} \in \hat{V}$ for all $m \in \mathbb{N}_0$ a.s. is easily verified by induction: If $m = 0$, we have $\mathbb{x}_{x,m} = x \in \hat{V}$ a.s., and if $\mathbb{x}_{x,m} \in \hat{V}$ a.s. for an $m \in \mathbb{N}_0$, it follows from the invariance of $\hat{V}$ w.r.t. T that $T(\beta_{m+1})\mathbb{x}_{x,m} \in \hat{V}$ almost surely. Ergo, $\mathbb{x}_{x,m+1} = T(\beta_{m+1})\mathbb{x}_{x,m} + \eta_{m+1} \in \hat{V}$ with probability one, since $\hat{V}$ is a vector space. Thus, $\mathbb{x}_{x,m} \in \hat{V}$ for all $m \in \mathbb{N}_0$ with probability one. Finally, using this, the invariance of $\hat{V}$ w.r.t. T and $0 \in \hat{V}$, yields the remaining claim. □

Now let us introduce the two real-valued semigroups which will serve as examples in the chapters dealing with ACPRM-processes.

Remark 2.2.7. *Let $\rho_1 \in (0,1)$ and $\rho_2 \in (0,\infty)$ be given and consider the families of mappings $(T_{\rho_i}(t))_{t\geq 0}$, with $T_{\rho_i}(t) : \mathbb{R} \to \mathbb{R}$ for all $t \in [0,\infty)$, $i = 1,2$, defined by*

i) $T_{\rho_1}(t)v := sgn(v)\,(-t + |v|^{\rho_1})_+^{\frac{1}{\rho_1}}$ for all $v \in \mathbb{R}$ and $t \in [0,\infty)$, where $(\cdot)_+ := \max(\cdot, 0)$, and

ii) $T_{\rho_2}(t)v := sgn(v)\left(t + |v|^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$ for all $v \in \mathbb{R}$ and $t \in [0,\infty)$. (If $v = 0$, set $(t + |v|^{-\frac{1}{\rho_2}})^{-\rho_2} := 0$ which is reasonable, since: For any $x \in [0,\infty)$ the mapping $(0,\infty) \ni y \mapsto \left(x + y^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$ can be extended continuously by zero in $y = 0$.)

The families of mappings $(T_{\rho_1}(t))_{t\geq 0}$ and $(T_{\rho_2}(t))_{t\geq 0}$ are time-continuous, contractive semigroups on $\mathbb{R}$. As a warm up, let us verify that $(T_{\rho_1}(t))_{t\geq 0}$ has in fact these properties. (For $(T_{\rho_2}(t))_{t\geq 0}$ this works analogously and will be omitted.)
Firstly, for $v \in \mathbb{R}$ and $t, h \geq 0$ we get

$$T_{\rho_1}(t)(T_{\rho_1}(h)v) = sgn(T_{\rho_1}(h)v)\,(-t + |T_{\rho_1}(h)v|^{\rho_1})_+^{\frac{1}{\rho_1}} = sgn(v)\,(-t - h + |v|^{\rho_1})_+^{\frac{1}{\rho_1}} = T_{\rho_1}(t+h)v,$$

and it is trivial that $T_{\rho_1}(0)v = v$. Thus, $(T_{\rho_1}(t))_{t\geq 0}$ is a semigroup, and it is plain that it is time continuous. Now, fix $v, \hat{v} \in \mathbb{R}$ and introduce $f : [0,\infty) \to [0,\infty)$ by $f(t) := (T_{\rho_1}(t)v - T_{\rho_1}(t)\hat{v})^2$ for all $t \geq 0$, then f is continuously differentiable on $[0,\infty)$, and one verifies (by differentiating) that f is decreasing on $[0,\infty)$. Thus we get

$$|T_{\rho_1}(t)v - T_{\rho_1}(t)\hat{v}| = \sqrt{f(t)} \leq \sqrt{f(0)} = |v - \hat{v}|, \ \forall t \geq 0,$$

which yields the desired contractivity.

Now, let us conclude this section with some remarks clarifying our conventions regarding L^q-spaces of (vector-valued) functions, σ-algebras generated by (vector-valued) random variables, and laws of (vector-valued) random variables:

Remark 2.2.8. *Let (K, Σ, μ) denote a σ-finite measure space, then $L^q(K, \Sigma, \mu; V)$ denotes, for any $q \in [1, \infty)$, the set of all (equivalence classes of) functions $f \in \mathcal{M}(K, \Sigma; V)$, such that*

$$\int\limits_K ||f||_V^q d\mu < \infty.$$

For any $f \in L^q(K, \Sigma, \mu; V)$, the integral $\int\limits_K f d\mu$ is understood as a Bochner integral; for an introduction to Bochner integrability, see [31, Section 2.1]. Particularly, note that the separability of V yields that the notions of measurability, weak measurability and strong measurability agree, see [31, p. 6]
As usually, if $V = \mathbb{R}$ we may simply write $L^q(K, \Sigma, \mu)$, and define $L^\infty(K, \Sigma, \mu)$ as the space of equivalence classes of μ-essentially bounded Σ-$\mathfrak{B}(\mathbb{R})$-measurable functions.
Finally, we introduce the short-cut notations

$$L^q(\Omega; V) := L^q(\Omega, \mathcal{F}, \mathbb{P}; V) \text{ and } L^q(\Omega) := L^q(\Omega, \mathcal{F}, \mathbb{P}).$$

Remark 2.2.9. *Let I be an index-set. Moreover, introduce for each $i \in I$ a separable Banach space $(V_i, || \cdot ||_{V_i})$ and a V_i-valued random variable $Y_i : \Omega \to V_i$. Then $\sigma(Y_j; j \in I) \subseteq \mathcal{F}$ denotes the smallest σ-Algebra, such that each Y_i is $\sigma(Y_j; j \in I) - \mathfrak{B}(V_i)$-measurable. In addition, $\sigma_0(Y_j; j \in I)$ denotes its completion, i.e.*

$$\sigma_0(Y_j; j \in I) := \{A \in \mathcal{F} : \ \exists B \in \sigma(Y_j; j \in I), \text{ such that } \mathbb{P}(A \Delta B) = 0\},$$

where Δ denotes the symmetric difference. It is easily verified that the right-hand-side of the previous equation is indeed a σ-Algebra and the smallest one containing all $\mathbb{P}$-null-sets as well as all elements of $\sigma(Y_j; j \in I)$.

Remark 2.2.10. *Whenever, $Y \in \mathcal{M}(\Omega; V)$, then $\mathbb{P}_Y$ denotes its law, i.e. $\mathbb{P}_Y : \mathfrak{B}(V) \to [0, 1]$, with $\mathbb{P}_Y(B) := \mathbb{P}(Y \in B)$ for all $B \in \mathfrak{B}(V)$.*

Chapter 3

Asymptotic Results for the weighted p-Laplacian evolution Equation

3.1 Outline & Highlights

In this chapter, we will derive numerous asymptotic results for the weighted p-Laplacian evolution equation given by

$$\begin{cases} u'(t) = \operatorname{div}\left(\gamma|\nabla u(t)|^{p-2}\nabla u(t)\right) & \text{on } S, \\ \gamma|\nabla u(t)|^{p-2}\nabla u(t)\cdot\Upsilon = 0 & \text{on } \partial S, \\ u(0) = v, \end{cases} \tag{3.1}$$

for a.e. $t \in (0,\infty)$; where: $\emptyset \neq S \subseteq \mathbb{R}^n$ is a bounded, open and connected set of class C^1, $n \in \mathbb{N}\setminus\{1\}$, $p \in (1,\infty)\setminus\{2\}$, Υ is the unit outer normal on ∂S, $v : S \to \mathbb{R}$ is an integrable initial and $\gamma : S \to (0,\infty)$ is an almost everywhere bounded, $\mathfrak{B}(S)$-$\mathfrak{B}(0,\infty)$-measurable weight function, which can be extended to a p-Muckenhoupt weight on $\mathbb{R}^n$ and fulfills $\int\limits_S \gamma^{\frac{1}{1-p}}\,d\lambda < \infty$.

Existence and uniqueness results for this problem have been studied in [3, Section 3]. Before outlining our asymptotic results, let us give a brief summary of [3, Section 3]: There, the authors introduce a single-valued p-Laplace operator $A_p : D(A_p) \to L^1(S)$, where we denote by $L^q(S) := L^q(S,\mathfrak{B}(S),\lambda;\mathbb{R})$, for any $q \in [1,\infty]$, the usual Lebesgue spaces. Then they prove that the closure of A_p, which we will denote by $\mathcal{A}_p : D(\mathcal{A}_p) \to 2^{L^1(S)}$, is m-accretive and densely defined. Thus, we can introduce $(T_{\mathcal{A}_p}(t))_{t\geq 0}$, where $T_{\mathcal{A}_p}(t) : L^1(S) \to L^1(S)$ for all $t \geq 0$, as the semigroup associated to $\mathcal{A}_p$, see Definition 2.1.8.
In fact, the authors of [3, Section 3] even show that the initial value problem

$$0 \in u'(t) + \mathcal{A}_p u(t),\ \text{for a.e. } t \in (0,\infty)\ u(0) = v,$$

has for any $v \in L^1(S)$ a uniquely determined strong solution, which of course coincides with $T_{\mathcal{A}_p}(\cdot)v$, see Remark 2.1.5.

In Section 3.2, we are going to recall these results in greater detail; particularly, A_p as well as $\mathcal{A}_p$ will be introduced there, and we also use this section to deduce some basic properties of $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ which are direct consequences of the results in Section 2.1.
Afterwards, we will establish that $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ conserves mass, that is $\overline{(T_{\mathcal{A}_p}(t)v)}_S = \overline{(v)}_S$ for every $t \geq 0$ and $v \in L^1(S)$, where $\overline{(v)}_S := \frac{1}{\lambda(S)} \int\limits_S v d\lambda$ denotes the average of any $v \in L^1(S)$.
This result builds the basis for our investigation of the asymptotic behavior of $(T_{\mathcal{A}_p}(t))_{t\geq 0}$. We will demonstrate that

$$\lim_{t\to\infty} ||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^q(S)} = 0, \tag{3.2}$$

for any $v \in L^q(S)$ and $q \in [1,\infty)$; as well as

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)} \leq C_1(p,\gamma,S)||v - \overline{(v)}_S||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \tag{3.3}$$

for all $v \in L^2(S)$ and $t \in (0,\infty)$, where $C_1(p,\gamma,S) > 0$ is a constant (being determined explicitly later) depending only on p, S and γ.[1] Actually, it will turn out that (3.3) is a corollary of a slightly stronger result which is more technical to formulate and will be postponed until Section 3.4.
Deriving further asymptotic results, requires a way of measuring how close γ is to zero, which is done by

$$p_0 := \inf\{q > 1 : \gamma^{\frac{1}{1-q}} \in L^1(S)\}.$$

The first highlight of this chapter is an L^∞-L^p-contraction principle, which reads: If $p > np_0$, then

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^\infty(S)} \leq C_2(p,\gamma,S,n)||v - \overline{(v)}_S||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \tag{3.4}$$

for all $t \in (0,\infty)$ and $v \in L^p(S)$.
Additionally, an extinction principle will be proven, i.e. if $p \in \left(\frac{p_0(n-2)}{n+2} + p_0, 2\right) \neq \emptyset$, then

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}||_{L^2(S)}^{2-p} \leq (-C_3(p,\gamma,S,n)t + ||v - \overline{(v)}||_{L^2(S)}^{2-p})_+, \tag{3.5}$$

for all $t \geq 0$ and $v \in L^2(S)$. In particular, $T_{\mathcal{A}_p}(t)v = \overline{(v)}$ for all $t \geq \frac{||v-\overline{(v)}||_{L^2(S)}^{2-p}}{C_3(p,\gamma,S,n)}$ and $v \in L^2(S)$.

[1] In this section, several constants $C_i(...)$ occur. They are all positive, will be determined explicitly later, and solely depend on the quantities inside of the parenthesis.

Moreover, we will see that: If $p \in (2, \infty)$ and $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$ (which is true if $p > 2p_0$), then

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)} \leq C_4(p, S, \gamma) \left(\frac{1}{t}\right)^{\frac{1}{p-2}}, \forall v \in L^1(S),\ t > 0 \tag{3.6}$$

as well as

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^2(S)} \leq \left(C_5(p, S, \gamma)t + ||v - \overline{(v)}_S||_{L^2(S)}^{2-p}\right)^{\frac{1}{2-p}},\ \forall v \in L^2(S),\ t \geq 0. \tag{3.7}$$

Note that if $n = 2$ and $p_0 = 1$, then (3.4), (3.6) and (3.7) can be applied if $p > 2$; and (3.5) can be applied, if $p \in (1, 2)$; given that the initial fulfills the stated integrability assumption. Hereby, $p_0 = 1$ is fulfilled if for instance, $\gamma \geq c$ a.e. on S for a constant $c > 0$. Moreover, as described in Section 1.1, the p-Laplacian semigroup $(T_{\mathcal{A}_p}(t))_{t \geq 0}$ can be used to model the evolution of a fluvial landscape, in which case $n = 2$.

In fact, there are some further asymptotic results proven in this chapter, which we left out to keep this section more concise, and since they are in a similar spirit to one of the results we have already stated.

This chapter is structured as follows: The basic notation needed in this chapter and a summary of the highlights of [3, Section 3] can be found in Section 3.2. Then, in Section 3.3 we prove our conservation of mass principle, and some further elementary properties of T_{A_p}, A_p and $\mathcal{A}_p$. In Section 3.4, we derive the results (3.2), (3.3) and (3.4). Finally, in Section 3.5 we employ differential inequality techniques to prove (3.5), (3.6) and (3.7).

Moreover, the asymptotic results proven here, will enable us to apply the general results regarding ACPRM-processes that will be developed in Chapters 4, 5 and 6, to $(T_{\mathcal{A}_p}(t))_{t \geq 0}$.

3.2 Assumptions, Notation and preliminary Results

Throughout this entire chapter, let $n \in \mathbb{N} \setminus \{1\}$, $p \in (1, \infty) \setminus \{2\}$ and let $\emptyset \neq S \subseteq \mathbb{R}^n$ be an open, connected and bounded set of class C^1.

Now let us introduce some general notations: λ denotes the Lebesgue measure on $\mathbb{R}^n$, $|\cdot|$ the euclidean norm on $\mathbb{R}^n$, and $x \cdot y$ is the canonical inner product of any $x, y \in \mathbb{R}^n$. In addition, we introduce the usual Lebesgue spaces $L^q(S) := L^q(S, \mathfrak{B}(S), \lambda; \mathbb{R})$ and $L^q(S; \mathbb{R}^n) := L^q(S, \mathfrak{B}(S), \lambda; \mathbb{R}^n)$, for all $q \in [1, \infty]$. As usually, $W^{1,1}_{\text{Loc}}(S)$ denotes the space of weakly differentiable functions and ∇f denotes the weak derivative of any $f \in W^{1,1}_{\text{Loc}}(S)$. In addition, for any $q \in [1, \infty)$, $W^{1,q}(S)$ denotes the Sobolev space of once weakly differentiable functions, such that $\varphi \in L^q(S)$ and $\nabla \varphi \in L^q(S; \mathbb{R}^n)$; and as usually $C_c^\infty(S)$ is the space of infinitely often continuous differentiable, compactly supported functions $\varphi : S \to \mathbb{R}$.

Remark 3.2.1. *Let $q \in (1,\infty)$. Then $M_q(\mathbb{R}^n)$ denotes the class of q-Muckenhoupt weights, that is: $M_q(\mathbb{R}^n)$ consists of all $\mathfrak{B}(\mathbb{R}^n)$-$\mathfrak{B}(\mathbb{R})$-measurable functions $\gamma_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\gamma_0 > 0$ a.e., γ_0 is (w.r.t. the Lebesgue measure) locally integrable on $\mathbb{R}^n$ and*

$$\sup_{\substack{B\subseteq\mathbb{R}^n \\ B \text{ is a ball}}} \Big[\frac{1}{\lambda(B)}\int\limits_B \gamma_0 d\lambda\Big(\frac{1}{\lambda(B)}\int\limits_B \gamma_0^{\frac{1}{1-q}} d\lambda\Big)^{q-1}\Big] < \infty.$$

Now, let us state the assumptions regarding γ which are needed throughout this chapter: Let $\gamma : S \to (0,\infty)$ be such that $\gamma \in L^\infty(S)$, $\gamma^{\frac{1}{1-p}} \in L^1(S)$ and assume that there is a $\gamma_0 \in M_p(\mathbb{R}^n)$ such that $\gamma_0|_S = \gamma$ a.e. on S.
Finally, we set $L^p(S,\gamma,\mathbb{R}^n) := L^p(S,\mathfrak{B}(S),\nu_\gamma;\mathbb{R}^n)$, where $\nu_\gamma : \mathfrak{B}(S) \to [0,\infty)$ is the measure induced by γ, i.e. $\nu_\gamma(B) := \int\limits_B \gamma d\lambda$ for all $B \in \mathfrak{B}(S)$, and introduce the weighted Sobolev space

$$W^{1,p}_\gamma(S) := \{f \in L^p(S) : \ \nabla f \in L^p(S,\gamma;\mathbb{R}^n)\}. \tag{3.8}$$

These notations enable us to introduce the following p-Laplace operator:

Definition 3.2.2. *Let $A_p : D(A_p) \to 2^{L^1(S)}$ be defined by: $(f,\hat{f}) \in A_p$ if and only if the following assertions hold.*

i) $f \in W^{1,p}_\gamma(S) \cap L^\infty(S)$.

ii) $\hat{f} \in L^1(S)$.

iii) $\int\limits_S \gamma|\nabla f|^{p-2}\nabla f \cdot \nabla\varphi d\lambda = \int\limits_S \hat{f}\varphi d\lambda$ *for all* $\varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S)$.

As introductory claimed, we will see in the next section that A_p is single-valued. Now, we would like to introduce the closure of A_p, which requires to generalize the concept of weak differentiability:

Remark 3.2.3. *In the sequel, $\tau_k : \mathbb{R} \to \mathbb{R}$, where $k \in (0,\infty)$, denotes the standard truncation function, i.e. $\tau_k(s) := s$, if $|s| < k$ and $\tau_k(s) := ksgn(s)$, if $|s| \geq k$. Moreover, if $f : S \to \mathbb{R}$ is Borel measurable and fulfills $\tau_k(f) \in W^{1,1}_{Loc}(S)$ for all $k \in (0,\infty)$, then $\tilde{\nabla} f : S \to \mathbb{R}^n$, denotes the (up to equality a.e.) uniquely determined function fulfilling*

$$\nabla\tau_k(f) = \tilde{\nabla} f \mathbb{1}_{\{|f|<k\}}, \ \forall k \in (0,\infty) \tag{3.9}$$

a.e. on S. The function $\tilde{\nabla} f$ is called the generalized weak derivative of f. Note that if $f : S \to \mathbb{R}$ is generalized weakly differentiable, then $f \in W^{1,1}_{Loc}(S)$ if and only if $\tilde{\nabla} f$ is (w.r.t. the Lebesgue measure) locally integrable on S; and in this case $\tilde{\nabla} f = \nabla f$. Cf. [6], for these and further properties.

Definition 3.2.4. *Let $\mathcal{A}_p : D(\mathcal{A}_p) \to 2^{L^1(S)}$ be defined by: $(f,\hat{f}) \in \mathcal{A}_p$ if and only if the following assertions hold.*

i) $f, \hat{f} \in L^1(S)$.

ii) $\tau_k(f) \in W^{1,p}_\gamma(S)$ *for all* $k \in (0,\infty)$.

iii) $\int\limits_S \gamma|\tilde{\nabla} f|^{p-2}\tilde{\nabla} f \cdot \nabla(\tau_k(f-\varphi))d\lambda \leq \int\limits_S \hat{f}\tau_k(f-\varphi)d\lambda$ *for all* $k \in (0,\infty)$ *and* $\varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S)$.

Now let us extract the following result from [3, Section 3]:

Theorem 3.2.5. *The operator $\mathcal{A}_p$ is m-accretive, complete accretive, densely defined and positively homogeneous of degree $p-1$. Consequently, for any $v \in V$, the initial value problem*

$$0 \in u'(t) + \mathcal{A}_p u(t) \text{ for a.e. } t \in (0,\infty) \text{ and } u(0) = v. \tag{3.10}$$

has precisely one strong solution. Moreover, $\mathcal{A}_p$ is the closure of A_p, and even $D(A_p)$ is a dense subset of $(L^1(S), ||\cdot||_{L^1(S)})$.

Remark 3.2.6. *Throughout this chapter, $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ denotes the semigroup associated to $\mathcal{A}_p$, see Definition 2.1.8. Consequently, as mild solutions are according to Remark 2.1.5 also strong ones, we infer from Theorem 3.2.5 that: For any $v \in V$, $T_{\mathcal{A}_p}(\cdot)v : [0,\infty) \to L^1(S)$ is the uniquely determined strong solution of (3.10), i.e. $T_{\mathcal{A}_p}(\cdot)v$ is continuous on $[0,\infty)$, $(T_{\mathcal{A}_p}(\cdot)v)|_{(0,\infty)} \in W^{1,1}_{Loc}((0,\infty); L^1(S))$ and last but not least*

$$0 \in T'_{\mathcal{A}_p}(t)v + \mathcal{A}_p T_{\mathcal{A}_p}(t)v \text{ for a.e. } t \in (0,\infty) \text{ and } T_{\mathcal{A}_p}(0)v = v. \tag{3.11}$$

For the reader's convenience let us state some further useful properties of $(T_{\mathcal{A}_p}(t))_{t\geq 0}$, which can be inferred from the properties of $\mathcal{A}_p$ stated in Theorem 3.2.5 and the results in Section 2.1:

i) $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ *is a jointly-continuous, contractive semigroup. (See Theorem 2.1.7)*

ii) *For any* $v \in D(\mathcal{A}_p)$ *and* $t > 0$, $(T_{\mathcal{A}_p}(\cdot))|_{[0,t]}$ *is Lipschitz continuous. (See Lemma 2.1.13.)*

iii) $||T_{\mathcal{A}_p}(t)v_1 - T_{\mathcal{A}_p}(t)v_2||_{L^q(S)} \leq ||v_1 - v_2||_{L^q(S)}$ *for all* $v_1, v_2 \in L^q(S)$, $q \in [1,\infty]$ *and* $t \geq 0$. *(See Theorem 2.1.18.i) and Remark 2.1.16)*

iv) $||T_{\mathcal{A}_p}(t)v||_{L^q(S)} \leq ||v||_{L^q(S)}$ *for all* $v \in L^q(S)$, $q \in [1,\infty]$ *and* $t \geq 0$. *(See Theorem 2.1.18.ii) and Remark 2.1.16.)*

v) $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ *is domain invariant and admits an infinitesimal generator* $\mathcal{A}^\circ_p : L^1(S) \to L^1(S)$. *(See Theorem 2.1.18.)*

vi) *Let* $\mathcal{A}^\circ_p$ *denote the infinitesimal generator of* $(T_{\mathcal{A}_p}(t))_{t\geq 0}$, $q \in [1,\infty]$, $t, h > 0$ *and* $v \in L^q(S)$, *then we have* $\frac{1}{h}||T_{\mathcal{A}_p}(t+h)v - T_{\mathcal{A}_p}(t)v||_{L^q(S)} \leq ||\mathcal{A}^\circ_p T_{\mathcal{A}_p}(t)v||_{L^q(S)} \leq \frac{2}{|p-2|}\frac{1}{t}||v||_{L^q(S)}$. *(See 2.1.18.iv,v and Remark 2.1.16.)*

3.3 Basic Results and Conservation of Mass

In this section, we will derive some basic properties of $T_{\mathcal{A}_p}$, A_p and $\mathcal{A}_p$, among them is the introductory mentioned conservation of mass principle, which is stated and proven in Lemma 3.3.5.

Lemma 3.3.1. *A_p is single-valued. Moreover, if $f \in D(\mathcal{A}_p) \cap L^\infty(S)$ and $\hat{f} \in \mathcal{A}_p f$, then $f \in D(A_p)$ and $\hat{f} = A_p f$.*

Proof. It is plain that A_p is single-valued, since $(f,\hat{f}),\ (f,\tilde{f}) \in A_p$ implies

$$\int_S (\hat{f} - \tilde{f})\varphi d\lambda = 0,\ \forall \varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S),$$

and obviously $C_c^\infty(S) \subseteq W^{1,p}_\gamma(S) \cap L^\infty(S)$.
Now let $f \in D(\mathcal{A}_p) \cap L^\infty(S)$ and $\hat{f} \in \mathcal{A}_p f$, then $\tau_k(f) \in W^{1,p}_\gamma(S)$ for all $k \in (0,\infty)$. Consequently, by choosing $k > ||f||_{L^\infty(S)}$, we get $f \in W^{1,p}_\gamma(S) \cap L^\infty(S)$. Hence the claim follows if

$$\int_S \gamma |\nabla f|^{p-2} \nabla f \cdot \nabla \varphi d\lambda = \int_S \hat{f} \varphi d\lambda,\ \forall \varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S). \tag{3.12}$$

Proof of (3.12). It follows from the definition of $\mathcal{A}_p$ that

$$\int_S \gamma |\tilde{\nabla} f|^{p-2} \tilde{\nabla} f \cdot \nabla(\tau_k(f-\varphi)) d\lambda \le \int_S \hat{f} \tau_k(f-\varphi) d\lambda,$$

for all $\varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S)$ and $k \in (0,\infty)$.
Observe that $f \in W^{1,p}_\gamma(S) \cap L^\infty(S)$ implies $\tilde{\nabla} f = \nabla f$ on S (see Remark 3.2.3) and that $\varphi = f - \tilde{\varphi}$, where $\tilde{\varphi} \in W^{1,p}_\gamma(S) \cap L^\infty(S)$, is a valid choice as a test function in the previous equation; hence

$$\int_S \gamma |\nabla f|^{p-2} \nabla f \cdot \nabla(\tau_k(\tilde{\varphi})) d\lambda \le \int_S \hat{f} \tau_k(\tilde{\varphi}) d\lambda, \tag{3.13}$$

for all $\tilde{\varphi} \in W^{1,p}_\gamma(S) \cap L^\infty(S)$ and $k \in (0,\infty)$.
Now (3.13) yields, by choosing $k > ||\tilde{\varphi}||_{L^\infty(S)}$ for a given $\tilde{\varphi} \in W^{1,p}_\gamma(S) \cap L^\infty(S)$, that

$$\int_S \gamma |\nabla f|^{p-2} \nabla f \cdot \nabla \tilde{\varphi} d\lambda \le \int_S \hat{f} \tilde{\varphi} d\lambda,\ \forall \tilde{\varphi} \in W^{1,p}_\gamma(S) \cap L^\infty(S). \tag{3.14}$$

Conclusively the claim follows since $\tilde{\varphi}$ can be replaced by $-\tilde{\varphi}$ as a test function in (3.14). □

Lemma 3.3.2. *Let $v \in L^\infty(S)$. Then we have $T_{\mathcal{A}_p}(t)v \in D(A_p)$ and $-T'_{\mathcal{A}_p}(t)v = A_p T_{\mathcal{A}_p}(t)v$ for almost every $t \in (0,\infty)$.*

Proof. We already know that $0 \in T'_{\mathcal{A}_p}(t)v + \mathcal{A}_p T_{\mathcal{A}_p}(t)v$ for a.e. $t \in (0,\infty)$ and (by domain invariance) even $T_{\mathcal{A}_p}(t)v \in D(\mathcal{A}_p)$ for every $t \in (0,\infty)$. Thus, employing the services of Remark 3.2.6.iv) (with $q = \infty$) and Lemma 3.3.1 yields the claim. □

Lemma 3.3.3. *Let $v \in L^1(S)$ and $\varphi : S \to \mathbb{R}$ be a constant function. Then*

$$T_{\mathcal{A}_p}(t)(v + \varphi) = T_{\mathcal{A}_p}(t)(v) + \varphi,\ \forall t \in [0,\infty). \tag{3.15}$$

Consequently, if $T_{\mathcal{A}_p}(\cdot)v$ is differentiable in $t \in (0,\infty)$, then $T_{\mathcal{A}_p}(\cdot)(v+\varphi)$ is differentiable in t and $T'_{\mathcal{A}_p}(t)(v+\varphi) = T'_{\mathcal{A}_p}(t)v$.

Proof. Let $v \in L^\infty(S)$, let $\varphi : S \to \mathbb{R}$ be a constant function and introduce $f : [0,\infty) \to L^1(S)$ by $f(t) := T_{\mathcal{A}_p}(t)(v) + \varphi$.
It is clear that $f(0) = v+\varphi$ and also that f is continuous on $[0,\infty)$ and an element of $W^{1,1}_{\text{Loc}}((0,\infty); L^1(S))$, since $T_{\mathcal{A}_p}(\cdot)v$ has these properties.
Now observe that obviously $f'(t) = T'_{\mathcal{A}_p}(t)v$ for a.e. $t \in (0,\infty)$. Moreover, we have for any $\varphi \in W^{1,p}_\gamma(S) \cap L^v(S)$ that

$$\int_S \gamma |\nabla f(t)|^{p-2} \nabla f(t) \cdot \nabla \varphi d\lambda = \int_S \gamma |\nabla T_{\mathcal{A}_p}(t)v|^{p-2} \nabla T_{\mathcal{A}_p}(t)v \cdot \nabla \varphi d\lambda$$

which implies, together with $f'(t) = T'_{\mathcal{A}_p}(t)v$ for a.e. $t \in (0,\infty)$ and Lemma 3.3.2, that $f(t) \in D(A_p)$ and $-f'(t) = A_p f(t)$ for a.e. $t \in (0,\infty)$. Consequently (3.15) is verified for initial values $v \in L^\infty(S)$.
Conclusively, applying Remark 3.2.6.iii (with $q=1$) yields that both sides of (3.15) depend continuously on v. Thus, as $L^\infty(S)$ is dense in $(L^1(S), ||\cdot||_{L^1(S)})$, (3.15) holds for all $v \in L^1(S)$.
Finally observe that (3.15) clearly implies the remaining part of the claim. □

Remark 3.3.4. *We denote for any $v \in L^1(S)$, its average by $\overline{(v)}_S$, that means $\overline{(v)}_S := \frac{1}{\lambda(S)} \int_S v d\lambda$. By slightly abusing notation, the constant function mapping from S to $\mathbb{R}$, which takes only the value $\overline{(v)}_S$ will also be denoted by $\overline{(v)}_S$. In addition, we set*

$$L^q_0(S) := \{v \in L^q(S) : \overline{(v)}_S = 0\},$$

for all $q \in [1,\infty]$. Moreover, we equip $L^q_0(S)$ with $||\cdot||_{L^q(S)}$ as a norm. Then $(L^q_0(S), ||\cdot||_{L^q(S)})$ is a Banach space, for any $q \in [1,\infty]$, and it is separable, if $q \neq \infty$.

Lemma 3.3.5. *Let $v \in L^1(S)$, then $\overline{(T_{\mathcal{A}_p}(t)v)}_S = \overline{(v)}_S$ for every $t \geq 0$.*

Proof. As, $L^1(S) \ni v \mapsto \overline{(v)}_S$ is clearly continuous, employing Remark 3.2.6.iii) (with $q=1$) yields that $L^1(S) \ni v \mapsto \overline{(T_{\mathcal{A}_p}(t)v)}_S$ is continuous as well. Thus, it suffices to prove the claim for $v \in D(A_p)$, since this is according to Theorem 3.2.5 a dense subset of $(L^1(S)||\cdot||_{L^1(S)})$. So let $v \in D(A_p)$ be given. Moreover, introduce $\tau \in (0,\infty)$ and $f : [0,\tau] \to \mathbb{R}$ by $f(t) := \int_S T_{\mathcal{A}_p}(t) v d\lambda$, for all $t \in [0,\tau]$.
According to Remark 3.2.6.ii), $(T_{\mathcal{A}_p}(\cdot)v)|_{[0,\tau]}$ is Lipschitz continuous which obviously implies that f is Lipschitz continuous as well. Moreover, it is plain that $f'(t) = \int_S T'_{\mathcal{A}_p}(t) v d\lambda$.
In addition, note that $D(A_p) \subseteq L^\infty(S)$ which yields by the aid of Lemma 3.3.2 that

$$f'(t) = -\int_S \gamma |\nabla T_{\mathcal{A}_p}(t)v|^{p-2} \nabla T_{\mathcal{A}_p}(t)v \cdot \nabla \varphi d\lambda = 0,$$

where $\varphi : S \to \mathbb{R}$ denotes the function which is constantly one.

Consequently, f is constant and therefore $\overline{(v)} = \overline{(T_{\mathcal{A}_p}(t)v)}$ for all $t \in [0, \tau]$ which gives the claim as τ is arbitrary. $\square$

Lemma 3.3.6. *For each $q \in [1, \infty)$, the space $L_0^q(S)$ is invariant w.r.t. $T_{\mathcal{A}_p}$. Moreover, the restriction of $T_{\mathcal{A}_p}$ to $L_0^q(S)$ is a time-continuous, contractive semigroup on $(L_0^q(S), ||\cdot||_{L^q(S)})$ which fulfills $T_{\mathcal{A}_p}(t)0 = 0$ for all $t \in [0, \infty)$.*

Proof. The invariance follows from Lemma 3.3.5 and Remark 3.2.6.iv). In addition, the contractivity follows from Remark 3.2.6.iii). Moreover, $T_{\mathcal{A}_p}(t)0 = 0$ is easily inferred from $0 \in D(A_p)$, $A_p 0 = 0$. In addition, it is plain that the semigroup property still holds on $L_0^q(S) \subseteq L^1(S)$.
Thus, it remains to prove the time-continuity. So let $v \in L_0^q(S)$ and introduce a null-sequence $(h_m)_{m\in\mathbb{N}} \subseteq \mathbb{R}$, as well as $t \geq 0$ and $\varepsilon > 0$. Moreover, assume w.l.o.g. that $t + h_m \geq 0$ for all $m \in \mathbb{N}$. Moreover, choose $\hat{v} \in L_0^\infty(S)$ such that $||v - \hat{v}||_{L^q(S)} < \frac{\varepsilon}{2}$. Then we get by the time continuity of $T_{\mathcal{A}_p}$, and by passing to a subsequence if necessary, that $\lim\limits_{m\to\infty} T_{\mathcal{A}_p}(t + h_m)\hat{v} = T_{\mathcal{A}_p}(t)\hat{v}$ almost everywhere. In addition, invoking Remark 3.2.6.iv) gives $||T_{\mathcal{A}_p}(t + h_m)\hat{v}||_{L^\infty(S)} \leq ||\hat{v}||_{L^\infty(S)}$ for all $m \in \mathbb{N}$ and employing dominated convergence yields $\lim\limits_{m\to\infty} T_{\mathcal{A}_p}(t + h_m)\hat{v} = T_{\mathcal{A}_p}(t)\hat{v}$ w.r.t. $||\cdot||_{L^q(S)}$. Conclusively, we get by contractivity that

$$\lim_{m\to\infty} ||T_{\mathcal{A}_p}(t + h_m)v - T_{\mathcal{A}_p}(t)v||_{L^q(S)} \leq 2||v - \hat{v}||_{L^q(S)} + \lim_{m\to\infty} ||T_{\mathcal{A}_p}(t + h_m)\hat{v} - T_{\mathcal{A}_p}(t)\hat{v}||_{L^q(S)} \leq \varepsilon,$$

which yields the desired time continuity. $\square$

Now, let us conclude this section with the following lemma which is not needed in the Sections 3.4 and 3.5, but in Section 4.4.

Lemma 3.3.7. *Let $\varphi \in W_\gamma^{1,p}(S) \cap L^\infty(S)$ and $v \in L^\infty(S)$. Moreover, let $\mathcal{A}_p^\circ : L^1(S) \to L^1(S)$ denote the infinitesimal generator of $T_{\mathcal{A}_p}$. Then, the mapping $(0, \infty) \ni \tau \mapsto \int_S \varphi \mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda$ is $\mathfrak{B}((0, \infty))$-$\mathfrak{B}(\mathbb{R})$-measurable and*

$$\int_0^t \left| \int_S \varphi \mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda \right| d\tau < \infty. \tag{3.16}$$

for all $t > 0$.

Proof. Firstly, Remark 3.2.6.vi) yields that $\varphi \mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v \in L^\infty(S) \subseteq L^1(S)$ for all $\tau > 0$; thus, the integral $\int_S \varphi \mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda$ exists. Moreover, Remark 3.2.6.v) yields

$$\int_S \varphi \mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda = -\lim_{h\downarrow 0} \frac{1}{h} \left(\int_S \varphi T_{\mathcal{A}_p}(\tau + h)v d\lambda - \int_S \varphi T_{\mathcal{A}_p}(\tau)v d\lambda \right), \ \forall \tau > 0.$$

In addition, for any $h > 0$, the mapping $(0, \infty) \ni \tau \mapsto \frac{1}{h} \left(\int_S \varphi T_{\mathcal{A}_p}(\tau + h)v d\lambda - \int_S \varphi T_{\mathcal{A}_p}(\tau)v d\lambda \right)$ is continuous, and therefore a fortiori $\mathfrak{B}((0, \infty))$-$\mathfrak{B}(\mathbb{R})$-measurable. Consequently, $(0, \infty) \ni \tau \mapsto$

$\int\limits_S \varphi\mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda$ is $\mathfrak{B}((0,\infty))$-$\mathfrak{B}(\mathbb{R})$-measurable, and it remains to prove (3.16).
Firstly, Lemma 3.3.2 yields $\mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v = A_p T_{\mathcal{A}_p}(\tau)v$, for a.e. $\tau > 0$. Thus, by employing Cauchy-Schwarz' and Hölder's inequality we get

$$\begin{aligned}\left|\int\limits_S \varphi\mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda\right| &= \left|\int\limits_S \gamma|\nabla T_{\mathcal{A}_p}(\tau)v|_n^{p-2}\nabla T_{\mathcal{A}_p}(\tau)v\cdot\nabla\varphi d\lambda\right| \\ &\leq \left(\int\limits_S \gamma|\nabla T_{\mathcal{A}_p}(\tau)v|_n^p d\lambda\right)^{\frac{p-1}{p}}\left(\int\limits_S \gamma|\nabla\varphi|_n^p d\lambda\right)^{\frac{1}{p}} \\ &= \left(\int\limits_S T_{\mathcal{A}_p}(\tau)v A_p T_{\mathcal{A}_p}(\tau)v d\lambda\right)^{\frac{p-1}{p}}\left(\int\limits_S \gamma|\nabla\varphi|_n^p d\lambda\right)^{\frac{1}{p}}\end{aligned}$$

for a.e. $\tau \in (0,\infty)$. Consequently, Remark 3.2.6.iv,vi) enable us to deduce that

$$\left|\int\limits_S \varphi\mathcal{A}_p^\circ T_{\mathcal{A}_p}(\tau)v d\lambda\right| \leq \left(\frac{1}{\tau}2\lambda(S)\frac{1}{|p-2|}||v||^2_{L^\infty(S)}\right)^{\frac{p-1}{p}}\left(\int\limits_S \gamma|\nabla\varphi|_n^p d\lambda\right)^{\frac{1}{p}},$$

for a.e. $\tau \in (0,\infty)$. But, the preceding inequality obviously implies (3.16). □

3.4 An L^∞-L^p-Contraction Principle

The purpose of this section is to prove the results (3.2), (3.3) and (3.4) mentioned in the introduction. Actually, it will turn out that (3.3) is a corollary of a slightly stronger result.
Among other things, the (proofs of the) asymptotic results we obtain, heavily rely on Poincareé's inequality:

Remark 3.4.1. *We denote for any $q \in [1,\infty)$, the Poincaré constant of S in $L^q(S)$ by $C_{S,q}$, that is: $C_{S,q} \in (0,\infty)$ is the smallest constant depending only on S and q, such that*

$$||f - \overline{(f)}_S||_{L^q(S)} \leq C_{S,q}||\nabla f||_{L^q(S;\mathbb{R}^n)},\ \forall f \in W^{1,q}(S).$$

Note that S is assumed to be open, bounded, connected and of class C^1. Consequently, Poincaré's inequality implies the existence of $C_{S,q}$.

Remark 3.4.2. *Throughout this chapter, let $p_0 \in [1,p]$ be the constant defined by*

$$p_0 := \inf\{q > 1 : \gamma^{\frac{1}{1-q}} \in L^1(S)\}.$$

Since $\gamma^{\frac{1}{1-p}} \in L^1(S)$ by assumption, it is clear that indeed $p_0 \leq p$.

Roughly speaking, p_0 gets as closer to one, as further away γ is from zero. Moreover, as it turns out, the closer p_0 is to one, the better our asymptotic results get. In addition, note the following special case:

Remark 3.4.3. *If there is a constant $c > 0$ such that $\gamma \geq c$ a.e. on S, then $p_0 = 1$, since: We then have for all $q > 1$ that $\int_S \gamma^{\frac{1}{1-q}} d\lambda \leq \lambda(S) c^{\frac{1}{1-q}} < \infty$.*
Thus, particularly if γ is constant a.e. on S, then $p_0 = 1$.

Lemma 3.4.4. *If $q > p_0$ then $\gamma^{\frac{1}{1-q}} \in L^1(S)$. Moreover, $p_0 < p$.*

Proof. Let $q > p_0$, then there is $\tilde{q} \in [p_0, q) \setminus \{1\}$ such that $\gamma^{\frac{1}{1-\tilde{q}}} \in L^1(S)$. Since trivially $\frac{1-q}{1-\tilde{q}} > 1$, Hölder's inequality yields

$$\int_S \gamma^{\frac{1}{1-q}} d\lambda \leq \lambda(S)^{\frac{\tilde{q}-q}{1-q}} \left(\int_S \gamma^{\frac{1}{1-\tilde{q}}} d\lambda \right)^{\frac{1-\tilde{q}}{1-q}} < \infty,$$

which implies $\gamma^{\frac{1}{1-q}} \in L^1(S)$.
By assumption there is $\gamma_0 \in M_p(\mathbb{R}^n)$ such that $\gamma = \gamma_0$ a.e. on S. Moreover, there is an $\varepsilon \in (0, p-1)$ such that $\gamma_0 \in M_{p-\varepsilon}(\mathbb{R}^n)$. (See [40, Ch. IX Prop. 4.3 and Theorem 5.5].)
Since S is bounded, there is a ball $B \subseteq \mathbb{R}^n$ containing S which implies $\gamma_0^{\frac{1}{1-(p-\varepsilon)}} \in L^1(S)$. This implies $p_0 < p$, since $\gamma = \gamma_0$ a.e. on S. □

Lemma 3.4.5. *Let $0 \leq \delta < \frac{p-p_0}{p_0}$ and $f \in W^{1,p}_\gamma(S)$, then $f \in W^{1,1+\delta}(S)$ and*

$$||\nabla f||_{L^{1+\delta}(S;\mathbb{R}^n)} \leq \left(\int_S \gamma^{\frac{1+\delta}{1+\delta-p}} d\lambda \right)^{\frac{p-1-\delta}{p(1+\delta)}} ||\nabla f||_{L^p(S,\gamma;\mathbb{R}^n)} < \infty. \tag{3.17}$$

Proof. Let $0 \leq \delta < \frac{p-p_0}{p_0}$. (Note that $p_0 < p$, thus such a δ does indeed exists.)
In addition, let $f \in W^{1,p}_\gamma(S)$, then obviously $f \in W^{1,1}_{\text{Loc}}(S)$ as well as $f \in L^p(S)$.
Moreover, note that $1 + \delta < 1 + \frac{p-p_0}{p_0} \leq p$ and thus $f \in L^{1+\delta}(S)$, since $\lambda(S) < \infty$. Conclusively, the claim follows once (3.17) is proven.
First of all $1 + \delta - p \neq 0$. Secondly $\frac{p}{1+\delta} > p_0$, thus Lemma 3.4.4 yields

$$\int_S \gamma^{\frac{1+\delta}{1+\delta-p}} d\lambda = \int_S \gamma^{\frac{1}{1-\frac{p}{1+\delta}}} d\lambda < \infty. \tag{3.18}$$

Finally, (3.17) follows from the following estimate, where Hölder's inequality is used.

$$||\nabla f||_{L^{1+\delta}(S;\mathbb{R}^n)} = \left(\int_S |\nabla f|^{1+\delta} \gamma^{\frac{1+\delta}{p}} \gamma^{-\frac{1+\delta}{p}} d\lambda \right)^{\frac{1}{1+\delta}}$$

$$\leq \left(\left(\int_S |\nabla f|^p \gamma d\lambda \right)^{\frac{1+\delta}{p}} \left(\int_S \gamma^{\frac{1+\delta}{1+\delta-p}} d\lambda \right)^{\frac{p-1-\delta}{p}} \right)^{\frac{1}{1+\delta}}$$

$$= ||\nabla f||_{L^p(S,\gamma;\mathbb{R}^n)} \left(\int_S \gamma^{\frac{1+\delta}{1+\delta-p}} d\lambda \right)^{\frac{p-1-\delta}{p(1+\delta)}},$$

which is finite due to (3.18). □

The preceding lemma is a slight modification of [23, Prop. 2.1]. There, an analogues result is proven for Sobolev spaces, where the function and its weak derivative need to be integrable with respect to the same measure and not to different ones as in our setting.

Lemma 3.4.6. *Let $v \in L^2(S) \cap L^p(S)$, then $T_{\mathcal{A}_p}(t)v \in L^2(S) \cap W^{1,p}_\gamma(S)$ for a.e. $t \in (0,\infty)$ and*

$$||\nabla T_{\mathcal{A}_p}(t)v||_{L^p(S,\gamma;\mathbb{R}^n)} \leq \left(\frac{2}{|p-2|} \right)^{\frac{1}{p}} ||v - \overline{(v)}_S||^{\frac{2}{p}}_{L^2(S)} \left(\frac{1}{t} \right)^{\frac{1}{p}} \tag{3.19}$$

for a.e. $t \in (0,\infty)$.

Proof. Thanks to Lemma 3.3.3 it suffices to prove the claim for $v \in L^2_0(S) \cap L^p_0(S)$. So let $v \in L^2_0(S) \cap L^p_0(S)$ be given and introduce $t \in (0,\infty)$ such that $0 \in T'_{\mathcal{A}_p}(t)v + \mathcal{A}_p T_{\mathcal{A}_p}(t)v$. According to (3.11) almost every value in $(0,\infty)$ is a valid choice for t.
Firstly, note that $T_{\mathcal{A}_p}(t)v \in L^2_0(S) \cap L^p_0(S)$ by Lemma 3.3.6. Moreover, $T_{\mathcal{A}_p}(t)v$ is generalized weakly differentiable. Consequently, if

$$\int_S \gamma |\tilde{\nabla} T_{\mathcal{A}_p}(t)v|^p d\lambda \leq \frac{2}{|p-2|} ||v||^2_{L^2(S)} \frac{1}{t}, \tag{3.20}$$

then obviously $\tilde{\nabla} T_{\mathcal{A}_p}(t)v \in L^p(S,\gamma;\mathbb{R}^n) \subseteq L^1(S;\mathbb{R}^n)$ and therefore, apppealing to Remark 3.2.3 yields $\tilde{\nabla} T_{\mathcal{A}_p}(t)v = \nabla T_{\mathcal{A}_p}(t)v$ a.e. on S.
Hence, if (3.20) holds, then also (3.19) as well as $T_{\mathcal{A}_p}(t)v \in L^2(S) \cap W^{1,p}_\gamma(S)$.
Proof of (3.20). Firstly, Remark 3.2.6.vi) implies

$$||T'_{\mathcal{A}_p}(t)v||_{L^2(S)} \leq \frac{2}{|p-2|t} ||v||_{L^2(S)}. \tag{3.21}$$

Moreover, Fatou's lemma yields

$$\int_S \gamma |\tilde{\nabla} T_{\mathcal{A}_p}(t)v|^p d\lambda \leq \liminf_{k\to\infty} \int_S -T'_{\mathcal{A}_p}(t)v \tau_k \left(T_{\mathcal{A}_p}(t)v \right) d\lambda.$$

Consequently, by Cauchy Schwarz' inequality, (3.21) and Lebesgue's theorem we get

$$\int_S \gamma|\tilde{\nabla} T_{\mathcal{A}_p}(t)v|^p d\lambda \leq \frac{2}{|p-2|t}||v||_{L^2(S)}||T_{\mathcal{A}_p}(t)v||_{L^2(S)}.$$

Finally, (3.20) follows by applying Remark 3.2.6.iv). $\square$

Theorem 3.4.7. *Let $0 \leq \delta < \frac{p-p_0}{p_0}$ and $v \in L^2(S) \cap L^{1+\delta}(S)$, then*

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^{1+\delta}(S)} \leq C_{S,1+\delta}\Gamma_{\delta,p}||v - \overline{(v)}_S||^{\frac{2}{p}}_{L^2(S)} \left(\frac{1}{t}\right)^{\frac{1}{p}} \tag{3.22}$$

for every $t \in (0, \infty)$, where

$$\Gamma_{\delta,p} := \left(\int_S \gamma^{\frac{1+\delta}{1+\delta-p}} d\lambda\right)^{\frac{p-1-\delta}{p(1+\delta)}} \left(\frac{2}{|p-2|}\right)^{\frac{1}{p}} < \infty. \tag{3.23}$$

Proof. Let $0 \leq \delta < \frac{p-p_0}{p_0}$ and $v \in L^2_0(S) \cap L^p_0(S)$.
Moreover, let $t \in (0, \infty)$ be such that the assertions of Lemma 3.4.6 hold. Since $T_{\mathcal{A}_p}(t)v \in W^{1,p}_\gamma(S)$, Lemma 3.4.5 yields $T_{\mathcal{A}_p}(t)v \in W^{1,1+\delta}(S)$ and thus Lemma 3.3.5, Poincaré's inequality, (3.17) and (3.19) imply

$$\begin{aligned} ||T_{\mathcal{A}_p}(t)v||_{L^{1+\delta}(S)} &= ||T_{\mathcal{A}_p}(t)v - \overline{(T_{\mathcal{A}_p}(t)v)}_S||_{L^{1+\delta}(S)} \\ &\leq C_{S,1+\delta}||\nabla T_{\mathcal{A}_p}(t)v||_{L^{1+\delta}(S;\mathbb{R}^n)} \\ &\leq C_{S,1+\delta}\Gamma_{\delta,p}||v||^{\frac{2}{p}}_{L^2(S)} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \end{aligned}$$

i.e. (3.22) holds for $v \in L^2_0(S) \cap L^p_0(S)$ and almost every $t \in (0, \infty)$. Thus, employing Lemma 3.3.6 yields (3.22) for $v \in L^2_0(S) \cap L^p_0(S)$ and each $t \in (0, \infty)$.
Now, let $t \in (0, \infty)$ be arbitrary and let $v \in L^2_0(S) \cap L^{1+\delta}_0(S)$.
Moreover, let $(v_m)_{m\in\mathbb{N}} \subseteq L^2_0(S) \cap L^p_0(S)$ be such that $\lim_{m\to\infty} v_m = v$ w.r.t. $||\cdot||_{L^2(S)}$ and $||\cdot||_{1+\delta(S)}$. Then, it follows from Remark 3.2.6.iii) that $\lim_{m\to\infty} T_{\mathcal{A}_p}(t)v_m = T_{\mathcal{A}_p}(t)v$ w.r.t. $||\cdot||_{1+\delta(S)}$. Hence

$$\begin{aligned} ||T_{\mathcal{A}_p}(t)v||_{L^{1+\delta}(S)} &= \lim_{m\to\infty} ||T_{\mathcal{A}_p}(t)v_m||_{L^{1+\delta}(S)} \\ &\leq \lim_{m\to\infty} C_{S,1+\delta}\Gamma_{\delta,p}||v_m||^{\frac{2}{p}}_{L^2(S)} \left(\frac{1}{t}\right)^{\frac{1}{p}} \\ &= C_{S,1+\delta}\Gamma_{\delta,p}||v||^{\frac{2}{p}}_{L^2(S)} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \end{aligned}$$

which implies (3.22) for every $t \in (0, \infty)$ and $v \in L^2_0(S) \cap L^{1+\delta}_0(S)$.
Finally, for arbitrary $v \in L^2(S) \cap L^{1+\delta}(S)$, (3.22) follows from Lemma 3.3.3. $\square$

Remark 3.4.8. *Whenever δ is given such that $0 \leq \delta < \frac{p-p_0}{p_0}$, then $\Gamma_{\delta,p}$ denotes the quantity introduced in* (3.23).

Corollary 3.4.9. *Let $v \in L^2(S)$, then*

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)} \leq C_{S,1}\Gamma_{0,p}||v - \overline{(v)}_S||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \tag{3.24}$$

for every $t \in (0,\infty)$.

If $0 \leq \delta < \frac{p-p_0}{p_0}$, then δ can be chosen as bigger as smaller p_0 gets, i.e. Theorem 3.4.7 yields the most general result if $p_0 = 1$. The reader is referred to Remark 3.4.3 for assumptions implying $p_0 = 1$. By virtue of Sobolev's embedding theorem we obtain the main result of this section:

Theorem 3.4.10. *Let $v \in L^p(S)$ and assume $p_0 < \frac{p}{n}$, then $T_{\mathcal{A}_p}(t)v \in L^\infty(S)$ for every $t \in (0,\infty)$. Moreover, if $n-1 < \delta < \frac{p-p_0}{p_0}$, then $T_{\mathcal{A}_p}(t)v \in W^{1,1+\delta}(S)$ and there is a constant $C^*_{S,\delta} \in [0,\infty)$, depending only on S and δ, such that*

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^\infty(S)} \leq C^*_{S,\delta}\Gamma_{\delta,p}||v - \overline{(v)}_S||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \tag{3.25}$$

for every $t \in (0,\infty)$.
In addition, $C^*_{S,\delta}$ can be chosen as $C^*_{S,\delta} = \tilde{C}_{S,1+\delta}\left(C_{S,1+\delta}^{1+\delta} + 1\right)^{\frac{1}{1+\delta}}$, where $\tilde{C}_{S,1+\delta}$ is the operator norm of the continuous injection $W^{1,1+\delta}(S) \hookrightarrow L^\infty(S)$.

Proof. Thanks to Lemma 3.3.3 it suffices to prove the claim for $v \in L^p_0(S)$. Moreover, note that if $p_0 < \frac{p}{n}$, then $\frac{p-p_0}{p_0} > n-1$, consequently $(n-1, \frac{p-p_0}{p_0}) \neq \emptyset$.
So let $n-1 < \delta < \frac{p-p_0}{p_0}$ and $v \in L^p_0(S)$ which implies $v \in L^2_0(S)$, since $p > np_0 \geq n \geq 2$.
Moreover, by appealing to Lemma 3.4.6 and Lemma 3.3.5 we get $T_{\mathcal{A}_p}(t)v \in L^2_0(S) \cap W^{1,p}_\gamma(S)$ for a.e. $t \in (0,\infty)$. Consequently, Lemma 3.4.5 yields $T_{\mathcal{A}_p}(t)v \in W^{1,1+\delta}(S)$ and

$$||\nabla T_{\mathcal{A}_p}(t)v||_{L^{1+\delta}(S;\mathbb{R}^n)} \leq \Gamma_{\delta,p}||v||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}, \tag{3.26}$$

for a.e. $t \in (0,\infty)$.
Since $T_{\mathcal{A}_p}(t)v \in W^{1,1+\delta}(S)$ and $1+\delta > n$, employing Sobolev's embedding theorem yields

$$||T_{\mathcal{A}_p}(t)v||_{L^\infty(S)} \leq \tilde{C}_{S,1+\delta}||T_{\mathcal{A}_p}(t)v||_{W^{1,1+\delta}(S)}, \tag{3.27}$$

for almost every $t \in (0,\infty)$, where $\tilde{C}_{S,1+\delta}$ is the operator norm of the continuous injection $W^{1,1+\delta}(S) \hookrightarrow L^\infty(S)$.
Hence it follows by virtue of Theorem 3.4.7, and the inequalities (3.26) and (3.27) that

$$\left(\frac{1}{\tilde{C}_{S,1+\delta}}||T_{\mathcal{A}_p}(t)v||_{L^\infty(S)}\right)^{1+\delta} \leq ||T_{\mathcal{A}_p}(t)v||_{W^{1,1+\delta}(S)}^{1+\delta}$$

$$
\begin{aligned}
&= ||T_{\mathcal{A}_p}(t)v||^{1+\delta}_{L^{1+\delta}(S)} + ||\nabla T_{\mathcal{A}_p}(t)v||^{1+\delta}_{L^{1+\delta}(S;\mathbb{R}^n)} \\
&\leq \left(C^{1+\delta}_{S,1+\delta}+1\right)\left(\Gamma_{\delta,p}||v||^{\frac{2}{p}}_{L^2(S)}\left(\frac{1}{t}\right)^{\frac{1}{p}}\right)^{1+\delta}
\end{aligned}
$$

Consequently, if one defines $C^*_{S,\delta} := \tilde{C}_{S,1+\delta}\left(C^{1+\delta}_{S,1+\delta}+1\right)^{\frac{1}{1+\delta}}$, then the preceding estimate yields the claim for almost every $t \in (0,\infty)$.
Now let $t \in (0,\infty)$ and choose a monotonically increasing sequence $(t_m)_{m\in\mathbb{N}} \subseteq (0,\infty)$ such that $\lim_{m\to\infty} t_m = t$, $t_m < t$ and such that (3.25) holds for each $m \in \mathbb{N}$. Then Remark 3.2.6.iv), yields

$$
||T_{\mathcal{A}_p}(t)v||_{L^\infty(S)} \leq ||T_{\mathcal{A}_p}(t_m)v||_{L^\infty(S)},
$$

for every $m \in \mathbb{N}$, which verifies (3.25) and $T_{\mathcal{A}_p}(t)v \in L^\infty(S)$ for every $t \in (0,\infty)$. Finally, invoking Remark 3.2.6.v) yields $T_{\mathcal{A}_p}(t)v \in D(\mathcal{A}_p)$ and thus (by Lemma 3.3.1) we get $T_{\mathcal{A}_p}(t)v \in D(A_p) \subseteq W^{1,p}_\gamma(S) \subseteq W^{1,1+\delta}(S)$ for every $t \in (0,\infty)$, where the last inclusion follows from Lemma 3.4.5. □

Remark 3.4.11. *Assume $v \in L^p(S)$ and $p_0 < \frac{p}{n}$. Moreover, let $n-1 < \delta < \frac{p-p_0}{p_0}$ and $t > 0$. Then the preceding theorem states particularly that $T_{\mathcal{A}_p}(t)v \in W^{1,1+\delta}(S)$. Consequently, Sobolev's embedding theorem also yields that $T_{\mathcal{A}_p}(t)v$ is Hölder continuous of order $1-\frac{n}{1+\delta}$, or more accurately that there is a representative in the equivalence class which is Hölder continuous of this order.*

Remark 3.4.12. *It is clear that Corollary 3.4.9 implies*

$$
\lim_{t\to\infty} ||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)} = 0,\ \forall v \in L^2(S).
$$

Moreover, Theorem 3.4.10 yields that even $\lim_{t\to\infty} ||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^\infty(S)} = 0$, if $v \in L^p(S)$ and $p_0 < \frac{p}{n}$. It is beyond the scope of this thesis to obtain a uniform convergence result under more general assumptions. But it will be proven that L^q-convergence holds under more general assumptions for any $q \in [1,\infty)$.

Theorem 3.4.13. *Let $q \in [1,\infty)$ and $v \in L^q(S)$, then*

$$
\lim_{t\to\infty} T_{\mathcal{A}_p}(t)v = \overline{(v)}_S \text{ in } L^q(S). \tag{3.28}
$$

Proof. Again, thanks to Lemma 3.3.3 it suffices to prove the claim for $v \in L^q_0(S)$. So let, $v \in L^q_0(S)$ be given and let $\tau_k : \mathbb{R} \to \mathbb{R}$ denote the standard truncation function, for each $k \in (0,\infty)$.
Moreover, let $(\tilde{t}_m)_{m\in\mathbb{N}} \subseteq [0,\infty)$ be an arbitrary sequence such that $\lim_{m\to\infty} \tilde{t}_m = \infty$. In addition, $(t_m)_{m\in\mathbb{N}}$ is a subsequence such that

$$
\lim_{m\to\infty} T_{\mathcal{A}_p}(t_m)\tau_k(v) = \overline{(\tau_k(v))}_S, \text{ a.e. on } S. \tag{3.29}
$$

(Corollary 3.4.9 ensures the existence of such a subsequence, since $\tau_k(v) \in L^2(S)$.)

Now observe that Remark 3.2.6.iv) implies

$$||T_{\mathcal{A}_p}(t_m)\tau_k(v) - \overline{(\tau_k(v))}_S||_{L^\infty(S)} \leq 2k,$$

for all $m \in \mathbb{N}$ and $k \in (0, \infty)$. Consequently, this, together with (3.29) yields by virtue of dominated convergence that $\lim\limits_{m\to\infty} T_{\mathcal{A}_p}(t_m)\tau_k(v) = \overline{(\tau_k(v))}_S$ in $L^q(S)$ and therefore

$$\lim_{t\to\infty} ||T_{\mathcal{A}_p}(t)\tau_k(v) - \overline{(\tau_k(v))}_S||_{L^q(S)} = 0, \ \forall k \in (0, \infty). \tag{3.30}$$

Observe that clearly $\lim\limits_{k\to\infty} \tau_k(v) = v$ a.e. on S and that $|\tau_k(v) - v|^q \leq (2|v|)^q$ for all $k \in (0, \infty)$. Consequently, Lebesgue's theorem yields

$$\lim_{k\to\infty} \tau_k(v) = v, \text{ in } L^q(S). \tag{3.31}$$

Now let $\varepsilon > 0$ and choose $k_0 \in (0, \infty)$ sufficiently large such that

$$\max(||\tau_{k_0}(v) - v||_{L^q(S)}, |\overline{(\tau_{k_0}(v))}_S|) < \frac{\varepsilon}{3}, \tag{3.32}$$

which is possible, due to (3.31).
Moreover, (3.30) yields the existence of a $t_0 \in (0, \infty)$ such that

$$||T_{\mathcal{A}_p}(t)\tau_{k_0}(v) - \overline{(\tau_{k_0}(v))}_S||_{L^q(S)} < \frac{\varepsilon}{3}, \ \forall t \geq t_0. \tag{3.33}$$

Finally, it follows by combining (3.32), (3.33) and by using Remark 3.2.6.iii), that $||T_{\mathcal{A}_p}(t)v||_{L^q(S)} < \varepsilon$ for all $t \geq t_0$. $\square$

3.5 Asymptotic Results obtained by differential Inequality Techniques

Let us open this section by stating and proving two differential inequality results which will be exploited to prove (3.5), (3.6) and (3.7).
The first of these results is stated, but not proven in [16, Lemma 2.2]. For the sake of completeness, we will give a proof. The second one seems to be in use, but we were not able to find it rigorously stated anywhere in the literature, even though it might be available somewhere.

Lemma 3.5.1. *Let $\rho_1 \in (0, 1)$, $\kappa_1 \in (0, \infty)$ and let $f : [0, \infty) \to [0, \infty)$ be locally Lipschitz continuous, i.e. $f|_{[0,t]}$ is Lipschitz continuous for any $t \in (0, \infty)$. Moreover, assume*

$$f'(t) + \frac{\kappa_1}{\rho_1} f(t)^{1-\rho_1} \leq 0,$$

for a.e. $t \in (0,\infty)$. *Then we have*

$$f(t)^{\rho_1} \le (-\kappa_1 t + f(0)^{\rho_1})_+, \ \forall t \ge 0.$$

In particular, $f(t) = 0$ *for all* $t \in [t^*, \infty)$, *where* $t^* := \frac{1}{\kappa_1} f(0)^{\rho_1}$.

Proof. Set $a := f(0)$ and introduce $\tilde{f} : [0,\infty) \to [0,\infty)$ as $\tilde{f}(t) := (-\kappa_1 t + a^{\rho_1})_+^{\frac{1}{\rho_1}}$, for all $t \in [0,\infty)$. Then, $\tilde{f}$ is Lipschitz continuous, and a direct calculation verifies that

$$\tilde{f}'(t) + \frac{\kappa_1}{\rho_1}\tilde{f}(t)^{1-\rho_1} = 0,$$

for all $t \in (0,\infty)$.
Now, let us prove (by superposition) that $0 \le f(t) \le \tilde{f}(t)$ for all $t \in [0,\infty)$ which obviously implies all claims.
So, assume there is a $t_1 > 0$ such that $f(t_1) > \tilde{f}(t_1)$, then there is, since both functions are continuous and since $f(0) = \tilde{f}(0)$, a $t_0 \in [0, t_1)$ such that

$$f(t) > \tilde{f}(t), \ \forall t \in (t_0, t_1] \text{ and } \tilde{f}(t_0) = f(t_0). \tag{3.34}$$

But this implies

$$\begin{aligned}
\tilde{f}(t_1) - f(t_1) &= \tilde{f}(t_1) - f(t_1) - (\tilde{f}(t_0) - f(t_0)) \\
&= \int_{t_0}^{t_1} \tilde{f}'(t) - f'(t) dt \\
&\ge \int_{t_0}^{t_1} -\frac{\kappa_1}{\rho_1}\tilde{f}(t)^{1-\rho_1} + \frac{\kappa_1}{\rho_1} f(t)^{1-\rho_1} dt \\
&\ge 0,
\end{aligned}$$

which yields $\tilde{f}(t_1) \ge f(t_1)$ and therefore contradicts (3.34). □

The next differential inequality result can be proven with the same technique as Lemma 3.5.1. But, there is also another interesting technique available, which we will employ.

Lemma 3.5.2. *Let* $f : [0,\infty) \to [0,\infty)$ *be locally Lipschitz continuous on* $[0,\infty)$. *Moreover, assume that there are constants* $\kappa_2, \rho_2 \in (0,\infty)$ *such that*

$$f'(t) + \kappa_2 \rho_2 f(t)^{1+\frac{1}{\rho_2}} \le 0, \tag{3.35}$$

for a.e. $t \in (0,\infty)$. *Then we have*

$$f(t) \le \left(\kappa_2 t + f(0)^{-\frac{1}{\rho_2}}\right)^{-\rho_2}, \tag{3.36}$$

for all $t \in [0, \infty)$.

Proof. Firstly, as f is locally Lipschitz continuous, (3.35) yields that f is monotonically decreasing. Now set $I := \inf\{t \geq 0 : \; f(t) = 0\}$. If $I = 0$, then $f(t) = 0$ for all $t > 0$ and by continuity for all $t \geq 0$. Consequently, in this case (3.36) trivially holds. So assume $I > 0$ and let $\tilde{I} \in [0, I)$ be arbitrary but fixed and introduce $F : [0, \tilde{I}] \to [0, \infty)$ with $F(t) := f(t)^{-\frac{1}{\rho_2}}$. As $f(t) \geq f(\tilde{I}) > 0$ for all $t \in [0, \tilde{I}]$, F is, as is the composition of Lipschitz continuous functions, itself Lipschitz continuous. Consequently, we get

$$F(t) - F(0) = \int_0^t F'(\tau) d\tau = -\frac{1}{\rho_2} \int_0^t f(\tau)^{-\frac{1}{\rho_2}-1} f'(\tau) d\tau \geq \kappa_2 t, \;\forall t \in [0, \tilde{I}].$$

Thus (3.36) holds on $[0, \tilde{I}]$ and as $\tilde{I}$ was arbitrary, it holds on $t \in [0, I)$. Finally, if $I = \infty$ the proof is complete and if $I < \infty$, the infimum is (by continuity) a minimum and by monotonicity $f = 0$ on $[I, \infty)$, in which case (3.36) is trivial. $\square$

Throughout this section, let $f_{u,v} : [0, \infty) \to [0, \infty)$ be defined by

$$f_{u,v}(t) := \int_S \left(T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v\right)^2 d\lambda,$$

for all $t \in [0, \infty)$ and $u, v \in L^2(S)$.
Now let us demonstrate that $f_{u,v}$ is locally Lipschitz continuous, if $u, v \in D(A_p)$ and calculate its almost everywhere existing derivative. Afterwards, we will prove a technical approximation result, and then proceed by employing Lemma 3.5.1 and Lemma 3.5.2 to get the desired asymptotic results.

Lemma 3.5.3. *Let* $u, v \in D(A_p)$. *Then* $f_{u,v}$ *is locally Lipschitz continuous. Thus, it is differentiable almost everywhere. Moreover, we have* $T_{\mathcal{A}_p}(t)u, T_{\mathcal{A}_p}(t)v \in W^{1,p}_\gamma(S)$ *as well as*

$$f'_{u,v}(t) = -2 \int_S \gamma \left(|\nabla T_{\mathcal{A}_p}(t)u|^{p-2} \nabla T_{\mathcal{A}_p}(t)u - |\nabla T_{\mathcal{A}_p}(t)v|^{p-2} \nabla T_{\mathcal{A}_p}(t)v\right) \cdot (\nabla T_{\mathcal{A}_p}(t)u - \nabla T_{\mathcal{A}_p}(t)v) d\lambda,$$

for a.e. $t \in (0, \infty)$.

Proof. Firstly, let us verify the desired local Lipschitz continuity. To this end, fix $c > 0$ and note that $[0, c] \ni t \mapsto T_{\mathcal{A}_p}(t)u$ and $[0, c] \ni t \mapsto T_{\mathcal{A}_p}(t)v$ are by Remark 3.2.6.ii), w.r.t. $|| \cdot ||_{L^1(S)}$, Lipschitz continuous. So let $C_u, C_v \geq 0$ denote their Lipschitz constants. Then, Remark 3.2.6.iv) (with $q = \infty$) yields

$$|f_{u,v}(t_1) - f_{u,v}(t_2)| \leq |t_1 - t_2|(C_u + C_v)(2||u||_{L^\infty(S)} + 2||v||_{L^\infty(S)}),$$

for every $t_1, t_2 \in [0, c]$.
Consequently, f is locally Lipschitz continuous and as it is real-valued, it is also differentiable almost

everywhere.
Now, recall that for all $w \in D(A_p)$ we have $T_{\mathcal{A}_p}(t)w \in D(A_p)$ and $-T'_{\mathcal{A}_p}(t)w = A_p T_{\mathcal{A}_p}(t)w$ for a.e. $t \in (0,\infty)$, see Lemma 3.3.2. Thus, as $D(A_p) \subseteq W^{1,p}_{\gamma}(S)$, it remains to prove the formula regarding $f'_{u,v}$.
Let $(h_m)_{m\in\mathbb{N}} \subseteq (0,\infty)$ be a null-sequence. As $T_{\mathcal{A}_p}(t)u, T_{\mathcal{A}_p}(t)v \in D(A_p)$, $-T'_{\mathcal{A}_p}(t)u = A_p T_{\mathcal{A}_p}(t)u$ and $-T'_{\mathcal{A}_p}(t)v = A_p T_{\mathcal{A}_p}(t)v$, we get (by passing to a subsequence if necessary) that

$$\lim_{m\to\infty} \frac{1}{h_m} \left(\left(T_{\mathcal{A}_p}(t+h_m)u - T_{\mathcal{A}_p}(t+h_m)v\right) - \left(T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v\right)\right) = -\left(A_p T_{\mathcal{A}_p}(t)u - A_p T_{\mathcal{A}_p}(t)v\right),$$

a.e. on S, for a.e. $t \in (0,\infty)$. Now Remark 3.2.6.vi) (with $q = \infty$) enables us to conclude from Lebesgue's theorem that the preceding convergence also holds w.r.t. $||\cdot||_{L^2(S)}$.
Moreover, we also have (by passing to a subsequence if necessary) that

$$\lim_{m\to\infty} \left(\left(T_{\mathcal{A}_p}(t+h_m)u - T_{\mathcal{A}_p}(t+h_m)v\right) + \left(T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v\right)\right) = 2\left(T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v\right)$$

a.e. on S, for a.e. $t \in (0,\infty)$, and by virtue of Lebesgue's theorem (which is thanks to Remark 3.2.6.iii) applicable) also w.r.t. $||\cdot||_{L^2(S)}$ for a.e. $t \in (0,\infty)$. Thus, combining the preceding two equations yields

$$\lim_{m\to\infty} \frac{1}{h_m}(f_{u,v}(t+h_m) - f_{u,v}(t)) = -2\int\limits_S (T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v)(A_p T_{\mathcal{A}_p}(t)u - A_p T_{\mathcal{A}_p}(t)v)d\lambda,$$

for a.e. $t \in (0,\infty)$. Consequently, as we already know that $f_{u,v}$ is differentiable almost everywhere, we get

$$f'_{u,v}(t) = -2\int\limits_S (T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v)(A_p T_{\mathcal{A}_p}(t)u - A_p T_{\mathcal{A}_p}(t)v)d\lambda, \tag{3.37}$$

for a.e. $t \in (0,\infty)$. Finally, (3.37) implies, by using $(T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v)$ as a test function in the definition of A_p, that

$$f'_{u,v}(t) = -2\int\limits_S \gamma\left(|\nabla T_{\mathcal{A}_p}(t)u|^{p-2}\nabla T_{\mathcal{A}_p}(t)u - |\nabla T_{\mathcal{A}_p}(t)v|^{p-2}\nabla T_{\mathcal{A}_p}(t)v\right)\cdot(\nabla T_{\mathcal{A}_p}(t)u - \nabla T_{\mathcal{A}_p}(t)v)d\lambda,$$

for a.e. $t \in (0,\infty)$; which completes the proof. □

Lemma 3.5.4. *$D(A_p)$ is a dense subset of $(L^2(S), ||\cdot||_{L^2(S)})$ and $D(A_p) \cap L^2_0(S)$ is a dense subset of $(L^2_0(S), ||\cdot||_{L^2(S)})$.*

Proof. Let us start by proving the first assertion. Firstly, it suffices to prove that there is for each $h \in L^\infty(S)$ a sequence $(v_m)_{m\in\mathbb{N}} \subseteq D(A_p)$ such that

$$\lim_{m\to\infty} v_m = h \text{ in } L^2(S),$$

since $L^\infty(S)$ is a dense subspace of $L^2(S)$.
So, let $h \in L^\infty(S)$ be arbitrary but fixed.
Since $\mathcal{A}_p$ is m-accretive there are for each $m \in \mathbb{N}$ functions $v_m \in D(\mathcal{A}_p)$, $\hat{v}_m \in \mathcal{A}_p v_m$, such that

$$h = v_m + \frac{1}{m}\hat{v}_m \text{ a.e. on } S, \tag{3.38}$$

for all $m \in \mathbb{N}$.
Moreover, by complete accretivity we get

$$||v_m||_{L^\infty(S)} \leq ||h||_{L^\infty(S)} < \infty, \ \forall m \in \mathbb{N}. \tag{3.39}$$

Consequently $v_m \in L^\infty(S)$ and therefore $v_m \in D(A_p)$ for all $m \in \mathbb{N}$.
Moreover, (3.39) also implies that the sequence $(||v_m||_{L^2(S)})_{m\in\mathbb{N}}$ is bounded. Hence, by passing to a subsequence if necessary, there is an $\tilde{h} \in L^2(S)$ such that

$$\text{w - }\lim_{m\to\infty} v_m = \tilde{h} \text{ in } L^2(S). \tag{3.40}$$

Now observe that

$$\lim_{m\to\infty} \frac{1}{m} \int_S \gamma |\nabla v_m|^{p-2} \nabla v_m \cdot \nabla \varphi d\lambda = 0, \ \forall \varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S), \tag{3.41}$$

since we obtain for all $\varphi \in W^{1,p}_\gamma(S) \cap L^\infty(S)$ and $q := \frac{p}{p-1}$ that

$$\begin{aligned}
& \left| \left(\frac{1}{m}\right)^{\frac{1}{q}} \int_S \gamma |\nabla v_m|^{p-2} \nabla v_m \cdot \nabla \varphi d\lambda \right| \\
\leq \ & \left(\frac{1}{m}\right)^{\frac{1}{q}} \left(\int_S \gamma |\nabla v_m|^{p-2} \nabla v_m \cdot \nabla v_m d\lambda \right)^{\frac{1}{q}} ||\nabla \varphi||_{L^p(S,\gamma;\mathbb{R}^n)} \\
= \ & \left(\int_S (h - v_m) v_m d\lambda \right)^{\frac{1}{q}} ||\nabla \varphi||_{L^p(S,\gamma;\mathbb{R}^n)} \\
\leq \ & \left(\int_S (||h||_{L^\infty(S)} + ||h||_{L^\infty(S)}) ||h||_{L^\infty(S)} d\lambda \right)^{\frac{1}{q}} ||\nabla \varphi||_{L^p(S,\gamma;\mathbb{R}^n)} \\
= \ & \left(2\lambda(S) ||h||^2_{L^\infty(S)} \right)^{\frac{1}{q}} ||\nabla \varphi||_{L^p(S,\gamma;\mathbb{R}^n)},
\end{aligned}$$

where Cauchy Schwarz inequality, Hölder's inequality, $\hat{v}_m = A_p v_m$, (3.38) and (3.39) were used.

Moreover, (3.41) yields

$$\begin{aligned}\int_S (h-\tilde{h})\varphi d\lambda &= \lim_{m\to\infty}\int_S (h-v_m)\varphi d\lambda \\ &= \lim_{m\to\infty}\int_S \frac{1}{m}\hat{v}_m\varphi d\lambda \\ &= \lim_{m\to\infty}\frac{1}{m}\int_S \gamma|\nabla v_m|^{p-2}\nabla v_m\cdot\nabla\varphi d\lambda \\ &= 0,\end{aligned}$$

for all $\varphi\in W^{1,p}_\gamma(S)\cap L^\infty(S)$ and therefore $h=\tilde{h}$.
Moreover, by complete accretivity we have $||v_m||_{L^2(S)}\leq ||v_m+\frac{1}{m}\hat{v}_m||_{L^2(S)}$ and thus we also get $||v_m||_{L^2(S)}\leq ||h||_{L^2(S)}=||\tilde{h}||_{L^2(S)}$ for all $m\in\mathbb{N}$, which implies particularly that

$$\limsup_{m\to\infty}||v_m||_{L^2(S)}\leq ||\tilde{h}||_{L^2(S)}.$$

Conclusively, this, (3.40) and the uniform convexity of $(L^2(S),||\cdot||_{L^2(S)})$ yield $\lim\limits_{m\to\infty}v_m=\tilde{h}=h$, in $L^2(S)$, which proves the first assertion.
Now, the second assertions is easily deduced from the first one: Let $h\in L^2_0(S)$, then there is a sequence $(h_m)_{m\in\mathbb{N}}\subseteq D(A_p)$, such that $\lim\limits_{m\to\infty}h_m=h$ in $L^2(S)$. Now, one instantly verifies that $h_m-\overline{(h_m)}\in D(A_p)$, with $A_p(h_m-\overline{(h_m)})=A_ph_m$. Consequently, $h_m-\overline{(h_m)}\in D(A_p)\cap L^2_0(S)$ for all $m\in\mathbb{N}$ and clearly $\lim\limits_{m\to\infty}h_m-\overline{(h_m)}=h$, in $L^2(S)$ since $\overline{(h)}=0$. ☐

Remark 3.5.5. *The proof of Lemma 3.5.4 has revealed the following: Let $h\in L^\infty(S)$ and introduce $v_m:=(Id+\frac{1}{m}\mathcal{A}_p)^{-1}h$.*[2] *Then we have*

$$\lim_{m\to\infty}v_m=h,\ \textit{in}\ L^2(S).$$

This fact is not needed in this section, but will be useful in Chapter 7. Moreover, the technique which we employed to prove Lemma 3.5.4 is a standard technique to prove such density results. It is for example also used in [4, Prop. 5.1].

Theorem 3.5.6. *Assume that the interval $\left(\frac{p_0(n-2)}{n+2}+p_0,2\right)$ is non-empty and that $p\in\left(\frac{p_0(n-2)}{n+2}+p_0,2\right)$. In addition, introduce*

$$\kappa_1:=(2-p)\left(\tilde{C}^p_S\left(C^{\frac{2n}{n+2}}_{S,\frac{2n}{n+2}}+1\right)^{\frac{np+2p}{2n}}\left(\int_S\gamma^{\frac{2n}{2n-np-2p}}d\lambda\right)^{\frac{np+2p-2n}{2n}}\right)^{-1}\in(0,\infty), \tag{3.42}$$

[2] See Remark 2.1.9 for the definition of the resolvent.

where $\tilde{C}_S$ is the operator norm of the continuous injection $W^{1,\frac{2n}{n+2}} \hookrightarrow L^2(S)$. Then we have

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}||^{2-p}_{L^2(S)} \leq (-\kappa_1 t + ||v - \overline{(v)}||^{2-p}_{L^2(S)})_+, \tag{3.43}$$

for all $t \geq 0$ and $v \in L^2(S)$. In particular, $T_{\mathcal{A}_p}(t)v = \overline{(v)}$ for all $t \geq \frac{||v-\overline{(v)}||^{2-p}_{L^2(S)}}{\kappa_1}$ and $v \in L^2(S)$.

Proof. Firstly, it is plain that (3.43) implies $T_{\mathcal{A}_p}(t)v = \overline{(v)}$ for all $t \geq \frac{||v-\overline{(v)}||^{2-p}_{L^2(S)}}{\kappa_1}$. Secondly, Lemma 3.3.3 yields that it suffices to prove (3.43) for $v \in L^2_0(S)$. Moreover, the right-hand-side of (3.43) clearly depends continuously on v, and appealing to Remark 3.2.6.iii) yields that the left-hand-side also depends continuously on v. Thus by invoking Lemma 3.5.4, we get that it suffices to prove (3.43) for $v \in D(A_p) \cap L^2_0(S)$.
So let $v \in D(A_p) \cap L^2_0(S)$ be given and assume $p \in \left(\frac{p_0(n-2)}{n+2} + p_0, 2\right) \neq \emptyset$. Moreover, note that $\frac{2n}{n+2} < n$, since $n \neq 1$. Consequently, Sobolev's embedding theorem yields that there is a continuous injection $W^{1,\frac{2n}{n+2}}(S) \hookrightarrow L^2(S)$. So let $\tilde{C}_S$ denote its operator norm.
Now, introduce $f_v : [0,\infty) \to [0,\infty)$, by

$$f_v(t) := f_{0,v}(t) = ||T_{\mathcal{A}_p}(t)v||^2_{L^2(S)},$$

for all $t \geq 0$.
Moreover, note that

$$0 \leq \frac{2n}{n+2} - 1 = \frac{1}{p_0}\left(\frac{p_0(n-2)}{n+2} + p_0\right) - 1 < \frac{p}{p_0} - 1 = \frac{p-p_0}{p_0}. \tag{3.44}$$

In addition, appealing to Lemma 3.5.3 yields that f_v is locally Lipschitz continuous and differentiable a.e. with

$$f_v'(t) = -2||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^p(S,\gamma;\mathbb{R}^n)} \tag{3.45}$$

for almost every $t \in (0,\infty)$. Moreover, as $T_{\mathcal{A}_p}(t)v \in W^{1,p}_\gamma(S)$ for a.e. $t \in (0,\infty)$, we get by employing Lemma 3.4.5 and (3.44) that $T_{\mathcal{A}_p}(t)v \in W^{1,\frac{2n}{n+2}}(S)$ as well as

$$||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^{\frac{2n}{n+2}}(S;\mathbb{R}^n)} \leq \left(\int_S \gamma^{\frac{2n}{2n-np-2p}}\, d\lambda\right)^{\frac{np+2p-2n}{2n}} ||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^p(S,\gamma;\mathbb{R}^n)} \tag{3.46}$$

and in particular, we get

$$\int_S \gamma^{\frac{2n}{2n-np-2p}}\, d\lambda < \infty,$$

which implies that the integral occurring in the definition of κ_1 is finite. Now, it follows from Sobolev's embedding theorem, Poincaré's inequality and (3.46) that

$$\begin{aligned}
f_v(t)^{\frac{p}{2}} &= ||T_{\mathcal{A}_p}(t)v||^p_{L^2(S)} \\
&\leq \tilde{C}_S^p ||T_{\mathcal{A}_p}(t)v||^p_{W^{1,\frac{2n}{n+2}}(S)} \\
&= \tilde{C}_S^p \left(||T_{\mathcal{A}_p}(t)v||^{\frac{2n}{n+2}}_{L^{\frac{2n}{n+2}}(S)} + ||\nabla T_{\mathcal{A}_p}(t)v||^{\frac{2n}{n+2}}_{L^{\frac{2n}{n+2}}(S;\mathbb{R}^n)} \right)^{\frac{np+2p}{2n}} \\
&\leq \tilde{C}_S^p \left(C^{\frac{2n}{n+2}}_{S,\frac{2n}{n+2}} ||\nabla T_{\mathcal{A}_p}(t)v||^{\frac{2n}{n+2}}_{L^{\frac{2n}{n+2}}(S;\mathbb{R}^n)} + ||\nabla T_{\mathcal{A}_p}(t)v||^{\frac{2n}{n+2}}_{L^{\frac{2n}{n+2}}(S;\mathbb{R}^n)} \right)^{\frac{np+2p}{2n}} \\
&= \tilde{C}_S^p \left(C^{\frac{2n}{n+2}}_{S,\frac{2n}{n+2}} + 1 \right)^{\frac{np+2p}{2n}} ||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^{\frac{2n}{n+2}}(S;\mathbb{R}^n)} \\
&\leq \tilde{C}_S^p \left(C^{\frac{2n}{n+2}}_{S,\frac{2n}{n+2}} + 1 \right)^{\frac{np+2p}{2n}} \left(\int\limits_S \gamma^{\frac{2n}{2n-np-2p}} \, d\lambda \right)^{\frac{np+2p-2n}{2n}} ||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^p(S,\gamma;\mathbb{R}^n)} \\
&= \frac{2-p}{\kappa_1} ||\nabla T_{\mathcal{A}_p}(t)v||^p_{L^p(S,\gamma;\mathbb{R}^n)},
\end{aligned}$$

for a.e. $t \in (0,\infty)$. Thus, employing (3.45) yields

$$f_v'(t) + \frac{\kappa_1}{\rho_1} f_v(t)^{1-\rho_1} \leq 0,$$

for a.e. $t \in (0,\infty)$, where $\rho_1 := \frac{2-p}{2}$. Consequently, invoking Lemma 3.5.1 yields

$$f_v(t)^{\rho_1} \leq (-\kappa_1 t + f_v(0)^{\rho_1})_+,$$

for all $t \geq 0$, which completes our proof. $\square$

Remark 3.5.7. *Note that if $n = 2$ and $p_0 = 1$, then we can either apply Theorem 3.5.6 or Theorem 3.4.10, i.e. depending on the value of p, either (3.43) or (3.25) holds. Thus, if $n = 2$ this works particularly if γ is bounded from below away from zero, see Remark 3.4.3.*

Remark 3.5.8. *The following proposition (and the results we deduce from it), rely on the assumptions $p \in (2,\infty)$ and $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$. Note that $\frac{2}{2-p} = \frac{1}{1-\frac{p}{2}}$. Thus, employing Lemma 3.4.4 yields: If $p_0 < \frac{p}{2}$, then $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$. Moreover, it is easily verified by Hölder's inequality that for $p \in (2,\infty)$, the assumption $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$ implies $\int\limits_S \gamma^{\frac{1}{1-p}} d\lambda < \infty$.*

Proposition 3.5.9. *Assume $p \in (2,\infty)$ and $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$. Moreover, introduce $u, v \in L^2_0(S)$. Then we have*

$$||T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v||_{L^2(S)} \leq \left(\kappa_2 t + ||u-v||^{2-p}_{L^2(S)} \right)^{\frac{1}{2-p}}, \ \forall t \geq 0, \tag{3.47}$$

where $\kappa_2 := (p-2)2^{-(p-2)} \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{2-p}{2}} C_{S,2}^{-p}$.

Proof. Thanks to Remark 3.2.6.iii) and Lemma 3.5.4 it suffices to prove (3.47) for $u, v \in L_0^2(S) \cap D(A_p)$. Moreover, we have $W_\gamma^{1,p}(S) \subseteq W^{1,2}(S)$, since appealing to Hölder's inequality gives

$$\int\limits_S |\nabla\varphi|^2 d\lambda = \int\limits_S \gamma^{-\frac{2}{p}} \gamma^{\frac{2}{p}} |\nabla\varphi|^2 d\lambda \leq \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{p-2}{p}} \left(\int\limits_S \gamma|\nabla\varphi|^p d\lambda\right)^{\frac{2}{p}} < \infty, \ \forall \varphi \in W_\gamma^{1,p}(S).$$

Consequently, employing Poincaré's inequality yields

$$\int\limits_S \gamma|\nabla\varphi|^p d\lambda \geq C_{S,2}^{-p} \left(\int\limits_S \varphi^2 d\lambda\right)^{\frac{p}{2}} \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{2-p}{2}}, \ \forall \varphi \in W_\gamma^{1,p}(S) \cap L_0^2(S). \tag{3.48}$$

Moreover, it is well known that $(|x|^{p-2}x - |y|^{p-2}y) \cdot (x-y) \geq 2^{2-p}|x-y|^p$ for all $x, y \in \mathbb{R}^n$, see [11, Lemma 3.6]. By Lemma 3.3.2, we get $T_{\mathcal{A}_p}(t)u, T_{\mathcal{A}_p}(t)v \in W_\gamma^{1,p}(S)$ for a.e. $t \in (0,\infty)$ and by the aid of Lemma 3.3.6 we then obtain $T_{\mathcal{A}_p}(t)u - T_{\mathcal{A}_p}(t)v \in W_\gamma^{1,p}(S) \cap L_0^2(S)$ for a.e. $t \in (0,\infty)$. These observations enable us to conclude from (3.48) and Lemma 3.5.3 that

$$f'_{u,v}(t) \leq -2^{3-p} \int\limits_S \gamma|\nabla T_{\mathcal{A}_p}(t)u - \nabla T_{\mathcal{A}_p}(t)v|^p d\lambda \leq -2^{3-p} C_{S,2}^{-p} \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{2-p}{2}} f_{u,v}(t)^{\frac{p}{2}},$$

for a.e. $t \in (0,\infty)$. Thus, by setting $\rho_2 := \frac{2}{p-2}$, we get $f'_{u,v}(t) + \kappa_2\rho_2 f_{u,v}(t)^{1+\frac{1}{\rho_2}} \leq 0$ for a.e. $t \in (0,\infty)$. Hence, invoking Lemma 3.5.2 yields $f_{u,v}(t) \leq (\kappa_2 t + f_{u,v}(0)^{-\frac{1}{\rho_2}})^{-\rho_2}$; thus by taking the square root and noting that $u, v \in L_0^2(S) \cap D(A_p)$ were arbitrary, we get (3.47) for all $u, v \in L_0^2(S) \cap D(A_p)$ and the proof is complete. $\square$

Theorem 3.5.10. *Assume* $p \in (2,\infty)$ *and* $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$. *Then we have*

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)} \leq \lambda(S)^{\frac{1}{2}} \kappa_2^{\frac{1}{2-p}} \left(\frac{1}{t}\right)^{\frac{1}{p-2}}, \tag{3.49}$$

for all $t > 0$ *and* $v \in L^1(S)$, *where* $\kappa_2 := (p-2)2^{-(p-2)} \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{2-p}{2}} C_{S,2}^{-p}$.

Proof. By Lemma 3.3.3 it suffices to prove the claim for $v \in L_0^1(S)$. Moreover, by Remark 3.2.6.iii) it suffices the prove the claim for $v \in L_0^2(S)$. But then, the claim is trivial, since appealing to Proposition 3.5.9 (with $u = 0$) and Hölder's inequality yield

$$||T_{\mathcal{A}_p}(t)v||_{L^1(S)} \leq \lambda(S)^{\frac{1}{2}} ||T_{\mathcal{A}_p}(t)v||_{L^2(S)} \leq \lambda(S)^{\frac{1}{2}} \left(\kappa_2 t + ||v||_{L^2(S)}^{2-p}\right)^{\frac{1}{2-p}} \leq \lambda(S)^{\frac{1}{2}} (\kappa_2 t)^{\frac{1}{2-p}},$$

for all $t > 0$ $\square$

Moreover, the following corollary is also easily deduced from Proposition 3.5.9.

Corollary 3.5.11. *Assume* $p \in (2, \infty)$ *and* $\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda < \infty$. *Then we have*

$$||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^2(S)} \leq \left(\kappa_2 t + ||v - \overline{(v)}_S||_{L^2(S)}^{2-p}\right)^{\frac{1}{2-p}}, \ \forall t \geq 0, \tag{3.50}$$

for all $v \in L^2(S)$, *where* $\kappa_2 := (p-2)2^{-(p-2)} \left(\int\limits_S \gamma^{\frac{2}{2-p}} d\lambda\right)^{\frac{2-p}{2}} C_{S,2}^{-p}$.

The advantage of (3.50) compared to (3.49) is that (3.50) is also sharp for $t \searrow 0$, whereas the right-hand-side of (3.49) diverges to $+\infty$ for $t \searrow 0$. On the other hand, the advantage of (3.49) is that it is applicable for all $v \in L^1(S)$ and not just $v \in L^2(S)$.
Moreover, note that the order of convergence (for $t \to \infty$) in (3.49) and (3.50) is better than in (3.25). Of course, the clear advantage of (3.25) is that this is a bound w.r.t $|| \cdot ||_{L^\infty(S)}$.

In fact, it would have been possible (and easier) to directly prove Corollary 3.5.11 instead of Proposition 3.5.9, and to then deduce Theorem 3.5.10 from this corollary. The reason why we undertook this detour is that Proposition 3.5.9 is needed in Section 6.4.

Chapter 4

Abstract Cauchy Problems driven by random Measures: Existence and Uniqueness

4.1 Outline & Highlights

In this chapter, we will set up an existence and uniqueness theory for random evolution inclusions of the form

$$\eta(t,z)N_\Theta(dt\otimes z)\in dX(t)+\mathcal{A}X(t)dt, \tag{ACPRM}$$

where $\mathcal{A}\,:\,D(\mathcal{A})\to 2^V$ is a multi-valued operator, $(V,||\cdot||_V)$ is a separable Banach space, $N_\Theta\,:\,(\mathfrak{B}((0,\infty))\otimes\mathcal{Z})\times\Omega\to\mathbb{N}_0\cup\{\infty\}$ is the random counting measure induced by a finite, simple point process Θ, $(Z,\mathcal{Z})$ is a measurable space, $\eta:(0,\infty)\times Z\times\Omega\to V$ is a (jointly measurable) drift function, $(\Omega,\mathcal{F},\mathbb{P})$ denotes our complete probability space and X is a V-valued stochastic process supposed to fulfill (1.5) in some sense. Equations of the above form will be called abstract Cauchy problem driven by a random measure.
There is a comprehensive existence and uniqueness theory for abstract Cauchy problems of the form

$$0\in u'(t)+\mathcal{A}u(t),\ \text{for a.e. }t\in(0,\infty), \tag{4.1}$$

which we briefly outlined in Section 2.1. We would like to obtain criteria which are similar to the existence/uniqueness results of Section 2.1, for (ACPRM).
Surprisingly, it seems that there are very few results connecting these areas.

The first step in deriving an existence/uniqueness theory for (ACPRM) is of course setting up a notion of a solution. In fact, we will introduce to different kinds of solutions: Strong and mild ones. Instead of simply giving the definition of a strong/mild solution, let us give the reader an intuition on how to set up such a notion: At first, one would try to define a solution of (ACPRM) as a process $X : [0,\infty) \times \Omega \to V$ which is sufficiently regular and fulfills

$$\int_{(0,t]\times Z} \eta(\tau, z) N_\Theta(d\tau \otimes z) \in X(t) - x + \int_0^t \mathcal{A}X(\tau)d\tau, \ \forall t > 0$$

where $x : \Omega \to V$ is an $\mathcal{F}$-$\mathfrak{B}(V)$-measurable initial, i.e. $X(0) = x$. The obvious issue is that $\mathcal{A}$ takes values in the power set of V. Consequently, one either has to somehow define the set-valued integral, or one has to "pick" for each τ and ω an element of $\mathcal{A}X(\tau,\omega)$ by some rule. We choose to do the latter. To define this rule, assume that $\mathcal{A}$ is m-accretive and densely defined and let $T_\mathcal{A}$ denote the semigroup associated to $\mathcal{A}$, see Definition 2.1.8. Moreover, assume that $T_\mathcal{A}$ admits an infinitesimal generator, which we denoted by $\mathcal{A}^\circ : V \to V$, see Definition 2.1.11.
Consequently, we have found a rule and would like to define a solution as a process fulfilling

$$\int_{(0,t]\times Z} \eta(\tau, z) N_\Theta(d\tau \otimes z) = X(t) - x + \int_0^t \mathcal{A}^\circ X(\tau)d\tau, \ \forall t > 0.$$

The issue with this equation is that one needs that the Bochner integral $\int_0^t \mathcal{A}^\circ X(\tau)d\tau$ exists for all $t > 0$ with probability one; which is unfortunately not necessarily fulfilled. To get an existence result as applicable as possible, we will therefore formulate the preceding equation in a weak sense; more precisely, we will coin the term strong solution, as a process X fulfilling

$$\int_{(0,t]\times Z} \langle \Psi, \eta(\tau, z)\rangle_V N_\Theta(d\tau \otimes z) = \langle \Psi, X(t) - x\rangle_V + \int_0^t \langle \Psi, \mathcal{A}^\circ X(\tau)\rangle_V d\tau, \ \forall t > 0 \tag{4.2}$$

for all $\psi \in V^*$, where V' denotes the dual of V, $\langle \cdot, \cdot \rangle_V$ the duality between V and V' and $V^* \subseteq V'$ is a set which separates points. Of course, the process X also has to fulfill some regularity assumptions, which mainly serve to make sure the uniqueness of solutions.
In addition, we will introduce a "mild solution of (ACPRM)", as a process which can be approximated in some sense by strong solutions.

Having done so, we shall see that (ACPRM) has for any $\mathcal{F}$-$\mathfrak{B}(V)$-measurable initial $x : \Omega \to V$ a unique mild solution, if: $\mathcal{A}$ is densely defined and m-accretive, $T_\mathcal{A}$ is domain invariant and admits an infinitesimal generator, and there is a dense subset $\mathcal{V} \subseteq V$, which is invariant w.r.t. $T_\mathcal{A}$ and such that $\langle \Psi, \mathcal{A}^\circ T_\mathcal{A}(\cdot)v\rangle_V \in L^1(0,t)$ for all $t > 0$, $v \in \mathcal{V}$ and $\Psi \in V^*$. Particularly, this result only requires that η and x are measurable. Moreover, it will be demonstrated that mild solutions depend Lipschitz continuously on the initial x and the drift η. Furthermore, if $x \in \mathcal{V}$ and $\eta(t,z) \in \mathcal{V}$ for all $t > 0$,

$z \in Z$ almost surely, then the mild solution is even a strong one. Moreover, we will see that the solution process is the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}})$ in V, where: $(\beta_m)_{m\in\mathbb{N}}$ is the sequence of inter-arrival times of Θ and $\eta_m(\omega) := \eta(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, where $\alpha_m := \sum_{k=1}^{m} \beta_k$ for all $m \in \mathbb{N}$. In addition, these results will be exemplified by the weighted p-Laplacian evolution equation considered in Chapter 3 as well as by the two one-dimensional semigroups introduced in Remark 2.2.7.

The main advantages of employing the theory of m-accretive operators to solve (ACPRM) is that this works on any separable Banach space. Moreover, the fairly lean assumptions on $\mathcal{A}$ allow to consider a large group of operators.

That all of this works is highly owed to the fact that the noise term "$\eta(t,z)N_\Theta(dt \otimes z)$" is a pure jump noise. However, it might be possible that one can extend these results to more general noise terms by applying the theory of m-accretive operators for inhomogeneous Cauchy problems.

This chapter is structured as follows: In Section 4.2, we give a very brief introduction to point processes. Section 4.3 is this chapter's centerpiece; all general results regarding existence and uniqueness are proven there. And last but not least, the applicability of these results to the weighted p-Laplacian evolution equation and the semigroups considered in Remark 2.2.7 is demonstrated in Section 4.4.

4.2 Intermezzo: Point Processes

Let $(Z, \mathcal{Z})$ be a measurable space and recall that $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. We call a mapping $\theta : D(\theta) \to Z$, where $D(\theta) \subseteq (0,\infty)$ is countable, a point function. Moreover, $\pi(Z)$ denotes the set of all point functions mapping into Z and we equip this space with the σ-algebra

$$\Pi(Z) := \sigma\big(\big\{\theta \in \pi(Z) : \#\{t \in D(\theta) : (t,\theta(t)) \in U\} = k\big\};\ k \in \mathbb{N}_0,\ U \in \mathfrak{B}((0,\infty)) \otimes \mathcal{Z}\big).$$

In addition, a mapping $\Theta : \Omega \to \pi(Z)$ which is $\mathcal{F} - \Pi(Z)$-measurable, is called a random point function, or point process. Moreover, for a point process $\Theta : \Omega \to \pi(Z)$, we introduce the mapping $N_\Theta : (\mathfrak{B}((0,\infty)) \otimes \mathcal{Z}) \times \Omega \to \mathbb{N}_0 \cup \{\infty\}$ by

$$N_\Theta(U,\omega) := \#\{t \in D(\Theta(\omega)) : (t, \Theta(\omega)(t)) \in U\},\ \forall U \in \mathfrak{B}((0,\infty)) \otimes \mathcal{Z},\ \omega \in \Omega$$

and refer to it as the counting measure induced by Θ.
It is plain to verify that the mapping $\mathfrak{B}((0,\infty)) \otimes \mathcal{Z} \ni U \mapsto N_\Theta(U,\omega)$ is a measure for each $\omega \in \Omega$ and that $\Omega \ni \omega \mapsto N_\Theta(U,\omega)$ is a (extended) real-valued random variable for each $\omega \in \Omega$. (Hereby extended refers to the fact that this random variable might take the value $+\infty$.)
Note that, by definition, any point process Θ is simple, i.e. $N_\Theta(\{t \times z\},\omega) \le 1$ for all $(t,z) \in (0,\infty) \times Z$ and $\omega \in \Omega$.

A point process $\Theta : \Omega \to \pi(Z)$, or the random measure N_Θ induced by Θ, is called finite if $\mathbb{E}N_\Theta((0,t] \times Z) < \infty$ for all $\forall t > 0$. It is easy to infer that this implies $N_\Theta((0,t] \times Z) < \infty$ for all $t \in (0,\infty)$ with probability one.

Remark 4.2.1. *Let N_Θ be the counting measure induced by a finite point process $\Theta : \Omega \to \Pi(Z)$. Then there is a $\mathbb{P}$-null-set $M \in \mathcal{F}$, such that $N_\Theta((0,t] \times Z, \omega) < \infty$ for all $t > 0$ and $\omega \in \Omega \setminus M$. Hence, $D(\Theta(\omega)) \cap (0,t]$ contains only finitely many elements for any $t > 0$; which yields that $D(\Theta(\omega))$ is an isolated set for any $\omega \in \Omega \setminus M$. Therefore, we can find a sequence of mappings $(\alpha_m)_{m\in\mathbb{N}}$, with $\alpha_m : \Omega \to (0,\infty)$, such that*

i) $D(\Theta(\omega)) = \{\alpha_1(\omega), \alpha_2(\omega), ...\}$ for all $\omega \in \Omega \setminus M$ and

ii) $0 < \alpha_m(\omega) < \alpha_{m+1}(\omega) < \infty$ for all $m \in \mathbb{N}$ and $\omega \in \Omega \setminus M$.

The sequence of mappings $(\alpha_m)_{m\in\mathbb{N}}$ fulfilling these two assertions is obviously unique on $\Omega \setminus M$. We will refer to the (up to a $\mathbb{P}$-null-set) uniquely determined sequence fulfilling the assertions i)-ii), as the sequence of hitting times induced by Θ. Moreover, the sequence $(\beta_m)_{m\in\mathbb{N}}$, with $\beta_m : \Omega \to (0,\infty)$, fulfilling $\beta_1 = \alpha_1$ and $\beta_m = \alpha_m - \alpha_{m-1}$ for all $m \in \mathbb{N} \setminus \{1\}$ on $\Omega \setminus M$ is called the sequence of inter-arrival times induced by Θ.
One instantly verifies that each α_m (and thus also each β_m) is $\mathcal{F}$-$\mathfrak{B}((0,\infty))$-measurable and that $\lim\limits_{m\to\infty} \alpha_m = \infty$ almost surely. Moreover, with slightly more effort one verifies that the mapping defined by $\Omega \ni \omega \mapsto \Theta(\omega)(\alpha_m(\omega))$ is $\mathcal{F} - \mathcal{Z}$-measurable.

For a function $f : (0,\infty) \times Z \times \Omega \to \mathbb{R}$ which is $\mathfrak{B}((0,\infty)) \otimes \mathcal{Z} \otimes \mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable and a finite point measure N_Θ, we introduce

$$\left(\int\limits_{(0,t]\times Z} f(\tau,z) N_\Theta(d\tau \otimes dz) \right)(\omega) := \int\limits_{(0,t]\times Z} f(\tau,z,\omega) N_\Theta(d\tau \otimes dz, \omega),\ \forall t > 0,\ \mathbb{P}-\text{a.e. } \omega \in \Omega. \quad (4.3)$$

Hereby the right hand side is understood as a Lebesgue integral w.r.t. the measure $N(\cdot,\omega)$. Let us conclude this section with the following lemma, which states in particular that the right-hand-side integral in (4.3) is in fact finite and well-defined:

Lemma 4.2.2. *Let $\Theta : \Omega \to \pi(Z)$ be a finite point process and $N_\Theta : (\mathfrak{B}((0,\infty)) \otimes \mathcal{Z}) \times \Omega \to \mathbb{N}_0 \cup \{\infty\}$ the counting measure induced by Θ. Moreover, let $(\alpha_m)_{m\in\mathbb{N}}$ be the sequence of hitting times induced by Θ. Now, let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set such that*

$$D(\Theta(\omega)) = \{\alpha_1(\omega),\ \alpha_2(\omega), ...\},\ 0 < \alpha_m(\omega) < \alpha_{m+1}(\omega),\ \forall m \in \mathbb{N} \text{ and } \lim_{m\to\infty} \alpha_m(\omega) = \infty, \quad (4.4)$$

for all $\omega \in \Omega \setminus M$. Finally, introduce $f : (0,\infty) \times Z \times \Omega \to \mathbb{R}$ and assume that it is $\mathfrak{B}((0,\infty)) \otimes \mathcal{Z} \otimes \mathcal{F}$-$\mathfrak{B}(\mathbb{R})$-measurable.
Then, for any $m \in \mathbb{N}$, the mapping $\Omega \ni \omega \mapsto f(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega) := f_m(\omega)$ is well defined on

$\Omega \setminus M$ and $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable. In addition, the Lebesgue integral

$$\int\limits_{(0,t]\times Z} f(\tau, z, \omega) N(d\tau \otimes z, \omega) \tag{4.5}$$

exists and is finite for all $t > 0$ and $\omega \in \Omega \setminus M$. Moreover, the mapping defined by $\Omega \ni \omega \mapsto \int\limits_{(0,t]\times Z} f(\tau, z, \omega) N(d\tau \otimes z, \omega)$ is well-defined on $\Omega \setminus M$, and it is $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable for all $t > 0$. Finally, the assertion

$$\int\limits_{(0,t]\times Z} f(\tau, z, \omega) N(d\tau \otimes z, \omega) = \sum_{m=1}^{\infty} \sum_{k=1}^{m} f_k(\omega) \mathbb{1}_{[\alpha_m(\omega), \alpha_{m+1}(\omega))}(t) \tag{4.6}$$

is valid for all $t > 0$ and $\omega \in \Omega \setminus M$.

Proof. For notational convenience introduce $\alpha_0 : \Omega \to \mathbb{R}$, with $\alpha_0 := 0$.
Firstly, employing Remark 4.2.1 yields that each f_m is the composition of measurable functions and consequently $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable.
Now note that it is plain that the mapping defined by $(0,t]\times Z \ni (\tau, z) \mapsto f(\tau, z, \omega)$ is $\mathfrak{B}((0,t]) \otimes \mathcal{Z} - \mathfrak{B}(\mathbb{R})$ measurable for all $\omega \in \Omega$ and $t > 0$. Consequently, it follows that the Lebesgue integral considered in (4.5) is well defined and finite, if

$$\int\limits_{(0,t]\times Z} |f(\tau, z, \omega)| N(d\tau \otimes z, \omega) < \infty, \ \forall t > 0, \ \omega \in \Omega \setminus M. \tag{4.7}$$

To this end, note that

$$N_\Theta\Big(\big(\alpha_m(\omega), \alpha_{m+1}(\omega)\big) \times Z, \omega\Big) = 0, \ \forall m \in \mathbb{N}_0, \ \omega \in \Omega \setminus M \tag{4.8}$$

as well as

$$N_\Theta\big(\{\alpha_m(\omega)\} \times Z, \omega\big) = N_\Theta\big(\{\alpha_m(\omega)\} \times \{\Theta(\omega)(\alpha_m(\omega))\}, \omega\big) = 1, \ \forall m \in \mathbb{N}, \ \omega \in \Omega \setminus M. \tag{4.9}$$

Moreover, for a given $t > 0$ and $\omega \in \Omega\setminus M$ there is an $m \in \mathbb{N}$, such that $t < \alpha_k(\omega)$ for all $k \in \mathbb{N}\setminus\{1, ..., m\}$. This combined with the preceding two equalities clearly yields (4.7).
Moreover, note that the right-hand-side of (4.6) defines an $\mathcal{F}$-$\mathfrak{B}(R)$-measurable mapping. Consequently, as $\mathcal{F}$ is complete, the claim follows as soon as (4.6) is proven. This is easily deduced from (4.8) and (4.9), since these two equations yield

i) $\int\limits_{(0,\alpha_m(\omega)]\times Z} f(\tau, z, \omega) N_\Theta(d\tau \otimes dz, \omega) = \sum\limits_{k=1}^{m} f_k(\omega)$ for all $m \in \mathbb{N}$,

ii) $\int\limits_{(0,t]\times Z} f(\tau, z, \omega) N_\Theta(d\tau \otimes dz, \omega) \mathbb{1}_{[0,\alpha_1(\omega))}(t) = 0$ for all $t > 0$ and

iii) $\int\limits_{(0,t]\times Z} f(\tau,z,\omega)N_\Theta(d\tau\otimes dz,\omega)\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t) = \sum\limits_{k=1}^{m} f_k(\omega)\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t)$, for all $m \in \mathbb{N}$ and $t > 0$,

for all $\omega \in \Omega \setminus M$. □

Similar versions of the preceding result can be found in the literature. For example a similar result (for the case that N_Θ is a Poisson random measure) can be found in [24, Corollary 3.4], nevertheless we were unable to find it stated precisely as above anywhere in the literature.

4.3 General Existence and Uniqueness Results

Throughout this section, $(V, ||\cdot||_V)$ denotes a real, separable Banach space with dual space V'. Moreover, let $\langle\cdot,\cdot\rangle_V$ denote the duality between V and V'. As usually, a subset $V^* \subseteq V'$ is said to separate points, if for all $v \in V$ we have that $\langle\Phi, v\rangle_V = 0$ for all $\Phi \in V^*$ implies $v = 0$.
In addition, let

$$W^{1,1}([a,b];V) := \{f : [a,b] \to V :\ f \text{ is absolutely continuous and differentiable a.e.}\},$$

for any interval $[a,b] \subseteq (0,\infty)$; and $L^1(0,t) := L^1((0,t),\mathfrak{B}((0,t)),\lambda;\mathbb{R})$, for any $t > 0$, where λ is the one-dimensional Lebesgue measure.
As previously, $(Z,\mathcal{Z})$ is a measurable space, and we introduce the short-cut notation

$$\mathcal{M}((0,\infty)\times Z\times\Omega;V) := \mathcal{M}((0,\infty)\times Z\times\Omega,\mathfrak{B}((0,\infty))\otimes\mathcal{Z}\otimes\mathcal{F};V),$$

and $\mathcal{M}(\Omega;V) := \mathcal{M}(\Omega,\mathcal{F};V)$, see Remark 2.2.1.
Moreover, $\Theta : \Omega \to \pi(Z)$ denotes a finite point process and $N_\Theta : (\mathfrak{B}((0,\infty))\otimes\mathcal{Z})\times\Omega \to \mathbb{N}_0\cup\{\infty\}$ denotes the counting measure induced by Θ. Furthermore, $(\alpha_m)_{m\in\mathbb{N}}$ denotes the sequence of hitting times induced by Θ and $(\beta_m)_{m\in\mathbb{N}}$ the sequence of inter-arrival times induced by Θ; and for notational convenience we also introduce $\alpha_0 : \Omega \to \mathbb{R}$, with $\alpha_0 := 0$.

Last but not least, $\mathcal{A} : D(\mathcal{A}) \to 2^V$ is a densely defined, m-accretive operator, $(T_\mathcal{A}(t))_{t\geq 0}$ denotes the semigroup associated to $\mathcal{A}$ and we assume that this semigroup is domain invariant and admits an infinitesimal generator $\mathcal{A}^\circ : V \to V$.[1]

Now we are in the position to rigorously define the notions of mild and strong solutions of (ACPRM). After this is achieved we will demonstrate that mild solutions are unique (Corollary 4.3.5) and also derive an upper bounds for the mild solution, see Theorem 4.3.6. Thereafter, we will turn our focus on showing that there is indeed a mild/strong solution, and that the mild/strong solution must be an ACPRM-

[1] Note that the p-Laplace operator $\mathcal{A}_p$ and the associated semigroup $T_{\mathcal{A}_p}$ considered in Chapter 3 fulfill all of these assertions, see Theorem 3.2.5 and Remark 3.2.6.v).

process. Hereby, our main result regarding the existence of a strong solution is Proposition 4.3.10 and our main result ensuring the existence of mild solutions is Theorem 4.3.12.

Definition 4.3.1. *Let $\eta \in \mathcal{M}((0,\infty)\times Z\times\Omega;V)$ and $x \in \mathcal{M}(\Omega;V)$. In addition, let $V^* \subseteq V'$ be a set that separates points. Then a $\mathfrak{B}([0,\infty))\otimes\mathcal{F}$-$\mathfrak{B}(V)$-measurable stochastic process $X:[0,\infty)\times\Omega\to V$ is called a strong solution of (ACPRM)$\{x,\eta,V^*\}$ if all of the following assertions hold for $\mathbb{P}$-a.e. $\omega\in\Omega$.*

i) $X(0,\omega) = x(\omega)$,

ii) *the mapping* $[0,\infty)\ni t\mapsto X(t,\omega)$ *is càdlàg,*

iii) $X(t,\omega)\in D(\mathcal{A}),\ \forall t\in(0,\infty)\setminus\{\alpha_m(\omega): m\in\mathbb{N}\}$,

iv) $\forall m\in\mathbb{N}_0,\ \forall [a,b]\subseteq(\alpha_m(\omega),\alpha_{m+1}(\omega)):\ X(\cdot,\omega)|_{[a,b]}\in W^{1,1}([a,b];V)$,

v) $\langle\Psi,\mathcal{A}^\circ X(\cdot,\omega)\rangle_V\in L^1(0,t),\ \forall t>0,\ \Psi\in V^*$ *and*

vi) $\langle\Psi,X(t,\omega)-x(\omega)\rangle_V+\int\limits_0^t\langle\Psi,\mathcal{A}^\circ X(\tau,\omega)\rangle_V d\tau = \int\limits_{(0,t]\times Z}\langle\Psi,\eta(\tau,z,\omega)\rangle_V N_\Theta(d\tau\otimes z,\omega),\ \forall t>0,\ \Psi\in V^*.$

In addition, a $\mathfrak{B}([0,\infty))\otimes\mathcal{F}$-$\mathfrak{B}(V)$-measurable stochastic process $Y:[0,\infty)\times\Omega\to V$ is called a mild solution of (ACPRM)$\{x,\eta,V^*\}$, if it fulfills conditions i-iv) with probability one and if there are sequences $(x_m)_{m\in\mathbb{N}}$, $(\eta_m)_{m\in\mathbb{N}}$ and $(X_m)_{m\in\mathbb{N}}$ such that

vii) $x_m\in\mathcal{M}(\Omega;V)$ *and* $\eta_m\in\mathcal{M}((0,\infty)\times Z\times\Omega;V)$ *for all* $m\in\mathbb{N}$,

viii) $X_m:\Omega\times[0,\infty)\to V$ *is a strong solution of (ACPRM)*$\{x_m,\eta_m,V^*\}$ *for all* $m\in\mathbb{N}$,

ix) $\lim\limits_{m\to\infty}\sup\limits_{\tau\in[0,t]}||X_m(\tau)-Y(\tau)||_V=0$ *for all* $t>0$ *almost surely and*

x) $\lim\limits_{m\to\infty}\int\limits_{(0,t]\times Z}||\eta_m(\tau,z)-\eta(\tau,z)||_V N_\Theta(d\tau\otimes z)=0$ *for all* $t>0$ *almost surely.*

Lemma 4.3.2. *Let $V^*\subseteq V'$ be a set that separates points, $\eta\in\mathcal{M}((0,\infty)\times Z\times\Omega;V)$, $x\in\mathcal{M}(\Omega;V)$ and introduce $\eta_k(\omega):=\eta(\alpha_k(\omega),\Theta(\omega)(\alpha_k(\omega)),\omega)$ for all $k\in\mathbb{N}$ and $\mathbb{P}$-a.e. $\omega\in\Omega$. Then $\eta_k\in\mathcal{M}(\Omega;V)$ for all $k\in\mathbb{N}$ and a $\mathfrak{B}([0,\infty))\otimes\mathcal{F}$-$\mathfrak{B}(V)$-measurable stochastic process $X:[0,\infty)\times\Omega\to V$ is a strong solution of (ACPRM)$\{x,\eta,V^*\}$ if and only if it fulfills 4.3.1.i-v) and*

$$\langle\Psi,X(t)-x\rangle_V+\int\limits_0^t\langle\Psi,\mathcal{A}^\circ X(\tau)\rangle_V d\tau=\sum_{m=1}^{\infty}\sum_{k=1}^{m}\langle\Psi,\eta_k\rangle_V\mathbb{1}_{[\alpha_m,\alpha_{m+1})}(t),\ \forall t>0,\ \Psi\in V^*. \tag{4.10}$$

almost surely.

Proof. Firstly, appealing to Remark 4.2.1 yields that each η_k is, up to a $\mathbb{P}$-null-set, well-defined and that η_k is the composition of measurable functions and consequently $\mathcal{F}$-$\mathfrak{B}(V)$-measurable.
Lemma 4.2.2 yields that there is a $\mathbb{P}$-null-set $M\in\mathcal{F}$ such that for all $\Psi\in V^*$, we have

$$\int\limits_{(0,t]\times Z}\langle\Psi,\eta(\tau,z,\omega)\rangle_V N(d\tau\otimes z,\omega)=\sum_{m=1}^{\infty}\sum_{k=1}^{m}\langle\Psi,\eta_k(\omega)\rangle_V\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))},\ \forall t>0,\ \omega\in\Omega\setminus M.$$

Consequently, we get that 4.3.1.vi) holds almost surely if and only if (4.10) does, which concludes the proof. $\square$

Proposition 4.3.3. *Let $V^* \subseteq V'$ be a set that separates points, $\eta_1,\ \eta_2 \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and $x_1,\ x_2 \in \mathcal{M}(\Omega; V)$. Moreover, assume $X_i : [0,\infty) \times \Omega \to V$ is a strong solution of (ACPRM)$\{x_i, \eta_i, V^*\}$ for $i = 1, 2$. Then we have*

$$||X_1(t) - X_2(t)||_V \leq ||x_1 - x_2||_V + \int\limits_{(0,t]\times Z} ||\eta_1(\tau, z) - \eta_2(\tau, z)||_V N_\Theta(d\tau \otimes z),\ \forall t \geq 0, \tag{4.11}$$

almost surely.

Proof. Firstly, by Lemma 4.3.2 and Remark 4.2.1 we get that there is a $\mathbb{P}$-null-set $M \in \mathcal{F}$ such that

$$D(\Theta(\omega)) = \{\alpha_1(\omega),\ \alpha_2(\omega), ...\},\ 0 < \alpha_m(\omega) < \alpha_{m+1}(\omega),\ \forall m \in \mathbb{N} \text{ and } \lim_{m\to\infty} \alpha_m(\omega) = \infty, \tag{4.12}$$

and

i) $X_i(0, \omega) = x_i(\omega)$,

ii) the mapping $[0,\infty) \ni t \mapsto X_i(t, \omega)$ is càdlàg,

iii) $X_i(t, \omega) \in D(\mathcal{A}),\ \forall t \in (0,\infty) \setminus \{\alpha_m(\omega) : m \in \mathbb{N}\}$,

iv) $\forall m \in \mathbb{N}_0,\ \forall [a, b] \subseteq (\alpha_m(\omega), \alpha_{m+1}(\omega)) :\ X_i(\cdot, \omega)|_{[a,b]} \in W^{1,1}([a, b]; V)$,

v) $\langle \Psi, \mathcal{A}^\circ X_i(\cdot, \omega)\rangle_V \in L^1(0, t),\ \forall t > 0,\ \Psi \in V^*$ and

vi) $\langle \Psi, X_i(t, \omega) - x_i(\omega)\rangle_V + \int\limits_0^t \langle \Psi, \mathcal{A}^\circ X_i(\tau, \omega)\rangle_V d\tau = \sum\limits_{m=1}^{\infty} \sum\limits_{k=1}^{m} \langle \Psi, \eta_{i,k}(\omega)\rangle_V \mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t),\ \forall t > 0$, $\Psi \in V^*$, where $\eta_{i,k}(\omega) := \eta_i(\alpha_k(\omega), \Theta(\omega)(\alpha_k(\omega)), \omega)$ for all $k \in \mathbb{N}$,

for all $\omega \in \Omega \setminus M$ and $i = 1, 2$.
Moreover, Lemma 4.2.2 yields that it suffices to prove that

$$||X_1(t, \omega) - X_2(t, \omega)||_V \leq ||x_1(\omega) - x_2(\omega)||_V + \sum_{m=1}^{\infty} \sum_{k=1}^{m} ||\eta_{1,k}(\omega) - \eta_{2,k}(\omega)||_V \mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t),$$

for all $t \geq 0$ and $\omega \in \Omega \setminus M$.
To this end, let $\omega \in \Omega \setminus M$ be arbitrary but fixed and introduce

$$\hat{\alpha}_0 := 0,\ \hat{\alpha}_m := \alpha_m(\omega),\ \hat{\eta}_{i,m} := \eta_{i,m}(\omega),\ \hat{x}_i := x_i(\omega) \text{ and } \hat{X}_i(t) := X_i(t, \omega),$$

for all $t \geq 0$, $m \in \mathbb{N}$ and $i = 1, 2$.
Let us start tackling the task ahead of us, by proving that

$$\lim_{\varepsilon \searrow 0} \hat{X}_i(\hat{\alpha}_{\tilde{m}}) - \hat{X}_i(\hat{\alpha}_{\tilde{m}} - \varepsilon) = \hat{\eta}_{i,\tilde{m}},\ \forall \tilde{m} \in \mathbb{N} \text{ and } i = 1, 2. \tag{4.13}$$

in norm. Let $\tilde{m} \in \mathbb{N}$ and $i \in \{1,2\}$ be arbitrary but fixed and note that v) yields

$$\lim_{\varepsilon \searrow 0} \int_0^{\hat{\alpha}_{\tilde{m}}-\varepsilon} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V d\tau - \int_0^{\hat{\alpha}_{\tilde{m}}} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V d\tau = 0,\ \forall \Psi \in V^*.$$

Consequently, we get by invoking vi) that

$$\lim_{\varepsilon \searrow 0} \langle \Psi, \hat{X}_i(\hat{\alpha}_{\tilde{m}}) - \hat{X}_i(\hat{\alpha}_{\tilde{m}} - \varepsilon)\rangle_V = \sum_{k=1}^{\tilde{m}} \langle \Psi, \hat{\eta}_{i,k}\rangle_V - \sum_{k=1}^{\tilde{m}-1} \langle \Psi, \hat{\eta}_{i,k}\rangle_V = \langle \Psi, \hat{\eta}_{i,\tilde{m}}\rangle_V,$$

for all $\Psi \in V^*$. Moreover, ii) implies that there is a $u \in V$ such that

$$\lim_{\varepsilon \searrow 0} ||\hat{X}_i(\hat{\alpha}_{\tilde{m}}) - \hat{X}_i(\hat{\alpha}_{\tilde{m}} - \varepsilon) - u||_V = 0. \tag{4.14}$$

Consequently, as convergence in norm implies weak convergence, we have

$$\langle \Psi, \hat{\eta}_{i,\tilde{m}} - u\rangle_V = \lim_{\varepsilon \searrow 0} \langle \Psi, \hat{X}_i(\hat{\alpha}_{\tilde{m}}) - \hat{X}_i(\hat{\alpha}_{\tilde{m}} - \varepsilon)\rangle_V - \langle \Psi, \hat{X}_i(\hat{\alpha}_{\tilde{m}}) - \hat{X}_i(\hat{\alpha}_{\tilde{m}} - \varepsilon)\rangle_V = 0,\ \forall \Psi \in V^*,$$

which yields $\hat{\eta}_{i,\tilde{m}} = u$, since V^* separates points. Consequently, (4.14) implies (4.13).

We will proceed by proving that

$$||\hat{X}_1(t) - \hat{X}_2(t)||_V \leq ||\hat{X}_1(\hat{\alpha}_m) - \hat{X}_2(\hat{\alpha}_m)||_V,\ \forall m \in \mathbb{N}_0,\ t \in [\hat{\alpha}_m, \hat{\alpha}_{m+1}). \tag{4.15}$$

Proving (4.15) is divided into several intermediate steps and requires some notations. To this end, fix $m \in \mathbb{N}_0$, and introduce $\varepsilon \in (0, \hat{\alpha}_{m+1} - \hat{\alpha}_m)$ arbitrary but fixed, $b_\varepsilon := \hat{\alpha}_{m+1} - \hat{\alpha}_m - \varepsilon$, $F_i : [0, b_\varepsilon] \to V$ by $F_i := \hat{X}_i(\cdot + \hat{\alpha}_m)$ and $u_i := \hat{X}_i(\hat{\alpha}_m)$ for $i = 1, 2$.
Firstly, note that

$$F_i|_{[a,b]} \in W^{1,1}([a,b];V),\ \forall [a,b] \subseteq (0, b_\varepsilon),\ i \in \{1,2\}, \tag{4.16}$$

since: For $[a,b] \subseteq (0, b_\varepsilon)$ and $t \in [a,b]$ we have

$$\hat{\alpha}_m < \hat{\alpha}_m + a \leq \hat{\alpha}_m + t \leq \hat{\alpha}_m + b < \hat{\alpha}_{m+1} - \varepsilon,$$

which yields by appealing to iv) that $\hat{X}_i \in W^{1,1}([\hat{\alpha}_m + a, \hat{\alpha}_m + b]; V)$; and hence (4.16), by the definition of F_i.
Secondly, we will prove that

$$F_i \in C([0, b_\varepsilon]; V),\ i \in \{1,2\}, \tag{4.17}$$

where $C(S;V)$ denotes the space of continuous functions, mapping from S into V, for any open or closed

set $S \subseteq \mathbb{R}$.
Note that (4.16) already yields $F_i \in C((0,b_\varepsilon);V)$ which then gives $F_i \in C([0,b_\varepsilon);V)$, since F_i inherits the right-continuity of $\hat{X}_i$. Consequently, (4.17) follows if F_i is left-continuous in b_ε. As $\hat{X}_i$ is càdlàg, we have that there is a $w_i \in V$ such that

$$\lim_{h\searrow 0} F_i(b_\varepsilon - h) - F_i(b_\varepsilon) = w_i.$$

in norm. Moreover, note that $b_\varepsilon + \hat{\alpha}_m \in (\hat{\alpha}_m, \hat{\alpha}_{m+1})$, which yields by invoking vi) that

$$\langle w_i, \Psi\rangle_V = \lim_{h\searrow 0} \int\limits_0^{b_\varepsilon+\hat{\alpha}_m} \langle \Psi, \mathcal{A}^\circ X_i(\tau,\omega)\rangle_V \, d\tau - \int\limits_0^{b_\varepsilon+\hat{\alpha}_m - h} \langle \Psi, \mathcal{A}^\circ X_i(\tau,\omega)\rangle_V \, d\tau = 0$$

for all $\Psi \in V^*$. As V^* separates points, this is only possible if $w_i = 0$, which establishes the desired left continuity and (4.17) follows.
The last intermediate step necessary to prove (4.15) is

$$0 \in F_i'(t) + \mathcal{A}F_i(t), \quad \text{a.e. } t \in (0,b_\varepsilon), \;\; F_i(0) = u_i, \;\; i = 1,2. \tag{4.18}$$

To this end, note that (4.16) yields that there is for each $i \in \{1,2\}$ a Lebesgue null-set $M(F_i) \subseteq (0,b_\varepsilon)$ such that F_i is differentiable (in norm) on $(0,b_\varepsilon) \setminus M(F_i)$.
Now let $V_c^* \subseteq V^*$ be a countable set which separates points; such a set exists due to [41, Lemma 2.1 and Theorem 2.1].
By virtue of the fundamental theorem of calculus for Lebesgue integrals, there is for each $\Psi \in V_c^*$ and $i \in \{1,2\}$ a Lebesgue null-set $M(\Psi,i) \subseteq (0,b_\varepsilon)$ such that

$$\lim_{h\to 0} \frac{1}{h} \left(\int\limits_0^{t+\hat{\alpha}_m+h} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V \, d\tau - \int\limits_0^{t+\hat{\alpha}_m} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V \, d\tau \right) = \langle \Psi, \mathcal{A}^\circ \hat{X}_i(t+\hat{\alpha}_m)\rangle_V$$

for all $t \in (0,b_\varepsilon) \setminus M(\Psi,i)$, $i = 1,2$.
Consequently, employing the previous equality, the differentiability a.e. of F_i and vi) yields

$$\begin{aligned}
& \langle F_i'(t) + \mathcal{A}^\circ F_i(t), \Psi\rangle_V \\
= \;& \lim_{h\to 0} \frac{1}{h} \langle F_i(t+h) - F_i(t), \Psi\rangle_V + \langle \mathcal{A}^\circ F_i(t), \Psi\rangle_V \\
= \;& -\lim_{h\to 0} \frac{1}{h} \left(\int\limits_0^{t+\hat{\alpha}_m+h} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V \, d\tau - \int\limits_0^{t+\hat{\alpha}_m} \langle \Psi, \mathcal{A}^\circ \hat{X}_i(\tau)\rangle_V \, d\tau \right) + \langle \mathcal{A}^\circ F_i(t), \Psi\rangle_V \\
= \;& -\langle \Psi, \mathcal{A}^\circ \hat{X}_i(t+\hat{\alpha}_m)\rangle_V + \langle \mathcal{A}^\circ F_i(t), \Psi\rangle_V \\
= \;& 0
\end{aligned}$$

for all $\Psi \in V_c^*$, $i \in \{1,2\}$ and $t \in (0,b_\varepsilon) \setminus (M(F_i) \cup M(\Psi,i))$.

Now let $M_i := \bigcup_{\Psi \in V_c^*} M(\Psi, i) \cup M(F_i)$, which is still a Lebesgue null-set since V_c^* is countable. Then the previous calculation implies $\langle F_i'(t) + \mathcal{A}^\circ F_i(t), \Psi \rangle_V = 0$ for all $\Psi \in V_c^*$ and $t \in (0, b_\varepsilon) \setminus M_i$. Consequently, as V_c^* separates points, we get $0 = F_i'(t) + \mathcal{A}^\circ F_i(t)$ for every $t \in (0, b_\varepsilon) \setminus M_i$. Finally, iii) yields $F_i(t) \in D(\mathcal{A})$ for all $t \in (0, b_\varepsilon)$ and consequently $\mathcal{A}^\circ F_i(t) \in \mathcal{A}F_i(t)$ for all $t \in (0, b_\varepsilon)$. Combining this with $0 = F_i'(t) + \mathcal{A}^\circ F(t)$ for every $t \in (0, b_\varepsilon) \setminus M_i$ gives (4.18).
The results (4.16)-(4.18) enable us to prove (4.15). By (4.16)-(4.18), we have that F_i is a strong solution of the initial value problem

$$0 \in U_i'(t) + \mathcal{A}U(t), \quad \text{a.e. } t \in (0, b_\varepsilon), \ U(0) = u_i, \tag{4.19}$$

for $i = 1, 2$. Consequently F_i is also a mild solution of (4.19), see Remark 2.1.5. Moreover, as $\mathcal{A}$ is m-accretive and densely defined (4.19) has precisely one mild solution, see Theorem2.1.7. This necessarily implies $F_i(t) = T_\mathcal{A}(t)u_i$ for $t \in [0, b_\varepsilon]$ and $i = 1, 2$. Therefore, as $(T_\mathcal{A}(t))_{t \geq 0}$ is contractive (see Theorem 2.1.7) we get

$$||\hat{X}_1(t + \hat{\alpha}_m) - \hat{X}_2(t + \hat{\alpha}_m)||_V = ||T_\mathcal{A}(t)u_1 - T_\mathcal{A}(t)u_2||_V \leq ||u_1 - u_2||_V = ||\hat{X}_1(\hat{\alpha}_m) - \hat{X}_2(\hat{\alpha}_m)||_V,$$

for all $t \in [0, b_\varepsilon] = [0, \hat{\alpha}_{m+1} - \hat{\alpha}_m - \varepsilon]$. As $\varepsilon \in (0, \hat{\alpha}_{m+1} - \hat{\alpha}_m)$ can be chosen arbitrarily small, this holds for all $t \in [0, \hat{\alpha}_{m+1} - \hat{\alpha}_m)$ which proves (4.15).

The next (and last) intermediate step enables us to prove the claim and reads as follows: For all $m \in \mathbb{N}$, all $t \in [\hat{\alpha}_m, \hat{\alpha}_{m+1})$ and all $\varepsilon \in (0, \min(\hat{\alpha}_1 - \hat{\alpha}_0, .., \hat{\alpha}_m - \hat{\alpha}_{m-1}))$, we have

$$||\hat{X}_1(t) - \hat{X}_2(t)||_V \leq ||\hat{x}_1 - \hat{x}_2||_V + \sum_{k=1}^{m} ||\hat{X}_1(\hat{\alpha}_k) - \hat{X}_1(\hat{\alpha}_k - \varepsilon) - \hat{X}_2(\hat{\alpha}_k) + \hat{X}_2(\hat{\alpha}_k - \varepsilon)||_V. \tag{4.20}$$

This will be proven inductively. Let $m = 1$, $t \in [\hat{\alpha}_1, \hat{\alpha}_2)$ and $\varepsilon \in (0, \hat{\alpha}_1 - \hat{\alpha}_0)$. Then appealing to (4.15) and i) yields

$$\begin{aligned} ||\hat{X}_1(t) - \hat{X}_2(t)||_V &\leq ||\hat{X}_1(\hat{\alpha}_1) - \hat{X}_2(\hat{\alpha}_1)||_V \\ &\leq ||\hat{x}_1 - \hat{x}_2||_V + ||\hat{X}_1(\hat{\alpha}_1) - \hat{X}_1(\hat{\alpha}_1 - \varepsilon) - \hat{X}_2(\hat{\alpha}_1) + \hat{X}_2(\hat{\alpha}_1 - \varepsilon)||_V. \end{aligned}$$

Induction step: Let $t \in [\hat{\alpha}_{m+1}, \hat{\alpha}_{m+2})$ and $\varepsilon \in (0, \min(\hat{\alpha}_1 - \hat{\alpha}_0, .., \hat{\alpha}_{m+1} - \hat{\alpha}_m))$. Firstly, note that $\hat{\alpha}_{m+1} - \varepsilon \in [\hat{\alpha}_m, \hat{\alpha}_{m+1})$ and that particularly $\varepsilon \in (0, \min(\hat{\alpha}_1 - \hat{\alpha}_0, .., \hat{\alpha}_m - \hat{\alpha}_{m-1}))$. Consequently, the induction hypothesis yields that

$$||\hat{X}_1(\hat{\alpha}_{m+1} - \varepsilon) - \hat{X}_2(\hat{\alpha}_{m+1} - \varepsilon)||_V \leq ||\hat{x}_1 - \hat{x}_2||_V + \sum_{k=1}^{m} ||\hat{X}_1(\hat{\alpha}_k) - \hat{X}_1(\hat{\alpha}_k - \varepsilon) - \hat{X}_2(\hat{\alpha}_k) + \hat{X}_2(\hat{\alpha}_k - \varepsilon)||_V$$

Conclusively, appealing to (4.15), the triangle inequality and the preceding estimate gives

$$||\hat{X}_1(t) - \hat{X}_2(t)||_V \leq ||\hat{x}_1 - \hat{x}_2||_V + \sum_{k=1}^{m+1} ||\hat{X}_1(\hat{\alpha}_k) - \hat{X}_1(\hat{\alpha}_k - \varepsilon) - \hat{X}_2(\hat{\alpha}_k) + \hat{X}_2(\hat{\alpha}_k - \varepsilon)||_V,$$

which implies (4.20).
Now the (from here on short) proof the claim will be derived: If $t \in [0, \hat{\alpha}_1)$ we have

$$||\hat{X}_1(t) - \hat{X}_2(t)||_V \leq ||\hat{x}_1 - \hat{x}_2||_V = ||\hat{x}_1 - \hat{x}_2||_V + \sum_{m=1}^{\infty} \sum_{k=1}^{m} ||\hat{\eta}_{1,k} - \hat{\eta}_{2,k}||_V \mathbb{1}_{[\hat{\alpha}_m, \hat{\alpha}_{m+1})}(t),$$

by (4.15) and i). If $t \in [\hat{\alpha}_1, \infty)$, then (4.12) yields that there is an $\tilde{m} \in \mathbb{N}$ such that $t \in [\hat{\alpha}_{\tilde{m}}, \hat{\alpha}_{\tilde{m}+1})$. Finally, employing (4.20) and (4.13) gives

$$\begin{aligned}
||\hat{X}_1(t) - \hat{X}_2(t)||_V &\leq ||\hat{x}_1 - \hat{x}_2||_V + \lim_{\varepsilon \searrow 0} \sum_{k=1}^{\tilde{m}} ||\hat{X}_1(\hat{\alpha}_k) - \hat{X}_1(\hat{\alpha}_k - \varepsilon) - \hat{X}_2(\hat{\alpha}_k) + \hat{X}_2(\hat{\alpha}_k - \varepsilon)||_V \\
&= ||\hat{x}_1 - \hat{x}_2||_V + \sum_{k=1}^{\tilde{m}} ||\hat{\eta}_{1,k} - \hat{\eta}_{2,k}||_V \\
&= ||\hat{x}_1 - \hat{x}_2||_V + \sum_{m=1}^{\infty} \sum_{k=1}^{m} ||\hat{\eta}_{1,k} - \hat{\eta}_{2,k}||_V \mathbb{1}_{[\hat{\alpha}_m, \hat{\alpha}_{m+1})}(t),
\end{aligned}$$

which concludes the proof. □

Theorem 4.3.4. *Let $V^* \subseteq V'$ be a set that separates points, $\eta_1,\ \eta_2 \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and introduce $x_1,\ x_2 \in \mathcal{M}(\Omega; V)$. Moreover, assume that $X_i : [0,\infty) \times \Omega \to V$ is a mild solution of (ACPRM)$\{x_i, \eta_i, V^*\}$ for $i = 1, 2$. Then we have*

$$||X_1(t) - X_2(t)||_V \leq ||x_1 - x_2||_V + \int_{(0,t] \times Z} ||\eta_1(\tau, z) - \eta_2(\tau, z)||_V N_\Theta(d\tau \otimes z),\ \forall t \geq 0, \tag{4.21}$$

almost surely.

Proof. Let $x_{i,m} \in \mathcal{M}(\Omega; V)$, $\eta_{i,m} \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and $X_{i,m} : \Omega \times [0,\infty) \to V$ be such that

i) $X_{i,m}$ is a strong solution of (ACPRM)$\{x_{i,m}, \eta_{i,m}, V^*\}$ for all $m \in \mathbb{N}$ and $i \in \{1, 2\}$,

ii) $\lim\limits_{m \to \infty} \sup\limits_{\tau \in [0,t]} ||X_{i,m}(\tau) - X_i(\tau)||_V = 0$ for all $t > 0$, $i \in \{1, 2\}$ almost surely,

iii) $\lim\limits_{m \to \infty} \int\limits_{(0,t] \times Z} ||\eta_{i,m}(\tau, z) - \eta_i(\tau, z)||_V N_\Theta(d\tau \otimes z) = 0$ for all $t > 0$, $i \in \{1, 2\}$ almost surely, and

iv) $||X_{1,m}(t) - X_{2,m}(t)||_V \leq ||x_{1,m} - x_{2,m}||_V + \int\limits_{(0,t] \times Z} ||\eta_{1,m}(\tau, z) - \eta_{2,m}(\tau, z)||_V N_\Theta(d\tau \otimes z)$ for all $t \geq 0$, $m \in \mathbb{N}$, $i \in \{1, 2\}$ almost surely.

Proposition 4.3.3 (and the definition of mild solution) guarantee the existence of these quantities. Consequently, we have

$$\begin{aligned}
&||X_1(t) - X_2(t)||_V \\
= \quad & \lim_{m\to\infty} ||X_{1,m}(t) - X_{2,m}(t)||_V \\
\leq \quad & \lim_{m\to\infty} ||x_{1,m} - x_{2,m}||_V + \int\limits_{(0,t]\times Z} ||\eta_{1,m}(\tau, z) - \eta_{2,m}(\tau, z)||_V N_\Theta(d\tau \otimes z) \\
= \quad & ||x_1 - x_2||_V + \int\limits_{(0,t]\times Z} ||\eta_1(\tau, z) - \eta_2(\tau, z)||_V N_\Theta(d\tau \otimes z)
\end{aligned}$$

for all $t \geq 0$, with probability one. $\square$

Theorem 4.3.4 has two important consequences: Uniqueness of mild solutions of (ACPRM) and an upper bound for the solution.

Corollary 4.3.5. *(ACPRM) has at most one mild solution; more precisely: Let $V^* \subseteq V'$ be such that it separates points, let $x \in \mathcal{M}(\Omega; V)$, $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and assume that $X_1,\ X_2 : [0,\infty) \times \Omega \to V$ are mild solutions of (ACPRM)$\{x, \eta, V^*\}$, then X_1 and X_2 are indistinguishable.*

Theorem 4.3.6. *Let $V^* \subseteq V'$ be such that it separates points, let $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$, $x \in \mathcal{M}(\Omega; V)$ and assume that $X : [0,\infty) \times \Omega \to V$ is a mild solution of (ACPRM)$\{x, \eta, V^*\}$. Finally, assume that $(0,0) \in \mathcal{A}0$. Then we have*

$$||X(t)||_V \leq ||x||_V + \int\limits_{(0,t]\times Z} ||\eta(\tau, z)||_V N_\Theta(d\tau \otimes z),\ \forall t \geq 0, \tag{4.22}$$

almost surely.

Proof. As $0 \in \mathcal{A}0$, it is plain that $T_\mathcal{A}(t)0 = 0$ for all $t \geq 0$. Consequently, we have $\mathcal{A}^\circ 0 = 0$. This implies that the stochastic process which is constantly zero, is a strong (and therefore also mild) solution of (ACPRM)$\{0, 0, V^*\}$. Consequently, the claim follows from Theorem 4.3.4. $\square$

Now we will turn to the question of existence. Firstly, the assumptions imposed on $\mathcal{A}$ and $T_\mathcal{A}$ enable us to apply Theorem 2.1.14, which yields:

Remark 4.3.7. *Let $v \in V$ be arbitrary but fixed. Then $T_\mathcal{A}(\cdot)v$ is locally Lipschitz continuous on $(0,\infty)$ and differentiable a.e. with $-\mathcal{A}^\circ T_\mathcal{A}(t)v = T'_\mathcal{A}(t)v$ for a.e. $t \in (0,\infty)$.*

Lemma 4.3.8. *Let $t > 0$, $v \in V$ and $\Psi \in V'$. Moreover, assume that $\langle \Psi, \mathcal{A}^\circ T_\mathcal{A}(\cdot)v\rangle_V \in L^1((0,t))$. Then we have*

$$\int\limits_0^t \langle \Psi, \mathcal{A}^\circ T_\mathcal{A}(\tau)v\rangle_V d\tau = -\langle \Psi, T_\mathcal{A}(t)v - v\rangle_V.$$

Proof. Let $\varepsilon \in (0,t)$ be arbitrary but fixed. Firstly, Remark 4.3.7 obviously implies that the mapping $(\varepsilon,t) \ni \tau \mapsto \langle \Psi, T_{\mathcal{A}}(\tau)v\rangle_V$ is Lipschitz continuous and differentiable almost everywhere with

$$\frac{\partial}{\partial \tau}\langle \Psi, T_{\mathcal{A}}(\tau)v\rangle_V = -\langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\tau)v\rangle_V, \text{ a.e. } \tau \in (\varepsilon,t).$$

Consequently, we have

$$\int_{\varepsilon}^{t} \langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\tau)v\rangle_V d\tau = -\langle \Psi, T_{\mathcal{A}}(t)v - T_{\mathcal{A}}(\varepsilon)v\rangle_V \tag{4.23}$$

Now the claim follows from (4.23) by taking limit, more precisely: By Theorem 2.1.7, $T_{\mathcal{A}}$ is a fortiori time continuous, ergo

$$\lim_{\varepsilon \searrow 0} -\langle \Psi, T_{\mathcal{A}}(t)v - T_{\mathcal{A}}(\varepsilon)v\rangle_V = -\langle \Psi, T_{\mathcal{A}}(t)v - v\rangle_V.$$

Moreover, dominated convergence yields that

$$\int_0^t \langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\tau)v\rangle_V d\tau = \int_0^t \lim_{\varepsilon \searrow 0}\langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\tau)v\rangle_V \mathbb{1}_{(\varepsilon,t)}(\tau)d\tau = \lim_{\varepsilon \searrow 0}\int_{\varepsilon}^t \langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\tau)v\rangle_V d\tau,$$

which is applicable since $\langle \Psi, \mathcal{A}^{\circ}T_{\mathcal{A}}(\cdot)v\rangle_V \in L^1((0,t))$ by assumption. □

Lemma 4.3.9. *Let $x \in \mathcal{M}(\Omega;V)$, $\eta \in \mathcal{M}((0,\infty)\times Z \times \Omega;V)$, set $\eta_k(\omega) := \eta(\alpha_k(\omega),\Theta(\omega)(\alpha_k(\omega)),\omega)$ for all $k \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, and let $\mathbb{X}_x$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}},(\eta_m)_{m\in\mathbb{N}},x,T_{\mathcal{A}})$ in V. Then the following assertions hold for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

i) $\mathbb{X}_x(t,\omega) \in D(\mathcal{A}),\ \forall t \in (0,\infty)\setminus\{\alpha_m(\omega): m \in \mathbb{N}\}$,

ii) $\forall m \in \mathbb{N}_0,\ \forall [a,b] \subseteq (\alpha_m(\omega),\alpha_{m+1}(\omega)):\ \mathbb{X}_x(\cdot,\omega)|_{[a,b]} \in W^{1,1}([a,b];V)$ *and*

iii) $\mathcal{A}^{\circ}\mathbb{X}_x(\cdot,\omega)$ *is* $\mathfrak{B}((0,\infty)) - \mathfrak{B}(V)$*-measurable.*

Proof. Firstly, let $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ be the sequence generated by $((\beta_m)_{m\in\mathbb{N}},(\eta_m)_{m\in\mathbb{N}},x,T_{\mathcal{A}})$ in V. In addition, let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set such that $0 = \alpha_0(\omega) < \alpha_1(\omega) < \alpha_2(\omega) < ...$ as well as $\lim_{m\to\infty}\alpha_m(\omega) = \infty$ and $\alpha_m(\omega) \in D(\Theta(\omega))$ for all $\omega \in \Omega \setminus M$ and $m \in \mathbb{N}$. Now, i)-iii) will be proven for all $\omega \in \Omega \setminus M$.
So, fix $\omega \in \Omega\setminus M$ and let us prove i). To this end, introduce $t \in (0,\infty)\setminus\{\alpha_m(\omega):\ m \in \mathbb{N}\}$, and note that there is precisely one $m \in \mathbb{N}_0$ such that $t \in (\alpha_m(\omega),\alpha_{m+1}(\omega))$ and thus $\mathbb{X}_x(t,\omega) = T_{\mathcal{A}}(t-\alpha_m(\omega))\mathbb{x}_{x,m}(\omega)$. Consequently, as $T_{\mathcal{A}}$ is domain invariant, we have $\mathbb{X}_x(t,\omega) \in D(\mathcal{A})$.
Proof of ii). Let $m \in \mathbb{N}$ and $[a,b] \subseteq (\alpha_m(\omega),\alpha_{m+1}(\omega))$. Then it is plain that $\mathbb{X}_x(\cdot,\omega)|_{[a,b]} = T_{\mathcal{A}}(\cdot - \alpha_m(\omega))\mathbb{x}_{x,m}(\omega)$. But the local absolute continuity and differentiability almost everywhere of this mapping follow trivially from Remark 4.3.7.

Proof of iii). Let $(h_k)_{k\in\mathbb{N}} \subseteq (0,\infty)$ be a null sequence. Moreover, introduce $f_{k,m} : (0,\infty) \to V$ by

$$f_{k,m}(t) := \frac{T_{\mathcal{A}}((t-\alpha_m(\omega))_+ + h_k)\mathbb{x}_{x,m}(\omega) - T_{\mathcal{A}}((t-\alpha_m(\omega))_+)\mathbb{x}_{x,m}(\omega)}{h_k} \mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t),$$

for all $t \in (0,\infty)$, $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$.
Then we have

$$\lim_{k\to\infty} f_{k,m}(t) = -\mathcal{A}^\circ(T_{\mathcal{A}}((t-\alpha_m(\omega))_+)\mathbb{x}_{x,m}(\omega))\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t), \tag{4.24}$$

for all $m \in \mathbb{N}_0$ and $t \in (0,\infty) \setminus \{\alpha_j(\omega) : j \in \mathbb{N}\}$, since: If $t \notin [\alpha_m(\omega), \alpha_{m+1}(\omega))$, for a given $m \in \mathbb{N}_0$, then (4.24) is trivial and if $t \in (\alpha_m(\omega), \alpha_{m+1}(\omega))$, we have by domain invariance of $T_{\mathcal{A}}$ that

$$\begin{aligned}\lim_{k\to\infty} f_{k,m}(t) &= \lim_{k\to\infty} \frac{T_{\mathcal{A}}(h_k)T_{\mathcal{A}}(t-\alpha_m(\omega))\mathbb{x}_{x,m}(\omega) - T_{\mathcal{A}}(t-\alpha_m(\omega))\mathbb{x}_{x,m}(\omega)}{h_k} \\ &= -\mathcal{A}^\circ T_{\mathcal{A}}(t-\alpha_m(\omega))\mathbb{x}_{x,m}(\omega).\end{aligned}$$

In addition, the joint continuity of $T_{\mathcal{A}}$ yields that each $f_{k,m}$ is $\mathfrak{B}((0,\infty)) - \mathfrak{B}(V)$-measurable. Consequently, (4.24) yields that $(0,\infty) \ni t \mapsto -\mathcal{A}^\circ(T_{\mathcal{A}}((t-\alpha_m(\omega))_+)\mathbb{x}_{x,m}(\omega))\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t)$ is also $\mathfrak{B}(0,\infty) - \mathfrak{B}(V)$-measurable for all $m \in \mathbb{N}_0$, since it is (except for a countable set) the pointwise limit of $\mathfrak{B}(0,\infty) - \mathfrak{B}(V)$-measurable functions.
Finally, it is plain that

$$\mathcal{A}^\circ \mathbb{X}_x(t,\omega) = \sum_{m=0}^{\infty} \mathcal{A}^\circ(T_{\mathcal{A}}((t-\alpha_m(\omega))_+)\mathbb{x}_{x,m})\mathbb{1}_{[\alpha_m(\omega),\alpha_{m+1}(\omega))}(t),\ \forall t \in (0,\infty),$$

which implies the desired measurability. □

The preceding lemma enables us to give a condition ensuring that (ACPRM) has a (uniquely determined) strong solution. Afterwards, just one more approximation lemma is needed to formulate this chapter's central result: A criteria ensuring the existence of a unique mild solution of (ACPRM).

Proposition 4.3.10. *Let $\mathcal{V} \subseteq V$ be a subspace of V and let $V^* \subseteq V'$ be a subset which separates points. Moreover, let $x \in \mathcal{M}(\Omega; V)$, $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and let $\mathbb{X}_x$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}})$ in V, where $\eta_m(\omega) := \eta(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. In addition, assume that $x \in \mathcal{V}$ a.s. and $\eta(t,z) \in \mathcal{V}$ for all $t \in (0,\infty)$ and $z \in Z$ with probability one. Finally, assume that $\mathcal{V}$ is an invariant space w.r.t. $T_{\mathcal{A}}$ and that $\langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\cdot)u\rangle_V \in L^1((0,t))$ for all $t > 0$, $u \in \mathcal{V}$ and $\Psi \in V^*$. Then the stochastic process $\mathbb{X}_x$ is the unique strong solution of (ACPRM)$\{x, \eta, V^*\}$.*

Proof. Firstly, let $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ denote the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}})$ in V.

In light of Lemma 4.3.9, Lemma 2.2.6.i),v),vi) and Lemma 4.3.2, it remains to verify

$$\int_0^t |\langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau)\rangle_V| d\tau < \infty, \ \forall t > 0, \ \Psi \in V^* \tag{4.25}$$

a.s. and

$$\langle \Psi, \mathbb{X}_x(t) - x\rangle_V + \int_0^t \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau)\rangle_V d\tau = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \langle \Psi, \eta_k\rangle_V \mathbb{1}_{[\alpha_m, \alpha_{m+1})}(t), \ \forall t > 0, \ \Psi \in V^* \tag{4.26}$$

almost surely.

Moreover, by Corollary 4.3.5 we get that this strong solution is unique. (The Corollary is indeed applicable, since every strong solution is obviously also a mild one.)

Now, let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set such that $0 = \alpha_0(\omega) < \alpha_1(\omega) < \alpha_2(\omega) < ...$ as well as $\lim_{m\to\infty} \alpha_m(\omega) = \infty$, $\alpha_m(\omega) \in D(\Theta(\omega))$ for all $m \in \mathbb{N}$, $\eta(t, z, \omega) \in \mathcal{V}$ for all $t \in (0, \infty)$ and $z \in Z$, $x(\omega) \in \mathcal{V}$, such that $t \mapsto \mathbb{X}_x(t, \omega)$ is cadlag, such that Lemma 4.3.9.i-iii) and such that

$$\mathbb{x}_{x,m}(\omega) \in \mathcal{V}, \ \forall m \in \mathbb{N}_0. \tag{4.27}$$

for all $\omega \in \Omega \setminus M$. (Invoking Lemma 2.2.6.vii) yields that M can indeed be chosen such that (4.27) holds.)

Now, fix an $\omega \in \Omega \setminus M$ and let us prove (4.25). For a given $t \in [0, \infty)$ there is an $m \in \mathbb{N}_0$ such that $t \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$. This yields

$$\begin{aligned} \int_0^t |\langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega)\rangle_V| d\tau &\leq \int_0^{\alpha_{m+1}(\omega)} |\langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega)\rangle_V| d\tau \\ &= \sum_{k=0}^{m} \int_{\alpha_k(\omega)}^{\alpha_{k+1}(\omega)} |\langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\tau - \alpha_k(\omega)) \mathbb{x}_{x,k}(\omega)\rangle_V| d\tau \\ &= \sum_{k=0}^{m} \int_0^{\beta_{k+1}(\omega)} |\langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\tau) \mathbb{x}_{x,k}(\omega)\rangle_V| d\tau. \end{aligned}$$

Moreover, invoking (4.27) gives

$$\int_0^{\beta_{k+1}(\omega)} |\langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\tau) \mathbb{x}_{x,k}(\omega)\rangle_V| d\tau < \infty$$

for all $k = 0, ..., m$, which concludes the proof of (4.25).

Proof of (4.26). Let $t \in (0, \infty)$ and (as usually) let $m \in \mathbb{N}_0$ such that $t \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$.

If $m = 0$, we have $\sum\limits_{j=1}^{\infty} \sum\limits_{k=1}^{j} \langle \Psi, \eta_k(\omega) \rangle_V \mathbb{1}_{[\alpha_j(\omega), \alpha_{j+1}(\omega))}(t) = 0$ and $\mathbb{X}_x(t, \omega) = T_{\mathcal{A}}(t)x(\omega)$. Hence in this case (4.26) follows from Lemma 4.3.8, which is applicable since $x(\omega) \in \mathcal{V}$.
Now assume $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\int_0^t \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau \\
= \quad & \sum_{k=0}^{m-1} \int_{\alpha_k(\omega)}^{\alpha_{k+1}(\omega)} \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau + \int_{\alpha_m(\omega)}^{t} \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau \\
= \quad & \sum_{k=0}^{m-1} \int_0^{\beta_{k+1}(\omega)} \langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\tau) \mathbb{x}_{x,k}(\omega) \rangle_V d\tau + \int_0^{t-\alpha_m(\omega)} \langle \Psi, \mathcal{A}^\circ T_{\mathcal{A}}(\tau) \mathbb{x}_{x,m}(\omega) \rangle_V d\tau.
\end{aligned}$$

In addition, (4.27) enables us to use Lemma 4.3.8 now. Doing so, and having in mind that $T_{\mathcal{A}}(\alpha_{k+1}(\omega) - \alpha_k(\omega)) \mathbb{x}_{x,k}(\omega) = \mathbb{x}_{x,k+1}(\omega) - \eta_{k+1}(\omega)$ for all $k \in \mathbb{N}_0$ gives

$$\begin{aligned}
&\int_0^t \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau \\
= \quad & -\sum_{k=0}^{m-1} \langle \Psi, \mathbb{x}_{x,k+1}(\omega) - \eta_{k+1}(\omega) - \mathbb{x}_{x,k}(\omega) \rangle_V - \langle \Psi, T_{\mathcal{A}}(t - \alpha_m(\omega)) \mathbb{x}_{x,m}(\omega) - \mathbb{x}_{x,m}(\omega) \rangle_V.
\end{aligned}$$

Now it is plain to deduce that also

$$\int_0^t \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau = \sum_{k=0}^{m-1} \langle \Psi, \eta_{k+1}(\omega) \rangle_V + \langle \Psi, x(\omega) \rangle_V - \langle \Psi, \mathbb{X}_x(t, \omega) \rangle_V.$$

Finally, the previous equation yields

$$\langle \Psi, \mathbb{X}_x(t, \omega) - x \rangle_V + \int_0^t \langle \Psi, \mathcal{A}^\circ \mathbb{X}_x(\tau, \omega) \rangle_V d\tau = \sum_{k=1}^{m} \langle \Psi, \eta_k(\omega) \rangle_V = \sum_{j=1}^{\infty} \sum_{k=1}^{j} \langle \Psi, \eta_k(\omega) \rangle_V \mathbb{1}_{[\alpha_j(\omega), \alpha_{j+1}(\omega))}(t),$$

which gives (4.26). □

Lemma 4.3.11. *Let $\mathcal{V} \subseteq V$ be a dense subspace of V. Then there is a sequence of mappings $(\Gamma_n)_{n \in \mathbb{N}}$, with $\Gamma_n : V \to V$, such that the following assertions hold.*

i) $\Gamma_n(V) \subseteq \mathcal{V}$ *for all* $n \in \mathbb{N}$,

ii) Γ_n *is* $\mathfrak{B}(V) - \mathfrak{B}(V)$*-measurable for all* $n \in \mathbb{N}$ *and*

iii) $\lim\limits_{n \to \infty} \Gamma_n(v) = v$ *for all* $v \in V$.

Proof. As $\mathcal{V}$ is dense and V is separable, we can find a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{V}$ such that

$$\overline{\{v_n : n \in \mathbb{N}\}} = \overline{\mathcal{V}} = V. \tag{4.28}$$

Now introduce

$$V_{j,n} := \{v \in V : \ ||v - v_j||_V = \min_{k=1,..,n} ||v - v_k||_V\}, \ \forall j \in \{1, ..., n\} \text{ and } n \in \mathbb{N},$$

set $\tilde{V}_{1,n} := V_{1,n}$ for all $n \in \mathbb{N}$ and

$$\tilde{V}_{j,n} := V_{j,n} \setminus (V_{1,n} \cup .. \cup V_{j-1,n}), \ \forall j \in \{2, .., n\} \text{ and } n \in \mathbb{N} \setminus \{1\}.$$

Then it is plain that for each $n \in \mathbb{N}$ the system of sets $(\tilde{V}_{j,n})_{j=1,..,n}$ is a disjoint cover of V.
Now introduce $\Gamma_n : V \to V$ by

$$\Gamma_n(v) := \sum_{j=1}^{n} v_j \mathbb{1}_{\tilde{V}_{j,n}}(v), \ \forall v \in V, , \ n \in \mathbb{N}.$$

Then it is plain that each Γ_n only takes values in the set $\{v_1, .., v_n\} \subseteq \mathcal{V}$ which gives i). In addition, we have that each $V_{j,n}$ is closed and therefore $V_{j,n} \in \mathfrak{B}(V)$ which implies $\tilde{V}_{j,n} \in \mathfrak{B}(V)$; this yields ii).
Finally, let us prove iii). To this end, fix $v \in V$ and note that for all $n \in \mathbb{N}$ there is precisely one $j(n) \in \{1, .., n\}$ such that $v \in \tilde{V}_{j(n),n}$ and hence $\Gamma_n(v) = v_{j(n)}$. Since also $v \in V_{j(n),n}$, we obtain

$$||v - \Gamma_n(v)||_V = ||v - v_{j(n)}||_V = \min_{k=1,..,n} ||v - v_k||_V, \ \forall n \in \mathbb{N}.$$

Finally, (4.28) yields that there is for a given $\varepsilon > 0$ an $n_0 \in \mathbb{N}$ such that $||v - v_{n_0}||_V < \varepsilon$ and consequently

$$||v - \Gamma_n(v)||_V = \min_{k=1,..,n} ||v - v_k||_V < \varepsilon, \ \forall n \geq n_0,$$

which concludes the proof. □

Theorem 4.3.12. *Let $\mathcal{V} \subseteq V$ be a dense subspace of V and let $V^* \subseteq V'$ be a subset which separates points. Moreover, let $x \in \mathcal{M}(\Omega; V)$, $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ and let $\mathbb{X}_x$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}})$ in V, where $\eta_m(\omega) := \eta(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Finally, assume that $\mathcal{V}$ is an invariant space w.r.t. $T_{\mathcal{A}}$ and that $\langle \Psi, \mathcal{A}^{\circ} T_{\mathcal{A}}(\cdot) u\rangle_V \in L^1((0,t))$ for all $t > 0$, $u \in \mathcal{V}$ and $\Psi \in V^*$.*
Then the stochastic process $\mathbb{X}_x$ is the unique mild solution of (ACPRM)$\{x, \eta, V^*\}$. Moreover, if in addition $(0,0) \in \mathcal{A}$, we have

$$||\mathbb{X}_x(t)||_V \leq ||x||_V + \int_{(0,t]\times Z} ||\eta(\tau, z)||_V N_\Theta(d\tau \otimes z), \ \forall t \geq 0, \tag{4.29}$$

with probability one.

Proof. Let $(\Gamma_n)_{n\in\mathbb{N}}$, where $\Gamma_n : V \to V$, be such that $\Gamma_n(V) \subseteq \mathcal{V}$, Γ_n is $\mathfrak{B}(V) - \mathfrak{B}(V)$-measurable and $\lim_{n\to\infty} \Gamma_n(v) = v$ for all $v \in V$. In addition, let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set such that

$$0 = \alpha_0(\omega) < \alpha_1(\omega) < \alpha_2(\omega) < ..., \; D(\Theta(\omega)) = \{\alpha_1(\omega), \alpha_2(\omega), ..\} \text{ and } \lim_{m\to\infty} \alpha_m(\omega) = \infty,$$

for all $\omega \in \Omega \setminus M$. In addition, let $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ be the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}})$ in V. Finally, for each $n \in \mathbb{N}$, let $(y_{n,k})_{k\in\mathbb{N}_0}$ and Y_n be the sequence and the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\Gamma_n(\eta_m))_{m\in\mathbb{N}}, \Gamma_n(x), T_{\mathcal{A}})$ in V.
Firstly, note that $\Gamma_n(x) \in \mathcal{M}(\Omega; V)$ and $\Gamma_n(\eta) \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$, for all $n \in \mathbb{N}$, since the composition of measurable functions remains measurable. Moreover, it is plain that $\Gamma_n(x), \; \Gamma_n(\eta) \in \mathcal{V}$ for all $n \in \mathbb{N}$ a.s. Consequently, we get by invoking Proposition 4.3.10 that Y_n is the strong solution of (ACPRM)$\{\Gamma_n(x), \Gamma_n(\eta), V^*\}$ for all $n \in \mathbb{N}$. Hence, it follows from Lemma 4.3.9.i),ii) and Lemma 2.2.6.i),v),vi) that $\mathbb{X}_x$ is a mild solution of (ACPRM)$\{x, \eta, V^*\}$, if

$$\lim_{n\to\infty} \int_{(0,t]\times Z} ||\Gamma_n(\eta(\tau, z, \omega)) - \eta(\tau, z, \omega)||_V N_\Theta(d\tau \otimes z, \omega) = 0, \; \forall t > 0 \text{ and } \omega \in \Omega \setminus M \tag{4.30}$$

and

$$\lim_{n\to\infty} \sup_{\tau\in[0,t]} ||Y_n(\tau, \omega) - \mathbb{X}_x(\tau, \omega)||_V = 0, \; \forall t > 0 \text{ and } \omega \in \Omega \setminus M. \tag{4.31}$$

Now let $t > 0$ and $\omega \in \Omega \setminus M$ be arbitrary but fixed and let $\tilde{m} \in \mathbb{N}_0$ be such that $t \in [\alpha_{\tilde{m}}(\omega), \alpha_{\tilde{m}+1}(\omega))$. (4.30) is trivial, since Lemma 4.2.2 gives that

$$\lim_{n\to\infty} \int_{(0,t]\times Z} ||\Gamma_n(\eta(\tau, z, \omega)) - \eta(\tau, z, \omega)||_V N_\Theta(d\tau \otimes z, \omega) = \lim_{n\to\infty} \sum_{k=1}^{\tilde{m}} ||\Gamma_n(\eta_k(\omega)) - \eta_k(\omega)||_V = 0.$$

Proof of (4.31). Firstly, it will be proven inductively that

$$||y_{n,m}(\omega) - \mathbb{x}_{x,m}(\omega)||_V \leq ||\Gamma_n(x(\omega)) - x(\omega)||_V + \sum_{k=1}^{m} ||\Gamma_n(\eta_k(\omega)) - \eta_k(\omega)||_V, \; \forall m \in \mathbb{N}_0 \tag{4.32}$$

and all $n \in \mathbb{N}$. If $m = 0$, (4.32) is trivial and if (4.32) holds for an $m \in \mathbb{N}$, then the contractivity of $T_{\mathcal{A}}$ and the induction hypothesis enable us to conclude that

$$\begin{aligned} ||y_{n,m+1}(\omega) - \mathbb{x}_{x,m+1}(\omega)||_V &\leq ||y_{n,m}(\omega) - \mathbb{x}_{x,m}(\omega)||_V + ||\Gamma_n(\eta_{m+1}(\omega)) - \eta_{m+1}(\omega)||_V \\ &\leq ||\Gamma_n(x(\omega)) - x(\omega)||_V + \sum_{k=1}^{m+1} ||\Gamma_n(\eta_k(\omega)) - \eta_k(\omega)||_V, \end{aligned}$$

which proves (4.32). Now note that for each $\tau \in [0, t]$ there is an $m_\tau \in \{0, ..., \tilde{m}\}$, such that

$\tau \in [\alpha_{m_\tau}(\omega), \alpha_{m_\tau+1}(\omega))$. Consequently, exploiting the contractivity of $T_{\mathcal{A}}$ and (4.32) yields

$$\begin{aligned}
||Y_n(\tau,\omega) - \mathbb{X}_x(\tau,\omega)||_V &= ||T_{\mathcal{A}}(\tau - \alpha_{m_\tau}(\omega))y_{n,m_\tau}(\omega) - T_{\mathcal{A}}(\tau - \alpha_{m_\tau}(\omega))\mathbb{x}_{x,m_\tau}(\omega)||_V \\
&\leq ||y_{n,m_\tau}(\omega) - \mathbb{x}_{x,m_\tau}(\omega)||_V \\
&\leq \max_{m=0,..,\tilde{m}} ||y_{n,m}(\omega) - \mathbb{x}_{x,m}(\omega)||_V \\
&\leq ||\Gamma_n(x(\omega)) - x(\omega)||_V + \sum_{k=1}^{\tilde{m}} ||\Gamma_n(\eta_k(\omega)) - \eta_k(\omega)||_V
\end{aligned}$$

As this upper bound is independent of $\tau \in [0,t]$, we get

$$\lim_{n\to\infty} \sup_{\tau\in[0,t]} ||Y_n(\tau,\omega) - \mathbb{X}_x(\tau,\omega)||_V \leq \lim_{n\to\infty} ||\Gamma_n(x(\omega)) - x(\omega)||_V + \sum_{k=1}^{\tilde{m}} ||\Gamma_n(\eta_k(\omega)) - \eta_k(\omega)||_V = 0,$$

which proves (4.31). Consequently, $\mathbb{X}_x$ is a mild solution of (ACPRM)$\{x, \eta, V^*\}$. Finally, Corollary 4.3.5 yields the uniqueness and Theorem 4.3.6 gives (4.29). □

In Chapter 5 and Chapter 6, we simply consider a time continuous, contractive semigroup and do not necessarily assume that this semigroup is associated to an m-accretive, densely defined operator. Note that this is a weaker assumption, since a semigroup associated to an m-accretive, densely defined operator is always time continuous and contractive, see Theorem 2.1.7.
Moreover, we will simply consider sequences $(\beta_m)_{m\in\mathbb{N}}$, where $\beta_m : \Omega \to (0,\infty)$ is $\mathcal{F}$-$\mathfrak{B}((0,\infty))$-measurable, and $(\eta_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$, and not point processes Θ and functions $\eta \in \mathcal{M}((0,\infty)\times Z\times\Omega; V)$. So, let us conclude this section with a remark connecting these different approaches:

Remark 4.3.13. *Let $(T(t))_{t\geq 0}$ be a time-continuous, contractive semigroup on V, and assume that the sequence of inter-arrival times $(\beta_m)_{m\in\mathbb{N}}$ induced by Θ, is an arbitrary sequence of independent and identically distributed random variables, with $\beta_m : \Omega \to (0,\infty)$. Then we have*

$$\mathbb{E}N_\Theta((0,t]\times Z) = \mathbb{E}\sum_{m=1}^{\infty} \mathbb{1}\left\{\sum_{k=1}^{m} \beta_k \leq t\right\} = \sum_{m=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{m} \beta_k \leq t\right) < \infty,$$

where the finiteness follows from [30, Theorem 1.6]. Thus, without any further assumptions regarding $(\beta_m)_{m\in\mathbb{N}}$, N_Θ is necessarily finite. In other words: For any sequence of $(0,\infty)$-valued i.i.d. random variables $(\beta_m)_{m\in\mathbb{N}}$, we can find a necessarily finite point process Θ, such that $(\beta_m)_{m\in\mathbb{N}}$ is the sequence of inter-arrival times induced by Θ. (Of course, $\Theta(\omega)(\alpha_m(\omega))$ has to be chosen such that Θ is $\mathcal{F}$-$\Pi(Z)$-measurable; this is always possible: For example $\Theta(\omega)(\alpha_m(\omega)) := z$ for all $\omega \in \Omega$ and $m \in \mathbb{N}$, where $z \in Z$ is fix, has the desired measurability property.)
Now, let $(\eta_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ and $x \in \mathcal{M}(\Omega; V)$. Moreover, let $\mathbb{X}_x$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V and introduce

$$\eta(t,z,\omega) := \sum_{k=1}^{\infty} \eta_k(\omega)\mathbb{1}_{[\alpha_k(\omega),\alpha_{k+1}(\omega))}(t),$$

for all $t > 0$, $z \in Z$ and $\omega \in \Omega$. Then it is obvious that indeed $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$.
Now, if one assumes that $T(t) = T_{\mathcal{A}}(t)$ for all $t \geq 0$ and that $\langle \Psi, \mathcal{A}^{\circ} T_{\mathcal{A}}(\cdot)u \rangle_V \in L^1((0,t))$ for all $t > 0$, $u \in \mathcal{V}$ and $\Psi \in V^*$, where $\mathcal{V} \subseteq V$ is a dense and invariant space w.r.t. $T_{\mathcal{A}}$, and $V^* \subseteq V'$ separates points; then $\mathbb{X}_x$ is the unique mild solution of (ACPRM)$\{x, \eta, V^*\}$.
In conclusion, if one has given: A time-continuous, contractive semigroup $(T(t))_{t\geq 0}$ on V, a sequence $(\eta_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$, another sequence $(\beta_m)_{m\in\mathbb{N}}$, of $(0,\infty)$-valued i.i.d. random variables and an initial $x \in \mathcal{M}(\Omega; V)$; then: One can choose $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; V)$ such that the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V is for some $V^* \subseteq V'$ the unique mild solution of (ACPRM)$\{x, \eta, V^*\}$, if $(T(t))_{t\geq 0}$ fulfills the assumptions we just outlined.

4.4 Examples

The purpose of this section is to demonstrate the applicability of the developed existence and uniqueness results to the one-dimensional examples introduced in Remark 2.2.7 and to the weighted p-Laplacian evolution equation with Neumann boundary conditions considered in Chapter 3.

Throughout this section, $(Z, \mathcal{Z})$ is a measurable space, $\Theta : \Omega \to \pi(Z)$ denotes a finite point process and $N_\Theta : (\mathfrak{B}((0,\infty)) \otimes \mathcal{Z}) \times \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is the counting measure induced by Θ. Furthermore, $(\alpha_m)_{m\in\mathbb{N}}$ denotes the sequence of hitting times induced by Θ and $(\beta_m)_{m\in\mathbb{N}}$ the sequence of inter-arrival times induced by Θ; and for notational convenience we also introduce $\alpha_0 : \Omega \to \mathbb{R}$, with $\alpha_0 := 0$.
Now let us start with the one-dimensional examples: (To do that, we will have to formulate fairly simple nonlinear ODEs in the language of the general existence/uniqueness theory introduced in Section 2.1. To the reader familiar with this theory, this will probably seem like a huge overkill, but to the unfamiliar reader, this might be helpful.)

Remark 4.4.1. *Let $\rho_1 \in (0,1)$ and $\rho_2 \in (0,\infty)$ be given and let T_{ρ_1}, T_{ρ_2} denote the semigroups introduced in Remark 2.2.7, i.e. $T_{\rho_1}(t)v := sgn(v)\left(-t + |v|^{\rho_1}\right)_+^{\frac{1}{\rho_1}}$ and $T_{\rho_2}(t)v := sgn(v)\left(t + |v|^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$ for all $v \in \mathbb{R}$ and $t \in [0,\infty)$.*
Now introduce the mappings $A_{\rho_1}, A_{\rho_2} : \mathbb{R} \to \mathbb{R}$ defined by

$$A_{\rho_1}(v) := \frac{1}{\rho_1} sgn(v)|v|^{1-\rho_1} \text{ and } A_{\rho_2}(v) := \rho_2 v|v|^{\frac{1}{\rho_2}},$$

for all $v \in \mathbb{R}$. Now, in the remainder of this remark we fix $i \in \{1,2\}$ and note that A_{ρ_i} can be viewed as a single-valued operator, with domain $D(A_{\rho_i}) = \mathbb{R}$.
Plainly, A_{ρ_i} is monotonically increasing. Consequently, A_{ρ_i} is accretive. Moreover, A_{ρ_i} is continuous and $\lim\limits_{v\to\infty} v + \alpha A_{\rho_i}(v) = \infty$ as well as $\lim\limits_{v\to-\infty} v + \alpha A_{\rho_i}(v) = -\infty$ for all $\alpha \in (0,\infty)$. Thus, A_{ρ_i} is also m-accretive.
Consequently, as we already know that A_{ρ_i} is densely defined, it follows from Theorem 2.1.7 that the

initial value problem

$$0 \in u'(t) + A_{\rho_i} u(t), \text{ for a.e. } t \in (0,\infty),\ u(0) = v, \tag{4.33}$$

has, for any $v \in \mathbb{R}$ precisely one mild solution. Moreover, for every $v \in \mathbb{R}$ we have: $T_{\rho_i}(\cdot)v$ is continuously differentiable on $[0,\infty)$, thus it is a fortiori continuous on $[0,\infty)$, locally absolutely continuous as well as differentiable almost everywhere on $(0,\infty)$, and a direct calculation verifies that $T'_{\rho_i}(t)v = -A_{\rho_i}(T_{\rho_i}(t)v)$ for all $t \in [0,\infty)$. Thus $T_{\rho_i}(\cdot)v$ is for any $v \in \mathbb{R}$ a strong solution of (4.33). And as any strong solution is a mild one (see Remark 2.1.5), $T_{\rho_i}(\cdot)v$ is the unique strong solution of (4.33); and in our terminology: $(T_{\rho_i}(t))_{t\geq 0}$ is the semigroup associated to A_{ρ_i}, see Definition 2.1.8.
Moreover, as A_{ρ_i} is everywhere defined, $(T_{\rho_i}(t))_{t\geq 0}$ is domain invariant. And as $T_{\rho_i}(\cdot)v$ is, for any $v \in \mathbb{R}$, continuously differentiable on $[0,\infty)$ it admits an infinitesimal generator which coincides with $\mathcal{A}_{\rho_i}$.
Now, we are in the position to Proposition 4.3.10 to T_{ρ_i}: Choose $V = \mathbb{R}$ and $\mathcal{V} = \mathbb{R}$. Of course, we identify the dual space of $\mathbb{R}$ with $\mathbb{R}$ and choose $V^* = \mathbb{R}$. As, $[0,\infty) \ni t \mapsto A_{\rho_i} T_{\rho_i}(t)v$ is a continuous map, the integrability assumption stated in Proposition 4.3.10 is clearly fulfilled, and we infer from Proposition 4.3.10 that: For any $x \in \mathcal{M}(\Omega;\mathbb{R})$ and $\eta \in \mathcal{M}((0,\infty)\times Z \times \Omega;\mathbb{R})$, (ACPRM)$\{x,\eta,\mathbb{R}\}$ has a uniquely determined strong solution, which is given by the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\rho_i})$ in $\mathbb{R}$, where $\eta_m(\omega) := \eta(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

Now let us turn to the p-Laplacian example: Firstly, let us recall some notations introduced in Section 3.2: Throughout the remainder of this section, let $n \in \mathbb{N}\setminus\{1\}$ and $p \in (1,\infty)\setminus\{2\}$. Moreover, $\emptyset \neq S \subseteq \mathbb{R}^n$ denotes a non-empty, open, connected and bounded sets of class C^1, and λ is the Lebesgue measure on $\mathfrak{B}(S)$. Moreover, $\gamma : S \to (0,\infty)$, denotes the weight function, i.e. we assume $\gamma \in L^\infty(S)$, $\gamma^{\frac{1}{1-p}} \in L^1(S)$ and that there is a $\gamma_0 : \mathbb{R}^n \to \mathbb{R}$ with $\gamma_0 \in M^p(\mathbb{R}^n)$ such that $\gamma_0|_S = \gamma$ a.e. on S. Moreover, as previously we set

$$W^{1,p}_\gamma(S) := \{f \in L^p(S) :\ \nabla f \in L^p(S,\gamma;\mathbb{R}^n)\}.$$

In addition, $A_p : D(A_p) \to 2^{L^1(S)}$ denotes the p-Laplace operator introduced in Definition 3.2.2, and $\mathcal{A}_p : D(\mathcal{A}_p) \to 2^{L^1(S)}$ denotes its closure, see Definition 3.2.4 for the definition and Theorem 3.2.5 for the fact that this is the closure of A_p. In addition, note that $\mathcal{A}_p$ is m-accretive and densely defined, see Theorem 3.2.5.
Finally, $(T_{\mathcal{A}_p}(t))_{t\geq 0}$, where $T_{\mathcal{A}_p} : L^1(S) \to L^1(S)$ for all $t \geq 0$, denotes the semigroup associated to $\mathcal{A}_p$, see Remark 3.2.6.

Now, let us apply the results of Section 4.3 to the current setting: As the reader probably guessed correctly, the Banach Space V considered there has to be chosen as $V = L^1(S)$. As usually, we identify

V' with $L^\infty(S)$. Note that in this case, the duality $\langle \cdot, \cdot \rangle_{L^1(S)}$ reduces to an integral, i.e.

$$\langle f, h \rangle_{L^1(S)} = \int\limits_S fhd\lambda,$$

for any $f \in L^1(S)$ and $h \in L^\infty(S)$.

Theorem 4.4.2. *Let $x \in \mathcal{M}(\Omega; L^1(S))$ and $\eta \in \mathcal{M}((0,\infty) \times Z \times \Omega; L^1(S))$. Moreover, introduce $\mathbb{X}_x^{(p)} : [0,\infty) \times \Omega \to L^1(S)$ as the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}_p})$ in $L^1(S)$, where $\eta_m(\omega) := \eta(\alpha_m(\omega), \Theta(\omega)(\alpha_m(\omega)), \omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Then $\mathbb{X}_x^{(p)}$ is the uniquely determined mild solution of (ACPRM)$\{x, \eta, W_\gamma^{1,p}(S) \cap L^\infty(S)\}$, and we have*

$$||\mathbb{X}_x^{(p)}(t)||_{L^1(S)} \leq ||x||_{L^1(S)} + \int\limits_{(0,t]\times Z} ||\eta(\tau, z)||_{L^1(S)} N_\Theta(d\tau \otimes z), \ \forall t \geq 0,$$

with probability one. Moreover, if in addition $x \in L^\infty(S)$ and $\eta(t,z) \in L^\infty(S)$ for all $t > 0$ and $z \in Z$ a.s., then $\mathbb{X}_x^{(p)}$ is even the uniquely determined strong solution of (ACPRM)$\{x, \eta, W_\gamma^{1,p}(S) \cap L^\infty(S)\}$.

Proof. All claims follow at once from Proposition 4.3.10 and Theorem 4.3.12, by choosing $V := L^1(S)$, $\mathcal{V} := L^\infty(S)$ and $V^* := W_\gamma^{1,p}(S) \cap L^\infty(S)$ there; more precisely: Firstly, we already know that $T_{\mathcal{A}_p}$ is domain invariant and admits an infinitesimal generator, see Remark 3.2.6.v). Moreover, $\mathcal{V}$ is indeed an invariant space w.r.t. $T_{\mathcal{A}_p}$ (Remark 3.2.6.iv), with $q = \infty$) and it is well-known that $\mathcal{V}$ is dense in $(V, || \cdot ||_V)$. Furthermore, as $C_c^\infty(S) \subseteq V^*$, it is clear that V^* separates points. Finally, the needed integrability assumption stated in Proposition 4.3.10 and Theorem 4.3.12 was already proven in Lemma 3.3.7, and it is clear that $(0,0) \in \mathcal{A}_p$. $\square$

Chapter 5

Asymptotic Results for ACPRM-Processes in the finite extinction Case

5.1 Outline & Highlights

The purpose of this chapter is to prove an SLLN and a CLT for ACPRM-processes, if the underlying semigroup extincts in finite time. Outlining this in greater detail requires to introduce some notations: Firstly, recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and introduce a separable Banach space $(V, ||\cdot||_V)$. Moreover, let $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$ be $(0,\infty)$-valued and V-valued sequences of random variables, respectively. In addition, let x be a V-valued random variable, introduce $\alpha_m := \sum\limits_{k=1}^{m} \beta_m$ and $\alpha_0 := 0$. Finally, let $(T(t))_{t\geq 0}$ denote a time-continuous, contractive semigroup on V, and let $\mathbb{X}_x$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V.

Now, the finite extinction assumption that we will have to assume throughout this chapter reads as follows: There are constants $\kappa \in (0,\infty)$ and $\rho \in (0,1)$ such that

$$||T(t)v||_{V_1}^{\rho} \leq (-\kappa t + ||v||_{V_1}^{\rho})_+ \tag{5.1}$$

for all $t \geq 0$ and $v \in V_1$, where $(V_1, ||\cdot||_{V_1}) \subseteq V$ is another separable Banach space which is an invariant space w.r.t. T and such that the injection $V_1 \hookrightarrow V$ is continuous.
The reason why we introduce V_1 is to make the results more applicable, since it is possible that one has a semigroup which is defined on a (separable) Banach space V but that the finite extinction property (5.1) only holds on a subspaces, or is only known on a subspace.

The most important stochastic assumptions needed to achieve this, are that $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$

are both i.i.d. sequences, which are independent of each other, independent of the initial x and that β_m is in some sense (to be made precise later) "larger" than η_m.
It will then be possible to show that, for a class of functionals $\Xi : V \to W$, where $(W, ||\cdot||_W)$ is another separable Banach space, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Xi(\mathbb{X}_x(\tau))d\tau = \nu_\Xi, \tag{SLLN}$$

with probability one, where $\nu_\Xi \in W$ will be made precise later; and that if $(W, ||\cdot||_W)$ is in addition a type 2 Banach space, we have

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \Xi(\mathbb{X}_x(\tau))d\tau - t\nu_\Xi \right) = Z, \tag{CLT}$$

in distribution, where $Z : \Omega \to W$ is a centered, Gaussian W-valued random variables, whose covariance will be determined explicitly.
Particularly, the class of functionals is sufficiently large, such that $\Xi(\mathbb{X}_x(t))$ in (SLLN) and (CLT) can be replaced by $\mathbb{X}_x(t)$. Moreover, Ξ depends on another separable Banach space $(V_2, ||\cdot||_{V_2}) \subseteq (V, ||\cdot||_V)$, with continuous injection and invariant w.r.t. T. This makes it possible to replace $\Xi(\mathbb{X}_x(t))$ in (SLLN) and (CLT) by $||\mathbb{X}_x(t)||_{V_2}$.

Moreover, our theoretical results will be applied to the one-dimensional semigroup introduced in Remark 2.2.7.i) and to the weighted p-Laplacian evolution equation for "small" p; more accurately, p has to be as in Theorem 3.5.6. For the latter semigroup, we will see that all L^q-norms, where $q \in [1, \infty)$, are a valid choice for $||\cdot||_{V_2}$ and that (SLLN) as well as (CLT) also hold for $\mathbb{X}_x$ itself.

There are besides the examples we consider, many other nonlinear semigroups which extinct in finite time. For another concrete example, see [2, Chapter 4] and for a general survey on the finite extinction property, containing many examples, including the (unweighted) p-Laplacian case, see [12].
Of course, there are plenty of semigroups that do not extinct in finite time. Therefore, in the next chapter we continue our investigation regarding the asymptotic properties of $\mathbb{X}_x$. Particularly, the theoretical results obtained there will be applied to the p-Laplacian semigroup for "large" p and to the semigroup introduced in Remark 2.2.7.ii).

Note that one inevitably has to draw some assumption regarding the asymptotic behavior of T, if one wants a result like (SLLN) or (CLT) to hold: If $\eta_k = 0$ for all $k \in \mathbb{N}$ a.s., then (thanks to the semigroup property) we have $\mathbb{X}_x(t) = T(t)x$ for all $t \geq 0$ with probability one. Thus, if the general assumptions on η_k are such that at least $\eta_k = 0$ for all $k \in \mathbb{N}$ is a valid choice for the η_k's then a result like (SLLN) can only hold if the limit $\lim_{t\to\infty} \frac{1}{t} \int_0^t \Xi(T(t)x)d\tau$ exists almost surely.
Of course, assuming that the semigroup extincts in finite time, and that this finite extinction is pre-

cisely due to (5.1) is a strong assumption. But, on the other hand note that the results we obtain are very strong; particular, that we obtain both the SLLN and the CLT for vector-valued functionals. One possible (and frequently used) way to prove an inequality like (5.1) is employing Lemma 3.5.1 - which is precisely how we derived such a bound for the weighted p-Laplacian evolution equation in Section 3.5.

The basic technique to prove this chapter's general results is to introduce a certain sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$, such that $\int\limits_0^{\tau_m} \Xi(\mathbb{X}_x(\tau))d\tau$ can be decomposed into an i.i.d. sum; and then to use approximation techniques to replace τ_m by t. Moreover, this chapter relies strongly on the theory of random variables taking values in separable Banach spaces. A comprehensive introduction to this topic can be found in [26].

This chapter is structured as follows: The mentioned general results are proven in Section 5.2 and the applicability of these results will be demonstrated in Section 5.3.
Moreover, Section 5.2 contains a type 2 Banach space version of Anscombe's CLT, which we did not find in the literature and might be of independent interest to some readers. It can be found in Theorem 5.2.22 and is written as self-contained as possible.

5.2 The SLLN and the CLT

The purpose of this section is to prove the introductory mentioned results (SLLN) and (CLT). At first we will state the needed assumptions, as well as some additional notations. As this section is quite long, a detailed outline is given after all the assumptions and notations have been stated, see Remark 5.2.6. There the technique employed to prove (SLLN) and (CLT) is also described in greater detail.

Throughout this section, $(V, ||\cdot||_V)$ denotes a separable Banach space and $(T(t))_{t\geq 0}$ denotes a time-continuous, contractive semigroup on V. (Consequently, $(T(t))_{t\geq 0}$ is also jointly continuous.)
In addition, the reader is reminded that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and that we use the short-cut notations $\mathcal{M}(\Omega; V) := \mathcal{M}(\Omega, \mathcal{F}; V)$ and $L^q(\Omega; V) := L^q(\Omega, \mathcal{F}, \mathbb{P}; V)$ for all $q \in [1, \infty)$.
Moreover, $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ denote i.i.d. sequences, where $\eta_m : \Omega \to V$ and $\beta_m : \Omega \to (0, \infty)$ are $\mathcal{F}$-$\mathfrak{B}(V)$-measurable and $\mathcal{F}$-$\mathfrak{B}((0, \infty))$-measurable, respectively. In addition, assume that $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ are independent of each other. Furthermore, introduce $(\alpha_m)_{m\in\mathbb{N}_0}$, where $\alpha_m : \Omega \to [0, \infty)$, by $\alpha_0 := 0$ and

$$\alpha_m := \sum_{k=1}^{m} \beta_k, \ \forall m \in \mathbb{N}.$$

Finally, for any $x \in \mathcal{M}(\Omega; V)$, $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ and $\mathbb{X}_x : [0, \infty) \times \Omega \to V$ denote the sequence and the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V.
Now, the following functional analytic assumption is drawn:

Assumption 5.2.1. *There are separable Banach spaces $(V_1, ||\cdot||_{V_1})$ and $(V_2, ||\cdot||_{V_2})$, with $V_i \subseteq V$, such that the injections $V_i \hookrightarrow V$ are continuous for $i = 1, 2$. In addition, the following assertions hold.*

i) *V_i is an invariant space w.r.t. $(T(t))_{t\geq 0}$, for $i = 1, 2$.*

ii) *There are constants $\kappa \in (0, \infty)$ and $\rho \in (0, 1)$ such that $||T(t)v||^{\rho}_{V_1} \leq (-\kappa t + ||v||^{\rho}_{V_1})_+$ for all $t \geq 0$ and $v \in V_1$, where $(\cdot)_+ := \max(\cdot, 0)$.*

iii) *$||T(t)v||_{V_2} \leq ||v||_{V_2}$ for all $v \in V_2$.*

Throughout this entire section, Assumption 5.2.1 is assumed to hold and $(V_1, ||\cdot||_{V_1})$, $(V_2, ||\cdot||_{V_2})$ as well as $\kappa \in (0, \infty)$, $\rho \in (0, 1)$ are as in this assumption.

Stating our stochastic assumption, requires the following remark regarding measurability:

Remark 5.2.2. *Let $(\hat{V}, ||\cdot||_{\hat{V}}) \subseteq (V, ||\cdot||_V)$ be another separable Banach space and assume that the injection $\hat{V} \hookrightarrow V$ is continuous. Then Lusin-Souslin's Theorem (see [22, Theorem 15.1]) yields $f(B) \in \mathfrak{B}(V)$ for all $B \in \mathfrak{B}(\hat{V})$ and $f : \hat{V} \to V$ which are continuous and injective. Consequently, we get $\mathfrak{B}(\hat{V}) \subseteq \mathfrak{B}(V)$. Particularly, for $|\cdot|_{\hat{V}} : V \to [0, \infty)$, with $|v|_{\hat{V}} := ||v||_{\hat{V}}$ for all $v \in \hat{V}$ and $|v|_{\hat{V}} := 0$ for all $v \in V \setminus \hat{V}$, we have that $|\cdot|_{\hat{V}}$ is $\mathfrak{B}(V)$-$\mathfrak{B}([0, \infty))$-measurable.*
Hence, if $y : \Omega \to V$ is $\mathcal{F}$-$\mathfrak{B}(V)$-measurable, with $\mathbb{P}(y \in \hat{V}) = 1$, then $||y||_{\hat{V}}$ is $\mathcal{F}$-$\mathfrak{B}([0, \infty))$-measurable.

Assumption 5.2.3. *Throughout this section, the following assertions hold for all $m \in \mathbb{N}$.*

i) *$\eta_m \in V_i$ for $i = 1, 2$ with probability one.*

ii) *$\mathbb{E}||\eta_m||^4_{V_2} < \infty$, and there is a $\hat{\varepsilon} > 0$, such that $\mathbb{E}||\eta_m||^{\rho(11+\hat{\varepsilon})}_{V_1} < \infty$ and $\mathbb{E}\beta_m^{11+\hat{\varepsilon}} < \infty$.*

iii) *$-\kappa \mathbb{E}\beta_m + \mathbb{E}||\eta_m||^{\rho}_{V_1} < 0$.*

Throughout this section $\hat{\varepsilon} > 0$ is as in the preceding assumption.

Notation 5.2.4. *We write $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$, whenever the following assertions hold.*

i) *$(W, ||\cdot||_W)$ is a separable Banach space.*

ii) *$\Xi : V \to W$ is $\mathfrak{B}(V) - \mathfrak{B}(W)$-measurable.*

iii) *Ξ is sub-linear in the following sense: There are constants c_1, $c_2 \in [0, \infty)$ such that $||\Xi(v)||_W \leq c_1||v||_{V_2} + c_2$, for all $v \in V_2$.*

Definition 5.2.5. *A mapping $x : \Omega \to V$ is called an independent initial leading to extinction, if the following assertions hold.*

i) *$x \in \mathcal{M}(\Omega; V)$.*

ii) *$x \in V_i$ for $i = 1, 2$ with probability one.*

iii) *$\mathbb{E}||x||^{2\rho}_{V_1} < \infty$.*

iv) x *is jointly independent of* $(\beta_m)_{m\in\mathbb{N}}$ *and* $(\eta_m)_{m\in\mathbb{N}}$.

Moreover, if $x:\Omega\to V$ *is an independent initial leading to extinction, we denote by* $(e_x(n))_{n\in\mathbb{N}}$*, where* $e_x(n):\Omega\to\mathbb{N}\cup\{\infty\}$*, the sequence of extinction times, defined by*

v) $e_x(1) := \min(m\in\mathbb{N}:\ T(\beta_m)\mathbb{x}_{x,m-1}=0)$ *and*

vi) $e_x(n) := \min\big(m\in\mathbb{N}:\ T(\beta_m)\mathbb{x}_{x,m-1}=0,\ m>e_x(n-1)\big)$ *for all* $n\in\mathbb{N}\setminus\{1\}$.

Finally, introduce the filtrations[1] $(\mathcal{F}^x_j)_{j\in\mathbb{N}}$ *and* $(\tilde{\mathcal{F}}^x_m)_{m\in\mathbb{N}_0}$*, by*

vii) $\mathcal{F}^x_1 := \sigma_0(x,\beta_1)$, $\tilde{\mathcal{F}}^x_0 := \sigma_0(x)$ *and*

viii) $\mathcal{F}^x_j := \sigma_0(x,\beta_1,...,\beta_j,\eta_1,...,\eta_{j-1})$ *for all* $j\in\mathbb{N}\setminus\{1\}$ *and* $\tilde{\mathcal{F}}^x_m := \sigma_0(x,\beta_1,...,\beta_m,\eta_1,...,\eta_m)$ *for all* $m\in\mathbb{N}$.

Remark 5.2.6. *Let* $x\in\mathcal{M}(\Omega;V)$ *be an independent initial leading to extinction and* $\Xi\in SL_{V_2}(V)$*. The centerpiece of the proof of the SLLN as well as the CLT, which are both proven in Theorem 5.2.23, is the fact that the sequence* $\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}}\Xi(\mathbb{X}_x(\tau))d\tau\right)_{n\in\mathbb{N}}$ *is i.i.d., square integrable and for each* $n\in\mathbb{N}$ *in distribution equal to* $\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}\Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau$*, where* $\overline{x}\in\mathcal{M}(\Omega;V)$ *is specified in Remark 5.2.14.*
Before one can prove these results, one of course needs that $\mathbb{P}(e_x(n)<\infty,\ \forall n\in\mathbb{N})=1$ *and that the occurring integrals exist and are well-defined, which is subject to Proposition 5.2.10 and Lemma 5.2.12. The stated i.i.d. and square integrability assertions are then proven in Proposition 5.2.17 and Lemma 5.2.18.*
Even though $\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}}\Xi(\mathbb{X}_x(\tau))d\tau\right)_{n\in\mathbb{N}}$ *is i.i.d., it remains so far open how one gets from there to Theorem 5.2.23. A similar obstacle occurs for discrete time Markov chains possessing an atom; and the technique we employ to overcome it is somehow similar to the one used in [29, Theorems 17.2.1 and 17.2.2]. It is just "somehow" similar, since we are not in discrete time, consider vector-valued instead of real-valued functionals and last but not least* $T(\beta_m)\mathbb{x}_{x,m-1}=0$*, means* $\mathbb{x}_{x,m}=\eta_m$*, i.e. we do not stop the sequence* $(\mathbb{x}_{x,m})_{m\in\mathbb{N}}$ *at deterministic states, but at a "random state"; moreover, note that even though* $(\mathbb{x}_{x,m})_{m\in\mathbb{N}}$ *is a Markov chain,* $\mathbb{X}_x$ *is not necessarily*[2] *a Markov process.*
Moreover, Corollary 5.2.24 is a useful application of Theorem 5.2.23 for special choices of $(\Xi,(W,||\cdot||_W))$*.*
In addition, Theorem 5.2.22 is a vector-valued version of Anscombe's CLT.
The remaining results, which have not been mentioned explicitly in this remark, solely serve to keep the exposition more clean and the proofs more accessible, but are not of independent interest out of this section.

Lemma 5.2.7. *Let* $x:\Omega\to V$ *be an independent initial leading to extinction. Then all of the following assertions hold.*

[1] See Remark 2.2.9 for our conventions regarding σ-Algebras.

[2] In Section 6.2, we shall see that $\mathbb{X}_x$ is a Markov process, if x, $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ are jointly independent, both sequences are i.i.d. and each β_m is exponentially distributed. The present author conjectures that this is the only (non-trivial) distribution for β_m turning $\mathbb{X}_x$ into a Markov process.

i) $\mathbb{x}_{x,m}$ *is* $\tilde{\mathcal{F}}^x_m$*-*$\mathfrak{B}(V)$*-measurable for all* $m \in \mathbb{N}_0$*.*

ii) $e_x(n) + 1 \leq e_x(n+1)$ *and* $e_x(n) \geq n$ *for all* $n \in \mathbb{N}$*.*

iii) $\{e_x(n) = j\} \in \mathcal{F}^x_j$ *for all* $n, j \in \mathbb{N}$*.*

Proof. Let us start by proving i) inductively. We have $\mathbb{x}_{x,0} = x$, which is obviously $\sigma_0(x)$-$\mathfrak{B}(V)$-measurable. Now assume that i) holds for an $m \in \mathbb{N}_0$ and note that $\mathbb{x}_{x,m+1} = T(\beta_{m+1})\mathbb{x}_{x,m} + \eta_{m+1}$. As $\mathbb{x}_{x,m}$ is by the induction hypothesis a fortiori $\tilde{\mathcal{F}}^x_{m+1}$-$\mathfrak{B}(V)$-measurable and since β_{m+1} is obviously $\tilde{\mathcal{F}}^x_{m+1}$-$\mathfrak{B}([0,\infty))$-measurable, Remark 2.2.5 yields that $T(\beta_{m+1})\mathbb{x}_{x,m}$ is $\tilde{\mathcal{F}}^x_{m+1}$-$\mathfrak{B}(V)$-measurable. As η_{m+1} has this property as well, i) follows.
Now note that it is plain that $e_x(n)+1 \leq e_x(n+1)$, which gives $e_x(n) \geq n$, since $e_x(1) \geq 1$, by definition. Consequently, ii) holds as well.
Proof of iii). This statement is proven inductively w.r.t. $n \in \mathbb{N}$. We have for any $j \in \mathbb{N}$ that

$$\{e_x(1) \leq j\} = \{\exists k \in \{1, ..., j\} : \; T(\beta_k)\mathbb{x}_{x,k-1} = 0\} = \bigcup_{k=1}^{j} \{T(\beta_k)\mathbb{x}_{x,k-1} = 0\} \in \mathcal{F}^x_j,$$

by Remark 2.2.5 and i). Consequently, as $\{e_x(1) = j\} = \{e_x(1) \leq j\} \setminus \{e_x(1) \leq j-1\}$ and $\mathcal{F}^x_{j-1} \subseteq \mathcal{F}^x_j$, iii) holds if $n = 1$.
Now assume that iii) holds for an $n \in \mathbb{N}$. If $j < n+1$, we have $\{e_x(n+1) \leq j\} = \emptyset$, by ii). So let $j \geq n+1$. Note that on $\{e_x(n+1) \leq j\}$, we have $n \leq e_x(n) < j$, by ii).
Consequently, we have

$$\{e_x(n+1) \leq j\} = \bigcup_{i=n}^{j-1} \{\exists \; k \in \{i+1, ..., j\} : \; T(\beta_k)\mathbb{x}_{x,k-1} = 0, \; e_x(n) = i\}.$$

Moreover, the induction hypothesis yields $\{e_x(n) = i\} \in \mathcal{F}^x_i \subseteq \mathcal{F}^x_j$, for all $i = n, ..., j-1$ and combining Remark 2.2.5 and i) gives $\{T(\beta_k)\mathbb{x}_{x,k-1} = 0\} \in \mathcal{F}^x_k \subseteq \mathcal{F}^x_j$ for all $k = n+1, ..., j$.
Consequently, we get $\{e_x(n+1) \leq j\} \in \mathcal{F}^x_j$ for all $j \in \mathbb{N}$ and therefore also $\{e_x(n+1) = j\} \in \mathcal{F}^x_j$. □

Lemma 5.2.8. *Let* $x : \Omega \to V$ *be an independent initial leading to extinction. Then the following assertions hold.*

i) $\mathbb{x}_{x,m} \in V_i$ *for all* $m \in \mathbb{N}_0$ *and* $i \in \{1, 2\}$ *almost surely.*

ii) $||\mathbb{x}_{x,m}||^{\rho}_{V_1} \leq (-\kappa\beta_m + ||\mathbb{x}_{x,m-1}||^{\rho}_{V_1})_+ + ||\eta_m||^{\rho}_{V_1}$ *for all* $m \in \mathbb{N}$ *almost surely.*

Proof. Thanks to Lemma 2.2.6.vii), the first assertion is trivial.
The second assertion is also easily verified: Appealing to Assumption 5.2.1.ii), while having in mind i), gives

$$||\mathbb{x}_{x,m}||^{\rho}_{V_1} \leq ||T(\beta_m)\mathbb{x}_{x,m-1}||^{\rho}_{V_1} + ||\eta_m||^{\rho}_{V_1} \leq (-\kappa\beta_m + ||\mathbb{x}_{x,m-1}||^{\rho}_{V_1})_+ + ||\eta_m||^{\rho}_{V_1}$$

for all $m \in \mathbb{N}$ almost surely, where the well-known inequality $(a+b)^\rho \leq a^\rho + b^\rho$ for all $a, b \geq 0$, was used. $\square$

Lemma 5.2.9. *Let $x : \Omega \to V$ be an independent initial leading to extinction and introduce $m, n \in \mathbb{N}$, with $m < n$. Then the inclusion*

$$\{-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^\rho > 0,\ \forall k = m, ..., n\} \subseteq \{-\kappa \sum_{k=m}^{n} \beta_k + \sum_{k=m}^{n-1} ||\eta_k||_{V_1}^\rho + ||\mathbb{x}_{x,m-1}||_{V_1}^\rho > 0\}$$

holds up to a $\mathbb{P}$-null-set.

Proof. Fix $n \in \mathbb{N} \setminus \{1\}$ and let us prove inductively that

$$\{-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^\rho > 0, \forall k = n-j, .., n\} \subseteq \{-\kappa \sum_{k=n-j}^{n} \beta_k + \sum_{k=n-j}^{n-1} ||\eta_k||_{V_1}^\rho + ||\mathbb{x}_{x,n-j-1}||_{V_1}^\rho > 0\} \quad (5.2)$$

for all $j = 1, ..., n-1$ almost surely, which obviously yields the claim.
So let $j = 1$. Firstly, invoking Lemma 5.2.8.ii) gives $||\mathbb{x}_{x,n-1}||_{V_1}^\rho \leq (-\kappa\beta_{n-1} + ||\mathbb{x}_{x,n-2}||_{V_1}^\rho)_+ + ||\eta_{n-1}||_{V_1}^\rho$ a.s. and therefore

$$\{-\kappa\beta_n + ||\mathbb{x}_{x,n-1}||_{V_1}^\rho > 0\} \subseteq \{-\kappa\beta_n + (-\kappa\beta_{n-1} + ||\mathbb{x}_{x,n-2}||_{V_1}^\rho)_+ + ||\eta_{n-1}||_{V_1}^\rho > 0\} \quad (5.3)$$

almost surely. Using this yields

$$\begin{aligned}
& \{-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^\rho > 0,\ \forall k = n-1, ..., n\} \\
\subseteq\ & \{-\kappa\beta_{n-1} + ||\mathbb{x}_{x,n-2}||_{V_1}^\rho > 0,\ -\kappa\beta_n + (-\kappa\beta_{n-1} + ||\mathbb{x}_{x,n-2}||_{V_1}^\rho)_+ + ||\eta_{n-1}||_{V_1}^\rho > 0\} \\
\subseteq\ & \{-\kappa \sum_{k=n-1}^{n} \beta_k + \sum_{k=n-1}^{n-1} ||\eta_k||_{V_1}^\rho + ||\mathbb{x}_{x,n-2}||_{V_1}^\rho > 0\}
\end{aligned}$$

almost surely, and consequently (5.2) holds for $j = 1$.
Now assume (5.2) holds for a $j \in \{1, ..., n-2\}$ (and w.l.o.g. that $n > 2$). Firstly, using the induction hypothesis yields

$$\begin{aligned}
& \{-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^\rho > 0, \forall k = n-(j+1), ..., n\} \\
\subseteq\ & \{-\kappa \sum_{k=n-j}^{n} \beta_k + \sum_{k=n-j}^{n-1} ||\eta_k||_{V_1}^\rho + ||\mathbb{x}_{x,n-j-1}||_{V_1}^\rho > 0\} \cap \{-\kappa\beta_{n-j-1} + ||\mathbb{x}_{x,n-j-2}||_{V_1}^\rho > 0\}
\end{aligned}$$

almost surely. Appealing to Lemma 5.2.8.ii) once more, yields

$$\{-\kappa \sum_{k=n-j}^{n} \beta_k + \sum_{k=n-j}^{n-1} ||\eta_k||_{V_1}^\rho + ||\mathbb{x}_{x,n-j-1}||_{V_1}^\rho > 0\}$$

$$\subseteq \quad \{-\kappa \sum_{k=n-j}^{n} \beta_k + \sum_{k=n-j}^{n-1} ||\eta_k||_{V_1}^{\rho} + (-\kappa\beta_{n-j-1} + ||\mathbb{x}_{x,n-j-2}||_{V_1}^{\rho})_+ + ||\eta_{n-j-1}||_{V_1}^{\rho} > 0\}$$

almost surely. Finally, combining the former and the latter inclusion gives the claim. $\square$

Proposition 5.2.10. *Let $x : \Omega \to V$ be an independent initial leading to extinction. Then we have*

$$\mathbb{P}(e_x(i) < \infty, \ \forall i \in \mathbb{N}) = 1.$$

Proof. It obviously suffices to prove that $e_x(i) < \infty$ a.s. for all $i \in \mathbb{N}$. This will be proven inductively. Firstly, employing the σ-continuity of probability measures from above yields

$$\mathbb{P}(e_x(1) = \infty) = \lim_{n\to\infty} \mathbb{P}(T(\beta_k)\mathbb{x}_{x,k-1} \neq 0, \ \forall k = 1, ..., n).$$

Moreover, appealing to Lemma 5.2.8.i) gives $\mathbb{x}_{x,k-1} \in V_1$ for all $k \in \mathbb{N}$ a.s. Consequently, Assumption 5.2.1.ii) gives

$$\{T(\beta_k)\mathbb{x}_{x,k-1} \neq 0\} \subseteq \{(-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^{\rho})_+ > 0\} = \{-\kappa\beta_k + ||\mathbb{x}_{x,k-1}||_{V_1}^{\rho} > 0\}, \ \forall k \in \mathbb{N} \tag{5.4}$$

a.s. Using this, while having in mind Lemma 5.2.9 yields

$$\mathbb{P}(e_x(1) = \infty) \leq \lim_{n\to\infty} \mathbb{P}\left(-\kappa \sum_{k=1}^{n} \beta_k + \sum_{k=1}^{n-1} ||\eta_k||_{V_1}^{\rho} + ||x||_{V_1}^{\rho} > 0\right). \tag{5.5}$$

Now note that $||x||_{V_1}^{\rho}, \beta_k, \ ||\eta_k||_{V_1}^{\rho} \in L^2(\Omega)$. Consequently, we can introduce

$$\nu_n := \mathbb{E}\left(-\kappa \sum_{k=1}^{n} \beta_k + \sum_{k=1}^{n-1} ||\eta_k||_{V_1}^{\rho} + ||x||_{V_1}^{\rho}\right) = n\left(-\kappa\mathbb{E}(\beta_1) + \mathbb{E}||\eta_1||_{V_1}^{\rho}\right) - \mathbb{E}||\eta_1||_{V_1}^{\rho} + \mathbb{E}||x||_{V_1}^{\rho}.$$

Moreover, appealing to Assumption 5.2.3.iii) yields $\nu_n < 0$ for all n sufficiently large. Consequently, by invoking (5.5) and employing Tschebyscheff's inequality, we get

$$\begin{aligned} \mathbb{P}(e_x(1) = \infty) &\leq \lim_{n\to\infty} \frac{1}{\nu_n^2} \mathrm{Var}\left(-\kappa \sum_{k=1}^{n} \beta_k + \sum_{k=1}^{n-1} ||\eta_k||_{V_1}^{\rho} + ||x||_{V_1}^{\rho}\right) \\ &= \lim_{n\to\infty} \frac{1}{\nu_n^2} \left(\kappa^2 \mathrm{Var}(\beta_1)n + (n-1)\mathrm{Var}(||\eta_1||_{V_1}^{\rho}) + (\mathrm{Var}||x||_{V_1}^{\rho})\right) \\ &= 0, \end{aligned}$$

which proves $\mathbb{P}(e_x(1) < \infty) = 1$.
Now assume $e_x(i) < \infty$ a.s. for a given $i \in \mathbb{N}$. Then there is a set $M_i \subseteq \mathbb{N}$, such that $\mathbb{P}(e_x(i) \in M_i) = 1$

and $\mathbb{P}(e_x(i)=m)>0$ for all $m\in M_i$. This implies

$$\mathbb{P}(e_x(i+1)=\infty)=\sum_{m\in M_i}\mathbb{P}(e_x(i+1)=\infty,\ e_x(i)=m).$$

Consequently, it suffices to prove that $\mathbb{P}(e_x(i+1)=\infty,\ e_x(i)=m)=0$ for all $m\in M_i$. So let $m\in M_i$ be given. Then we have

$$\mathbb{P}(e_x(i+1)=\infty,\ e_x(i)=m)=\mathbb{P}(T(\beta_k)\mathbb{x}_{x,k-1}\neq 0,\ \forall k>m,\ e_x(i)=m).$$

Consequently, employing the σ-continuity of probability measures, (5.4) and Lemma 5.2.9 gives

$$\mathbb{P}(e_x(i+1)=\infty,\ e_x(i)=m)\leq \lim_{n\to\infty}\mathbb{P}\left(-\kappa\sum_{k=m+1}^{n}\beta_k+\sum_{k=m+1}^{n-1}||\eta_k||_{V_1}^{\rho}+||\mathbb{x}_{x,m}||_{V_1}^{\rho}>0,\ e_x(i)=m\right)$$

Moreover, it is plain that $\mathbb{x}_{x,m}=\eta_m$ on $\{e_x(i)=m\}$ which implies

$$\begin{aligned}\mathbb{P}(e_x(i+1)=\infty,\ e_x(i)=m) &\leq \lim_{n\to\infty}\mathbb{P}\left(-\kappa\sum_{k=m+1}^{n}\beta_k+\sum_{k=m+1}^{n-1}||\eta_k||_{V_1}^{\rho}+||\eta_m||_{V_1}^{\rho}>0\right)\\ &= \lim_{n\to\infty}\mathbb{P}\left(-\kappa\sum_{k=1}^{n-m}\beta_k+\sum_{k=1}^{n-m}||\eta_k||_{V_1}^{\rho}>0\right),\end{aligned}$$

where the last equality follows from the fact that the η_k's as well as the β_k's are i.i.d. and independent of each other. Analogously to the induction beginning, one now easily verifies by the aid of Tschebyscheff's inequality that the last limit converges to zero and the claim follows. □

Remark 5.2.11. *The following observations will be useful in the sequel, and follow directly from the definition of $SL_{V_2}(V)$.*

i) If $(\Xi,(W,||\cdot||_W))\in SL_{V_2}(V)$, then $(||\Xi||_W,\mathbb{R})\in SL_{V_2}(V)$.

ii) If $(\Xi,(W,||\cdot||_W))\in SL_{V_2}(V)$ and $w\in W$, then $(\Xi_w,(W,||\cdot||_W))\in SL_{V_2}(V)$, where we set $\Xi_w(v):=\Xi(v)+w$ for all $v\in V$.

Lemma 5.2.12. *Let $(\Xi,(W,||\cdot||_W))\in SL_{V_2}(V)$ and let $x:\Omega\to V$ be an independent initial leading to extinction. Then we have*

i) $\mathbb{P}(\mathbb{X}_x(t)\in V_i,\ \forall t\geq 0)=1$, where $i\in\{1,2\}$.

ii) The mapping defined by $[0,\infty)\times\Omega\ni(t,\omega)\mapsto\Xi(\mathbb{X}_x(t,\omega))$ is $\mathfrak{B}([0,\infty))\otimes\mathcal{F}$-$\mathfrak{B}(W)$-measurable.

iii) $\mathbb{P}\left(\int_0^t||\Xi(\mathbb{X}_x(\tau))||_W d\tau<\infty,\ \forall t\geq 0\right)=1$.

Consequently, the Bochner integral $\int_0^t \Xi(\mathbb{X}_x(\tau))d\tau$ *is (up-to a* $\mathbb{P}$*-null-set which is independent of* t*) well-defined, for all* $t \geq 0$*, and the stochastic process defined by* $[0,\infty) \times \Omega \ni (t,w) \mapsto \int_0^t \Xi(\mathbb{X}_x(\tau,\omega))d\tau$ *is* $\mathcal{F} \otimes \mathfrak{B}([0,\infty))$*-*$\mathfrak{B}(W)$*-measurable.*

Proof. The first assertions follows directly from Lemma 2.2.6.vii); and the second follows from the measurability of Ξ and Lemma 2.2.6.v).
Proof of iii). Let $t > 0$ and $\omega \in \Omega \setminus M$, where M is a $\mathbb{P}$-null-set such that $\lim_{m\to\infty} \alpha_m(\omega) = \infty$, and such that $\mathbb{X}_x(t,\omega) \in V_i$ for $i = 1,2$ and all $t \geq 0$. Moreover, introduce $m \in \mathbb{N}$ such that $t < \alpha_m(\omega)$. Then there are constants $c_1,\ c_2 \in [0,\infty)$ such that

$$\begin{aligned}
\int_0^t ||\Xi(\mathbb{X}_x(\tau,\omega))||_W d\tau &\leq \sum_{k=0}^{m-1} \int_{\alpha_k(\omega)}^{\alpha_{k+1}(\omega)} ||\Xi(T(\tau-\alpha_k(\omega))\mathbb{x}_{x,k}(\omega))||_W d\tau \\
&\leq \sum_{k=0}^{m-1} \int_{\alpha_k(\omega)}^{\alpha_{k+1}(\omega)} c_1||T(\tau-\alpha_k(\omega))\mathbb{x}_{x,k}(\omega)||_{V_2} d\tau + c_2\beta_{k+1}(\omega) \\
&\leq \sum_{k=0}^{m-1} \beta_{k+1}(\omega)(c_1||\mathbb{x}_{x,k}(\omega)||_{V_2} + c_2),
\end{aligned}$$

where the last inequality follows from Assumption 5.2.1.iii). Consequently, iii) is proven, since the $\mathbb{P}$-null-set M is indeed independent of $t \geq 0$.
Moreover, it follows from ii) that $[0,\infty) \ni t \mapsto \Xi(\mathbb{X}_x(t,\omega))$ is $\mathfrak{B}([0,\infty))$-$\mathfrak{B}(W)$-measurable for all $\omega \in \Omega$. This and (the proof of) iii) yield that the Bochner integral $\int_0^t \Xi(\mathbb{X}_x(\tau,\omega))d\tau$ exists for all $\omega \in \Omega \setminus M$ and $t \geq 0$.
Finally, [31, Lemma 2.2.4] yields that $[0,\infty) \times \Omega \ni (t,\omega) \mapsto \int_0^t \Xi(\mathbb{X}_x(\tau,\omega))d\tau := I(t,\omega)$ is (almost surely) continuous and that each $I(t)$ is $\mathcal{F}$-$\mathfrak{B}(W)$-measurable; which implies that I is $\mathcal{F} \otimes \mathfrak{B}([0,\infty))$-$\mathfrak{B}(W)$-measurable, by [31, Proposition 2.2.3]. (The results in [31] are formulated for filtered probability spaces, chose the filtration which is constantly $\mathcal{F}$ while applying [31, Lemma 2.2.4, Proposition 2.2.3].) $\square$

The preceding lemma yields in particular that $\Omega \ni \omega \mapsto \int_{a_1(\omega)}^{a_2(\omega)} \Xi(\mathbb{X}_x(\omega,\tau))d\tau$ is well-defined and $\mathcal{F}$-$\mathfrak{B}(W)$-measurable, whenever $a_i : \Omega \to [0,\infty)$ are $\mathcal{F}$-$\mathfrak{B}([0,\infty))$-measurable, $x : \Omega \to V$ is an independent initial leading to extinction and $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$.
Our next goal is to establish that the sequence defined by $\left(\int_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)_{n\in\mathbb{N}}$ is i.i.d.

Remark 5.2.13. *Whenever* $x : \Omega \to V$ *is an independent initial leading to extinction, then* $(\mathcal{F}^x_{e_x(n)})_{n\in\mathbb{N}}$ *denotes the stopped filtration, defined by*

$$\mathcal{F}^x_{e_x(n)} := \{A \in \mathcal{F} :\ A \cap \{e_x(n) = j\} \in \mathcal{F}^x_j,\ \forall j \in \mathbb{N}\},$$

for all $n \in \mathbb{N}$.
Note that $(\mathcal{F}^x_j)_{j\in\mathbb{N}}$ is trivially a filtration. Moreover, invoking Lemma 5.2.7.iii) yields that each $e_x(n)$ is a stopping time w.r.t. $(\mathcal{F}^x_j)_{j\in\mathbb{N}}$ and that $e_x(n) \leq e_x(n+1)$ for all $n \in \mathbb{N}$. Consequently, it is standard that each $\mathcal{F}^x_{e_x(n)}$ is indeed a σ-algebra and that $\mathcal{F}^x_{e_x(n)} \subseteq \mathcal{F}^x_{e_x(n+1)}$ for all $n \in \mathbb{N}$. In addition, it is plain that $(\mathcal{F}^x_{e_x(n)})_{n\in\mathbb{N}}$ inherits the completeness of $(\mathcal{F}^x_j)_{j\in\mathbb{N}}$.

Remark 5.2.14. *In all that follows $\overline{x} \in \mathcal{M}(\Omega; V)$, denotes a mapping fulfilling*

i) $\overline{x} = \eta_1$ in distribution and

ii) $\overline{x}$ is jointly independent of $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$.

Note that this implies $\overline{x} \in V_i$ a.s. for $i = 1, 2$. Moreover, as $0 < 2\rho < \rho(11 + \hat{\varepsilon})$, we also have $\mathbb{E}||\overline{x}||^{2\rho}_{V_1} < \infty$, which gives that $\overline{x}$ is an independent initial leading to extinction.

Lemma 5.2.15. *Let $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$ and $x : \Omega \to V$ be an independent initial leading to extinction. Then we have*

$$\mathbb{E}\left(f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \right) \middle| \mathcal{F}^x_{e_x(n)} \right) = \mathbb{E}f\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau \right),$$

for all $n \in \mathbb{N}$ and $f : W \to \mathbb{R}$ which are $\mathfrak{B}(W)$-$\mathfrak{B}(\mathbb{R})$-measurable and bounded.

Proof. Let $A \in \mathcal{F}^x_{e_x(n)}$ be given and introduce $A_i := \{\omega \in A : \ e_x(n)(\omega) = i\}$ for all $i \in \mathbb{N}$, with $i \geq n$. At first, it will be shown that

$$\mathbb{E}\mathbb{1}_{A_i}\hat{f}_j(\mathbb{x}_{x,i}, ..., \mathbb{x}_{x,i+j-1}, \beta_{i+1}, ..., \beta_{i+j}) = \mathbb{P}(A_i)\mathbb{E}\hat{f}_j(\mathbb{x}_{\overline{x},0}, ..., \mathbb{x}_{\overline{x},j-1}, \beta_1, ..., \beta_j), \tag{5.6}$$

for all $i \in \mathbb{N}$, with $i \geq n$, all $j \in \mathbb{N}$ and $\hat{f}_j : V^j \times [0,\infty)^j \to \mathbb{R}$ which are bounded and $\mathfrak{B}(V^j)\otimes\mathfrak{B}([0,\infty)^j)$-$\mathfrak{B}(\mathbb{R})$-measurable.
Now let us prove (5.6) inductively w.r.t. $j \in \mathbb{N}$.
Let $j = 1$, $i \geq n$ and $\hat{f}_1 : V \times [0,\infty) \to \mathbb{R}$ be bounded and measurable. Note that $T(\beta_i)\mathbb{x}_{x,i-1} = 0$ on A_i. Consequently, we get $\mathbb{E}\mathbb{1}_{A_i}\hat{f}_1(\mathbb{x}_{x,i}, \beta_{i+1}) = \mathbb{E}\mathbb{1}_{A_i}\hat{f}_1(\eta_i, \beta_{i+1})$. Moreover, appealing to Remark 5.2.13 yields that $A_i \in \mathcal{F}^x_i = \sigma_0(x, \beta_1, ..., \beta_i, \eta_1, ..., \eta_{i-1})$. Hence, A_i is independent of $\hat{f}_1(\eta_i, \beta_{i+1})$, which gives

$$\mathbb{E}\mathbb{1}_{A_i}\hat{f}_1(\mathbb{x}_{x,i}, \beta_{i+1}) = \mathbb{P}(A_i)\mathbb{E}\hat{f}_1(\eta_i, \beta_{i+1}) = \mathbb{P}(A_i)\mathbb{E}\hat{f}_1(\mathbb{x}_{\overline{x},0}, \beta_1),$$

where the last inequality follows from the fact that $(\mathbb{x}_{\overline{x},0}, \beta_1) = (\overline{x}, \beta_1)$, which is in distribution equal to (η_i, β_{i+1}). Hence, (5.6) holds for $j = 1$.
Now assume that it holds for an $j \in \mathbb{N}$, let $i \geq n$ and $\hat{f}_{j+1} : V^{j+1} \times [0,\infty)^{j+1} \to \mathbb{R}$ be bounded and $\mathfrak{B}(V^{j+1}) \otimes \mathfrak{B}([0,\infty)^{j+1})$-$\mathfrak{B}(\mathbb{R})$-measurable. Moreover, for any $\tilde{\beta} \in [0,\infty)$, $\tilde{\eta} \in V$, introduce $\hat{f}_{\tilde{\beta},\tilde{\eta}} : V^j \times [0,\infty)^j \to \mathbb{R}$, by

$$\hat{f}_{\tilde{\beta},\tilde{\eta}}(y_0, ..., y_{j-1}, b_1, ..., b_j) := \hat{f}_{j+1}(y_0, ..., y_{j-1}, T(b_j)y_{j-1} + \tilde{\eta}, b_1, ..., b_j, \tilde{\beta}),$$

for all $y_0,...,y_{j-1},\tilde{\eta}\in V$ and $b_1,...,b_j,\tilde{\beta}\in[0,\infty)$. Then $\hat{f}_{\tilde{\beta},\tilde{\eta}}$ inherits the boundedness of $\hat{f}_{j+1}$. Moreover, invoking Remark 2.2.5, gives that $\hat{f}_{\tilde{\beta},\tilde{\eta}}$ is $\mathfrak{B}(V^j)\otimes\mathfrak{B}([0,\infty)^j)$-$\mathfrak{B}(\mathbb{R})$-measurable, as it is the composition of measurable functions, for all $\tilde{\beta}\in[0,\infty)$ and $\tilde{\eta}\in V$. Consequently, the induction hypothesis yields

$$\mathbb{E}\mathbb{1}_{A_i}\hat{f}_{\tilde{\beta},\tilde{\eta}}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},\beta_{i+1},...,\beta_{i+j})d\mathbb{P}=\mathbb{P}(A_i)\mathbb{E}\hat{f}_{\tilde{\beta},\tilde{\eta}}(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j),$$

which gives

$$\begin{aligned}&\mathbb{E}\mathbb{1}_{A_i}\hat{f}_{j+1}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},T(\beta_{i+j})\mathbb{x}_{x,i+j-1}+\tilde{\eta},\beta_{i+1},...,\beta_{i+j},\tilde{\beta})\\ =\;&\mathbb{P}(A_i)\mathbb{E}\hat{f}_{j+1}(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},T(\beta_j)\mathbb{x}_{\overline{x},j-1}+\tilde{\eta},\beta_1,...,\beta_j,\tilde{\beta}),\end{aligned}$$

for all $i\geq n$, $\tilde{\beta}\in[0,\infty)$ and $\tilde{\eta}\in V$.
Moreover, Lemma 5.2.7 yields, that $(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},\beta_{i+1},...,\beta_{i+j})$ is $\mathcal{F}^x_{i+j}$-$\mathfrak{B}(V^j)\otimes\mathfrak{B}([0,\infty)^j)$-measurable and, a fortiori, that $\mathbb{1}_{A_i}$ is $\mathcal{F}^x_{i+j}$-$\mathfrak{B}(\mathbb{R})$-measurable. Consequently, as $(\beta_{i+j+1},\eta_{i+j})$ is independent of $\mathcal{F}^x_{i+j}$ and as $(\beta_{i+j+1},\eta_{i+j})=(\beta_{j+1},\eta_j)$ in distribution, we get

$$\begin{aligned}&\mathbb{E}\mathbb{1}_{A_i}\hat{f}_{j+1}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j},\beta_{i+1},...,\beta_{i+j+1})\\ =\;&\mathbb{E}(\mathbb{1}_{A_i}\hat{f}_{j+1}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},T(\beta_{i+j})\mathbb{x}_{x,i+j-1}+\eta_{i+j},\beta_{i+1},...,\beta_{i+j},\beta_{i+j+1}))\\ =\;&\int\limits_{[0,\infty)\times V}\mathbb{E}(\mathbb{1}_{A_i}\hat{f}_{j+1}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},T(\beta_{i+j})\mathbb{x}_{x,i+j-1}+\tilde{\eta},\beta_{i+1},...,\beta_{i+j},\tilde{\beta}))d\mathbb{P}_{(\beta_{i+j+1},\eta_{i+j})}(\tilde{\beta},\tilde{\eta})\\ =\;&\int\limits_{[0,\infty)\times V}\mathbb{P}(A_i)\mathbb{E}\hat{f}_{j+1}(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},T(\beta_j)\mathbb{x}_{\overline{x},j-1}+\tilde{\eta},\beta_1,...,\beta_j,\tilde{\beta})d\mathbb{P}_{(\beta_{i+j+1},\eta_{i+j})}(\tilde{\beta},\tilde{\eta})\\ =\;&\int\limits_{[0,\infty)\times V}\mathbb{P}(A_i)\mathbb{E}\hat{f}_{j+1}(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},T(\beta_j)\mathbb{x}_{\overline{x},j-1}+\tilde{\eta},\beta_1,...,\beta_j,\tilde{\beta})d\mathbb{P}_{(\beta_{j+1},\eta_j)}(\tilde{\beta},\tilde{\eta}).\end{aligned}$$

Now, appealing to Lemma 5.2.7 yields, that $(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)$ is $\mathcal{F}^{\overline{x}}_j$-$\mathfrak{B}(V^j)\otimes[0,\infty)^j$-measurable. (Note that this is indeed possible, since $\overline{x}$ is also an independent initial leading to extinction, see Remark 5.2.14.) Moreover, it is plain that (β_{j+1},η_j) is independent of $\mathcal{F}^{\overline{x}}_j$. Consequently, we get

$$\mathbb{E}\mathbb{1}_{A_i}\hat{f}_{j+1}(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j},\beta_{i+1},...,\beta_{i+j+1})=\mathbb{P}(A_i)\mathbb{E}\hat{f}_{j+1}(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\mathbb{x}_{\overline{x},j},\beta_1,...,\beta_j,\beta_{j+1}),$$

which gives (5.6).
Now the actual claim is proven by the aid of (5.6). Firstly, appealing to Lemma 5.2.7.ii) and Proposition 5.2.10 yields

$$\mathbb{E}\left(\mathbb{1}_A f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}}\Xi(\mathbb{X}_x(\tau))d\tau\right)\right)=\sum_{i=n}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}\left(\mathbb{1}_{A_i}\mathbb{1}_{\{e_x(n+1)=i+j\}}f\left(\int\limits_{\alpha_i}^{\alpha_{i+j}}\Xi(\mathbb{X}_x(\tau))d\tau\right)\right).$$

In addition, we have

$$\int_{\alpha_i}^{\alpha_{i+j}} \Xi(\mathbb{X}_x(\tau))d\tau = \sum_{k=i}^{i+j-1} \int_0^{\beta_{k+1}} \Xi(T(\tau)\mathbb{x}_{x,k})d\tau = \sum_{k=0}^{j-1} \int_0^{\beta_{k+i+1}} \Xi(T(\tau)\mathbb{x}_{x,k+i})d\tau$$

Combining the former and the latter equality implies

$$\mathbb{E}\left(\mathbb{1}_A f\left(\int_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\right) = \sum_{i=n}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{A_i}\mathbb{1}_{\{e_x(n+1)=i+j\}} f\left(\sum_{k=0}^{j-1}\int_0^{\beta_{k+i+1}} \Xi(T(\tau)\mathbb{x}_{x,k+i})d\tau\right)\right).$$

For all $j \in \mathbb{N}$, introduce $\hat{h}_j : V^j \times [0,\infty)^j \times \mathbb{R}$, by

$$\hat{h}_j(y_0, ..., y_{j-1}, b_1, ..., b_j) := f\left(\sum_{k=0}^{j-1}\int_0^{b_{k+1}} \Xi(T(\tau)y_k)\mathbb{1}_{V_2}(y_k)d\tau\right).$$

Invoking Remark 2.2.5 and Remark 5.2.2, gives that $[0,\infty)\times V \ni (\tau, y) \mapsto \Xi(T(\tau)y)\mathbb{1}_{V_2}(y)$ is $\mathfrak{B}([0,\infty))\otimes V$-$\mathfrak{B}(W)$-measurable. Moreover, working as in the proof of Lemma 5.2.12 yields that $\Xi(T(\cdot)y)\mathbb{1}_{V_2}(y) \in L^1([0,t];W)$ for all $t > 0$ and $y \in V$. Consequently, [31, Proposition 2.1.3] yields that $(y,t) \mapsto \int_0^t \Xi(T(\tau)y)\mathbb{1}_{V_2}(y)d\tau$ is, for each y, as mapping in t continuous, and by [31, Proposition 2.1.4] it is for each $t \in [0,\infty)$, as a mapping in y, $\mathfrak{B}(V)$-$\mathfrak{B}(W)$-measurable. Consequently, this mapping is $\mathfrak{B}(V)\otimes\mathfrak{B}([0,\infty))$-$\mathfrak{B}(W)$-measurable, see [1, Lemma 4.51].
Using these observations, it is plain to deduce that $\hat{h}_j$ is $\mathfrak{B}(V^j)$-$\mathfrak{B}([0,\infty)^j)$-$\mathfrak{B}(\mathbb{R})$-measurable for all $j \in \mathbb{N}$. Moreover, each $\hat{h}_j$ is obviously bounded.
For all $j \in \mathbb{N}$, introduce $\hat{g}_j : V^j \times [0,\infty)^j \times \mathbb{R}$, by

$$\hat{g}_j(y_0, ..., y_{j-1}, b_1, ..., b_j) := \mathbb{1}\{T(b_k)y_{k-1} \neq 0, \forall k = 1, ..., j-1,\ T(b_j)y_{j-1} = 0\},\ \forall j \in \mathbb{N}\setminus\{1\}$$

and $\hat{g}_1(y_0, b_1) := \mathbb{1}\{T(b_1)y_0 = 0\}$. Then $\hat{g}_j$ is obviously bounded, and by the aid of Remark 2.2.5 also $\mathfrak{B}(V^j)\otimes\mathfrak{B}([0,\infty)^j)$-$\mathfrak{B}(\mathbb{R})$-measurable.
Moreover, appealing to Lemma 5.2.8.i) yields

$$\hat{h}_j(\mathbb{x}_{x,i}, ..., \mathbb{x}_{x,i+j-1}, \beta_{i+1}, ..., \beta_{i+j}) = f\left(\sum_{k=0}^{j-1}\int_0^{\beta_{i+k+1}} \Xi(T(\tau)\mathbb{x}_{x,i+k})d\tau\right),\ \forall i \geq n,\ j \in \mathbb{N}$$

almost surely. In addition, for all $\omega \in A_i$, we have

$$\hat{g}_j(\mathbb{x}_{x,i}, ..., \mathbb{x}_{x,i+j-1}, \beta_{i+1}, ..., \beta_{i+j})(\omega) = \mathbb{1}_{\{e_x(n+1)=i+j\}}(\omega),\ \forall i \geq n,\ j \in \mathbb{N}.$$

Consequently, putting it all together yields

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_A f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\right)\\
= &\sum_{i=n}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}\left(\mathbb{1}_{A_i}\hat{g}_j(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},\beta_{i+1},...,\beta_{i+j})\hat{h}_j(\mathbb{x}_{x,i},...,\mathbb{x}_{x,i+j-1},\beta_{i+1},...,\beta_{i+j})\right)\\
= &\sum_{i=n}^{\infty}\sum_{j=1}^{\infty}P(A_i)\mathbb{E}\left(\hat{g}_j(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)\hat{h}_j(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)\right)\\
= &P(A)\sum_{j=1}^{\infty}\mathbb{E}\left(\hat{g}_j(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)\hat{h}_j(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)\right).
\end{aligned}$$

In addition, it is straightforward that

$$\hat{g}_j(\mathbb{x}_{\overline{x},0},...,\mathbb{x}_{\overline{x},j-1},\beta_1,...,\beta_j)(\omega) = \mathbb{1}_{\{e_{\overline{x}}(1)=j\}}(\omega).$$

Using this, while having in mind Lemma 5.2.8.i), gives

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_A f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\right) &= P(A)\sum_{j=1}^{\infty}\mathbb{E}\left(\mathbb{1}_{\{e_{\overline{x}}(1)=j\}} f\left(\sum_{k=0}^{j-1}\int\limits_0^{\beta_{k+1}}\Xi(T(\tau)\mathbb{x}_{\overline{x},k})d\tau\right)\right)\\
&= P(A)\sum_{j=1}^{\infty}\mathbb{E}\left(\mathbb{1}_{\{e_{\overline{x}}(1)=j\}} f\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}\Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right)\right)
\end{aligned}$$

Finally, as $e_{\overline{x}}(1) \in \mathbb{N}$ a.s. and as $A \in \mathcal{F}^x_{e_x(n)}$ was arbitrary, we obtain

$$\mathbb{E}\left(\mathbb{1}_A f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\right) = P(A)\mathbb{E}\left(f\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}\Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right)\right),$$

for all $A \in \mathcal{F}^x_{e_x(n)}$, which implies the claim, by the very definition of the conditional expectation. □

Lemma 5.2.16. *Let* $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$*,* $n \in \mathbb{N}\setminus\{1\}$ *and* $x : \Omega \to V$ *an independent initial leading to extinction. Then the mapping defined by*

$$\Omega \ni \omega \mapsto \int\limits_{\alpha_{e_x(n-1)(\omega)}}^{\alpha_{e_x(n)(\omega)}} \Xi(\mathbb{X}_x(\tau,\omega))d\tau,$$

is $\mathcal{F}^x_{e_x(n)}$*-*$\mathfrak{B}(W)$*-measurable.*

Proof. As $(\mathcal{F}^x_{e_x(m)})_{m\in\mathbb{N}}$ is a filtration, it suffices to prove that $\int\limits_0^{\alpha_{e_x(n)}} \Xi(\mathbb{X}_x(\tau))d\tau$ is $\mathcal{F}^x_{e_x(n)}$-$\mathfrak{B}(W)$-measurable, for all $n \in \mathbb{N}$. To this end, introduce $j \in \mathbb{N}$ as well as $B \in \mathfrak{B}(W)$ and observe that

$$\left\{ \int\limits_0^{\alpha_{e_x(n)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B \right\} \cap \{e_x(n) = j\} = \left\{ \sum_{k=0}^{j-1} \int\limits_0^{\beta_{k+1}} \Xi(T(\tau)\mathbb{x}_{x,k})d\tau \in B \right\} \cap \{e_x(n) = j\}. \tag{5.7}$$

As demonstrated in the proof of Lemma 5.2.15, $(t, v) \mapsto \int\limits_0^t \Xi(T(\tau)v)\mathbb{1}_{V_2}(v)d\tau$ is $\mathfrak{B}([0,\infty)) \otimes \mathfrak{B}(V)$-$\mathfrak{B}(W)$-measurable. Consequently, since $\mathbb{x}_{x,k}$ and β_{k+1} are $\mathcal{F}^x_{k+1}$-$\mathfrak{B}(V)$-measurable and $\mathcal{F}^x_{k+1}$-$\mathfrak{B}([0,\infty))$-measurable, resp., for all $k = 0, ..., j-1$, we get that

$$\sum_{k=0}^{j-1} \int\limits_0^{\beta_{k+1}} \Xi(T(\tau)\mathbb{x}_{x,k})d\tau = \sum_{k=0}^{j-1} \int\limits_0^{\beta_{k+1}} \Xi(T(\tau)\mathbb{x}_{x,k})\mathbb{1}_{V_2}(\mathbb{x}_{x,k})d\tau$$

is $\mathcal{F}^x_j$-$\mathfrak{B}(W)$-measurable, where the equality holds almost surely. This gives, while having in mind (5.7) as well as Lemma 5.2.7.iii) that

$$\left\{ \int\limits_0^{\alpha_{e_x(n)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B \right\} \cap \{e_x(n) = j\} \in \mathcal{F}^x_j$$

and the claim follows. □

Proposition 5.2.17. *Let* $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$ *and let* $x : \Omega \to V$ *be an independent initial leading to extinction. Then the sequence* $\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \right)_{n\in\mathbb{N}}$ *is i.i.d., with*

$$\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau = \int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau \tag{5.8}$$

in distribution, for all $n \in \mathbb{N}$.

Proof. Let $B \in \mathfrak{B}(W)$ be given, and set $f := \mathbb{1}_B$, where $f : W \to \mathbb{R}$. Then f is obviously bounded and $\mathfrak{B}(W)$-$\mathfrak{B}(\mathbb{R})$-measurable. Consequently, appealing to Lemma 5.2.15 yields

$$\mathbb{P}\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B \right) = \mathbb{E}f\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \right) = \mathbb{E}f\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau \right),$$

which implies (5.8).

Consequently, it remains to show that

$$\mathbb{P}\left(\int\limits_{\alpha_{e_x(1)}}^{\alpha_{e_x(2)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_1, ..., \int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_n\right) = \prod_{k=1}^{n} \mathbb{P}\left(\int\limits_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_k\right) \quad (5.9)$$

for all $B_1, ..., B_n \in \mathfrak{B}(W)$ and $n \in \mathbb{N}$.
(5.9) is trivial if $n = 1$. So assume it holds for $n - 1 \in \mathbb{N}$ and let us prove it for n. To this end, introduce $B_1, ..., B_n \in \mathfrak{B}(W)$ and $f_k := \mathbb{1}_{B_k}$. Then employing Lemma 5.2.15, Lemma 5.2.16, (5.9) and (5.8) yields

$$\begin{aligned}
&\mathbb{P}\left(\int\limits_{\alpha_{e_x(1)}}^{\alpha_{e_x(2)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_1, ..., \int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_n\right) \\
= \;& \mathbb{E}\left(\prod_{k=1}^{n-1} f_k\left(\int\limits_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\mathbb{E}\left(f_n\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau\right)\Big|\mathcal{F}^x_{e_x(n)}\right)\right) \\
= \;& \prod_{k=1}^{n} \mathbb{P}\left(\int\limits_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in B_k\right)
\end{aligned}$$

and the claim follows. □

Lemma 5.2.18. *Let* $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$ *and let* $x : \Omega \to V$ *be an independent initial leading to extinction. Then, the assertion*

$$\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \in L^2(\Omega; W)$$

is valid for all $n \in \mathbb{N}$.

Proof. The desired measurability follows a fortiori from Lemma 5.2.16. Moreover, employing Proposition 5.2.17 yields that it suffices to prove that

$$\mathbb{E}\left|\left|\int\limits_{0}^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 < \infty.$$

To this end, note that

$$\mathbb{E}\left|\left|\int\limits_{0}^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \leq \mathbb{E}\left(\int\limits_{0}^{\alpha_{e_{\overline{x}}(1)}} ||\Xi(\mathbb{X}_{\overline{x}}(\tau))||_W \, d\tau\right)^2 \leq \mathbb{E}\left(\sum_{k=0}^{e_{\overline{x}}(1)-1} \beta_{k+1}\left(c_1||\mathbb{x}_{\overline{x},k}||_{V_2} + c_2\right)\right)^2,$$

where the second inequality follows from Lemma 5.2.12.i), Assumption 5.2.1.iii) and Lemma 5.2.8.i).

Now introduce $\eta_0 := \overline{x}$, for notational conveniences. Moreover, by the aid of Assumption 5.2.1.iii) and Lemma 5.2.8.i), it is easy to verify inductively that

$$||\mathbb{x}_{\overline{x},k}||_{V_2} \leq \sum_{j=0}^{k} ||\eta_k||_{V_2}, \ \forall k \in \mathbb{N}_0. \tag{5.10}$$

Consequently, we get

$$\mathbb{E}\left|\left|\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \leq \mathbb{E}\left(\sum_{k=0}^{e_{\overline{x}}(1)-1} \beta_{k+1}(c_1\sum_{j=0}^{k}||\eta_k||_{V_2}+c_2)\right)^2.$$

Hence, we also have

$$\mathbb{E}\left|\left|\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \leq \sum_{m=1}^{\infty} \mathbb{E}\left(\left(\sum_{k=0}^{m-1} \beta_{k+1}(c_1\sum_{j=0}^{k}||\eta_k||_{V_2}+c_2)\right)^2 \mathbb{1}_{\{e_{\overline{x}}(1)=m\}}\right).$$

Consequently, appealing to Cauchy-Schwarz' inequality implies

$$\mathbb{E}\left|\left|\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \leq \sum_{m=1}^{\infty}\left(\mathbb{E}\left(\sum_{k=0}^{m-1} \beta_{k+1}(c_1\sum_{j=0}^{k}||\eta_k||_{V_2}+c_2)\right)^4\right)^{\frac{1}{2}} \mathbb{P}(e_{\overline{x}}(1)=m)^{\frac{1}{2}}. \tag{5.11}$$

Now upper bounds for each factor of each summand of the preceding series will be derived.
So let $m \in \mathbb{N}$ be arbitrary but fixed. Then the triangle inequality, the independence of $(\beta_k)_{k\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$ as well as the fact that each of these sequences is identically distributed, yields

$$\begin{aligned}
\left(\mathbb{E}\left(\sum_{k=0}^{m-1} \beta_{k+1}(c_1\sum_{j=0}^{k}||\eta_k||_{V_2}+c_2)\right)^4\right)^{\frac{1}{4}} &\leq \sum_{k=0}^{m-1}||\beta_{k+1}||_{L^4(\Omega)}\left(c_1\sum_{j=0}^{k}||\ ||\eta_k||_{V_2}||_{L^4(\Omega)}+c_2\right) \\
&= ||\beta_1||_{L^4(\Omega)}c_1||\ ||\eta_1||_{V_2}||_{L^4(\Omega)}\frac{m(m+1)}{2}+||\beta_1||_{L^4(\Omega)}c_2 m \\
&\leq m^2\left(||\beta_1||_{L^4(\Omega)}c_1||\ ||\eta_1||_{V_2}||_{L^4(\Omega)}+||\beta_1||_{L^4(\Omega)}c_2\right).
\end{aligned}$$

Note that $||\beta_1||_{L^4(\Omega)} < \infty$ and $||\ ||\eta_1||_{V_2}||_{L^4(\Omega)} < \infty$, by Assumption 5.2.3.ii).
Consequently, by introducing $C := \left(||\beta_1||_{L^4(\Omega)}c_1||\ ||\eta_1||_{V_2}||_{L^4(\Omega)}+||\beta_1||_{L^4(\Omega)}c_2\right)^2 < \infty$, we get

$$\left(\mathbb{E}\left(\sum_{k=0}^{m-1} \beta_{k+1}(c_1\sum_{j=0}^{k}||\eta_k||_{V_2}+c_2)\right)^4\right)^{\frac{1}{2}} \leq Cm^4, \ \forall m \in \mathbb{N}. \tag{5.12}$$

Now for all $m \in \mathbb{N} \setminus \{1\}$ we have

$$\mathbb{P}(e_{\overline{x}}(1) = m) \leq \mathbb{P}(T(\beta_k)\mathbb{x}_{\overline{x},k-1} \neq 0,\ \forall k = 1, ..., m-1).$$

Consequently, employing Assumption 5.2.1.ii), which is possible due to Lemma 5.2.8.i), yields

$$\mathbb{P}(e_{\overline{x}}(1) = m) \leq \mathbb{P}(-\kappa\beta_k + ||\mathbb{x}_{\overline{x},k-1}||_{V_1}^{\rho} > 0,\ \forall k = 1, ..., m-1)$$

Hence by appealing to Lemma 5.2.9 we get

$$\mathbb{P}(e_{\overline{x}}(1) = m) \leq \mathbb{P}\left(-\kappa \sum_{k=1}^{m-1} \beta_k + \sum_{k=1}^{m-2} ||\eta_k||_{V_1}^{\rho} + ||\eta_0||_{V_1}^{\rho} > 0\right) = \mathbb{P}\left(\sum_{k=1}^{m-1} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} > 0\right),$$

for all $m \in \mathbb{N} \setminus \{1\}$. Now let $\nu := \mathbb{E}(-\kappa\beta_1 + ||\eta_0||_{V_1}^{\rho})$, which is negative by Assumption 5.2.3.iii). Consequently, we have

$$\mathbb{P}(e_{\overline{x}}(1) = m) \leq \mathbb{P}\left(\left|\sum_{k=1}^{m-1} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} - \nu(m-1)\right| > |\nu|(m-1)\right) \tag{5.13}$$

for all $m \in \mathbb{N} \setminus \{1\}$. Hence, combining (5.11), (5.12) and (5.13) yields

$$\mathbb{E}\left|\left|\int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \leq C + \sum_{m=2}^{\infty} Cm^4 \mathbb{P}\left(\left|\sum_{k=1}^{m-1} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} - \nu(m-1)\right| > |\nu|(m-1)\right)^{\frac{1}{2}}$$

Moreover, it is plain that $m \leq 2(m-1)$ for all $m \geq 2$ and consequently $m^4 \leq 16(m-1)^4$, which yields by employing Cauchy Schwarz' inequality that

$$\begin{aligned}
&\mathbb{E}\left|\left|\int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right|\right|_W^2 \\
\leq\ & C + 16C \sum_{m=1}^{\infty} m^4 \mathbb{P}\left(\left|\sum_{k=1}^{m} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} - \nu m\right| > |\nu| m\right)^{\frac{1}{2}} \\
\leq\ & C + 16C \left(\sum_{m=1}^{\infty} m^{-1-\hat{\varepsilon}}\right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} m^{9+\hat{\varepsilon}} \mathbb{P}\left(\left|\sum_{k=1}^{m} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} - \nu m\right| > |\nu| m\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

It is common knowledge that the first series in the preceding expression is finite. Consequently, the claim follows if the second is finite as well. To this end, note that the sequence $(-\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho})_{k\in\mathbb{N}}$ is i.i.d. with mean ν. Consequently, [21, Theorem 1] yields

$$\sum_{m=1}^{\infty} m^{9+\hat{\varepsilon}} \mathbb{P}\left(\left|\sum_{k=1}^{m} -\kappa\beta_k + ||\eta_{k-1}||_{V_1}^{\rho} - \nu m\right| > |\nu| m\right) < \infty,$$

if (and only if) $-\kappa\beta_1 + ||\eta_0||_{V_1}^{\rho} \in L^{11+\hat{\varepsilon}}(\Omega)$, which is true by Assumption 5.2.3.ii). $\square$

Note that $(\varphi, \mathbb{R}) \in SL_{V_2}(V)$, where $\varphi : V \to \mathbb{R}$ is the function which is constantly one. This plain fact, together with Proposition 5.2.17 and Lemma 5.2.18 yields the following quite useful corollary.

Corollary 5.2.19. *Let $x : \Omega \to V$ be an independent initial leading to extinction. Then the sequence $(\alpha_{e_x(n+1)} - \alpha_{e_x(n)})_{n\in\mathbb{N}}$ is square integrable and i.i.d with $\alpha_{e_x(n+1)} - \alpha_{e_x(n)} = \alpha_{e_{\overline{x}}(1)}$ in distribution.*

Proving our SLLN requires the following very simple lemma, which might be available somewhere in the literature:

Lemma 5.2.20. *Let $(U, ||\cdot||_U)$ be a separable Banach space. Moreover, let $(Y_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; U)$ be such that there is a $Y \in \mathcal{M}(\Omega; U)$, with $\lim\limits_{m\to\infty} Y_m = Y$ almost surely. Finally, let $(N_t)_{t\geq 0}$, with $N_t : \Omega \to \mathbb{N}$, be such that each N_t is $\mathcal{F}$-$2^{\mathbb{N}}$-measurable and $\lim\limits_{t\to\infty} N_t = \infty$ almost surely. Then the convergence $\lim\limits_{t\to\infty} Y_{N_t} = Y$ takes place with probability one.*

Proof. Let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set such that $\lim\limits_{m\to\infty} Y_m(\omega) = Y(\omega)$ and $\lim\limits_{t\to\infty} N_t(\omega) = \infty$ for all $\omega \in \Omega \setminus M$. Now fix one of these $\omega \in \Omega \setminus M$ and note that there is for each $\varepsilon > 0$ an $m_0 \in \mathbb{N}$ such that $||Y_m(\omega) - Y(\omega)||_U < \varepsilon$ for all $m \geq m_0$. In addition, we can find a $t_0 \in [0, \infty)$ such that $N_t(\omega) \geq m_0$ for all $t \in [t_0, \infty)$. Consequently, we get $||Y_{N_t(\omega)}(\omega) - Y(\omega)||_U < \varepsilon$ for all $t \geq t_0$, which yields the claim. $\square$

The results proven so far already enable us to prove the desired SLLN. But, to also prove our CLT, a version of Anscombe's CLT in type 2 Banach spaces and some clarifications regarding Gaussian random variables taken values in separable Banach spaces are needed:

Remark 5.2.21. *Let $(W, ||\cdot||_W)$ be a separable Banach space with dual space W'. Moreover, $\langle\cdot,\cdot\rangle_W$ denotes the duality between W and W'.*
$(W, ||\cdot||_W)$ is said to be of type 2, if: There is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $X_1, ..., X_n \in L^2(\Omega; W)$ which are centered and independent, we have

$$\mathbb{E}\left|\left|\sum_{k=1}^{n} X_k\right|\right|_W^2 \leq C \sum_{k=1}^{n} \mathbb{E}||X_k||_W^2.$$

The main feature of such Banach spaces is that these are precisely the Banach spaces where every centered, square integrable i.i.d. sequence still fulfills the CLT, see [26, Theorem 10.5].
Now let $Y \in \mathcal{M}(\Omega; W)$. Then Y is called Gaussian, if $\langle Y, \psi\rangle_W$ is Gaussian for all $\psi \in W'$. (Note that by this definition constant random variables are Gaussian as well.) In addition, for a (not necessarily Gaussian) random variable $Y \in L^2(\Omega; W)$, we call the mapping $Cov_W(Y) : W' \times W' \to \mathbb{R}$, where

$$Cov_W(Y)(\psi_1, \psi_2) := \mathbb{E}(\langle Y - \mathbb{E}Y, \psi_1\rangle_W \langle Y - \mathbb{E}Y, \psi_2\rangle_W), \ \forall \psi_1, \ \psi_2 \in W',$$

the covariance of Y. It is plain to verify that the right-hand-side expectation in the preceding equation indeed exists.

Moreover, if $Y \in \mathcal{M}(\Omega; W)$ is Gaussian, then particularly $Y \in L^2(\Omega; W)$, see [37, p. 5]. In addition, analogously to the real-valued case, the distribution of Y is still uniquely determined by $\mathbb{E}Y$ and $Cov_W(Y)$, see [37, p. 5].
In the sequel, it will be written $Y \sim N_W(\mu, Q)$ whenever $Y \in \mathcal{M}(\Omega; W)$ is Gaussian, with mean μ and covariance Q. Of course, if $W = \mathbb{R}$ this is abbreviated by $N(\mu, \sigma^2)$, where $\sigma^2 := Q(Id, Id)$ is the variance of Y.
Last but not least, let us remark, that as usually we say that $\lim_{m\to\infty} Y_m = Y$ in distribution, where $Y_m,\ Y \in \mathcal{M}(\Omega; W)$, if $\lim_{m\to\infty} \mathbb{E}f(Y_m) = \mathbb{E}f(Y)$, for all $f : W \to \mathbb{R}$ which are continuous and bounded.

The following theorem is a version of Anscombe's theorem in type 2 Banach spaces. This theorem might be available somewhere in the literature, but the present author was unable to find it. The proof works analogously to the proof of the real-valued version of Anscombe's theorem, since the standard CLT as well as Kolmogorov's inequality both hold if (and only if) the underlying Banach space is of type 2. Since it is just from the theorem itself far from obvious that its proof carries on to random variables taking values in type 2 Banach spaces, the proof will be given. For a treatment of the real-valued case, see [17, Theorem 3.2].
Afterwards, this section's main result will be formulated.

Theorem 5.2.22. *Let $(W, ||\cdot||_W)$ be a separable Banach space of type 2, introduce $(Y_m)_{m\in\mathbb{N}} \subseteq L^2(\Omega; W)$, $(N_n)_{n\in\mathbb{N}}$, where $N_n : \Omega \to \mathbb{N}$ is $\mathcal{F}$-$2^{\mathbb{N}}$-measurable and $(\theta_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$. Moreover, assume that*

i) $(Y_m)_{m\in\mathbb{N}}$ is i.i.d. and $\mathbb{E}Y_1 = 0$ and

ii) $\lim_{n\to\infty} \theta_n = \infty$ and $\lim_{n\to\infty} \frac{N_n}{\theta_n} = 1$ in probability.

Then the convergence

$$\lim_{n\to\infty} \frac{1}{\sqrt{\theta_n}} \sum_{k=1}^{N_n} Y_k = \lim_{n\to\infty} \frac{1}{\sqrt{N_n}} \sum_{k=1}^{N_n} Y_k = Z, \tag{5.14}$$

takes place in distribution, where $Z \sim N_W(0, Cov_W(Y_1))$.

Proof. Firstly, the claim is trivial if $Y_1 = 0$ a.s., so assume w.l.o.g. $Y_1 \neq 0$. Now, introduce $S_n := \sum_{k=1}^{n} Y_k$, $\hat{S}_n := \frac{1}{\sqrt{n}} S_n$ for all $n \in \mathbb{N}$ and let us start by proving the second equality in (5.14). Appealing to the CLT in type 2 Banach spaces, see [19, Corollary 3.3 and Remark 1.1], yields $\lim_{n\to\infty} \hat{S}_n = Z$ in distribution. Now set $\tilde{\theta}_n := \lceil \theta_n \rceil$ and note that clearly $\lim_{n\to\infty} \hat{S}_{\tilde{\theta}_n} = Z$ in distribution and $\lim_{n\to\infty} \frac{N_n}{\tilde{\theta}_n} = 1$ in probability. Moreover, as $\hat{S}_{N_n} = (\hat{S}_{\tilde{\theta}_n} + \frac{S_{N_n} - S_{\tilde{\theta}_n}}{\sqrt{\tilde{\theta}_n}})\sqrt{\frac{\tilde{\theta}_n}{N_n}}$ for all $n \in \mathbb{N}$, Slutsky's theorem[3] yields that the second equality in (5.14) follows, if

$$\lim_{n\to\infty} \frac{S_{N_n} - S_{\tilde{\theta}_n}}{\sqrt{\tilde{\theta}_n}} = 0, \tag{5.15}$$

[3] We were unable to find a direct reference for Slutsky's theorem in the Banach space setting. However, this is easily deduced from [10, Theorem 3.9] and the continuous mapping theorem.

in probability.
So let us prove (5.15). To this end, let $\varepsilon > 0$, $\delta \in (0,1)$, $r_n := \lceil \tilde{\theta}_n(1-\delta) \rceil$ and $R_n := \lfloor \tilde{\theta}_n(1+\delta) \rfloor$ for all $n \in \mathbb{N}$. And note that it is plain that

$$\mathbb{P}\left(||S_{N_n} - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n} \right) \leq \mathbb{P}\left(||S_{N_n} - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n},\ \left| \frac{N_n}{\tilde{\theta}_n} - 1 \right| \leq \delta \right) + \mathbb{P}\left(\left| \frac{N_n}{\tilde{\theta}_n} - 1 \right| > \delta \right)$$

for all $n \in \mathbb{N}$. Moreover, as $\left| \frac{N_n}{\tilde{\theta}_n} - 1 \right| \leq \delta$ if and only if $N_n \in [r_n, R_n]$, we get

$$\mathbb{P}\left(||S_{N_n} - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n},\ \left| \frac{N_n}{\tilde{\theta}_n} - 1 \right| \leq \delta \right) \leq \mathbb{P}\left(\max_{m=r_n,..,R_n} ||S_m - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n} \right),$$

for all $n \in \mathbb{N}$. In addition, note that

$$\begin{aligned} \max_{m=r_n,..,R_n} ||S_m - S_{\tilde{\theta}_n}||_W &\leq \max_{m=r_n,..,\tilde{\theta}_n - 1} \left|\left| \sum_{k=m+1}^{\tilde{\theta}_n} Y_k \right|\right|_W + \max_{m=\tilde{\theta}_n+1,..,R_n} \left|\left| \sum_{k=\tilde{\theta}_n+1}^{m} Y_k \right|\right|_W \\ &= \max_{m=1,..,\tilde{\theta}_n - r_n} \left|\left| \sum_{k=1}^{m} Y_{\tilde{\theta}_n+1-k} \right|\right|_W + \max_{m=1,..,R_n - \tilde{\theta}_n} \left|\left| \sum_{k=1}^{m} Y_{k+\tilde{\theta}_n} \right|\right|_W \end{aligned}$$

where we set $\max\limits_{m=a,..,b}(\cdot) := 0$, if $a > b$.
Using this, together with the well known inequality $P(X_1 + X_2 > t) \leq \mathbb{P}(2X_1 > t) + \mathbb{P}(2X_2 > t)$, for any $X_1, X_2 \in \mathcal{M}(\Omega; \mathbb{R})$, $t > 0$ and Kolmogorov's inequality in type 2 Banach spaces (see [19, Theorem 6.1]), yields that there is a constant $C > 0$ such that

$$\mathbb{P}\left(\max_{m=r_n,..,R_n} ||S_m - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n} \right) \leq \frac{4C\mathbb{E}||Y_1||_W^2}{\varepsilon^2 \tilde{\theta}_n}(\tilde{\theta}_n - r_n) + \frac{4C\mathbb{E}||Y_1||_W^2}{\varepsilon^2 \tilde{\theta}_n}(R_n - \tilde{\theta}_n),$$

for all $n \in \mathbb{N}$. Now let $\varepsilon' > 0$ be arbitrary but fixed and choose $0 < \delta < \min\left(\frac{\varepsilon^2 \varepsilon'}{8C\mathbb{E}||Y_1||_W^2}, 1 \right)$, then we get

$$\mathbb{P}\left(\max_{m=r_n,..,R_n} ||S_m - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n} \right) \leq \frac{4C\mathbb{E}||Y_1||_W^2}{\varepsilon^2 \tilde{\theta}_n}(R_n - r_n) \leq \frac{8C\mathbb{E}||Y_1||_W^2}{\varepsilon^2}\delta \leq \varepsilon'.$$

Conclusively, putting it all together yields $\limsup\limits_{n\to\infty} \mathbb{P}\left(||S_{N_n} - S_{\tilde{\theta}_n}||_W > \varepsilon \sqrt{\tilde{\theta}_n} \right) \leq \varepsilon'$, which implies (5.15), since $\varepsilon' > 0$ can be chosen arbitrarily small. Finally, the first inequality in (5.14) now follows from the second one and Slutsky's theorem. $\square$

Theorem 5.2.23. *Let $(\Xi, (W, ||\cdot||_W)) \in SL_{V_2}(V)$ and let $x : \Omega \to V$ be an independent initial leading to extinction. Moreover, introduce $\nu := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau \right)$. Then the convergence*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Xi(\mathbb{X}_x(\tau))d\tau = \nu, \tag{5.16}$$

takes place almost surely in $(W, ||\cdot||_W)$[4]*. Moreover, if* $(W, ||\cdot||_W)$ *is of type* 2*, then*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \Xi(\mathbb{X}_x(\tau))d\tau - t\nu \right) = Z, \tag{5.17}$$

in distribution, as elements of $(W, ||\cdot||_W)$[5]*, where* $Z \sim N_W(0, Q)$ *and the covariance is given by* $Q := Cov_W \left(\sqrt{\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}} \int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau)) - \nu d\tau \right)$.

Proof. Until explicitly stated otherwise, $(W, ||\cdot||_W)$ is not necessarily of type 2.
Firstly, note that both expectations occurring in the definition of ν are indeed finite by Proposition 5.2.17, Lemma 5.2.18 and Corollary 5.2.19. Now, introduce $\Xi_\nu : V \to W$, by $\Xi_\nu(v) := \Xi(v) - \nu$ for all $v \in V$; and $(Y_k)_{k\in\mathbb{N}_0}$, with $Y_k : \Omega \to W$ for all $k \in \mathbb{N}_0$, by $Y_k := \int_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}} \Xi_\nu(\mathbb{X}_x(\tau))d\tau$ for all $k \in \mathbb{N}$ and $Y_0 := \int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi_\nu(\mathbb{X}_{\overline{x}}(\tau))d\tau$. Finally, let $L(t) : \Omega \to \mathbb{N}_0$ be defined by $L(t) := \max(k \in \mathbb{N}_0 :\ \alpha_{e_x(k)} \le t)$ for all $t \ge 0$, where $e_x(0) := 0$
Now we will proceed by proving the following assertions, from which (5.16) as well as (5.17) will follow quickly.

i) $\mathbb{E}\alpha_{e_{\overline{x}}(1)} > 0$ and $\lim_{t\to\infty} \frac{L(t)+1}{t} = \frac{1}{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}$ almost surely.

ii) $\Xi_\nu \in SL_{V_2}(V)$ and consequently $(Y_m)_{m\in\mathbb{N}} \subseteq L^2(\Omega; W)$ is centered, i.i.d. and $Y_m = Y_0$ in distribution for all $m \in \mathbb{N}$.

iii) $\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \Xi_\nu(\mathbb{X}_x(\tau))d\tau - \sum_{k=1}^{L(t)+1} Y_k \right) = 0$ almost surely.

Proof of i). Firstly, note that $\mathbb{P}(L(t) < \infty,\ \forall t \ge 0) = 1$, since: Employing Corollary 5.2.19 and the usual SLLN on the line yields

$$\lim_{k\to\infty} \frac{1}{k}\alpha_{e_x(k)} = \lim_{k\to\infty} \frac{1}{k}\alpha_{e_x(1)} + \frac{k-1}{k}\frac{1}{k-1}\sum_{j=1}^{k-1}(\alpha_{e_x(j+1)} - \alpha_{e_x(j)}) = \mathbb{E}\alpha_{e_{\overline{x}}(1)} > 0, \tag{5.18}$$

almost surely, where the last inequality follows from $\alpha_{e_{\overline{x}}(1)} \ge \alpha_1 > 0$ almost surely. Consequently, if there were a $t \ge 0$ such that $\mathbb{P}(L(t) = \infty) > 0$, then

$$0 < \mathbb{P}(L(t) = \infty) = \mathbb{P}(\alpha_{e_x(k)} \le t,\ \forall k \in \mathbb{N}) \le \mathbb{P}(\lim_{k\to\infty} \frac{1}{k}\alpha_{e_x(k)} - \frac{t}{k} \le 0) = 0.$$

Hence, $\mathbb{P}(L(t) < \infty) = 1$ for a given t, which yields $\mathbb{P}(L(t) < \infty,\ \forall t \ge 0) = 1$, as the paths of $L(t)$ are clearly increasing with probability one.

[4] This of course means convergence for $\mathbb{P}$-a.e. $\omega \in \Omega$ with respect to $||\cdot||_W$. So far it seems redundant to write "almost surely w.r.t. $||\cdot||_W$", instead of just "almost surely". But later on we will choose W as a subspace of V, which makes it necessary to emphasize w.r.t. which norm the almost sure convergence is taking place.

[5] Again, in the next theorem it becomes clear why we emphasize on the fact that these are elements of $(W, ||\cdot||_W)$.

Moreover, it is plain to verify that the simple but quite useful inequality

$$\alpha_{e_x(L(t))} \leq t \leq \alpha_{e_x(L(t)+1)}, \ \forall t \geq 0 \tag{5.19}$$

takes place with probability one. Particularly, we have

$$\frac{\alpha_{e_x(L(t))}}{L(t)+1} \leq \frac{t}{L(t)+1} \leq \frac{\alpha_{e_x(L(t)+1)}}{L(t)+1}$$

for all $t \geq 0$, almost surely. Furthermore, thanks to (5.18), it is plain that also $\lim\limits_{k\to\infty} \frac{1}{k}\alpha_{e_x(k-1)} = \mathbb{E}\alpha_{e_{\overline{x}}(1)}$ almost surely. Consequently, if $\lim\limits_{t\to\infty} L(t)+1 = \infty$ a.s., then employing (5.18), Lemma 5.2.20, the previous inequality as well as the sandwich lemma give i). Hence, i) follows once $\lim\limits_{t\to\infty} L(t) = \infty$ a.s. is proven.

To this end, let $M \in \mathcal{F}$ be a $\mathbb{P}$-null-set, such that $\alpha_{e_x(k)}(\omega)$ is well-defined for all $k \in \mathbb{N}_0$ and such that $\lim\limits_{k\to\infty} \frac{1}{k}\alpha_{e_x(k)}(\omega) = \mathbb{E}\alpha_{e_{\overline{x}}(1)}$, for all $\omega \in \Omega \setminus M$. Now fix one these ω and note that there is for a given $\varepsilon > 0$ a $k_0 \in \mathbb{N}$, such that $\left|\frac{1}{k}\alpha_{e_x(k)}(\omega) - \mathbb{E}\alpha_{e_{\overline{x}}(1)}\right| < \varepsilon$ for all $k \geq k_0$. Hence, choosing $\varepsilon = \mathbb{E}\alpha_{e_{\overline{x}}(1)}$ yields the existence of a $k_0 \in \mathbb{N}$, with $0 < \alpha_{e_x(k)}(\omega) < 2k\mathbb{E}\alpha_{e_{\overline{x}}(1)}$ for all $k \geq k_0$, and hence

$$\sup_{t\geq 0} L(t)(\omega) \geq \sup_{k\geq k_0} L(2k\mathbb{E}\alpha_{e_{\overline{x}}(1)})(\omega) \geq \sup_{k\geq k_0} k = \infty.$$

Finally, this implies $\lim\limits_{t\to\infty} L(t) = \infty$ a.s., since M is a $\mathbb{P}$-null-set and L has paths that increase with probability one.

Proof of ii). Employing Remark 5.2.11.ii) yields that $\Xi_\nu \in SL_{V_2}(V)$. Consequently, appealing to Lemma 5.2.18 as well as Proposition 5.2.17 yields all claims in ii), except for $\mathbb{E}Y_k = 0$ for all $k \in \mathbb{N}_0$. But this is plain, since $Y_k = Y_0$ in distribution gives

$$\mathbb{E}Y_k = \mathbb{E}Y_0 = \mathbb{E}\left(\int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi_\nu(\mathbb{X}_{\overline{x}}(\tau))d\tau\right) = \mathbb{E}\left(\int_0^{\alpha_{e_{\overline{x}}(1)}} \Xi(\mathbb{X}_{\overline{x}}(\tau))d\tau\right) - \nu\mathbb{E}\alpha_{e_{\overline{x}}(1)} = 0,$$

for all $k \in \mathbb{N}_0$.

Proof of iii). Let us start by proving that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_{\alpha_{e_x(L(t))}}^{\alpha_{e_x(L(t)+2)}} ||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau = 0 \tag{5.20}$$

with probability one. Firstly, ii) and Remark 5.2.11.i) yield $(||\Xi_v||_W, \mathbb{R}) \in SL_{V_2}(V)$. Consequently, invoking Lemma 5.2.18 and Proposition 5.2.17 yields that $\left(\left(\int_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} ||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\right)^2\right)_{n\in\mathbb{N}}$ is integrable and i.i.d. Hence by appealing to the SLLN on the line we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\left(\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\right)^2\\
=&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\int\limits_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\right)^2-\frac{n-1}{n}\frac{1}{n-1}\sum_{k=1}^{n-1}\left(\int\limits_{\alpha_{e_x(k)}}^{\alpha_{e_x(k+1)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\right)^2\\
=&\ 0
\end{aligned}$$

almost surely. Consequently, we also get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int\limits_{\alpha_{e_x(n-1)}}^{\alpha_{e_x(n+1)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\\
=&\lim_{n\to\infty}\sqrt{\frac{n-1}{n}}\frac{1}{\sqrt{n-1}}\int\limits_{\alpha_{e_x(n-1)}}^{\alpha_{e_x(n)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau+\frac{1}{\sqrt{n}}\int\limits_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau\\
=&\ 0.
\end{aligned}$$

almost surely. In addition, i) enables us to apply Lemma 5.2.20 to the preceding equality, which gives

$$\lim_{t\to\infty}\frac{1}{\sqrt{L(t)+1}}\int\limits_{\alpha_{e_x(L(t))}}^{\alpha_{e_x(L(t)+2)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W d\tau=0$$

almost surely; this yields (5.20) by employing i) once more. Finally, appealing to (5.19), while having in mind (5.20), yields

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{\sqrt{t}}\left|\left|\int\limits_0^t\Xi_\nu(\mathbb{X}_x(\tau))d\tau-\sum_{k=1}^{L(t)+1}Y_k\right|\right|_W\\
\le&\lim_{t\to\infty}\frac{1}{\sqrt{t}}\left|\left|\int\limits_0^t\Xi_\nu(\mathbb{X}_x(\tau))d\tau-\int\limits_0^{\alpha_{e_x(L(t)+2)}}\Xi_\nu(\mathbb{X}_x(\tau))d\tau\right|\right|_W+\frac{1}{\sqrt{t}}\left|\left|\int\limits_0^{\alpha_{e_x(1)}}\Xi_\nu(\mathbb{X}_x(\tau))d\tau\right|\right|_W\\
\le&\lim_{t\to\infty}\frac{1}{\sqrt{t}}\int\limits_{\alpha_{e_x(L(t))}}^{\alpha_{e_x(L(t)+2)}}||\Xi_\nu(\mathbb{X}_x(\tau))||_W\, d\tau\\
=&\ 0,
\end{aligned}$$

with probability one.

Now (5.16) will be proven. Firstly, ii) and the SLLN in separable Banach spaces, see [26, Corollary

7.10], enable us to conclude that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 0$ a.s. Using this, as well as Lemma 5.2.20 and i) gives

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{L(t)+1} Y_k = \lim_{t\to\infty} \frac{L(t)+1}{t} \frac{1}{L(t)+1} \sum_{k=1}^{L(t)+1} Y_k = 0,$$

with probability one. Conclusively, Appealing to the previous equality, while having in mind iii), implies

$$\begin{aligned} & \lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t \Xi(\mathbb{X}_x(\tau))d\tau - \nu \right\|_W \\ \leq \quad & \lim_{t\to\infty} \frac{1}{\sqrt{t}} \frac{1}{\sqrt{t}} \left\| \int_0^t \Xi_\nu(\mathbb{X}_x(\tau))d\tau - \sum_{k=1}^{L(t)+1} Y_k \right\|_W + \left\| \frac{1}{t} \sum_{k=1}^{L(t)+1} Y_k \right\|_W \\ = \quad & 0, \end{aligned}$$

with probability one, which proves (5.16).

Finally, let us prove (5.17). Consequently, from now on it is assumed that $(W, ||\cdot||_W)$ is a type 2 Banach space. Let $(t_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ be such that $\lim_{n\to\infty} t_n = \infty$ and $(\theta_n)_{n\in\mathbb{N}}$, by $\theta_n := \frac{t_n}{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}$ for all $n \in \mathbb{N}$ and note that i) yields $\lim_{n\to\infty} \frac{L(t_n)+1}{\theta_n} = 1$ almost surely, and particularly in probability. Moreover, in light of ii), it is obvious that the sequence $(\frac{1}{\sqrt{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}} Y_n)_{n\in\mathbb{N}}$ is also centered, square integrable, i.i.d. and that each $\frac{1}{\sqrt{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}} Y_n$ is distributed as $\frac{1}{\sqrt{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}} Y_0$. These results enable us to employ Theorem 5.2.22, which yields

$$\lim_{n\to\infty} \frac{1}{\sqrt{t_n}} \sum_{k=1}^{L(t_n)+1} Y_k = \lim_{n\to\infty} \frac{1}{\sqrt{\theta_n}} \sum_{k=1}^{L(t_n)+1} \frac{1}{\sqrt{\mathbb{E}\alpha_{e_{\overline{x}}(1)}}} Y_k = Z,$$

in distribution. Finally, invoking iii) yields

$$\lim_{n\to\infty} \frac{1}{\sqrt{t_n}} \left(\int_0^{t_n} \Xi(\mathbb{X}_x(\tau))d\tau - t_n\nu \right) - \frac{1}{\sqrt{t_n}} \sum_{k=1}^{L(t_n)+1} Y_k = 0,$$

almost surely and consequently

$$\lim_{n\to\infty} \frac{1}{\sqrt{t_n}} \left(\int_0^{t_n} \Xi(\mathbb{X}_x(\tau))d\tau - t_n\nu \right) = Z,$$

in distribution, by [10, Theorem 3.1], which gives the claim as $(t_n)_{n\in\mathbb{N}}$ was arbitrary. (By the very definition of convergence in distribution it is clear that it suffices to consider sequences.) $\square$

Now note that for $\Xi : V \to V_2$ with $\Xi(v) := v$, if $v \in V_2$ and $\Xi(v) := 0$, if $v \in V \setminus V_2$, it is easy to verify that $(\Xi, (V_2, ||\cdot||_{V_2})) \in SL_{V_2}(V)$. Moreover, for $\xi : V \to \mathbb{R}$ with $\xi(v) := ||v||_{V_2}$ if $v \in V_2$ and

$\xi(v) := 0$ for $v \in V \setminus V_2$, we also get $(\xi, \mathbb{R}) \in SL_{V_2}(V)$. Using these facts together with the preceding theorem and Lemma 5.2.12.i) yields the following corollary.

Corollary 5.2.24. *Let $x : \Omega \to V$ be an independent initial leading to extinction. Then the following assertions hold.*

i) $\lim\limits_{t\to\infty} \frac{1}{t} \int\limits_0^t \mathbb{X}_x(\tau) d\tau = \nu_1$ *almost surely in* $(V_2, ||\cdot||_{V_2})$*, where* $\nu_1 := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})} \mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \mathbb{X}_{\overline{x}}(\tau) d\tau \right)$.

ii) $\lim\limits_{t\to\infty} \frac{1}{t} \int\limits_0^t ||\mathbb{X}_x(\tau)||_{V_2} d\tau = \nu_2$ *almost surely, where* $\nu_2 := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})} \mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} ||\mathbb{X}_{\overline{x}}(\tau)||_{V_2} d\tau \right)$.

iii) $\lim\limits_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int\limits_0^t ||\mathbb{X}_x(\tau)||_{V_2} d\tau - t\nu_2 \right) = Z_2$ *in distribution, where* $Z_2 \sim N(0, \sigma^2)$ *and* $\sigma^2 \in [0, \infty)$ *is given by* $\sigma^2 := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})} \mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} ||\mathbb{X}_{\overline{x}}(\tau)||_{V_2} - \nu_2 d\tau \right)^2$.

iv) If $(V_2, ||\cdot||_{V_2})$ *is in addition of type 2, then* $\lim\limits_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int\limits_0^t \mathbb{X}_x(\tau) d\tau - t\nu_1 \right) = Z_1$ *in distribution, as elements of* $(V_2, ||\cdot||_{V_2})$*, where* $Z_1 \sim N_{V_2}(0, Q)$ *and* $Q := Cov_{V_2}\left(\sqrt{\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}} \int\limits_0^{\alpha_{e_{\overline{x}}(1)}} \mathbb{X}_{\overline{x}}(\tau) - \nu_1 d\tau \right)$.

5.3 Examples

Let us start this section by applying the results of the previous section to the semigroup introduced in Remark 2.2.7.i).
Before doing so, let us remark, that as in the previous section $(\beta_m)_{m\in\mathbb{N}}$ denotes a sequence of $(0, \infty)$-valued i.i.d. random variables. Moreover, we set $\alpha_m := \sum\limits_{k=1}^m \beta_k$. for all $m \in \mathbb{N}$ and $\alpha_0 := 0$.

Remark 5.3.1. *Let $\rho_1 \in (0,1)$ and set $T_{\rho_1}(t)v := sgn(v) \left(-t + |v|^{\rho_1}\right)_+^{\frac{1}{\rho_1}}$ for all $v \in \mathbb{R}$ and $t \in [0, \infty)$. This is the family of mappings introduced in Remark 2.2.7.i). Note that, by this same remark $(T_{\rho_1}(t))_{t\geq 0}$ is a time-continuous, contractive semigroup on $\mathbb{R}$. Now, we apply the results of the previous section to $(T_{\rho_1}(t))_{t\geq 0}$, with $V = V_1 = V_2 = \mathbb{R}$, $\kappa = 1$ and $\rho = \rho_1$.*
Firstly, note that Assumption 5.2.1 is obviously fulfilled. Moreover, introduce a sequence $(\eta_m)_{m\in\mathbb{N}}$ of real-valued i.i.d. random variables, assume that $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$ are independent of each other, and that the following moment conditions hold: $\mathbb{E}\eta_m^4 < \infty$, there is a $\hat{\varepsilon} > 0$, such that $\mathbb{E}|\eta_m|^{\rho_1(11+\hat{\varepsilon})} < \infty$ and $\mathbb{E}\beta_m^{11+\hat{\varepsilon}} < \infty$, and last but not least that $-\mathbb{E}\beta_m + \mathbb{E}|\eta_m|^{\rho_1} < 0$. Then, Assumption 5.2.3 holds by construction.
Moreover, let $x \in \mathcal{M}(\Omega; \mathbb{R})$ be an independent initial leading to extinction, i.e. assume that x is jointly independent of $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$, and $\mathbb{E}|x|^{2\rho_1} < \infty$. Furthermore, let $\overline{x} \in \mathcal{M}(\Omega; \mathbb{R})$ be as in Remark 5.2.14 and $e_{\overline{x}}(1)$ be as in Definition 5.2.5.v).
Finally, $(\mathbb{X}_x^{(\rho_1)}(t))_{t\geq 0}$ and $(\mathbb{X}_{\overline{x}}^{(\rho_1)}(t))_{t\geq 0}$ denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\rho_1})$ in $\mathbb{R}$

and $((\beta_m)_{m\in\mathbb{N}},(\eta_m)_{m\in\mathbb{N}},\overline{x},T_{\rho_1})$ *in* $\mathbb{R}$*, respectively. Then, the following assertions instantly follow from Corollary 5.2.24:*

i) $\lim\limits_{t\to\infty}\frac{1}{t}\int\limits_0^t \mathbb{X}_x^{(\rho_1)}(\tau)d\tau=\nu_1$ *almost surely, where* $\nu_1:=\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}\mathbb{X}_{\overline{x}}^{(\rho_1)}(\tau)d\tau\right)$.

ii) $\lim\limits_{t\to\infty}\frac{1}{t}\int\limits_0^t |\mathbb{X}_x^{(\rho_1)}(\tau)|d\tau=\nu_2$ *almost surely, where* $\nu_2:=\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}|\mathbb{X}_{\overline{x}}^{(\rho_1)}(\tau)|d\tau\right)$.

iii) $\lim\limits_{t\to\infty}\frac{1}{\sqrt{t}}\left(\int\limits_0^t |\mathbb{X}_x^{(\rho_1)}(\tau)|d\tau-t\nu_2\right)=Z_2$ *in distribution, where* $Z_2\sim N(0,\sigma_2^2)$ *and* $\sigma_2^2\in[0,\infty)$ *is given by* $\sigma_2^2:=\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}|\mathbb{X}_{\overline{x}}^{(\rho_1)}(\tau)|-\nu_2 d\tau\right)^2$.

iv) $\lim\limits_{t\to\infty}\frac{1}{\sqrt{t}}\left(\int\limits_0^t \mathbb{X}_x^{(\rho_1)}(\tau)d\tau-t\nu_1\right)=Z_1$ *in distribution, where* $Z_1\sim N(0,\sigma_1^2)$ *and* $\sigma_1^2\in[0,\infty)$ *is given by* $\sigma_1^2:=\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}}\mathbb{X}_{\overline{x}}^{(\rho_1)}(\tau)-\nu_1 d\tau\right)^2$.

Note that even though the semigroup considered in Remark 5.3.1 acts on $\mathbb{R}$, none of the statements 5.3.1.i)-iv) is trivial. But, of course we did not go through all the trouble of proving the results in Section 5.2 for separable Banach spaces, to then just considered a semigroup acting on $\mathbb{R}$. So let us turn to our more sophisticated example: The weighted p-Laplacian evolution equation for "small" p.

Firstly, let us recall some notations introduced in Section 3.2: Let $n\in\mathbb{N}\setminus\{1\}$, $p\in(1,\infty)\setminus\{2\}$ and let $\emptyset\neq S\subseteq\mathbb{R}^n$ be an open, connected and bounded sets of class C^1. Moreover, $\gamma:S\to(0,\infty)$, denotes the weight function, i.e. we assume $\gamma\in L^\infty(S)$, $\gamma^{\frac{1}{1-p}}\in L^1(S)$ and that there is a $\gamma_0:\mathbb{R}^n\to\mathbb{R}$ with $\gamma_0\in M^p(\mathbb{R}^n)$ such that $\gamma_0|_S=\gamma$ a.e. on S. In addition, we set

$$p_0:=\inf\{q>1:\gamma^{\frac{1}{1-q}}\in L^1(S)\}.$$

Moreover, $A_p:D(A_p)\to 2^{L^1(S)}$ denotes the p-Laplace operator introduced in Definition 3.2.2, and $\mathcal{A}_p:D(\mathcal{A}_p)\to 2^{L^1(S)}$ denotes its closure, see Definition 3.2.4 for the definition and Theorem 3.2.5 for the fact that this is the closure of A_p. In addition, note that $\mathcal{A}_p$ is m-accretive and densely defined, see Theorem 3.2.5.
Finally, $(T_{\mathcal{A}_p}(t))_{t\geq0}$, where $T_{\mathcal{A}_p}:L^1(S)\to L^1(S)$ for all $t\geq0$, denotes the semigroup associated to $\mathcal{A}_p$, see Remark 3.2.6.

Of course, the main difficulty in applying the results of Section 5.2 to a semigroup, is verifying Assumption 5.2.1.ii). In our case, this is thanks to Theorem 3.5.6 possible, if

$$p\in\left(\frac{p_0(n-2)}{n+2}+p_0,2\right)\neq\emptyset. \tag{5.21}$$

Throughout the remainder of this section, we assume that (5.21) holds. Note that both statements in (5.21) are assumptions, i.e. it has to be assumed that the interval in (5.21) is non-empty and that p lies in this interval.
Consequently, thanks to Theorem 3.5.6, we have

$$||T_{\mathcal{A}_p}(t)v||_{L^2(S)}^{2-p} \leq (-\kappa_1 t + ||v||_{L^2(S)}^{2-p})_+,$$

for all $t \geq 0$ and $v \in L_0^2(S)$, where the constant $\kappa_1 \in (0, \infty)$ denotes, throughout the remainder of this section, the constant introduced in (3.42). (The space $L_0^2(S)$ was introduced in Remark 3.3.4.)

Now let us state the needed stochastic assumptions/notations: Let, $(\eta_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; L^1(S))$ denote an i.i.d. sequence which is independent of $(\beta_m)_{m\in\mathbb{N}}$, assume $\eta_m \in L_0^2(S)$ almost surely and that the following moment conditions hold: There is a constant $\hat{\varepsilon} > 0$ such that $\beta_m^{11+\hat{\varepsilon}}, ||\eta_m||_{L^2(S)}^{(2-p)(11+\hat{\varepsilon})} \in L^1(\Omega)$ and $-\kappa_1 \mathbb{E}\beta_m + \mathbb{E}||\eta_m||_{L^2(S)}^{(2-p)} < 0$.
Furthermore, let $\overline{x} \in \mathcal{M}(\Omega; L^1(S))$ and $e_{\overline{x}}(1)$ be as in Remark 5.2.14 and Definition 5.2.5.v), respectively. Now, let $x \in \mathcal{M}(\Omega; L^1(S))$ be jointly independent of $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$; assume that $x \in L_0^2(S)$ a.s. and $||x||_{L^2(S)}^{2(2-p)} \in L^1(\Omega)$. Finally, $\mathbb{X}_x^{(p)} : [0, \infty) \times \Omega \to L^1(S)$ and $\mathbb{X}_{\overline{x}}^{(p)} : [0, \infty) \times \Omega \to L^1(S)$, denote the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\mathcal{A}_p})$ in $L^1(S)$; and $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, \overline{x}, T_{\mathcal{A}_p})$ in $L^1(S)$, respectively.

Theorem 5.3.2. *Assume $||\eta_m||_{L^2(S)} \in L^4(\Omega)$ and introduce $\nu := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int_0^{\alpha_{e_{\overline{x}}(1)}} \mathbb{X}_{\overline{x}}^{(p)}(\tau)d\tau\right)$. Moreover, recall that we assume (5.21). Then the convergence*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{X}_x^{(p)}(\tau)d\tau = \nu,$$

takes place almost surely in $(L^2(S), ||\cdot||_{L^2(S)})$. Moreover, we have

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \mathbb{X}_x^{(p)}(\tau)d\tau - t\nu\right) = Z,$$

in distribution, as elements of $(L^2(S), ||\cdot||_{L^2(S)})$, where $Z \sim N_{L^2(S)}(0, Q)$ and the covariance is given by $Q := Cov_{L^2(S)}\left(\sqrt{\frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}} \int_0^{\alpha_{e_{\overline{x}}(1)}} \mathbb{X}_{\overline{x}}^{(p)}(\tau) - \nu d\tau\right)$.

Proof. This follows from Corollary 5.2.24.i),iv), more precisely: Choose $(V, ||\cdot||_V) = (L^1(S), ||\cdot||_{L^1(S)})$, $(V_1, ||\cdot||_{V_1}) = (L_0^2(S), ||\cdot||_{L^2(S)})$, $(V_2, ||\cdot||_{V_2}) = (L^2(S), ||\cdot||_{L^2(S)})$, $\kappa = \kappa_1$ and $\rho = 2 - p$ in Section 5.2, then: Remark 3.2.6.iv) yields that $L^2(S)$ is an invariant space w.r.t to $T_{\mathcal{A}_p}$ and thanks to Lemma 3.3.5 $L_0^2(S)$ is also an invariant space w.r.t. $T_{\mathcal{A}}$. Thus, Assumption 5.2.1.i) is fulfilled. Moreover, Remark 3.2.6.iv) yields Assumption 5.2.1.iii). In addition, Assumption 5.2.1.ii) follows from Theorem 3.5.6, and it is plain that the injections $L^2(S) \hookrightarrow L^1(S)$ and $L_0^2(S) \hookrightarrow L^1(S)$ are continuous. Thus, Assumption

5.2.1 holds. Moreover, Assumption 5.2.3 holds by construction. Finally, it is well known that $L^2(S)$ is a type 2 Banach space, see [19, Theorem 3.4], and thus both claims follow from Corollary 5.2.24.i),iv). $\square$

Theorem 5.3.3. *Let $q \in [1, \infty)$ be given. Moreover, assume $x, \eta_m \in L^q(S)$ a.s., $||\eta_m||_{L^q(S)} \in L^4(\Omega)$, introduce $\nu := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} ||\mathbb{X}_{\overline{x}}^{(p)}(\tau)||_{L^q(S)} d\tau\right)$ and recall that we assume (5.21). Then the convergence*

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t ||\mathbb{X}_x^{(p)}(\tau)||_{L^q(S)} d\tau = \nu,$$

takes place almost surely. Moreover,

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int\limits_0^t ||\mathbb{X}_x^{(p)}(\tau)||_{L^q(S)} d\tau - t\nu\right) = Z$$

in distribution, where $Z \sim N(0, \sigma^2)$ and $\sigma^2 \in [0, \infty)$ is given by

$$\sigma^2 := \frac{1}{\mathbb{E}(\alpha_{e_{\overline{x}}(1)})}\mathbb{E}\left(\int\limits_0^{\alpha_{e_{\overline{x}}(1)}} ||\mathbb{X}_{\overline{x}}^{(p)}(\tau)||_{L^q(S)} - \nu d\tau\right)^2.$$

Proof. Analogously, all assertions follow at once from Corollary 5.2.24.ii),iii), by choosing $(V, ||\cdot||_V) = (L^1(S), ||\cdot||_{L^1(S)})$, $(V_1, ||\cdot||_{V_1}) = (L_0^2(S), ||\cdot||_{L^2(S)})$, $(V_2, ||\cdot||_{V_2}) = (L^q(S), ||\cdot||_{L^q(S)})$, $\kappa = \kappa_1$ and $\rho = 2 - p$ in Section 5.2. $\square$

Remark 5.3.4. *Let us emphasize once more that we formulated Theorem 5.3.2 and Theorem 5.3.3 under the assumption that (5.21) holds. Note that, if $n = 2$ and $p_0 = 1$[6], then $\left(\frac{p_0(n-2)}{n+2} + p_0, 2\right) = (1, 2)$. Thus, in this case both theorems are applicable for all $p \in (1, 2)$.*
In the next chapter, we will obtain a SLLN and a CLT by assuming (among other things) that a semigroup fulfills a decay assumption which is in the spirit of Proposition 3.5.9. As expected, we then also apply these theoretical results to $T_{\mathcal{A}_p}$ and obtain a SLLN and a CLT for $p \in (2, \infty)$.

[6]See Remark 3.4.3 for a simple criteria ensuring $p_0 = 1$.

Chapter 6

A Markov Process Approach to ACPRM-Processes

6.1 Outline & Highlights

In this chapter, we establish that ACPRM-processes are time-homogeneous Markov processes, if the involved random variables fulfill certain distributional assumptions. Moreover, we exploit this fact to prove an SLLN and a CLT, if the underlying semigroup fulfills in addition a polynomial decay assumption.

For being able to outline this in greater detail, let us introduce some notations: Let $(V, ||\cdot||_V)$ be a separable Banach space and let $(T(t))_{t\geq 0}$ be a time-continuous, contractive semigroup on V. Moreover, recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, let $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ be i.i.d. sequences that are independent of each other; where the former consists of V-valued random variables and the latter of $(0,\infty)$-valued, exponentially distributed random variables. Moreover, let x be a V-valued random variable which is jointly independent of $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$. Finally, $\mathbb{X}_x$ denotes the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V.

This chapter's first highlight is that $(\mathbb{X}_x(t))_{t\geq 0}$ is (w.r.t. the completion of its natural filtration) a time-continuous Markov process - which is a property the fulfilled by the solutions of many SPDEs, see [27, Chapter 4]. For proving this, it is crucial that $(\beta_m)_{m\in\mathbb{N}}$ is not any i.i.d. sequence, but one consisting of exponentially distributed random variables. Moreover, due to the contractivity and time-continuity of $(T(t))_{t\geq 0}$, the transition semigroup of $(\mathbb{X}_x(t))_{t\geq 0}$ has the e-property and the Feller property. For proving these results, we only need the assumptions that have been stated in this introduction so far. But obtaining more sophisticated results, requires the following polynomial decay assumption regarding $(T(t))_{t\geq 0}$: There is a dense and separable Banach space $(W, ||\cdot||_W) \subseteq V$ with continuous injection, which is an invariant space w.r.t. T and such that there are constants $\kappa, \rho \in (0,\infty)$ such that

$$||T(t)w_1 - T(t)w_2||_W \leq \left(\kappa t + ||w_1 - w_2||_W^{-\frac{1}{\rho}}\right)^{-\rho}, \ \forall t \in [0,\infty) \tag{6.1}$$

and $w_1, w_2 \in W$. Moreover, we have to assume that $||\eta_k||_V \in L^2(\Omega)$ and that $T(t)0 = 0$ for all $t \in [0,\infty)$. The latter is due to the nonlinearity indeed not necessarily true, but it is "usually" easily verified whether it holds.
As we shall see, (6.1) enables us to derive upper bounds for $||\mathbb{X}_x(t)||_V$ and $||\mathbb{X}_x(t) - \mathbb{X}_y(t)||_V$, where y is another V-valued random variable which is jointly independent of $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$. These bounds, together with the e-property allow us to conclude by the aid of the results in [39], that the transition function of $(\mathbb{X}_x(t))_{t\geq 0}$ possesses a unique invariant probability measure $\bar{\mu} : \mathfrak{B}(V) \to [0,1]$. From there, we infer that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\mathbb{X}_x(\tau))d\tau = \overline{(\psi)} := \int_V \psi(v)\bar{\mu}(dv), \tag{SLLN}$$

with probability one, for any Lipschitz continuous $\psi : V \to \mathbb{R}$. Once this is achieved we will employ the results in [18] to prove that: If, in addition the constant ρ appearing in (6.1) fulfills $\rho > \frac{1}{2}$, then there is a $\sigma^2(\psi) \in [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \psi(\mathbb{X}_x(\tau))d\tau - t\overline{(\psi)} \right) = Y \sim N(0, \sigma^2(\psi)), \tag{CLT}$$

in distribution, for any Lipschitz continuous $\psi : V \to \mathbb{R}$.
Finally, we will illustrate the applicability of these results with two examples: The weighted p-Laplacian evolution equation for "large" p, and the semigroup $(T_{\rho_2}(t))_{t\geq 0}$ introduced in Remark 2.2.7.ii). By the aid of the latter semigroup, we will see that (CLT) can fail if (6.1) only holds for a $\rho \in (0, \frac{1}{2}]$; particularly, even if $\rho = \frac{1}{2}$.
In our p-Laplacian example, we will prove that (SLLN) holds for any $p \in (2,\infty)$ and that (CLT) holds if $p \in (2,4)$.

As mentioned, proving that $\mathbb{X}_x$ is a time-homogeneous Markov process possessing the Feller and the e-property, works without any asymptotic assumptions on the involved semigroup. Thus, these facts might be exploited by others to derive long time results regarding $\mathbb{X}_x$, under different asymptotic assumptions on the involved semigroup.

Finally, let us briefly outline this chapter's structure: In Section 6.2 we demonstrate that $(\mathbb{X}_x(t))_{t\geq 0}$ is a time-homogeneous Markov process and show that it possesses the Feller and the e-property. Section 6.3 is this section's centerpiece, it is proven there that the transition function of $(\mathbb{X}_x(t))_{t\geq 0}$ has a unique invariant probability measure (Proposition 6.3.3) that it fulfills the SLLN (Theorem 6.3.6) as well as the CLT (Theorem 6.3.10). Finally, in Section 6.4 we illustrate the applicability of these results at hand of the aforementioned examples.

6.2 The Markov Property

Throughout this section $(V, ||\cdot||_V)$ denotes a separable (real) Banach space and $(T(t))_{t\geq 0}$ denotes a time-continuous, contractive semigroup on V. Moreover, recall that $\mathcal{M}(\Omega; V) := \mathcal{M}(\Omega, \mathcal{F}; V)$.
In addition, we introduce the spaces $\mathrm{BM}(V)$, $C_b(V)$, $\mathrm{Lip}_b(V)$ and $\mathrm{Lip}(V)$ as the spaces of all functions $\psi : V \to \mathbb{R}$ which are bounded and measurable, continuous and bounded, Lipschitz continuous and bounded, and Lipschitz continuous, respectively. In addition, for any Lipschitz continuous function ψ, we denote its Lipschitz constant by L_ψ.

Throughout this section, $(\eta_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ denotes an i.i.d. sequence. In addition $(\beta_m)_{m\in\mathbb{N}}$, where $\beta_m : \Omega \to (0, \infty)$, is an i.i.d. sequence of exponentially distributed random variables, with parameter $\theta \in (0, \infty)$. Furthermore, we assume that $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ are independent of each other.
Now, set $\alpha_m := \sum\limits_{k=1}^{m} \beta_k$ for all $m \in \mathbb{N}$, introduce $\alpha_0 := 0$ and $N : [0, \infty) \times \Omega \to \mathbb{N}_0$ by

$$N(t) := \sum_{m=0}^{\infty} m\mathbb{1}\{\alpha_m \leq t < \alpha_{m+1}\},\ \forall t \in [0, \infty). \tag{6.2}$$

Moreover, for any $x \in \mathcal{M}(\Omega; V)$, we denote by $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ the sequence and by $\mathbb{X}_x : [0, \infty) \times \Omega \to V$ the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V. In addition, we introduce the filtration $(\mathcal{F}_t^x)_{t\geq 0}$, by $\mathcal{F}_t^x := \sigma_0(\mathbb{X}_x(\tau);\ \tau \in [0, t])$,[1] for all $t \in [0, \infty)$ and any $x \in \mathcal{M}(\Omega; V)$. Moreover, whenever necessary we identify a $v \in V$ with the random variable which is constantly v, and use this convention to introduce $P : [0, \infty) \times V \times \mathfrak{B}(V) \to [0, 1]$ as

$$P(t, v, B) := \mathbb{P}(\mathbb{X}_v(t) \in B) = \mathbb{P}_{X_v(t)}(B), \tag{6.3}$$

for all $v \in V$, $t \in [0, \infty)$ and $B \in \mathfrak{B}(V)$. Finally, an $x \in \mathcal{M}(\Omega; V)$ is called an independent initial, if x is jointly independent of $(\eta_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$. Note that any $v \in V$ is obviously an independent initial.

The purpose of this section is to show that $\mathbb{X}_x$ is for any independent initial $x \in \mathcal{M}(\Omega; V)$ a time homogeneous Markov process with transition function P and initial distribution $\mathbb{P}_x$. In addition, we will establish some basic properties of these quantities.

Remark 6.2.1. *Let $x \in \mathcal{M}(\Omega; V)$. Appealing to the strong law of large numbers yields $\lim\limits_{m\to\infty} \alpha_m = \infty$ almost surely. Consequently, on a set $\tilde{\Omega} \in \mathcal{F}$ of full $\mathbb{P}$-measure, we can find for each $\omega \in \tilde{\Omega}$ and $t \in [0, \infty)$*

[1] See Remark 2.2.9 for our conventions regarding σ-algebras.

precisely one $m \in \mathbb{N}_0$*, s.t.* $t \in [\alpha_m(\omega), \alpha_{m+1}(\omega))$*. Thus, we get*

$$\mathbb{P}\left(\mathbb{X}_x(t) = T(t - \alpha_{N(t)})\mathbb{x}_{x,N(t)},\ \forall t \geq 0\right) = 1.$$

In addition, it is well known that $(N(t))_{t\geq 0}$ *is a homogeneous Poisson process with intensity* θ*.*

The following continuity result will be frequently employed in this (and the next) section:

Lemma 6.2.2. *Let* $(\hat{\eta}_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ *and* $x,\ \hat{x} \in \mathcal{M}(\Omega; V)$*. Moreover, let* $\hat{\mathbb{X}}_{\hat{x}}$ *be the processes generated by* $((\beta_m)_{m\in\mathbb{N}}, (\hat{\eta}_m)_{m\in\mathbb{N}}, \hat{x}, T)$ *in* V*. Then the assertion*

$$||\mathbb{X}_x(t) - \hat{\mathbb{X}}_{\hat{x}}(t)||_V \leq ||x - \hat{x}||_V + \sum_{k=1}^{N(t)} ||\eta_k - \hat{\eta}_k||_V,\ \forall t \in [0, \infty) \tag{6.4}$$

holds on Ω*.*

Proof. Let $t \in [0, \infty)$ be given and set $M_t := \{\omega \in \Omega : t < \sup_{m\in\mathbb{N}} \alpha_m(\omega)\}$. Note that $\mathbb{X}_x(t) = \hat{\mathbb{X}}_{\hat{x}}(t) = 0$ on $\Omega \setminus M_t$, thus (6.4) holds on $\Omega \setminus M_t$. Moreover, on M_t we have by contractivity of $(T(t))_{t\geq 0}$ that

$$||\mathbb{X}_x(t) - \hat{\mathbb{X}}_{\hat{x}}(t)||_V = ||T(t - \alpha_{N(t)})\mathbb{x}_{x,N(t)} - T(t - \alpha_{N(t)})\hat{\mathbb{x}}_{\hat{x},N(t)}||_V \leq ||\mathbb{x}_{x,N(t)} - \hat{\mathbb{x}}_{\hat{x},N(t)}||_V,$$

where $(\hat{\mathbb{x}}_{\hat{x},m})_{m\in\mathbb{N}_0}$ denote the sequences generated by $((\beta_m)_{m\in\mathbb{N}}, (\hat{\eta}_m)_{m\in\mathbb{N}}, \hat{x}, T)$ in V, respectively. Consequently, it suffices to prove that

$$||\mathbb{x}_{x,m} - \hat{\mathbb{x}}_{\hat{x},m}||_V \leq ||x - \hat{x}||_V + \sum_{k=1}^{m} ||\eta_k - \hat{\eta}_k||_V,\ \forall m \in \mathbb{N}_0.$$

For $m = 0$ this is clear and for $m \in \mathbb{N}_0$ we get $||\mathbb{x}_{x,m+1} - \hat{\mathbb{x}}_{,\hat{x},m+1}||_V \leq ||\mathbb{x}_{x,m} - \hat{\mathbb{x}}_{\hat{x},m}||_V + ||\eta_{m+1} - \hat{\eta}_{m+1}||_V$, which yields the claim by induction. ☐

Lemma 6.2.3. *The mapping defined by* $[0, \infty) \times V \times \Omega \ni (t, v, \omega) \mapsto \mathbb{X}_v(t, \omega)$ *is* $\mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(V) \otimes \mathcal{F}$*-*$\mathfrak{B}(V)$*-measurable. Consequently, if* $\psi \in BM(V)$*, then* $[0, \infty) \times V \ni (t, v) \mapsto \mathbb{E}\psi(\mathbb{X}_v(t))$ *is* $\mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(V)$*-*$\mathfrak{B}(\mathbb{R})$*-measurable*

Proof. Invoking Lemma 2.2.6.v) yields that $\mathbb{X}_v$ is $\mathfrak{B}([0, \infty)) \otimes \mathcal{F}$-$\mathfrak{B}(V)$-measurable, for any $v \in V$. Moreover, we infer from Lemma 6.2.2 that $V \ni v \mapsto \mathbb{X}_v(t, \omega)$ is continuous for all $t \in [0, \infty)$ and $\omega \in \Omega$. Consequently, $[0, \infty) \times V \times \Omega \ni (t, v, \omega) \mapsto \mathbb{X}_v(t, \omega)$ is $\mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(V) \otimes \mathcal{F}$-$\mathfrak{B}(V)$-measurable, by [1, Lemma 4.51].
Finally, let $\psi \in \mathrm{BM}(V)$, then the boundedness of ψ yields that the expectation at hand exists; and the already proven measurability result (together with [31, Prop. 2.1.4]) enables us to conclude the remaining claim. ☐

Theorem 6.2.4. *Let $x \in \mathcal{M}(\Omega; V)$ be an independent initial. Then $(\mathbb{X}_x(t))_{t\geq 0}$ is a Markov process with respect to $(\mathcal{F}_t^x)_{t\geq 0}$, i.e.*

$$\mathbb{P}(\mathbb{X}_x(t+h) \in B | \mathcal{F}_t^x) = \mathbb{P}(\mathbb{X}_x(t+h) \in B | \mathbb{X}_x(t)), \tag{6.5}$$

almost surely, for all $t, h \in [0, \infty)$ and $B \in \mathfrak{B}(V)$. In addition, P is a time homogeneous transition function, that is

i) $\mathfrak{B}(V) \ni B \mapsto P(t, v, B)$ is a probability measure on $(V, \mathfrak{B}(V))$, for all $t \in [0, \infty)$ and $v \in V$,

ii) $\mathbb{P}(0, v, B) = \mathbb{1}_B(v)$ for all $v \in V$ and $B \in \mathfrak{B}(V)$,

iii) $P(\cdot, \cdot, B)$ is $\mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(V)$-$\mathfrak{B}([0, 1])$-measurable for any $B \in \mathfrak{B}(V)$ and

iv) P fulfills has the Chapman-Kolmogorov property, i.e. $P(t+h, v, B) = \int_V P(h, \hat{v}, B) dP(t, v, d\hat{v})$ for all $t, h \in [0, \infty)$, $v \in V$ and $B \in \mathfrak{B}(V)$.

Moreover, $(\mathbb{X}_x(t))_{t\geq 0}$ is time homogeneous (with initial distribution $\mathbb{P}_x$) and transition function P, i.e.

$$\mathbb{P}(\mathbb{X}_x(t+h) \in B | \mathcal{F}_t^x) = P(h, \mathbb{X}_x(t), B), \tag{6.6}$$

almost surely, for all $t, h \in [0, \infty)$ and $B \in \mathfrak{B}(V)$.

Proof. The assertions i) and ii) are trivial. Moreover, the third follows from Lemma 6.2.3.
Proving the remaining assertions is more involved and will occupy us for some time. Let us start with some preparatory observations. To this end, let $t, h \in [0, \infty)$, $v \in V$ and $B \in \mathfrak{B}(V)$ be given; and introduce $F_m : V \times [0, \infty)^m \times V^m \to V$, for all $m \in \mathbb{N}$, by $F_1(y, b, n) := T(b)y + n$ and $F_m(y, b_1, .., b_m, n_1, .., n_m) := T(b_m)F_{m-1}(y, b_1, .., b_{m-1}, n_1, .., n_{m-1}) + n_m$ for all $y, n, n_1, .., n_m \in V$, $b_1, .., b_m \in [0, \infty)$ and $m \in \mathbb{N} \setminus \{1\}$.
Appealing to Remark 2.2.5 yields that F_1 is continuous and it then follows inductively that each F_m has this property and is therefore $\mathfrak{B}(V) \otimes \mathfrak{B}([0, \infty)^m) \otimes \mathfrak{B}(V^m)$-$\mathfrak{B}(V)$-measurable.
Now, for the sake of space let $\hat{\eta}_{\tau,m} := (\eta_{N(\tau)+1}, .., \eta_{N(\tau)+m})$, for all $m \in \mathbb{N}$, $\tau \in [0, \infty)$ and $\hat{\beta}_{\tau,m} := (\alpha_{N(\tau)+1} - \tau, \beta_{N(\tau)+2}, .., \beta_{N(\tau)+m})$ if $m \geq 2$ and $\hat{\beta}_{\tau,1} := \alpha_{N(\tau)+1} - \tau$ for all $\tau \in [0, \infty)$ and let us prove inductively that

$$\mathbb{x}_{x,N(\tau)+m} = F_m(\mathbb{X}_x(\tau), \hat{\beta}_{\tau,m}, \hat{\eta}_{\tau,m}), \ \forall \tau \in [0, \infty), \tag{6.7}$$

almost surely for all $m \in \mathbb{N}$.
If $m = 1$, we get by the semigroup property and Remark 6.2.1 that

$$\mathbb{x}_{x,N(\tau)+1} = T(\alpha_{N(\tau)+1} - \tau)\mathbb{X}_x(\tau) + \eta_{N(\tau)+1} = F_1(\mathbb{X}_x(\tau), \alpha_{N(\tau)+1} - \tau, \eta_{N(\tau)+1})$$

almost surely. Moreover, if (6.7) holds for an $m \in \mathbb{N}$ we get

$$\mathbb{x}_{x,N(\tau)+m+1} \quad = \quad T(\beta_{N(\tau)+m+1})\mathbb{x}_{x,N(\tau)+m} + \eta_{N(\tau)+m+1}$$

$$
\begin{aligned}
&= T(\beta_{N(\tau)+m+1})F_m(\mathbb{X}_x(\tau),\hat{\beta}_{\tau,m},\hat{\eta}_{\tau,m}) + \eta_{N(\tau)+m+1} \\
&= F_{m+1}(\mathbb{X}_x(\tau),\hat{\beta}_{\tau,m},\beta_{N(\tau)+m+1},\hat{\eta}_{\tau,m},\eta_{N(\tau)+m+1}),
\end{aligned}
$$

which yields (6.7). Consequently, on $\{N(\tau+h)=N(\tau)\}$, we have

$$
\mathbb{X}_x(\tau+h) = T(\tau+h-\alpha_{N(\tau)})\mathbb{x}_{x,N(\tau)} = T(h)\mathbb{X}_x(\tau),\ \forall \tau,h\in[0,\infty). \tag{6.8}
$$

up-to a $\mathbb{P}$-null-set and on $\{N(\tau+h)=N(\tau)+m\}$, where $m\in\mathbb{N}$, we have

$$
\mathbb{X}_x(\tau+h) = T(\tau+h-\alpha_{N(\tau)+m})F_m(\mathbb{X}_x(\tau),\hat{\beta}_{\tau,m},\hat{\eta}_{\tau,m}),\ \forall \tau,h\in[0,\infty), \tag{6.9}
$$

up-to a $\mathbb{P}$-null-set. These two results will turn out to be useful to prove (6.5) and (6.6). But before we can do so some distribution results have to be established, namely

I) For all $m\in\mathbb{N}$, we have that $(\alpha_{N(t)+1}-t,..,\alpha_{N(t)+m}-t,\eta_{N(t)+1},..,\eta_{N(t)+m},N(t+h)-N(t))$ is in distribution equal to $(\alpha_1,..,\alpha_m,\eta_1,..,\eta_m,N(h))$.

II) For all $m\in\mathbb{N}$, we have that $(\alpha_{N(t)+1},\beta_{N(t)+2},..,\beta_{N(t)+m},\eta_{N(t)+1},..,\eta_{N(t)+m},N(t+h)-N(t))$ is independent of $\mathcal{F}_t^x$.

Proof of I). Let $z_1,..,z_m\in[0,\infty)$, $B_1,..,B_m\in\mathfrak{B}(V)$ and $C\subseteq\mathbb{N}_0$. Then, as $(\eta_m)_{m\in\mathbb{N}}$ is i.i.d and independent of $(\alpha_m)_{m\in\mathbb{N}}$ we get

$$
\begin{aligned}
&\mathbb{P}(\alpha_{N(t)+k}-t\le z_k,\ \eta_{N(t)+k}\in B_k,\ k=1,..,m,\ N(t+h)-N(t)\in C) \\
=\ &\mathbb{P}(\eta_k\in B_k,\ k=1,..,m)\sum_{j=0}^{\infty}\mathbb{P}(\alpha_{j+k}-t\le z_k,\ k=1,..,m,\ N(t+h)-N(t)\in C,\ N(t)=j) \\
=\ &\mathbb{P}(\eta_k\in B_k,\ k=1,..,m)\mathbb{P}(N(z_k+t)-N(t)\ge k,\ k=1,..,m,\ N(t+h)-N(t)\in C)
\end{aligned}
$$

where the last equality follows from

$$
\{\alpha_k\le\tau\}=\{N(\tau)\ge k\},\ \forall\tau\in[0,\infty),\ k\in\mathbb{N}_0, \tag{6.10}
$$

up to a $\mathbb{P}$-null-set. Since $(N(t))_{t\ge0}$ is a homogeneous Poisson process, it is now easily verified that the distribution of $(N(z_1+t)-N(t),..,N(z_m+t)-N(t),N(t+h)-N(t))$ is independent of t. Using this and (6.10) yields

$$
\mathbb{P}(N(z_k+t)-N(t)\ge k,\ k=1,..,m,\ N(t+h)-N(t)\in C)=\mathbb{P}(\alpha_k\le z_k,\ k=1,..,m,\ N(h)\in C).
$$

Combining the preceding two calculations, while having in mind the independence of $(\eta_m)_{m\in\mathbb{N}}$ and $(\alpha_m)_{m\in\mathbb{N}}$ gives i).
Proof of II). Since $\beta_{N(t)+k}=\alpha_{N(t)+k}-\alpha_{N(t)+k-1}$ for all $k\in\mathbb{N}\setminus\{1\}$, it suffices to prove that $(\alpha_{N(t)+1},..,\alpha_{N(t)+m},\eta_{N(t)+1},..,\eta_{N(t)+m},N(t+h)-N(t))$ is independent of $\mathcal{F}_t^x$. The latter is obviously true if $(\alpha_{N(t)+1}-t,..,\alpha_{N(t)+m}-t,\eta_{N(t)+1},..,\eta_{N(t)+m},N(t+h)-N(t))$ is independent of $\mathcal{F}_t^x$.

Now introduce $\Sigma_\tau := \sigma(A \cap B : \ A \in \Sigma_\tau^N, \ B \in \sigma_0(\eta_k, \ k \in \mathbb{N}_0))$, for all $\tau \in [0, \infty)$, where $(\Sigma_\tau^N)_{\tau \geq 0}$ denotes the completion of the natural filtration of $(N(\tau))_{\tau \geq 0}$ and $\eta_0 := x$ and let us prove that

$$\mathcal{F}_t^x \subseteq \Sigma_t \text{ and } \eta_{N(t)+j} \text{ is } \Sigma_t - \mathfrak{B}(V) - \text{measurable for all } j \in \mathbb{N}. \tag{6.11}$$

The second assertion is clearly true, since

$$\{\eta_{N(t)+j} \in B\} = \bigcup_{k=0}^{\infty} \{\eta_{k+j} \in B, \ N(t) = k\} \in \Sigma_t, \ \forall B \in \mathfrak{B}(V).$$

Now, note that Σ_t contains by construction every $\mathbb{P}$-null-set. Consequently, the first assertion follows if $\mathbb{X}_x(s)$ is Σ_t-$\mathfrak{B}(V)$-measurable, for each $s \in [0, t]$, which will be verified now: So let $s \in [0, t]$, then appealing to (6.8) and (6.9) (with $\tau = 0$ and $h = s$ there) yields, for a given $B \in \mathfrak{B}(V)$, that

$$\{X_x(s) \in B\} = \{T(s)x \in B, N(s) = 0\} \cup \left(\bigcup_{n=1}^{\infty} \{T(s - \alpha_n) F_n(x, \beta_1, .., \beta_n, \eta_1, .., \eta_n) \in B, \ N(s) = n\} \right),$$

up to a $\mathbb{P}$-null-set. It is plain that the first set in the preceding equation is an element of Σ_t. Moreover, for each $k \in \{1, .., n\}$ and $z \in [0, \infty)$ we have $\{\alpha_k \leq z, \ N(s) = n\} = \{N(z) \geq k, N(s) = n\}$. If $z \leq s$, this set is clearly in Σ_s^N and if $z > s$, we have $N(z) \geq N(s)$, which gives $\{N(z) \geq k, N(s) = n\} = \{N(s) = n\} \in \Sigma_s^N$; thus in any case $\{\alpha_k \leq z, \ N(s) = n\} \in \Sigma_t$. Consequently, as $\beta_k = \alpha_k - \alpha_{k-1}$, we obtain that $\beta_k \mathbb{1}_{\{N(s)=n\}}$ is Σ_t-$\mathfrak{B}([0, \infty))$-measurable, for all $k = 1, .., n$ and $n \in \mathbb{N}$.
Hence, we get

$$\begin{aligned} & \{T(s - \alpha_n) F_n(x, \beta_1, .., \beta_n, \eta_1, .., \eta_n) \in B, \ N(s) = n\} \\ = \ & \{T(s - (\beta_1 + .. + \beta_n) \mathbb{1}_{\{N(s)=n\}}) F_n(x, \beta_1 \mathbb{1}_{\{N(s)=n\}}, .., \beta_n \mathbb{1}_{\{N(s)=n\}}, \eta_1, .., \eta_n) \in B, \ N(s) = n\}, \end{aligned}$$

is in Σ_t, for all $n \in \mathbb{N}$ by the measurability of F_n and T, which concludes the proof of (6.11).
Now let $n \in \mathbb{N}$ $z_1, .., z_m \in [0, \infty)$, $B_1, ..B_m, D_1, .., D_n \in \mathfrak{B}(V)$, $C \subseteq \mathbb{N}_0$ and $s_1, .., s_n \in [0, t]$. As $(N(\tau))_{\tau \geq 0}$ is a Poisson process, it is clear that $(N(z_1 + t) - N(t), .., N(z_m + t) - N(t), N(t + h) - N(t))$ is independent of Σ_t^N. Consequently, as $\sigma_0(\eta_k, \ k \in \mathbb{N}_0)$ is independent of all Σ_τ^N, for $\tau \in [0, \infty)$, we get that $(N(z_1 + t) - N(t), .., N(z_m + t) - N(t), N(t + h) - N(t))$ is independent of Σ_t and that this random vector's distribution does not depend on t. Hence, employing (6.10) and (6.11) yields

$$\begin{aligned} & \mathbb{P}(\alpha_{N(t)+k} - t \leq z_k, \eta_{N(t)+k} \in B_k, k = 1, .., m, \ N(t + h) - N(t) \in C, \ \mathbb{X}_x(s_j) \in D_j, j = 1, .., n) \\ = \ & \mathbb{P}(N(z_k + t) - N(t) \geq k, \eta_{N(t)+k} \in B_k, k = 1, .., m, \ N(t + h) - N(t) \in C, \mathbb{X}_x(s_j) \in D_j, j = 1, .., n) \\ = \ & \mathbb{P}(N(z_k) \geq k, k = 1, .., m, \ N(h) \in C) \mathbb{P}(\eta_{N(t)+k} \in B_k, k = 1, .., m, \ \mathbb{X}_x(s_j) \in D_j, j = 1, .., n) \\ = \ & \mathbb{P}(\alpha_k \leq z_k, \ k = 1, .., m, N(h) \in C) \mathbb{P}(\eta_{N(t)+k} \in B_k, k = 1, .., m, \ \mathbb{X}_x(s_j) \in D_j, j = 1, .., n). \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&\mathbb{P}(\eta_{N(t)+k} \in B_k, k = 1,..,m, \mathbb{X}_x(s_j) \in D_j, j = 1,..,n) \\
= &\sum_{i=0}^{\infty} \sum_{i_1,.,i_n=0}^{i} \mathbb{P}(\eta_{i+k} \in B_k, k = 1,..,m,\ N(t) = i, N(s_j) = i_j, T(s_j - \alpha_{i_j})\mathbb{x}_{x,i_j} \in D_j, j = 1,..,n)
\end{aligned}
$$

Now it is easily verified that $\mathbb{x}_{x,i}$ is $\sigma(x, \beta_1,..,\beta_i, \eta_1,..,\eta_i)$ for any $i \in \mathbb{N}_0$. (For $i = 0$ this is trivial, for $i \in \mathbb{N}$ this follows from (6.7) by putting $\tau = 0$ there.)
Consequently, since $i_1,..,i_n \in \{0,..,i\}$ in the sum of the preceding calculation, we get by the imposed independence assumptions

$$
\mathbb{P}(\eta_{N(t)+k} \in B_k, k = 1,..,m, \mathbb{X}_x(s_j) \in D_j, j = 1,..,n) = \mathbb{P}(\eta_k \in B_k, k = 1,..,m)\mathbb{P}(\mathbb{X}_x(s_j) \in D_j, j = 1,..,n).
$$

Finally, putting it all together while having in mind (I) yields

$$
\begin{aligned}
&\mathbb{P}(\alpha_{N(t)+k} - t \leq z_k, \eta_{N(t)+k} \in B_k, k = 1,..,m,\ N(t+h) - N(t) \in C,\ \mathbb{X}_x(s_j) \in D_j, j = 1,..,n) \\
= &\mathbb{P}(\alpha_k \leq z_k, \eta_k \in B_k\ k = 1,..,m, N(h) \in C,)\mathbb{P}(\mathbb{X}_x(s_j) \in D_j, j = 1,..,n) \\
= &\mathbb{P}(\alpha_{N(t)+k} - t \leq z_k, \eta_{N(t)+k} \in B_k\ k = 1,..,m, N(t+h) - N(t) \in C,)\mathbb{P}(\mathbb{X}_x(s_j) \in D_j, j = 1,..,n),
\end{aligned}
$$

which proves (II).
Now (6.5) and (6.6) will be deduced from I)-II) as well as (6.8) and (6.9).
Firstly, I) enables us to conclude that $(t + h - \alpha_{N(t)+m}, \hat{\beta}_{t,m}, \hat{\eta}_{t,m}, N(t+h) - N(t))$ is in distribution equal to $(h - \alpha_m, \hat{\beta}_{0,m}, \hat{\eta}_{0,m}, N(h))$, since $\beta_{N(t)+m} = (\alpha_{N(t)+m} - t) - (\alpha_{N(t)+m-1} - t)$, for all $m \in \mathbb{N}$.
Now, thanks to II) and I) we can apply well known properties of conditional probabilities (cf. [15, Prop. 1.43]) in the following two calculations; where in the first line of the first calculation (6.8) and in the first line of the second one (6.9) is used.

$$
\begin{aligned}
\mathbb{E}(\mathbb{1}_B(\mathbb{X}_x(t+h))\mathbb{1}_{\{N(t+h)=N(t)\}}|\mathcal{F}_t^x)(\omega) &= \mathbb{E}(\mathbb{1}_B(T(h)\mathbb{X}_x(t))\mathbb{1}_{\{N(t+h)=N(t)\}}|\mathcal{F}_t^x)(\omega) \\
&= \int_\Omega \mathbb{1}_B(T(h)\mathbb{X}_x(t,\omega))\mathbb{1}_{\{N(t+h)=N(t)\}}(\tilde{\omega})\mathbb{P}(d\tilde{\omega}) \\
&= \int_\Omega \mathbb{1}_B(T(h)\mathbb{X}_x(t,\omega))\mathbb{1}_{\{N(h)=0\}}(\tilde{\omega})\mathbb{P}(d\tilde{\omega})
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}(\mathbb{1}_B(\mathbb{X}_x(t+h))\mathbb{1}_{\{N(t+h)=N(t)+m\}}|\mathcal{F}_t^x)(\omega) \\
= &\mathbb{E}(\mathbb{1}_B(T(t+h-\alpha_{N(t)+m})F_m(\mathbb{X}_x(t), \hat{\beta}_{t,m}, \hat{\eta}_{t,m}))\mathbb{1}_{\{N(t+h)=N(t)+m\}}|\mathcal{F}_t^x)(\omega) \\
= &\int_\Omega \mathbb{1}_B(T(t+h-\alpha_{N(t)+m}(\tilde{\omega}))F_m(\mathbb{X}_x(t,\omega), \hat{\beta}_{t,m}(\tilde{\omega}), \hat{\eta}_{t,m}(\tilde{\omega})))\mathbb{1}_{\{N(t+h)=N(t)+m\}}(\tilde{\omega})\mathbb{P}(d\tilde{\omega})
\end{aligned}
$$

$$= \int_{\Omega} \mathbb{1}_B(T(h-\alpha_m(\tilde{\omega}))F_m(\mathbb{X}_x(t,\omega),\hat{\beta}_{0,m}(\tilde{\omega}),\hat{\eta}_{0,m}(\tilde{\omega})))\mathbb{1}_{\{N(h)=m\}}(\tilde{\omega})\mathbb{P}(d\tilde{\omega}),$$

for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.
Moreover, for any $v \in V$, it is easily verified by induction that $\mathbb{x}_{v,m} = F_m(v,\hat{\beta}_{0,m},\hat{\eta}_{0,m})$ for all $m \in \mathbb{N}$ a.s. Consequently, we get

$$\begin{aligned} P(h,v,B) &= \sum_{m=0}^{\infty} \mathbb{E}\left(\mathbb{1}_B(T(h-\alpha_m)\mathbb{x}_{v,m})\mathbb{1}_{\{N(h)=m\}}\right) \\ &= \mathbb{E}\left(\mathbb{1}_B(T(h)v)\mathbb{1}_{\{N(h)=0\}}\right) + \sum_{m=1}^{\infty} \mathbb{E}\left(\mathbb{1}_B(T(h-\alpha_m)F_m(v,\hat{\beta}_{0,m},\hat{\eta}_{0,m}))\mathbb{1}_{\{N(h)=m\}}\right), \end{aligned}$$

for all $v \in V$. Hence, combining the preceding three calculations yields

$$\mathbb{P}(X_x(t+h) \in B|\mathcal{F}_t^x)(\omega) = \sum_{m=0}^{\infty} \mathbb{E}(\mathbb{1}_B(\mathbb{X}_x(t+h))\mathbb{1}_{\{N(t+h)=N(t)+m\}}|\mathcal{F}_t^x)(\omega) = P(h,\mathbb{X}_x(t,\omega),B)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, which proves (6.6). Consequently, invoking iii) gives that the random variable $\mathbb{P}(X_x(t+h) \in B|\mathcal{F}_t^x)$ is (after a possible modification on a $\mathbb{P}$-null-set) $\sigma(\mathbb{X}_x(t))$-$\mathfrak{B}([0,1])$-measurable, which yields $\mathbb{P}(\mathbb{X}_x(t+h) \in B|\mathcal{F}_t^x) = \mathbb{E}(\mathbb{P}(\mathbb{X}_x(t+h) \in B|\mathcal{F}_t^x)|\mathbb{X}_x(t))$ a.s., which implies (6.5) by the tower property of conditional expectations.
Finally, as x was arbitrary, (6.6) holds for all independent initials, which is well known to imply iv) - for the sake of completeness: Appealing to (6.6) yields

$$P(t+h,v,B) = \mathbb{E}(\mathbb{P}(\mathbb{X}_v(t+h) \in B|\mathcal{F}_t^v)) = \mathbb{E}P(h,\mathbb{X}_v(t),B) = \int_V, P(h,\hat{v},B)P(t,v,d\hat{v}),$$

for all $v \in V$, where the equality of the third and the fourth expression follow from the change of measure formula for expectations, which also holds for vector-valued random variables, see [10, p. 25]. □

Remark 6.2.5. *Throughout this chapter, let $(Q(t))_{t\geq 0}$, where $Q(t) : BM(V) \to BM(V)$, denote the family of mappings, defined by*

$$(Q(t)\psi)(v) := \mathbb{E}\psi(\mathbb{X}_v(t)) = \int_V \psi(\hat{v})P(t,v,d\hat{v}), \tag{6.12}$$

for all $\psi \in BM(V)$, $v \in V$ and $t \in [0,\infty)$.

Now, this section concludes by deriving some basic properties of our Markov process. Particularly, the e-property established in the following lemma, opens the door to useful results which enable one to conclude that a (transition function of a) Markov process on a polish state space possesses a unique invariant probability measure, see [39] for more details.

Lemma 6.2.6. *The family of mappings $(Q(t))_{t\geq 0}$ has the Feller and the e-property, that is* *i) Feller Property:* $Q(t)\psi \in C_b(V)$ *for all* $\psi \in C_b(V)$.

ii) e-property: For all $\psi \in Lip_b(V)$, $v \in V$ *and* $\varepsilon > 0$, *there is a* $\delta > 0$, *such that for all* $\hat{v} \in V$, *with* $||\hat{v} - v||_V < \delta$, *we have* $|(Q(t)\psi)(v) - (Q(t)\psi)(\hat{v})| < \varepsilon$ *for all* $t \geq 0$.

Moreover, the following assertions hold for any independent initial $x \in \mathcal{M}(\Omega; V)$.

iii) $(\mathbb{X}_x(t))_{t\geq 0}$ *has almost surely càdlàg paths and is continuous in probability.*

iv) The mapping $[0, \infty) \ni t \mapsto \mathbb{E}\psi(\mathbb{X}_x(t))$ *is continuous, whenever* $\psi \in C_b(V)$*; in particular,* $(Q(\cdot)\psi)(v)$ *is continuous for all* $\psi \in C_b(V)$ *and* $v \in V$.

v) The filtration $(\mathcal{F}_t^x)_{t\geq 0}$ *fulfills the usual conditions, i.e. it is complete and right-continuous.*

vi) The stochastic process $(\mathbb{X}_x(t))_{t\geq 0}$ *is* $(\mathcal{F}_t^x)_{t\geq 0}$*-progressive.*

Proof. The required boundedness in i) is plain and the desired continuity follows from Lemma 6.2.2 and dominated convergence.
Proof of ii). Let $\psi \in \mathrm{Lip}_b(V)$ and assume that it is not constantly zero, since the claim is trivial in this case. Moreover, let $v \in V$ and $\varepsilon > 0$ be given and introduce $\delta := \frac{\varepsilon}{2L_\psi}$. Then employing the services of Lemma 6.2.2 once more yields

$$|(Q(t)\psi)(v) - (Q(t)\psi)(\hat{v})| \leq L_\psi \mathbb{E}||\mathbb{X}_v(t) - \mathbb{X}_{\hat{v}}(t)||_V \leq L_\psi ||v - \hat{v}||_V < \varepsilon, \ \forall t \geq 0$$

for all $\hat{v} \in V$, with $||v - \hat{v}||_V < \delta$, which proves ii).
Proof of iii). Thanks to Remark 6.2.1, it follows from Lemma 2.2.6.vi) that $\mathbb{X}_x$ has almost surely càdlàg paths. In light of this, it remains to prove that $\mathbb{X}_x$ is left-continuous in probability. So let $t_0 \in (0, \infty)$, $t \in [0, t_0]$ and $\varepsilon > 0$ and note that

$$\mathbb{P}(||\mathbb{X}_x(t) - \mathbb{X}_x(t_0)||_V > \varepsilon) \leq \mathbb{P}(||\mathbb{X}_x(t) - \mathbb{X}_x(t_0)||_V > \varepsilon, \ N(t) = N(t_0)) + P(|N(t) - N(t_0)| \geq 1).$$

Moreover, the contractivity of $(T(t))_{t\geq 0}$ yields

$$\begin{aligned} ||\mathbb{X}_x(t) - \mathbb{X}_x(t_0)||_V &= ||T(t - \alpha_{N(t_0)})\mathbb{x}_{x,N(t_0)} - T(t - \alpha_{N(t_0)})T(t_0 - t)\mathbb{x}_{x,N(t_0)}||_V \\ &\leq ||\mathbb{x}_{x,N(t_0)} - T(t_0 - t)\mathbb{x}_{x,N(t_0)}||_V \end{aligned}$$

on $\{N(t) = N(t_0)\}$, up-to a $\mathbb{P}$-null-set. Conclusively, as Poisson processes are well-known to be stochastically continuous and as $(T(t))_{t\geq 0}$ is time-continuous, we get

$$\lim_{t \nearrow t_0} \mathbb{P}(||\mathbb{X}_x(t) - \mathbb{X}_x(t_0)||_V > \varepsilon) \leq \lim_{t \nearrow t_0} \mathbb{P}(||\mathbb{x}_{x,N(t_0)} - T(t_0 - t)\mathbb{x}_{x,N(t_0)}||_V > \varepsilon, \ N(t) = N(t_0)) = 0,$$

which proves iii).
Proof of iv). Let $(t_m)_{m\in\mathbb{N}}$ be converging to a given $t \in [0, \infty)$. Then, we get by iii) (and by passing to a subsequence if necessary) that $\lim\limits_{m\to\infty} ||\mathbb{X}_x(t_m) - \mathbb{X}_x(t)||_V = 0$ almost surely. Consequently,

$\lim_{m\to\infty} \psi(\mathbb{X}_x(t_m)) = \psi(\mathbb{X}_x(t))$ a.s. and by dominated convergence also in $L^1(\Omega)$, which gives iv).
Finally, the desired completeness in v) holds by construction, the right-continuity follows from [13, Theorem, p. 556], which is indeed applicable due to i) and Lemma 2.2.6.iv); and vi) follows from v) and Lemma 2.2.6.iv), by [31, Prop. 2.2.3]. □

6.3 The SLLN and the CLT

Let the notations of the previous section prevail, that means: $(V, ||\cdot||_V)$ is a separable Banach space and $((\eta_m)_{m\in\mathbb{N}}, (\beta_m)_{m\in\mathbb{N}}, T)$ is a fixed triplet, where $(\eta_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ is an i.i.d. sequence, $(\beta_m)_{m\in\mathbb{N}}$ is an i.i.d. sequence which is independent of $(\eta_m)_{m\in\mathbb{N}}$ and each β_m is exponentially distributed with parameter $\theta \in (0,\infty)$, and $(T(t))_{t\geq 0}$ is a time-continuous contractive semigroup on V. Moreover, $(N(t))_{t\geq 0}$ is the Poisson process arising from $(\beta_m)_{m\in\mathbb{N}}$ and $(\alpha_m)_{m\in\mathbb{N}_0}$ is the process' sequence of arrival times.
Again we refer to an $x \in \mathcal{M}(\Omega; V)$ which is independent of $((\eta_m)_{m\in\mathbb{N}}, (\beta_m)_{m\in\mathbb{N}})$ as an independent initial, and denote by $\mathbb{X}_x$ and $(\mathbb{x}_{x,m})_{m\in\mathbb{N}_0}$ the sequence and the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T)$ in V. Moreover, set $P(t, v, B) := \mathbb{P}(\mathbb{X}_v(t) \in B)$ and $(Q(t)\psi)(v) := \mathbb{E}\psi(\mathbb{X}_v(t))$, for every $v \in V$, $t \in [0,\infty)$, $B \in \mathfrak{B}(V)$ and $\psi \in \mathrm{BM}(V)$.
In addition, we assume throughout this entire section that

$$||\eta_k||_V \in L^2(\Omega),\ \forall k \in \mathbb{N}, \tag{6.13}$$

where $L^q(\Omega) := L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for every $q \in [1,\infty)$. Moreover, we impose the following assumption regarding $(T(t))_{t\geq 0}$.

Assumption 6.3.1. *There is a separable Banach space $(W, ||\cdot||_W)$, with $W \subseteq V$, such that the following assertions hold.*

i) The injection $W \hookrightarrow V$ is continuous and W is dense in $(V, ||\cdot||_V)$.

ii) W is an invariant space with respect to T.

iii) There are constants $\kappa, \rho \in (0,\infty)$ such that $||T(t)w_1 - T(t)w_2||_W \leq \left(\kappa t + ||w_1 - w_2||_W^{-\frac{1}{\rho}}\right)^{-\rho}$ for all $w_1, w_2 \in W$ and $t \in [0,\infty)$.

iv) $T(t)0 = 0$ for all $t \in [0,\infty)$.

Throughout this entire section, Assumption 6.3.1 is assumed to hold. Particularly, $(W, ||\cdot||_W)$ and $\kappa, \rho \in (0,\infty)$ are such that 6.3.1.i)-iii) are fulfilled. In addition, $C > 0$ denotes the operator norm of the injection $W \hookrightarrow V$; hereby, we exclude the trivial case $C = 0$, since $C = 0$ implies $W = \{0\}$ and by density $V = \{0\}$.

The following estimates will play a fundamental role in this entire section, it is needed in the proofs of all of our main results, which are: Proposition 6.3.3, Theorem 6.3.6, Theorem 6.3.10 and Corollary 6.3.11. The remaining results of this section simply serve to keep the exposition more structured, but

are probably not of independent interest.
As mentioned introductory, proving the CLT requires the additional assumption $\rho > \frac{1}{2}$. It will be stated explicitly whenever this additional assumption is needed.

Lemma 6.3.2. *Let $x \in \mathcal{M}(\Omega; V)$ be an independent initial. Then the inequality*

$$||\mathbb{X}_x(t)||_V \leq C\kappa^{-\rho}(t - \alpha_{N(t)})^{-\rho}, \ \forall t > 0, \tag{6.14}$$

takes place with probability one. In addition, if $y \in \mathcal{M}(\Omega; V)$ is another independent initial we have

$$||\mathbb{X}_x(t) - \mathbb{X}_y(t)||_V \leq C\kappa^{-\rho}t^{-\rho}, \ \forall t > 0, \tag{6.15}$$

almost surely.

Proof. Let us start by proving (6.14). To this end, let $(\tilde{\eta}_m)_{m\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; V)$ and $\tilde{x} \in \mathcal{M}(\Omega; V)$, assume $\tilde{\eta}_m, \tilde{x} \in W$ for all $m \in \mathbb{N}$, almost surely and introduce $(\tilde{X}(t))_{t\geq 0}$ and $(\tilde{x}_m)_{m\in\mathbb{N}_0}$ as the process and the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\tilde{\eta}_m)_{m\in\mathbb{N}}, \tilde{x}, T)$ in V, respectively.
Then, appealing to Lemma 2.2.6.vii) yields $\tilde{x}_m \in W$ for all $m \in \mathbb{N}_0$ and $\tilde{X}(t) \in W$ for all $t \geq 0$ almost surely.
Hence, employing Assumption 6.3.1.iii) and iv) yields

$$||\tilde{X}(t)||_V \leq C||T(t - \alpha_{N(t)})\tilde{x}_{N(t)}||_W \leq C\left(\kappa(t - \alpha_{N(t)}) + ||\tilde{x}_{N(t)}||_W^{-\frac{1}{\rho}}\right)^{-\rho} \leq C\kappa^{-\rho}(t - \alpha_{N(t)})^{-\rho} \tag{6.16}$$

for all $t > 0$ almost surely.
Now let us infer (6.14) from (6.16). To this end, let $(\Gamma_n)_{n\in\mathbb{N}}$, where $\Gamma_n : V \to V$, be a sequence of $\mathfrak{B}(V)$-$\mathfrak{B}(V)$-measurable mappings, such that

$$\Gamma_n(V) \subseteq W, \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \Gamma_n(v) = v, \ \forall v \in V. \tag{6.17}$$

Since W is dense in $(V, ||\cdot||_V)$, such a sequence exists, see Lemma 4.3.11. Now, for every $n \in \mathbb{N}$, let $(X^n(t))_{t\geq 0}$ be the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\Gamma_n(\eta_m))_{m\in\mathbb{N}}, \Gamma_n(x), T)$ in V. Then, as $\Gamma_n(V) \subseteq W$, (6.16) yields $||X^n(t)||_V \leq C\kappa^{-\rho}(t - \alpha_{N(t)})^{-\rho}$ for all $t > 0$ and $n \in \mathbb{N}$ almost surely. (If one sets $(\tilde{\eta}_m)_{m\in\mathbb{N}} = (\Gamma_n(\eta_m))_{m\in\mathbb{N}}$ and $\tilde{x} = \Gamma_n(x)$ for a given $n \in \mathbb{N}$, then $\tilde{X} = X^n$). Moreover, appealing to Lemma 6.2.2, while having in mind (6.17), yields

$$||\mathbb{X}_x(t)||_V \leq \lim_{n\to\infty} ||x - \Gamma_n(x)||_V + \sum_{m=1}^{N(t)} ||\eta_m - \Gamma_n(\eta_m)||_V + C\kappa^{-\rho}(t - \alpha_{N(t)})^{-\rho} = C\kappa^{-\rho}(t - \alpha_{N(t)})^{-\rho},$$

for all $t > 0$, with probability one. Consequently, (6.14) is proven and it remains to verify (6.15).
In addition, to the existing notations, let $\tilde{y} \in \mathcal{M}(\Omega; V)$, assume $\tilde{y} \in W$ almost surely and introduce $(\tilde{Y}(t))_{t\geq 0}$ and $(\tilde{y}_m)_{m\in\mathbb{N}_0}$ as the process and the sequence generated by $((\beta_m)_{m\in\mathbb{N}}, (\tilde{\eta}_m)_{m\in\mathbb{N}}, \tilde{y}, T)$ in V, respectively.

Of course, we then also have $\tilde{Y}(t), \tilde{y}_m \in W$ for all $t \geq 0$ and $m \in \mathbb{N}_0$, with probability one. Now let us verify inductively that

$$||\tilde{x}_m - \tilde{y}_m||_W \leq \left(\kappa\alpha_m + ||\tilde{x} - \tilde{y}||_W^{-\frac{1}{\rho}}\right)^{-\rho}, \text{ a.s. } \forall m \in \mathbb{N}_0. \tag{6.18}$$

If $m = 0$, (6.18) is even an equality. And if it holds for an $m \in \mathbb{N}_0$ we get by applying Assumption 6.3.1.iii) and then the induction hypothesis that

$$||\tilde{x}_{m+1} - \tilde{y}_{m+1}||_W \leq \left(\kappa\beta_{m+1} + ||\tilde{x}_m - \tilde{y}_m||_W^{-\frac{1}{\rho}}\right)^{-\rho} \leq \left(\kappa\alpha_{m+1} + ||\tilde{x} - \tilde{y}||_W^{-\frac{1}{\rho}}\right)^{-\rho},$$

with probability one, which proves (6.18). Using this, while employing the services of 6.3.1.iii) once more gives

$$\begin{aligned}
||\tilde{X}(t) - \tilde{Y}(t)||_V &\leq C||T(t - \alpha_{N(t)})\tilde{x}_{N(t)} - T(t - \alpha_{N(t)})\tilde{y}_{N(t)}||_W \\
&\leq C\left(\kappa(t - \alpha_{N(t)}) + ||\tilde{x}_{N(t)} - \tilde{y}_{N(t)}||_W^{-\frac{1}{\rho}}\right)^{-\rho} \\
&\leq C\left(\kappa(t - \alpha_{N(t)}) + \kappa\alpha_{N(t)} + ||\tilde{x} - \tilde{y}||_W^{-\frac{1}{\rho}}\right)^{-\rho} \\
&\leq C\,(\kappa t)^{-\rho},
\end{aligned}$$

for all $t > 0$ with probability one. Now, for every $n \in \mathbb{N}$, let $(Y^n(t))_{t\geq 0}$ be the process generated by $((\beta_m)_{m\in\mathbb{N}}, (\Gamma_n(\eta_m))_{m\in\mathbb{N}}, \Gamma_n(y), T)$ in V. Then, as $\Gamma_n(V) \subseteq W$, the preceding calculation yields $||X^n(t) - Y^n(t)||_V \leq C\,(\kappa t)^{-\rho}$ for all $t > 0$ and $n \in \mathbb{N}$ almost surely. Finally, Lemma 6.2.2 enables us to conclude that

$$||\mathbb{X}_x(t) - \mathbb{X}_y(t)||_V \leq C\,(\kappa t)^{-\rho} + ||x - \Gamma_n(x)||_V + ||y - \Gamma_n(y)||_V + 2\sum_{m=1}^{N(t)} ||\eta_m - \Gamma_n(\eta_m)||_V,$$

for all $t > 0$ and $n \in \mathbb{N}$ with probability one, which yields the claim by recalling (6.17) and letting n to infinity. □

Proposition 6.3.3. *The transition function P possesses a unique invariant probability measure, i.e. there is one, and only one, probability measure $\mu : \mathfrak{B}(V) \to [0,1]$, such that*

$$\int_V P(t, v, B)\mu(dv) = \mu(B), \ \forall t \geq 0, \ B \in \mathfrak{B}(V). \tag{6.19}$$

Proof. Appealing to Theorem 6.2.4 as well as Lemma 6.2.6.i)-iii) yields, by virtue of [39, Theorem 1], the existence of a unique invariant probability measure, if we can prove that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t P(||\mathbb{X}_v(t)||_V < \varepsilon) d\tau > 0, \ \forall \varepsilon > 0, \ v \in V.$$

So fix $\varepsilon > 0$ as well as $v \in V$ and recall the well-known fact that $\mathbb{P}(\tau - \alpha_{N(\tau)} > q) = \exp(-\theta q)\mathbb{1}_{[0,\tau)}(q)$ for all $\tau, q \in [0, \infty)$. Now, introduce $q := \kappa^{-1}\varepsilon^{-\frac{1}{\rho}}C^{\frac{1}{\rho}}$.
Then we get by Lemma 6.3.2 that

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{t} \int_0^t P(||\mathbb{X}_v(\tau)||_V < \varepsilon) d\tau &\geq \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(C\kappa^{-\rho}(\tau - \alpha_{N(\tau)})^{-\rho} < \varepsilon) d\tau \\
&= \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(\tau - \alpha_{N(\tau)} > q) d\tau \\
&= \exp(-\theta q),
\end{aligned}$$

which is obviously strictly positive. $\square$

Remark 6.3.4. *In the remainder of this section, $\bar{\mu} : \mathfrak{B}(V) \to [0, 1]$, denotes the uniquely determined probability measure fulfilling (6.19). Moreover, we call an $\bar{x} \in \mathcal{M}(\Omega; V)$ which is an independent initial with $\mathbb{P}(\bar{x} \in B) = \bar{\mu}(B)$, for all $B \in \mathfrak{B}(V)$, an independent, stationary initial.*
As $\bar{\mu}$ is unique, it is ergodic, see [38, Theorem 3.2.6] for a proof and [38, Theorem 3.2.4] for a couple of useful equivalent definitions of ergodicity, commonly used in the literature.
Furthermore, if $\bar{x} \in \mathcal{M}(\Omega; V)$ is an independent, stationary initial, then the Markov process $(\mathbb{X}_{\bar{x}}(t))_{t\geq 0}$ is strictly stationary, see [20, Lemma 8.11]. Moreover, $(\mathbb{X}_{\bar{x}}(t))_{t\geq 0}$ is also ergodic (in the sense that the shift invariant σ-algebra is $\mathbb{P}$-trivial), which one easily deduces from [9, Prop. 2.2] by appealing to [38, Theorem 3.2.4.ii)].
Finally, $L^2(\bar{\mu}) := L^2(V, \mathfrak{B}(V), \bar{\mu})$ and for any $\psi \in L^2(\bar{\mu})$ we set $\overline{(\psi)} := \int_V \psi(v)\bar{\mu}(dv)$ and introduce $L^2_0(\bar{\mu}) := \{\psi \in L^2(\bar{\mu}) : \overline{(\psi)} = 0\}$.

Lemma 6.3.5. *Let $x \in \mathcal{M}(\Omega; V)$ be an independent initial. Then $||\mathbb{X}_x(t)||_V \in L^2(\Omega)$ for all $t \in (0, \infty)$. In particular, the following assertions hold.*

i) $\psi(\mathbb{X}_{\bar{x}}(t)) \in L^2(\Omega)$, for all $t \in [0, \infty)$, $\psi \in Lip(V)$ and independent stationary initials $\bar{x} \in \mathcal{M}(\Omega; V)$.

ii) $Lip(V) \subseteq L^2(\bar{\mu})$.

Proof. Let $t > 0$ and $x \in \mathcal{M}(\Omega; V)$ be an independent initial. Then we get by employing the services of Lemma 6.2.2 and Lemma 6.3.2 that

$$||\mathbb{X}_x(t)||_V \leq ||\mathbb{X}_x(t) - \mathbb{X}_0(t)||_V + ||\mathbb{X}_0(t)||_V \leq C\kappa^{-\rho}t^{-\rho} + \sum_{m=1}^{N(t)} ||\eta_m||_V$$

almost surely. Consequently, $||\mathbb{X}_x(t)||_V \in L^2(\Omega)$ holds, if $\sum_{m=1}^{N(t)} ||\eta_m||_V \in L^2(\Omega)$. But the latter is true by the Blackwell-Girshick equation, which is applicable since $(||\eta_k||)_{k\in\mathbb{N}} \subseteq L^2(\Omega)$ is i.i.d. and independent of $(N(t))_{t\geq 0}$, which is (as it is a Poisson process) in particular square integrable.
Now, note that, due to stationary, 6.3.5.i) holds for one $t \in [0,\infty)$ if and only if, it holds for every $t \in [0,\infty)$. So assume $t > 0$, then we get $|\psi(\mathbb{X}_{\bar{x}}(t))| \leq L_\psi ||\mathbb{X}_{\bar{x}}(t)||_V + |\psi(0)|$, which is already known to be square integrable. Finally, 6.3.5.ii) follows from 6.3.5.i), since $||\psi||^2_{L^2(\bar{\mu})} = \mathbb{E}\left(\psi(\bar{x})^2\right)$. □

Theorem 6.3.6. *Let $\psi \in Lip(V)$ and $x \in \mathcal{M}(\Omega;V)$ be an independent initial. Then the convergence*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\mathbb{X}_x(\tau))d\tau = \overline{(\psi)}, \tag{6.20}$$

takes place with probability one.

Proof. Firstly, note that the left hand side integral exists, since Lemma 6.2.3 and Lemma 6.2.6.iii) yield that $[0,t] \ni \tau \mapsto \psi(\mathbb{X}_x(\tau,\omega))$ is $\mathfrak{B}([0,t])$-$\mathfrak{B}(\mathbb{R})$-measurable and for $\mathbb{P}$-a.e. $\omega \in \Omega$ bounded, respectively. Now let $\bar{x} \in \mathcal{M}(\Omega;V)$ be an independent stationary initial. Then appealing to [38, Theorem 3.3.1] yields

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\mathbb{X}_{\bar{x}}(\tau))d\tau = \overline{(\psi)}, \tag{6.21}$$

almost surely, for all $\psi \in Lip(V)$. (This theorem is indeed applicable, since $\bar{\mu}$ is ergodic, $(\mathbb{X}_{\bar{x}}(t))_{t\geq 0}$ is stationary, stochastically continuous and since $Lip(V) \subseteq L^2(\bar{\mu})$.)
Conclusively, recalling Lemma 6.3.2 gives

$$\lim_{t\to\infty} \left| \frac{1}{t} \int_0^t \psi(\mathbb{X}_{\bar{x}}(\tau)) - \psi(\mathbb{X}_x(\tau))d\tau \right| \leq L_\psi C \kappa^{-\rho} \lim_{t\to\infty} \frac{1}{t} \int_1^t \tau^{-\rho} d\tau = 0,$$

almost surely, which yields combined with (6.21) the claim. □

The task ahead of us that remains is proving the CLT, which will be achieved by the results in [18]. Applying the results in [18] requires to extend the family of mappings $(Q(t))_{t\geq 0}$ to a linear, time-continuous, contractive semigroup on $L^2(\bar{\mu})$. To aid the reader who is not too familiar with Markov processes, let us outline why this is possible.

Remark 6.3.7. *Let $t \in [0,\infty)$ be given. Then for any $\hat{V} \in \mathfrak{B}(V)$, with $\bar{\mu}(\hat{V}) = 1$, we get by the invariance of $\bar{\mu}$ that there is a set $\tilde{V} \in \mathfrak{B}(V)$, with $\mu(\tilde{V}) = 1$, such that $\mathbb{P}(\mathbb{X}_v(t) \in \hat{V}) = 1,\ \forall v \in \tilde{V}$. Moreover, if $\psi = \mathbb{1}_B$, where $B \in \mathfrak{B}(V)$, then the invariance of $\bar{\mu}$ gives*

$$\int_V \mathbb{E}\psi(\mathbb{X}_v(t))\bar{\mu}(dv) = \int_V \psi(v)\bar{\mu}(dv). \tag{6.22}$$

Moreover, by linearity in ψ, (6.22) also holds for all step functions. Now let $\psi \in L^2(\bar{\mu})$ be arbitrary, then there are step functions $(\psi_m)_{m\in\mathbb{N}}$ with $\lim\limits_{m\to\infty} \psi_m = \psi$ in $L^2(\bar{\mu})$ and $\bar{\mu}$-a.e. Hence, for $\bar{\mu}$-a.e. $v \in V$ we get $\lim\limits_{m\to\infty} \psi_m(\mathbb{X}_v(t)) = \psi(\mathbb{X}_v(t))$ a.s. Consequently, applying Fatou's Lemma (twice) and (6.22) yields

$$\int_V \mathbb{E}\left(\psi(\mathbb{X}_v(t))^2\right)\bar{\mu}(dv) \leq \liminf_{m\to\infty} \int_V \mathbb{E}\left(\psi_m(\mathbb{X}_v(t))^2\right)\bar{\mu}(dv) = \liminf_{m\to\infty} \int_V \psi_m(v)^2 \bar{\mu}(dv) = ||\psi||^2_{L^2(\bar{\mu})} < \infty.$$

Hence, for $\bar{\mu}$-almost every $v \in V$, $\mathbb{E}\psi(\mathbb{X}_v(t))$ exists and we infer from Jensen's inequality that

$$\int_V \left(\mathbb{E}\psi(\mathbb{X}_v(t))\right)^2 \bar{\mu}(dv) \leq ||\psi||^2_{L^2(\bar{\mu})}, \ \forall \psi \in L^2(\bar{\mu}). \tag{6.23}$$

Consequently, we can extend the domain of each $Q(t)$ to $L^2(\bar{\mu})$, i.e. from now on $Q(t) : L^2(\bar{\mu}) \to L^2(\bar{\mu})$, with $(Q(t)\psi)(v) := \mathbb{E}\psi(\mathbb{X}_v(t))$, for all $t \in [0,\infty)$, $v \in V$ and $\psi \in L^2(\bar{\mu})$.
Using this and Theorem 6.2.4.iv) yields that $(Q(t))_{t\geq 0}$ is a linear, contractive semigroup on $L^2(\bar{\mu})$, see [42, Theorem 1, p. 381] for a detailed proof. (Hereby, linear of course means $Q(t)(a\psi_1 + b\psi_2) = aQ(t)\psi_1 + bQ(t)\psi_2$ for all $\psi_1, \psi_2 \in L^2(\bar{\mu})$, $a, b \in \mathbb{R}$ and $t \geq 0$.)

It seems to be mathematical common knowledge that this semigroup is (due to stochastic continuity and contractivity) time-continuous. But, the present author was unable to find any proof of this assertion, therefore let's do that:

Lemma 6.3.8. *The family of mappings $(Q(t))_{t\geq 0}$ is a linear, time-continuous contractive semigroup on $L^2(\bar{\mu})$.*

Proof. In light of Remark 6.3.7, it remains to prove the time continuity. So let $(h_m)_{m\in\mathbb{N}}$ be a null-sequence, let $t \in [0,\infty)$ and assume w.l.o.g. that $t + h_m \geq 0$ for all $m \in \mathbb{N}$. Now let $\psi \in L^2(\bar{\mu})$, choose $\varepsilon > 0$ and $\varphi \in C_b(V)$ such that $||\psi - \varphi||_{L^2(\bar{\mu})} < \frac{\varepsilon}{2}$. Then, by stochastic continuity of $(\mathbb{X}_v(t))_{t\geq 0}$, and passing to a subsequence if necessary, we have $\lim\limits_{m\to\infty} \varphi(\mathbb{X}_v(t + h_m)) = \varphi(\mathbb{X}_v(t))$ almost surely. Thus, the boundedness of φ yields (by dominated convergence) that $\lim\limits_{m\to\infty} (Q(t + h_m)\varphi)(v) = (Q(t)\varphi)(v)$ for all $v \in V$. Consequently, employing Lebesgue's theorem once more gives $\lim\limits_{m\to\infty} Q(t + h_m)\varphi = Q(t)\varphi$ in $L^2(\bar{\mu})$. Using this, as well as the contractivity of Q gives

$$\lim_{m\to\infty} ||Q(t + h_m)\psi - Q(t)\psi||_{L^2(\bar{\mu})} \leq 2||\psi - \varphi||_{L^2(\bar{\mu})} + \lim_{m\to\infty} ||Q(t + h_m)\varphi - Q(t)\varphi||_{L^2(\bar{\mu})} \leq \varepsilon,$$

which yields the desired time continuity, as $\varepsilon > 0$ was arbitrary. □

Lemma 6.3.9. *Let $\psi \in Lip(V)$ and set $\psi_c := \psi - \overline{(\psi)}$. Then $\psi_c \in L^2_0(\bar{\mu})$ and*

$$||Q(t)\psi_c||_{L^2(\bar{\mu})} \leq L_\psi C \kappa^{-\rho} t^{-\rho},$$

for all $t > 0$.

Proof. Clearly, $\psi_c \in Lip(V)$, thus $\psi_c \in L^2(\bar{\mu})$ by Lemma 6.3.5.ii). Moreover, ψ_c is obviously centered. In addition, by stationary we get $\overline{(\psi)} = \mathbb{E}\psi(\mathbb{X}_{\bar{x}}(t))$, where $\bar{x} \in \mathcal{M}(\Omega; V)$ is an independent, stationary initial. Using this and invoking Lemma 6.3.2 yields

$$||Q(t)\psi_c||^2_{L^2(\bar{\mu})} = \int\limits_V \left(\mathbb{E}[\psi(\mathbb{X}_v(t)) - \psi(\mathbb{X}_{\bar{x}}(t))]\right)^2 \bar{\mu}(dv) \leq (L_\psi C\kappa^{-\rho}t^{-\rho})^2$$

and the claim follows. $\square$

Theorem 6.3.10. *Assume $\rho > \frac{1}{2}$, let $\psi \in Lip(V)$ and $x \in \mathcal{M}(\Omega; V)$ be an independent initial. Then there is a $\sigma^2(\psi) \in [0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int\limits_0^t \psi(\mathbb{X}_x(\tau))d\tau - t\overline{(\psi)} \right) = Y \sim N(0, \sigma^2(\psi)), \tag{6.24}$$

in distribution. Moreover, we have

$$\sigma^2(\psi) := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left(\int\limits_0^t \psi_c(\mathbb{X}_{\bar{x}}(\tau))d\tau \right)^2 = \lim_{t\to\infty} \frac{1}{t}\, Var\left(\int\limits_0^t \psi(\mathbb{X}_{\bar{x}}(\tau))d\tau \right), \tag{6.25}$$

where $\bar{x} \in \mathcal{M}(\Omega; V)$ is an arbitrary stationary, independent initial and $\psi_c := \psi - \overline{(\psi)}$.

Proof. Appealing to Lemma 6.3.9 gives $\psi_c \in L^2_0(\bar{\mu})$ as well as

$$\int\limits_1^\infty \frac{1}{\sqrt{t}} ||Q(t)\psi_c||_{L^2(\bar{\mu})}dt \leq L_\psi C\kappa^{-\rho} \int\limits_1^\infty t^{-\rho-\frac{1}{2}}dt,$$

which is finite, since $\rho > \frac{1}{2}$. Consequently, as we already know that $(\mathbb{X}_{\bar{x}}(t))_{t\geq 0}$ is a stationary, ergodic, $(\mathcal{F}^{\bar{x}}_t)_{t\geq 0}$-progressive Markov process with time-continuous, contractive semigroup $(Q(t))_{t\geq 0}$, we get by [18, Corollary 3.2 and Theorem 3.1] that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int\limits_0^t \psi_c(\mathbb{X}_{\bar{x}}(\tau))d\tau = Y \sim N(0, \sigma^2(\psi)), \tag{6.26}$$

in distribution and that $\sigma^2(\psi)$ is indeed given by the first equality in (6.25). Moreover, the second equality in (6.25) is trivial, since $\psi_c(\mathbb{X}_{\bar{x}}(\tau)) = \psi(\mathbb{X}_{\bar{x}}(\tau)) - \mathbb{E}(\psi(\mathbb{X}_{\bar{x}}(\tau)))$ by stationarity.
Now, note that clearly

$$\frac{1}{\sqrt{t}} \left(\int\limits_0^t \psi(\mathbb{X}_x(\tau))d\tau - t\overline{(\psi)} \right) = \frac{1}{\sqrt{t}} \int\limits_0^t \psi(\mathbb{X}_x(\tau)) - \psi(\mathbb{X}_{\bar{x}}(\tau))d\tau + \frac{1}{\sqrt{t}} \int\limits_0^t \psi_c(\mathbb{X}_{\bar{x}}(\tau))d\tau,\ \forall t > 0$$

which yields, in light of (6.26), that (6.24) holds, if the first summand in the previous express converges almost surely to zero. But recalling that $\rho > \frac{1}{2}$ and invoking Lemma 6.3.2 yields

$$\lim_{t\to\infty}\left|\frac{1}{\sqrt{t}}\int_0^t \psi(\mathbb{X}_x(\tau)) - \psi(\mathbb{X}_{\bar{x}}(\tau))d\tau\right| \leq L_\psi C\kappa^{-\rho}\lim_{t\to\infty}\frac{1}{\sqrt{t}}\int_1^t \tau^{-\rho}d\tau = 0,$$

with probability one. □

Now this section concludes by summarizing Theorem 6.3.6 and Theorem 6.3.10 for the probably most prominent Lipschitz continuous map from V to $\mathbb{R}$, namely $||\cdot||_V$.

Corollary 6.3.11. *Let $x \in \mathcal{M}(\Omega; V)$ be an independent initial and $\bar{x} \in \mathcal{M}(\Omega; V)$ a stationary independent initial. Then the following assertions hold.*

i) $\lim_{t\to\infty}\frac{1}{t}\int_0^t ||\mathbb{X}_x(\tau)||_V d\tau = \nu$ *with probability one, where* $\nu := \int_V ||v||_V \bar{\mu}(dv) = \mathbb{E}||\bar{x}||_V$.

ii) If $\rho > \frac{1}{2}$, *then* $\lim_{t\to\infty}\frac{1}{\sqrt{t}}\left(\int_0^t ||\mathbb{X}_x(\tau)||_V d\tau - t\nu\right) = Y \sim N(0,\sigma^2)$ *in distribution, where* $\sigma^2 \in [0,\infty)$, *with* $\sigma^2 = \lim_{t\to\infty}\frac{1}{t}\, Var\left(\int_0^t ||\mathbb{X}_{\bar{x}}(\tau)||_V d\tau\right)$.

6.4 Examples

Let us start demonstrating the applicability of the results from Sections 6.2 and 6.3 by considering the semigroup introduced in Remark 2.2.7.ii). The example considered there, also serves to demonstrate that the assumption "$\rho > \frac{1}{2}$" in Theorem 6.3.10 cannot be dropped.

Example 6.4.1. *Let $\rho_2 \in (0,\infty)$ and let $(T_{\rho_2}(t))_{t\geq 0}$ denote the family of mappings introduced in Remark 2.2.7.ii), i.e. $T_{\rho_2}(t)v := sgn(v)\left(t + |v|^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$ for all $v \in \mathbb{R}$ and $t \geq 0$. Then $(T_{\rho_2}(t))_{t\geq 0}$ is according to this same remark a time-continuous, contractive semigroup on $\mathbb{R}$.*
Now let us verify Assumption 6.3.1 with $V = W = \mathbb{R}$, $\rho = \rho_2$ and $\kappa := 2^{-\frac{1}{\rho_2}}$. Firstly, Assumption 6.3.1.i),ii) are trivial. Verifying Assumption 6.3.1.iii) is slightly more involved, and requires to prove

i) $T_{\rho_2}(t)v_1 + T_{\rho_2}(t)v_2 \leq \left(\kappa t + (v_1+v_2)^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$, *for all* $t \in [0,\infty)$, $v_1, v_2 \geq 0$ *and*

ii) $T_{\rho_2}(t)v_1 - T_{\rho_2}(t)v_2 \leq \left(\kappa t + (v_1-v_2)^{-\frac{1}{\rho_2}}\right)^{-\rho_2}$, *for all* $t \in [0,\infty)$, $v_1, v_2 \geq 0$ *with* $v_1 \geq v_2$.

Proof of i). Firstly, the convexity of $[0,\infty) \ni x \mapsto x^{1+\frac{1}{\rho_2}}$ yields $x^{1+\frac{1}{\rho_2}} + y^{1+\frac{1}{\rho_2}} \geq 2^{-\frac{1}{\rho_2}}(x+y)^{1+\frac{1}{\rho_2}}$ for all $x, y \in [0,\infty)$. Now set $f(t) := T_{\rho_2}(t)v_1 + T_{\rho_2}(t)v_2$, for all $t \in [0,\infty)$. Then we get

$$f'(t) = -\rho_2\left((T_{\rho_2}(t)v_1)^{1+\frac{1}{\rho_2}} + (T_{\rho_2}(t)v_2)^{1+\frac{1}{\rho_2}}\right) \leq -\rho_2\kappa\left(T_{\rho_2}(t)v_1 + T_{\rho_2}(t)v_2\right)^{1+\frac{1}{\rho_2}} = -\rho_2\kappa f(t)^{1+\frac{1}{\rho_2}},$$

for all $t > 0$. Consequently, as f is (particularly locally) Lipschitz continuous, i) follows from Lemma 3.5.2.

Proof of ii). Firstly, it is easily verified that $x^{1+\frac{1}{\rho_2}} - y^{1+\frac{1}{\rho_2}} \geq (x-y)^{1+\frac{1}{\rho_2}} \geq \kappa(x-y)^{1+\frac{1}{\rho_2}}$ *for all* $x \geq y \geq 0$. *Moreover, note that* $T_{\rho_2}(t)v_1 \geq T_{\rho_2}(t)v_2$, *since* $v_1 \geq v_2 \geq 0$. *Now, set* $f(t) := T_{\rho_2}(t)v_1 - T_{\rho_2}(t)v_2$, *then we get*

$$f'(t) = -\rho_2 \left((T_{\rho_2}(t)v_1)^{1+\frac{1}{\rho_2}} - (T_{\rho_2}(t)v_2)^{1+\frac{1}{\rho_2}} \right) \leq -\rho_2\kappa \left(T_{\rho_2}(t)v_1 - T_{\rho_2}(t)v_2\right)^{1+\frac{1}{\rho_2}} = -\rho_2\kappa f(t)^{1+\frac{1}{\rho_2}},$$

for all $t > 0$. *Consequently, employing Lemma 3.5.2 once more yields ii).*
Now, one easily infers from i), ii) and $T_{\rho_2}(t)(-v) = -T_{\rho_2}(t)v$, *for all* $v \in \mathbb{R}$ *that*

$$|T_{\rho_2}(t)v_1 - T_{\rho_2}(t)v_2| \leq \left(\kappa t + |v_1 - v_2|^{-\frac{1}{\rho_2}}\right)^{-\rho_2}, \ \forall t \in [0,\infty), \ v_1, v_2 \in \mathbb{R}.$$

Finally, it is plain that $T_{\rho_2}(t)0 = 0$ *for all* $t \geq 0$; *thus,* $(T_{\rho_2}(t))_{t\geq 0}$ *is a time-continuous, contractive semigroup fulfilling Assumption 6.3.1 with* $V = W = \mathbb{R}$, $\rho = \rho_2$ *and* $\kappa = 2^{-\frac{1}{\rho_2}}$. *Now, let* $(\eta_m)_{m\in\mathbb{N}} \subseteq L^2(\Omega)$ *be an i.i.d. sequence and let* $(\beta_m)_{m\in\mathbb{N}}$ *be an i.i.d. sequence of* $Exp(\theta)$*-distributed random variables, where* $\theta \in (0,\infty)$. *In addition, assume that both sequences are independent of each other and let, for any independent initial* $x \in \mathcal{M}(\Omega;\mathbb{R})$, $\mathbb{X}_x^{(\rho_2)} : [0,\infty) \times \Omega \to \mathbb{R}$ *denote the process generated by* $((\beta_m)_{m\in\mathbb{N}}, (\eta_m)_{m\in\mathbb{N}}, x, T_{\rho_2})$ *in* $\mathbb{R}$. *Then, as the identity is Lipschitz continuous, it follows from Theorem 6.3.6 and Theorem 6.3.10 that*

iii) $\lim\limits_{t\to\infty} \frac{1}{t} \int\limits_0^t \mathbb{X}_x^{(\rho_2)}(\tau)d\tau = \mathbb{E}\bar{x}$ *a.s., for any independent initial* $x \in \mathcal{M}(\Omega;\mathbb{R})$ *where* $\bar{x} \in \mathcal{M}(\Omega;\mathbb{R})$ *is a stationary, independent initial, and*

iv) if in addition $\rho_2 > \frac{1}{2}$, *then we have* $\lim\limits_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int\limits_0^t \mathbb{X}_x^{(\rho_2)}(\tau)d\tau - t\mathbb{E}\bar{x} \right) = Y \sim N(0,\sigma^2)$ *in distribution, for any independent initial* $x \in \mathcal{M}(\Omega;\mathbb{R})$, *where* $\sigma^2 = \lim\limits_{t\to\infty} \frac{1}{t} Var\left(\int\limits_0^t \mathbb{X}_{\bar{x}}^{(\rho_2)}(\tau)d\tau \right)$.

Now let us demonstrate that the assumption $\rho_2 > \frac{1}{2}$ *in iv) cannot be dropped. To this end, assume* $\eta_k = 0$ *for all* $k \in \mathbb{N}$, *then* $\mathbb{X}_x^{(\rho_2)}(t) = T_{\rho_2}(t)x$ *for any independent initial* $x \in \mathcal{M}(\Omega;\mathbb{R})$. *Since* $T_{\rho_2}(t)0 = 0$ *for all* $t \geq 0$, $\bar{x} = 0$ *is the (in this case even almost surely unique) stationary, independent initial. Consequently, we have* $\mathbb{E}\bar{x} = Var\left(\int\limits_0^t \mathbb{X}_{\bar{x}}^{(\rho_2)}(\tau)d\tau \right) = 0$ *and iv), with* $x = 1$ *and without additional assuming* $\rho_2 > \frac{1}{2}$, *would imply*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int\limits_0^t (\tau+1)^{-\rho_2} \, d\tau = 0, \ \forall \rho_2 > 0, \tag{6.27}$$

which is now, due to the lack of randomness, simply convergence in $\mathbb{R}$. *But obviously, (6.27) is true if and only if* $\rho_2 > \frac{1}{2}$.

Even though the semigroup considered in the previous example only acted on $\mathbb{R}$ and not an infinite dimensional Banach space, it is worth mentioning that neither 6.4.1.iii) nor 6.4.1.iv) are trivial.

Now, let us turn to our p-Laplacian semigroup for large p. Firstly, let us recall some notations: Let $n \in \mathbb{N} \setminus \{1\}$, $p \in (1,\infty) \setminus \{2\}$ and let $\emptyset \neq S \subseteq \mathbb{R}^n$ be an open, connected and bounded sets of class C^1. Moreover, $\gamma : S \to (0,\infty)$ denotes the weight function, i.e. we assume $\gamma \in L^\infty(S)$, $\gamma^{\frac{1}{1-p}} \in L^1(S)$ and that there is a $\gamma_0 : \mathbb{R}^n \to \mathbb{R}$ with $\gamma_0 \in M^p(\mathbb{R}^n)$ such that $\gamma_0|_S = \gamma$ a.e. on S.
Moreover, $A_p : D(A_p) \to 2^{L^1(S)}$ denotes the p-Laplace operator introduced in Definition 3.2.2, and $\mathcal{A}_p : D(\mathcal{A}_p) \to 2^{L^1(S)}$ denotes its closure, see Definition 3.2.4 for the definition and Theorem 3.2.5 for the fact that this is the closure of A_p. In addition, note that $\mathcal{A}_p$ is m-accretive and densely defined, see Theorem 3.2.5.
Finally, $(T_{\mathcal{A}_p}(t))_{t\geq 0}$, where $T_{\mathcal{A}_p} : L^1(S) \to L^1(S)$ for all $t \geq 0$, denotes the semigroup associated to $\mathcal{A}_p$, see Remark 3.2.6.
Now, in the remainder of this section we assume

$$p \in (2,\infty) \text{ and } \int\limits_S \gamma^{\frac{2}{2-p}} \, d\lambda < \infty. \tag{6.28}$$

Note that(6.28) already implies $\int\limits_S \gamma^{\frac{1}{1-p}} d\lambda < \infty$, see Remark 3.5.8. Moreover, by this same Remark, (6.28) holds if $\frac{p}{2} > p_0$. (See Remark 3.4.2 for the definition of p_0.)
Finally, set

$$\kappa_2 := (p-2)2^{-(p-2)} \left(\int\limits_S \gamma^{\frac{2}{2-p}} \, d\lambda \right)^{\frac{2-p}{2}} C_{S,2}^{-p},$$

where $C_{S,2}$ denotes the Poincaré constant of S in $L^2(S)$, see Remark 3.4.1.

Thanks to Lemma 3.3.6 and Proposition 3.5.9, we can apply the results of Section 6.2 and Section 6.3 now. Hereby, note the following: For any $q \in [1,\infty)$, we have: If we restrict each $T_{\mathcal{A}_p}(t)$ to $L^q_0(S)$, then (thanks to Lemma 3.3.6) $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ is a time-continuous contractive semigroup on $L^q_0(S)$. Hereby we just have made (and will continue to make) the following minor abuse of notation: We denote $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ viewed as a semigroup on $L^1(S)$ and $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ viewed as a semigroup on $L^q_0(S)$, by the same latter, namely $(T_{\mathcal{A}_p}(t))_{t\geq 0}$.

Theorem 6.4.2. *Let $q \in [1,2]$ and let $(\eta_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(\Omega; L^q_0(S))$ be an i.i.d. sequence. Moreover, let $(\beta_m)_{m\in\mathbb{N}}$ be another i.i.d. sequence which is independent of $(\eta_k)_{k\in\mathbb{N}}$ and assume that $\beta_m \sim Exp(\theta)$ for all $m \in \mathbb{N}$, where $\theta \in (0,\infty)$. In addition, assume $||\eta_k||_{L^q(S)} \in L^2(\Omega)$ for all $k \in \mathbb{N}$. Moreover, let $x \in \mathcal{M}(\Omega; L^q_0(S))$ be an independent initial, i.e. independent of $((\eta_k)_{k\in\mathbb{N}}, (\beta_k)_{k\in\mathbb{N}})$ and let $\mathbb{X}^{(p)}_x : [0,\infty) \times \Omega \to L^q_0(S)$ be the process generated by $((\beta_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}}, x, T_{\mathcal{A}_p})$ in $L^q_0(S)$, where $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ is viewed as a semigroup on $L^q_0(S)$. Finally, note that we assume (6.28).*
Then $(\mathbb{X}^{(p)}_x(t))_{t\geq 0}$ is a time-homogeneous Markov process (w.r.t. the completion of its natural filtration) which possesses a unique invariant probability measure $\bar{\mu} : \mathfrak{B}(L^q_0(S)) \to [0,1]$. In addition, for any

$\psi \in Lip(L_0^q(S))$, the convergence

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\mathbb{X}_x^{(p)}(\tau))d\tau = \int_{L_0^q(S)} \psi(v)\bar{\mu}(dv) := \overline{(\psi)}, \tag{6.29}$$

takes place with probability one, and if additionally $p \in (2,4)$, then there is a $\sigma^2(\psi) \in [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\int_0^t \psi(\mathbb{X}_x^{(p)}(\tau))d\tau - t\overline{(\psi)} \right) = Y \sim N(0,\sigma^2(\psi)), \tag{6.30}$$

in distribution.

Proof. By Lemma 3.3.6, $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ is a time continuous, contractive semigroup on $L_0^q(S)$. Consequently, by choosing $V = L_0^q(S)$ in Section 6.2 it follows from Theorem 6.2.4 that $\mathbb{X}_x^{(p)}$ is a time-continuous Markov process.
Moreover, it follows from Lemma 3.3.6 and Proposition 3.5.9 that $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ fulfills Assumption 6.3.1, where we choose $V = L_0^q(S)$, $W = L_0^2(S)$, $\rho := \frac{1}{p-2}$ and $\kappa = \kappa_2$. Consequently, appealing to Proposition 6.3.3 yields the existence of a unique invariant probability measure and Theorem 6.3.6 implies (6.29). Finally, (6.30) follows from Theorem 6.3.10, since $p \in (2,4)$ implies $\rho > \frac{1}{2}$. ☐

Remark 6.4.3. *In accordance with Corollary 6.3.11, (6.29) and (6.30) hold (under the assumptions of Theorem 6.4.2) of course particularly for $\psi := ||\cdot||_{L^q(S)}$.*

Remark 6.4.4. *As demonstrated in Remark 6.4.1, the assumption "$\rho > \frac{1}{2}$" in Theorem 6.3.10 cannot be dropped. This gives some evidence that (6.30) might also fail if $p \notin (2,4)$. But let us point out that we were unable to find a concrete counterexample showing that (6.30) fails if $p \notin (2,4)$.*

Remark 6.4.5. *Note that Theorem 6.4.2 was formulated under Assumption (6.28); and that Theorem 5.3.2 as well as Theorem 5.3.3 were formulated under Assumption (5.21). Let us compare these two assumptions:*
Firstly, it is clear that for a given value of p, at most one of these two assumptions can hold. Moreover, note that, if $n = 2$ and $\gamma \geq c$ a.e. on S, for a constant $c > 0$, then (5.21) reduces to $p \in (1,2)$ and (6.28) reduces to $p \in (2,\infty)$. Thus, in this case, we can either apply Theorem 5.3.2 and 5.3.3 or Equation (6.29), for any possible value of $p \in (1,\infty) \setminus \{2\}$, if the random quantities $(\beta_m)_{m\in\mathbb{N}}$, $(\eta_m)_{m\in\mathbb{N}}$ and x fulfill the respective assumptions stated in these theorems.

Chapter 7

The randomized weighted p-Laplacian evolution Equation

7.1 Outline & Highlights

The purpose of this chapter is to extend the results developed for the deterministic weighted p-Laplacian evolution equation in Chapter 3, to the randomized case. By that we mean that we will replace the occurring weight function as well as the initial by a vector-valued random variable. Consequently, we will study the problem

$$\begin{cases} U'(t)(\omega) = \operatorname{div}\left(g(\omega)|\nabla U(t)(\omega)|^{p-2}\nabla U(t)(\omega)\right) \text{ on } S, \\ g(\omega)|\nabla U(t)(\omega)|^{p-2}\nabla U(t)(\omega)\cdot\Upsilon = 0 \text{ on } \partial S, \\ U(0)(\omega) = u(\omega), \end{cases} \tag{7.1}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and a.e. $t \in (0,\infty)$, where: $(\Omega, \mathcal{F}, \mathbb{P})$ is (as usually) a complete probability space, $p \in (1,\infty) \setminus \{2\}$, $S \subseteq \mathbb{R}^n$ is a non-empty, open, bounded and connected set of class C^1, $n \in \mathbb{N} \setminus \{1\}$, $g \in L^1(\Omega; L^1(S))$ fulfills $0 < g_1 \leq g \leq g_2 < \infty$ almost surely, for some constants $g_1, g_2 \in (0,\infty)$, Υ is the unit outer normal on ∂S, and $u \in L^1(\Omega; L^1(S)$ is an initial. Hereby, we set $L^1(S) := L^1(S, \mathfrak{B}(S), \lambda; \mathbb{R})$, where λ is the n-dimensional Lebesgue measure, and $L^q(\Omega; V) := L^q(\Omega, \mathcal{F}, \mathbb{P}; V)$ for any $q \in [1,\infty)$ and separable Banach space $(V, ||\cdot||_V)$.
We will employ nonlinear semigroup theory to establish that this equation has a unique solution and derive asymptotic properties of the solution.

Before being able to rigorously describe our results, we have to fix some notations: Note that if $\gamma \in L^1(S)$, with $0 < g_1 \leq \gamma \leq g_2$, then γ fulfills all assumptions the weight function in Chapter 3 had to fulfill, and in this case, the weighted Sobolev space $W^{1,p}_\gamma(S)$ introduced in Section 3.2 (Equation (3.8)), is equal to $W^{1,p}(S)$. Now, let $A^d_p(\gamma) : D(A^d_p(\gamma)) \to L^1(S)$ denote the p-Laplace operator introduced in

Definition 3.2.2, i.e.: $(f, \hat{f}) \in A_p^d(\gamma)$, if $f, \hat{f} \in L^1(S)$, $f \in W^{1,p}(S) \cap L^\infty(S)$ and

$$\int_S \gamma |\nabla f|^{p-2} \nabla f \cdot \nabla \varphi d\lambda = \int_S \hat{f} \varphi d\lambda, \ \forall \varphi \in W_\gamma^{1,p}(S) \cap L^\infty(S).$$

This operator is indeed single-valued, see Lemma 3.3.1.
The reason why we denote this operator now by $A_p^d(\gamma)$ and no longer simply by A_p, is that we need a notation that indicates that this operator depends on γ. Moreover, the superscribe "d" (for deterministic) is necessary to be able to better distinguish it from the operator introduced next. We are aware that this is an inconsistency in our own notation and that we could have denoted this operator in all preceding chapters by $A_p^d(\gamma)$ instead of A_p. We chose note to do so, since doing that would have caused an unnecessarily long and inconvenient notation.
In this chapter, we will introduce a random p-Laplace operator $A_p^r : D(A_p^r) \to L^1(\Omega; L^1(S))$ and demonstrate that this operator is characterized by: For any $f, \hat{f} \in L^1(\Omega; L^1(S))$ we have $(f, \hat{f}) \in A_p^r$ if and only if $(f(\omega), \hat{f}(\omega)) \in A_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We use this operator to show that: The closure of A_p^r, which we denote by $\mathcal{A}_p^r : D(\mathcal{A}_p^r) \to 2^{L^1(\Omega; L^1(S))}$, is m-accretive and densely defined. Thus (thanks to Theorem 2.1.7) the initial value problem

$$0 \in U'(t) + \mathcal{A}_p^r U(t), \text{ for a.e. } t \in (0, \infty), \ U(0) = u, \tag{7.2}$$

has, for any $u \in L^1(\Omega; L^1(S))$ a uniquely determined mild solution.
Once this is achieved, we proceed by proving that (7.2) also has a uniquely determined strong solution. To be able to outline how this works, let us introduce some more notations: Let $(T_{\mathrm{ra}}(t))_{t \geq 0}$ denote the semigroup associated to $\mathcal{A}_p^r$; thus, for any $u \in L^1(\Omega; L^1(S))$, $T_{\mathrm{ra}}(\cdot)u$ is the unique mild solution of (7.2). Moreover, for any $\gamma \in L^1(S)$, with $g_1 \leq \gamma \leq g_2$, let $(T_{\det}(t, \gamma))_{t \geq 0}$ denote the semigroup associated to $\mathcal{A}_p^d(\gamma)$, where $\mathcal{A}_p^d(\gamma)$ is the closure of $A_p^d(\gamma)$.
We will prove that

$$\mathbb{P}\left(\{\omega \in \Omega : \ (T_{\mathrm{ra}}(t)u)(\omega) = T_{\det}(t, g(\omega))u(\omega)\}\right) = 1, \tag{7.3}$$

for all $u \in L^1(\Omega; L^1(S))$ and $t \in [0, \infty)$. This, together with the properties of $(T_{\det}(t, \gamma))_{t \geq 0}$ developed in Chapter 3 will enable us to conclude by the aid of Theorem 2.1.12 that $T_{\mathrm{ra}}(\cdot)u$ is, for any $u \in L^1(\Omega; L^1(S))$, also the uniquely determined strong solution of (7.2).

Besides these existence/uniqueness results, we will also derive interesting asymptotic results. Firstly, (7.3) enables us to transfer the results from Chapter 3 , to the current setting: For example, we have:

i) If $t \in [0, \infty)$ and $u \in L^1(\Omega; L^1(S))$, then $\overline{(T_{\mathrm{ra}}(t)u)}_S = \overline{(u)}_S$ almost surely, where $\overline{(v)}_S := \frac{1}{\lambda(S)} \int_S v d\lambda$ for all $v \in L^1(S)$ and $\overline{(u)}_S(\omega) := \overline{(u(\omega))}_S$ for all $\omega \in \Omega$. (Follows directly from Lemma 3.3.5.)

ii) If $q \in [1, \infty)$ and $u \in L^q(\Omega; L^q(S))$, then $\lim_{t \to \infty} T_{\mathrm{ra}}(t)u = \overline{(u)}_S$ in $L^q(\Omega; L^q(S))$. (Follows fairly directly Theorem 3.4.13.)

iii) If $u \in L^1(\Omega; L^1(S))$, with $u \in L^2(S)$ a.s., then $||T_{\text{ra}}(t)u - \overline{(u)}_S||_{L^1(S)} \leq c_1 ||u - \overline{(u)}_S||_{L^2(S)}^{\frac{2}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}$, a.s. for all $t > 0$, where $c_1 > 0$ can be determined explicitly. (Follows directly from Corollary 3.4.9.)

Of course, we can (and will) also extend many other asymptotic results from Chapter 3 to the current setting, such as Theorem 3.4.10, Theorem 3.5.6 and Theorem 3.5.10.

Besides these results, which are direct consequences of the results in Chapter 3, we are also going to derive upper bounds for the tail function of $||T(t)u - \overline{(u)}||_{L^2(S)}^2$, assuming that $p \in [\frac{2n}{n+2}, 2) \setminus \{1\}$; more precisely: Introduce $u \in L^1(\Omega; L^1(S))$ with $u \in L^2(S)$ almost surely. Then we have: If $r \in [1, \infty)$ and $\Delta_u \in L^{2r}(\Omega)$, where $\Delta_u := ||u - \overline{(u)}_S||_{L^2(S)}^2$, then

$$\mathbb{P}\left(\int_S (T_{\text{ra}}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \left(\frac{1}{t}\right)^r \frac{2}{\log(\alpha+1)} c_2 \left(\mathbb{E}(\Delta_u)\mathbb{E}((1+\Delta_u)^{2r})\right)^{\frac{1}{2}}, \tag{7.4}$$

for any $\alpha,\ t \in (0, \infty)$; and if there is even an $\varepsilon > 0$ such that $e^{\varepsilon \Delta_u} \in L^1(\Omega)$, we have

$$\mathbb{P}\left(\int_S (T_{\text{ra}}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \exp\left(-t^{\frac{1}{2}} c_3\right) \frac{2\exp\left(\frac{\varepsilon}{2}\right)}{\log(\alpha+1)} \left(\mathbb{E}(\Delta_u)\mathbb{E}\left(\exp\left(\varepsilon\Delta_u\right)\right)\right)^{\frac{1}{2}}, \tag{7.5}$$

for any $\alpha,\ t \in (0, \infty)$.
Hereby $c_2,\ c_3 \geq 0$ are constants which can be determined explicitly.

This chapter is structured as follows: Section 7.2 contains the basic assumptions/notations needed in this chapter. Particularly, the needed operators are introduced there. In Section 7.3 we develop some basic properties of these operators and prove that (7.2) has a unique mild solution. Then, we proceed in Section 7.4 by deriving the identity (7.3) and prove that the mild solution is also a strong one. Afterwards, in Section 7.5 we establish the asymptotic results i)-iii). Finally, Section 7.6 deals with the tail function bounds (7.4) and (7.5).
Moreover, this chapter contains two appendices: In the first, we answer some technical measurability questions which occur while defining A_p^r and derive a result regarding the measurability of the $L^\infty(S)$-norm of vector-valued random variables; and in the second we provide some delicate results about $A_p^d(\gamma)$, its closure and their resolvents as well as a certain denseness result, which are needed to prove the existence and uniqueness of mild solutions of (7.2).

7.2 Notation

Let us start by recalling some notations that where also used in Chapter 3: Throughout this entire chapter, let $n \in \mathbb{N} \setminus \{1\}$, $p \in (1, \infty) \setminus \{2\}$ and let $\emptyset \neq S \subseteq \mathbb{R}^n$ be an open, connected and bounded set

of class C^1.
Moreover, λ denotes the Lebesgue measure on $\mathbb{R}^n$, $|\cdot|$ the euclidean norm on $\mathbb{R}^n$, and $x \cdot y$ is the canonical inner product of any $x, y \in \mathbb{R}^n$. In addition, we introduce the short-cut notations $L^q(S) := L^q(S, \mathfrak{B}(S), \lambda; \mathbb{R})$ and $L^q(S; \mathbb{R}^n) := L^q(S, \mathfrak{B}(S), \lambda; \mathbb{R}^n)$, for all $q \in [1, \infty]$. As usually, $W^{1,1}_{\text{Loc}}(S)$ denotes the space of weakly differentiable functions and ∇f denotes the weak derivative of any $f \in W^{1,1}_{\text{Loc}}(S)$. In addition, for any $q \in [1, \infty)$, $W^{1,q}(S)$ denotes the Sobolev space of once weakly differentiable functions, such that $\varphi \in L^q(S)$ and $\nabla\varphi \in L^q(S; \mathbb{R}^n)$; and as usually $C^\infty_c(S)$ is the space of infinitely often continuous differentiable, compactly supported functions $\varphi : S \to \mathbb{R}$.
Finally, the reader is reminded that $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space and that

$$L^q(\Omega; V) := L^q(\Omega, \mathcal{F}, \mathbb{P}; V)$$

for any separable Banach space $(V, ||\cdot||_V)$ and all $q \in [1, \infty)$. This is furhter abbreviated by $L^q(\Omega)$, if $V = \mathbb{R}$.

Remark 7.2.1. *Now let us introduce some new notations used throughout this chapter: For all* $0 < \varepsilon_1 < \varepsilon_2 < \infty$*, we set*

i) $L^1_{\varepsilon_1,\varepsilon_2}(S) := \{f \in L^1(S) : \varepsilon_1 \leq f \leq \varepsilon_2$ *a.e. on* $S\}$ *and*

ii) $L^1_{\varepsilon_1,\varepsilon_2}(\Omega; L^1(S)) := \{f \in L^1(\Omega; L^1(S)) : \ \mathbb{P}(\{\omega \in \Omega : f(\omega) \in L^1_{\varepsilon_1,\varepsilon_2}(S)\}) = 1\}$.

Note that $L^1_{\varepsilon_1,\varepsilon_2}(S)$ *is closed w.r.t.* $||\cdot||_{L^1(S)}$*, thus* $L^1_{\varepsilon_1,\varepsilon_2}(S) \in \mathfrak{B}(L^1(S))$.
Furthermore, $\tau_k : \mathbb{R} \to \mathbb{R}$*, where* $k \in (0, \infty)$*, always denotes the standard truncation function, i.e.* $\tau_k(x) := x$*, if* $|x| < k$ *and* $\tau_k(x) := k sgn(x)$*. Moreover, for any* $f : S \to \mathbb{R}$*,* $\tau_k(f) : S \to \mathbb{R}$ *is defined by* $\tau_k(f) := \tau_k(f(\cdot))$ *and for* $f : \Omega \to L^1(S)$*,* $\tau_k(f) : \Omega \to L^1(S)$ *is defined by* $\tau_k(f)(\omega) := \tau_k(f(\omega))$.
In addition, we introduce the spaces

iii) $\tau(L^1(\Omega; L^1(S))) := \{\tau_k(f)|\ f \in L^1(\Omega; L^1(S)),\ k \in (0, \infty)\}$ *and*

iv) $L^{1,\infty}(\Omega; L^1(S)) := \{f \in L^1(\Omega; L^1(S))|\ \mathbb{P}(f \in L^\infty(S)) = 1\}$.

Finally, throughout this chapter, let $0 < g_1 \leq g_2 < \infty$ be fixed constants and let $g \in L^1_{g_1,g_2}(\Omega; L^1(S))$ be a fixed function.

Now, note the following: If $\gamma \in L^1_{g_1,g_2}(S)$, then $\gamma \in L^\infty(S)$, $\gamma > 0$, $\int\limits_S \gamma^{\frac{1}{1-p}} d\lambda < \infty$, and: If we set $\gamma_0 : \mathbb{R}^n \to (0, \infty)$, by $\gamma_0 := \gamma$ on S, and $\gamma_0 := g_1$ on $\mathbb{R}^n \setminus S$, then $\gamma_0 \in M_p(R^n)$ (see Remark 3.2.1), with $\gamma = \gamma_0$ on S. Thus, any $\gamma \in L^1_{g_1,g_2}(S)$ fulfills all assumptions, the fixed weight function in Chapter 3 had to fulfill. In particular, $g(\omega)$ is for $\mathbb{P}$-a.e. $\omega \in \Omega$, a suitable choice as a weight function in Chapter 3. Moreover, note that $W^{1,p}_\gamma(S) = W^{1,p}(S)$ for all $\gamma \in L^1_{g_1,g_2}(S)$, where $W^{1,p}_\gamma(S)$ is the Sobolev space introduced in (3.8) .

Now, let us re-introduce the operators considered in Chapter 3 using the introductory mentioned new notation:

Definition 7.2.2. *Let $\gamma \in L^1_{g_1,g_2}(S)$ and introduce the single-valued operator $A_p^d(\gamma) : D(A_p^d(\gamma)) \to L^1(S)$ by: $(f, \hat{f}) \in A_p^d(\gamma)$ if and only if the following assertions hold.*

i) $f \in W^{1,p}(S) \cap L^\infty(S)$.

ii) $\hat{f} \in L^1(S)$.

iii) $\int\limits_S \gamma |\nabla f|^{p-2} \nabla f \cdot \nabla \varphi d\lambda = \int\limits_S \hat{f} \varphi d\lambda$ *for all* $\varphi \in W^{1,p}(S) \cap L^\infty(S)$.

Moreover, let $\mathcal{A}_p^d(\gamma) : D(\mathcal{A}_p^d(\gamma)) \to L^1(S)$ be the closure of $A_p^d(\gamma)$, i.e. $(f, \hat{f}) \in \mathcal{A}_p^d(\gamma)$, if there are sequences $((f_m, \hat{f}_m))_{m \in \mathbb{N}} \subseteq A_p^d(\gamma)$ such that $\lim\limits_{m \to \infty} f_m = f$ in $L^1(S)$ and $\lim\limits_{m \to \infty} \hat{f}_m = \hat{f}$ in $L^1(S)$.

Thus, for any $\gamma \in L^1_{g_1,g_2}(S)$, we can apply the results of Chapter 3 to $A_p^d(\gamma)$ and $\mathcal{A}_p^d(\gamma)$. Particularly, $A_p^d(\gamma)$ is indeed single-valued, see Lemma 3.3.1.
Of course, we could have given the explicit definition of $\mathcal{A}_p^d(\gamma)$ as in Definition 3.2.4, but this technical definition is not needed in this chapter.
Now, let us define the random counterparts of the operators introduced in the previous definition. Some questions concerning measurability occur during their definition. Answers to these question can be found in Appendix 7.G.

Definition 7.2.3. *Let $A_p^r : D(A_p^r) \to 2^{L^1(\Omega; L^1(S))}$ be such that $(f, \hat{f}) \in A_p^r$ if and only if the following assertions hold.*

i) $f \in L^1(\Omega; L^1(S))$ *and* $\mathbb{P}(f \in W^{1,p}(S) \cap L^\infty(S)) = 1$.

ii) $\hat{f} \in L^1(\Omega; L^1(S))$.

iii) $\mathbb{P}\left(\int\limits_S g |\nabla f|^{p-2} \nabla f \cdot \nabla \varphi d\lambda = \int\limits_S \hat{f} \varphi d\lambda,\ \forall \varphi \in W^{1,p}(S) \cap L^\infty(S) \right) = 1.$

Moreover, let $\mathcal{A}_p^r : D(\mathcal{A}_p^r) \to 2^{L^1(\Omega; L^1(S))}$ be the closure of A_p^r.

7.3 Mild Solutions of the randomized weighted p-Laplacian evolution Equation

The purpose of this section is to prove that the initial value problem (7.2) has for any $u \in L^1(\Omega; L^1(S))$ precisely one mild solution. This will be achieved by the aid of Theorem 2.1.7. That means, we have to verify that $\mathcal{A}_p^r$ is densely defined and m-accretive.

For proving this, some technical properties of $A_p^d(\gamma)$ have to be established. These technical results and their proofs have been moved to Appendix 7.H. Moreover, the denseness of the spaces $\tau(L^1(\Omega; L^1(S)))$ and $L^{1,\infty}(\Omega; L^1(S))$ in $L^1(\Omega; L^1(S))$ is also proven in Appendix 7.H.
Particularly, none of the proofs in Appendix 7.H relies on any result in this section.

The Lemmata 7.3.1-7.3.4 are essentially a collection of useful properties of the considered operators. These results are on the one hand of extreme importance for the next sections, and on the other hand, they build the path to an estimate which yields particularly the accretivity of $\mathcal{A}_p^r$, see Proposition 7.3.5. Lemma 7.3.6, together with Appendix 7.H, brings us in the position to prove that $\mathcal{A}_p^r$ is m-accretive, which is achieved in Theorem 7.3.7. Finally, it will be established that A_p^r, and a fortiori also $\mathcal{A}_p^r$, has dense domain, which then implies the first main result of this chapter, namely the existence of unique mild solutions of (7.2).

Lemma 7.3.1. *The operator A_p^r is single-valued.*

Proof. Let $(f,\hat{f}),\ (f,\tilde{f}) \in A_p^r$. Then we get

$$\int\limits_S (\hat{f}-\tilde{f})\varphi d\lambda = 0,\ \forall \varphi \in W^{1,p}(S)\cap L^\infty(S)$$

with probability one. Consequently, $\tilde{f}=\hat{f}$ a.e. on S with probability one, i.e. $\hat{f}=\tilde{f}$ as elements of $L^1(\Omega, L^1(S))$. □

Lemma 7.3.2. *Let $f,\ \hat{f} \in L^1(\Omega; L^1(S))$. The following assertions are equivalent.*

i) $(f,\hat{f}) \in A_p^r$

ii) $\mathbb{P}\left(\left\{\omega \in \Omega :\ (f(\omega),\hat{f}(\omega)) \in A_p^d(g(\omega))\right\}\right) = 1$

Proof. Let $f,\ \hat{f} \in L^1(\Omega; L^1(S))$. Then we have

$$\begin{aligned}
& \{\omega :\ (f(\omega),\hat{f}(\omega)) \in A_p^d(g(\omega))\} \\
= \quad & \{\omega :\ f \in W^{1,p}(S)\cap L^\infty(S),\ \int\limits_S g|\nabla f|^{p-2}\nabla f\cdot\nabla\varphi d\lambda = \int\limits_S \hat{f}\varphi d\lambda,\ \forall\varphi \in W^{1,p}(S)\cap L^\infty(S)\},
\end{aligned}$$

which yields, by invoking Lemma 7.G.1 and Lemma 7.G.2 that the event $\{\omega :\ (f(\omega),\hat{f}(\omega)) \in A_p^d(g(\omega))\}$ is measurable. Moreover, the former equality also yields the equivalence of i) and ii). □

Lemma 7.3.3. *Let $(f,\hat{f}) \in \mathcal{A}_p^r$. Then we have*

$$\mathbb{P}\left(\left\{\omega \in \Omega :\ (f(\omega),\hat{f}(\omega)) \in \mathcal{A}_p^d(g(\omega))\right\}\right) = 1.$$

Proof. As $(f,\hat{f}) \in \mathcal{A}_p^r$, there is, by passing to a subsequence if necessary, a sequence $((f_m,\hat{f}_m))_{m\in\mathbb{N}} \subseteq A_p^r$ such that

$$\lim_{m\to\infty}(f_m(\omega),\hat{f}_m(\omega)) = (f(\omega),\hat{f}(\omega)),\ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,\ \text{in } L^1(S)^2.$$

Consequently, Lemma 7.3.2 yields that we have, up to a $\mathbb{P}$-nullset

$$\{\omega \in \Omega :\ (f(\omega),\hat{f}(\omega)) \in \mathcal{A}_p^d(g(\omega))\}$$

$$
\begin{aligned}
&= \left\{\omega \in \Omega : \exists (F_m(\omega), \hat{F}_m(\omega))_{m\in\mathbb{N}} \subseteq A_p^d(g(\omega)),\ \lim_{m\to\infty}(F_m(\omega), \hat{F}_m(\omega)) = (f(\omega), \hat{f}(\omega)),\ \text{in } L^1(S)^2\right\} \\
&\supseteq \left\{\omega \in \Omega : \lim_{m\to\infty}(f_m(\omega), \hat{f}_m(\omega)) = (f(\omega), \hat{f}(\omega)),\ \text{in } L^1(S)^2\right\} \\
&= \Omega.
\end{aligned}
$$

Hence $\{\omega \in \Omega : (f(\omega), \hat{f}(\omega)) \in \mathcal{A}_p^d(g(\omega))\}$ is, up to a $\mathbb{P}$-nullset, equal to Ω. Therefore this event is $\mathcal{F}$-measurable, because $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, and occurs with probability one. $\square$

Lemma 7.3.4. *Let $(f, \hat{f}) \in \mathcal{A}_p^r$ and assume $f \in L^{1,\infty}(\Omega; L^1(S))$. Then $(f, \hat{f}) \in A_p^r$.*

Proof. Invoking Lemma 7.3.3 yields that $(f(\omega), \hat{f}(\omega)) \in \mathcal{A}_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. As also $f(\omega) \in L^\infty(S)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have, by virtue of Lemma 3.3.1 that $(f(\omega), \hat{f}(\omega)) \in A_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. This yields the claim by Lemma 7.3.2. $\square$

Proposition 7.3.5. *Let $(f, \hat{f}),\ (h, \hat{h}) \in \mathcal{A}_p^r$, $\alpha \in (0, \infty)$, $q \in [1, \infty]$ and assume that $f - h + \alpha(\hat{f} - \hat{h}) \in L^q(S)$ with probability one. Then we have*

$$
\mathbb{P}\left(\left\{\omega \in \Omega : ||f(\omega) - h(\omega)||_{L^q(S)} \leq ||f(\omega) - h(\omega) + \alpha(\hat{f}(\omega) - \hat{h}(\omega))||_{L^q(S)}\right\}\right) = 1. \tag{7.6}
$$

Particularly, A_p^r as well as $\mathcal{A}_p^r$, are accretive.

Proof. We have, by virtue of Lemma 7.3.3, that $(f(\omega), \hat{f}(\omega)), (h(\omega), \hat{h}(\omega)) \in \mathcal{A}_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. This implies (7.6), since $\mathcal{A}_p^d(g(\omega))$ is for $\mathbb{P}$-a.e. $\omega \in \Omega$, a completely accretive operator, see Theorem 3.2.5.
Moreover, (7.6) yields particularly that $||f - h||_{L^1(\Omega;L^1(S))} \leq ||f - h + \alpha(\hat{f} - \hat{h})||_{L^1(\Omega;L^1(S))}$, i.e. $\mathcal{A}_p^r$ is accretive. This obviously implies that A_p^r is accretive as well. $\square$

Lemma 7.3.6. *Assume that g is simple, i.e. there is an $m \in \mathbb{N}$, $\gamma_1, ..., \gamma_m \in L^1(S)$ and disjoint sets $\Omega_1, ..., \Omega_m \in \mathcal{F}$, such that $\bigcup_{k=1}^{m} \Omega_k = \Omega$ and*

$$
g(\cdot) = \sum_{k=1}^{m} \gamma_k \mathbb{1}_{\Omega_k}(\cdot). \tag{7.7}
$$

Moreover, let $h \in L^{1,\infty}(\Omega; L^1(S))$. Then the mapping defined by

$$
\Omega \ni \omega \mapsto (Id + A_p^d(g(\omega)))^{-1} h(\omega)
$$

is $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable.[1]

Proof. Let $h \in L^{1,\infty}(\Omega; L^1(S))$ be arbitrary but fixed and assume that g is given by (7.7). Moreover, assume w.l.o.g. that none of the Ω_k is a $\mathbb{P}$-nullset.
Since $g_1 \leq g \leq g_2$ a.e. on S with probability one, it is clear that $g_1 \leq \gamma_k \leq g_2$ a.e. on S for each $k = 1, ..., m$.

[1] See Remark 2.1.9 for the definition of the resolvent $(Id + A_p^d(g(\omega)))^{-1}$.

Moreover, as $h(\omega) \in L^\infty(S)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, Lemma 7.H.1.i) yields that the mapping $\varphi : \Omega \to L^1(S)$ defined by

$$\varphi(\omega) := (Id + A_p^d(g(\omega)))^{-1} h(\omega), \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega$$

is well-defined.
For a given $k \in \{1, ..., m\}$ and all $\omega \in \Omega_k$ except for a $\mathbb{P}$-null-set, we have $\varphi(\omega) = (Id + A_p^d(\gamma_k))^{-1} h(\omega)$. Consequently, Lemma 7.H.1.ii) yields

$$\varphi(\omega) = \sum_{k=1}^m \mathbb{1}_{\Omega_k}(\omega)(Id + A_p^d(\gamma_k))^{-1} h(\omega) = \sum_{k=1}^m \mathbb{1}_{\Omega_k}(\omega)(Id + \mathcal{A}_p^d(\gamma_k))^{-1} h(\omega), \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Moreover, $(Id + \mathcal{A}_p^d(\gamma_k))^{-1}$ is $L^1(S)$-continuous for all $k = 1, ..., m$, see Lemma 7.H.1.iii). Hence, the mapping $(Id + \mathcal{A}_p^d(\gamma_k))^{-1}$ is $\mathfrak{B}(L^1(S)) - \mathfrak{B}(L^1(S))$-measurable. As $h \in L^1(\Omega; L^1(S))$, it follows that $\Omega \ni \omega \mapsto (Id + \mathcal{A}_p^d(\gamma_k))^{-1} h(\omega)$ is $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable as it is the composition of measurable functions. Consequently, $\Omega \ni \omega \mapsto \mathbb{1}_{\Omega_k}(\omega)(Id + \mathcal{A}_p^d(\gamma_k))^{-1} h(\omega)$ is also $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable, which yields the measurability of φ. □

Theorem 7.3.7. *We have*

$$L^{1,\infty}(\Omega; L^1(S)) \subseteq R(Id + A_p^r). \tag{7.8}$$

Consequently, the following assertions hold.

i) $R(Id + A_p^r)$ *is a dense subset of* $L^1(\Omega; L^1(S))$.

ii) $\mathcal{A}_p^r$ *is m-accretive.*

Proof. Lemma 7.H.3 yields that (7.8) implies i). Moreover, it is plain that $R(Id + A_p^r) \subseteq R(Id + \mathcal{A}_p^r)$. Consequently, (7.8) implies that $R(Id + \mathcal{A}_p^r)$ is also dense. Moreover, as $\mathcal{A}_p^r$ is accretive and closed, we have that $R(Id + \mathcal{A}_p^r)$ is closed, cf. [8, Proposition 2.18]. Consequently, (7.8) implies i) as well as ii).
Now prove inclusion (7.8). Let $h \in L^{1,\infty}(\Omega; L^1(S))$.
Let $f : \Omega \to L^1(S)$ be defined by $f(\omega) := (Id + A_p^d(g(\omega)))^{-1} h(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. This is well-defined by Lemma 7.H.1.i).
Now introduce $\hat{f} : \Omega \to L^1(S)$ by $\hat{f}(\omega) := A_p^d(g(\omega)) f(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.
Trivially, $h = f + \hat{f}$ by construction. Consequently the claim follows if $(f, \hat{f}) \in A_p^r$. Proving this result is divided in the following steps.

(I) f as well as $\hat{f}$ are $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable.
(II) $f,\ \hat{f} \in L^1(\Omega; L^1(S))$.
(III) $(f, \hat{f}) \in A_p^r$.

Proof of (I). Since $\hat{f} = h - f$ it suffices to prove that f is $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable.
As $g \in L^1_{g_1,g_2}(\Omega; L^1(S))$, there is a sequence of simple functions $(\gamma_m)_{m\in\mathbb{N}} \subseteq L^1_{g_1,g_2}(\Omega; L^1(S))$ such that $\lim_{m\to\infty} \gamma_m(\omega) = g(\omega)$ in $L^1(S)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Consequently, it follows by virtue of Lemma 7.H.2 that

$$f(\omega) = (Id + A_p^d(g(\omega)))^{-1}h(\omega) = \text{w - }\lim_{m\to\infty}(Id + A_p^d(\gamma_m(\omega)))^{-1}h(\omega), \text{ in } L^1(S) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

i.e. f is, by Lemma 7.3.6, almost surely the $L^1(S)$-weak limit of $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable functions and consequently it is itself $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable.
Proof of (II). As particularly $h \in L^1(\Omega; L^1(S))$, it suffices to prove that $f \in L^1(\Omega; L^1(S))$.
The needed measurability condition has been proven in (I).
As $(f(\omega), \hat{f}(\omega)) \in A_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and as $A_p^d(g(\omega))$ is completely accretive, we obtain in particular

$$\int_S |f(\omega)| d\lambda \leq \int_S |f(\omega) + \hat{f}(\omega)| d\lambda = \int_S |h(\omega)| d\lambda \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

This obviously implies $\int_\Omega ||f(\omega)||_{L^1(S)} d\mathbb{P}(\omega) \leq \int_\Omega ||h(\omega)||_{L^1(S)} d\mathbb{P}(\omega) < \infty$, i.e. $f \in L^1(\Omega; L^1(S))$.
Proof of (III). We have $f, \hat{f} \in L^1(\Omega; L^1(S))$ and trivially $(f(\omega), \hat{f}(\omega)) \in A_p^d(g(\omega))$ $\mathbb{P}$-a.e. $\omega \in \Omega$, which yields (III) by Lemma 7.3.2. □

Lemma 7.3.8. *$D(A_p^r)$ as well as $D(\mathcal{A}_p^r)$ are dense subsets of $(L^1(\Omega; L^1(S)), ||\cdot||_{L^1(\Omega;L^1(S))})$.*

Proof. As $D(A_p^r) \subseteq D(\mathcal{A}_p^r)$, it suffices to prove the claim for $D(A_p^r)$. Moreover, Lemma 7.H.3 yields that it suffices to prove that

$$\tau(L^1(\Omega; L^1(S))) \subseteq \overline{D(A_p^r)}^{L^1(\Omega;L^1(S))}.$$

Let $h \in \tau(L^1(\Omega; L^1(S)))$ and introduce $k \in (0, \infty)$, $\tilde{h} \in L^1(\Omega; L^1(S))$ such that $h = \tau_k(\tilde{h})$.
As $\mathcal{A}_p^r$ is m-accretive there is for each $m \in \mathbb{N}$ a uniquely determined pair of functions $(f_m, \hat{f}_m) \in \mathcal{A}_p^r$, such that

$$h = f_m + \frac{1}{m}\hat{f}_m.$$

Moreover, the last equation yields, by observing that obviously $(0,0) \in \mathcal{A}_p^r$ and by recalling Proposition 7.3.5 that

$$||f_m(\omega)||_{L^\infty(S)} \leq ||f_m(\omega) + \frac{1}{m}\hat{f}_m(\omega)||_{L^\infty(S)} = ||h(\omega)||_{L^\infty(S)} \leq k, \ \forall m \in \mathbb{N} \ \text{ and for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Consequently, we have in particular $f_m \in L^{1,\infty}(\Omega; L^1(S))$ and hence it follows, by invoking Lemma 7.3.4 that $(f_m, \hat{f}_m) \in A_p^r$ for each $m \in \mathbb{N}$.
Hence the claim follows if we prove that

$$\lim_{m\to\infty} f_m = h, \text{ in } L^1(\Omega; L^1(S)). \tag{7.9}$$

Firstly, $(f_m, \hat{f}_m) \in A_p^r$ yields $(f_m(\omega), \hat{f}_m(\omega)) \in A_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, cf. Lemma 7.3.2. Moreover, $h(\omega) \in L^\infty(S)$, $h(\omega) = f_m(\omega) + \frac{1}{m}\hat{f}(\omega)$, i.e. $f_m(\omega) = (Id + \frac{1}{m}A_p^d(g(\omega)))^{-1}h(\omega)$ for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Consequently, we obtain by virtue of Remark 3.5.5 and Lemma 7.H.1.ii) that

$$\lim_{m\to\infty} ||f_m(\omega) - h(\omega)||_{L^1(S)} = 0, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Finally, observe that $||f_m(\omega) - h(\omega)||_{L^1(S)} \leq 2k\lambda(S)$ for all $m \in \mathbb{N}$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, which yields (7.9), by virtue of dominated convergence. □

Theorem 7.3.9. *$\mathcal{A}_p^r$ is densely defined and m-accretive. Thus, for any $u \in L^1(\Omega; L^1(S))$, the initial value problem (7.2) has a uniquely determined mild solution.*

Proof. Follows from Theorem 2.1.7, Theorem 7.3.7 and Lemma 7.3.8. □

7.4 Strong Solutions of the randomized weighted p-Laplacian evolution Equation

Throughout everything which follows $(T_{\mathrm{ra}}(t))_{t\geq 0}$ denotes the semigroup associated to $\mathcal{A}_p^r$, see Definition 2.1.8. Moreover, for any $\gamma \in L^1_{g_1,g_2}(S)$, we denote by $(T_{\mathrm{det}}(t,\gamma))_{t\geq 0}$ the semigroup associated to $\mathcal{A}_p^d(\gamma)$. Consequently, $(T_{\mathrm{det}}(t,\gamma))_{t\geq 0}$ is precisely the semigroup we considered in Chapter 3 - A fact which will be exploited frequently in all of the following sections.

The prime objective of this section is to establish that $t \mapsto T_{\mathrm{ra}}(t)u$ is, for any $u \in L^1(\Omega; L^1(S))$, not only a mild, but also a strong solution of (7.2), which will be achieved by the aid of Theorem 2.1.12 and the results in Chapter 3. Moreover, we will then also derive some basic properties of $(T_{\mathrm{ra}}(t))_{t\geq 0}$.
Let us start with the following useful result connecting the deterministic and the random semigroup:

Theorem 7.4.1. *Let $u \in L^1(\Omega; L^1(S))$ and $t \in [0,\infty)$. Then we have*

$$\mathbb{P}\left(\{\omega \in \Omega : \ (T_{ra}(t)u)(\omega) = T_{det}(t, g(\omega))u(\omega)\}\right) = 1.$$

Proof. Let $u \in L^1(\Omega; L^1(S))$ and $\tilde{t} \in (0,\infty)$.
Firstly, it will be proven inductively that

$$((Id + \tilde{t}\mathcal{A}_p^r)^{-m}u)(\omega) = (Id + \tilde{t}\mathcal{A}_p^d(g(\omega)))^{-m}(u(\omega)), \ \forall m \in \mathbb{N} \text{ and for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{7.10}$$

So let $m = 1$ and introduce $f := (Id + \tilde{t}\mathcal{A}_p^r)^{-1}u$. Consequently, there is an $\hat{f} \in \mathcal{A}_p^r f$ such that $f + \tilde{t}\hat{f} = u$. As $(f, \hat{f}) \in \mathcal{A}_p^r$, we have $(f(\omega), \hat{f}(\omega)) \in \mathcal{A}_p^d(g(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, see Lemma 7.3.3.
Since obviously $f(\omega) + \tilde{t}\hat{f}(\omega) = u(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ we obtain that $f(\omega) = (Id + \tilde{t}\mathcal{A}_p^d(g(\omega)))^{-1}(u(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and consequently

$$((Id + \tilde{t}\mathcal{A}_p^r)^{-1}u)(\omega) = f(\omega) = (Id + \tilde{t}\mathcal{A}_p^d(g(\omega)))^{-1}(u(\omega)) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

i.e. (7.10) is proven for $m = 1$. The proof of the induction step works analogously and will be skipped.

Now let $t \in [0, \infty)$ be given and choose $\tilde{t} := \frac{t}{m}$ in (7.10). Then we get

$$\left(\left(Id + \frac{t}{m}\mathcal{A}_p^r\right)^{-m} u\right)(\omega) = \left(Id + \frac{t}{m}\mathcal{A}_p^d(g(\omega))\right)^{-m} u(\omega),\ \forall m \in \mathbb{N} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{7.11}$$

Moreover, the exponential formula (Theorem 2.1.10) yields, by passing to a subsequence if necessary, that

$$\lim_{m\to\infty} \left(\left(Id + \frac{t}{m}\mathcal{A}_p^r\right)^{-m} u\right)(\omega) = (T_{\mathrm{ra}}(t)u)(\omega),\ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ in } L^1(S).$$

Analogously, we also have by virtue of the exponential formula that

$$\lim_{m\to\infty} \left(Id + \frac{t}{m}\mathcal{A}_p^d(g(\omega))\right)^{-m} u(\omega) = T_{\det}(t, g(\omega))u(\omega),\ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \text{ in } L^1(S),$$

which yields the claim. $\square$

Theorem 7.4.2. *Let $u \in L^1(\Omega; L^1(S))$. Then $(T_{ra}(\cdot)u)|_{(0,\infty)}$ is locally Lipschitz continuous and right differentiable. Thus, $(T_{ra}(\cdot)u)|_{(0,\infty)} \in W^{1,1}_{Loc}((0,\infty); L^1(\Omega; L^1(S)))$ and*

$$0 \in T_{ra}'(t)u + \mathcal{A}_p^r T_{ra}(t)u,\ \text{for a.e. } t \in (0,\infty),\ T_{ra}(0)u = u,$$

i.e. $T_{ra}(\cdot)u$ is not only the mild, but also the uniquely determined strong solution of (7.2).

Proof. Thanks to Theorem 2.1.12 (and Theorem 7.3.9) it indeed suffices to prove the local Lipschitz continuity and the right differentiability on $(0, \infty)$.
The desired Lipschitz continuity follows directly from Remark 3.2.6.vi) (with $q = 1$) and Theorem 7.4.1, more precisely: Let $[\varepsilon_1, \varepsilon_2] \subseteq (0, \infty)$, then the two aforementioned results yield: For all $t, t+h \in [\varepsilon_1, \varepsilon_2]$, where $h \geq 0$ we have

$$||(T_{\mathrm{ra}}(t+h)u)(\omega) - (T_{\mathrm{ra}}(t)u)(\omega)||_{L^1(S)} \leq h\frac{2}{|p-2|t}||u(\omega)||_{L^1(S)} \leq h\frac{2}{|p-2|\varepsilon_1}||u(\omega)||_{L^1(S)} \tag{7.12}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$; which of course implies

$$||T_{\mathrm{ra}}(t+h)u - T_{\mathrm{ra}}(t)u||_{L^1(\Omega;L^1(S))} \leq h\frac{2}{|p-2|\varepsilon_1}||u||_{L^1(\Omega;L^1(S))},$$

for all $t, t+h \in [\varepsilon_1, \varepsilon_2]$, with $h \geq 0$.
To prove the desired right differentiability, let $(h_m)_{m\in\mathbb{N}} \subseteq (0,\infty)$ be a null-sequence, and let, for any $\gamma \in L^1_{g_1,g_2}(S)$, $\mathcal{A}_p^d(\gamma)^\circ : L^1(S) \to L^1(S)$ denote the infinitesimal generator of $T_{\det}$, which exists, see Remark 3.2.6.v).

Firstly, note that the domain invariance of $T_{\text{det}}(\cdot,\gamma)$ yields

$$\lim_{m\to\infty}\frac{1}{h_m}\left(T_{\text{det}}(t+h_m,\gamma)v - T_{\text{det}}(t,\gamma)v\right) = -\mathcal{A}_p^d(\gamma)^{\circ}T_{\text{det}}(t,\gamma)v, \tag{7.13}$$

in $L^1(S)$, for all $v \in L^1(S)$, $\gamma \in L^1_{g_1,g_2}(S)$ and $t > 0$. Now introduce the mapping $\zeta : (0,\infty) \to L^1(\Omega, L^1(S))$ by $\zeta(t)(\omega) := -\mathcal{A}_p^d(g(\omega))^{\circ}T_{\text{det}}(t, g(\omega))u(\omega)$, for all $t > 0$ and $\mathbb{P}$-a.e, $\omega \in \Omega$, which is indeed well-defined, since: For each $t > 0$ and $\omega \in \Omega$, with $g_1 \le g(\omega) \le g_2$, $\zeta(\tau)(\omega)$ exists. Moreover, (7.13), together with Theorem 7.4.1, implies

$$\lim_{m\to\infty}\left|\left|\frac{1}{h_m}\Big((T_{\text{ra}}(t+h_m)u)(\omega) - (T_{\text{ra}}(t)u)(\omega)\Big) - \zeta(t)(\omega)\right|\right|_{L^1(S)} = 0,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and any fixed $t > 0$. Thus, each $\zeta(t)$ is $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable, since it is the almost sure limit of $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable functions and $(\Omega,\mathcal{F},\mathbb{P})$ is complete. Moreover, thanks to (7.12) and $u \in L^1(\Omega; L^1(S))$, we can apply dominated convergence to the preceding equation, which yields that indeed $\zeta(t) \in L^1(\Omega; L^1(S))$ and

$$\lim_{m\to\infty}\left|\left|\frac{1}{h_m}\Big(T_{\text{ra}}(t+h_m)u - T_{\text{ra}}(t)u\Big) - \zeta(t)\right|\right|_{L^1(\Omega;L^1(S))} = 0,$$

for all $t > 0$, which implies that $T_{\text{ra}}(\cdot)u$ is right differentiable on $(0,\infty)$. □

Proposition 7.4.3. *Let $u, u_1, u_2 \in L^1(\Omega; L^1(S))$, $q \in [1,\infty]$ and $t \in [0,\infty)$. Moreover, assume that $u, u_1, u_2 \in L^q(S)$ a.s. Then the following assertions hold.*

i) $\mathbb{P}\left(\left\{\omega \in \Omega :\ ||(T_{ra}(t)u_1)(\omega) - (T_{ra}(t)u_2)(\omega)||_{L^q(S)} \le ||u_1(\omega) - u_2(\omega)||_{L^q(S)}\right\}\right) = 1,$

ii) $\mathbb{P}\left(\left\{\omega \in \Omega :\ ||(T_{ra}(t)u)(\omega)||_{L^q(S)} \le ||u(\omega)||_{L^q(S)}\right\}\right) = 1.$

Proof. Follows trivially from Remark 3.2.6.iii),iv) and Theorem 7.4.1. □

Theorem 7.4.4. *Let $u \in L^{1,\infty}(\Omega; L^1(S))$. Then we have*

$$-T'_{ra}(t)u = A_p^r T_{ra}(t)u, \text{ for a.e. } t \in (0,\infty). \tag{7.14}$$

Proof. Theorem 7.4.2 yields that $(T_{\text{ra}}(t)u, -T'_{\text{ra}}(t)u) \in \mathcal{A}_p^r$ for a.e. $t \in (0,\infty)$. Consequently, it follows by virtue of Lemma 7.3.4 that it suffices to prove $\mathbb{P}(T_{\text{ra}}(t)u \in L^\infty(S)) = 1$, for a.e. $t \in (0,\infty)$. But this is a trivial consequence of Proposition 7.4.3.ii). □

Theorem 7.4.4 finishes the discussion on existence and uniqueness results. The remaining part of this chapter is devoted to determine the asymptotic behavior of $(T_{\text{ra}}(t))_{t\ge 0}$.

7.5 Stability Results

This section opens the investigation on asymptotic results regarding T_{ra}. Thanks to Theorem 7.4.1, it is straightforward to transfer the asymptotic results from Chapter 3 to the current setting.

As in Chapter 3, we denote by $\overline{(v)}_S$ the average of any $v \in L^1(S)$, i.e. $\overline{(v)}_S := \frac{1}{\lambda(S)} \int_S v d\lambda$. Moreover, for any $q \in [1, \infty)$, we denote by $C_{S,q}$ the Poincaré constant of S in $L^q(S)$, see Remark 3.4.1.

Now, for any $u : \Omega \to L^1(S)$, introduce the real-valued random variable $\overline{(u)}_S : \Omega \to \mathbb{R}$, by $\overline{(u)}_S(\omega) := \overline{(u(\omega))}_S$. Moreover, for any $u \in L^1(\Omega; L^1(S))$, with $\mathbb{P}(u \in L^2(S)) = 1$, we denote by $\Delta_u : \Omega \to [0, \infty)$ the real-valued random variable defined by

$$\Delta_u(\omega) := ||u(\omega) - \overline{(u(\omega))}_S||^2_{L^2(S)},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Theorem 7.5.1. *Let $u \in L^1(\Omega; L^1(S))$. Then $\overline{(T_{ra}(t)u)}_S = \overline{(u)}_S$ almost surely, for all $t \in [0, \infty)$. Moreover, all of the following assertions hold.*

i) If $u \in L^2(S)$ a.s., then $||T_{ra}(t)u - \overline{(u)}_S||_{L^1(S)} \leq C_{S,1} \lambda(S)^{\frac{p-1}{p}} \left(\frac{2}{g_1|p-2|}\right)^{\frac{1}{p}} \Delta_u^{\frac{1}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}$ a.s. for all $t > 0$.

ii) If $u \in L^p(S)$ a.s. and $p > n$, then $||T_{ra}(t)u - \overline{(u)}_S||_{L^\infty(S)} \leq C^*_{S,\delta} \lambda(S)^{\frac{1}{1+\delta}} \left(\frac{2}{\lambda(S) g_1 |p-2|}\right)^{\frac{1}{p}} \Delta_u^{\frac{1}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}}$, a.s. for all $t > 0$ and $\delta \in (n-1, p-1)$ where $C^*_{S,\delta} = \tilde{C}_{S,1+\delta} \left(C^{1+\delta}_{S,1+\delta} + 1\right)^{\frac{1}{1+\delta}}$, and $\tilde{C}_{S,1+\delta}$ is the operator norm of the continuous injection $W^{1,1+\delta}(S) \hookrightarrow L^\infty(S)$.

iii) If $p \in \left(\frac{(n-2)}{n+2} + 1, 2\right) \neq \emptyset$ and $u \in L^2(S)$ a.s., then $||T_{ra}(t)u - \overline{(u)}_S||^{2-p}_{L^2(S)} \leq \left(-\hat{\kappa}_1 t + \Delta_u^{\frac{2-p}{2}}\right)_+$ a.s., for all $t > 0$, where $\hat{\kappa}_1 := (2-p) \left(\tilde{C}_S^p \left(C^{\frac{2n}{n+2}}_{S,\frac{2n}{n+2}} + 1\right)^{\frac{np+2p}{2n}} \lambda(S)^{\frac{np+2p-2n}{2n}} g_1^{-1}\right)^{-1}$ and $\tilde{C}_S$ is the operator norm of the continuous injection $W^{1,\frac{2n}{n+2}} \hookrightarrow L^2(S)$.

iv) If $p \in (2, \infty)$, then $||T_{ra}(t)u - \overline{(u)}_S||_{L^1(S)} \leq \lambda(S)^{\frac{1}{2}} \hat{\kappa}_2^{\frac{1}{2-p}} \left(\frac{1}{t}\right)^{\frac{1}{p-2}}$ a.s., for all $t > 0$, where we set $\hat{\kappa}_2 := (p-2) 2^{2-p} g_1 \lambda(S)^{\frac{2-p}{2}} C^{-p}_{S,2}$.

Proof. That $\overline{(T_{\mathrm{ra}}(t)u)}_S = \overline{(u)}_S$ a.s., for all $t \in [0, \infty)$, follows from Lemma 3.3.5 and Theorem 7.4.1. Now, let us prove i). Fix $t > 0$ and assume $u \in L^2(S)$ a.s. Then it follows from Theorem 7.4.1 and Corollary 3.4.9 that

$$||(T_{\mathrm{ra}}(t)u)(\omega) - \overline{(u)}_S(\omega)||_{L^1(S)} \leq C_{S,1} \left(\int_S g(\omega)^{\frac{1}{1-p}} d\lambda\right)^{\frac{p-1}{p}} \left(\frac{2}{|p-2|}\right)^{\frac{1}{p}} \Delta_u(\omega)^{\frac{1}{p}} \left(\frac{1}{t}\right)^{\frac{1}{p}},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, which yields i) since $g \geq g_1$ almost surely.
The claims ii)-iv) are all proven identically to the proof of i): One uses $g \geq g_1$ a.s., together with Theorem 7.4.1 to deduce: ii) from Theorem 3.4.10, iii) from Theorem 3.5.6 and iv) from 3.5.10. $\square$

In contrary to the deterministic case Theorem 7.5.1.iii) does not imply that the semigroup extincts in finite time, i.e.: Under the assumptions of Theorem 7.5.1.iii), there of course is for $\mathbb{P}$-a.e. $\omega \in \Omega$, a time $t(\omega) \in (0,\infty)$, which depends on ω, such that $||T_{\mathrm{ra}}(t(\omega))u(\omega) - \overline{(u)}_S(\omega)||_{L^2(S)} = 0$, but this does not necessarily imply $||T_{\mathrm{ra}}(t^*)u - \overline{(u)}_S||_{L^1(\Omega;L^2(S))} = 0$ for a deterministic constant $t^* \in (0,\infty)$.
Now let us conclude this short section by deriving an analogous version of Theorem 3.4.13:

Theorem 7.5.2. *Let $q \in [1,\infty)$ and $u \in L^q(\Omega; L^q(S))$. Then we have*

$$\lim_{t\to\infty} T_{ra}(t)u = \overline{(u)}_S, \text{ in } L^q(\Omega; L^q(S)). \tag{7.15}$$

Proof. Let $(t_m)_{m\in\mathbb{N}} \subseteq (0,\infty)$ be such that $\lim_{m\to\infty} t_m = \infty$. Then appealing to Theorem 3.4.13 and Theorem 7.4.1 yields

$$\lim_{m\to\infty} ||T_{\mathrm{ra}}(t_m)u - \overline{(u)}_S||^q_{L^q(S)} = 0,$$

almost surely. Moreover, Remark 3.2.6.iv) and Theorem 7.4.1 enable us to conclude that

$$||T_{\mathrm{ra}}(t_m)u - \overline{(u)}_S||^q_{L^q(S)} \leq (||u||_{L^q(S)} + ||\overline{(u)}_S||_{L^q(S)})^q, \ \forall m \in \mathbb{N},$$

almost surely. Thus, as the right hand side of the previous inequality is an element of $L^1(\Omega)$, it follows from Lebesgue's theorem that

$$\lim_{m\to\infty} \int_\Omega ||T_{\mathrm{ra}}(t_m)u - \overline{(u)}_S||^q_{L^q(S)} d\mathbb{P} = 0,$$

which means that (7.15) holds. $\square$

7.6 Decay Estimates of the Tail Function for "small" p.

The purposes of this Section is to prove the estimates (7.4) and (7.5).

Remark 7.6.1. *Let $v \in L^2(S)$ and $\gamma \in L^1_{g_1,g_2}(S)$. Throughout the remaining part of this section $h_{v,\gamma} : [0,\infty) \to [0,\infty)$ denotes the function defined by*

$$h_{v,\gamma}(t) := \log\left(\int_S \left(T_{det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda + 1\right)$$

for any $t \in [0,\infty)$.

The basic technique to obtain a bound on the tail function of $||T_{\mathrm{ra}}(t)u - \overline{(u)}_S||^2_{L^2(S)}$ is as follows: We use Markov's inequality to bound the tail function by $\frac{\mathbb{E}(\log(||T_{\mathrm{ra}}(t)u - u\overline{(u)}_S||^2_{L^2(S)}+1))}{\log(\alpha+1)}$. And afterwards we use Theorem 7.4.1 together with an upper bound on $h_{v,\gamma}$ to get an upper bound on the tail function of $||T_{\mathrm{ra}}(t)u - u\overline{(u)}_S||^2_{L^2(S)}$. Finally, some technical calculations yield the results (7.4) and (7.5).
The following well known lemma (which is a version of Grönwall's inequality) builds the foundation for bounding $h_{v,\gamma}$. The proof works exactly like the proof of Lemma 3.5.1 and will be omitted.

Lemma 7.6.2. *Let $h : [0,\infty) \to [0,\infty)$ be locally Lipschitz continuous. Moreover, set $b := h(0)$ and assume that there is a $\beta > 0$ such that*

$$h'(t) + \beta h(t) \le 0, \text{ for a.e. } t \in (0,\infty). \tag{7.16}$$

Then we have

$$h(t) \le b\exp(-\beta t)$$

for all $t \in [0,\infty)$.

Remark 7.6.3. *Recall that $C_{S,q}$ denotes the Poincaré constant of S in $L^q(S)$, $q \in [1,\infty)$. In addition, $\tilde{C}_{S,\frac{2n}{n+2}}$ denotes the operator norm of the continuous injection $W^{1,\frac{2n}{n+2}}(S) \hookrightarrow L^2(S)$. Note that $\frac{2n}{n+2} < n$, consequently Sobolev's embedding theorem yields the existence of such an injection.*

Lemma 7.6.4. *Let $\gamma \in L^1_{g_1,g_2}(S)$ and introduce $v \in D(A^d_p(\gamma))$. Then $h_{v,\gamma}$ is locally Lipschitz continuous. Moreover, $T_{det}(t,\gamma)v \in W^{1,p}(S)$ for every $t \in (0,\infty)$ and*

$$h'_{v,\gamma}(t) \le -2\int_S \gamma|\nabla T_{det}(t,\gamma)v|^p d\lambda \left(\int_S \left(v - \overline{(v)}_S\right)^2 d\lambda + 1\right)^{-1} \tag{7.17}$$

for a.e. $t \in (0,\infty)$.

Proof. At first the local Lipschitz continuity will be established. Let $\tau > 0$ be given. Then appealing to Lemma 3.5.3 and Lemma 3.3.3 gives that the mapping defined by $[0,\tau] \ni t \mapsto \int_S \left(T_{\mathrm{det}}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda$ is Lipschitz continuous. This, together with the commonly known inequality

$$|\log(x+1) - \log(y+1)| \le |x-y|, \ \forall x,y \in [0,\infty).$$

yields the Lipschitz continuity of $h_{v,\gamma}|_{[0,\tau]}$.

As $v \in D(A^d_p(\gamma)) \subseteq L^\infty(S)$, we get by Remark 3.2.6.iv),v) that $T_{\mathrm{det}}(t,\gamma)v \in D(\mathcal{A}^d_p(\gamma)) \cap L^\infty(S)$ for all $t \in (0,\infty)$. Thus, employing Lemma 3.3.1 yields $T_{\mathrm{det}}(t,\gamma)v \in D(A) \subseteq W^{1,p}(S)$ for all $t \in (0,\infty)$.

Consequently, it remains to prove (7.17). Firstly, we infer from Lemma 3.5.3 and Lemma 3.3.3 that

$$\frac{\partial}{\partial t}\int\limits_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda = -2\int\limits_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda, \text{ for a.e. } t\in(0,\infty). \tag{7.18}$$

Moreover, it follows from Lemma 3.3.3 and Remark 3.2.6.iv) that

$$\int\limits_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda \le \int\limits_S \left(v - \overline{(v)}_S\right)^2 d\lambda, \ \forall t\in[0,\infty).$$

This, together with (7.18), yields

$$\frac{\partial}{\partial t}h_{v,\gamma}(t) = \frac{-2\int_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda}{\int_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda + 1} \le -2\int\limits_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda \left(\int\limits_S \left(v - \overline{(v)}_S\right)^2 d\lambda + 1\right)^{-1}$$

for a.e. $t\in(0,\infty)$. □

Lemma 7.6.5. *Let $\gamma \in L^1_{g_1,g_2}(S)$, introduce $v\in D(A^d_p(\gamma))$ and set $m := \frac{2n}{n+2}$. Moreover, assume $p\in[m,2)\setminus\{1\}$. Then we have*

$$h_{v,\gamma}(t) \le -h'_{v,\gamma}(t)\frac{1}{p}\max\left(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}}\lambda(S)^{\frac{p-m}{p}}, 1\right)\left(\int\limits_S \left(v-\overline{(v)}_S\right)^2 d\lambda + 1\right),$$

for a.e. $t\in(0,\infty)$.

Proof. Firstly, we infer from Sobolev's embedding theorem and $\overline{(T_{\det}(t,\gamma)v)}_S = \overline{(v)}_S$ that

$$\int\limits_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda \le \tilde{C}^2_{S,m}\left(||T_{\det}(t,\gamma)v - \overline{(v)}_S||^m_{L^m(S)} + ||\nabla T_{\det}(t,\gamma)v||^m_{L^m(S;\mathbb{R}^n)}\right)^{\frac{2}{m}}.$$

Using this and Poincaré's inequality yields

$$\int\limits_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda \le \tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}}\left(\int\limits_S |\nabla T_{\det}(t,\gamma)v|^m d\lambda\right)^{\frac{2}{m}}, \ \forall t\in[0,\infty), \tag{7.19}$$

which is finite as $p\ge m$ and $T_{\det}(t,\gamma)v\in W^{1,p}(S)$, by Lemma 7.6.4.
Consequently, it follows from $p\ge m$, $\gamma\ge g_1$, (7.19) as well as Hölder's inequality that

$$h_{v,\gamma}(t) \le \log\left(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}}\lambda(S)^{\frac{p-m}{p}}\left(\int\limits_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda\right)^{\frac{2}{p}} + 1\right), \ \forall t\in[0,\infty).$$

Now it is plain that $\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}} \lambda(S)^{\frac{p-m}{p}} \le \max(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}} \lambda(S)^{\frac{p-m}{p}}, 1)$ and hence employing Bernoulli's inequality yields

$$h_{v,\gamma}(t) \le \max\left(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}} \lambda(S)^{\frac{p-m}{p}}, 1\right) \log\left(\left(\int_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda\right)^{\frac{2}{p}} + 1\right) \tag{7.20}$$

for all $t \in [0,\infty)$.
Consequently, (7.20) and the well known inequalities $x^{\frac{2}{p}} + 1 = x^{\frac{2}{p}} + 1^{\frac{2}{p}} \le (x+1)^{\frac{2}{p}}$ and $\log(x+1) \le x$ for all $x \ge 0$ imply

$$h_{v,\gamma}(t) \le \max\left(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}} \lambda(S)^{\frac{p-m}{p}}, 1\right) \frac{2}{p} \int_S \gamma|\nabla T_{\det}(t,\gamma)v|^p d\lambda.$$

Finally, the claim follows from the preceding inequality and (7.17). $\square$

Lemma 7.6.6. *Let $\gamma \in L^1_{g_1,g_2}(S)$, introduce $v \in L^2(S)$, set $m := \frac{2n}{n+2}$ and assume $p \in [m,2) \setminus \{1\}$. Then we have*

$$\log\left(\int_S \left(T_{det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda + 1\right) \le 2||v - \overline{(v)}_S||_{L^2(S)} \exp\left(\frac{-C^*_{S,m,p,g_1} t}{1 + ||v - \overline{(v)}_S||^2_{L^2(S)}}\right), \tag{7.21}$$

for every $t \in [0,\infty)$, where

$$C^*_{S,m,p,g_1} = p\left(\max\left(\tilde{C}^2_{S,m}\left(C^m_{S,m}+1\right)^{\frac{2}{m}} g_1^{-\frac{2}{p}} \lambda(S)^{\frac{p-m}{p}}, 1\right)\right)^{-1}. \tag{7.22}$$

Proof. Firstly, assume $v \in D(A^d_p(\gamma))$ and introduce $\beta := C^*_{S,m,p,g_1}\left(1 + ||v - \overline{(v)}_S||^2_{L^2(S)}\right)^{-1}$. Then we have, by recalling Lemma 7.6.5, that $h'_{v,\gamma}(t) + \beta h_{v,\gamma}(t) \le 0$ for a.e. $t \in (0,\infty)$ which yields, by invoking Lemma 7.6.2 that $h_{v,\gamma}(t) \le h_{v,\gamma}(0)\exp(-\beta t)$ for every $t \in [0,\infty)$ and therefore

$$\log\left(\int_S \left(T_{\det}(t,\gamma)v - \overline{(v)}_S\right)^2 d\lambda + 1\right) \le \log\left(\int_S \left(v - \overline{(v)}_S\right)^2 d\lambda + 1\right) \exp\left(\frac{-C^*_{S,m,p,g_1} t}{1 + ||v - \overline{(v)}_S||^2_{L^2(S)}}\right).$$

Consequently, as $\log(x^2+1) \le 2x$ for all $x \ge 0$ we obtain

$$\log\left(\int_S \left(v - \overline{(v)}_S\right)^2 d\lambda + 1\right) = \log(||v - \overline{(v)}_S||^2_{L^2(S)} + 1) \le 2||v - \overline{(v)}_S||_{L^2(S)}.$$

Thus, combining the preceding two inequalities yields (7.21) for $v \in D(A^d_p(\gamma))$.
Now let $v \in L^2(S)$ and introduce $(v_k)_{k\in\mathbb{N}} \subseteq D(A^d_p(\gamma))$ such that $\lim_{k\to\infty} v_k = v$ in $L^2(S)$. Such a sequence exists, se Lemma 3.5.4.

Then trivially $\lim\limits_{k\to\infty} \overline{(v_k)}_S = \overline{(v)}_S$, and we get by contractivity that $\lim\limits_{k\to\infty} T_{\text{det}}(t,\gamma)v_k = T_{\text{det}}(t,\gamma)v$ in $L^2(S)$.
As the mappings $[0,\infty) \ni x \mapsto \log(x+1)$ and $[0,\infty) \ni x \mapsto \exp(-(x+1)^{-1}C^*_{S,m,p,g_1}t)$ are continuous, the claim follows. $\square$

Remark 7.6.7. *In the sequel C^*_{S,m,p,g_1} denotes the constant defined in (7.22). The previous lemma brings us in the position to prove the main result of this section.*

Theorem 7.6.8. *Let $u \in L^1(\Omega; L^1(S))$, with $u \in L^2(S)$ a.s., $t \in (0,\infty)$, $\alpha > 0$ and assume that $p \in [m,2) \setminus \{1\}$, where $m := \frac{2n}{n+2}$. Then all of the following assertions hold.*
If $\Delta_u \in L^1(\Omega)$, then

$$\mathbb{P}\left(\int\limits_S (T_{ra}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \frac{2}{\log(\alpha+1)} \left(\mathbb{E}(\Delta_u)\mathbb{E}\left(\exp\left(\frac{-2tC^*_{S,m,p,g_1}}{1+\Delta_u}\right)\right)\right)^{\frac{1}{2}}. \tag{7.23}$$

If there is an $r \in [1,\infty)$ such that $\Delta_u \in L^{2r}(\Omega)$, then

$$\mathbb{P}\left(\int\limits_S (T_{ra}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \left(\frac{1}{t}\right)^r \frac{2}{\log(\alpha+1)} \left(\frac{r}{2C^*_{S,m,p,g_1}}\right)^r \left(\mathbb{E}(\Delta_u)\mathbb{E}((1+\Delta_u)^{2r})\right)^{\frac{1}{2}}. \tag{7.24}$$

If there is an $\varepsilon > 0$ such that $e^{\varepsilon\Delta_u} \in L^1(\Omega)$, then

$$\mathbb{P}\left(\int\limits_S (T_{ra}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \exp\left(-t^{\frac{1}{2}}\left(\frac{\varepsilon C^*_{S,m,p,g_1}}{2}\right)^{\frac{1}{2}}\right) \frac{2\exp(\frac{\varepsilon}{2})}{\log(\alpha+1)} \left(\mathbb{E}(\Delta_u)\mathbb{E}\left(\exp\left(\varepsilon\Delta_u\right)\right)\right)^{\frac{1}{2}}.$$

Proof. Proof of (7.23). Firstly, note that $[0,\infty) \ni x \mapsto \log(x+1)$ is obviously nonnegative, increasing and strictly positive on $(0,\infty)$. Consequently, we have by virtue of Markov's inequality and by recalling Lemma 7.6.6 as well as Theorem 7.4.1 that

$$\mathbb{P}\left(\int\limits_S (T_{\text{ra}}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right) \leq \frac{2}{\log(\alpha+1)} \int\limits_\Omega \Delta_u(\omega)^{\frac{1}{2}} \exp\left(\frac{-C^*_{S,m,p,g_1}t}{1+\Delta_u(\omega)}\right) d\mathbb{P}(\omega)$$

which verifies (7.23) by applying Cauchy-Schwarz' inequality. (Moreover, note that the assumption on Δ_u ensures that the first expectation exists and the the second one exists trivially.)

Throughout the remaining part of this proof, let $\tilde{\Delta}_u := \frac{1}{1+\Delta_u}$.
Now inequality (7.24) follows from the succeeding estimate, where relation (7.23) is used.

$$t^r\mathbb{P}\left(\int\limits_S (T_{\text{ra}}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right)$$

$$
\begin{aligned}
&\leq \quad t^r \frac{2}{\log(\alpha+1)} \left(\mathbb{E}(\Delta_u)\mathbb{E}\left(\exp\left(\frac{-2tC^*_{S,m,p,g_1}}{1+\Delta_u}\right)\right)\right)^{\frac{1}{2}}\\
&= \quad \frac{2(\mathbb{E}(\Delta_u))^{\frac{1}{2}}}{\log(\alpha+1)} \left(\mathbb{E}\left((2r^{-1}\tilde{\Delta}_u C^*_{S,m,p,g_1})^{-2r}\exp\left(r\log\left((2r^{-1}\tilde{\Delta}_u C^*_{S,m,p,g_1}t)^2\right) - 2tC^*_{S,m,p,g_1}\tilde{\Delta}_u\right)\right)\right)^{\frac{1}{2}}\\
&\leq \quad \frac{2(\mathbb{E}(\Delta_u))^{\frac{1}{2}}}{\log(\alpha+1)} \left(\mathbb{E}\left((2r^{-1}\tilde{\Delta}_u C^*_{S,m,p,g_1})^{-2r}\exp\left(r2r^{-1}\tilde{\Delta}_u C^*_{S,m,p,g_1}t - 2tC^*_{S,m,p,g_1}\tilde{\Delta}_u\right)\right)\right)^{\frac{1}{2}}\\
&= \quad \frac{2}{\log(\alpha+1)} \left(\frac{r}{2C^*_{S,m,p,g_1}}\right)^r \left(\mathbb{E}(\Delta_u)\mathbb{E}\left((1+\Delta_u)^{2r}\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, it remains to verify the last tail bound. For the sake of brevity, let $\beta := \left(\frac{1}{2}\varepsilon C^*_{S,m,p,g_1}\right)^{\frac{1}{2}}$. The desired estimate easily follows from (7.23), more precisely: Thanks to (7.23) we have

$$
\begin{aligned}
&\exp\left(t^{\frac{1}{2}}\beta\right)\mathbb{P}\left(\int\limits_S (T_{\mathrm{ra}}(t)u - \overline{(u)}_S)^2 d\lambda > \alpha\right)\\
\leq \quad & \frac{2\left(\mathbb{E}(\Delta_u)\right)^{\frac{1}{2}}}{\log(\alpha+1)} \left(\mathbb{E}\left(\exp\left(2t^{\frac{1}{2}}\left(\beta - t^{\frac{1}{2}}C^*_{S,m,p,g_1}\tilde{\Delta}_u\right)\mathbb{1}\{\beta > t^{\frac{1}{2}}C^*_{S,m,p,g_1}\tilde{\Delta}_u\}\right)\right)\right)^{\frac{1}{2}}\\
\leq \quad & \frac{2\left(\mathbb{E}(\Delta_u)\right)^{\frac{1}{2}}}{\log(\alpha+1)} \left(\mathbb{E}\left(\exp\left(2t^{\frac{1}{2}}\beta\mathbb{1}\left\{t^{\frac{1}{2}} < \frac{\beta}{C^*_{S,m,p,g_1}\tilde{\Delta}_u}\right\}\right)\right)\right)^{\frac{1}{2}}\\
\leq \quad & \frac{2\left(\mathbb{E}(\Delta_u)\right)^{\frac{1}{2}}}{\log(\alpha+1)} \left(\mathbb{E}\left(\exp\left(2\,\frac{\beta^2}{C^*_{S,m,p,g_1}}(1+\Delta_u)\right)\right)\right)^{\frac{1}{2}}\\
= \quad & \frac{2}{\log(\alpha+1)}\exp\left(\frac{\varepsilon}{2}\right)\left(\mathbb{E}(\Delta_u)\mathbb{E}\left(\exp\left(\varepsilon\Delta_u\right)\right)\right)^{\frac{1}{2}},
\end{aligned}
$$

which completes our proof. □

Remark 7.6.9. *Let $u \in L^1(\Omega; L^1(S))$, with $\mathbb{P}(u \in L^2(S)) = 1$ and assume $p \in [\frac{2n}{n+2}, 2) \setminus \{1\}$. Then we have: If there is an $\varepsilon > 0$ such that $e^{\varepsilon\Delta_u} \in L^1(\Omega)$, then $\Delta_u \in L^{2r}(\Omega)$ for all $r \in [1, \infty)$, and of course also $\Delta_u \in L^1(\Omega)$.*
Thus, if $e^{\varepsilon\Delta_u} \in L^1(\Omega)$ for an $\varepsilon > 0$, then all three bounds in Theorem 7.6.8 are applicable.
Particularly, if $u : \Omega \to L^1(S)$ is Gaussian with $\mathbb{P}(u \in L^2(S)) = 1$, then it is well-known that there is an $\varepsilon > 0$, such that $e^{\varepsilon\Delta_u} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Consequently, we can apply all three bounds if u is Gaussian and $\mathbb{P}(u \in L^2(S)) = 1$.

7.G Appendix: Measurability Questions concerning A_p^r

The following two lemmas reveal that all events occurring in the definition of A_p^r are indeed measurable and that all occurring integrals are well-defined as well as finite. The remaining two results of this section are concerned with the measurability of the $L^q(S)$-norm of vector-valued random variables.

Lemma 7.G.1. *The set $W^{1,p}(S) \cap L^\infty(S)$ is $\mathfrak{B}(L^1(S))$-measurable. Let $f \in L^1(\Omega; L^1(S))$ and assume $\mathbb{P}(f \in W^{1,p}(S) \cap L^\infty(S)) = 1$. Then the following assertions hold.*

i) f is $\mathcal{F}$-$\mathfrak{B}(L^p(S))$-measurable.

ii) ∇f is $\mathcal{F}$-$\mathfrak{B}(L^p(S;\mathbb{R}^n))$-measurable.

iii) The mapping $\Phi : L^p(S;\mathbb{R}^n) \to L^{\tilde{p}}(S;\mathbb{R}^n)$ defined by $\Phi(\varphi) := |\varphi|^{p-2}\varphi$ for all $\varphi \in L^p(S;\mathbb{R}^n)$ is continuous, where $\tilde{p} := \frac{p}{p-1}$.

iv) The mapping defined by $\Omega \ni \omega \mapsto g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega) \in L^{\tilde{p}}(S;\mathbb{R}^n)$ is $\mathcal{F}$-$\mathfrak{B}(L^{\tilde{p}}(S;\mathbb{R}^n))$-measurable, where $\tilde{p} := \frac{p}{p-1}$.

Proof. At first it will be proven that $L^\infty(S) \in \mathfrak{B}(L^1(S))$.
Introduce $K(\kappa) := \{f \in L^\infty(S) : \ ||f||_{L^\infty(S)} \leq \kappa\}$ and note that obviously $L^\infty(S) = \bigcup_{\kappa \in \mathbb{N}} K(\kappa)$ and that each of the $K(\kappa)$ is closed w.r.t. $||\cdot||_{L^1(S)}$. Consequently, $L^\infty(S)$ is the countable union of $L^1(S)$-closed sets and therefore $L^\infty(S) \in \mathfrak{B}(L^1(S))$.
Moreover, as the injection $W^{1,p}(S) \hookrightarrow L^1(S)$ is continuous, and $(W^{1,p}(S), ||\cdot||_{W^{1,p}(S)})$ is separable, $W^{1,p}(S) \in \mathfrak{B}(L^1(S))$ follows from Remark 5.2.2; in fact, we even have $\mathfrak{B}(W^{1,p}(S)) \subseteq \mathfrak{B}(L^1(S))$.
Consequently, as $L^\infty(S),\ W^{1,p}(S) \in \mathfrak{B}(L^1(S))$, we get $L^\infty(S) \cap W^{1,p}(S) \in \mathfrak{B}(L^1(S))$.
Proof of i). Follows from Remark 5.2.2, which is applicable since the injection $L^p(S) \hookrightarrow L^1(S)$ is continuous and $L^p(S)$ is separable.
Proof of ii). It follows from Remark 5.2.2, that f is $\mathcal{F}$-$\mathfrak{B}(W^{1,p}(S))$-measurable. Thus, as $\nabla : W^{1,p}(S) \to L^p(S;\mathbb{R}^n)$ is continuous, ii) holds as well.
Proof of iii). Let $\varphi \in L^p(S;\mathbb{R}^n)$ and let $(\varphi_m)_{m\in\mathbb{N}} \subseteq L^p(S;\mathbb{R}^n)$, such that $\lim_{m\to\infty} \varphi_m = \varphi$ in $L^p(S;\mathbb{R}^n)$.
We have, by passing to a subsequence if necessary, that there is an $h \in L^p(S)$ such that $\lim_{m\to\infty} \varphi_m = \varphi$ a.e. on S and $|\varphi_m| \leq |h|$ a.e. on S for each $m \in \mathbb{N}$. Moreover, the continuity of $\mathbb{R}^n \ni x \mapsto |x|^{p-2}x$ yields

$$\lim_{m\to\infty} \Phi(\varphi_m) = \Phi(\varphi) \text{ a.e. on } S. \tag{7.25}$$

In addition,

$$|\Phi(\varphi_m) - \Phi(\varphi)|^{\tilde{p}} \leq (|\varphi_m|^{p-1} + |\varphi|^{p-1})^{\tilde{p}} \leq 2^{\tilde{p}}|h|^p \in L^1(S),\ \forall m \in \mathbb{N}.$$

This yields, by virtue of dominated convergence, that $\lim_{m\to\infty} \Phi(\varphi_m) = \Phi(\varphi)$ in $L^{\tilde{p}}(S;\mathbb{R}^n)$.
Proof of iv). It is obvious that $\mathbb{P}(g \in L^p(S)) = 1$. In addition, g is by assumption $\mathcal{F}$-$\mathfrak{B}(L^1(S))$-measurable. Consequently, we get that g is $\mathcal{F}$-$\mathfrak{B}(L^p(S))$-measurable. Moreover, ii) and iii) yield that $\Omega \ni \omega \mapsto |\nabla f(\omega)|^{p-2}\nabla f(\omega)$ is $\mathcal{F}$-$L^{\tilde{p}}(S;\mathbb{R}^n)$-measurable.
Moreover, it is now easily verified that $g|\nabla f|^{p-2}\nabla f$ is $\mathcal{F}$-$\mathfrak{B}(L^1(S;\mathbb{R}^n))$-measurable.
In addition, as $|\nabla f|^{p-2}\nabla f \in L^{\tilde{p}}(S;\mathbb{R}^n)$ a.s. and particularly $g \in L^\infty(S)$ almost surely, we get $g|\nabla f|^{p-2}\nabla f \in L^{\tilde{p}}(S;\mathbb{R}^n)$ a.s. and iv) follows from Remark 5.2.2. $\square$

Lemma 7.G.2. *Let $f,\ \hat{f} \in L^1(\Omega; L^1(S))$ and assume $\mathbb{P}(f \in W^{1,p}(S) \cap L^\infty(S)) = 1$. Then the Lebesgue integrals*

$$\int\limits_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega) \cdot \nabla\varphi d\lambda \text{ and } \int\limits_S \hat{f}(\omega)\varphi d\lambda \tag{7.26}$$

exist for any given $\varphi \in W^{1,p}(S) \cap L^\infty(S)$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover, the mappings defined by

$$\Omega \ni \omega \mapsto \int\limits_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega) \cdot \nabla\varphi d\lambda \text{ and } \Omega \ni \omega \mapsto \int\limits_S \hat{f}(\omega)\varphi d\lambda \tag{7.27}$$

are $\mathcal{F}$-$\mathfrak{B}(\mathbb{R})$-measurable for any $\varphi \in W^{1,p}(S) \cap L^\infty(S)$.
Finally, we have

$$\left\{\omega \in \Omega : \int\limits_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega) \cdot \nabla\varphi d\lambda = \int\limits_S \hat{f}(\omega)\varphi d\lambda,\ \forall \varphi \in W^{1,p}(S) \cap L^\infty(S)\right\} \in \mathcal{F},$$

and A_p^r is well-defined.

Proof. Firstly, note that the assertions concerning $\hat{f}$ stated in (7.26) and (7.27) are trivial.
Moreover, for $\varphi \in W^{1,p}(S) \cap L^\infty(S)$, we have a fortiori $\nabla\varphi \in L^p(S; \mathbb{R}^n)$ which yields, by virtue of Lemma 7.G.1.iv), that the left-hand-side integral in (7.26) exists with probability one and also that the left-hand-side mapping in (7.27) is $\mathcal{F}$-$\mathfrak{B}(\mathbb{R})$-measurable.
Now the final assertion in this lemma will be proven. Firstly, note that

$$\left\{\omega \in \Omega : \int\limits_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega) \cdot \nabla\varphi d\lambda = \int\limits_S \hat{f}(\omega)\varphi d\lambda\right\} \in \mathcal{F} \tag{7.28}$$

for any given $\varphi \in W^{1,p}(S) \cap L^\infty(S)$.
Introduce $L_k^\infty(S) := \{f \in L^\infty(S) : \ ||f||_{L^\infty(S)} \leq k\}$ for every $k \in \mathbb{N}$. One verifies immediately that $W^{1,p}(S) \cap L_k^\infty(S)$ is a closed subset of $W^{1,p}(S)$ w.r.t. $||\cdot||_{W^{1,p}(S)}$. Moreover, it is well known that $(W^{1,p}(S), ||\cdot||_{W^{1,p}(S)})$ is separable and that subsets of separable spaces are separable as well. Consequently, for each $k \in \mathbb{N}$ there is a countable set $\mathcal{D}(k) \subseteq W^{1,p}(S) \cap L_k^\infty(S)$ fulfilling

$$\overline{\mathcal{D}(k)} = W^{1,p}(S) \cap L_k^\infty(S),$$

where the closure is taken w.r.t. $||\cdot||_{W^{1,p}(S)}$.
Now introduce $\gamma \in L^1_{g_1,g_2}(S)$ and $(F, \hat{F}) \in (W^{1,p}(S) \cap L^\infty(S)) \times L^1(S)$. It will be proven that: For a given $k \in \mathbb{N}$,

$$\int\limits_S \gamma|\nabla F|^{p-2}\nabla F \cdot \nabla\varphi d\lambda = \int\limits_S \hat{F}\varphi d\lambda,\ \forall \varphi \in W^{1,p}(S) \cap L_k^\infty(S), \tag{7.29}$$

if and only if

$$\int_S \gamma|\nabla F|^{p-2}\nabla F\cdot\nabla\varphi d\lambda = \int_S \hat{F}\varphi d\lambda,\ \forall\varphi\in\mathcal{D}(k). \tag{7.30}$$

Firstly, note that (7.29) obviously implies (7.30).
Now assume that (7.30) holds. Let $\varphi \in W^{1,p}(S)\cap L_k^\infty(S)$ be arbitrary but fixed and introduce $(\varphi_m)_{m\in\mathbb{N}}\subseteq\mathcal{D}(k)$ such that $\lim\limits_{m\to\infty}\varphi_m=\varphi$ in $W^{1,p}(S)$.
As $\gamma|\nabla F|^{p-2}\nabla F\in L^{\tilde{p}}(S;\mathbb{R}^n)$ and as particularly $\lim\limits_{m\to\infty}\nabla\varphi_m=\nabla\varphi$ in $L^p(S;\mathbb{R}^n)$ we obtain that

$$\lim_{m\to\infty}\int_S \gamma|\nabla F|^{p-2}\nabla F\cdot\nabla\varphi_m d\lambda = \int_S \gamma|\nabla F|^{p-2}\nabla F\cdot\nabla\varphi d\lambda.$$

Moreover, as particularly $\lim\limits_{m\to\infty}\varphi_m=\varphi$ in $L^p(S)$ we obtain, by passing to a subsequence if necessary, that $\lim\limits_{m\to\infty}(\varphi_m-\varphi)\hat{F}=0$ a.e. on S. Since clearly $|(\varphi_m-\varphi)\hat{F}|\leq 2k|\hat{F}|\in L^1(S)$ we obtain by virtue of dominated convergence that

$$\lim_{m\to\infty}\int_S \hat{F}\varphi_m d\lambda = \int_S \hat{F}\varphi d\lambda.$$

This yields that (7.30) implies (7.29).

Finally, we obtain by using $W^{1,p}(S)\cap L^\infty(S)=\bigcup\limits_{k\in\mathbb{N}}(W^{1,p}(S)\cap L_k^\infty(S))$ and the equivalence of (7.29) and (7.30) that

$$\begin{aligned}
&\left\{\omega\in\Omega:\ \int_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega)\cdot\nabla\varphi d\lambda=\int_S \hat{f}(\omega)\varphi d\lambda,\ \forall\varphi\in W^{1,p}(S)\cap L^\infty(S)\right\}\\
=\ &\bigcap_{k\in\mathbb{N}}\left\{\omega\in\Omega:\ \int_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega)\cdot\nabla\varphi d\lambda=\int_S \hat{f}(\omega)\varphi d\lambda,\ \forall\varphi\in\mathcal{D}(k)\right\}\\
=\ &\bigcap_{k\in\mathbb{N}}\ \bigcap_{\varphi\in\mathcal{D}(k)}\left\{\omega\in\Omega:\ \int_S g(\omega)|\nabla f(\omega)|^{p-2}\nabla f(\omega)\cdot\nabla\varphi d\lambda=\int_S \hat{f}(\omega)\varphi d\lambda\right\},
\end{aligned}$$

which implies, using (7.28), the claim as $\mathcal{D}(k)$ is countable for each $k\in\mathbb{N}$. □

Remark 7.G.3. *Let $q\in(1,\infty)$, $f\in L^1(\Omega;L^1(S))$ and assume $\mathbb{P}(f\in L^q(S))=1$. Then, thanks to the separability of $(L^q(S),||\cdot||_{L^q(S)})$, it follows from Lusin-Souslin's Theorem (see [22, Theorem 15.1]) that $\Omega\ni\omega\mapsto||f(\omega)||_{L^q(S)}$ is $\mathcal{F}-\mathfrak{B}(\mathbb{R})$-measurable. This holds, as the following lemma reveals, also for $q=\infty$.*

Lemma 7.G.4. *Let $f\in L^1(\Omega;L^1(S))$ and assume $\mathbb{P}(f\in L^\infty(S))=1$. Then the mapping defined by $\Omega\ni\omega\mapsto||f(\omega)||_{L^\infty(S)}$ is $\mathcal{F}-\mathfrak{B}(\mathbb{R})$-measurable.*

Proof. As $\mathbb{P}(f \in L^\infty(S)) = 1$, we have particularly $\mathbb{P}(f \in L^m(S),\ \forall m \in \mathbb{N}) = 1$. Consequently f is $\mathcal{F} - \mathfrak{B}(L^m(S))$ measurable for any $m \in \mathbb{N}$. This yields that $\Omega \ni \omega \mapsto ||f(\omega)||_{L^m(S)}$ is $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable. Moreover, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$ that

$$||f(\omega)||_{L^\infty(S)} = \lim_{m\to\infty} ||f(\omega)||_{L^m(S)}.$$

Consequently, $\Omega \ni \omega \mapsto ||f(\omega)||_{L^\infty(S)}$ is the almost sure limit of $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable functions and therefore itself $\mathcal{F} - \mathfrak{B}(\mathbb{R})$-measurable. □

7.H Appendix: Technical Results to prove the Existence and Uniqueness of mild Solutions

All results of this section serve to prove that (7.2) has a unique mild solution. Hereby, some simple properties of $A_p^d(\gamma)$, its closure and their resolvents are collected in Lemma 7.H.1. In Lemma 7.H.2, we prove that the resolvent of $A_p^d(\gamma)$ depends in some sense continuously on $\gamma \in L^1_{g_1,g_2}(S)$, and in the last lemma, we prove two density results.

Lemma 7.H.1. *Let $\gamma \in L^1_{g_1,g_2}(S)$. The following assertions hold.*

i) $L^\infty(S) \subseteq R(Id + A_p^d(\gamma))$.

ii) $(Id + A_p^d(\gamma))^{-1}|_{L^\infty(S)} = (Id + \mathcal{A}_p^d(\gamma))^{-1}|_{L^\infty(S)}$.

iii) $(Id + \mathcal{A}_p^d(\gamma))^{-1}$ *is $L^1(S)$-continuous.*

Proof. A proof of i) can be found in [3, Prop. 3.5].
Proof of ii). Let $h \in L^\infty(S)$ and let $(f, \hat{f}) \in A_p^d(\gamma)$ and $(F, \hat{F}) \in \mathcal{A}_p^d(\gamma)$, be the uniquely determined functions fulfilling $h = f + \hat{f}$ and $h = F + \hat{F}$, i.e. $f = (Id + A_p^d(\gamma))^{-1}h$ and $F = (Id + \mathcal{A}_p^d(\gamma))^{-1}h$.
The complete accretivity of $\mathcal{A}_p^d(\gamma)$ yields $F << F + \hat{F}$ and consequently $F << h$. This implies $F \in L^\infty(S)$, since $h \in L^\infty(S)$. Hence, it follows by virtue of Lemma 3.3.1 that $(F, \hat{F}) \in A_p^d(\gamma)$. Conclusively, we have, by uniqueness, that $f = F$.
Finally, iii) is an immediate consequence of the accretivity of $\mathcal{A}_p^d(\gamma)$. □

Lemma 7.H.2. *Let $(\gamma_m)_{m\in\mathbb{N}} \subseteq L^1_{g_1,g_2}(S)$ and assume that there is $\gamma \in L^1(S)$ such that*

$$\lim_{m\to\infty} \gamma_m = \gamma, \text{ in } L^1(S).$$

Then

$$w\text{ - }\lim_{m\to\infty} (Id + A_p^d(\gamma_m))^{-1}h = (Id + A_p^d(\gamma))h, \text{ in } L^1(S),$$

for any $h \in L^\infty(S)$.

The following proof is long and technical. Moreover, the proof works similar to the one of [3, Prop. 3.5] which states the range condition $L^\infty(S) \subseteq R(Id + A_p^d(\gamma))$. Proving the range condition works by showing that a certain resolvent converges. As in our case, it is easy to see that the resolvent converges to a limit, but it is very challenging to show that this is the correct limit. And the delicate technique which is used to show that the limit is the correct one, is the same as in [3]. As it is, on a first glance, not that obvious that these proofs work similar, the proof of Lemma 7.H.2 will be given here.

Proof. Firstly, observe that, by passing to a subsequence if necessary, $\lim_{m\to\infty} \gamma_m = \gamma$ a.e. on S. Moreover, it is clear that $|\gamma| \leq g_2$ a.e. on S. This implies $|\gamma_m - \gamma|^{\max(p,\tilde{p})} \leq (2g_2)^{\max(p,\tilde{p})}$, where $\tilde{p} := \frac{p}{p-1}$. Consequently, since $\lambda(S) < \infty$ it follows by virtue of dominated convergence that

$$\lim_{m\to\infty} \gamma_m = \gamma, \text{ in } L^p(S) \text{ and in } L^{\tilde{p}}(S). \tag{7.31}$$

Let $f_m := (Id + A_p^d(\gamma_m))^{-1}h$ for each $m \in \mathbb{N}$ and $f := (Id + A_p^d(\gamma))^{-1}h$. Additionally introduce $\hat{f}_m := A_p^d(\gamma_m)f_m$ for all $m \in \mathbb{N}$ and $\hat{f} := A_p^d(\gamma)f$.
Note that by construction $f + \hat{f} = h = f_m + \hat{f}_m$ for each $m \in \mathbb{N}$. Moreover, we have

$$f_m << f_m + \hat{f}_m = h, \ \forall m \in \mathbb{N}, \tag{7.32}$$

by complete accretivity.
Consequently, $||f_m||_{L^p(S)} \leq ||h||_{L^p(S)} < \infty$ for each $m \in \mathbb{N}$. As $L^p(S)$ is reflexive this implies, by passing to a subsequence if necessary, that there is an $F \in L^p(S)$ such that

$$\text{w - }\lim_{m\to\infty} f_m = F \text{ in } L^p(S). \tag{7.33}$$

Now introduce $\hat{F} := h - F$. Then

$$\text{w - }\lim_{m\to\infty} \hat{f}_m = \hat{F} \text{ in } L^p(S), \tag{7.34}$$

as $\hat{f}_m = h - f_m$ for each $m \in \mathbb{N}$.
Now it will be verified that

$$F \in W^{1,p}(S) \cap L^\infty(S). \tag{7.35}$$

Proof of (7.35). First of all it follows from (7.32), (7.33) and by the virtue of [7, Corollary 2.7] that $F << h$ and consequently $F \in L^\infty(S)$. Hence, particularly $F \in L^p(S)$.
Moreover, $(f_m, \hat{f}_m) \in A_p^d(\gamma_m)$ together with (7.32) yields

$$||\nabla f_m||^p_{L^p(S;\mathbb{R}^n)} = \int_S |\nabla f_m|^p d\lambda \leq \frac{1}{g_1}\int_S \gamma_m|\nabla f_m|^p d\lambda = \frac{1}{g_1}\int_S f_m\hat{f}_m d\lambda \leq \frac{2}{g_1}\lambda(S)||h||^2_{L^\infty(S)}. \tag{7.36}$$

Consequently, by passing to a subsequence if necessary, there is an $\mathbb{F} = (\mathbb{F}_1, ..., \mathbb{F}_n) \in L^p(S;\mathbb{R}^n)$ such that $\text{w-}\lim_{m\to\infty} \nabla f_m = \mathbb{F}$ in $L^p(S;\mathbb{R}^n)$. This, together with (7.33), implies for all $\varphi \in C_c^\infty(S)$ that

$$\int_S F\frac{\partial}{\partial x_j}\varphi d\lambda = \lim_{m\to\infty}\int_S f_m \frac{\partial}{\partial x_j}\varphi d\lambda = \lim_{m\to\infty} -\int_S \varphi\frac{\partial}{\partial x_j} f_m d\lambda = -\int_S \varphi \mathbb{F}_j d\lambda,$$

i.e. $F \in W^{1,1}_{\text{Loc}}(S)$ and $\nabla F = \mathbb{F}$. Consequently, (7.35) holds and also

$$\text{w-}\lim_{m\to\infty} \nabla f_m = \nabla F \text{ in } L^p(S;\mathbb{R}^n). \tag{7.37}$$

Now observe that (7.36) yields

$$||\ |\nabla f_m|^{p-2}\nabla f_m||^{\tilde p}_{L^{\tilde p}(S;\mathbb{R}^n)} = \int_S |\nabla f_m|^{(p-1)\tilde p} d\lambda = \int_S |\nabla f_m|^p d\lambda \le \frac{2}{g_1}\lambda(S)||h||^2_{L^\infty(S)}.$$

Consequently, by passing to a subsequence if necessary, there is a $\zeta \in L^{\tilde p}(S;\mathbb{R}^n)$ such that

$$\text{w-}\lim_{m\to\infty} |\nabla f_m|^{p-2}\nabla f_m = \zeta \text{ in } L^{\tilde p}(S;\mathbb{R}^n). \tag{7.38}$$

Now it will be proven that

$$\int_S \gamma\zeta\cdot\nabla\varphi d\lambda = \int_S \hat F \varphi d\lambda,\ \forall \varphi \in W^{1,p}(S)\cap L^\infty(S). \tag{7.39}$$

For $\varphi \in C^\infty(\overline{S})$[2], (7.36) implies that

$$\int_S \left||\nabla f_m|^{p-2}\nabla f_m\cdot\nabla\varphi\right|^{\tilde p} d\lambda \le \int_S |\nabla f_m|^p|\nabla\varphi|^{\tilde p} d\lambda \le ||\ |\nabla\varphi|^{\tilde p}||_{L^\infty(S)}\frac{2}{g_1}\lambda(S)||h||^2_{L^\infty(S)} < \infty,$$

for all $m \in \mathbb{N}$.
This yields, by virtue of Hölder's inequality and (7.31) that

$$\lim_{m\to\infty}\left|\int_S (\gamma_m-\gamma)|\nabla f_m|^{p-2}\nabla f_m\cdot\nabla\varphi d\lambda\right| \le \lim_{m\to\infty} ||\gamma_m-\gamma||_{L^p(S)}||\ |\nabla f_m|^{p-2}\nabla f_m\cdot\nabla\varphi||_{L^{\tilde p}(S)} = 0$$

for all $\varphi \in C^\infty(\overline{S})$.
Using this, (7.34) as well as (7.38) yields (7.39) for $\varphi \in C^\infty(\overline{S})$. Moreover, for arbitrary $\varphi \in W^{1,p}(S)\cap L^\infty(S)$, there is, as S is of class C^1, a sequence $(\varphi_m)_{m\in\mathbb{N}} \subseteq C^\infty(\overline{S})$ such that $\lim_{m\to\infty}\varphi_m = \varphi$

[2] $C^\infty(\overline{S})$ denotes the space of all functions $\varphi : S \to \mathbb{R}$ which are infinitely often continuously differentiable, such that φ and all its partial derivatives can be extended continuously to the boundary of S

in $W^{1,p}(S)$. Hence, as $\gamma\zeta \in L^{\tilde{p}}(S;\mathbb{R}^n)$ and $\hat{F} = h - F \in L^\infty(S) \subseteq L^{\tilde{p}}(S)$, we obtain

$$\int_S \hat{F}\varphi d\lambda = \lim_{m\to\infty} \int_S \hat{F}\varphi_m d\lambda = \lim_{m\to\infty} \int_S \gamma\zeta \cdot \nabla\varphi_m d\lambda = \int_S \gamma\zeta \cdot \nabla\varphi d\lambda,$$

which verifies (7.39).
Now observe the following: If

$$\zeta = |\nabla F|^{p-2}\nabla F, \tag{7.40}$$

then (7.39) yields

$$\int_S \gamma|\nabla F|^{p-2}\nabla F \cdot \varphi d\lambda = \int_S \hat{F}\varphi d\lambda,\ \forall \varphi \in W^{1,p}(S) \cap L^\infty(S).$$

This, together with (7.35) implies $(F, \hat{F}) \in A_p^d(\gamma)$.
Since it has been already established that $h = F + \hat{F}$ and since also $h = f + \hat{f}$ as well as $(f, \hat{f}) \in A_p^d(\gamma)$ the accretivity of $A_p^d(\gamma)$ yields

$$||f - F||_{L^1(S)} \leq ||f - F + \hat{f} - \hat{F}||_{L^1(S)} = ||h - h||_{L^1(S)} = 0.$$

Consequently, (7.40) implies $f = F$ and it follows by virtue of (7.33) that

$$\text{w - }\lim_{m\to\infty}(Id + A_p^d(\gamma_m))^{-1}h = \text{w - }\lim_{m\to\infty} f_m = F = f = (Id + A_p^d(\gamma))^{-1}h \text{ in } L^p(S).$$

Conclusively, since $L^p(S)$-weak convergence implies $L^1(S)$-weak convergence, the claim follows once (7.40) is proven.
The delicate proof of (7.40) is preceded by the proofs of the following four statements.
We have, by passing to a subsequence if necessary:

(I) $\limsup_{m\to\infty} \int_S \gamma_m |\nabla f_m|^p \leq \int_S F\hat{F} d\lambda$.
(II) $\text{w - }\lim_{m\to\infty} \gamma_m \nabla f_m = \gamma\nabla F$ in $L^p(S)$.
(III) $\text{w - }\lim_{m\to\infty} \gamma_m |\nabla f_m|^{p-2}\nabla f_m = \gamma\zeta$ in $L^{\tilde{p}}(S;\mathbb{R}^n)$.
(IV) $\lim_{m\to\infty} \int_S (\gamma - \gamma_m)|\varphi|^p d\lambda = 0$ for all $\varphi \in L^p(S;\mathbb{R}^n)$.

Proof of (I). Firstly, (7.32) implies that $||f_m||_{L^2(S)} \leq ||h||_{L^2(S)}$. Consequently, $(f_m)_{m\in\mathbb{N}}$ has, by passing to a subsequence if necessary, an $L^2(S)$-weakly convergent subsequence, converging to an $\tilde{f} \in L^2(S)$. Moreover, it is plain that $\tilde{f} = F$ and therefore

$$\int_S F^2 d\lambda \leq \liminf_{m\to\infty} \int_S f_m^2 d\lambda. \tag{7.41}$$

Conclusively, it follows by virtue of (7.33) as well as (7.41) that

$$\limsup_{m\to\infty} \int\limits_S \gamma_m |\nabla f_m|^p d\lambda \leq \limsup_{m\to\infty} \int\limits_S f_m h d\lambda - \liminf_{m\to\infty} \int\limits_S f_m^2 d\lambda = \int\limits_S F\hat{F} d\lambda.$$

Proof of (II). Firstly, note that (7.36) yields

$$||\gamma_m \nabla f_m||^p_{L^p(S;\mathbb{R}^n)} = \int\limits_S \gamma_m^p |\nabla f_m|^p d\lambda \leq \frac{g_2^p}{g_1} 2\lambda(S)||h||^2_{L^\infty(S)} < \infty, \ \forall m \in \mathbb{N}.$$

Consequently, by passing to a subsequence if necessary, there is an $\alpha \in L^p(S;\mathbb{R}^n)$ such that

$$\text{w - }\lim_{m\to\infty} \gamma_m \nabla f_m = \alpha \text{ in } L^p(S;\mathbb{R}^n). \tag{7.42}$$

Moreover, we have for any $\varphi \in L^\infty(S;\mathbb{R}^n)$, by virtue of Hölder's inequality, Cauchy-Schwarz' inequality, (7.36) and (7.31) that

$$\lim_{m\to\infty} \left| \int\limits_S (\gamma_m - \gamma)\nabla f_m \cdot \varphi d\lambda \right| \leq \lim_{m\to\infty} ||\gamma_m - \gamma||_{L^{\tilde{p}}(S)} || \, |\varphi| \, ||_{L^\infty(S)} \left(\frac{2}{g_1} \lambda(S)||h||_{L^\infty(S)} \right)^{\frac{1}{p}} = 0.$$

Consequently, we obtain for any $\varphi \in L^\infty(S;\mathbb{R}^n)$ by invoking (7.37) and (7.42) that

$$\int\limits_S (\alpha - \gamma\nabla F) \cdot \varphi d\lambda = \lim_{m\to\infty} \int\limits_S \gamma_m \nabla f_m \cdot \varphi - \nabla f_m \cdot \varphi\gamma d\lambda = 0.$$

This clearly implies $\alpha = \gamma\nabla F$. Hence, (7.42) implies (II).
Proof of (III). Firstly, note that it follows by virtue of (7.36) that

$$||\gamma_m |\nabla f_m|^{p-2} \nabla f_m||^{\tilde{p}}_{L^{\tilde{p}}(S;\mathbb{R}^n)} \leq g_2^{\tilde{p}} \int\limits_S |\nabla f_m|^p d\lambda \leq \frac{g_2^{\tilde{p}}}{g_1} 2\lambda(S)||h||^2_{L^\infty(S)}.$$

Consequently there is, by passing to a subsequence if necessary, an $\alpha \in L^{\tilde{p}}(S;\mathbb{R}^n)$ such that

$$\text{w - }\lim_{m\to\infty} \gamma_m |\nabla f_m|^{p-2} \nabla f_m = \alpha \text{ in } L^{\tilde{p}}(S;\mathbb{R}^n).$$

Moreover, we have for any $\varphi \in L^\infty(S;\mathbb{R}^n)$, by virtue of Hölder's inequality, Cauchy-Schwarz' inequality, (7.36) and (7.31) that

$$\lim_{m\to\infty} \left| \int\limits_S (\gamma_m - \gamma)|\nabla f_m|^{p-2} \nabla f_m \cdot \varphi d\lambda \right| = 0.$$

Consequently, we obtain for any $\varphi \in L^\infty(S;\mathbb{R}^n)$ by recalling (7.38) that

$$\int_S (\alpha - \gamma\zeta)\cdot\varphi d\lambda = \lim_{m\to\infty}\int_S (\gamma_m|\nabla f_m|^{p-2}\nabla f_m - \gamma|\nabla f_m|^{p-2}\nabla f_m)\cdot\varphi d\lambda = 0.$$

This implies $\alpha = \gamma\zeta$.

Proof of (IV). Since particularly $\lim_{m\to\infty}\gamma_m - \gamma = 0$ in $L^1(S)$ we have, by passing to a subsequence if necessary, that $\lim_{m\to\infty}(\gamma - \gamma_m)|\varphi|^p = 0$ a.e. on S for any given $\varphi \in L^p(S;\mathbb{R}^n)$. Since plainly $|(\gamma-\gamma_m)|\varphi|^p| \leq 2g_2|\varphi|^p \in L^1(S)$, dominated convergence yields (IV).

Proof of (7.40). Let $\varphi \in L^p(S;\mathbb{R}^n)$. First of all it is a direct consequence of Cauchy-Schwarz' inequality for the Euclidean norm that

$$\gamma_m(|\nabla f_m|^{p-2}\nabla f_m - |\varphi|^{p-2}\varphi)\cdot(\nabla f_m - \varphi) \geq 0,\ \forall m \in \mathbb{N}$$

and consequently

$$\int_S \gamma_m|\varphi|^{p-2}\varphi\cdot(\nabla f_m - \varphi)d\lambda \leq \int_S \gamma_m|\nabla f_m|^{p-2}\nabla f_m\cdot(\nabla f_m - \varphi)d\lambda,\ \forall m\in\mathbb{N},\ \varphi \in L^p(S;\mathbb{R}^n).$$

(Hereby the existence of both integrals is a direct consequence of the boundedness of γ_m and Hölder's inequality.)

The last yields, by using at first (II) and (IV), then the last inequality, and finally (I) as well as (III) that

$$\int_S \gamma|\varphi|^{p-2}\varphi\cdot(\nabla F - \varphi)d\lambda \leq \int_S F\hat{F} - \gamma\zeta\cdot\varphi d\lambda,\ \forall\varphi \in L^p(S;\mathbb{R}^n), \tag{7.43}$$

Now note that it follows from $f_m \in D(A_p^d(\gamma_m))$ that particularly $f_m \in W^{1,p}(S)\cap L^\infty(S)$. Consequently, by (7.39) we get

$$\int_S \gamma\zeta\cdot\nabla f_m d\lambda = \int_S \hat{F} f_m d\lambda,\ \forall m \in \mathbb{N}. \tag{7.44}$$

Now note that combining (7.33), (7.37) and (7.44) yields

$$\int_S \gamma\zeta\cdot\nabla F d\lambda = \int_S \hat{F} F d\lambda$$

Consequently, we infer from (7.43) that

$$\int_S \gamma|\varphi|^{p-2}\varphi\cdot(\nabla F - \varphi)d\lambda \leq \int_S \gamma\zeta\cdot(\nabla F - \varphi)d\lambda,\ \forall\varphi\in L^p(S;\mathbb{R}^n). \tag{7.45}$$

Now note that $\nabla F \in L^p(S;\mathbb{R}^n)$ which implies that $\nabla F - \alpha\varphi$ is, for any $\alpha \in (0,\infty)$ and $\varphi \in L^p(S;\mathbb{R}^n)$ a valid choice as a test function in (7.45). Hence, using $\nabla F - \alpha\varphi$ as a test function in (7.45) and dividing the resulting equation by α yields

$$\int_S \gamma|\nabla F - \alpha\varphi|^{p-2}(\nabla F - \alpha\varphi)\cdot\varphi d\lambda \leq \int_S \gamma\zeta\cdot\varphi d\lambda,\ \forall\varphi \in L^p(S;\mathbb{R}^n),\ \alpha \in (0,\infty). \tag{7.46}$$

It is obvious that

$$\lim_{\alpha\downarrow 0}\gamma|\nabla F - \alpha\varphi|^{p-2}(\nabla F - \alpha\varphi)\cdot\varphi = \gamma|\nabla F|^{p-2}\nabla F\cdot\varphi \text{ a.e. on } S$$

for a given $\varphi \in L^p(S;\mathbb{R}^n)$. Now let $\varepsilon > 0$ and $\alpha \in (0,\varepsilon)$. Then one instantly verifies that

$$|\gamma|\nabla f - \alpha\varphi|^{p-2}(\nabla F - \alpha\varphi)\cdot\varphi| \leq g_2(|\nabla F| + \varepsilon|\varphi|)^{p-1}|\varphi| \text{ a.e. on } S$$

for any $\varphi \in L^p(S;\mathbb{R}^n)$. Now it is a direct consequence of Hölder's inequality that the right-hand-side of the last inequality is in $L^1(S)$ for any $\varphi \in L^p(S;\mathbb{R}^n)$. Hence, it follows by virtue of dominated convergence that

$$\lim_{\alpha\downarrow 0}\int_S \gamma|\nabla F - \alpha\varphi|^{p-2}(\nabla f - \alpha\varphi)\cdot\varphi d\lambda = \int_S \gamma|\nabla F|^{p-2}\nabla F\cdot\varphi d\lambda.$$

Consequently, it follows by recalling (7.46) that

$$\int_S \gamma|\nabla F|^{p-2}\nabla F\cdot\varphi d\lambda \leq \int_S \gamma\zeta\cdot\varphi d\lambda,\ \forall\varphi \in L^p(S;\mathbb{R}^n).$$

Conclusively, replacing φ by $-\varphi$ implies

$$\int_S (\gamma|\nabla F|^{p-2}\nabla F - \gamma\zeta)\cdot\varphi d\lambda = 0,\ \forall\varphi \in L^p(S;\mathbb{R}^n).$$

Finally, this yields $\gamma|\nabla F|^{p-2}\nabla F - \gamma\zeta = 0$ a.e. on S which implies (7.40) since particularly $\gamma \neq 0$ a.e. on S. □

Lemma 7.H.3. *$\tau(L^1(\Omega;L^1(S)))$ as well as $L^{1,\infty}(\Omega;L^1(S))$ are dense subsets of $L^1(\Omega;L^1(S))$.*

Proof. Firstly, note that clearly $\tau(L^1(\Omega;L^1(S))) \subseteq L^{1,\infty}(\Omega;L^1(S))$ which implies that it suffices to prove the claim for $\tau(L^1(\Omega;L^1(S)))$.
Now let $f \in L^1(\Omega;L^1(S))$ and introduce $f_k := \tau_k(f)$ for each $k \in \mathbb{N}$.
As $\lim_{k\to\infty}\tau_k(s) = s$ for each $s \in \mathbb{R}$ it is clear that $\lim_{k\to\infty} f_k(\omega) = f(\omega)$ a.e. on S for every given $\omega \in \Omega$, up to a $\mathbb{P}$-nullset.

Since $|f_k(\omega) - f(\omega)| \leq 2|f(\omega)| \in L^1(S)$, Lebesgue's theorem yields

$$\lim_{k \to \infty} ||f_k(\omega) - f(\omega)||_{L^1(S)} = 0, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Moreover, $||f_k(\cdot) - f(\cdot)||_{L^1(S)} \leq 2||f(\cdot)||_{L^1(S)} \in L^1(\Omega)$, which implies, by applying Lebesgue's Theorem again, that $\lim_{k \to \infty} f_k = f$ in $L^1(\Omega; L^1(S))$. $\square$

Chapter 8

Summary & Outlook

In this thesis, we have established numerous existence, uniqueness and asymptotic results. At first, we demonstrated that the p-Laplacian semigroup $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ fulfills an L^∞-L^p-contraction principle for "large" p (Theorem 3.4.10), that $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ extincts in finite time for "small" p (Theorem 3.5.6); and we have derived a decay estimate for $||T_{\mathcal{A}_p}(t)v - \overline{(v)}_S||_{L^1(S)}$ which is independent of the initial $v \in L^1(S)$, if $p \in (2,\infty)$ and γ is sufficiently integrable, see Theorem 3.5.10. Moreover, as a side-effect of our L^∞-L^p-contraction principle, we also obtained a regularity result on the Hölder continuity of this semigroup, see Remark 3.4.11.
Even though these are strong results, it might be possible to improve them. Particularly, investigating the space (or time) regularity of solutions in greater detail is an interesting way to continue the research on the weighted p-Laplacian evolution equation.

Afterwards, we had developed an existence/uniqueness and asymptotic theory for ACPRM-processes. In Chapter 4, we have introduced the notions of strong and mild solutions of (ACPRM) and derived convenient criteria guaranteeing the existence of a unique strong/mild solution, see Proposition 4.3.10 and Theorem 4.3.12. Moreover, we exemplified the applicability of these results at hand of $(T_{\mathcal{A}_p}(t))_{t\geq 0}$ as well as the two real-valued semigroups introduced in Remark 2.2.7. In addition, we have seen that the mild/strong solution of (ACPRM) must be an ACPRM-process. This fact is highly owed to the structure of the noise term "$\eta(t,z)N_\Theta(dt \otimes z)$", which is a pure-jump noise. It might be possible, to extend this to continuous noise by employing the theory of inhomogeneous abstract Cauchy problems, but in this case it seems very unlikely that the solution still admits a representation formula which is as nice as it is in the current setting. Nevertheless, doing that is an interesting (and probably challenging) task.
In addition, we have devoted two chapters of this thesis to the asymptotic behavior of ACPRM-processes. In Chapter 5 we exploited a finite extinction assumption on the involved semigroup, to derive an SLLN and a CLT for vector-valued functionals of ACPRM-processes, see Theorem 5.2.23 and Corollary 5.2.24. This required, among other things, that the noise terms $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$ are i.i.d. sequences, which are independent of each other and jointly independent of the initial x. Even though our proofs

heavily relied on these i.i.d.-assumptions, there is a chance that they can be relaxed: As outlined in Remark 5.2.6 the results in Chapter 5, rely on the fact that the sequence $\left(\int_{\alpha_{e_x(n)}}^{\alpha_{e_x(n+1)}} \Xi(\mathbb{X}_x(\tau))d\tau \right)_{n\in\mathbb{N}}$ is an i.i.d.-sequence. This raises the intriguing question, whether some weaker distributional assumptions on $(\beta_m)_{m\in\mathbb{N}}$, $(\eta_m)_{m\in\mathbb{N}}$ and x imply that the sequence of integrals fulfill some similar weaker conditions, which are still good enough to derive an SLLN and a CLT. Of course, one runs the risk, that one cannot prove an SLLN and a CLT for vector-valued functionals anymore, but only for real-valued ones.

Moreover, in Chapter 6 we demonstrated that ACPRM-processes are time-homogeneous Markov processes, if the noise terms $(\beta_m)_{m\in\mathbb{N}}$ and $(\eta_m)_{m\in\mathbb{N}}$ are i.i.d. sequences, which are independent of each other and jointly independent of the initial x, and if each β_m is exponentially distributed, see Theorem 6.2.4. The present author does not believe, that any of these conditions can be dropped. As demonstrated, these results enable one to prove an SLLN (see Theorem 6.3.6), if the underlying semigroup decays polynomially, and even a CLT (see Theorem 6.3.10), if this polynomial decay is sufficiently fast - but in contrary to the results in Chapter 5 only for real-valued, Lipschitz continuous functionals. Hereby, it is worth noting that establishing the Markov property, did not require any decay assumptions on the involved semigroup, and that the proofs in Section 6.3 relied on the general results in [39] and [18]. Thus, it might be possible that one can prove an SLLN and a CLT under different decay assumptions, employing techniques similar to those used in Section 6.3.

Finally, in Chapter 7 we demonstrated that the weight function occurring in the weighted p-Laplacian evolution equation, can be replaced by an $L^1(S)$-valued random variable g, fulfilling $0 < g_1 \leq g \leq g_2$, for some constants $g_1, g_2 \in (0,\infty)$. We managed to derive existence and uniqueness of strong solutions for the resulting equation (see Theorem 7.4.2), were able to transfer the asymptotic results for the p-Laplacian semigroup to the new randomized case (Section 7.5), and last but not least we derived bounds for the tail function if p is "small", see Theorem 7.6.8. One obvious way of extending these results, is trying to derive tail function bounds for "large" p. Another (probably way more challenging) generalization would be to weaken the assumptions on g, such that $g(\omega)$ fulfills for $\mathbb{P}$-a.e. $\omega \in \Omega$, the assumptions the weight function γ in Chapter 3 had to fulfill. This change is not as innocuous as it seems on a first glance: Firstly, we then would have to work with a "random" Sobolev space $W^{1,p}_{g(\omega)}(S)$, and secondly the (long and delicate) proof of Lemma 7.H.2 fails in this case.

In conclusion, we have gained new insights into the asymptotic behavior of the p-Laplacian semigroup, developed an existence/uniqueness and asymptotic theory for ACPRM-processes, and have extended the weighted p-Laplacian evolution equation to the randomized case.

The present author hopes that others will apply the results developed here, that the reader found the theory presented in this monograph appealing; and, of course that this thesis might encouraged one, to refine the developed results in one of the aforementioned ways.

Notation

A_p	p-Laplace operator, see Definition 3.2.2
$\mathcal{A}_p$	Closure of A_p, see Definition 3.2.4
$\mathfrak{B}(M)$	Borel σ-Algebra on a topological space (M, τ)
λ	Lebesgue measure
$L^q(K, \Sigma, \mu; V)$	Usual Bochner spaces, see Remark 2.2.8
$L^q_0(S)$	Elements of $L^q(S, \mathfrak{B}(S), \lambda; \mathbb{R})$ which are centered, see Remark 3.3.4
$\mathcal{M}(K, \Sigma; M)$	Space of Σ-$\mathfrak{B}(M)$-measurable functions, see Remark 2.2.1
∇	$\nabla\varphi$ is the vector of weak derivatives of a weakly diff. function φ, see Section 3.2
$(\Omega, \mathcal{F}, \mathbb{P})$	Complete Probability Space
$\mathbb{P}_Y$	The law of a random variable Y, see Remark 2.2.10
$(T_{\mathcal{A}_p}(t))_{t\geq 0}$	The p-Laplacian semigroup, see Remark 3.2.6
$(T_{\rho_i}(t))_{t\geq 0}$	See Remark 2.2.7
$W^{1,1}_{\mathrm{Loc}}((0,\infty); V)$	V-valued functions which are loc. abs. cont. and diff. a.e., See Definition 2.1.2
$\mathbb{X}_x$	An ACPRM-process, see Definition 2.2.2 and Remark 2.2.4
$x \cdot y$	Inner product of any $x, y \in \mathbb{R}^m$, where $m \in \mathbb{N}$.
$\lvert \cdot \rvert$	Euclidean norm on $\mathbb{R}^m$, where $m \in \mathbb{N}$.

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- A. Nerlich, *Abstract Cauchy Problems in separable Banach Spaces driven by random Measures: Existence and Uniqueness* (Submitted in 2017: Under Review); Arxiv: https://arxiv.org/pdf/1710.01796.pdf
- A. Nerlich, *Abstract Cauchy Problems in separable Banach Spaces driven by random Measures: Asymptotic Results in the finite extinction Case* (Submitted in 2017: Under Review); Arxiv: https://arxiv.org/pdf/1710.01795.pdf
- A. Nerlich, *A randomized weighted p-Laplacian evolution Equation with Neumann Boundary Conditions*, Nonlinear Differential Equations and Applications, 2018
- A. Nerlich, *A Markov Process Approach to the asymptotic Theory of abstract Cauchy Problems driven by Poisson Processes*, (Submitted in 2018); Arxiv: https://arxiv.org/pdf/1801.05726.pdf

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- A. Nerlich, *Asymptotic Results for Solutions of a weighted p-Laplacian evolution Equation with Neumann Boundary Conditions*, Nonlinear Differential Equations and Applications, 2017
- A. Nerlich, *A randomized weighted p-Laplacian evolution Equation with Neumann Boundary Conditions*, Nonlinear Differential Equations and Applications, 2018

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