Embeddings and decompositions of graphs and hypergraphs

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften der

Universität Ulm

vorgelegt von
Stefan Ehard
aus Roth

2020

Amtierender Dekan:	Prof. Dr. Martin Müller
1. Gutachter:	Prof. Dr. Dieter Rautenbach
2. Gutachter:	Prof. Dr. Felix Joos
3. Gutachterin:	Prof. Dr. Maria Axenovich
Tag der Promotion:	16.11.2020

Overview

The theme of decomposing mathematical objects appears in nearly every field of mathematics. Not surprisingly, it is also a vibrant research area in discrete mathematics and combinatorics with many fundamental questions that are still unanswered. For instance, early work dates already back to the study of Latin squares by Leonhard Euler in the 18th century as well as the study of Steiner (triple) systems by Jakob Steiner and Thomas Kirkman in the 19th century. Recently, there has been some exciting progress in this area. This thesis adds to this body of research and contains various new results on embeddings and decompositions of graphs and hypergraphs. Our motivating meta-question can be phrased as follows:

Suppose we are given a (hyper)graph G and a (hyper)graph H.

(1) *Can we find H as a subgraph in G?*

(2) *Can we even decompose the edge set of G into edge-disjoint copies of H?*

Question (1) is well-studied in many aspects. Early instances include Turán's and Ramsey's theorem as well as Hall's and Tutte's characterizations for a graph to contain a perfect matching. Frequently studied problems concern the setting when G and H have roughly the same number of vertices. Classical results are Dirac's theorem on the minimum degree threshold for the existence of a Hamilton cycle and the celebrated Hajnal-Szemerédi theorem.

In 1997, Komlós, Sárközy and Szemerédi developed a powerful tool that reshaped the landscape of extremal combinatorics and extremal graph theory, called the 'blow-up lemma'. It answers question (1) in the affirmative for general bounded degree graphs H, given that G is sufficiently large and quasirandom. That is, the blow-up lemma informally states that (multipartite) quasirandom graphs behave as if they were complete for the purpose of embedding spanning bounded degree graphs. The power of the blow-up lemma arises in combination with Szemerédi's regularity lemma, which allows to 'regularise' any large enough graph into a bounded number of such quasirandom pairs.

Naturally, decomposition problems of the type of question (2) are usually harder. Besides the classical theorem of Walecki from the 1890s on decompositions of the complete graph into Hamilton cycles, most achievements on question (2) have been obtained more recently. In 2019, Kim, Kühn, Osthus and Tyomkyn greatly extended the blow-up lemma by proving a 'blow-up lemma for approximate decompositions'. It says that also question (2) is approximately true for general bounded degree graphs H, given that G is sufficiently large and quasirandom. In more detail, the blow-up lemma for approximate decompositions states that the edge set of a multipartite quasirandom graph can be almost decomposed into any collection of bounded degree graphs with the same multipartite structure and slightly fewer edges.

This thesis contains three novel results concerning questions (1) and (2): we prove a 'rainbow blow-up lemma for almost optimally bounded edge-colourings', we give a short proof of the aforementioned blow-up lemma for approximate decompositions, and most notably, we lift the blow-up lemma for approximate decompositions to the setting of hypergraphs. We explain these results in more detail in the following.

A natural variant of question (1) is to impose further restrictions or additional properties on H and G. One way to incorporate this is to assign colours to the edges of the host graph G. A subgraph of an edge-coloured graph is called *rainbow* if all its edges have different colours. Many combinatorial problems can be phrased as a rainbow subgraph problem, as for instance the famous Ryser–Brualdi–Stein conjecture on partial transversals in Latin squares as well as the graceful labelling conjecture. Recently, Montgomery, Pokrovskiy and Sudakov proved a long-standing conjecture of Ringel on graph decompositions into trees by reducing it to a rainbow subgraph problem. In this thesis we prove a rainbow version of the original blow-up lemma that applies to almost optimally bounded colourings. Our result implies that there exists a rainbow copy of any bounded degree spanning subgraph H in a quasirandom host graph G, assuming that the edge-colouring of G is such that there are only a few more colours present than actually needed so that H can be rainbow. We apply our rainbow blow-up lemma to obtain new results for graph decompositions, orthogonal double covers and graph labellings following the work of Montgomery, Pokrovskiy and Sudakov.

The second main contribution is a significantly shorter proof of the blow-up lemma for approximate decompositions. In fact, we prove a more general theorem that yields approximate decompositions with stronger quasirandom properties, which lead to an easier applicability of the theorem. It can be applied in combination with Keevash's results on designs such that we obtain new perfect decomposition results of quasirandom graphs into regular spanning subgraphs. Our proof method also gives rise to decompositions of directed graphs.

All previously mentioned results apply only to the setting of graphs. In fact, only very few results have been obtained that concern hypergraph decompositions into (essentially) spanning structures. Here, we extend the blow-up lemma for approximate decompositions to hypergraphs. This answers question (2) in the affirmative for general bounded degree hypergraphs H, given that G is sufficiently large and quasirandom. That is, we prove that any quasirandom hypergraph G can be approximately decomposed into any collection of bounded degree hypergraphs with almost as many edges. The result also applies to multipartite hypergraphs and even to the sparse setting when the edge density of G tends to 0 in terms of the number of vertices of G. This answers and addresses questions of Kim, Kühn, Osthus and Tyomkyn as well as Keevash.

A key ingredient for the proofs of the blow-up lemmas in this thesis is a result on pseudorandom hypergraph matchings. A celebrated theorem of Pippenger states that any almost regular hypergraph with small pair-degrees has an almost perfect matching. It is based on Rödl's resolution of the Erdős-Hanani conjecture by the invention of the nowadays famous 'Rödl nibble'. We show that one can find such an almost perfect matching which is random-like with respect to a collection of weight functions.

Acknowledgement

I had the great pleasure to enjoy not only one but two truly excellent supervisors during my time as a (PhD) student. I am deeply grateful to Dieter Rautenbach and Felix Joos — to Dieter for giving me all possible opportunities, for all your support, for many nice projects and discussions we had, and for many laughs together; to Felix for endless hours of nice discussions, for your guidance and knowledge, for all the motivation (even for exhausting problems), and for the generous invitations to Birmingham.

I always enjoyed my time working at the institute and I was very lucky to have great colleagues and friends around me. Thanks for many lunch breaks, conferences and (non-math) talks spent together.

I am also thankful to all my coauthors [14, 15, 20, 26, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. It was a pleasure working with you.

This thesis would not exist without the huge support and the love from my family, my friends and my girlfriend. Thank you!

Finally, I feel very grateful to the Studienstiftung des Deutschen Volkes for all the support that I received since the early stages of my studies.

Contents

Chapter 1

Introduction

The theme of decomposing 'large' objects into 'smaller' objects or finding a maximal number of specified 'small' objects in a 'larger' object is among the key topics in mathematics and there are fundamental decomposition theorems in almost all areas of mathematics, such as the decomposition of an integer into its prime factors, various results on matrix decompositions into a product of matrices, Doob decompositions of stochastic processes and martingales, Lebesgue, Hahn and Jordan decompositions of measures, Helmholtz decompositions of vector fields, decomposition of manifolds, and decomposition of groups and Lie groups. In discrete mathematics and combinatorics, problems on decompositions already appear in Euler's question from 1782 for which n there exist pairs of orthogonal Latin squares of order n, in Steiner's questions for Steiner systems from the 1850s which cumulated in the 'existence of designs' question, in Walecki's theorem on decompositions of complete graphs into edge-disjoint Hamilton cycles from the 1890s, and in Kirkman's famous 'school girl problem'. These questions and results set off an entire branch of combinatorics and design theory. It is nowadays a vibrant research area with several beautiful results and conjectures as well as some exciting progress in the last decades.

In this thesis, I present several new results on embeddings and decompositions of graphs and hypergraphs. In particular, the main results are versatile tools for attacking such embedding and decomposition problems. I briefly introduce the main results in Section 1.3.

1.1 Decompositions of graphs and hypergraphs

This thesis is on packings, embeddings and decompositions of graphs and hypergraphs. I will highlight some of the history and recent advances in Section 1.1.2. But what is a *graph* or *hypergraph*? What is a *packing*, *embedding* or *decomposition* of a (hyper)graph? Let us make this precise first.

1.1.1 Terminology

A *hypergraph* G consists of a vertex set $V(G)$ and an edge set $E(G)$ which is a set of subsets of $V(G)$, and we denote by $v(G)$ and $e(G)$ the number of vertices and edges of G, respectively. We say G is k-uniform or simply a k-graph if all edges have size k. A 2-graph is simply called a *graph*. For a collection $\mathcal{H}$ of (hyper)graphs and a (hyper)graph G, we say there is a *packing* of $\mathcal{H}$ into G if there are edge-disjoint copies of the members of $\mathcal{H}$ in G. If $\mathcal{H}$ consists only of a single (hyper)graph H, we often call such a packing an *embedding* of H into G. The collection $\mathcal{H}$ *decomposes* G if additionally $\sum_{H\in\mathcal{H}} e(H) = e(G)$. We also say that there is an H-*decomposition* of G

into copies of a (hyper)graph H if the edge set of G can be partitioned into copies of H.

1.1.2 History — some beautiful conjectures and results

Rather than providing a complete historical background, I will highlight some famous advances and conjectures in the area of graph and hypergraph decompositions. There is a large collection of surveys emphasising certain aspects of the area, to which I refer the reader for a more comprehensive outline of further results and proof techniques [48, 54, 55, 89, 91, 93, 113, 121, 122, 124].

Designs and Steiner systems

Let us first come back to Steiner's question for Steiner systems from the 1850s and the question of the 'existence of designs'. An (n, q, r, λ)-*design* is a set B of q-element subsets (called 'blocks') of some n-element set V, such that every r-element subset of V belongs to exactly λ blocks of B. It can be easily seen that there are some obvious and necessary 'divisibility conditions' for the existence of a design. An $(n, q, r, 1)$-design is also called an (n, q, r)-*Steiner system*. In the 1850s Jakob Steiner asked for which parameters these systems exist which cumulated in the analogous 'existence of designs' question. In more detail, the 'existence conjecture' states that for given parameters q, r, λ, the necessary divisibility conditions are also sufficient for the existence of an (n, q, r, λ)-design except for a finite number of n.

It took over a century, until Wilson [118, 119, 120] famously established this question for $r = 2$. For larger values of r, only very little was known until recently. In particular, it was even open whether there exist infinitely many Steiner systems for $r \geq 4$, and for $r \geq 6$, not a single example of a Steiner system was known. Rödl [111] gave an approximate solution to the existence conjecture which established a conjecture of Erdős and Hanani. Not only the result itself but also the proof method known as the 'Rödl nibble' became a cornerstone in the area. In a phenomenal achievement in 2014, Keevash [71] established the existence of designs in general. In fact, Keevash considered the more general setting of decomposing sufficiently large and quasirandom hypergraphs into cliques of fixed size. Note that an (n, q, r)-Steiner system is equivalent to the decomposition of the complete r-graph $K_n^{(r)}$ on n vertices into r-uniform cliques $K_q^{(r)}$ on q vertices. The result of Keevash was generalized by Glock, Kühn, Lo and Osthus [52] to the setting of hypergraph decompositions into arbitrary hypergraphs of fixed size, before Keevash lifted his result to a more general framework in [72]. This includes the decomposition into H-factors of fixed size,[1] which relates to *resolvable designs*, and the decomposition of multipartite quasirandom hypergraphs into hypergraphs of fixed size. A resolvable design is the general form of Kirkman's famous 'school girl problem'. I refer to [117] for more history on this topic. Further, note that Ray-Chaudhuri and Wilson [108, 109] previously established the existence for resolvable designs for $r = 2$.

Hamilton decompositions of graphs

Whereas the results in the previous section deal with H-decompositions of (hyper)-graphs G into (hyper)graphs H of fixed size, there is also a large amount of results on H-decompositions for graphs where the number of vertices of H equals the number of vertices of G, or is at least comparable. This dates already back to the classical

[1] An H-factor in a hypergraph G is a set of vertex-disjoint copies of H that together cover all vertices of G.

result of Walecki from the 1890s on decompositions of the complete graph K_{2n+1} into Hamilton cycles. There are plenty of further results on Hamilton decompositions and I want to highlight two exciting results that have been established in the last decade.

Kühn and Osthus [92] famously resolved Kelly's conjecture stating that every regular tournament[2] has a decomposition into Hamilton cycles. In fact, they proved a more general decomposition result based on robust expansion.

Nash-Williams [105, 106] raised the problem of decomposing a D-regular graph for even D into Hamilton cycles. This was answered in more general by Csaba, Kühn, Lo, Osthus and Treglown [25] who resolved the Hamilton decomposition conjecture, that is, the decomposition of every D-regular graph on n vertices for large n and $D \geq \lfloor n/2 \rfloor$ into Hamilton cycles and at most one perfect matching.

Recent advances on graph decompositions

The following three prominent conjectures have been wide open for the last decades, until recently some striking advances have been achieved. The first two conjectures concern decompositions of the complete graph into trees.

Conjecture 1.1 (Ringel, 1963, [110])**.** *For all n and all trees T on $n+1$ vertices, there is a decomposition of K_{2n+1} into $2n+1$ copies of T.*

Conjecture 1.2 (Tree packing conjecture – Gyárfás and Lehel, 1976, [58])**.** *For all n and all sequences of trees $T_1, \dots, T_n$ where T_i has i vertices, there is a decomposition of K_n into $T_1, \dots, T_n$.*

The following third conjecture concerns the decomposition of the complete graph into cycle factors. It was posed by Ringel at an Oberwolfach meeting in 1967, where he asked whether there exists a seating chart for $2n+1$ people at n diners around round tables such that every person sits next to every other person exactly once.

Conjecture 1.3 (Oberwolfach problem – Ringel, 1967, cf. [94])**.** *For all odd n and all 2-regular graphs F on n vertices, there is a decomposition of K_n into copies of F.*

Most of the recent progress on these conjectures is based on very elaborated analyses of random processes. This includes the resolution of Conjectures 1.1 and 1.2 for bounded degree trees [65], the complete resolutions of Conjecture 1.1 in [76, 103] and of Conjecture 1.3 in [51, 75], and the resolution of Conjecture 1.2 for families of trees with many leaves [5]. All these results hold for sufficiently large n.

In this thesis I present several new results on embeddings and decompositions of graphs and hypergraphs. In particular, the main results are versatile tools for attacking embedding and decomposition problems. I briefly introduce these results in Section 1.3.

1.2 The blow-up lemma

The blow-up lemma by Komlós, Sárközy and Szemerédi [82] is a powerful result in extremal combinatorics for embedding a spanning bounded degree graph into a dense host graph. Clearly, the task of finding one specific spanning subgraph becomes trivial if the host graph is complete. Informally, the blow-up lemma states that dense quasirandom bipartite graphs behave as complete bipartite graphs for the purpose of embedding a bounded degree graph. The power of the blow-up lemma therefore arises from the combination with Szemerédi's regularity lemma [116], which provides such a partition

[2] A tournament is an orientation of the complete graph. It is regular if the indegree of every vertex equals its outdegree.

of any host graph into quasirandom bipartite graphs. Roughly speaking, Szemerédi's regularity lemma guarantees a partition of any large enough graph into a bounded number of bipartite pairs that behave random-like, and the blow-up lemma enables to embed any (spanning) bounded degree graph into such random-like host graphs. We refer to Theorem 4.5 in Chapter 4 for an explicit statement of the blow-up lemma.

The blow-up lemma has had major impact on extremal graph theory and lead to a series of very strong results: the existence of spanning trees in dense graphs [86], the proofs of the Pósa-Seymour conjecture [83] and the Seymour conjecture [84] on the minimum degree condition for the existence of the kth power of a Hamilton cycle, the proof of the Alon-Yuster conjecture [85] on the minimum degree condition for the existence of an H-factor, the proof of the bandwith conjecture of Bollobás and Komlós [19], and the proofs of Kühn and Osthus of Kelly's conjecture [92] as well as for the minimum degree threshold for perfect graph packings [90]. We also refer to the surveys [89, 113, 124] for further results and developments.

Many variations of the blow-up lemma have been obtained over the years (see also [18, 24, 112]), which includes versions for sparse host graphs due to Allen, Böttcher, Hàn, Kohayakawa and Person [6], a hypergraph version due to Keevash [70], and a version suitable for graph decompositions due to Kim, Kühn, Osthus and Tyomkyn [79]. The latter one is a far-reaching generalization of the original blow-up lemma. Kim, Kühn, Osthus and Tyomkyn proved that one cannot only find one single copy of a bounded degree graph H in G (as in the usual blow-up lemma), but in fact, G can almost be decomposed into copies of H. This result immediately implies various decomposition conjectures approximately such as Conjecture 1.3, as well as Conjectures 1.1 and 1.2 for bounded degree trees; but even more: combining this approximate decomposition result with certain absorbing techniques allows for complete decompositions. Therefore, the blow-up lemma for approximate decompositions is also one of the key tools for the resolutions of Conjectures 1.1 and 1.2 for bounded degree trees [65] and implicitly also for the complete resolution of Conjecture 1.3 in [51]. However, the proof of the blow-up lemma for approximate decompositions is quite involved, very long and complex.

1.3 Overview of the main results

The main results of this thesis are three generalizations of the original blow-up lemma presented in Chapters 3–5 that we briefly introduce in the following Sections 1.3.1–1.3.3, respectively. In Section 1.3.4, I present the result of Chapter 2 on pseudorandom hypergraph matchings that will be used as a key ingredient in the proofs of our blow-up lemmas. The corresponding chapters contain the explicit statements of the corresponding results as well as further discussions and related work on the individual problem.

1.3.1 A rainbow blow-up lemma

A subgraph of an edge-coloured graph is called *rainbow* if all its edges have different colours. Rainbow colourings appear in many contexts of combinatorics, and many problems beyond graph colouring can be translated into a rainbow subgraph problem, where one aims to find a rainbow subgraph in an edge-coloured host graph. For instance, in a recent breakthrough, Montgomery, Pokrovskiy and Sudakov [103] completely resolved Conjecture 1.1 by reducing it to a rainbow embedding problem. We further discuss problems on rainbow embeddings in Section 3.1.

In Chapter 3 we prove a rainbow version of the original blow-up lemma that applies to 'almost optimally bounded edge-colourings', that is, when the host graph is edge-coloured with only a few more colours than needed. A corollary of this is that there exists a rainbow copy of any bounded degree spanning subgraph H in a quasirandom host graph G, assuming that the edge-colouring of G fulfils a boundedness condition that is asymptotically best possible (cf. Theorems 3.2 and 3.3); that is, there are only a few more colours present in G than actually needed so that H can be rainbow.

In Section 3.7 we discuss applications of this result to graph decompositions, graph labellings and orthogonal double covers following the recent work of Montgomery, Pokrovskiy and Sudakov [104].

These results are presented in Chapter 3 and are joint work together with Stefan Glock and Felix Joos based on [30].

1.3.2 The blow-up lemma for approximate decompositions

As mentioned in Section 1.2, Kim, Kühn, Osthus and Tyomkyn [79] extended the usual blow-up lemma by proving a 'blow-up lemma for approximate decompositions', which can also be applied in combination with Szemerédi's regularity lemma. This result states that for a multipartite quasirandom graph G and a bounded degree graph H, one cannot only find one single copy of H in G, but in fact, G can be almost decomposed into copies of H. In more detail, they even prove that if G is a multipartite quasirandom graph and $\mathcal{H}$ is a collection of bounded degree graphs with the same multipartite structure and slightly fewer edges than G, then $\mathcal{H}$ packs into G. This extends results of Böttcher, Hladkỳ, Piguet and Taraz [16], of Messuti, Rödl and Schacht [100], and of Ferber, Lee and Mousset [41]. The multipartite framework can also be used to obtain results for the non-partite setting; let us state this (simpler) version. For $\varepsilon > 0$ and $d \in (0, 1]$, we say an n-vertex graph G is *(ε, d)-quasirandom* if $|N_G(u)| = (d \pm \varepsilon)n$ and $|N_G(u) \cap N_G(v)| = (d^2 \pm \varepsilon)n$ for all distinct $u, v \in V(G)$.

Theorem 1.4 (Kim, Kühn, Osthus, Tyomkyn [79])**.** *For all $\alpha > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose G is an (ε, d)-quasirandom graph on n vertices and $\mathcal{H}$ is a collection of graphs on at most n vertices with maximum degree at most α^{-1}, $|\mathcal{H}| \leq \alpha^{-1}n$ and $\sum_{H\in\mathcal{H}} e(H) \leq (1-\alpha)e(G)$. Then there is a packing ϕ of $\mathcal{H}$ into G.*

Let us note at this point that Kim, Kühn, Osthus, and Tyomkyn also raised the following question, to which we come back in the next section.

Question 1.5 (Kim, Kühn, Osthus, Tyomkyn [79])**.** *Does Theorem 1.4 also hold when $d \to 0$ for $n \to \infty$?*

In Chapter 4 we present a new and significantly shorter proof of the blow-up lemma for approximate decompositions. In fact, we prove a more general theorem that yields approximate decompositions with stronger quasirandom properties and allow for an easier handling of exceptional vertices (cf. Theorems 4.2 and 4.3 in Chapter 4). This allows for an easier applicability such that one can combine this with Keevash's results on designs [72].

We discuss applications of this result in Section 4.6. This also includes a result for approximate decompositions of directed graphs.

These results are presented in Chapter 4 and are joint work together with Felix Joos based on the preprint [32].

1.3.3 A hypergraph blow-up lemma for approximate decompositions

Chapter 5 contains the most technically involved but also most powerful result of this thesis: We prove that any quasirandom uniform hypergraph G can be approximately decomposed into any collection of bounded degree hypergraphs with almost as many edges. Let us state a simplified version of this result. We use the following notion of quasirandomness as also used by Keevash in [72]. To that end, for a k-graph G and a $(k-1)$-set S of vertices of G, let $N_G(S)$ be the set of vertices that form an edge together with S. For $\varepsilon > 0$, $t \in \mathbb{N}$, $d \in (0,1]$, we say the k-graph G on n vertices is *(ε,t,d)-typical* if $|\bigcap_{S\in\mathcal{S}} N_G(S)| = (1\pm\varepsilon)d^{|\mathcal{S}|}n$ for all sets $\mathcal{S}$ of $(k-1)$-sets of vertices of G with $|\mathcal{S}| \leq t$. We refer to the (vertex) degree of a vertex v as the number of edges containing v.

Theorem 1.6 (Ehard, Joos [31]). *For all $\alpha > 0$, there exist $n_0, t \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose G is an (ε,t,d)-typical k-graph on n vertices with $k \leq \alpha^{-1}$, $d \geq n^{-\varepsilon}$ and $H_1,\dots,H_\ell$ is a collection of k-graphs on n vertices with maximum vertex degree at most α^{-1} such that $\sum_{i\in[\ell]} e(H_i) \leq (1-\alpha)e(G)$. Then $H_1,\dots,H_\ell$ pack into G.*

Our results also apply to the sparse setting when the edge-density of the host hypergraph G tends to 0 in terms of the number of vertices (cf. that we allow for $d \geq n^{-\varepsilon}$ in Theorem 1.6). This answers Question 1.5 of Kim, Kühn, Osthus and Tyomkyn in a strong form. Further, our main result is also tailored for multipartite hypergraphs. Hence, we lift the hypergraph blow-up lemma for embedding one single hypergraph due to Keevash [70] as well as the blow-up lemma for approximate decompositions of graphs due to Kim, Kühn, Osthus and Tyomkyn to the setting of decomposing (sparse) quasirandom hypergraphs. They both explicitly asked for such a result in [73] and [79]. However, our notion of quasirandomness is a stronger assumption than hypergraph regularity as used in [70] (cf. the precise statements of our results in Section 5.1.1).

So far, there have been only few results for decompositions of hypergraphs into spanning structures, such as various types of Hamilton cycles [12, 45, 46] as well as Keevash's results on H-factors [72]. Our result is an extension to decompositions of quasirandom hypergraphs into any spanning hypergraph H with bounded maximum degree. Naturally, since we allow for decompositions into arbitrary hypergraphs H and do not require any further restrictions on the structure of H except that the maximum degree is bounded, one cannot hope to obtain a perfect decomposition result in general.

We discuss applications of the main results of Chapter 5 in Section 5.8. In particular, we pose analogues of Conjectures 1.1–1.3 for hypergraphs and our main results imply approximate versions thereof. Further, we illustrate applications to decompositions of simplicial complexes.

The results of Chapter 5 are joint work together with Felix Joos based on the preprint [31].

1.3.4 Pseudorandom hypergraph matchings

Hypergraph matchings are a versatile concept as many questions in combinatorics can be formulated as a matching problem in a hypergraph. Note that a matching in a hypergraph is a collection of pairwise disjoint edges. A celebrated theorem of Pippenger states that any almost regular hypergraph with small codegrees has an almost perfect matching.[3] This generalizes Rödl's result used for the solution of the Erdős-Hanani conjecture. In Chapter 2 we show that one can find such an almost perfect matching which is 'pseudorandom'. For instance, the matching contains as many edges from a given set of edges as we would expect in the setting of an idealized random matching. More generally, we allow to introduce weight functions that assign non-negative weight

[3] The codegree or pair-degree of two vertices in a hypergraph is the number of edges where these two vertices appear together.

to tuples of edges of the hypergraph and our result yields a matching that contains as much weight as we would expect in the setting of an idealized random matching. I refer to Theorems 2.2 and 2.3 in Chapter 2 for exact statements of this result. It will be a key ingredient in the main proofs of Chapters 3–5.

In Chapter 2 we provide more background on hypergraph matchings together and the main results, which are joint work together with Stefan Glock and Felix Joos based on [29].

1.4 A general proof outline

In this section we outline the general strategy and the high-level approach that underpin the proofs of the presented blow-up lemmas in Chapters 3–5. The proofs become gradually more complex starting from Chapter 3 to Chapter 5, and we give more detailed proof overviews in each individual chapter.

Suppose H and G are multipartite graphs with vertex partitions $(X_1, \dots, X_r)$ and $(V_1, \dots, V_r)$, respectively, with $|X_i| = |V_i|$, and we aim to embed H into G so that X_i is mapped onto V_i.

In the literature, there are two common approaches for proving blow-up lemmas. The original approach of Komlós, Sárközy and Szemerédi consists of a randomised sequential embedding algorithm, which embeds the vertices of H one-by-one, choosing each time a random image from all available ones. This strategy has also been used in [6, 18, 24, 70].

Shortly after the appearance of the blow-up lemma, Rödl and Ruciński [112] developed an alternative proof, where instead of embedding vertices one-by-one, the algorithm consists of only a constant number of steps. In the i-th step, the whole cluster X_i is embedded into V_i. The desired bijection is obtained as a perfect matching within a 'candidacy graph' A_i, which is an auxiliary bipartite graph between X_i and V_i where $xv \in E(A_i)$ only if v is still a suitable image for x. Although these candidacy graphs (of clusters not yet embedded) become sparser after each step, Rödl and Ruciński were able to show that one can maintain their super-regularity throughout the procedure. This approach was also employed in [50, 79] and also underpins our general proof strategy.

In the individual settings of the blow-up lemmas presented in this thesis, we have to impose additional restrictions on the matching that we want to find within the candidacy graph A_i; for instance, that the used edges in G when embedding the vertices X_i into V_i are rainbow, or that they are still edge-disjoint if we pack a collection of graphs H into G. We encode these restrictions in an auxiliary hypergraph $\mathcal{H}_{aux}$ such that a hypergraph matching in $\mathcal{H}_{aux}$ will correspond to a matching in A_i that satisfies the required restrictions. By employing our result for pseudorandom hypergraph matchings from Chapter 2, we can find a matching in $\mathcal{H}_{aux}$ that maps almost all vertices of X_i onto V_i and satisfies several pseudorandom properties. In particular, these properties will guarantee that the updated candidacy graphs for the next embedding rounds will still be super-regular such that we can iteratively find a pseudorandom hypergraph matchings that maps almost all vertices of X_i into V_i for each cluster of $(X_1, \dots, X_r)$. Note that this approximate procedure leaves some vertices of each cluster X_i unembedded, that is, they are not mapped onto V_i. However, by employing the pseudorandom properties of our hypergraph matching, we can sufficiently control this leftover and guarantee that it is well-behaved so that we can turn such an approximate embedding into a complete one. Therefore, each of the Chapters 3–5 will consist of two parts: a first part with an approximate embedding/packing lemma that performs one such embedding step of almost all vertices of X_i into V_i, which we then apply iteratively to map almost all vertices $(X_1, \dots, X_r)$ into $(V_1, \dots, V_r)$, and a second part for the completion that turns the approximate embedding into a complete one. The difficulties of the first part are to carefully control several important quantities during the approximate embedding/packing phase. We have to guarantee that we can iteratively apply our hypergraph matching result from Chapter 2 in order to map almost all vertices of X_i into V_i for each $i \in [r]$, and we have to guarantee that the vertices that are left unembedded are well-behaved. The difficulties of the second part are to suitably turn the approximate embedding/packing of the first stage into a complete one. To that end, in each of the settings of Chapters 3–5, we will reserve in the beginning an edge set of the host graph G for this completion step that we do not use for the approximate embedding/packing in the first stage. We will use the following completion techniques for the individual settings. For the completion step of our rainbow blow-up lemma for almost optimally bounded edge-colourings, we will employ another rainbow blow-up lemma for $o(n)$-bounded edge-colourings due to Glock and Joos [50]. For the completion step of our proof of the blow-up lemma for approximate decompositions, we will iteratively apply the usual blow-up lemma. For the completion step of our hypergraph blow-up lemma for approximate decompositions, we will use a random matching procedure.

1.5 Preliminaries

1.5.1 Notation

In this section we introduce some general notation and terminology for graphs and hypergraphs. Recall also Section 1.1.1 for the definitions of (hyper)graphs, packings, embeddings and decompositions.

Let us first collect some graph terminology. We consider finite, simple and undirected graphs and use standard graph terminology. For a graph G, we let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. We say $u \in V(G)$ is a G-vertex and it is a G-neighbour of $v \in V(G)$ if $uv \in E(G)$. As usual, $\deg_G(u)$ denotes the degree of a vertex u in G, and $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. For $u, v \in V(G)$, let $N_G(u, v) := N_G(u) \cap N_G(v)$ denote the common neighbourhood of u and v, and let $N_G[u] := N_G(u) \cup \{u\}$. For a set S, let $N_G(S) := \bigcup_{v \in S \cap V(G)} N_G(v)$. For disjoint subsets $A, B \subseteq V(G)$, let $G[A, B]$ denote the bipartite subgraph of G between A and B and $G[A]$ the subgraph in G induced by A. Let $e(G)$ be the number of edges of G and let $e_G(A, B)$ denote the number of edges of $G[A, B]$. For $m \in \mathbb{N}$, let G^m denote the m-th power of G, that is, the graph obtained from G by adding all edges between vertices whose distance in G is at most m. A subset $X \subseteq V(G)$ is 2-independent if it is independent in G^2. For (hyper)graphs G, H, we write $G - H$ to denote the (hyper)graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

Next, we introduce some terminology for hypergraphs. We will collect more specific notation for the statement of our hypergraph blow-up lemma for approximate decompositions in Section 5.3.1 of Chapter 5. A hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq 2^{V(H)}$. If all edges have size r, then H is called *r-uniform*, or simply an *r-graph*. For a hypergraph H, we denote by $v(H)$ and $e(H)$ the number of vertices and edges of H, respectively, and we define for vertices $u, v \in V(H)$, the degree $\deg_H(v) := |\{e \in E(H) \colon v \in e\}|$ and codegree $\deg_H(uv) := |\{e \in E(H) \colon \{u, v\} \subseteq e\}|$. Let

$$\Delta(H) := \max_{v \in V(H)} \deg_H(v), \ \delta(H) := \min_{v \in V(H)} \deg_H(v) \text{ and } \Delta^c(H) := \max_{u \neq v \in V(H)} \deg_H(uv)$$

denote the *maximum degree*, *minimum degree* and *maximum codegree* of H, respectively.

We collect some general notation. For $k \in \mathbb{N}$, we write $[k]_0 := [k] \cup \{0\} = \{0, 1, \ldots, k\}$, where $[0] = \emptyset$. For a finite set S and $k \in \mathbb{N}$, we write $\binom{S}{k}$ for the set of all subsets of S of size k and 2^S for the powerset of S. For a set $\{i, j\}$, we sometimes simply write ij. For $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ whenever $a \in [b - c, b + c]$. For $a, b, c \in (0, 1]$, we sometimes write $a \ll b \ll c$ in our statements meaning that there are increasing functions $f, g : (0, 1] \to (0, 1]$ such that whenever $a \leq f(b)$ and $b \leq g(c)$, then the subsequent result holds. For $a \in (0, 1]$ and $\mathbf{b} \in (0, 1]^k$, we write $a \ll \mathbf{b}$ whenever $a \ll b_i$ for all $b_i \in \mathbf{b}$, $i \in [k]$. For the sake of a clearer presentation, we avoid roundings and assume that large numbers are integers whenever it does not affect the argument.

1.5.2 Probabilistic tools

We will make use of several probabilistic arguments and employ concentration inequalities to establish the concentration of a random variable X. If X is the sum of independent Bernoulli variables, we use the following well-known Chernoff-type bound.

Lemma 1.7 (Chernoff's bound, see [63])**.** *Suppose $X_1, \ldots, X_m$ are independent random variables taking values in $\{0, 1\}$. Let $X := \sum_{i=1}^m X_i$. Then, for all $\lambda > 0$,*

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \lambda\right] \leq 2\exp\left(-\frac{\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right).$$

Similarly, if X is a function of several independent random variables and does not depend too much on any of the variables, we use the following 'bounded differences inequality'.

Lemma 1.8 (McDiarmid's inequality, see [99, Lemma 1.2])**.** *Suppose $X_1, \ldots, X_m$ are independent random variables and suppose $b_1, \ldots, b_m \in [0, B]$. Suppose X is a real-valued random variable determined by $X_1, \ldots, X_m$ such that changing the outcome of X_i changes X by at most b_i for all $i \in [m]$. Then, for all $\lambda > 0$, we have*

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \lambda\right] \leq 2\exp\left(-\frac{2\lambda^2}{B\sum_{i=1}^m b_i}\right).$$

Even though we will mostly use the concentration inequalities of Lemmas 1.7 and 1.8, we also consider exposure martingales at some points (for instance, in the proof of Theorem 2.5 in Section 2.3). That is, suppose we have a random variable X that is determined by independent random variables $Y_1, \ldots, Y_n$ and we define $X_t := \mathbb{E}[X \mid Y_1, \ldots, Y_t]$. Then it is well-known that $(X_t)_{t \geq 0}$ is a martingale, the so-called *exposure martingale* for X. Note that $X_0 = \mathbb{E}[X]$ and $X_n = X$. Now, Freedman's concentration inequality for martingales can be used to obtain concentration of X around its mean.

Lemma 1.9 (Freedman's inequality [44])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration of $\mathcal{F}$. Let $(X_t)_{t \geq 0}$ be a martingale adapted to $(\mathcal{F}_t)_{t \geq 0}$. Suppose $\sum_{t \geq 0} \mathbb{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq \sigma$ and that $|X_{t+1} - X_t| \leq C$ for all t. Then, for any $\lambda > 0$,*

$$\mathbb{P}\left[|X_t - X_0| \geq \lambda \text{ for some } t\right] \leq 2\exp\left(-\frac{\lambda^2}{2C(\lambda + \sigma)}\right).$$

The following more convenient form of Freedman's inequality will often suffice for our purposes and follows directly from Lemma 1.9.

Lemma 1.10. *Suppose $X, X_1, \dots, X_m$ are real-valued random variables with $X = \sum_{i\in[m]} X_i$ such that $|X_i| \le B$ and $\sum_{i\in[m]} \mathbb{E}'[|X_i|] \le \mu$, where $\mathbb{E}'[X_i]$ denotes the expectation conditional on any given values of X_j for $j < i$. Then*

$$\mathbb{P}[|X| > 2\mu] \le 2\exp\left(-\frac{\mu}{4B}\right).$$

1.5.3 Graph regularity

For a bipartite graph G with vertex partition (V_1, V_2), we define the *density* of W_1, W_2 with $W_i \subseteq V_i$ by $d_G(W_1, W_2) := e_G(W_1, W_2)/|W_1||W_2|$. Given $\varepsilon > 0$ and $d \in [0,1]$, we say G is (ε, d)*-regular* if $d_G(W_1, W_2) = d \pm \varepsilon$ for all $W_i \subseteq V_i$ with $|W_i| \ge \varepsilon|V_i|$, and G is (ε, d)*-super-regular* if in addition $|N_G(v) \cap V_{3-i}| = (d \pm \varepsilon)|V_{3-i}|$ for each $i \in [2]$ and $v \in V_i$. The following is one of the fundamental properties of ε-regularity.

Fact 1.11. *Let G be an (ε, d)-regular bipartite graph with partition (A, B), and let $Y \subseteq B$ with $|Y| \ge \varepsilon|B|$. Then all but at most $2\varepsilon|A|$ vertices of A have $(d \pm \varepsilon)|Y|$ neighbours in Y.*

We will also often use the fact that super-regularity is robust with respect to small vertex and edge deletions.

Fact 1.12. *Suppose $1/n \ll \varepsilon \ll d$. Let G be an (ε, d)-super-regular bipartite graph with partition (A, B), where $\varepsilon^{1/6}n \le |A|, |B| \le n$. If $\Delta(H) \le \varepsilon n$ and $X \subseteq A \cup B$ with $|X| \le \varepsilon n$, then $G[A \setminus X, B \setminus X] - E(H)$ is $(\varepsilon^{1/3}, d)$-super-regular.*

The following is essentially a result from [28]. (In [28] it is proved in the case when $|A| = |B|$ with $16\epsilon^{1/5}$ instead of $\epsilon^{1/6}$. The version stated below can be easily derived from this.)

Theorem 1.13. *Suppose $1/n \ll \varepsilon \ll \gamma, d$. Suppose G is a bipartite graph with vertex partition (A, B) such that $|A| = n$, $\gamma n \le |B| \le \gamma^{-1}n$ and at least $(1 - 5\varepsilon)n^2/2$ pairs $u, v \in A$ satisfy $\deg_G(u), \deg_G(v) \ge (d - \varepsilon)|B|$ and $|N_G(u, v)| \le (d + \varepsilon)^2|B|$. Then G is $(\varepsilon^{1/6}, d)$-regular.*

Chapter 2

Pseudorandom hypergraph matchings

The content of this chapter is based on [29] with Stefan Glock and Felix Joos.

2.1 Introduction to hypergraph matchings

A *matching* in a hypergraph H is a collection of pairwise disjoint edges, and a *cover of* H is a set of edges whose union contains all vertices. A matching is *perfect* if it is also a cover. These concepts are widely applicable, as "almost all combinatorial questions can be reformulated as either a matching or a covering problem of a hypergraph" [47], and their study is thus of great relevance in combinatorics and beyond.

Results like Hall's theorem and Tutte's theorem that characterize when a graph has a perfect matching are central in graph theory. However, for each $r \geq 3$, it is NP-complete to decide whether a given r-uniform hypergraph has a perfect matching [69]. It is thus of great importance to find sufficient conditions that guarantee a perfect matching in an r-uniform hypergraph. This problem has received a lot of attention over the years. For instance, one line of research has focused on minimum degree conditions that guarantee a perfect matching (see e.g. [2, 61, 74, 114] and the survey [113]). Another important direction has been to study perfect matchings in random hypergraphs. The so-called Shamir's problem, to determine the threshold for which the (binomial) random k-graph has a perfect matching with high probability, was open for over 25 years resisting numerous efforts, until famously solved by Johansson, Kahn and Vu [64]. Recently, Kahn [68] refined the asymptotics for this threshold. Moreover, Cooper, Frieze, Molloy and Reed [22] determined when regular hypergraphs have a perfect matching with high probability. It would be very interesting to obtain such results not only for random hypergraphs, but to find pseudorandom conditions that (deterministically) guarantee a perfect matching. Apart from some partial results (e.g. [45, 60, 95]), this seems wide open.

Many of the aforementioned results are proven by first obtaining an *almost* perfect matching, and then using some clever ideas to complete it. It turns out that almost perfect matchings often exist under weaker conditions. For example, in the minimum degree setting, the threshold for finding an almost perfect matching is often smaller than that of finding a perfect matching. Also, there is a well-known theorem that yields almost perfect matchings under astonishingly mild pseudorandomness conditions. Mostly referred to as Pippenger's theorem, any almost regular hypergraph with small codegrees has an almost perfect matching. Both the result itself and also its proof method, the so-called 'semi-random method' or 'Rödl nibble', have had a tremendous impact on combinatorics. We add to this body of research by showing the existence of 'pseudorandom' matchings in this setting. We note that our result does not improve previous bounds on the *size* of a matching that can be obtained. Rather, our focus is on the *structure* of such a matching within the hypergraph it is contained in.

In Section 2.1.1, we revisit Pippenger's theorem. In Section 2.1.2, we discuss a theorem of Alon and Yuster, which can be viewed as an intermediate step. In Section 2.1.3, we will motivate and state our main results.

2.1.1 Pippenger's theorem

Pippenger never published his theorem, and it was really the culmination of the efforts of various researchers in the 1980s. Most notably, in 1985, Rödl [111] proved a long-standing conjecture of Erdős and Hanani on approximate Steiner systems. A *(partial) (n,k,t)-Steiner system* is a set $\mathcal{S}$ of k-subsets of some n-set V such that every t-subset of V is contained in (at most) one k-set in $\mathcal{S}$. Steiner asked in 1853 for which parameters such systems exist, a question that has intrigued mathematicians for more than 150 years and was only answered recently by Keevash [71]. In 1963, Erdős and Hanani asked whether one can, for fixed k,t, always find an 'approximate Steiner system', that is, a partial (n,k,t)-Steiner system covering all but $o(n^t)$ of the t-sets, as $n \to \infty$. This was proved by Rödl using the celebrated 'nibble' method, with some ideas descending from [3, 81]. Frankl and Rödl [43] observed that in fact a much more general theorem holds, which applies to almost regular hypergraphs with small codegrees. Pippenger's version stated below is a slightly stronger and cleaner version.

Theorem 2.1 (Pippenger)**.** *For $r \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\mu > 0$ such that any r-uniform hypergraph H with $\delta(H) \geq (1-\mu)\Delta(H)$ and $\Delta^c(H) \leq \mu\Delta(H)$ has a matching that covers all but at most an ε-fraction of the vertices.*

To see why this generalizes Rödl's result, fix n,k,t and construct a hypergraph H with vertex set $\binom{[n]}{t}$ where every k-set $X \subseteq [n]$ induces the edge $\binom{X}{t}$. Note that perfect matchings in H correspond to (n,k,t)-Steiner systems. Clearly, H is $\binom{k}{t}$-uniform. Moreover, every vertex has degree $\binom{n-t}{k-t} = \Theta(n^{k-t})$ and $\Delta^c(H) = \binom{n-t-1}{k-t-1} = o(n^{k-t})$. Thus, for sufficiently large n, Pippenger's theorem implies the existence of a matching $\mathcal{M}$ in H that covers all but $o(n^t)$ of the vertices, which corresponds to a partial (n,k,t)-Steiner system which covers all but $o(n^t)$ of the t-sets. Frankl and Rödl [43] also applied (their version of) this theorem to obtain similar results for other combinatorial problems, for instance the existence of Steiner systems in vector spaces. Keevash [73] raised the meta question of whether there exists a general theorem that provides sufficient conditions for a sparse 'design-like' hypergraph to admit a perfect matching (for a notion of 'design-like' that captures for example Steiner systems, but hopefully many more structures). Since such hypergraphs will likely be (almost) regular and have small codegree, the existence of an almost perfect matching follows from Pippenger's theorem, and a natural approach would be to use the absorbing method to complete such a matching to a perfect one. This of course can be extremely challenging since the relevant auxiliary hypergraphs are generally very sparse.

2.1.2 The Alon–Yuster theorem

In the case of Steiner systems, the absorbing method has been successfully applied to answer Steiner's question [52, 71]. Very roughly speaking, the idea of an absorbing approach is to set aside a 'magic' absorbing structure, then to obtain an approximate Steiner system, and finally to employ the magic absorbing structure to clean up. One (minor, but still relevant) challenge is that the leftover of the approximate Steiner system must be 'well-behaved'. More precisely, instead of the global condition that the number of uncovered t-sets is $o(n^t)$, one needs the stronger local condition that for every fixed $(t-1)$-set, the number of uncovered t-sets containing this $(t-1)$-set is $o(n)$. Fortunately, Alon and Yuster [10, Theorem 1.2], by building on a theorem of Pippenger and Spencer [107], provided a tool achieving this. They showed that any almost regular hypergraph with small codegrees contains a matching that is 'well-behaved' in the sense that it not only covers all but a tiny proportion of the entire vertex set, but also has this property with respect to a specified collection of not too many not too small vertex subsets. The precise statement is technical and allows for certain tradeoffs between set sizes, number of sets, and degree conditions. To give a concrete example, if the r-uniform almost regular hypergraph H has N vertices, $\Delta^c(H) \leq \Delta(H)/\log^{9r} N$ and we consider a family $\mathcal{F}$ of at most $N^{\log N}$ vertex subsets, each of size at least $N^{2/5}$, then there exists a matching in H which covers all but $o(|F|)$ vertices from F for each $F \in \mathcal{F}$.

In the above application to Steiner systems, for every $(t-1)$-set S, consider the set $U_S \subseteq V(H)$ of all t-sets containing S. A matching in H which covers almost all vertices of U_S then corresponds to a partial Steiner system which covers all but $o(n)$ of the t-sets containing S, as desired.

2.1.3 Pseudorandom matchings

Our main contribution is to provide a tool that is (qualitatively) a generalization of the Alon–Yuster theorem and gives much more control on the matching obtained. This result will be a key ingredient for the proofs of the blow-up lemmas in Chapters 3–5 as sketched in Section 1.4. In [29], we gave a further application of our main result that shows that there exist approximate Steiner systems that behave 'pseudorandomly', that is, their subgraph statistics resemble the random model.

To motivate this, suppose for simplicity that we are given a D-regular hypergraph and want to find an (almost) perfect matching $\mathcal{M}$. Moreover, we wish $\mathcal{M}$ to be 'pseudorandom', that is, to have certain properties that we expect from an idealized random matching. In a perfect matching, at a fixed vertex, exactly one edge needs to be included in the matching, and assuming that each edge is equally likely to be chosen, we may heuristically expect that every edge of H is in a random perfect matching with probability $1/D$. Thus, given a (large) set $E \subseteq E(H)$ of edges, we expect $|E|/D$ matching edges in E. More generally, given a set X, a *weight function on* X is a function $\omega\colon X \to \mathbb{R}_{\geq 0}$. For a subset $X' \subseteq X$, we define $\omega(X') := \sum_{x\in X'} \omega(x)$. If ω is a weight function on $E(H)$, the above heuristic would imply that we expect from a 'pseudorandom' matching $\mathcal{M}$ that $\omega(\mathcal{M}) \approx \omega(E(H))/D$. The following is a simplified version of our main theorem (Theorem 2.3) which asserts that a hypergraph with small codegrees has a matching that is pseudorandom in the above sense.

Theorem 2.2 (Ehard, Glock, Joos [29]). *Suppose $\delta \in (0,1)$ and $r \in \mathbb{N}$ with $r \geq 2$, and let $\varepsilon := \delta/50r^2$. Then there exists Δ_0 such that for all $\Delta \geq \Delta_0$, the following holds: Let H be an r-uniform hypergraph with $\Delta(H) \leq \Delta$ and $\Delta^c(H) \leq \Delta^{1-\delta}$ as well as $e(H) \leq \exp(\Delta^{\varepsilon^2})$. Suppose that $\mathcal{W}$ is a set of at most $\exp(\Delta^{\varepsilon^2})$ weight functions on $E(H)$. Then, there exists a matching $\mathcal{M}$ in H such that $\omega(\mathcal{M}) = (1 \pm \Delta^{-\varepsilon})\omega(E(H))/\Delta$ for all $\omega \in \mathcal{W}$ with $\omega(E(H)) \geq \max_{e\in E(H)} \omega(e)\Delta^{1+\delta}$.*

We remark that a similar statement when $\mathcal{W}$ has bounded size and without polynomial error bounds is implied by a theorem of Kahn [67]. It has later been observed that the proof in [67] also gives the more general statement (see e.g. [72]). Here, we prove a more general theorem which not only allows weight functions on edges, but on tuples of edges. This allows, for instance, to specify a set of pairs of edges, and control how many pairs will be contained in the matching. In particular, this provides a proof of Theorem 2.2 for completeness and convenient use in future research.

Let us discuss a few aspects of this theorem. First, note that we do not require H to be almost regular. The theorem can be applied with any (sufficiently large) Δ. Moreover, $v(H)$ plays no role in the parametrization of the theorem. If H is almost regular, an almost perfect matching can be obtained by applying the theorem with $\Delta = \Delta(H)$ to the weight function $\omega \equiv 1$. This yields that $|\mathcal{M}| \geq (1 - o(1))\frac{e(H)}{\Delta(H)} \geq (1 - o(1))v(H)/r$, where the last inequality uses that $re(H) = \sum_{x\in V(H)} \deg_H(x) = (1 \pm o(1))v(H)\Delta(H)$.

We remark that, while Pippenger's theorem only needs $\Delta^c(H) = o(\Delta)$, we need a stronger condition to apply concentration inequalities. For the same reason, we also need that $\omega(E(H))$ is not too small (relative to the maximum possible weight). As a result, our theorem also allows stronger conclusions in that the error term $\Delta^{-\varepsilon}$ decays polynomially with Δ.

Note that Theorem 2.2 is (qualitatively) more general than the Alon–Yuster theorem. Indeed, suppose H is an almost regular hypergraph and we are given a collection $\mathcal{V}$ of subsets $U \subseteq V(H)$ and want to ensure that $\mathcal{M}$ covers each $U \in \mathcal{V}$ almost completely. For each target subset $U \in \mathcal{V}$, we can define a weight function ω_U by setting $\omega_U(e) := |e \cap U|$. Note that $\omega_U(E(H)) = \sum_{x\in U} \deg_H(x) = (1 \pm o(1))|U|\Delta(H)$. Thus, since $\omega_U(\mathcal{M}) = (1 \pm o(1))\omega_U(E(H))/\Delta(H)$ by Theorem 2.2, we deduce that $|U \cap V(\mathcal{M})| = \omega_U(\mathcal{M}) = (1 \pm o(1))\omega_U(E(H))/\Delta(H) \geq (1 - o(1))|U|$, implying that almost all vertices of U are covered by $\mathcal{M}$. More generally, if we are given weight functions $p\colon V(H) \to \mathbb{R}_{\geq 0}$ (e.g. $p_U(v) := \mathbb{1}_{v\in U}$), then, setting $\omega_p(e) := \sum_{v\in e} p(v)$, we obtain that

$$\sum_{v\in V(\mathcal{M})} p(v) = (1 \pm o(1)) \sum_{v\in V(H)} p(v).$$

Note that the boundedness condition on the edge weight in Theorem 2.2 translates to the condition that $\max_{v\in V(H)} p(v) = o(\sum_{v\in V(H)} p(v))$.

We now state our main result, for which we need to introduce a bit more notation. Given a set X and an integer $\ell \in \mathbb{N}$, an *ℓ-tuple weight function on X* is a function $\omega\colon \binom{X}{\ell} \to \mathbb{R}_{\geq 0}$, that is, a weight function on $\binom{X}{\ell}$. For a subset $X' \subseteq X$, we then define $\omega(X') := \sum_{S\in\binom{X'}{\ell}} \omega(S)$. Moreover, if $\mathcal{X} \subseteq \binom{X}{\ell}$, we write $\omega(\mathcal{X})$ for $\sum_{S\in\mathcal{X}} \omega(S)$ as for usual weight functions. For $k \in [\ell]_0$ and a tuple $T \in \binom{X}{k}$, define

$$\omega(T) := \sum_{S\supseteq T} \omega(S), \text{ and let } \|\omega\|_k := \max_{T\in\binom{X}{k}} \omega(T). \tag{2.1.1}$$

Suppose H is an r-uniform hypergraph and ω is an ℓ-tuple weight function on $E(H)$. Clearly, if $\mathcal{M}$ is a matching, then a tuple of edges which do not form a matching will never contribute to $\omega(\mathcal{M})$. We thus say that ω is *clean* if $\omega(\mathcal{E}) = 0$ whenever $\mathcal{E} \in \binom{E(H)}{\ell}$ is not a matching.

The following is our main result, which readily implies Theorem 2.2.

Theorem 2.3 (Ehard, Glock, Joos [29]). *Suppose $\delta \in (0, 1)$ and $r, L \in \mathbb{N}$ with $r \geq 2$, and let $\varepsilon\leq\delta/50L^2r^2$. Then there exists Δ_0 such that for all $\Delta \geq \Delta_0$, the following holds: Let H be an r-uniform hypergraph with $\Delta(H) \leq \Delta$ and $\Delta^c(H) \leq \Delta^{1-\delta}$ as well as $e(H) \leq \exp(\Delta^{\varepsilon^2})$. Suppose that for each $\ell \in [L]$, we are given a set $\mathcal{W}_\ell$ of clean ℓ-tuple weight functions on $E(H)$ of size at most $\exp(\Delta^{\varepsilon^2})$, such that $\omega(E(H)) \geq \|\omega\|_k\Delta^{k+\delta}$ for all $\omega \in \mathcal{W}_\ell$ and $k \in [\ell]$.*

Then, there exists a matching $\mathcal{M}$ in H such that $\omega(\mathcal{M}) = (1 \pm \Delta^{-\varepsilon})\omega(E(H))/\Delta^\ell$ for all $\ell \in [L]$ and $\omega \in \mathcal{W}_\ell$.

We will prove Theorem 2.3 in Section 2.3, after stating some preliminary results in the next section.

2.2 Preliminaries

Our main tool is the next theorem of Molloy and Reed on the chromatic index of a hypergraph with small codegrees, improving on earlier work of Pippenger and Spencer as well as Kahn. Pippenger and Spencer [107] strengthened Theorem 2.1 by showing that under the same assumptions, one can even obtain an almost optimal edge-colouring of H, using $(1+o(1))\Delta$ colours. (The existence of an almost perfect matching follows then by averaging over the colour classes.) Kahn [66] generalized this to list colourings, and Molloy and Reed improved the $o(1)$-term. For simplicity, we only state their result for normal colourings.

Theorem 2.4 (Molloy and Reed [101, Theorem 2])**.** *Let $1/\Delta \ll \delta, 1/r$. Suppose H is an r-uniform hypergraph satisfying $\Delta^c(H) \leq \Delta^\delta$ and $\Delta(H) \leq \Delta$. Then, the edge set $E(H)$ can be decomposed into $\Delta + \Delta^{1-\frac{1-\delta}{r}} \log^5 \Delta$ edge-disjoint matchings.*

Note here that H is not required to be almost regular. In fact, this assumption can also be omitted from the Pippenger–Spencer theorem since any given r-uniform hypergraph H can be embedded into a $\Delta(H)$-regular hypergraph H' with $\Delta^c(H') = \Delta^c(H)$, and any colouring of H' induces a colouring of H with the same number of colours.

2.3 Proof

In this section we prove our main result Theorem 2.3.

2.3.1 Proof overview

We first sketch our proof. For simplicity, we first consider only the setting of Theorem 2.2. We split H randomly into p vertex-disjoint induced subgraphs $H_1, \dots, H_p$ and let H' be the union of those. With high probability, $\Delta(H_i) \approx \Delta(H)p^{-(r-1)}$ for each i, and for a given weight function ω, we have $\omega(E(H')) \approx \omega(E(H))p^{-(r-1)}$. After fixing such a partition, we utilize the theorem of Molloy and Reed to find, for each $i \in [p]$, a partition of $E(H_i)$ into $M \approx \Delta(H)p^{-(r-1)}$ matchings. Finally, we select a matching from each partition uniformly at random, and let $\mathcal{M}$ be the union of these matchings. Clearly, every edge in H' is contained in $\mathcal{M}$ with probability M^{-1}, so $\mathbb{E}\left[\omega(\mathcal{M})\right] = \omega(E(H'))M^{-1} \approx \omega(E(H))/\Delta(H)$. Moreover, the individual effect of the matching chosen in H_i is relatively small, so we could hope to use McDiarmid's inequality to establish concentration. So far, this approach is the same as taken by Alon and Yuster. However, the individual effects of the matchings chosen in H_i are in fact still too large in our setting to apply McDiarmid's inequality. One important new ingredient in our proof is that we partition each H_i further into edge-disjoint subgraphs $H_{i,1}, \dots, H_{i,q}$ such that $\omega(E(H_{i,j}))$ is of magnitude $\omega(E(H_i))/q$, and then apply Theorem 2.4 to each $H_{i,j}$. This gives, as above, a partition of H_i into matchings, from which we still choose one uniformly at random. However, the individual effect of each matching chosen has now been drastically reduced, which allows us to apply McDiarmid's inequality with the desired parameters.

2.3.2 Edge-slicing

In the setting of Theorem 2.2, the partition of each H_i into edge-disjoint subgraphs $H_{i,1}, \dots, H_{i,q}$ could be obtained easily with a generalized Chernoff bound. However, in the setting of Theorem 2.3, we are not aware of a conventional concentration inequality that suits our needs for this step (in particular, since q is rather large). Thus, we first prove a tool that will achieve this for us. Roughly speaking, what we require is the following: Let H be a 'directed' ℓ-graph on V, that is, a collection of ordered ℓ-subsets of V. Let $f\colon V \to [q]$ be obtained by choosing $f(v) \in [q]$ uniformly at random for each vertex v independently. For each directed edge $e = (v_1, \dots, v_\ell)$, let $f(e) := (f(v_1), \dots, f(v_\ell))$. For a fixed 'pattern' $\alpha \in [q]^\ell$, let X_α denote the number of $e \in E(H)$ with $f(e) = \alpha$. Clearly, for each edge e, we have that $\mathbb{P}[f(e) = \alpha] = q^{-\ell}$, thus, $\mathbb{E}[X_\alpha] = q^{-\ell} e(H)$. We would like to know that X_α is concentrated around its mean, even when q is quite large.

For simplicity, we will actually only consider the case when H is an ℓ-graph, the vertex set V is ordered, and each edge of H obtains its direction from the ordering of V. Thus, our setup is as follows. Let $(V, <)$ be an ordered set. Let $f\colon V \to [q]$ be obtained by choosing $f(v) \in [q]$ uniformly at random for each $v \in V$ independently. For each ℓ-set $e = \{v_1, \dots, v_\ell\}$ with $v_1 < \cdots < v_\ell$, let $f(e) := (f(v_1), \dots, f(v_\ell))$. For a fixed 'pattern' $\alpha \in [q]^\ell$, let $E_\alpha = E_\alpha(f)$ denote the (random) set of all $e \in \binom{V}{\ell}$ with $f(e) = \alpha$. Given an ℓ-tuple weight function ω on V, the following theorem shows that the random variable $\omega(E_\alpha)$ is concentrated around its mean.

Theorem 2.5 (Ehard, Glock, Joos [29])**.** *Suppose $(V, <)$, f, ℓ, α, ω are as above. Suppose that $g \geq 24\ell^3(\ell + 1 + \log |V|)$. Define $M := q^{-\ell} \max_{k \in [\ell]} \{\|\omega\|_k q^k g^{k-1}\}$. Then for any $\lambda > 0$, we have*

$$\mathbb{P}[|\omega(E_\alpha) - \mathbb{E}[\omega(E_\alpha)]| \geq \lambda] \leq 2^\ell \exp\left(-\frac{\lambda^2}{12\ell^2 M(\lambda + \mathbb{E}[\omega(E_\alpha)])}\right) + \exp\left(-\frac{g}{24\ell^2}\right).$$

Proof. Let $n := |V|$ and let $v_1 < \cdots < v_n$ be the ordered elements of V and write $\alpha = (\alpha_1, \dots, \alpha_\ell)$. For $t \in [n]_0$, let

$$X_t := \mathbb{E}[\omega(E_\alpha) \mid f(v_1), \dots, f(v_t)]$$

(and $X_t := X_n$ for $t \geq n$). Hence $X = (X_t)_{t \geq 0}$ is the so-called exposure martingale for $\omega(E_\alpha)$, where the labels $f(v_i)$ are revealed one by one. In particular, $X_0 = \mathbb{E}[\omega(E_\alpha)]$ and $X_n = \omega(E_\alpha)$.

For $k \in [\ell]$ and a k-tuple weight function ω' on V, let

$$M_k(\omega') := q^{-k} \max_{i \in [k]} \{\|\omega'\|_i q^i g^{i-1}\}.$$

Note that we have

$$M_k(\omega') q^k \leq M_\ell(\omega') q^\ell. \tag{2.3.1}$$

Let $M_k := M_k(\omega)$ and note that $M = M_\ell$.

We prove the theorem by induction on ℓ (with $(V, <)$ and g being fixed). Thus, assume first that $\ell = 1$. (This case is also contained in the inductive step below with no inductive hypothesis being needed, but the short proof here may serve as a warm up.) Observe that $X_t(f) - X_{t-1}(f) = \omega(\{v_t\})(\mathbb{1}_{f(v_t) = \alpha_1} - 1/q)$ for $t \in [n]$. Hence, we can directly apply Freedman's inequality to obtain (observe that $M_1 = \|\omega\|_1$)

$$\begin{aligned}\mathbb{P}[|\omega(E_\alpha) - \mathbb{E}[\omega(E_\alpha)]| \geq \lambda] &\leq 2\exp\left(-\frac{\lambda^2}{2\|\omega\|_1(\lambda + \sum_{t \in [n]} 2\omega(\{v_t\})/q)}\right) \\ &\leq 2\exp\left(-\frac{\lambda^2}{4M_1(\lambda + \mathbb{E}[\omega(E_\alpha)])}\right),\end{aligned}$$

as desired.

Suppose now that $\ell \geq 2$. In order to apply induction, we need to introduce some more notation. For $t \in [n]$ and $k \in [\ell-1]_0$, let $\omega^{t,k}\colon \binom{V}{k} \to [0,\infty)$ be defined as (where $j_1 < \ldots < j_k$)

$$\omega^{t,k}(\{v_{j_1},\ldots,v_{j_k}\}) := \sum_{\substack{j_{k+1}<\ldots<j_\ell \\ j_k<j_{k+1}=t}} \omega(\{v_{j_1},\ldots,v_{j_\ell}\}).$$

Moreover, let $\omega^{\leq t,k}\colon \binom{V}{k} \to [0,\infty)$ be defined by $\omega^{\leq t,k}(S) := \sum_{s\leq t}\omega^{s,k}(S)$ for all $S \in \binom{V}{k}$. Note that

$$\omega^{\leq n,k}(V) = \omega(V) \text{ and } \|\omega^{\leq n,k}\|_i \leq \|\omega\|_i \text{ for all } i \in [k]. \tag{2.3.2}$$

For $k \in [\ell-1]_0$, let $\alpha[k] := (\alpha_1,\ldots,\alpha_k)$, and define $E_{\alpha[k]} = E_{\alpha[k]}(f)$ as the random set of all k-sets $\{v_{j_1},\ldots,v_{j_k}\}$ for which $f(v_{j_i}) = \alpha_i$ for all $i \in [k]$, where $j_1 < \cdots < j_k$. For clarity, we briefly discuss the case $k = 0$, when $\omega^{t,0}$ is the function that maps $\emptyset$ to $\sum_{t<j_2<\ldots<j_\ell}\omega(\{v_t, v_{j_2}\ldots,v_{j_\ell}\})$. In particular, we have for all $t \in [n]$ that

$$\omega^{t,0}(\emptyset) \leq \omega(\{v_t\}) \leq \|\omega\|_1 = M_1; \tag{2.3.3}$$

$$\omega^{\leq t,0}(\emptyset) \leq \omega(V). \tag{2.3.4}$$

Note also that $E_{\alpha[0]} = \{\emptyset\}$.

The purpose of these definitions lies in the following formula for the one-step change of the process X: for $t \in [n]$, we have

$$X_t(f) - X_{t-1}(f) = \sum_{k\in[\ell-1]_0} \omega^{t,k}(E_{\alpha[k]}(f)) \cdot (\mathbb{1}_{f(v_t)=\alpha_{k+1}} - 1/q)\cdot q^{-(\ell-(k+1))}.$$

Clearly, $|\mathbb{1}_{f(v_t)=\alpha_{k+1}} - 1/q| \leq 1$ and $\mathbb{E}\left[|\mathbb{1}_{f(v_t)=\alpha_{k+1}} - 1/q|\right] = 2(1-1/q)/q \leq 2/q$. Hence, for the absolute change and expected absolute change of the process X in one step we obtain the following bounds:

$$|X_t - X_{t-1}| \leq \sum_{k\in[\ell-1]_0} \omega^{t,k}(E_{\alpha[k]})\cdot q^{k+1-\ell}; \tag{2.3.5}$$

$$\mathbb{E}\left[|X_t - X_{t-1}| \mid f(v_1),\ldots,f(v_{t-1})\right] \leq \sum_{k\in[\ell-1]_0} 2\omega^{t,k}(E_{\alpha[k]})\cdot q^{k-\ell}. \tag{2.3.6}$$

Note that $\omega^{t,k}(E_{\alpha[k]})$ is itself a random variable, when $k > 0$. Unfortunately, its deterministic upper bound is not good enough to apply Freedman's inequality directly to the martingale $(X_t)_{t\geq 0}$. We apply a common trick by defining a stopped process $Y = (Y_t)_{t\geq 0}$ which is equal to X as long as the random variables $\omega^{t,k}(E_{\alpha[k]})$ behave nicely, and then 'freezes'. We can then apply Freedman's inequality to Y. Finally, we need to show that the process is unlikely to freeze, implying that the concentration result for Y transfers to X. For this, we employ the statement inductively with $\omega^{t,k}, \omega^{\leq n,k}, \alpha[k]$.

We define two types of stopping times for X. For $k \in [\ell-1]$, let

$$\tau'_k := \min_{t\in[n-1]}\{\omega^{\leq t+1,k}(E_{\alpha[k]}) \geq \omega(V)q^{-k} + \lambda q^{\ell-k}\} \wedge n. \tag{2.3.7}$$

Moreover, for $k \in [\ell-1]$ and $t \in [n-1]$, define

$$\tau^t_k := \begin{cases} t & \text{if } \omega^{t+1,k}(E_{\alpha[k]}) \geq 2M_{k+1}, \\ n & \text{otherwise.} \end{cases} \tag{2.3.8}$$

Let $\tau := \min_{t\in[n],k\in[\ell-1]}\{\tau'_k, \tau^t_k\}$. Note that $\omega^{t+1,k}(E_{\alpha[k]})$ is fully determined by $f(v_1), \dots, f(v_t)$, since $\omega^{t+1,k}(S) = 0$ whenever S contains a vertex v_j with $j \geq t+1$. Thus, τ is indeed a stopping time for X. We define $Y = (Y_t)_{t\geq 0}$ by $Y_t := X_{t\wedge\tau}$, and let $\Delta Y_t := Y_t - Y_{t-1}$. By the optional stopping theorem Y is also a martingale [80], and thus we can apply Freedman's inequality. To this end, we next bound the absolute and expected one step change for Y.

We claim that $|\Delta Y_t| \leq 2\ell M_\ell$ for all t. Indeed, if $t \geq \tau+1$, then trivially $|\Delta Y_t| = 0$ and whenever $t \leq \tau$, then

$$|\Delta Y_t| \overset{(2.3.5)}{\leq} \sum_{k\in[\ell-1]_0} \omega^{t,k}(E_{\alpha[k]}) \cdot q^{k+1-\ell} \overset{(2.3.3),(2.3.8)}{\leq} \sum_{k\in[\ell-1]_0} 2M_{k+1} \cdot q^{k+1-\ell} \overset{(2.3.1)}{\leq} 2\ell M_\ell.$$

Similarly,

$$\begin{aligned}
\sum_{t\geq 1} \mathbb{E}\left[|\Delta Y_t| \mid f(v_1), \dots, f(v_{t-1})\right] &\overset{(2.3.6)}{\leq} \sum_{t\in[\tau]} \sum_{k\in[\ell-1]_0} 2\omega^{t,k}(E_{\alpha[k]}) \cdot q^{k-\ell} \\
&= \sum_{k\in[\ell-1]_0} 2\omega^{\leq\tau,k}(E_{\alpha[k]}) \cdot q^{k-\ell} \\
&\overset{(2.3.4),(2.3.7)}{\leq} \sum_{k\in[\ell-1]_0} 2(\omega(V)q^{-k} + \lambda q^{\ell-k}) \cdot q^{k-\ell} \\
&= 2\ell(\omega(V)q^{-\ell} + \lambda).
\end{aligned}$$

Thus, we can apply Freedman's inequality to obtain

$$\begin{aligned}
\mathbb{P}\left[|Y_n - Y_0| \geq \lambda\right] &\leq 2\exp\left(-\frac{\lambda^2}{4\ell M_\ell(\lambda + 2\ell(\omega(V)q^{-\ell} + \lambda))}\right) \\
&\leq 2\exp\left(-\frac{\lambda^2}{12\ell^2 M_\ell(\lambda + \mathbb{E}\left[\omega(E_\alpha)\right])}\right).
\end{aligned}$$

It remains to show that $Y_n = X_n$ with high probability. We first consider the stopping times τ'_k. Fix $k \in [\ell-1]$ and note that $\mathbb{E}\left[\omega^{\leq n,k}(E_{\alpha[k]})\right] = \omega^{\leq n,k}(V)/q^k = \omega(V)/q^k$ by (2.3.2). We apply the induction hypothesis to $\omega^{\leq n,k}$, with $\lambda q^{\ell-k}$ and k playing the roles of λ and ℓ, and obtain

$$\begin{aligned}
\mathbb{P}\left[\tau'_k < n\right] &\leq \mathbb{P}\left[\omega^{\leq n,k}(E_{\alpha[k]}) \geq \mathbb{E}\left[\omega^{\leq n,k}(E_{\alpha[k]})\right] + \lambda q^{\ell-k}\right] \\
&\leq 2^k \exp\left(-\frac{\lambda^2 q^{2(\ell-k)}}{12k^2 M_k(\omega^{\leq n,k})(\lambda q^{\ell-k} + \mathbb{E}\left[\omega^{\leq n,k}(E_{\alpha[k]})\right])}\right) + \exp\left(-\frac{g}{24k^2}\right) \\
&\leq 2^k \exp\left(-\frac{\lambda^2}{12k^2 M_\ell(\lambda + \mathbb{E}\left[\omega(E_\alpha)\right])}\right) + \exp\left(-\frac{g}{24k^2}\right),
\end{aligned}$$

where we have used that $\mathbb{E}\left[\omega^{\leq n,k}(E_{\alpha[k]})\right] = q^{\ell-k}\mathbb{E}\left[\omega(E_\alpha)\right]$ and $M_k(\omega^{\leq n,k}) \leq M_k(\omega) \leq q^{\ell-k}M_\ell$ by (2.3.2) and (2.3.1).

Next, we consider the stopping times τ^t_k. Let $k \in [\ell-1]$ and $t \in [n-1]$. Observe that $\|\omega^{t,k}\|_i \leq \|\omega\|_{i+1}$ for all $i \in [k]$. Hence

$$\frac{M_{k+1}(\omega)}{M_k(\omega^{t,k})} = \frac{q^{-k-1}\max_{i\in[k+1]}\{\|\omega\|_i q^i g^{i-1}\}}{q^{-k}\max_{i\in[k]}\{\|\omega^{t,k}\|_i q^i g^{i-1}\}} \geq \frac{g\max_{i\in[k+1]}\{\|\omega\|_i q^i g^{i-1}\}}{\max_{i\in[k+1]\setminus\{1\}}\{\|\omega\|_i q^i g^{i-1}\}} \geq g.$$

Note that $\mathbb{E}\left[\omega^{t,k}(E_{\alpha[k]})\right] = q^{-k}\omega^{t,k}(V) \le q^{-k}\|\omega\|_1 \le M_{k+1}$. Thus, using induction for $\omega^{t,k}$ with M_{k+1} and k playing the roles of λ and ℓ, we deduce that

$$\begin{aligned}\mathbb{P}\left[\tau_k^t < n\right] \le \mathbb{P}\left[\omega^{t,k}(E_{\alpha[k]}) \ge 2M_{k+1}\right] &\le 2^k \exp\left(-\frac{M_{k+1}}{24k^2 M_k(\omega^{t,k})}\right) + \exp\left(-\frac{g}{24k^2}\right)\\ &\le (2^k+1)\exp\left(-\frac{g}{24k^2}\right).\end{aligned}$$

A union bound now implies that

$$\begin{aligned}\mathbb{P}[\tau < n] &\le \sum_{k=1}^{\ell-1}\left(2^k \exp\left(-\frac{\lambda^2}{12k^2 M_\ell(\lambda + \mathbb{E}[\omega(E_\alpha)])}\right) + (1 + n(2^k+1))\exp\left(-\frac{g}{24k^2}\right)\right)\\ &\le (2^\ell - 2)\exp\left(-\frac{\lambda^2}{12\ell^2 M_\ell(\lambda + \mathbb{E}[\omega(E_\alpha)])}\right) + 2^{\ell+1} n \exp\left(-\frac{g}{24(\ell-1)^2}\right).\end{aligned}$$

Since $(\ell-1)^{-2} - \ell^{-2} \ge \ell^{-3}$ and $g/24\ell^3 \ge \log(2^{\ell+1}n)$ by assumption, we can finally conclude that

$$\begin{aligned}\mathbb{P}[|\omega(E_\alpha) - \mathbb{E}[\omega(E_\alpha)]| > \lambda] &\le \mathbb{P}[|Y_n - Y_0| \ge \lambda] + \mathbb{P}[\tau < n]\\ &\le 2^\ell \exp\left(-\frac{\lambda^2}{12\ell^2 M_\ell(\lambda + \mathbb{E}[\omega(E_\alpha)])}\right) + \exp\left(-\frac{g}{24\ell^2}\right).\end{aligned}$$

This completes the proof. □

2.3.3 Proof of Theorem 2.3

We are now ready to prove Theorem 2.3. The proof proceeds in three steps as outlined in the beginning of this section.

Proof of Theorem 2.3. We can assume that $\varepsilon = \delta/50L^2r^2$.

Step 1. Random vertex partition

Let $p := \Delta^{20Lr\varepsilon}$. We will first partition $V(H)$ into p subsets $V_1, \ldots, V_p$. For each $i \in [p]$, let $H_i := H[V_i]$. For an edge $e \in E(H)$, let $\tau(e) = i$ if $e \in E(H_i)$, and let $\tau(e) = 0$ if no such i exists. For a tuple $\mathcal{E} = (e_1, \ldots, e_\ell) \in \binom{E(H)}{\ell}$, define the multiset $\tau(\mathcal{E}) := \{\tau(e_1), \ldots, \tau(e_\ell)\}$. Let $\mathcal{J}_\ell$ be the set of all multisets of size ℓ with elements in $[p]$. For $J \in \mathcal{J}_\ell$, let $\mathrm{supp}(J)$ be the underlying set. We further define $\pi(J)$ as the number of functions $f\colon [\ell] \to \mathrm{supp}(J)$ with $\{f(1), \ldots, f(\ell)\} = J$. For all $\ell \in [L]$ and $J \in \mathcal{J}_\ell$, we define E_J as the set of all $\mathcal{E} \in \binom{E(H)}{\ell}$ with $\tau(\mathcal{E}) = J$.

We claim that there exists a partition $V_1, \ldots, V_p$ of $V(H)$ such that the following hold:

(a) $\Delta(H_i) \le (1 + \Delta^{-2\varepsilon})\Delta/p^{r-1}$ for all $i \in [p]$;

(b) $\omega(E_J) = (1 \pm \Delta^{-2\varepsilon})\omega(E(H))\frac{\pi(J)}{p^{r\ell}}$ for all $\ell \in [L]$, $\omega \in \mathcal{W}_\ell$ and $J \in \mathcal{J}_\ell$.

This can be seen using a probabilistic argument. For every vertex $x \in V(H)$ independently, choose an index $i \in [p]$ uniformly at random and assign x to V_i. We now show that (a) and (b) hold with high probability, implying that such a partition exists.

For (a), consider a vertex $x \in V(H)$ and $i \in [p]$. Let X be the number of edges e containing x for which $e \setminus \{x\} \subseteq V_i$. For each edge e containing x, we have that $\mathbb{P}[e \setminus \{x\} \subseteq V_i] = (1/p)^{r-1}$. Thus, $\mathbb{E}[X] = \deg_H(x)/p^{r-1} \le \Delta/p^{r-1}$. Note that for any other vertex $y \neq x$, the random label that we choose for y affects X by at most

$\deg_H(xy) \leq \Delta^c(H)$. Note that $\sum_{y\in V(H)\setminus\{x\}} \deg_H(xy) = \deg_H(x)(r-1) \leq \Delta r$. Thus, using McDiarmid's inequality, we deduce that

$$\begin{aligned}\mathbb{P}\left[X - \mathbb{E}\left[X\right] \geq \Delta^{1-2\varepsilon}/p^{r-1}\right] &\leq 2\exp\left(-\frac{2\Delta^{2-4\varepsilon}}{\Delta^c(H)\Delta r p^{2r-2}}\right) \leq 2\exp\left(-\Delta^{\delta-45Lr^2\varepsilon}\right)\\ &\leq \exp(-\Delta^{\varepsilon}).\end{aligned}$$

With a union bound over all (non-isolated) vertices (there are at most $re(H) \leq r\exp(\Delta^{\varepsilon^2})$ non-isolated vertices) and $i \in [p]$, we can infer that with high probability (a) holds.

For (b), consider $\ell \in [L]$, $\omega \in \mathcal{W}_\ell$ and $J \in \mathcal{J}_\ell$. For an edge $e \in E(H)$ and $i \in [p]$, we have that $\mathbb{P}\left[e \in E(H_i)\right] = p^{-r}$. Thus, for $\mathcal{E} \in \binom{E(H)}{\ell}$, we have $\mathbb{P}\left[\tau(\mathcal{E}) = J\right] = \pi(J)p^{-r\ell}$ if the edges in $\mathcal{E}$ are pairwise disjoint, and $\omega(\mathcal{E}) = 0$ otherwise since ω is clean. Hence, $\mathbb{E}\left[\omega(E_J)\right] = \omega(E(H))\frac{\pi(J)}{p^{r\ell}}$. We now establish concentration. For any vertex x, the random label chosen for x affects $\omega(E_J)$ by at most $\omega(E_x^\ell)$, where E_x^ℓ is the set of all $\mathcal{E} \in \binom{E(H)}{\ell}$ for which x is contained in some edge of $\mathcal{E}$. Note that

$$\omega(E_x^\ell) \leq \Delta\|\omega\|_1 \text{ for all } x \in V(H), \text{ and } \sum_{x\in V(H)} \omega(E_x^\ell) = r\ell\omega(E(H)).$$

Thus, we can use McDiarmid's inequality to conclude that

$$\begin{aligned}\mathbb{P}\left[\omega(E_J) \neq (1 \pm \Delta^{-2\varepsilon})\mathbb{E}\left[\omega(E_J)\right]\right] &\leq 2\exp\left(-\frac{2\mathbb{E}\left[\omega(E_J)\right]^2}{\Delta\|\omega\|_1 r\ell\omega(E(H))\Delta^{4\varepsilon}}\right)\\ &\leq 2\exp\left(-\frac{\omega(E(H))}{\|\omega\|_1\Delta^{1+45L^2r^2\varepsilon}}\right)\\ &\leq 2\exp\left(-\Delta^{\delta-45L^2r^2\varepsilon}\right)\\ &\leq \exp(-\Delta^{\varepsilon}),\end{aligned}$$

which together with a union bound over all $\ell \in [L]$, $\omega \in \mathcal{W}_\ell$ and $J \in \mathcal{J}_\ell$ proves (b).

Step 2. Random edge partition

Let $H' := \bigcup_{i\in[p]} H_i$. For each $i \in [p]$, we now partition H_i further into $q := \Delta^{1-20(r-1+1/4L)Lr\varepsilon}$ edge-disjoint subgraphs $H_{i,1}, \ldots, H_{i,q}$. Note that

$$p^{r-1}q = \Delta^{1-5r\varepsilon} \text{ and } p^r q \geq \Delta^{1+15Lr\varepsilon}. \tag{2.3.9}$$

We do so (for all i at once) by choosing a function $f\colon E(H') \to [q]$ and then let $H_{i,j}$ consist of all edges $e \in E(H_i)$ with $f(e) = j$, for all $i \in [p], j \in [q]$.

For $\ell \in [L]$, $J \in \mathcal{J}_\ell$ and a function $\sigma\colon \mathrm{supp}(J) \to [q]$, let $E_{J,\sigma}$ be the set of all $\mathcal{E} \in E_J$ for which $\sigma(\tau(e)) = f(e)$ for all $e \in \mathcal{E}$.

We claim that there exists a choice of f such that the following hold:

(A) $\Delta(H_{i,j}) \leq (1 + 2\Delta^{-2\varepsilon})\Delta/qp^{r-1}$ for all $i \in [p], j \in [q]$;

(B) $\Delta^c(H_{i,j}) \leq \Delta^{\varepsilon}$ for all $i \in [p], j \in [q]$;

(C) $\omega(E_{J,\sigma}) \leq 2\ell!\omega(E(H))/q^\ell p^{r\ell}$ for all $\ell \in [L]$, $\omega \in \mathcal{W}_\ell$, $J \in \mathcal{J}_\ell$ and $\sigma\colon \mathrm{supp}(J) \to [q]$.

This again can be seen using a probabilistic argument. For each $e \in E(H')$ independently, choose $f(e) \in [q]$ uniformly at random.

For (A), fix $i \in [p], j \in [q]$ and a vertex $x \in V(H_i)$. Note that $\mathbb{E}\left[\deg_{H_{i,j}}(x)\right] = \deg_{H_i}(x)/q \leq (1+\Delta^{-2\varepsilon})\Delta/qp^{r-1}$ by (a). Thus, by Chernoff's bound, we have

$$\begin{aligned}\mathbb{P}\left[\deg_{H_{i,j}}(x) - \mathbb{E}\left[\deg_{H_{i,j}}(x)\right] \geq \Delta^{1-2\varepsilon}/qp^{r-1}\right] &\leq 2\exp\left(-\frac{\Delta^{1-4\varepsilon}}{3qp^{r-1}}\right) \\ &\overset{(2.3.9)}{\leq} \exp\left(-\Delta^{\varepsilon}\right).\end{aligned}$$

Similarly, for (B), fix $i \in [p], j \in [q]$ and two distinct vertices $x, y \in V(H_i)$. Note that $\mathbb{E}\left[\deg_{H_{i,j}}(xy)\right] = \deg_{H_i}(xy)/q \leq \Delta^c(H)/q \leq 1$. Thus, by Chernoff's bound, we have

$$\mathbb{P}\left[\deg_{H_{i,j}}(xy) \geq \Delta^{\varepsilon}\right] \leq 2\exp\left(-\Delta^{\varepsilon}\right).$$

To prove (C), consider $\ell \in [L]$, $\omega \in \mathcal{W}_\ell$, $J \in \mathcal{J}_\ell$ and $\sigma\colon \mathrm{supp}(J) \to [q]$. First note that $\mathbb{E}\left[\omega(E_{J,\sigma})\right] = \omega(E_J)/q^\ell \leq \frac{3}{2}\ell!\omega(E(H))/q^\ell p^{r\ell}$ by (b). We now aim to employ Theorem 2.5 with $E(H')$ playing the role of V. Let $<$ be an ordering of $E(H')$ in which the edges of H_i precede those of $H_{i'}$ whenever $i < i'$. Write $J = \{j_1, \ldots, j_\ell\}$ such that $j_1 \leq \cdots \leq j_\ell$ and define $\alpha := (\sigma(j_1), \ldots, \sigma(j_\ell)) \in [q]^\ell$. Hence, for $\mathcal{E} \in E_J$, we have $\mathcal{E} \in E_{J,\sigma}$ if and only if $f(e_i) = \sigma(j_i)$ for all $i \in [\ell]$, where $\mathcal{E} = \{e_1, \ldots, e_\ell\}$ with $e_1 < \cdots < e_\ell$. Consequently, with notation as in Theorem 2.5, we have $E_{J,\sigma} = E_J \cap E_\alpha$. Thus, $\omega(E_{J,\sigma}) = \omega_J(E_\alpha)$, where $\omega_J(\mathcal{E}) := \omega(\mathcal{E})\mathbb{1}_{\mathcal{E}\in E_J}$.

We now apply Theorem 2.5 with $E(H')$, ℓ, ω_J, $\frac{1}{2}\ell!\omega(E(H))/q^\ell p^{r\ell}$, $\Delta^{2\varepsilon}$ playing the roles of $V, \ell, \omega, \lambda, g$, respectively. For $k \in [\ell]$, we have that (recall that $\omega(E(H)) \geq \|\omega\|_k \Delta^{k+\delta}$ by assumption)

$$\|\omega_J\|_k q^k g^{k-1} \leq \|\omega\|_k \Delta^k \leq \omega(E(H))\Delta^{-\delta}.$$

Hence, we infer that (note $\mathbb{E}\left[\omega_J(E_\alpha)\right] + \lambda \leq 4\lambda$)

$$\begin{aligned}\mathbb{P}\left[\omega_J(E_\alpha) \geq \mathbb{E}\left[\omega_J(E_\alpha)\right] + \lambda\right] &\leq 2^\ell \exp\left(-\frac{\lambda}{48\ell^2 q^{-\ell}\omega(E(H))\Delta^{-\delta}}\right) + \exp\left(-\frac{\Delta^{2\varepsilon}}{24\ell^2}\right) \\ &\leq 2^\ell \exp\left(-\frac{\Delta^{\delta}}{96\ell p^{r\ell}}\right) + \exp\left(-\frac{\Delta^{2\varepsilon}}{24\ell^2}\right) \leq \exp\left(-\Delta^{\varepsilon}\right).\end{aligned}$$

A union bound implies that the random choice of f satisfies (A), (B) and (C) simultaneously with positive probability. From now, fix such a function f.

Step 3. *Random matchings*

Let $\widetilde{\Delta} := (1+2\Delta^{-2\varepsilon})\Delta/qp^{r-1} \geq \Delta^{5r\varepsilon}$ by (2.3.9) and $M := (1+\Delta^{-2\varepsilon})\widetilde{\Delta}$. Note that

$$p^{r-1}qM = (1 \pm 4\Delta^{-2\varepsilon})\Delta. \tag{2.3.10}$$

By (A), we have $\Delta(H_{i,j}) \leq \widetilde{\Delta}$. Moreover, by (B), $\Delta^c(H_{i,j}) \leq \Delta^{\varepsilon} \leq \widetilde{\Delta}^{1/5r}$. Thus, for all $i \in [p]$, $j \in [q]$, we can apply Theorem 2.4 (with $\delta = 1/2$, say) to obtain a partition of $E(H_{i,j})$ into M matchings. This yields a partition of each $E(H_i)$ into $q \cdot M$ matchings $\mathcal{M}_{i,1}, \ldots, \mathcal{M}_{i,qM}$.

Now, for each $i \in [p]$ independently, pick an index $s_i \in [qM]$ uniformly at random, and define

$$\mathcal{M} := \bigcup_{i\in[p]} \mathcal{M}_{i,s_i}.$$

Clearly, $\mathcal{M}$ is a matching in $H' \subseteq H$. Moreover, every edge of H' belongs to $\mathcal{M}$ with probability $1/qM$.

Now, consider $\ell \in [L]$ and $\omega \in \mathcal{W}_\ell$. We first determine the expected value of $\omega(\mathcal{M})$. By linearity,

$$\mathbb{E}\left[\omega(\mathcal{M})\right] = \sum_{\mathcal{E} \in \binom{E(H)}{\ell}} \omega(\mathcal{E}) \mathbb{P}\left[\mathcal{E} \subseteq \mathcal{M}\right].$$

We analyse this sum according to the different types of $\mathcal{E}$. For $k \in [\ell]$, let $\mathcal{J}_{\ell,k}$ be the set of all $J \in \mathcal{J}_\ell$ with $|\mathrm{supp}(J)| = k$. Consider $\mathcal{E} \in \binom{E(H)}{\ell}$ and let $J := \tau(\mathcal{E})$. Note that if $0 \in J$, then some edge in $\mathcal{E}$ does not belong to H' and hence $\mathbb{P}\left[\mathcal{E} \subseteq \mathcal{M}\right] = 0$. Hence, we can assume that $J \in \mathcal{J}_\ell$. If $J \in \mathcal{J}_{\ell,\ell}$, then the edges in $\mathcal{E}$ belong to $\mathcal{M}$ independently with probability $1/qM$, and hence $\mathbb{P}\left[\mathcal{E} \subseteq \mathcal{M}\right] = (qM)^{-\ell}$. Now, suppose $J \in \mathcal{J}_{\ell,k}$ for some $k \in [\ell - 1]$. By the definition of $\mathcal{M}$, if $e, e' \in \mathcal{E}$ with $\tau(e) = \tau(e')$, then $\mathbb{P}\left[\mathcal{E} \subseteq \mathcal{M}\right] = 0$ if $e \in E(H_{\tau(e),j})$ and $e' \in E(H_{\tau(e),j'})$ for distinct j, j'. Hence, we can further assume that $\mathcal{E} \in E_{J,\sigma}$ for some $\sigma \colon \mathrm{supp}(J) \to [q]$. We then have $\mathbb{P}\left[\mathcal{E} \subseteq \mathcal{M}\right] \in \{0, (qM)^{-k}\}$. Altogether, we deduce that

$$\mathbb{E}\left[\omega(\mathcal{M})\right] = \sum_{J \in \mathcal{J}_{\ell,\ell}} \omega(E_J)(qM)^{-\ell} \pm \sum_{k=1}^{\ell-1} \sum_{J \in \mathcal{J}_{\ell,k}, \sigma \colon \mathrm{supp}(J) \to [q]} \omega(E_{J,\sigma})(qM)^{-k}.$$

We will show that the first sum is the dominant term. Clearly, $|\mathcal{J}_{\ell,\ell}| = \binom{p}{\ell}$. Thus, using (b), we infer that

$$\begin{aligned}\sum_{J \in \mathcal{J}_{\ell,\ell}} \omega(E_J)(qM)^{-\ell} &= \binom{p}{\ell} \cdot (1 \pm \Delta^{-2\varepsilon}) \frac{\ell! \omega(E(H))}{p^{r\ell}} \cdot \frac{1}{(qM)^\ell} \\ &\overset{(2.3.10)}{=} (1 \pm \Delta^{-3\varepsilon/2}) \omega(E(H))/\Delta^\ell.\end{aligned}$$

For $k \in [\ell - 1]$, employing (C) and $|\mathcal{J}_{\ell,k}| = \binom{p}{k}\binom{\ell-1}{k-1}$, we deduce that

$$\sum_{J \in \mathcal{J}_{\ell,k}, \sigma \colon \mathrm{supp}(J) \to [q]} \omega(E_{J,\sigma})(qM)^{-k} \leq p^k 2^\ell q^k \frac{2\ell! \omega(E(H))}{q^\ell p^{r\ell}} \cdot \frac{1}{(qM)^k} \leq \frac{\omega(E(H))}{\Delta^{\ell+14\varepsilon}},$$

where in the last inequality we used that $\frac{p^k q^k}{q^\ell p^{r\ell}(qM)^k} = \frac{1}{(p^r q)^{\ell-k}(p^{r-1}qM)^k}$ together with $(p^r q)^{\ell-k} \geq \Delta^{\ell-k+15\varepsilon}$ by (2.3.9) and $(p^{r-1}qM)^k \geq \frac{1}{2}\Delta^k$ by (2.3.10). Putting everything together, we obtain that

$$\mathbb{E}\left[\omega(\mathcal{M})\right] = (1 \pm 2\Delta^{-3\varepsilon/2}) \omega(E(H))/\Delta^\ell.$$

Finally, we need to bound the effect of each random variable s_i. Note that each outcome of the variables $s_1, \ldots, s_p$ induces a function $\sigma \colon [p] \to [q]$, where $\sigma(i)$ is the unique $j \in [q]$ for which $\mathcal{M}_{i,s_i}$ was one of the matchings coming from $E(H_{i,j})$, and each tuple $\mathcal{E} \subseteq \mathcal{M}$ satisfies $\mathcal{E} \in E_{J,\sigma|_{\mathrm{supp}(J)}}$, where $J = \tau(\mathcal{E}) \in \mathcal{J}_\ell$. Since changing the value of s_i only affects those $\mathcal{E}$ with $i \in \tau(\mathcal{E})$, we have that the effect of s_i on $\omega(\mathcal{M})$ is at most

$$\max_{\sigma \colon [p] \to [q]} \sum_{J \in \mathcal{J}_\ell \colon i \in J} \omega(E_{J,\sigma|_{\mathrm{supp}(J)}}) \overset{(C)}{\leq} p^{\ell-1} \frac{2\ell! \omega(E(H))}{q^\ell p^{r\ell}} \overset{(2.3.9)}{=} \frac{2\ell! \omega(E(H))}{p\Delta^{(1-5r\varepsilon)\ell}} \leq \frac{\omega(E(H))}{\Delta^{\ell+14Lr\varepsilon}}.$$

Thus, using McDiarmid's inequality, we deduce that

$$\begin{aligned}\mathbb{P}\left[\omega(\mathcal{M}) \neq (1 \pm \Delta^{-2\varepsilon})\mathbb{E}\left[\omega(\mathcal{M})\right]\right] &\leq 2\exp\left(-\frac{2\Delta^{-4\varepsilon}\mathbb{E}\left[\omega(\mathcal{M})\right]^2}{p \cdot (\omega(E(H))/\Delta^{\ell+14Lr\varepsilon})^2}\right) \\ &\leq 2\exp\left(-\frac{\Delta^{28Lr\varepsilon-4\varepsilon}}{p}\right) \leq \exp\left(-\Delta^\varepsilon\right).\end{aligned}$$

A union bound over all $\ell \in [L]$ and $\omega \in \mathcal{W}_\ell$ completes the proof. $\square$

Chapter 3

A rainbow blow-up lemma

The content of this chapter is based on [30] with Stefan Glock and Felix Joos.

3.1 Introduction to rainbow embedding problems

We study rainbow embeddings of bounded-degree spanning subgraphs into quasi-random graphs with almost optimally bounded edge-colourings. Moreover, following the recent work of Montgomery, Pokrovskiy and Sudakov [104] on embedding rainbow trees, we present several applications to graph decompositions, graph labellings and orthogonal double covers.

Given a (not necessarily proper) edge-colouring of a graph, a subgraph is called *rainbow* if all its edges have different colours. Rainbow colourings appear in many different contexts of combinatorics, and many problems beyond graph colouring can be translated into a rainbow subgraph problem. What makes this concept so versatile is that it can be used to find 'conflict-free' subgraphs. More precisely, an edge-colouring of a graph G can be interpreted as a system of conflicts on $E(G)$, where two edges conflict if they have the same colour. A subgraph is then conflict-free if and only if it is rainbow. For instance, rainbow matchings in $K_{n,n}$ can be used to model transversals in Latin squares. The study of Latin squares dates back to the work of Euler in the 18th century and has since been a fascinating and fruitful area of research. The famous Ryser–Brualdi–Stein conjecture asserts that every $n \times n$ Latin square has a partial transversal of size $n-1$, which is equivalent to saying that any proper n-edge-colouring of $K_{n,n}$ admits a rainbow matching of size $n-1$. This problem is wide open and the currently best approximate result is due to Hatami and Shor [62].

As a second example, we consider a powerful application of rainbow colourings to graph decompositions. Perhaps one of the oldest decomposition results is Walecki's theorem from 1892 saying that K_{2n+1} can be decomposed into Hamilton cycles. His construction not only gives any decomposition, but a 'cyclic' decomposition based on a rotation technique, by finding one Hamilton cycle H^* in K_{2n+1} and a permutation π on $V(K_{2n+1})$ such that the permuted copies $\pi^i(H^*)$ of H^* for $i = 0, \dots, n-1$ are pairwise edge-disjoint (and thus decompose K_{2n+1}). The difficulty here is of course finding H^* given π, or vice versa. Unfortunately, for many other decomposition problems, this is not as easy, or indeed not possible at all.

In recent years, exciting progress has been made in the area of (hyper-)graph decompositions (as also discussed in Section 1.1). Many of these striking results are based on very different techniques, such as absorbing-type methods, randomised constructions and variations of Szemerédi's regularity technique. In a recent breakthrough, Montgomery, Pokrovskiy and Sudakov [103, 104] brought the use of the rotation technique back into focus and employed it as a key tool for proving Ringel's conjecture, by reducing it to a rainbow embedding problem.[1] A strengthening of Ringel's conjecture is due to Kotzig [87], who conjectured in 1973 that there even exists a cyclic decomposition. This can be phrased as a rainbow embedding problem as follows: Order the vertices of K_{2n+1} cyclically and colour each edge $\{i, j\} \in E(K_{2n+1})$ with its distance (that is, the distance of i, j in the cyclic ordering), which is a number between 1 and n. We call this edge-colouring the *near-distance colouring*. The simple but crucial observation is that if T is a rainbow subtree, then T can be rotated according to the cyclic vertex ordering, yielding $2n + 1$ edge-disjoint copies of T (and thus a cyclic decomposition if T has n edges). See Figure 3.1 for an illustration.

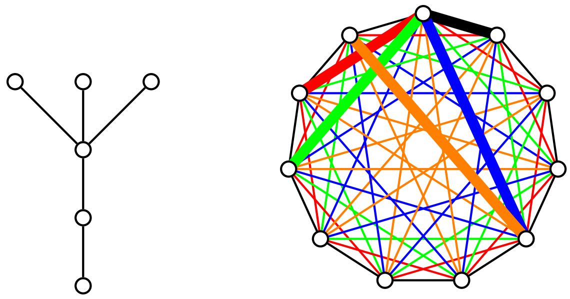


Figure 3.1: A tree T with $n = 5$ edges (left) and the K_{11} with its near-distance colouring (right). The thick edges in the complete graph show a rainbow copy of T. By cyclically rotating this rainbow copy of T, we can decompose the K_{11} into 11 disjoint copies of T. To that end, notice that the rotation shifts an edge to another edge of the same colour.

Montgomery, Pokrovskiy and Sudakov succeeded in proving that indeed every near-distance coloured K_{2n+1} contains a rainbow copy of any tree T on n edges for sufficiently large n. Thus, they completely resolved the conjectures of Ringel and Kotzig for sufficiently large n. Note that for each vertex v in a near-distance coloured complete graph and any given distance, there are only two vertices which have exactly this distance from v. More generally, an edge-colouring is called *locally k-bounded* if each colour class has maximum degree at most k. The following statement thus implies Kotzig's and Ringel's conjecture: Any locally 2-bounded edge-colouring of K_{2n+1} contains a rainbow copy of any tree with n edges. Along their way of proving Ringel's conjecture exactly [103], Montgomery, Pokrovskiy and Sudakov [104] first proved the following asymptotic version of this statement, which in turn yields asymptotic versions of these conjectures (all asymptotic terms are considered as $n \to \infty$).

Theorem 3.1 ([104])**.** *For fixed k, any locally k-bounded edge-colouring of K_n contains a rainbow copy of any tree with $(1 - o(1))n/k$ edges.*

Our main results are very similar in spirit. Roughly speaking, instead of dealing with trees, our results apply to general graphs H, but we require H to have bounded degree, whereas one of the great achievements of [104] is that no such requirement is necessary when dealing with trees. The following is a special case of our main result (Theorem 3.3). An edge-colouring is called *(globally) k-bounded* if any colour appears at most k times.

Theorem 3.2 (Ehard, Glock, Joos [30])**.** *Suppose H is a graph on at most n vertices with $\Delta(H) = \mathcal{O}(1)$. Then any locally $\mathcal{O}(1)$-bounded and globally $(1 - o(1))\binom{n}{2}/e(H)$-bounded edge-colouring of K_n contains a rainbow copy of H.*

[1] A similar approach has previously been used by Drmota and Lladó [27] in connection with a bipartite version of Ringel's conjecture posed by Graham and Häggkvist.

It is plain that any locally k-bounded colouring is (globally) $kn/2$-bounded. Thus, Theorem 3.2 implies Theorem 3.1 for bounded-degree trees. Note that the assumption that the colouring is $(1-o(1))\binom{n}{2}/e(H)$-bounded is asymptotically best possible in the sense that if the colouring was not $\binom{n}{2}/e(H)$-bounded, there might be less than $e(H)$ colours, making the existence of a rainbow copy of H impossible.

Beyond the approximate solution of Ringel's conjecture in [104], Montgomery, Pokrovskiy and Sudakov also provide applications of their result to graph labelling and orthogonal double covers. Our applications are very much inspired by theirs and are essentially proved analogously. We refer the discussion of these applications to Section 3.7.

Rainbow embedding problems have also been extensively studied for their own sake. For instance, Erdős and Stein asked for the maximal k such that any k-bounded edge-colouring of K_n contains a rainbow Hamilton cycle (cf. [39]). After several subsequent improvements, Albert, Frieze and Reed [4] showed that $k = \Omega(n)$. Theorem 3.2 implies that under the additional assumption that the colouring is locally $\mathcal{O}(1)$-bounded, we have $k = (1-o(1))n/2$, which is essentially best possible. This is not a new result but also follows from results in [78, 102]. However, the results in [78, 102] are limited to finding Hamilton cycles or F-factors (in fact, approximate decompositions into these structures). Theorem 3.2 allows the same conclusion if we seek an $\sqrt{n/2} \times \sqrt{n/2}$ grid, say, or any other bounded-degree graph with roughly n edges. For general subgraphs H, the best previous result is due to Böttcher, Kohayakawa and Procacci [17], who showed that given any $n/(51\Delta^2)$-bounded edge-colouring of K_n and any graph H on n vertices with $\Delta(H) \leq \Delta$, one can find a rainbow copy of H. For bounded-degree graphs, our Theorem 3.2 improves the global boundedness condition to an asymptotically best possible one, under the additional assumption that the colouring is locally $\mathcal{O}(1)$-bounded.

3.1.1 Main result

We now state a more general version of Theorem 3.2. We say that a graph G on n vertices is (ε, d)-*quasirandom* if for all $v \in V(G)$ we have $\deg_G(v) = (d \pm \varepsilon)n$, and for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$, we have $e_G(S,T) = (d \pm \varepsilon)|S||T|$.

Theorem 3.3 (Ehard, Glock, Joos [30]). *For all $d, \gamma \in (0,1]$ and $\Delta, \Lambda \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose G and H are graphs on n vertices, G is (ε, d)-quasirandom and $\Delta(H) \leq \Delta$. Then given any locally Λ-bounded and globally $(1-\gamma)e(G)/e(H)$-bounded edge-colouring of G, there is a rainbow copy of H in G.*

Clearly, Theorem 3.3 implies Theorem 3.2. We derive Theorem 3.3 from an even more general 'blow-up lemma' (Lemma 3.4). The original blow-up lemma of Komlós, Sárközy and Szemerédi [82] developed roughly 20 years ago, is a powerful tool to find spanning subgraphs and has found numerous important applications in extremal combinatorics [19, 51, 83, 84, 85, 90, 92]. Roughly speaking, it says that given a k-partite graph G that is 'super-regular' between any two vertex classes, and a k-partite bounded-degree graph H with a matching vertex partition, then H is a subgraph of G. Note that the conclusion is trivial if G is complete k-partite, so the crux here is that instead of requiring G to be complete between any two vertex classes, super-regularity suffices. Such a scenario can often be obtained in conjunction with Szemerédi's regularity lemma, which makes it widely applicable. Many variations of the blow-up lemma have been obtained over the years (e.g. [6, 18, 24, 70, 79, 112]). Recently, Glock and Joos [50] proved a rainbow blow-up lemma for $o(n)$-bounded edge-colourings which allows to find a rainbow embedding of H. The content of this chapter builds upon this result. The key novelty is that instead of requiring the colouring to be $o(n)$-bounded, our new result applies for almost optimally bounded colourings. (But we assume here that the colouring is locally $\mathcal{O}(1)$-bounded, which is not necessary in [50]).

In order to state our new rainbow blow-up lemma, we need to introduce some terminology. If $c\colon E(G) \to C$ is an edge-colouring of a graph G and $\alpha \in C$, denote by $e^\alpha(G)$ the number of α-coloured edges of G. Moreover, for disjoint $S, T \subseteq V(G)$, denote by $e^\alpha_G(S,T)$ the number of α-coloured edges of G with one endpoint in S and the other one in T.

We say that $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d)*-super-regular blow-up instance* if

- H and G are graphs, $(X_i)_{i\in[r]}$ is a partition of $V(H)$ into independent sets, $(V_i)_{i\in[r]}$ is a partition of $V(G)$, and $|X_i| = |V_i|$ for all $i \in [r]$, and
- for all $ij \in \binom{[r]}{2}$, the bipartite graph $G[V_i, V_j]$ is (ε, d)-super-regular.

We say that $\phi\colon V(H) \to V(G)$ is an *embedding of H into G* if ϕ is injective and $\phi(x)\phi(y) \in E(G)$ for all $xy \in E(H)$. We also write $\phi\colon H \to G$ in this case. We say that ϕ is *rainbow* if $\phi(H)$ is rainbow.

We now state our new rainbow blow-up lemma.

Lemma 3.4 (Ehard, Glock, Joos [30] – Rainbow blow-up lemma)**.** *For all $d, \gamma \in (0, 1]$ and $\Delta, \Lambda, r \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d)-super-regular blow-up instance. Assume further that*

(i) $\Delta(H) \leq \Delta$*;*

(ii) $|V_i| = (1 \pm \varepsilon)n$ *for all* $i \in [r]$*;*

(iii) $c\colon E(G) \to C$ *is a locally* Λ*-bounded edge-colouring such that the following holds for all* $\alpha \in C$*:*
$$\sum_{ij\in\binom{[r]}{2}} e^\alpha_G(V_i, V_j)e_H(X_i, X_j) \leq (1 - \gamma)dn^2.$$

Then there exists a rainbow embedding ϕ of H into G such that $\phi(x) \in V_i$ for all $i \in [r]$ and $x \in X_i$.

The boundedness condition in (iii) can often be simplified, for instance in the following natural situations: if $e_H(X_i, X_j)$ is the same for all pairs i, j, then c needs to be $(1 - \gamma)e(G[V_1, \ldots, V_r])/e(H)$-bounded. Similarly, if c is 'colour-split', that is, $e^\alpha_G(V_i, V_j) \in \{e^\alpha(G), 0\}$, then c needs to be $(1 - \gamma)e(G[V_i, V_j])/e(H[X_i, X_j])$-bounded for all $ij \in \binom{[r]}{2}$. Both conditions are easily seen to be asymptotically best possible. Condition (iii) is designed to work in the general setting of Lemma 3.4. In fact, we will deduce Lemma 3.4 in Section 3.6 from a reduced instance (Lemma 3.13) where the colouring c is colour-split. In the proof of Theorem 3.3, we will randomly partition $V(G)$ into equal-sized $(V_i)_{i\in[r]}$ and see that (iii) holds.

The blow-up lemma for $o(n)$-bounded colourings of Glock and Joos was applied in [50] to transfer the bandwidth theorem to the rainbow setting, using Szemerédi's regularity lemma. Unfortunately, it seems much more complicated to use the regularity lemma in the optimally bounded setting. In particular, if the number of clusters r may be much larger than ε^{-1} after an application of the regularity lemma, we are not aware how to perform the reduction steps in Section 3.4 to refine the partitioning and split the colours into groups such that the colouring is 'colour-split'. It would be interesting whether one could strengthen our result in this direction for an easier applicability in conjunction with Szemerédi's regularity lemma.

3.2 Proof overview

To sketch the approach for the proof our main result, let us consider the following simplified setup. Suppose $V(H)$ is partitioned into independent sets X_1, X_2, X_3 of size n and H consists of a perfect matching between X_1 and X_2, and a perfect matching between X_2 and X_3. Following our general strategy as explained in Section 1.4, we proceed cluster by cluster and in the ith step, we embed the vertices of X_i into V_i. The desired bijection is obtained as a matching within a 'candidacy graph' A_i, which is an auxiliary bipartite graph between X_i and V_i where $xv \in E(A_i)$ only if v is still a suitable image for x.

Suppose that we have already found an embedding $\phi_1 \colon X_1 \to V_1$, and next we want to embed X_2 into V_2. We define the bipartite graph A_2 between X_2 and V_2 by adding the edge xv if $\phi_1(y)v \in E(G)$, where y is the H-neighbour of x in X_1. Now, the aim is to find a perfect matching σ in A_2. Note that any such perfect matching yields a valid embedding of $H[X_1, X_2]$ into $G[V_1, V_2]$. Moreover, if we aim to find a rainbow embedding, this can be achieved as follows. For each $xv \in E(A_2)$, we colour xv with the colour of $\phi_1(y)v$. Observe that if σ is rainbow, then the embedding of $H[X_1, X_2]$ into $G[V_1, V_2]$ will be rainbow, too. Let us assume that A_2 is super-regular. It is well known that A_2 then has a perfect matching. One key ingredient in [50] was to combine this fact with a recent result of Coulson and Perarnau [23], based on the switching method, to even find a rainbow perfect matching. Unfortunately, the switching method relies upon the fact that the given colouring is $o(n)$-bounded, and is thus not applicable in the present setting. There are two key insights that will allow us to deal with almost optimally bounded colourings.

First, note that given a proper colouring of a graph G, if we take a random subset U of size $\mu|G|$, then with high probability, the colouring induced on U will be $(1 + o(1))\mu|U|$-bounded, and thus the rainbow blow-up lemma from [50] is applicable (on U). This gives hope to combine this with an 'approximate result' on $V(G) \setminus U$ to obtain the desired embedding. Such a combination of techniques has already been successfully used in [78]. In our simplified discussion, let us thus assume we do not need to find a perfect rainbow matching σ, but would be content if σ is almost perfect.

This leads us to the second main ingredient of our proof—matchings in hypergraphs. Given our candidacy graph A_2 and its (auxiliary) colouring $c_2 \colon E(A_2) \to C_2$, we define a hypergraph $\mathcal{H}$ on $X_2 \cup V_2 \cup C_2$ where for every edge $e \in E(A_2)$, we add the hyperedge $e \cup \{c(e)\}$ to $\mathcal{H}$. A simple but crucial observation is that there is a one-to-one correspondence between matchings in $\mathcal{H}$ and rainbow matchings in A_2. In particular, a matching $\mathcal{M}$ in $\mathcal{H}$ that covers almost all vertices of $X_2 \cup V_2$ would translate into our desired almost perfect rainbow matching σ in A_2. Here, we can make use of the hypergraph matching result from Chapter 2. At this point, we remark that since A_2 is super-regular, all vertices of $X_2 \cup V_2$ have roughly the same degree in $\mathcal{H}$, and if the degrees of the colours are not larger (that is, the colouring is appropriately bounded), this will suffice to find the desired matching in $\mathcal{H}$.

Moreover, note that we assumed that A_2 is super-regular and its colouring is appropriately bounded. After embedding X_2 according to σ, we have to *update* the candidacy graph A_3 as we updated A_2 after embedding X_1. Of course, whether A_3 will be super-regular and its colouring appropriately bounded depends heavily on σ. For the embedding not to get stuck, we need to find in A_2 not just *any* σ, but a *good* one. To achieve this, we make use of our result on hypergraph matchings (Theorem 2.2) which guarantees a matching $\mathcal{M}$ in $\mathcal{H}$ that is in many ways 'random-like'. This will allow us to find an almost perfect rainbow matching σ for which the updated candidacy graph A_3 will have the desired properties. In more detail, the conclusion of Theorem 2.2

allows to put weight functions on the edges of the hypergraph $\mathcal{H}$ and guarantees a matching $\mathcal{M}$ in $\mathcal{H}$ such that the weight covered by $\mathcal{M}$ is what we would expect from an idealized random matching. We employ such weight functions to provide that A_3 will still be super-regular and its colouring appropriately bounded. This is done in Section 3.5 where we prove an 'Approximate Embedding Lemma' (Lemma 3.12). As discussed, in the end we will make use of the rainbow blow-up lemma for $o(n)$-bounded edge-colourings from [50] to turn an approximate embedding into a complete one.

This simplified setup already presents the main ingredients for the proof of our rainbow blow-up lemma (Lemma 3.4). An important step in the approach of Rödl and Ruciński [112] is to refine the partition of H such that H only induces matchings between its refined partition classes using the Hajnal–Szemerédi theorem. We follow the same strategy and additionally find a subgraph G' of G such that the edge-colouring of G' is colour-split, that is, each colour only appears between one bipartite pair of the refined partition classes of G'. This enables us to deduce Lemma 3.4 from a reduced instance (Lemma 3.13) where we impose that H only induces matchings between its partition classes and the edge-colouring of G is colour-split. The main tools to perform these reductions are given in Section 3.4.

3.3 Preliminaries

3.3.1 Colouring notation

For a graph G and a a set C, a function $c\colon E(G) \to 2^C$ is called an *edge set colouring* of G. A colour $\alpha \in C$ *appears* on an edge e if $\alpha \in c(e)$. We define the *codegree* of c as the maximum number of edges on which any two fixed colours appear together. For a colour $\alpha \in C$, a vertex $v \in V(G)$, and disjoint sets $A, B \subseteq V(G)$, we define

- $\deg_G^\alpha(v) := |\{u \in N_G(v)\colon \alpha \text{ appears on } uv\}|$;
- $e_G^\alpha(A, B) := |\{ab \in E(G)\colon a \in A, b \in B, \text{ and } \alpha \text{ appears on } ab\}|$;
- $e^\alpha(G) := |\{e \in E(G)\colon \alpha \text{ appears on } e\}|$.

We say that

- c is *(globally) k-bounded* if each colour appears on at most k edges;
- c is *locally Λ-bounded* if each colour class has maximum degree at most Λ.

Given a partition $(V_i)_{i\in[r]}$ of $V(G)$, we say that c is *colour-split with respect to* $(V_i)_{i\in[r]}$ if for all $e, f \in E(G)$ we have $c(e) \cap c(f) = \emptyset$ whenever $e \in E(G[V_i, V_j])$ and $f \notin E(G[V_i, V_j])$. If the partition is clear from the context, we just say that c is colour-split. We call a subgraph G' of G *rainbow* if all the edges in G' have pairwise disjoint colour sets.

3.3.2 Another rainbow blow-up lemma

Our final tool is the following special case of the rainbow blow-up lemma from [50] for $o(n)$-bounded colourings. Even though the global boundedness condition is more restrictive there, it is still applicable on a random subset of vertices (see the discussion in Section 3.2). As such, it is the main tool in our proof to turn a partial rainbow embedding into a complete one.

We say that $(H, G, (X_i)_{i\in[r]_0}, (V_i)_{i\in[r]_0})$ is an *(ε, d)-super-regular blow-up instance with exceptional sets* (X_0, V_0) if X_0 is an independent set in H, $|V_0| = |X_0|$ and

$(H - X_0, G - V_0, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d)-super-regular blow-up instance. We call graphs $(A_i)_{i\in[r]}$ *candidacy graphs* if A_i is a bipartite graph with partition (X_i, V_i) for all $i \in [r]$.

Lemma 3.5 ([50, Lemma 5.2]). *Suppose $1/n \ll \varepsilon, \mu \ll d, 1/r, 1/\Delta$.*
Let $\mathcal{B} = (H, G, (X_i)_{i\in[r]_0}, (V_i)_{i\in[r]_0})$ be an (ε, d_G)-super-regular blow-up instance with exceptional sets (X_0, V_0) and (ε, d_A)-super-regular candidacy graphs $(A_i)_{i\in[r]}$, where $d_G, d_A \geq d$. Assume further that

(i) $\Delta(H) \leq \Delta$;

(ii) $|V_i| = n$ *for all* $i \in [r]$;

(iii) $H[X_i, X_j]$ *is a matching for all* $ij \in \binom{[r]}{2}$.

Let $c\colon E(G) \to C$ be a μn-bounded edge-colouring of G. Suppose a bijection $\psi_0\colon X_0 \to V_0$ is given such that

(iv) *for all $x \in X_0$, $i \in [r]$ and $x_i \in N_H(x) \cap X_i$, we have $N_{A_i}(x_i) \subseteq N_G(\psi_0(x))$;*

(v) *for all $i \in [r]$, $x \in X_i$, $v \in N_{A_i}(x)$ and distinct $x_0, x_0' \in N_H(x) \cap X_0$, we have $c(\psi_0(x_0)v) \neq c(\psi_0(x_0')v)$.*

Then there exists a rainbow embedding ψ of H into G which extends ψ_0 such that $\psi(x) \in N_{A_i}(x)$ for all $i \in [r]$ and $x \in X_i$.

3.4 Colour splitting

The goal of this section is to provide some useful lemmas to refine the partitions of a blow-up instance and split the colours into groups in order to obtain better control for the rainbow embedding. In particular, we will refine the partition of H using the Hajnal–Szemerédi theorem (Theorem 3.8) and we will refine the partition of G accordingly by a random procedure. This reduction is performed in Lemma 3.9. Additionally, we group the edges of G such that the edge-colouring of G is colour-split, which is based on a random procedure given in Lemma 3.6. To obtain better control on the boundedness condition of the edge-colouring of a blow-up instance when performing these reductions, we first group the edges of G such that G is colour-split (Lemma 3.7) and afterwards refine the partitions of H and G (Lemma 3.9).

The first lemma will guarantee that with high probability the resulting graph is still super-regular when we randomly split colours in order to obtain a colour-split colouring.

Lemma 3.6 ([30]). *Let $1/n \ll \varepsilon \ll \varepsilon' \ll \gamma, d, 1/\Lambda$. Suppose G is an (ε, d)-super-regular graph with vertex partition (A, B) such that $|A|, |B| = (1 \pm \varepsilon)n$, and $c\colon E(G) \to C$ is a locally Λ-bounded edge-colouring of G. Suppose $\{Y_\alpha\colon \alpha \in C\} \cup \{Z_e\colon e \in E(G)\}$ is a set of mutually independent Bernoulli random variables such that $\mathbb{P}\left[Y_{c(e)} + Z_e = 2\right] = \gamma$ for every $e \in E(G)$. Suppose G' is the random spanning subgraph of G where $e \in E(G)$ belongs to $E(G')$ whenever $Y_{c(e)} + Z_e = 2$. Then G' is $(\varepsilon', \gamma d)$-super-regular with probability at least $1 - 1/n^{10}$.*

Proof. We call a pair of distinct vertices $u, v \in A$ *good* if $|N_G(u, v)| = (d \pm \varepsilon)^2|B|$, and $|\{w \in N_G(u, v)\colon c(uw) = c(vw)\}| \leq \varepsilon|B|$. We first claim that almost all pairs are good.

Claim 1. There are at least $(1 - 7\varepsilon)|A|^2/2$ good pairs $u, v \in A$.

Proof of claim: Since G is (ε, d)-super-regular, at most $2\varepsilon|A|^2$ pairs $u, v \in A$ do not satisfy $|N_G(u,v)| = (d \pm \varepsilon)^2|B|$ by Fact 1.11.

We claim that the number of pairs $u, v \in A$ with $|\{w \in N_G(u,v) : c(uw) = c(vw)\}| \geq \varepsilon|B|$ is at most $\varepsilon|A|^2$. For this, we first count the number of monochromatic paths of length 2 in G with both ends in A. Each vertex $w \in B$ is contained in $\sum_{\alpha\in C}\binom{\deg^\alpha_G(w)}{2}$ monochromatic paths uwv in G. Since $\deg^\alpha_G(w) \leq \Lambda$ for every colour $\alpha \in C$ and $\sum_{\alpha\in C}\deg^\alpha_G(w) \leq |A|$, we have

$$\sum_{\alpha\in C}\binom{\deg^\alpha_G(w)}{2} \leq \sum_{\alpha\in C}\deg^\alpha_G(w)^2 \leq \Lambda|A|.$$

Hence, there are at most $\Lambda|A||B|$ monochromatic paths of length 2 in G with both ends in A. This implies that the number of pairs $u, v \in A$ with $|\{w \in N_G(u,v) : c(uw) = c(vw)\}| \geq \varepsilon|B|$ is at most

$$\frac{\Lambda|A||B|}{\varepsilon|B|} \leq \varepsilon|A|^2.$$

Thus, there are at least $\binom{|A|}{2} - 3\varepsilon|A|^2 \geq (1-7\varepsilon)|A|^2/2$ good pairs $u, v \in A$. $-$

We fix a vertex $x \in A \cup B$ and a good pair of vertices $u, v \in A$. Let $X_x := \deg_{G'}(x)$ and $X_{u,v} := |N_{G'}(u,v)|$. Clearly, X_x and $X_{u,v}$ are determined by $\{Y_\alpha : \alpha \in C\} \cup \{Z_e : e \in E(G)\}$. Note that if $w \in N_G(u,v)$ satisfies $c(uw) \neq c(vw)$, then $\mathbb{P}[w \in N_{G'}(u,v)] = \gamma^2$. Thus, we have

$$\mathbb{E}[X_x] = \gamma\deg_G(x) = \gamma dn \pm 3\varepsilon n \quad \text{and} \quad \mathbb{E}[X_{u,v}] = \gamma^2 d^2 n \pm 10\varepsilon n. \tag{3.4.1}$$

For all $\alpha \in C$ and $e \in E(G)$, let b_α and b_e be minimally chosen such that changing the outcome of Y_α changes X_x by at most b_α, and changing the outcome of Z_e changes X_x by at most b_e. Note that

$$\sum_{\alpha\in C} b_\alpha + \sum_{e\in E(G)} b_e \leq 2\deg_G(x) \leq 3n.$$

Moreover, we clearly have $b_e \leq 1$, and since the colouring c is locally Λ-bounded, $b_\alpha \leq \Lambda$. Using McDiarmid's inequality (Lemma 1.8), we obtain that

$$\mathbb{P}[|X_x - \mathbb{E}[X_x]| > \varepsilon n] \leq 2\exp\left(-\frac{\varepsilon^2 n^2}{\Lambda \cdot 3n}\right) < \frac{1}{n^{20}}. \tag{3.4.2}$$

With similar arguments one can show that

$$\mathbb{P}[|X_{u,v} - \mathbb{E}[X_{u,v}]| > \varepsilon n] < \frac{1}{n^{20}}. \tag{3.4.3}$$

A union bound over all $x \in A \cup B$ and all good pairs $u, v \in A$ yields together with (3.4.1), (3.4.2) and (3.4.3) that with probability at least $1 - 1/n^{10}$, we have $\deg_{G'}(x) = \gamma dn \pm 4\varepsilon n$ for all $x \in A \cup B$, and $|N_{G'}(u,v)| = \gamma^2 d^2 n \pm 11\varepsilon n$ for all good pairs $u, v \in A$. Given that, Theorem 1.13 implies that G' is $(\varepsilon', \gamma d)$-super-regular.

□

The next lemma states that we can split the colours of the host graph G into groups and obtain a subgraph G' which is still super-regular, and whose colouring is colour-split and appropriately bounded.

Lemma 3.7 ([30]). *Let $1/n \ll \varepsilon \ll \varepsilon' \ll d' \ll \gamma \ll d, 1/\Lambda, 1/r, 1/\Delta$. Suppose $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d)-super-regular blow-up instance. Assume further that*

(i) $\Delta(H) \le \Delta$ *and* $e_H(X_i, X_j) \ge \gamma^2 n$ *for all* $ij \in \binom{[r]}{2}$;

(ii) $|V_i| = (1 \pm \varepsilon)n$ *for all* $i \in [r]$;

(iii) $c\colon E(G) \to C$ *is locally* Λ*-bounded and the following holds for all* $\alpha \in C$:

$$\sum_{ij \in \binom{[r]}{2}} e^{\alpha}_G(V_i, V_j) e_H(X_i, X_j) \le (1-\gamma) dn^2.$$

Then there exists a spanning subgraph G' *of* G *such that*

(a) $(H, G', (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ *is an* (ε', d')*-super-regular blow-up instance;*

(b) c *restricted to* G' *is colour-split;*

(c) c *restricted to* $G'[V_i, V_j]$ *is* $(1 - \frac{\gamma}{2})\frac{e_{G'}(V_i,V_j)}{e_H(X_i,X_j)}$*-bounded for all* $ij \in \binom{[r]}{2}$.

Proof. Let $\hat{\varepsilon}$ be such that $\varepsilon \ll \hat{\varepsilon} \ll \varepsilon'$. The proof proceeds in three steps, where we iteratively define spanning subgraphs $G_3 \subseteq G_2 \subseteq G_1 \subseteq G$ such that G_3 satisfies the required properties of G' in the statement.

In the first step we suitably sparsify each bipartite subgraph $G[V_i, V_j]$. For every $ij \in \binom{[r]}{2}$, let

$$p_{ij} := \frac{e_H(X_i, X_j)}{2\Delta n}. \tag{3.4.4}$$

Note that $\gamma^2/(2\Delta) \le p_{ij} \le 1$ since $\gamma^2 n \le e_H(X_i, X_j) \le \Delta |X_i| \le 2\Delta n$. For every $ij \in \binom{[r]}{2}$, we keep each edge of $G[V_i, V_j]$ independently at random with probability p_{ij} and denote the resulting graph by $G_1[V_i, V_j]$. A simple application of Chernoff's inequality together with a union bound yields the following claim.

Claim 1. The following properties hold simultaneously with probability at least $1 - 1/n$ *for every* $ij \in \binom{[r]}{2}$.

(C1.1) $G_1[V_i, V_j]$ *is* $(2\varepsilon, p_{ij}d)$*-super-regular;*

(C1.2) $e^{\alpha}_{G_1}(V_i, V_j) \le e^{\alpha}_G(V_i, V_j) p_{ij} + \varepsilon n$ *for every colour* $\alpha \in C$.

Hence, by Claim 1, we may assume that G_1 is a spanning subgraph of G such that properties (C1.1)–(C1.2) hold. For every colour $\alpha \in C$, we obtain that

$$\begin{aligned}
&\sum_{ij \in \binom{[r]}{2}} e^{\alpha}_{G_1}(V_i, V_j) \frac{e_H(X_i, X_j)}{e_{G_1}(V_i, V_j)} \\
&\overset{(C1.1),(C1.2)}{\le} \sum_{ij \in \binom{[r]}{2}} (e^{\alpha}_G(V_i, V_j) p_{ij} + \varepsilon n) \frac{e_H(X_i, X_j)}{(1 - \varepsilon^{1/2}) p_{ij} dn^2} \\
&\le (1 + 2\varepsilon^{1/2}) \sum_{ij \in \binom{[r]}{2}} e^{\alpha}_G(V_i, V_j) \frac{e_H(X_i, X_j)}{dn^2} + \varepsilon n \sum_{ij \in \binom{[r]}{2}} \frac{2 e_H(X_i, X_j)}{p_{ij} dn^2} \\
&\overset{(iii),(3.4.4)}{\le} (1 + 2\varepsilon^{1/2})(1 - \gamma) + \varepsilon n \binom{r}{2} \frac{4\Delta n}{dn^2} \le 1 - \gamma + 3\varepsilon^{1/2} \le 1 - \frac{3\gamma}{4}.
\end{aligned} \tag{3.4.5}$$

Note that (3.4.4) and (C1.1) imply that

$$\frac{e_{G_1}(V_i, V_j)}{e_H(X_i, X_j)} = \frac{(d \pm \varepsilon^{1/2})n}{2\Delta}. \tag{3.4.6}$$

Hence, for every colour $\alpha \in C$, we obtain

$$e^{\alpha}(G_1) = \sum_{ij \in \binom{[r]}{2}} e^{\alpha}_{G_1}(V_i, V_j) \overset{(3.4.6)}{\leq} \frac{(d+\varepsilon^{1/2})n}{2\Delta} \sum_{ij \in \binom{[r]}{2}} e^{\alpha}_{G_1}(V_i, V_j) \frac{e_H(X_i, X_j)}{e_{G_1}(V_i, V_j)}$$

$$\overset{(3.4.5)}{\leq} \left(1 - \frac{3\gamma}{4}\right) \frac{(d+\varepsilon^{1/2})n}{2\Delta}. \tag{3.4.7}$$

In the next step we define a random subgraph $G_2 \subseteq G_1$. This will ensure that the final colouring is colour-split. We choose $\tau \colon C \to \binom{[r]}{2}$ where each $\tau(\alpha)$ is chosen independently at random according to some probability distribution $(q^{\alpha}_{ij})_{ij \in \binom{[r]}{2}}$, and for each $ij \in \binom{[r]}{2}$ and each edge e of $G_1[V_i, V_j]$, let Z_e be a Bernoulli random variable with parameter $\gamma^2/q^{c(e)}_{ij}$, all independent and independent of the choice of τ. Define G_2 by keeping each edge $e \in E_{G_1}(V_i, V_j)$ if $\tau(c(e)) = ij$ and $Z_e = 1$. Hence,

$$\text{for all } e \in E(G_1), \text{ we have } \mathbb{P}\left[e \in E(G_2)\right] = \gamma^2. \tag{3.4.8}$$

We define q^{α}_{ij} as follows. For all $\alpha \in C$, let

$$\mathcal{I}^{\alpha} := \left\{ ij \in \binom{[r]}{2} : e^{\alpha}_{G_1}(V_i, V_j) > \frac{\gamma^2 e^{\alpha}(G_1)}{1 - \binom{r}{2}\gamma^2} \right\} \quad \text{and} \quad \overline{\mathcal{I}^{\alpha}} := \binom{[r]}{2} \setminus \mathcal{I}^{\alpha}. \tag{3.4.9}$$

For $ij \in \overline{\mathcal{I}^{\alpha}}$, we set $q^{\alpha}_{ij} := \gamma^2$. For $ij \in \mathcal{I}^{\alpha}$, we set

$$q^{\alpha}_{ij} := \left(1 - \left|\overline{\mathcal{I}^{\alpha}}\right|\gamma^2\right) \frac{e^{\alpha}_{G_1}(V_i, V_j)}{\sum_{i'j' \in \mathcal{I}^{\alpha}} e^{\alpha}_{G_1}(V_{i'}, V_{j'})}. \tag{3.4.10}$$

Note that $\gamma^2 \leq q^{\alpha}_{ij} \leq 1$ for all $ij \in \binom{[r]}{2}$, and $\sum_{ij \in \binom{[r]}{2}} q^{\alpha}_{ij} = 1$.

Claim 2. The following properties hold simultaneously with probability at least $1 - 1/n$ *for every* $ij \in \binom{[r]}{2}$ *and every colour* $\alpha \in C$.

(C2.1) $G_2[V_i, V_j]$ *is* $(\hat{\varepsilon}, \gamma^2 p_{ij} d)$*-super-regular;*

(C2.2) $e^{\alpha}_{G_2}(V_i, V_j) \leq \frac{\gamma^2}{q^{\alpha}_{ij}} e^{\alpha}_{G_1}(V_i, V_j) + \varepsilon n$.

Proof of claim: For every $ij \in \binom{[r]}{2}$, by (3.4.8) and (C1.1), Lemma 3.6 with $Y_\alpha = \mathbb{1}_{\tau(\alpha)=ij}$ and Z_e as defined above implies that (C2.1) holds with probability at least $1 - 1/n^5$.

In order to verify (C2.2), note that for $ij \in \binom{[r]}{2}$ the colour α appears in $G_2[V_i, V_j]$ only if $\tau(\alpha) = ij$. Since we keep each α-coloured edge independently at random with probability γ^2/q^{α}_{ij}, a simple application of Chernoff's inequality yields that (C2.2) holds with probability at least $1 - 1/n^5$. —

Hence, by Claim 2, we may assume that G_2 is a spanning subgraph of G_1 such that properties (C2.1) and (C2.2) hold. By the construction of G_2, the restricted colouring $c|_{E(G_2)}$ is colour-split.

We show that also the required boundedness condition is satisfied, see (3.4.14) below. For $ij \in \binom{[r]}{2}$, we deduce from (3.4.4) and (C2.1) that

$$e_{G_2}(V_i, V_j) = \gamma^2 p_{ij}(d \pm \hat{\varepsilon}^{1/2})n^2 \overset{(3.4.4)}{=} \left(d \pm \hat{\varepsilon}^{1/2}\right) \frac{\gamma^2 n}{2\Delta} e_H(X_i, X_j). \tag{3.4.11}$$

For a colour $\alpha \in C$ and $ij \in \overline{\mathcal{I}^\alpha}$, as $G_2 \subseteq G_1$, we obtain that

$$e^\alpha_{G_2}(V_i, V_j) \le e^\alpha_{G_1}(V_i, V_j) \overset{(3.4.9)}{\le} \frac{\gamma^2}{1 - \binom{r}{2}\gamma^2} e^\alpha(G_1). \tag{3.4.12}$$

For a colour $\alpha \in C$ and $ij \in \mathcal{I}^\alpha$, we obtain with (C2.2) that

$$\begin{aligned} e^\alpha_{G_2}(V_i, V_j) &\overset{(3.4.10)}{\le} \frac{\gamma^2}{1 - |\overline{\mathcal{I}^\alpha}|\gamma^2} \cdot e^\alpha_{G_1}(V_i, V_j) \frac{\sum_{i'j' \in \mathcal{I}^\alpha} e^\alpha_{G_1}(V_{i'}, V_{j'})}{e^\alpha_{G_1}(V_i, V_j)} + \varepsilon n \\ &\le \frac{\gamma^2}{1 - \binom{r}{2}\gamma^2} \sum_{i'j' \in \binom{[r]}{2}} e^\alpha_{G_1}(V_{i'}, V_{j'}) + \varepsilon n = \frac{\gamma^2}{1 - \binom{r}{2}\gamma^2} e^\alpha(G_1) + \varepsilon n. \end{aligned} \tag{3.4.13}$$

Moreover, for every colour $\alpha \in C$ and every $ij \in \binom{[r]}{2}$, we conclude that

$$\begin{aligned} \frac{\gamma^2}{1 - \binom{r}{2}\gamma^2} e^\alpha(G_1) + \varepsilon n &\overset{(3.4.7)}{\le} \frac{1 - 3\gamma/4}{1 - \binom{r}{2}\gamma^2} \cdot \frac{\gamma^2 n}{2\Delta}(d + \varepsilon^{1/2}) + \varepsilon n \\ &\overset{(3.4.11)}{\le} \frac{1 - 3\gamma/4}{1 - \binom{r}{2}\gamma^2} \cdot \frac{e_{G_2}(V_i, V_j)}{e_H(X_i, X_j)} \cdot \frac{d + \varepsilon^{1/2}}{d - \hat{\varepsilon}^{1/2}} + \varepsilon n \quad \le \quad \left(1 - \frac{2\gamma}{3}\right) \frac{e_{G_2}(V_i, V_j)}{e_H(X_i, X_j)}, \end{aligned}$$

which implies together with (3.4.12) and (3.4.13) that for every colour $\alpha \in C$ and every $ij \in \binom{[r]}{2}$,

$$e^\alpha_{G_2}(V_i, V_j) \le \left(1 - \frac{2\gamma}{3}\right) \frac{e_{G_2}(V_i, V_j)}{e_H(X_i, X_j)}. \tag{3.4.14}$$

Let G_3 be a spanning subgraph of G_2 where for each bipartite pair $G_2[V_i, V_j]$ we keep each edge independently at random with probability $d'/(\gamma^2 p_{ij} d)$. As $G_2[V_i, V_j]$ is $(\hat{\varepsilon}, \gamma^2 p_{ij} d)$-super-regular, we may conclude by simple applications of Chernoff's inequality that with probability at least $1 - 1/n$ for all $ij \in \binom{[r]}{2}$, the graph $G_3[V_i, V_j]$ is (ε', d')-super-regular, and for every colour $\alpha \in C$, we have

$$e^\alpha_{G_3}(V_i, V_j) \le \left(1 - \frac{\gamma}{2}\right) \frac{e_{G_3}(V_i, V_j)}{e_H(X_i, X_j)}$$

due to (3.4.14). Clearly, also c restricted to G_3 is colour-split. Hence, we conclude that there is a spanning subgraph G_3 of G_2 satisfying properties (a)–(c), which implies the statement with G_3 playing the role of G'. □

The next lemma states that we can refine the partitions of a blow-up instance $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ where the edge-colouring of G is colour-split such that H only induces matchings between its refined partition classes and the bipartite pairs of G are still super-regular and colour-split. Similar as in the reduction in [112], we first apply the Hajnal–Szemerédi theorem to $H^2[X_i]$ for each cluster X_i to obtain a refined partition of H where every cluster is now 2-independent.[2] Accordingly, we refine the partition of G randomly to preserve the super-regularity. Additionally, we partition the colours into disjoint colour sets such that the colouring between the refined partitions of G is still colour-split.

We first state the classical Hajnal–Szemerédi theorem.

Theorem 3.8 ([59])**.** *Let G be a graph on n vertices with $\Delta(G) < k \le n$. Then $V(G)$ can be partitioned into k independent sets of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$.*

[2]Recall that we say a subset X of vertices in a graph G is 2-*independent* if it is independent in G^2.

Lemma 3.9 ([30]). *Let $1/n \ll \varepsilon \ll \varepsilon' \ll d' \ll \gamma \ll d, 1/\Lambda, 1/r, 1/\Delta$. Suppose $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d)-super-regular blow-up instance. Assume further that*

(i) *$\Delta(H) \le \Delta$ and $e_H(X_i, X_j) \ge \gamma^2 n$ for all $ij \in \binom{[r]}{2}$;*

(ii) *$|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$;*

(iii) *$c\colon E(G) \to C$ is a colour-split edge-colouring such that c is locally Λ-bounded and c restricted to $G[V_i, V_j]$ is $(1-\gamma)e_G(V_i, V_j)/e_H(X_i, X_j)$-bounded for all $ij \in \binom{[r]}{2}$.*

Then there exists an (ε', d')-super-regular blow-up instance $(H', G', (X_{i,j})_{i\in[r],j\in[\Delta^2]}, (V_{i,j})_{i\in[r],j\in[\Delta^2]})$ such that

(a) *$(X_{i,j})_{j\in[\Delta^2]}$ is partition of X_i and $(V_{i,j})_{j\in[\Delta^2]}$ is partition of V_i for every $i \in [r]$, and $|X_{i,j}| = |V_{i,j}| = (1 \pm \varepsilon')n/\Delta^2$ for all $i \in [r], j \in [\Delta^2]$;*

(b) *H' is a supergraph of H on $V(H)$ such that $H'[X_{i_1,j_1}, X_{i_2,j_2}]$ is a matching of size at least $\gamma^4 n/\Delta^2$ for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta^2], (i_1, j_1) \neq (i_2, j_2)$;*

(c) *G' is a graph on $V(G)$ such that $G'[V_{i_1,j_1}, V_{i_2,j_2}] \subseteq G[V_{i_1}, V_{i_2}]$ for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta^2]$;*

(d) *$c'\colon E(G') \to C'$ is an edge-colouring of G' such that $c'|_{E(G)\cap E(G')} = c|_{E(G)\cap E(G')}$, and c' is colour-split with respect to the partition $(V_{i,j})_{i\in[r],j\in[\Delta^2]}$, and c' is locally Λ-bounded, and c' restricted to $G'[V_{i_1,j_1}, V_{i_2,j_2}]$ is*

$$\left(1 - \frac{\gamma}{2}\right) \frac{e_{G'}(V_{i_1,j_1}, V_{i_2,j_2})}{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}\text{-bounded}$$

for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta^2], (i_1, j_1) \neq (i_2, j_2)$.

Proof. Since $c\colon E(G) \to C$ is colour-split, we may assume that c is the union of edge-colourings $c_{i_1i_2}\colon E(G[V_{i_1}, V_{i_2}]) \to C_{i_1i_2}$ for $i_1i_2 \in \binom{[r]}{2}$ where $C_{i_1i_2} \cap C_{i_1'i_2'} = \emptyset$ for distinct $i_1i_2, i_1'i_2' \in \binom{[r]}{2}$.

First, we apply Theorem 3.8 to $H^2[X_i]$ for every $i \in [r]$. Since $\Delta(H^2[X_i]) \le \Delta^2 - 1$, there exists a partition of X_i into 2-independent sets $X_{i,1}, \dots, X_{i,\Delta^2}$ in H each of size $|X_i|/\Delta^2 \pm 1 = (1 \pm 2\varepsilon)n'$, where $n' := n/\Delta^2$. Hence for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta^2]$, the bipartite graph $H[X_{i_1,j_1}, X_{i_2,j_2}]$ is a (possibly empty) matching. Clearly, we can add a minimal number of edges to H to obtain a supergraph H' such that $H'[X_{i_1,j_1}, X_{i_2,j_2}]$ is a matching of size at least $\gamma^4 n'$ for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta^2], (i_1, j_1) \neq (i_2, j_2)$, which yields (b).

In order to obtain (a), we refine the partition of $V(G)$ accordingly. We claim that the following partitions exist. For every $i \in [r]$, let $(V_{i,j})_{j\in[\Delta^2]}$ be a partition of V_i such that $|V_{i,j}| = |X_{i,j}|$ for every $j \in [\Delta^2]$, and such that for all distinct $i_1, i_2 \in [r]$, all $j_1, j_2 \in [\Delta^2]$, and $v \in V_{i_1,j_1} \cup V_{i_2,j_2}$, we have

$$\deg_{G[V_{i_1,j_1}, V_{i_2,j_2}]}(v) = (d \pm 3\varepsilon)n' \tag{3.4.15}$$

and

$$c|_{E(G[V_{i_1,j_1}, V_{i_2,j_2}])} \text{ is } (1 - \gamma + \varepsilon)\frac{(d + 3\varepsilon)n'^2}{e_H(X_{i_1}, X_{i_2})}\text{-bounded.} \tag{3.4.16}$$

That such a partition exists can be seen by a probabilistic argument as follows: For each $i \in [r]$, let $\tau_i\colon V_i \to [\Delta^2]$ where $\tau_i(v)$ is chosen uniformly at random for every $v \in V_i$, all independently, and let $V_{i,j} := \{v \in V_i\colon \tau_i(v) = j\}$ for every $j \in$

$[\Delta^2]$. McDiarmid's inequality together with a union bound implies that (3.4.15) and (3.4.16) hold with probability at least $1 - \mathrm{e}^{-\sqrt{n}}$. Moreover, standard properties of the multinomial distribution yield that $|V_{i,j}| = |X_{i,j}|$ for all $i \in [r], j \in [\Delta^2]$ with probability at least $\Omega(n^{-\Delta^2 r})$.

Thus, for every $i \in [r]$, there exists a partition $(V_{i,j})_{j\in[\Delta^2]}$ of V_i with the required properties.

Since $G[V_{i_1}, V_{i_2}]$ is (ε, d)-super-regular and due to (3.4.15), it follows that for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta^2]$, the graph $G[V_{i_1,j_1}, V_{i_2,j_2}]$ is $(2\Delta^2\varepsilon, d)$-super-regular. By the construction of the supergraph H', we have added at most $\gamma^4\Delta^2 n$ edges to each pair (X_{i_1}, X_{i_2}) in H. Hence for all distinct $i_1, i_2 \in [r]$,

$$e_H(X_{i_1}, X_{i_2}) \geq e_{H'}(X_{i_1}, X_{i_2}) - \gamma^4\Delta^2 n \geq e_{H'}(X_{i_1}, X_{i_2})(1 - \gamma^2\Delta^2), \tag{3.4.17}$$

where the last inequality holds since $e_{H'}(X_{i_1}, X_{i_2}) \geq e_H(X_{i_1}, X_{i_2}) \geq \gamma^2 n$. Now (3.4.16) and (3.4.17) imply that for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta^2]$, the colouring

$$c|_{E(G[V_{i_1,j_1}, V_{i_2,j_2}])} \text{ is } \left(1 - \frac{3\gamma}{4}\right)\frac{(d+3\varepsilon)n'^2}{e_{H'}(X_{i_1}, X_{i_2})}\text{-bounded.} \tag{3.4.18}$$

Next, we iteratively define spanning subgraphs $G_2 \subseteq G_1 \subseteq G$ and a supergraph $G' \supseteq G_2$ that satisfies the required properties in the statement.

First, we claim that there exists a spanning subgraph $G_1 \subseteq G$ that is colour-split with respect to the partition $(V_{i,j})_{i\in[r], j\in[\Delta^2]}$ and still super-regular. In order to see that such a subgraph exists, we use a probabilistic argument. For all distinct $i_1, i_2 \in [r]$, let $\tau_{i_1 i_2} \colon C_{i_1 i_2} \to [\Delta^2] \times [\Delta^2]$ where each $\tau_{i_1 i_2}(\alpha)$ is chosen independently at random according to the probability distribution $(p_{(i_1,j_1),(i_2,j_2)})_{j_1,j_2\in[\Delta^2]}$ with

$$p_{(i_1,j_1),(i_2,j_2)} := \frac{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}{e_{H'}(X_{i_1}, X_{i_2})} \geq \frac{\gamma^4 n'}{2\Delta^4 n'} \geq \gamma^5. \tag{3.4.19}$$

Define G_1 by keeping each edge $e \in E(G[V_{i_1,j_1}, V_{i_2,j_2}])$ if $\tau_{i_1 i_2}(c(e)) = (j_1, j_2)$. By Lemma 3.6 and since $G[V_{i_1,j_1}, V_{i_2,j_2}]$ is $(2\Delta^2\varepsilon, d)$-super-regular, there exists $G_1 \subseteq G$ such that the colouring of G_1 is colour-split and $G_1[V_{i_1,j_1}, V_{i_2,j_2}]$ is $(\varepsilon'/2, p_{(i_1,j_1),(i_2,j_2)}d)$-super-regular for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta^2]$.

For all distinct $i_1, i_2 \in [r]$, all $j_1, j_2 \in [\Delta^2]$, and every colour $\alpha \in C_{i_1 i_2}$, we obtain

$$\begin{aligned} e^{\alpha}_{G_1}(V_{i_1,j_1}, V_{i_2,j_2}) \leq e^{\alpha}_{G}(V_{i_1,j_1}, V_{i_2,j_2}) &\overset{(3.4.18)}{\leq} \left(1 - \frac{3\gamma}{4}\right)\frac{(d+3\varepsilon)n'^2}{e_{H'}(X_{i_1}, X_{i_2})} \\ &= \left(1 - \frac{3\gamma}{4}\right)\frac{p_{(i_1,j_1),(i_2,j_2)}(d+3\varepsilon)n'^2}{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}, \end{aligned}$$

and thus, since $G_1[V_{i_1,j_1}, V_{i_2,j_2}]$ is $(\varepsilon'/2, p_{(i_1,j_1),(i_2,j_2)}d)$-super-regular, we conclude that

$$e^{\alpha}_{G_1}(V_{i_1,j_1}, V_{i_2,j_2}) \leq \left(1 - \frac{2\gamma}{3}\right)\frac{e_{G_1}(V_{i_1,j_1}, V_{i_2,j_2})}{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}. \tag{3.4.20}$$

Let G_2 be the spanning subgraph of G_1 where for each bipartite pair $G_1[V_{i_1,j_1}, V_{i_2,j_2}]$, we keep each edge independently at random with probability $d'/(p_{(i_1,j_1),(i_2,j_2)}d)$. As $G_1[V_{i_1,j_1}, V_{i_2,j_2}]$ is $(\varepsilon'/2, p_{(i_1,j_1),(i_2,j_2)}d)$-super-regular, we may conclude by simple applications of Chernoff's inequality that with probability at least $1 - 1/n$ for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta^2]$, the graph $G_2[V_{i_1,j_1}, V_{i_2,j_2}]$ is (ε', d')-super-regular, and by (3.4.20) for every colour $\alpha \in C$, we have

$$e^{\alpha}_{G_2}(V_{i_1,j_1}, V_{i_2,j_2}) \leq \left(1 - \frac{\gamma}{2}\right)\frac{e_{G_2}(V_{i_1,j_1}, V_{i_2,j_2})}{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}. \tag{3.4.21}$$

Finally, we may add edges in the empty bipartite graphs $G_2[V_{i,j}, V_{i,j'}]$ for all $i \in [r]$ and all distinct $j, j' \in [\Delta^2]$ in such a way that we obtain a supergraph $G' \supseteq G_2$ where $G'[V_{i_1,j_1}, V_{i_2,j_2}]$ is (ε', d')-super-regular for all $i_1, i_2 \in [r]$ and $j_1, j_2 \in [\Delta^2]$, $(i_1, j_1) \neq (i_2, j_2)$. Hence, we conclude that $(H', G', (X_{i,j})_{i\in[r],j\in[\Delta^2]}, (V_{i,j})_{i\in[r],j\in[\Delta^2]})$ is an (ε', d')-super-regular blow-up instance that satisfies (c).

Let $c^{art} \colon \binom{V(G)}{2} \to C^{art}$ be a rainbow edge-colouring of all possible edges $\binom{V(G)}{2}$ such that $C^{art} \cap C = \emptyset$. By colouring the edges $E(G') \setminus E(G_2)$ using c^{art}, we may obtain an edge-colouring $c' \colon E(G') \to C \cup C^{art}$ which extends c and is clearly Λ-bounded. By the construction of G_2, the colouring c' is colour-split, and

$$\left(1 - \frac{\gamma}{2}\right) \frac{e_{G'}(V_{i_1,j_1}, V_{i_2,j_2})}{e_{H'}(X_{i_1,j_1}, X_{i_2,j_2})}\text{-bounded}$$

for each bipartite subgraph $G'[V_{i_1,j_1}, V_{i_2,j_2}]$ with $i_1, i_2 \in [r], j_1, j_2 \in [\Delta^2], (i_1, j_1) \neq (i_2, j_2)$ due to (3.4.21). This yields (d) and completes the proof. □

3.5 Approximate Embedding Lemma

In this section, we prove the 'Approximate Embedding Lemma' (Lemma 3.12), which allows us to embed a cluster X_i into V_i (here X_0, V_0) almost completely, while maintaining crucial properties of other clusters for future embedding rounds. As outlined in Section 3.2, we track these properties using 'candidacy graphs' A_i, which are auxiliary bipartite graphs between X_i and V_i where $xv \in E(A_i)$ only if v is still a suitable image for x given previous embedding rounds.

We say that $(H, G, (A_i)_{i\in[r]_0}, c)$ is an *embedding-instance* if

- H, G are graphs and A_i is a bipartite graph with vertex partition (X_i, V_i) for every $i \in [r]_0$ such that $(X_i)_{i\in[r]_0}$ is a partition of $V(H)$ into independent sets, $(V_i)_{i\in[r]_0}$ is a partition of $V(G)$, and $|X_i| = |V_i|$ for all $i \in [r]_0$;
- for all $i \in [r]$, the graph $H[X_0, X_i]$ is a matching;
- $c \colon E(G \cup \bigcup_{i\in[r]_0} A_i) \to 2^C$ is an edge set colouring that is colour-split with respect to the partition $(X_0, \ldots, X_r, V_0, \ldots, V_r)$ and satisfies $|c(e)| = 1$ for all $e \in E(G)$.

We say that $(H, G, (A_i)_{i\in[r]_0}, c)$ is an $(\varepsilon, (d_i^G)_{i\in[r]}, (d_i)_{i\in[r]_0}, t, \Lambda)$-*embedding-instance* if in addition, we have that

- $G[V_0, V_i]$ is (ε, d_i^G)-super-regular and c restricted to $G[V_0, V_i]$ is $(1 + \varepsilon)e_G(V_0, V_i)/e_H(X_0, X_i)$-bounded for all $i \in [r]$;
- A_i is (ε, d_i)-super-regular and c restricted to A_i is $(1 + \varepsilon)d_i|X_i|$-bounded for all $i \in [r]_0$;
- c is locally Λ-bounded and $|c(e)| \leq t$ for all $e \in \bigcup_{i\in[r]_0} E(A_i)$.

Here, X_0 is the cluster we want to embed into V_0 by finding an almost perfect rainbow matching σ in A_0, and t can be thought of as the number of clusters we have previously embedded. For a matching σ, we denote by $V(\sigma)$ the vertices contained in σ, and for convenience, we identify matchings σ between X_0 and V_0 with functions $\sigma \colon X_0^\sigma \to V_0^\sigma$, where $X_0^\sigma = V(\sigma) \cap X_0$ and $V_0^\sigma = V(\sigma) \cap V_0$. Whenever we write $xv \in E(A_i)$, we tacitly assume that $x \in X_i$ and $v \in V_i$.

The following two definitions encapsulate how the choice of σ affects the candidacy graphs $(A_i)_{i\in[r]}$ and their colouring for the next step (see Figure 3.2). Let $(H, G, (A_i)_{i\in[r]_0}, c)$ be an embedding-instance.

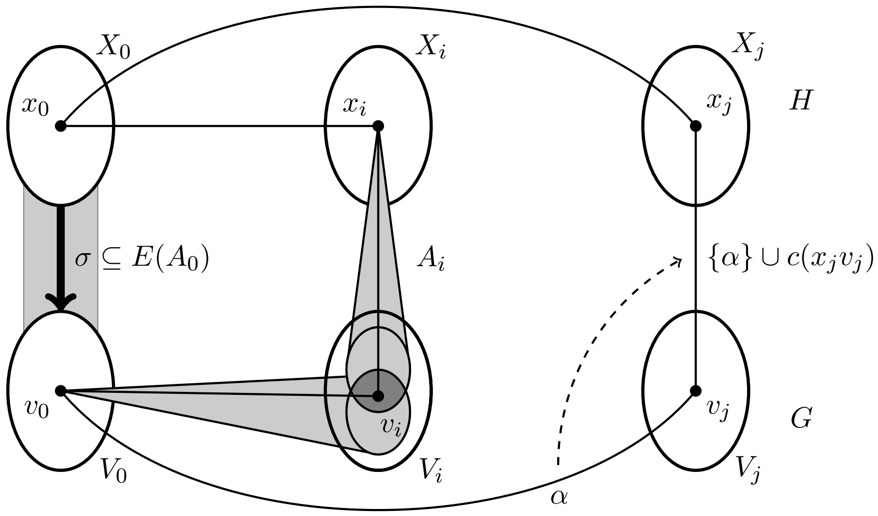


Figure 3.2: If x_0 is mapped to v_0 by σ, then only those candidates of x_i remain that are neighbours of v_0. Moreover, colour α of the edge $v_0 v_j$ is added to the the candidate edge $x_j v_j$, which captures the information that if x_j is later embedded at v_j, then this embedding uses α.

Definition 3.10 (Updated candidacy graphs)**.** *For a matching $\sigma\colon X_0^\sigma \to V_0^\sigma$ in A_0, we define $(A_i^\sigma)_{i\in[r]}$ as the* updated candidacy graphs (with respect to σ) *as follows: for every $i \in [r]$, let A_i^σ be the spanning subgraph of A_i containing precisely those edges $xv \in E(A_i)$ for which the following holds: if x has an H-neighbour $x_0 \in X_0^\sigma$ (which would be unique), then $\sigma(x_0)v \in E(G[V_0, V_i])$.*

This definition ensures that when we embed x in a future round, we are guaranteed that the H-edge x_0x is mapped to a G-edge. Note that this definition does not depend at all on the colouring c. Moreover, we also define updated colourings for the updated candidacy graphs, where we add up to one additional colour to the edges in the new candidacy graphs according to σ.

Definition 3.11 (Updated colouring)**.** *For a matching $\sigma\colon X_0^\sigma \to V_0^\sigma$ in A_0, we define the* updated edge set colouring c^σ *of the updated candidacy graphs as follows: for each $i \in [r]$ and $xv \in E(A_i^\sigma)$, when x has an H-neighbour $x_0 \in X_0^\sigma$, then set $c^\sigma(xv) := c(xv) \cup c(\sigma(x_0)v)$, and otherwise set $c^\sigma(xv) := c(xv)$.*

We now state and prove our Approximate Embedding Lemma.

Lemma 3.12 ([30] – Approximate Embedding Lemma)**.** *Let*

$$1/n \ll \varepsilon \ll \varepsilon' \ll (d_i^G)_{i\in[r]}, (d_i)_{i\in[r]_0}, 1/\Lambda, 1/r, 1/(t+1).$$

Suppose $(H, G, (A_i)_{i\in[r]_0}, c)$ is an $(\varepsilon, (d_i^G)_{i\in[r]}, (d_i)_{i\in[r]_0}, t, \Lambda)$-embedding-instance with $|V_0| = n$, $|V_i| = (1 \pm \varepsilon)n$, and $e_H(X_0, X_i) \geq \varepsilon' n$ for all $i \in [r]$. Suppose the codegree of c is $K \leq \sqrt{n}$.

Then there is a rainbow matching $\sigma\colon X_0^\sigma \to V_0^\sigma$ in A_0 of size at least $(1-\varepsilon')n$ such that for all $i \in [r]$, there exists a spanning subgraph A_i^{new} of the updated candidacy graph A_i^σ and

(I)$_{\text{L3.12}}$ *A_i^{new} is $(\varepsilon', d_i^G d_i)$-super-regular;*

(II)$_{\text{L3.12}}$ *the updated colouring* c^σ *restricted to* A_i^{new} *is* $(1+\varepsilon')d_i^G d_i|X_i|$*-bounded;*

(III)$_{\text{L3.12}}$ c^σ *restricted to* A_i^{new} *has codegree at most* $\max\{K, n^\varepsilon\}$.

We split the proof into three steps. In Step 1 we remove non-typical vertices and edges in order to guarantee that certain neighbourhoods intersect appropriately. In Step 2 we use a suitable hypergraph construction together with Theorem 2.2 to obtain the required rainbow matching σ. By defining certain weight functions in Step 3, we utilise the conclusions of Theorem 2.2 to show that σ can be chosen such that (I)$_{\text{L3.12}}$–(III)$_{\text{L3.12}}$ hold.

Proof. Without loss of generality we may assume that $|c(e)| = t$ for all $e \in E(A_0)$. (Otherwise, we may simply add new 'dummy' colours in such a way that the obtained colouring still satisfies the conditions of the lemma, and these colours can simply be deleted afterwards.)

We also choose a new constant $\hat{\varepsilon}$ such that $\varepsilon \ll \hat{\varepsilon} \ll \varepsilon'$.

Step 1. Removing non-typical vertices and edges

In this step we define subgraphs of G and $(A_i)_{i\in[r]_0}$ to achieve that certain neighbourhoods intersect appropriately (see properties (3.5.4)–(3.5.6)). Let H^+ be an auxiliary supergraph of H that is obtained by adding a maximal number of edges between X_0 and X_i for every $i \in [r]$ subject to $H^+[X_0, X_i]$ being a matching (note that $e_{H^+}(X_0, X_i) \geq (1-\varepsilon)n$).

Let A_0^{bad} be the spanning subgraph of A_0 such that an edge $x_0v_0 \in E(A_0)$ belongs to A_0^{bad} if there is some $i \in [r]$ with $\{x_i\} = N_{H^+}(x_0) \cap X_i$ and

$$|N_{A_i}(x_i) \cap N_G(v_0)| \neq (d_i^G d_i \pm 3\varepsilon)|V_i|. \tag{3.5.1}$$

For $i \in [r]$, let A_i^{bad} be the spanning subgraph of A_i such that an edge $x_iv_i \in E(A_i)$ belongs to A_i^{bad} if $\{x_0\} = N_{H^+}(x_i) \cap X_0$ and

$$|N_{A_0}(x_0) \cap N_G(v_i)| \neq (d_i^G d_0 \pm 3\varepsilon)|V_0|. \tag{3.5.2}$$

Let G^{bad} be the spanning subgraph of G such that an edge $v_0v_i \in E(G[V_0, V_i])$ belongs to $G^{bad}[V_0, V_i]$ for $i \in [r]$ whenever

$$e_H(N_{A_0}(v_0), N_{A_i}(v_i)) \neq (d_0d_i \pm 3\varepsilon)e_H(X_0, X_i). \tag{3.5.3}$$

Using Fact 1.11, it is easy to see that $\Delta(A_0^{bad}) \leq 3r\varepsilon n$ and $\Delta(A_i^{bad}) \leq 3\varepsilon|V_i|$ for each $i \in [r]$. We also claim that for each $i \in [r]_0$, there exists $V_i^{bad} \subseteq V_i$ with $|V_i^{bad}| \leq 3r\varepsilon n$, such that all vertices not in $V_0^{bad} \cup \dots \cup V_r^{bad}$ have degree at most $3r\varepsilon n$ in G^{bad}. Indeed, fix $i \in [r]$ and let $\widetilde{X}_0 := N_H(X_i)$ and $\widetilde{X}_i := N_H(X_0)$. Recall that $|\widetilde{X}_0| = |\widetilde{X}_i| = e_H(X_0, X_i) \geq \varepsilon' n$. Using Fact 1.11, there exists $V_i^{bad} \subseteq V_i$ with $|V_i^{bad}| \leq 3\varepsilon|V_i|$ such that all $v_i \in V_i \setminus V_i^{bad}$ satisfy $|N_{A_i}(v_i) \cap \widetilde{X}_i| = (d_i \pm \varepsilon)|\widetilde{X}_i|$. Now, fix such a vertex v_i. Let $U := N_H(N_{A_i}(v_i))$. Using Fact 1.11 again, we can see that all but at most $3\varepsilon n$ vertices $v_0 \in V_0$ satisfy $|N_{A_0}(v_0) \cap U| = (d_0 \pm \varepsilon)|U| = (d_0 \pm \varepsilon)(d_i \pm \varepsilon)e_H(X_0, X_i)$. Hence, $\deg_{G^{bad}}(v_i) \leq 3\varepsilon n$. Similarly, one can see that there exists $V_{0,i}^{bad} \subseteq V_0$ with $|V_{0,i}^{bad}| \leq 3\varepsilon n$ such that all $v_0 \in V_0 \setminus V_{0,i}^{bad}$ satisfy $|N_{G^{bad}}(v_0) \cap V_i| \leq 3\varepsilon n$. Let $V_0^{bad} := \bigcup_{i=1}^r V_{0,i}^{bad}$. Then $V_0^{bad}, \dots, V_r^{bad}$ are as desired.

Now, let

$$\begin{aligned} A_0' &:= A_0[X_0, V_0 \setminus V_0^{bad}] - E(A_0^{bad}), \quad A_i' := A_i - E(A_i^{bad}), \\ G_{0i}' &:= G[V_0 \setminus V_0^{bad}, V_i] - E(G^{bad}[V_0, V_i \setminus V_i^{bad}]), \quad \text{for all } i \in [r]. \end{aligned}$$

Since we only seek an almost perfect rainbow matching σ in A_0, we can remove the vertices V_0^{bad} from A_0 and find σ in A_0'. By keeping the vertices V_i^{bad} for $i \in [r]$ and the corresponding edges $E(G[V_0, V_i^{bad}])$ in G_{0i}', we can guarantee that the candidacy graphs A_i' are still spanning subgraphs of A_i.

By Fact 1.12, we have that G_{0i}' is $(\hat{\varepsilon}, d_i^G)$-super-regular, that A_0' is $(\hat{\varepsilon}, d_0)$-super-regular and that A_i' is $(\hat{\varepsilon}, d_i)$-super-regular. Crucially, we now have the following properties.

$$\begin{aligned}&|N_{A_i'}(x_i) \cap N_{G_{0i}'}(v_0)| = (d_i^G d_i \pm \hat{\varepsilon})|V_i|,\\ &\text{for all } x_0v_0 \in E(A_0') \text{ whenever } \{x_i\} = N_{H^+}(x_0) \cap X_i,\ i \in [r];\end{aligned} \tag{3.5.4}$$

$$\begin{aligned}&|N_{A_0'}(x_0) \cap N_{G_{0i}'}(v_i)| = (d_i^G d_0 \pm \hat{\varepsilon})|V_0|,\\ &\text{for all } x_iv_i \in E(A_i'),\ i \in [r], \text{ whenever } \{x_0\} = N_{H^+}(x_i) \cap X_0;\end{aligned} \tag{3.5.5}$$

$$\begin{aligned}&e_H(N_{A_0'}(v_0), N_{A_i'}(v_i)) = (d_0 d_i \pm \hat{\varepsilon})e_H(X_0, X_i),\\ &\text{for all } v_0v_i \in E(G_{0i}' - V_i^{bad}) \text{ and } i \in [r].\end{aligned} \tag{3.5.6}$$

Indeed, consider $x_0v_0 \in E(A_0')$ with $\{x_i\} = N_{H^+}(x_0) \cap X_i$. By (3.5.1), we have $|N_{A_i}(x_i) \cap N_G(v_0)| = (d_i^G d_i \pm 3\varepsilon)|V_i|$. Moreover, $v_0 \notin V_0^{bad}$. Hence, $\deg_{A_i^{bad}}(x_i)$, $\deg_{G^{bad}}(v_0) \leq 3r\varepsilon n$, which implies (3.5.4). Similar arguments hold for (3.5.5) and (3.5.6).

<u>*Step 2.*</u> *Constructing an auxiliary hypergraph*

We aim to apply Theorem 2.2 to find the required rainbow matching σ. To this end, let $f_e := e \cup c(e)$ for $e \in E(A_0')$ and let $\mathcal{H}$ be the $(t+2)$-uniform hypergraph $\mathcal{H}$ with vertex set $X_0 \cup V_0 \cup C$ and edge set $\{f_e \colon e \in E(A_0')\}$. A key property of the construction of $\mathcal{H}$ is a bijection between rainbow matchings M in A_0' and matchings $\mathcal{M}$ in $\mathcal{H}$ by assigning M to $\mathcal{M} = \{f_e \colon e \in M\}$.

In order to apply Theorem 2.2, we first establish upper bounds on $\Delta(\mathcal{H})$ and $\Delta^c(\mathcal{H})$. Since A_0' is $(\hat{\varepsilon}, d_0)$-super-regular, $|X_0| = n$, and c restricted to A_0' is $(1+\varepsilon)d_0 n$-bounded, we conclude that

$$\Delta(\mathcal{H}) \leq (d_0 + \hat{\varepsilon})n. \tag{3.5.7}$$

Let $\Delta := (d_0 + \hat{\varepsilon})n$. Since c is locally Λ-bounded, the codegree in $\mathcal{H}$ of a vertex in $X_0 \cup V_0$ and a colour in C is at most Λ. By assumption, the codegree in $\mathcal{H}$ of two colours in C is at most K. For two vertices in $X_0 \cup V_0$, the codegree in $\mathcal{H}$ is at most 1. Altogether, this implies that

$$\Delta^c(\mathcal{H}) \leq \sqrt{n} \leq \Delta^{1-\varepsilon^2}. \tag{3.5.8}$$

Suppose $\mathcal{W}$ is a set of given weight functions $\omega \colon E(A_0') \to [\Lambda]_0$ with $|\mathcal{W}| \leq n^5$ (which we will explicitly specify in Step 3 to establish $(\mathrm{I})_{\mathrm{L3.12}}$–$(\mathrm{III})_{\mathrm{L3.12}}$.) Note that every weight function $\omega \colon E(A_0') \to [\Lambda]_0$ naturally corresponds to a weight function $\omega_{\mathcal{H}} \colon E(\mathcal{H}) \to [\Lambda]_0$ by defining $\omega_{\mathcal{H}}(f_e) := \omega(e)$. If $\omega(E(A_0')) \geq n^{1+\varepsilon/2}$, define $\tilde{\omega} := \omega$. Otherwise, arbitrarily choose $\tilde{\omega} \colon E(A_0') \to [\Lambda]_0$ such that $\omega \leq \tilde{\omega}$ and $\tilde{\omega}(E(A_0')) = n^{1+\varepsilon/2}$. By (3.5.7) and (3.5.8), we can apply Theorem 2.2 (with $(d_0+\hat{\varepsilon})n, \varepsilon^2, t+2, \{\tilde{\omega}_{\mathcal{H}} : \omega \in \mathcal{W}\}$ playing the roles of $\Delta, \delta, r, \mathcal{W}$) to obtain a matching $\mathcal{M}$ in $\mathcal{H}$ that corresponds to a rainbow matching M in A_0' that satisfies the following property by the conclusion of Theorem 2.2:

$$\omega(M) = (1 \pm \hat{\varepsilon}^{1/2})\frac{\omega(E(A_0'))}{d_0 n}, \text{ for all } \omega \in \mathcal{W} \text{ with } \omega(E(A_0')) \geq n^{1+\varepsilon/2}; \tag{3.5.9}$$

$$\omega(M) \leq \max\{(1 + \hat{\varepsilon}^{1/2})\frac{\omega(E(A_0'))}{d_0 n}, n^{\varepsilon}\} \text{ for all } \omega \in \mathcal{W}. \tag{3.5.10}$$

Let $\sigma\colon X_0^\sigma \to V_0^\sigma$ be the function given by the matching M, where $X_0^\sigma = X_0 \cap V(M)$ and $V_0^\sigma = V_0 \cap V(M)$.

One way to exploit (3.5.9) is to control the number of edges in M between sufficiently large sets of vertices. To this end, for subsets $S \subseteq X_0$ and $T \subseteq V_0$ such that $|S|, |T| \geq 2\hat{\varepsilon}n$, we define a weight function $\omega_{S,T}\colon E(A_0') \to [\Lambda]_0$ with

$$\omega_{S,T}(e) := \begin{cases} 1 & \text{if } e \in E(A_0'[S, T \setminus V_0^{bad}]), \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.11}$$

That is, $\omega_{S,T}(M)$ counts the number of edges between S and T that lie in M. Since A_0' is $(\hat{\varepsilon}, d_0)$-super-regular, (3.5.9) implies (whenever $\omega_{S,T} \in \mathcal{W}$) that

(3.5.12)
$$|\sigma\big(S \cap X_0^\sigma\big) \cap T| = \omega_{S,T}(M) \overset{(3.5.9)}{=} (1 \pm \hat{\varepsilon}^{1/2}) \frac{e(A_0'[S, T \setminus V_0^{bad}])}{d_0 n} = (1 \pm 2\hat{\varepsilon}^{1/2}) \frac{|S||T|}{n}.$$

Step 3. Employing weight functions to conclude (I)$_{\text{L3.12}}$–(III)$_{\text{L3.12}}$

By Step 2, we may assume that (3.5.9) holds for a set of weight functions $\mathcal{W}$ that we will define during this step. We will show that for this choice of $\mathcal{W}$ the matching $\sigma\colon X_0^\sigma \to V_0^\sigma$ as obtained in Step 1 satisfies (I)$_{\text{L3.12}}$–(III)$_{\text{L3.12}}$. Similar as in Definition 3.10 (here with H replaced by H^+), we define subgraphs $(A_i^*)_{i\in[r]}$ of $(A_i')_{i\in[r]}$ as follows. For every $i \in [r]$, let A_i^* be the spanning subgraph of A_i' containing precisely those edges $xv \in E(A_i')$ for which the following holds: if $\{x_0\} = N_{H^+}(x) \cap X_0^\sigma$, then $\sigma(x_0)v \in E(G_{0i}')$. Since $A_i' \subseteq A_i$ and due to the construction of A_i^*, we conclude that A_i^* is a spanning subgraph of the updated candidacy graph A_i^σ (with respect to σ) for every $i \in [r]$ (see Definition 3.10). By taking a suitable subgraph of A_i^* we will later obtain the required candidacy graph A_i^{new}.

First, we show that the matching M has size at least $(1 - 2\hat{\varepsilon}^{1/2})n$. Adding ω_{X_0,V_0} as defined in (3.5.11) to $\mathcal{W}$ and using (3.5.12) yields

$$|M| \geq (1 - 2\hat{\varepsilon}^{1/2})n. \tag{3.5.13}$$

For every $i \in [r]$, define $X_i^H := N_{H^+}(X_0^\sigma) \cap X_i$. Note that $|X_i^H| = (1 \pm 3\hat{\varepsilon}^{1/2})|X_i| = (1 \pm 4\hat{\varepsilon}^{1/2})n$.

Step 3.1. Checking (I)$_{\text{L3.12}}$

In order to prove (I)$_{\text{L3.12}}$, we first show that $A_i^*[X_i^H, V_i]$ is super-regular for every $i \in [r]$. We will show that every vertex in $X_i^H \cup V_i$ has the appropriate degree, and that the common neighbourhood of most pairs of vertices in V_i has the correct size, such that we can employ Theorem 1.13 to guarantee the super-regularity of $A_i^*[X_i^H, V_i]$.

For all $i \in [r]$ and for every vertex $x \in X_i^H$ with $\{x_0\} = N_{H^+}(x) \cap X_0^\sigma$, we have $\deg_{A_i^*[X_i^H,V_i]}(x) = |N_{A_i'}(x) \cap N_{G_{0i}'}(\sigma(x_0))|$. Hence, (3.5.4) implies that

$$\deg_{A_i^*[X_i^H,V_i]}(x) = (d_i^G d_i \pm \hat{\varepsilon})|V_i|.$$

For $v \in V_i$, let $U_v := N_{A_i'}(v) \subseteq X_i$. Observe that

$$\deg_{A_i^*[X_i^H,V_i]}(v) = |\sigma(N_{H^+}(U_v) \cap X_0^\sigma) \cap N_{G_{0i}'}(v)|, \tag{3.5.14}$$

and $|N_{H^+}(U_v) \cap X_0| = |N_{A_i'}(v)| \pm \varepsilon n = (d_i \pm 2\hat{\varepsilon})n$, and $|N_{G_{0i}'}(v)| = (d_i^G \pm 2\hat{\varepsilon})n$. Adding for every $i \in [r]$ and every vertex $v \in V_i$, the weight function $\omega_{S,T}$ as defined in (3.5.11) for $S := N_{H^+}(U_v) \cap X_0$ and $T := N_{G_{0i}'}(v)$ to $\mathcal{W}$, we obtain that

(3.5.15)
$$\deg_{A_i^*[X_i^H,V_i]}(v) \overset{(3.5.12),(3.5.14)}{=} (1 \pm 2\hat{\varepsilon}^{1/2})|N_{H^+}(U_v) \cap X_0||N_{G_{0i}'}(v)|n^{-1} = (d_i^G d_i \pm \hat{\varepsilon}^{1/3})|X_i^H|.$$

Note that these are at most $2rn$ weight functions $\omega_{S,T}$ that we added to $\mathcal{W}$.

We will use Theorem 1.13 to show that $A_i^*[X_i^H, V_i]$ is super-regular. We call a pair of vertices $u, v \in V_i$ *good* if $|N_{A'_i}(u,v)| = (d_i \pm \hat{\varepsilon})^2|X_i|$, and $|N_{G'_{0i}}(u,v)| = (d_i^G \pm \hat{\varepsilon})^2 n$. By the $\hat{\varepsilon}$-regularity of A'_i and G'_{0i}, using Fact 1.11, there are at most $2\hat{\varepsilon}|V_i|^2$ pairs $u, v \in V_i$ which are not good. For every $i \in [r]$ and all good pairs $u, v \in V_i$, let $S_{u,v} := N_{H^+}(N_{A'_i}(u,v)) \cap X_0$ and $T_{u,v} := N_{G'_{0i}}(u,v)$. We add the weight function $\omega_{S_{u,v},T_{u,v}}$ as defined in (3.5.11) to $\mathcal{W}$. Observe that $|S_{u,v}| = |N_{A'_i}(u,v)| \pm \varepsilon n = (d_i \pm 2\hat{\varepsilon})^2|X_i|$ and $|T_{u,v}| = (d_i^G \pm \hat{\varepsilon})^2 n$. Note that these are at most rn^2 functions $\omega_{S_{u,v},T_{u,v}}$ that we add to $\mathcal{W}$ in this way. By (3.5.12), we obtain for all good pairs $u, v \in V_i$ that

$$\begin{aligned} |N_{A_i^*[X_i^H,V_i]}(u,v)| &= |\sigma(S_{u,v} \cap X_0^\sigma) \cap T_{u,v}| = (1 \pm 2\hat{\varepsilon}^{1/2})|S_{u,v}||T_{u,v}|n^{-1} \\ &\leq (d_i^G d_i + \hat{\varepsilon}^{1/3})^2 |X_i^H|. \end{aligned}$$

Together with (3.5.15), we can apply Theorem 1.13 and obtain that

$$A_i^*[X_i^H, V_i] \text{ is } \big(\hat{\varepsilon}^{1/18}, d_i^G d_i\big)\text{-super-regular for every } i \in [r]. \tag{3.5.16}$$

In order to complete the proof of $(\mathrm{I})_{\mathrm{L3.12}}$, for every $i \in [r]$, since $|X_i \setminus X_i^H| \leq 3\hat{\varepsilon}^{1/2}|X_i|$, we can easily find a spanning subgraph A_i^{new} of A_i^* that is $(\varepsilon', d_i^G d_i)$-super-regular by deleting from every vertex $x \in X_i \setminus X_i^H$ a suitable number of edges. This establishes $(\mathrm{I})_{\mathrm{L3.12}}$.

Step 3.2. *Checking* $(\mathrm{II})_{\mathrm{L3.12}}$

Next, we show that for every $i \in [r]$, the edge set colouring c^σ restricted to A_i^* is $(1+\varepsilon')d_i^G d_i|X_i|$-bounded, which implies $(\mathrm{II})_{\mathrm{L3.12}}$ because $A_i^{new} \subseteq A_i^*$. Recall that we defined c^σ (in Definition 3.11) such that for $xv \in E(A_i^*)$, we have $c^\sigma(xv) = c(xv) \cup c(\sigma(x_0)v)$ if x has an H-neighbour $x_0 \in X_0^\sigma$, and otherwise $c^\sigma(xv) = c(xv)$. Since c is colour-split, we may assume that $c_{A'_i}\colon E(A'_i) \to 2^{C_{A'_i}}$ is the edge set colouring c restricted to A'_i and $c_{G'_{0i}}\colon E(G'_{0i}) \to C_{G'_{0i}}$ is the edge-colouring c restricted to G'_{0i} such that $C_{A'_i} \cap C_{G'_{0i}} = \emptyset$ for all $i \in [r]$. Fix $i \in [r]$. We have to show that for all $\alpha \in C_{A_i} \cup C_{G'_{0i}}$, there are at most $(1+\varepsilon')d_i^G d_i|X_i|$ edges of A_i^* on which α appears.

First, consider $\alpha \in C_{A'_i}$. Let $E_\alpha \subseteq E(A'_i)$ be the edges of A'_i on which α appears. By assumption, $|E_\alpha| \leq (1+\varepsilon)d_i|X_i|$. We need to show that $|E_\alpha \cap E(A_i^*)| \leq (1+\varepsilon')d_i^G d_i|X_i|$. To this end, we define a weight function $\omega_\alpha\colon E(A'_0) \to [\Lambda]_0$ by setting

$$\omega_\alpha(xv) := \left|\{v_i \in N_{G'_{0i}}(v)\colon x_i v_i \in E_\alpha, x x_i \in E(H^+[X_0, X_i])\}\right|$$

for every $xv \in E(A'_0)$, and we add ω_α to $\mathcal{W}$. Note that

$$\begin{aligned} |E_\alpha \cap E(A_i^*)| &\leq \sum_{x_i \in X_i^H} \left|\{v_i \in N_{G'_{0i}}(\sigma(x)) \,:\, x_i v_i \in E_\alpha, x x_i \in E(H^+[X_0, X_i])\}\right| + \Lambda|X_i \setminus X_i^H| \\ &\leq \omega_\alpha(M) + 3\hat{\varepsilon}^{1/2}\Lambda|X_i|. \end{aligned}$$

We now obtain an upper bound for $\omega_\alpha(M)$ using (3.5.10). For every edge $x_i v_i \in E_\alpha$ with $x x_i \in E(H^+[X_0, X_i])$, condition (3.5.5) states that

$$|N_{A'_0}(x) \cap N_{G'_{0i}}(v_i)| = (d_i^G d_0 \pm \hat{\varepsilon})n.$$

Hence, every such edge contributes weight $(d_i^G d_0 \pm \hat{\varepsilon})n$ to $\omega_\alpha(E(A'_0))$. We obtain

$$\omega_\alpha(E(A'_0)) \leq (1+\varepsilon)d_i|X_i| \cdot (d_i^G d_0 + \hat{\varepsilon})n \leq (d_0 d_i d_i^G + 2\hat{\varepsilon})|X_i|n.$$

Now (3.5.10) implies that $\omega_\alpha(M) \leq (1+2\hat{\varepsilon}^{1/2})d_i^G d_i|X_i|$ and hence $|E_\alpha \cap E(A_i^*)| \leq (1+\varepsilon')d_i^G d_i|X_i|$.

Now, consider $\alpha \in C_{G'_{0i}}$. Let $E_\alpha \subseteq E(G'_{0i})$ be the set of edges of G'_{0i} on which α appears. We define a weight function $\omega_\alpha\colon E(A'_0) \to [\Lambda]_0$ by setting

$$\omega_\alpha(xv) := \left|\{v_i \in N_{G'_{0i}}(v)\colon vv_i \in E_\alpha, xx_i \in E(H[X_0, X_i]), x_iv_i \in E(A'_i)\}\right|$$

for every $xv \in E(A'_0)$, and we add ω_α to $\mathcal{W}$. Note that the number of edges of A_i^* on which α appears is at most $\omega_\alpha(M)$.

In order to bound $\omega_\alpha(M)$, we again use (3.5.10) and seek an upper bound for $\omega_\alpha(E(A'_0))$. Since c is $(1+\varepsilon)e_G(V_0,V_i)/e_H(X_0,X_i)$-bounded on $G[V_0,V_i]$ by assumption, we have $|E_\alpha| \leq (1+\varepsilon^{1/2})d_i^G|X_i|n/e_H(X_0,X_i)$.

For every edge $vv_i \in E_\alpha$ with $v_i \in V_i \setminus V_i^{bad}$, condition (3.5.6) implies that

$$e_H(N_{A'_0}(v), N_{A'_i}(v_i)) = (d_0d_i \pm \hat{\varepsilon})e_H(X_0,X_i).$$

Hence, every edge $vv_i \in E_\alpha$ with $v_i \in V_i \setminus V_i^{bad}$ contributes weight $(d_0d_i \pm \hat{\varepsilon})e_H(X_0,X_i)$ to $\omega_\alpha(E(A'_0))$. Since $\Delta(E_\alpha) \leq \Lambda$ and $|V_i^{bad}| \leq 3r\varepsilon n$, there are at most $3r\Lambda\varepsilon n$ edges $vv_i \in E_\alpha$ with $v_i \in V_i^{bad}$, each of which contributes weight at most n. We conclude that

$$\omega_\alpha(E(A'_0)) \leq \frac{(1+\varepsilon^{1/2})d_i^G|X_i|n}{e_H(X_0,X_i)} \cdot (d_0d_i + \hat{\varepsilon})e_H(X_0,X_i) + 3r\Lambda\varepsilon n^2 \leq (d_0d_id_i^G + 2\hat{\varepsilon})|X_i|n.$$

Now (3.5.10) implies that $\omega_\alpha(M) \leq (1+\varepsilon')d_i^G d_i|X_i|$, completing the proof of $\text{(II)}_{\text{L3.12}}$.

<u>*Step 3.3.*</u> *Checking* $\text{(III)}_{\text{L3.12}}$

Finally, we show that for all $i \in [r]$, $\alpha \in C_{G'_{0i}}$ and $\beta \in C_{A'_i}$, the pair $\{\alpha,\beta\}$ appears on at most n^ε edges of A_i^*. This implies $\text{(III)}_{\text{L3.12}}$, as the codegree of a pair in $C_{A'_i}$ is at most K by assumption, and the codegree of a pair in $C_{G'_{0i}}$ is 0. Fix $i \in [r]$, $\alpha \in C_{G'_{0i}}$ and $\beta \in C_{A'_i}$. Let

$$E_{\alpha,\beta} := \{v_0v_ix_i\colon v_0v_i \in E(G'_{0i}), x_iv_i \in E(A'_i), c(v_0v_i) = \{\alpha\}, \beta \in c(x_iv_i)\}$$

and define the weight function $\omega_{\alpha,\beta}\colon E(A'_0) \to [\Lambda]_0$ by setting

$$\omega_{\alpha,\beta}(xv) := \left|\{vv_ix_i \in E_{\alpha,\beta}\colon xx_i \in E(H^+[X_0,X_i])\}\right|.$$

Note that the number of edges of A_i^* on which $\{\alpha,\beta\}$ appears is at most $\omega_{\alpha,\beta}(M)$. In order to bound $\omega_{\alpha,\beta}(M)$, note that every triple $vv_ix_i \in E_{\alpha,\beta}$ contributes weight at most 1 to $\omega_{\alpha,\beta}(E(A'_0))$. By assumption, c is locally Λ-bounded and (globally) $(1+\varepsilon)d_i|X_i|$-bounded on A_i, which implies that $\omega_{\alpha,\beta}(E(A'_0)) \leq |E_{\alpha,\beta}| \leq (1+\varepsilon)d_i\Lambda|X_i| \leq 2\Lambda n$. Now, (3.5.10) implies that $\omega_{\alpha,\beta}(M) \leq n^\varepsilon$. Hence, for all $i \in [r]$, $\alpha \in C_{G'_{0i}}$ and $\beta \in C_{A'_i}$, we add the corresponding weight function $\omega_{\alpha,\beta}$ to $\mathcal{W}$, which implies $\text{(III)}_{\text{L3.12}}$. This completes the proof. $\square$

3.6 Proof of Lemma 3.4

In this section, we prove our rainbow blow-up lemma (Lemma 3.4). First, we will deduce Lemma 3.4 from a similar statement (Lemma 3.13), where we impose stronger conditions on G and H. This reduction utilises the results of Section 3.4. We will conclude with the proof of Lemma 3.13.

Lemma 3.13 ([30])**.** *Let $1/n \ll \varepsilon \ll \gamma, d, 1/r, 1/\Lambda$. Let $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ be an (ε, d)-super-regular blow-up instance. Assume further that*

(i) *$|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$;*

(ii) *for all $ij \in \binom{[r]}{2}$, the graph $H[X_i, X_j]$ is a matching of size at least $\gamma^2 n$;*

(iii) *$c\colon E(G) \to C$ is a colour-split edge-colouring of G such that c is locally Λ-bounded and c restricted to $G[V_i, V_j]$ is $(1-\gamma)e_G(V_i, V_j)/e_H(X_i, X_j)$-bounded for all $ij \in \binom{[r]}{2}$.*

Then there exists a rainbow embedding ϕ of H into G such that $\phi(x) \in V_i$ for all $i \in [r]$ and $x \in X_i$.

Proof of Lemma 3.4. We split the proof into three steps. In Step 1, we apply Lemma 3.7 in order to obtain a spanning subgraph $G_1 \subseteq G$ such that the restricted edge-colouring is colour-split. In Step 2, we apply Lemma 3.9 in order to refine the partitions of G_1 and H in such a way that the vertex classes of H are 2-independent. Then, in Step 3, we can apply Lemma 3.13 to complete the proof.

In view of the statement, we may assume that $1/n \ll \varepsilon \ll \gamma \ll d, 1/r, 1/\Delta, 1/\Lambda$. Choose new constants $\varepsilon_1, \varepsilon_2, \gamma', d_1, d_2$ with $\varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll d_2 \ll \gamma' \ll d_1 \ll \gamma$.

Step 1. Colour-splitting

First, let H_1 be a supergraph of H on $V(H)$ such that $e_{H_1-H}(X_i, X_j) \le \gamma^2 n \le e_{H_1}(X_i, X_j)$ for all $ij \in \binom{[r]}{2}$ and $\Delta(H_1) \le \Delta' := \Delta + r$. We claim that for all $\alpha \in C$, we have

$$\sum_{ij\in\binom{[r]}{2}} e^{\alpha}_G(V_i, V_j) e_{H_1}(X_i, X_j) \le \left(1 - \frac{\gamma}{2}\right) dn^2.$$

Indeed, since c is locally Λ-bounded, we obtain that

$$e^{\alpha}_G(V_i, V_j) e_{H_1-H}(X_i, X_j) \le 2\Lambda n \cdot \gamma^2 n \le r^{-2} \cdot \gamma dn^2/2$$

for each $ij \in \binom{[r]}{2}$. Hence, we can apply Lemma 3.7 to $(H_1, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ (with $\gamma/2, \Delta'$ playing the roles of γ, Δ), and obtain a spanning subgraph G_1 of G such that $(H_1, G_1, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε_1, d_1)-super-regular blow-up instance, and the colouring $c_1 := c|_{E(G_1)}$ is colour-split and

$$\left(1 - \frac{\gamma}{4}\right) \frac{e_{G_1}(V_i, V_j)}{e_{H_1}(X_i, X_j)}\text{-bounded}$$

for each bipartite subgraph $G_1[V_i, V_j]$. Clearly, a rainbow embedding of H_1 into G_1 also yields a rainbow embedding of H into G.

Step 2. Refining the vertex partitions

We can now apply Lemma 3.9 to the (ε_1, d_1)-super-regular blow-up instance $(H_1, G_1, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ with edge-colouring c_1 and γ', Δ' playing the roles of γ, Δ. Hence, we obtain an (ε_2, d_2)-super-regular blow-up instance

$$(H_2, G_2, (X_{i,j})_{i\in[r],j\in[\Delta'^2]}, (V_{i,j})_{i\in[r],j\in[\Delta'^2]})$$

such that for $n' := n/\Delta'^2$ we have that

(a) $(X_{i,j})_{j\in[\Delta'^2]}$ is partition of X_i and $(V_{i,j})_{j\in[\Delta'^2]}$ is partition of V_i for every $i \in [r]$, and $|X_{i,j}| = |V_{i,j}| = (1 \pm \varepsilon_2)n'$ for all $i \in [r], j \in [\Delta'^2]$;

(b) H_2 is a supergraph of H_1 on $V(H)$ such that $H_2[X_{i_1,j_1}, X_{i_2,j_2}]$ is a matching of size at least $\gamma'^4 n'$ for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta'^2], (i_1, j_1) \neq (i_2, j_2)$;

(c) G_2 is a graph on $V(G)$ such that $G_2[V_{i_1,j_1}, V_{i_2,j_2}] \subseteq G_1[V_{i_1}, V_{i_2}]$ for all distinct $i_1, i_2 \in [r]$ and all $j_1, j_2 \in [\Delta'^2]$;

(d) c_2 is an edge-colouring of G_2 such that $c_2|_{E(G_1)\cap E(G_2)} = c_1|_{E(G_1)\cap E(G_2)}$, and c_2 is colour-split with respect to the partition $(V_{i,j})_{i\in[r],j\in[\Delta'^2]}$, and c_2 is locally Λ-bounded, and c_2 restricted to $G_2[V_{i_1,j_1}, V_{i_2,j_2}]$ is

$$\left(1 - \frac{\gamma'}{2}\right) \frac{e_{G_2}(V_{i_1,j_1}, V_{i_2,j_2})}{e_{H_2}(X_{i_1,j_1}, X_{i_2,j_2})}\text{-bounded}$$

for all $i_1, i_2 \in [r], j_1, j_2 \in [\Delta'^2], (i_1, j_1) \neq (i_2, j_2)$.

Again, a c_2-rainbow embedding of H_2 into G_2 also yields a c_1-rainbow embedding of H_1 into G_1.

Step 3. *Applying Lemma 3.13*

We can now complete the proof by applying Lemma 3.13 as follows:

parameter	n'	ε_2	γ'^2	d_2	$r\Delta'^2$	Λ	H_2	G_2	$(X_{i,j})_{i\in[r],j\in[\Delta'^2]}$	$(V_{i,j})_{i\in[r],j\in[\Delta'^2]}$
replaces	n	ε	γ	d	r	Λ	H	G	$(X_i)_{i\in[r]}$	$(V_i)_{i\in[r]}$

This yields a rainbow embedding of H_2 into G_2, and hence of H in G. □

We now deduce Theorem 3.3 from Lemma 3.4 by partitioning H using the Hajnal–Szemerédi theorem (Theorem 3.8) and G randomly.

Proof of Theorem 3.3. Let $r := \Delta + 1$. We may assume that ε is sufficiently small and n is sufficiently large. By applying Theorem 3.8 to H, we obtain a partition $(X_i)_{i\in[r]}$ of $V(H)$ into independent sets with $|X_i| \in \{\lfloor \frac{n}{r} \rfloor, \lceil \frac{n}{r} \rceil\}$. We claim that there exists a partition $(V_i)_{i\in[r]}$ of $V(G)$ such that

(i) $G[V_i, V_j]$ is $(2r\varepsilon, d)$-super-regular for all $ij \in \binom{[r]}{2}$;

(ii) for all $\alpha \in C$ with $e^\alpha(G) \geq n^{3/4}$, we have $e^\alpha_G(V_i, V_j) = (1 \pm \varepsilon)2e^\alpha(G)/r^2$ for all $ij \in \binom{[r]}{2}$;

(iii) $|V_i| = |X_i|$ for all $i \in [r]$.

That such a partition exists can be seen using a probabilistic argument: For each $v \in V(G)$ independently, choose a label $i \in [r]$ uniformly at random and put v into V_i. Using Chernoff's inequality (Lemma 1.7) for (i) and McDiarmid's inequality (Lemma 1.8) for (ii), it is easy to check that (i) and (ii) are satisfied with probability at least $1 - \mathrm{e}^{-n^{1/3}}$. Moreover, (iii) holds with probability $\Omega(n^{-r/2})$. Hence, such a partition exists.

Therefore, we conclude that $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is a $(2r\varepsilon, d)$-super-regular blow-up instance. Consider $\alpha \in C$. If $e^\alpha(G) \leq n^{3/4}$, then condition (iii) in Lemma 3.4 clearly holds. If $e^\alpha(G) \geq n^{3/4}$, we use (ii) to see that

$$\begin{aligned}\sum_{ij\in\binom{[r]}{2}} e^\alpha_G(V_i, V_j)e_H(X_i, X_j) &= (1 \pm \varepsilon)2e^\alpha(G)e(H)/r^2 \\ &\leq (1 + \varepsilon)(1 - \gamma)2e(G)/r^2 \\ &\leq (1 - \gamma/2)d(n/r)^2.\end{aligned}$$

Thus, we can apply Lemma 3.4 and obtain a rainbow copy of H in G. □

It remains to prove Lemma 3.13. The proof splits into four steps as follows. In Step 1, we split G into two spanning subgraphs G_A and G_B with disjoint colour sets. In Step 2, we define the necessary 'candidacy graphs' that we track during the approximate embedding in Step 3. We then iteratively apply Lemma 3.12 in Step 3 to find approximate rainbow embeddings of X_i into V_i using only the edges of G_A. All those steps have to be performed carefully such that we can employ Lemma 3.5 in Step 4 and use the reserved set of colours of G_B to turn the approximate rainbow embedding into a complete one.

Proof of Lemma 3.13. In view of the statement, we may assume that $\gamma \ll d, 1/r, 1/\Lambda$. Choose new constants $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{r+1}, \mu$ with $\varepsilon \ll \varepsilon_0 \ll \varepsilon_1 \ll \cdots \ll \varepsilon_{r+1} \ll \mu \ll \gamma$. For $i \in [r]$, let

$$\mathcal{X}_i := \bigcup_{j\in[i]} X_j, \qquad\qquad \mathcal{V}_i := \bigcup_{j\in[i]} V_j.$$

<u>*Step 1.*</u> *Colour splitting*

In order to reserve an exclusive set of colours for the application of Lemma 3.5, we randomly partition the edges of G into two spanning subgraphs G_A and G_B as follows. For each colour class of G independently, we add its edges to G_A with probability $1-\gamma$ and otherwise to G_B. Let $d_A := (1-\gamma)d$ and $d_B := \gamma d$. By Lemma 3.6, we conclude that with probability at least $1 - 1/n$,

$$G_Z[V_i, V_j] \textit{ is } (\varepsilon_0^2, d_Z)\textit{-super-regular for all } ij \in \binom{[r]}{2},\ Z \in \{A, B\}. \tag{3.6.1}$$

Hence, we may assume that G is partitioned into G_A and G_B such that (3.6.1) holds.

<u>*Step 2.*</u> *Candidacy graphs*

We want to show that there is a partial rainbow embedding of $H[\mathcal{X}_r]$ into $G_A[\mathcal{V}_r]$ that maps almost all vertices of X_i into V_i for every $i \in [r]$. Moreover, we need to ensure certain conditions for the remaining unembedded vertices in order to finally apply Lemma 3.5. We will achieve this by iteratively applying the Approximate Embedding Lemma (Lemma 3.12) in Step 3. In order to formally state the induction hypothesis, we need some preliminary definitions.

For $t \in [r]_0$, we call $\phi_t\colon X_1^{\phi_t} \cup \ldots \cup X_t^{\phi_t} \to V_1^{\phi_t} \cup \ldots \cup V_t^{\phi_t}$ a *t-partial embedding* if $X_i^{\phi_t} \subseteq X_i$, $V_i^{\phi_t} \subseteq V_i$, and $\phi_t(X_i^{\phi_t}) = V_i^{\phi_t}$ for every $i \in [t]$, such that ϕ_t is an embedding of $H[X_1^{\phi_t} \cup \ldots \cup X_t^{\phi_t}]$ into $G_A[V_1^{\phi_t} \cup \ldots \cup V_t^{\phi_t}]$. For brevity, define

$$\mathcal{X}_t^{\phi_t} := \bigcup_{i\in[t]} X_i^{\phi_t}, \qquad\qquad \mathcal{V}_t^{\phi_t} := \bigcup_{i\in[t]} V_i^{\phi_t}.$$

Given a t-partial embedding ϕ_t, we define two kinds of bipartite auxiliary graphs: for each $i \in [r] \setminus [t]$, we define a graph $A_i(\phi_t)$ with bipartition (X_i, V_i) that tracks the still available images of a vertex $x \in X_i$ in G_A, which will be used to extend the t-partial rainbow embedding ϕ_t to a $(t+1)$-partial rainbow embedding ϕ_{t+1} via Lemma 3.12 in Step 3. Moreover, for each $i \in [r]$, we define a bipartite graph $B_i(\phi_t)$ that tracks the potential images of a vertex $x \in X_i$ in G_B, which will be used for the completion via Lemma 3.5 in Step 4. Here, we keep tracking potential images of vertices even if they have been embedded, since in Step 4, we will actually 'unembed' a few vertices.

When extending ϕ_t to ϕ_{t+1}, we intend to update the graphs $A_i(\phi_t)$ and $B_i(\phi_t)$ simultaneously using Lemma 3.12. In order to facilitate this, we define $B_i(\phi_t)$ on a copy (X_i^B, V_i^B) of the bipartition (X_i, V_i). For every $i \in [r]$, let X_i^B and V_i^B be disjoint copies of X_i and V_i, respectively. Let π be the bijection that maps a vertex in $\bigcup_{i\in[r]}(X_i \cup V_i)$ to its copy in $\bigcup_{i\in[r]}(X_i^B \cup V_i^B)$. Let G^+ and H^+ be supergraphs of G_A

and H with vertex partitions $(V_1, \dots, V_r, V_1^B, \dots, V_r^B)$ and $(X_1, \dots, X_r, X_1^B, \dots, X_r^B)$, respectively, and edge sets

$$E(G^+) := E(G_A) \cup \{u\pi(v), v\pi(u) \colon uv \in E(G_B)\} \cup E_G^*,$$
$$E(H^+) := E(H) \cup \{x\pi(y), y\pi(x) \colon xy \in E(H)\} \cup E_H^*,$$

where we added for convenience a suitable set $E_G^* \subseteq \bigcup_{i\in[r]}\{uv \colon u \in V_i, v \in V_i^B\}$ such that $G^+[V_i, V_i^B]$ is (ε_0, d_B)-super-regular for all $i \in [r]$, and the set $E_H^* := \{x\pi(x) \colon x \in V(H)\}$ so that $H^+[X_i, X_i^B]$ is a perfect matching for all $i \in [r]$. Note that $G^+[V_i, V_j] = G_A[V_i, V_j]$, whereas $G^+[V_i, V_j^B]$ and $G^+[V_i^B, V_j]$ are isomorphic to $G_B[V_i, V_j]$ for all $ij \in \binom{[r]}{2}$.

We now define $A_i(\phi_t)$ and $B_i(\phi_t)$. Let $X_i^A := X_i$ and $V_i^A := V_i$ for every $i \in [r]$. For $Z \in \{A, B\}$ and $i \in [r]$, we say that $v_i \in V_i^Z$ is a *candidate for* $x_i \in X_i^Z$ *(given* ϕ_t*)* if

$$\phi_t(N_{H^+}(x_i) \cap \mathcal{X}_t^{\phi_t}) \subseteq N_{G^+}(v_i), \tag{3.6.2}$$

and we define $Z_i(\phi_t)$ as the bipartite graph with partition (X_i^Z, V_i^Z) and edge set

$$E(Z_i(\phi_t)) := \big\{x_i v_i \colon x_i \in X_i^Z, v_i \in V_i^Z, \text{ and } v_i \text{ is a candidate for } x_i \text{ given } \phi_t\big\}.$$

We call any spanning subgraph of $Z_i(\phi_t)$ a *candidacy graph.*

Next, we define edge set colourings for these candidacy graphs. For $i \in [r] \setminus [t]$, we assign to every edge $e = x_i v_i \in E(A_i(\phi_t))$ a colour set $c_t(e)$ of size at most t, which represents the colours that would be used if we were to embed x_i at v_i in the next step. More precisely, for every $i \in [r] \setminus [t]$ and every edge $x_i v_i \in E(A_i(\phi_t))$, we set

$$c_t(x_i v_i) := c\big(E\big(G_A\big[\phi_t(N_H(x_i) \cap \mathcal{X}_t^{\phi_t}), \{v_i\}\big]\big)\big). \tag{3.6.3}$$

Tracking this set will help us to ensure that the embedding is rainbow when we extend ϕ_t to ϕ_{t+1}. Since $|N_H(x_i) \cap \mathcal{X}_t^{\phi_t}| \le t$ and $|c(e)| = 1$ for all $e \in E(G_A)$, we have $|c_t(x_i v_i)| \le t$.

For the candidacy graphs $B_i(\phi_t)$, we merely need to know that they maintain super-regularity during the inductive approximate embedding (see $\mathbf{S}(t)$ below). Hence, for convenience, we set $c_t(e) := \emptyset$ for every $e \in E(B_i(\phi_t))$.

We also assign artificial dummy colours to the edges of $E(G^+) \setminus E(G_A)$ as follows. Let $c^{art} \colon \binom{V(G^+)}{2} \to C^{art}$ be a rainbow edge-colouring of all possible edges in $V(G^+)$ such that $C^{art} \cap C = \emptyset$. Define c^+ on $E(G^+)$ by setting $c^+(e) := c(e)$ if $e \in E(G_A)$ and $c^+(e) := c^{art}(e)$ otherwise.

Step 3. Induction

We inductively prove the following statement $\mathbf{S}(t)$ for all $t \in [r]_0$.

$\mathbf{S}(t)$. There exists a t-partial rainbow embedding $\phi_t \colon \mathcal{X}_t^{\phi_t} \to \mathcal{V}_t^{\phi_t}$ with $|X_s^{\phi_t}| = |V_s^{\phi_t}| \ge (1 - \varepsilon_t)|X_s|$ for all $s \in [t]$, and for all $Z \in \{A, B\}$, there exists a candidacy graph $Z_i^t \subseteq Z_i(\phi_t)$ such that

(a) A_i^t is (ε_t, d_A^t)-super-regular for all $i \in [r] \setminus [t]$;

(b) B_i^t is (ε_t, d_B^t)-super-regular for all $i \in [r]$;

(c) the colouring c_t restricted to A_i^t is $(1 + \varepsilon_t)d_A^t|X_i|$-bounded and has codegree at most $n^{1/3}$ for all $i \in [r] \setminus [t]$.

The statement $\mathbf{S}(0)$ holds for ϕ_0 being the empty function: Clearly, for all $Z \in \{A, B\}$, $i \in [r]$, the candidacy graph $Z_i(\phi_0)$ is complete bipartite, and by (3.6.3), we have $c_0(e) = \emptyset$ for all $e \in E(A_i(\phi_0))$, implying $\mathbf{S}(0)$.

Hence, we may assume the truth of $\mathbf{S}(t)$ for some $t \in [r-1]_0$ and let $\phi_t \colon \mathcal{X}_t^{\phi_t} \to \mathcal{V}_t^{\phi_t}$ and A_i^t, B_i^t be as in $\mathbf{S}(t)$. We will now extend ϕ_t to a $(t+1)$-partial rainbow embedding ϕ_{t+1} such that $\mathbf{S}(t+1)$ holds. Note that any matching $\sigma \colon X_{t+1}^\sigma \to V_{t+1}^\sigma$ in A_{t+1}^t with $X_{t+1}^\sigma \subseteq X_{t+1}$ and $V_{t+1}^\sigma \subseteq V_{t+1}$ induces an embedding $\phi_{t+1} \colon \mathcal{X}_t^{\phi_t} \cup X_{t+1}^\sigma \to \mathcal{V}_t^{\phi_t} \cup V_{t+1}^\sigma$ which extends ϕ_t to a $(t+1)$-partial embedding as follows:

$$\phi_{t+1}(x) := \begin{cases} \phi_t(x) & \text{if } x \in \mathcal{X}_t^{\phi_t}, \\ \sigma(x) & \text{if } x \in X_{t+1}^\sigma. \end{cases} \tag{3.6.4}$$

The following is a key observation: Since c is colour-split and by definition of the candidacy graph A_{t+1}^t and the colouring c_t on $E(A_{t+1}^t)$, whenever σ is a *rainbow* matching in A_{t+1}^t, then ϕ_{t+1} is a $(t+1)$-partial *rainbow* embedding.

Now, we aim to apply Lemma 3.12 in order to obtain an almost perfect rainbow matching σ in A_{t+1}^t. Let $H^{t+1} := H^+ - \mathcal{X}_t$ and let $G^{t+1} := G^+ - \mathcal{V}_t$. We claim that

$$(H^{t+1}, G^{t+1}, \mathcal{A}, c^+ \cup c_t) \textit{ is an } (\varepsilon_t, \mathbf{d}, (d_A^t, \mathbf{d}^t), t, \Lambda)\textit{-embedding-instance,} \tag{3.6.5}$$

where $\mathcal{A} := (A_{t+1}^t, \dots, A_r^t, B_1^t, \dots, B_r^t)$ and $\mathbf{d} := (d_A, \dots, d_A, d_B, \dots, d_B)$ (d_A repeated $r-t-1$ times and d_B repeated r times).

First, note that the colouring $c^+ \cup c_t$ is locally Λ-bounded and colour-split with respect to the vertex partition

$$(X_{t+1}, \dots, X_r, X_1^B, \dots, X_r^B, V_{t+1}, \dots, V_r, V_1^B, \dots, V_r^B)$$

of $G^{t+1} \cup \bigcup_{i \in [r] \setminus [t]} A_i^t \cup \bigcup_{i \in [r]} B_i^t$. Moreover, the colour sets of G^{t+1}-edges have size 1 and the colour sets of candidacy graph edges have size at most t.

Further, the super-regularity of the G^{t+1}-pairs follows from (3.6.1) (and for the pair $G^{t+1}[V_{t+1}, V_{t+1}^B]$ from the choice of E_G^*). Moreover, combining (3.6.1) with assumption (iii), we infer that for every $i \in [r-t-1]$, the edge-colouring

$$c \text{ restricted to } G_A[V_{t+1}, V_{t+1+i}] \text{ is } (1+\varepsilon_t)\frac{e_{G_A}(V_{t+1}, V_{t+1+i})}{e_H(X_{t+1}, X_{t+1+i})}\text{-bounded.}$$

Finally, the super-regularity of the candidacy graphs and the boundedness of their colourings follows from $\mathbf{S}(t)$. We conclude that (3.6.5) holds. Hence, we can apply Lemma 3.12 to this instance with the following parameters:

parameter	$\lvert X_{t+1} \rvert$	ε_t	ε_{t+1}	t	$r-t-1+r$	Λ	$n^{1/3}$	$\mathbf{d}$	$(d_A^t, \mathbf{d}^t)$
replaces	n	ε	ε'	t	r	Λ	K	$(d_i^G)_{i \in [r]}$	$(d_i)_{i \in [r]_0}$

Let $\sigma \colon X_{t+1}^\sigma \to V_{t+1}^\sigma$ be the rainbow matching in A_{t+1}^t obtained from Lemma 3.12 with $|X_{t+1}^\sigma| \geq (1-\varepsilon_{t+1})|X_{t+1}|$. The matching σ extends ϕ_t to a $(t+1)$-partial rainbow embedding ϕ_{t+1} as defined in (3.6.4). By Definition 3.10, the updated candidacy graphs with respect to σ obtained from Lemma 3.12 are also updated candidacy graphs with respect to ϕ_{t+1} as defined in Step 2. (More precisely, we have $Z_i^{t,\sigma} \subseteq Z_i(\phi_{t+1})$ for $Z \in \{A, B\}$.) Hence, by Lemma 3.12, we obtain new candidacy graphs $A_i^{t+1} \subseteq A_i(\phi_{t+1})$ for $i \in [r] \setminus [t+1]$ and $B_i^{t+1} \subseteq B_i(\phi_{t+1})$ for $i \in [r]$ that satisfy (I)$_{\text{L3.12}}$–(III)$_{\text{L3.12}}$. By (I)$_{\text{L3.12}}$, we know that A_i^{t+1} is $(\varepsilon_{t+1}, d_A^{t+1})$-super-regular for every $i \in [r] \setminus [t+1]$, and B_i^{t+1} is $(\varepsilon_{t+1}, d_B^{t+1})$-super-regular for every $i \in [r]$, which implies $\mathbf{S}(t+1)$(a) and $\mathbf{S}(t+1)$(b). Moreover, the new colouring c_{t+1} as defined in (3.6.3) corresponds to the updated colouring as in Definition 3.11, so we can assume that c_{t+1} satisfies $(\mathrm{II})_{\mathrm{L3.12}}$ and $(\mathrm{III})_{\mathrm{L3.12}}$. Thus, for every $i \in [r] \setminus [t+1]$, the colouring c_{t+1} restricted to A_i^{t+1} is $(1+\varepsilon_{t+1})d_A^{t+1}|X_i|$-bounded by $(\mathrm{II})_{\mathrm{L3.12}}$, and has codegree at most $n^{1/3}$ by $(\mathrm{III})_{\mathrm{L3.12}}$. This implies $\mathbf{S}(t+1)$(c), and hence completes the inductive step.

Step 4. Completion

We may assume that $\phi_r\colon \mathcal{X}_r^{\phi_r} \to \mathcal{V}_r^{\phi_r}$ is an r-partial embedding fulfilling $\mathbf{S}(r)$ with (ε_r, d_B^r)-super-regular candidacy graphs $B_i^r \subseteq B_i(\phi_r)$. Recall that we defined the bipartite candidacy graphs B_i^r on copies (X_i^B, V_i^B) only to conveniently apply Lemma 3.12 in Step 3. We now identify B_i^r with a bipartite graph B_i' on (X_i, V_i) and edge set $E(B_i') := \{x_i v_i \colon \pi(x_i)\pi(v_i) \in E(B_i^r)\}$. Hence, for each $i \in [r]$, B_i' is (ε_r, d_B^r)-super-regular and for every edge $x_i v_i \in E(B_i')$, we deduce from (3.6.2) that

$$\phi_r(N_H(x_i) \cap \mathcal{X}_r^{\phi_r}) \subseteq N_{G_B}(v_i). \tag{3.6.6}$$

We want to apply Lemma 3.5 in order to complete the embedding using the edges in G_B and the candidacy graphs $(B_i')_{i\in[r]}$. For every $i \in [r]$, let $\overline{V}_i := V_i \setminus V_i^{\phi_r}$ and $\overline{X}_i := X_i \setminus X_i^{\phi_r}$ be the sets of unused/unembedded vertices. Note that we have no control over these sets except knowing that they are very small. To be able to apply Lemma 3.5, we now (randomly) add vertices that have already been embedded back to the unembedded vertices. That is, we will find sets $V_i' \supseteq \overline{V}_i$ and $X_i' \supseteq \overline{X}_i$ of size exactly $n_B := \lceil \mu n \rceil$ (same size required for condition (ii) in Lemma 3.5) such that $B_i'[X_i', V_i']$ is still super-regular.

For the application of Lemma 3.5, we also have to ensure that not only the colouring c restricted to $G_B[V_1' \cup \ldots \cup V_r']$ is sufficiently bounded (see property (c) below), but also that the colouring c restricted to G_B between already embedded sets $V_i \setminus V_i'$ and sets V_j' used for the completion is sufficiently bounded (see property (d) below). Therefore, for $i, j \in [r]$, let $G_B^{hit}[V_i \setminus V_i', V_j']$ be the spanning subgraph of $G_B[V_i \setminus V_i', V_j']$ containing those edges $v_i v_j \in E(G_B[V_i \setminus V_i', V_j'])$ for which $\phi_r^{-1}(v_i)$ has an H-neighbour in X_j'. That is, $G_B^{hit}[V_i \setminus V_i', V_j']$ contains all the edges between $V_i \setminus V_i'$ and V_j' that will potentially be used to extend the partial embedding when applying Lemma 3.5.

We claim that sets $\overline{V}_i^+ \subseteq V_i^{\phi_r}$ can be chosen such that, setting $\overline{X}_i^+ := \phi_r^{-1}(\overline{V}_i^+)$, $V_i' := \overline{V}_i \cup \overline{V}_i^+$, and $X_i' := \overline{X}_i \cup \overline{X}_i^+$, we have:

(a) $G_B[V_i', V_j']$ is (ε_{r+1}, d_B)-super-regular for all $ij \in \binom{[r]}{2}$;

(b) $B_i'[X_i', V_i']$ is $(\varepsilon_{r+1}, d_B^r)$-super-regular for every $i \in [r]$;

(c) the colouring c restricted to $G_B[V_1' \cup \ldots \cup V_r']$ is $\mu^{3/2}n$-bounded;

(d) the colouring c restricted to $G_B^{hit}[V_i \setminus V_i', V_j']$ is $\mu^{3/2}n$-bounded for all $i, j \in [r]$;

(e) $|V_i'| = |X_i'| = n_B$ for every $i \in [r]$.

This can be seen with a probabilistic argument. Independently for every $i \in [r]$ and $v \in V_i^{\phi_r}$, let v belong to $\overline{V}_i^+$ with probability $p_i := (n_B - |\overline{V}_i|)/|V_i^{\phi_r}|$. We now show that (a)–(e) hold simultaneously with positive probability.

Note that $p_i = \mu \pm \sqrt{\varepsilon_r}$. Recall that $G_B[V_i, V_j]$ is (ε_0, d_B)-super-regular, B_i' is (ε_r, d_B^r)-super-regular, $|\overline{V}_i| = |\overline{X}_i| \leq 2\varepsilon_r n$, and c is locally Λ-bounded. Using Chernoff's bound, it is routine to show that (a) and (b) hold with probability at least $1 - \mathrm{e}^{-\sqrt{n}}$, say. Note here that the regularity follows easily from the regularity of the respective supergraphs.

We show next that also (d) holds with high probability. Let $i, j \in [r]$ and let α be a colour. Let X be the number of α-coloured edges $v_i v_j$ in $G_B[V_i \setminus V_i', V_j'])$ for which

$v_j \in \overline{V}_j^+$ and $\phi_r^{-1}(v_i)$ has an H-neighbour in $\overline{X}_j^+$. Note that since $|\overline{V}_j| = |\overline{X}_j| \le 2\varepsilon_r n$ and c is locally Λ-bounded, the number of α-coloured edges v_iv_j in $G_B[V_i \setminus V_i', V_j']$) for which $v_j \in \overline{V}_j$ or $\phi_r^{-1}(v_i)$ has an H-neighbour in $\overline{X}_j$, is at most $4\Lambda\varepsilon_r n$. Now, consider an edge $v_iv_j \in E(G_B[V_i^{\phi_r}, V_j^{\phi_r}])$ with $\{x_j\} = N_H(\phi_r^{-1}(v_i)) \cap X_j^{\phi_r}$. Crucially, observe that $x_j \neq \phi_r^{-1}(v_j)$ because v_iv_j is an edge in G_B and therefore not in G_A. This implies that

$$\mathbb{P}\left[v_j \in \overline{V}_j^+, x_j \in \overline{X}_j^+\right] = p_j^2 \le 2\mu^2.$$

Since c is locally Λ-bounded, α appears on at most $2\Lambda n$ such edges v_iv_j and hence

$$\mathbb{E}[X] \le 4\Lambda\mu^2 n.$$

Since c is locally Λ-bounded, an application of McDiarmid's inequality yields that, with probability at least $1 - \mathrm{e}^{-n^{2/3}}$, we have $X \le 5\Lambda\mu^2 n$, which implies that the number of α-coloured edges in $G_B^{hit}[V_i \setminus V_i', V_j']$ is at most $\mu^{3/2}n$. Together with a union bound, we infer that (d) holds with probability at least $1 - \mathrm{e}^{-\sqrt{n}}$.

A similar (even simpler) argument using the local boundedness of c and McDiarmid's inequality also works for (c). Thus, a union bound implies that (a)–(d) hold simultaneously with probability at least $1 - 4\mathrm{e}^{-\sqrt{n}}$. Moreover, standard properties of the binomial distribution yield that $|\overline{V}_i^+| = n_B - |\overline{V}_i|$ (and thus, $|V_i'| = |X_i'| = n_B$) for all $i \in [r]$ with probability at least $\Omega(n^{-r/2})$. Hence, there exist such sets X_i' and V_i' for all $i \in [r]$ satisfying (a)–(e).

Let

$$\mathcal{X}_r' := \bigcup_{i\in[r]} X_i', \qquad \mathcal{V}_r' := \bigcup_{i\in[r]} V_i',$$
$$X_0' := \mathcal{X}_r \setminus \mathcal{X}_r', \qquad V_0' := \mathcal{V}_r \setminus \mathcal{V}_r'.$$

The restriction of ϕ_r to X_0' clearly yields a rainbow embedding $\psi_0 \colon X_0' \to V_0'$ of $H[X_0']$ into $G_A[V_0']$. Let $G' := G_B[\mathcal{V}_r'] \cup G_B^{hit}[V_0', \mathcal{V}_r']$, and let H' be the subgraph of H with partition $(X_i')_{i\in[r]_0}$ that arises from H by discarding all edges in $H[X_0']$. (This is feasible since edges within X_0' have already been embedded by ψ_0.) By (a) and (b), we have that $\mathcal{B}' := (H', G', (X_i')_{i\in[r]_0}, (V_i')_{i\in[r]_0})$ is an (ε_{r+1}, d_B)-super-regular blow-up instance with exceptional sets (X_0', V_0') and $(\varepsilon_{r+1}, d_B^r)$-super-regular candidacy graphs $(B_i'[X_i', V_i'])_{i\in[r]}$. Moreover, c restricted to G' is $\mu^{1/2}n_B$-bounded by (c) and (d), and all clusters have the same size n_B by (e). Further,

- from (3.6.6) and the definition of G_B^{hit}, it holds that for all $x \in X_0'$, $i \in [r]$ and $x_i \in N_{H'}(x) \cap X_i'$, we have $N_{B_i'}(x_i) \subseteq N_{G'}(\psi_0(x))$;
- for all $i \in [r]$, $x \in X_i'$, $v \in N_{B_i'}(x)$ and distinct $x_0, x_0' \in N_{H'}(x) \cap X_0'$, we have $c(\psi_0(x_0)v) \neq c(\psi_0(x_0')v)$ because $\psi_0(x_0)$ and $\psi_0(x_0')$ belong to different clusters of $(V_i)_{i\in[r]}$ and c is colour-split with respect to $(V_i)_{i\in[r]}$.

Hence, we can finally apply Lemma 3.5 as follows:

parameter	n_B	ε_{r+1}	$\mu^{1/2}$	d_B	d_B^r	r	$r-1$	$\mathcal{B}'$	$(B_i'[X_i', V_i'])_{i\in[r]}$
plays the role of	n	ε	μ	d_G	d_A	r	Δ	$\mathcal{B}$	$(A_i)_{i\in[r]}$

This yields a rainbow embedding ψ of H' into G' which extends ψ_0, such that $\psi(x) \in N_{B_i'}(x)$ for all $i \in [r]$ and $x \in X_i'$. Since the colours of c restricted to $G' \subseteq G_B$ are distinct from the colours already used by ψ_0, it holds that ψ is a valid rainbow embedding of H into G. This completes the proof. $\square$

3.7 Applications

In this section, we discuss applications of our main result to graph decompositions, graph labelling and orthogonal double covers. As mentioned before, these applications are inspired by recent work of Montgomery, Pokrovskiy and Sudakov [104], and basically transfer their applications from trees to general, yet bounded degree, graphs.

Graph decompositions

We briefly explain the general idea of utilizing rainbow edge-colourings to find graph decompositions, and then give two examples.

Suppose G is a graph and Γ is a subgroup of the automorphism group $Aut(G)$. If for some subgraph H of G, $\{\phi(H)\}_{\phi\in\Gamma}$ is a collection of edge-disjoint subgraphs of G, we call this a Γ*-generated* H*-packing in* G, and if every edge of G is covered, then it is a Γ*-generated* H*-decomposition of* G. For instance, in Walecki's theorem, G is the complete graph and Γ is generated by one permutation π. We say that a packing/decomposition of K_n is *cyclic* if Γ is isomorphic to $\mathbb{Z}_n$. Recall Kotzig's conjecture that for any given tree T with n edges, there exists a cyclic T-decomposition of K_{2n+1}. Note that there are two natural divisibility conditions for the existence of such a decomposition, one 'global' edge divisibility condition and one 'local' degree condition. First, the number of edges of K_{2n+1} is $(2n+1)n$ which is divisible by n. Secondly, every vertex of K_{2n+1} is supposed to play the role of every vertex of T exactly once, thus we need that $\sum_{v\in V(T)} d_T(v) = 2n$, which is true by the hand-shaking lemma. However, note that we have not used the fact that T is a tree. The same divisibility conditions hold for any graph with n edges. We thus propose the following conjecture as an analogue to Kotzig's conjecture for general (bounded degree) graphs.

Conjecture 3.14 ([30])**.** *For all $\Delta \in \mathbb{N}$, there exists n_0 such that for all $n \geq n_0$, the following is true. For any graph H with n edges and $\Delta(H) \leq \Delta$, there exists a cyclic H-decomposition of K_{2n+1}.*

We will provide some evidence for this conjecture below (Theorem 3.16). Before, we discuss in a general way how to use rainbow embeddings to find Γ-generated packings and decompositions. Let G and Γ be as above. Then Γ acts on G as a group action and every element $\phi \in \Gamma$ sends vertices onto vertices and edges onto edges. The *orbit* $\Gamma \cdot e$ of an edge e is defined as $\Gamma \cdot e := \{\phi(e) \colon \phi \in \Gamma\}$. It is well-known that two orbits are either disjoint or equal. Hence we may colour the edges of G according to which orbit they belong to. We refer to the *orbit colouring* c_o^Γ of G induced by Γ and define $c_o^\Gamma(e) := \Gamma \cdot e$ for all $e \in E(G)$.

The following simple lemma now asserts that if we can find a rainbow copy with respect to the orbit colouring, and all orbits have maximum size, then the copies of H obtained via Γ are pairwise edge-disjoint. The proof is immediate and thus omitted.

Lemma 3.15 ([30])**.** *Let G be a graph and let Γ be a subgroup of $Aut(G)$ such that $|\Gamma \cdot e| = |\Gamma|$ for all $e \in E(G)$. Suppose that H is a rainbow subgraph in G with respect to c_o^Γ. Then $\{\phi(H)\}_{\phi\in\Gamma}$ is a Γ-generated H-packing in G.*

In particular, if $|\Gamma| = e(G)/e(H)$, then this yields a Γ-generated H-decomposition of G.

Theorem 3.16 ([30])**.** *For all $\Delta \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose H is a graph with $|V(H)| \leq n$, $\Delta(H) \leq \Delta$ and at most $(1-\varepsilon)n/2$ edges. Then K_n contains a cyclic H-packing.*

Proof. Let G be the graph on vertex set $[n]$ that is the complete graph if n is odd and is otherwise obtained from the complete graph by deleting the edges $\{i, i+n/2\}$ for all $i \in [n/2]$. Consider the subgroup Γ of $Aut(G)$ that is generated by the automorphism which sends a vertex i to $i+1$ (modulo n). Clearly, $\Gamma \cong \mathbb{Z}_n$ and hence $|\Gamma| = n$. In addition, $|\Gamma \cdot e| = n$ for all $e \in E(G)$ and c_o^Γ is locally 2-bounded. Therefore, Theorem 3.3 yields a rainbow copy of H with respect to c_o^Γ in G, which by Lemma 3.15 yields a cyclic H-packing in $G \subseteq K_n$. □

We can also deduce a partite version of this. For simplicity, we only consider the bipartite case.

Theorem 3.17 ([30])**.** *For all $\Delta \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose H is a graph with $\Delta(H) \leq \Delta$ and at most $(1-\varepsilon)n$ edges, and $V(H)$ is partitioned into 2 independent sets of size n. Then the complete bipartite graph $K_{n,n}$ contains a $\mathbb{Z}_n$-generated H-packing.*

Proof. We proceed similarly as in Theorem 3.16. Let $K_{n,n}$ have vertex set $\{(1,i),(2,i) : i \in [n]\}$ and edge set $\{(1,i)(2,j) : i,j \in [n]\}$. Consider the subgroup Γ of $Aut(G)$ that is generated by the automorphism which sends each vertex (ℓ, i) to $(\ell, i+1)$ (modulo n in the second coordinate), for $\ell \in [2]$. Consequently, $\Gamma \cong \mathbb{Z}_n$. Moreover, $|\Gamma \cdot e| = n$ for all $e \in E(K_{n,n})$ and c_o^Γ is proper. Thus, Lemma 3.4 yields a rainbow copy of H in $K_{n,n}$ with respect to c_o^Γ. Then Lemma 3.15 completes the proof. □

These results demonstrate the usefulness of rainbow embeddings to decomposition problems. Clearly, the application is limited to decompositions of a host graph into copies of the same graph H. Approximate decomposition results which do not arise from a group action but from random procedures have been studied recently in great depth. At the expense that one does not obtain very symmetric (approximate) decompositions, it is possible to embed different graphs and not only many copies of a single graph. In particular, the blow-up lemma for approximate decompositions by Kim, Kühn, Osthus and Tyomkyn [79] yields approximate decompositions into bounded degree graphs of quasirandom multipartite graphs. Both this and another recent result of Allen, Böttcher, Hladký and Piguet [7] imply Conjecture 2.1 asymptotically for non-cyclic decompositions.

Orthogonal double covers

An *orthogonal double cover* of K_n by some graph F is a collection of n copies of F in K_n such that every edge of K_n is contained in exactly two copies, and each two copies have exactly one edge in common. Note that F must have exactly $n-1$ edges. For instance, an orthogonal double cover of $K_{\binom{k}{2}+1}$ by K_k is equivalent to a *biplane*, which is, roughly speaking, the orthogonal double cover version of a finite projective plane. Only a handful of such biplanes is known and it is a major open question whether there are infinitely many.

Another natural candidate for F is a spanning tree. Gronau, Mullin, Rosa conjectured the following.

Conjecture 3.18 (Gronau, Mullin, Rosa [57])**.** *Let T be an arbitrary tree with n vertices, $n \geq 2$, where T is not the path of length 3. Then there exists an orthogonal double cover of K_n by T.*

Montgomery, Pokrovskiy and Sudakov [104] proved an asymptotic version of this when n is a power of 2, using their Theorem 3.1. Similarly, our main theorem yields approximate orthogonal double covers by copies of any bounded degree graph with

$(1 - o(1))n$ edges whenever n is a power of 2. We omit the proof as it is verbatim the same as in [104].

Theorem 3.19 ([30]). *For all $\Delta \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ with $n = 2^k$ for some $k \in \mathbb{N}$. Suppose H is a graph with $|V(H)| \leq n$, $\Delta(H) \leq \Delta$ and at most $(1-\varepsilon)n$ edges. Then the complete graph K_n contains n copies of H such that every edge of K_n belongs to at most two copies, and any two copies have at most one edge in common.*

Graph labellings

The study of graph labellings began in the 1960s and has since produced a vast amount of different concepts, results and applications (see e.g. the survey [48]). Perhaps the most popular types of labellings are graceful labellings and harmonious labellings. The former were introduced by Rosa [115] in 1967. Given a graph H with q edges, a *graceful labelling of H* is an injection $f\colon V(H) \to [q+1]$ such that the induced edge labels $|f(x) - f(y)|$, $xy \in E(H)$, are pairwise distinct, and H is *graceful* if such a labelling exists. The Graceful tree conjecture asserts that all trees are graceful. Rosa [115] showed that this would imply the aforementioned Ringel–Kotzig conjecture. Despite extensive research, this conjecture remains wide open. Adamaszek, Allen, Grosu and Hladký [1] recently proved that almost all trees are almost graceful.

Harmonious labellings were introduced by Graham and Sloane [56] in 1980. Given a graph H and an abelian group Γ, a *Γ-harmonious labelling of H* is an injective map $f\colon V(H) \to \Gamma$ such that the induced edge labels $f(x) + f(y)$, $xy \in E(H)$, are pairwise distinct, and H is *Γ-harmonious* if such a labelling exists. Graham and Sloane asked which graphs H are $\mathbb{Z}_{e(H)}$-harmonious. Note that this necessitates that $|V(H)| \leq e(H)$. In the special case when H is a tree on n vertices, they conjectured that there exists an injective map $f\colon V(H) \to [n]$ such that the induced edge labels $f(x) + f(y)$, $xy \in E(H)$, are pairwise distinct modulo $n-1$. Żak [123] proposed a weakening of this. He conjectured that every tree on $n-o(n)$ vertices is $\mathbb{Z}_n$-harmonious. Montgomery, Pokrovskiy and Sudakov [104] proved Żak's conjecture as a corollary of Theorem 3.1. Using our Theorem 3.2, we can deduce a similar statement for general bounded degree graphs.

Theorem 3.20 ([30]). *For all $\Delta \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose H is a graph with at most n vertices, at most $(1-\varepsilon)n$ edges and $\Delta(H) \leq \Delta$. Let Γ be an abelian group of order n. Then H is Γ-harmonious.*

Proof. Consider the complete graph K_Γ on Γ. Define the edge-colouring $c\colon E(K_\Gamma) \to \Gamma$ by setting $c(ij) = i + j$, and note that c is proper and thus $n/2$-bounded. Hence, by Theorem 3.2, K_Γ contains a rainbow copy of H, which corresponds to a Γ-harmonious labelling of H. $\square$

Chapter 4

The blow-up lemma for approximate decompositions

The content of this chapter is based on the preprint [32] with Felix Joos.

4.1 Introduction to graph decompositions

Preceding the recent advances on Conjectures 1.1–1.3 (as mentioned in the introductory chapter), there has been a collection of approximate decomposition results, that is, a few edges of the host graph are not covered, under various and quite general conditions, see [7, 16, 41, 79, 100, 104]. We also refer to [21, 42, 77, 92] for further developments in the field. The importance of these approximate results should not be under estimated. In fact, numerous decomposition results combine approximate decomposition results with certain absorbing techniques. This includes [5, 52, 65, 71, 72, 75, 76, 103]. Having this in mind and in need of a powerful approximate decomposition result, Kim, Kühn, Osthus and Tyomkyn [79] proved a far-reaching generalization of the original blow-up lemma – a 'blow-up lemma for approximate decompositions'. This result can also be combined with Szemerédi's regularity lemma to obtain almost decompositions of graphs into bounded degree graphs.

The blow-up lemma for approximate decompositions has already exhibited its versatility. It has been applied in [65, 77, 88] and in [21] for a 'bandwidth theorem for approximate decompositions', which in turn is one of the key ingredients for the resolution of the Oberwolfach problem in [51]. However, its very complex and long proof is an obstacle for further generalizations. We overcome this and present a new and significantly shorter proof in this chapter.

Our approach makes it possible to include some more features: an easier handling of exceptional vertices, which results in a substantially easier applicability of the theorem, as well as stronger quasirandom properties for the approximate decompositions. To be more precise, the first yields shorter proofs of the main results in [21] and [65] as certain technically involved preprocessing steps are no longer needed; the latter permits to combine our main result with Keevash's recent results on designs [72]. We demonstrate this in Section 4.6 and obtain new results on decomposing quasirandom graphs into regular spanning graphs. Further, we illustrate in Section 4.6 how our proof methods also give rise to approximate decompositions for directed graphs.

4.1.1 The blow-up lemma for approximate decompositions

In this section, we first introduce some terminology and then state the blow-up lemma for approximate decompositions. We say that a collection/multiset of graphs $\mathcal{H} =$

$\{H_1, \ldots, H_s\}$ *packs* into a graph G if there is a function $\phi : \bigcup_{H\in\mathcal{H}} V(H) \to V(G)$ such that $\phi|_{V(H)}$ is injective and ϕ injectively maps edges onto edges. In such a case, we call ϕ a *packing of* $\mathcal{H}$ *into* G. Our general aim is to pack a collection $\mathcal{H}$ of multipartite graphs in a host graph G having the same multipartite structure which is captured by a so-called 'reduced graph' R. To this end, let $(H, G, R, \mathcal{X}, \mathcal{V})$ be a *blow-up instance* if

- H, G, R are graphs where $V(R) = [r]$ for some $r \geq 2$;
- $\mathcal{X} = (X_i)_{i\in[r]}$ is a vertex partition of H into independent sets, $\mathcal{V} = (V_i)_{i\in[r]}$ is a vertex partition of G such that $|V_i| = |X_i|$ for all $i \in [r]$;
- $H[X_i, X_j]$ is empty whenever $ij \in \binom{[r]}{2} \setminus E(R)$.

We also refer to $\mathscr{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ as a *blow-up instance* if $\mathcal{H}$ is a collection of graphs and $\mathcal{X}$ is a collection of vertex partitions $(X_i^H)_{i\in[r],H\in\mathcal{H}}$ so that $(H, G, R, (X_i^H)_{i\in[r]}, \mathcal{V})$ is a blow-up instance for every $H \in \mathcal{H}$.

The blow-up instance $\mathscr{B}$ is (ε, d)*-super-regular* if $G[V_i, V_j]$ is (ε, d)-super-regular for all $ij \in E(R)$, and $\mathscr{B}$ is Δ*-bounded* if $\Delta(R), \Delta(H) \leq \Delta$ for each $H \in \mathcal{H}$. Now we are ready to state the blow-up lemma for approximate decompositions.

Theorem 4.1 (Kim, Kühn, Osthus, Tyomkyn [79])**.** *For all $\alpha \in (0, 1]$ and $r \geq 2$, there exist $\varepsilon = \varepsilon(\alpha) > 0$ and $n_0 = n_0(\alpha, r)$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ is an (ε, d)-super-regular and α^{-1}-bounded blow-up instance such that $|V_i| = n$ for all $i \in [r]$, $|\mathcal{H}| \leq \alpha^{-1}n$, and $\sum_{H\in\mathcal{H}} e_H(X_i^H, X_j^H) \leq (1 - \alpha)dn^2$ for all $ij \in E(R)$. Then there is a packing ϕ of $\mathcal{H}$ into G such that $\phi(X_i^H) = V_i$ for all $i \in [r]$ and $H \in \mathcal{H}$.*

We remark that there are more general versions of Theorem 4.1 in [79], but omit the more technical statements here. Instead we state our main result and the interested reader can easily check that it generalizes[1] the more technical versions in [79].

4.1.2 Main result

Most blow-up lemmas exhibit their power if they are applied in conjunction with Szemerédi's regularity lemma. This, however, comes with the expense of a small set of vertices over which we have no control. Consequently, in such a setting, when embedding a graph H into G, it is often the case that some vertices of H are already embedded and the blow-up lemma is applied only to some nice part of G. To deal with such scenarios we consider extended blow-up instances. We say $(H, G, R, \mathcal{X}, \mathcal{V}, \phi_0)$ is an *extended blow-up instance* if

- H, G, R are graphs where $V(R) = [r]$ for some $r \geq 2$;
- $\mathcal{X} = (X_i)_{i\in[r]_0}$ is a vertex partition of H into independent sets, $\mathcal{V} = (V_i)_{i\in[r]_0}$ is a vertex partition of G such that $|V_i| = |X_i|$ for all $i \in [r]_0$;
- $H[X_i, X_j]$ is empty whenever $ij \in \binom{[r]}{2} \setminus E(R)$;
- ϕ_0 is an injective embedding of X_0 into V_0.

[1]Observe that we do not allow different densities between the cluster pairs in G. However, this technical complication could very easily be implemented by adding at numerous places extra indices. As this feature has never been used so far in applications, we omitted it for the sake of a clearer presentation.

This definition also extends as above to the case when H is replaced by a collection of graphs $\mathcal{H}$ in the obvious way as before. An extended blow-up instance is (ε, d)-*super-regular* if $G[V_i, V_j]$ is (ε, d)-super-regular for all $ij \in E(R)$.

Let $\mathscr{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V}, \phi_0)$ be an extended blow-up instance. We say $\mathscr{B}$ is (ε, α)-*linked* if

- at most $\varepsilon|X_i^H|$ vertices in X_i^H have a neighbour in X_0^H for all $i \in [r]$, $H \in \mathcal{H}$;
- $|V_i \cap \bigcap_{x_0 \in X_0^H \cap N_H(x)} N_G(\phi_0(x_0))| \geq \alpha|V_i|$ for all $x \in X_i^H, i \in [r], H \in \mathcal{H}$;
- $|\phi_0^{-1}(v)| \leq \varepsilon|\mathcal{H}|$ for all $v \in V_0$;
- $\sum_{H \in \mathcal{H}} |N_H(x_0^H) \cap N_H(x_0'^H) \cap X_i^H| \leq \varepsilon|V_i|^{1/2}$ for all $i \in [r]$ and distinct $v_0, v_0' \in V_0$ where $x_0^H = \phi_0^{-1}(v_0) \cap X_0^H$ and $x_0'^H = \phi_0^{-1}(v_0') \cap X_0^H$ for $H \in \mathcal{H}$.

One feature of our result is that one can replace 'blow-up instance' in Theorem 4.1 by 'extended blow-up instance that is (ε, α)-linked'. We remark that the above conditions are easily met in applications known to us and are similar to conditions found elsewhere for this purpose.

Next, we define two types of structures for $\mathscr{B}$ and our main result yields a packing that behaves as we would expect it from an idealised typical random packing with respect to these structures. We say $(W, Y_1, \ldots, Y_k)$ is an ℓ-*set tester* for $\mathscr{B}$ if $k \leq \ell$ and there exist $i \in [r]$ and distinct $H_1, \ldots, H_k \in \mathcal{H}$ such that $W \subseteq V_i$ and $Y_j \subseteq X_i^{H_j}$ for all $j \in [k]$. We say (v, ω) is an ℓ-*vertex tester* for $\mathscr{B}$ if $v \in V_i$ and $\omega : \bigcup_{H \in \mathcal{H}} X_i^H \to [0, \ell]$ for some $i \in [r]$. For a weight function ω on a finite set X, we define $\omega(X') := \sum_{x \in X'} \omega(x)$ for any $X' \subseteq X$. The following theorem is our main result.

Theorem 4.2 (Ehard, Joos [32]). *For all $\alpha \in (0, 1]$ and $r \geq 2$, there exist $\varepsilon = \varepsilon(\alpha) > 0$ and $n_0 = n_0(\alpha, r)$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V}, \phi_0)$ is an (ε, d)-super-regular, α^{-1}-bounded and (ε, α)-linked extended blow-up instance, $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$, $|\mathcal{H}| \leq \alpha^{-1}n$, and $\sum_{H \in \mathcal{H}} e_H(X_i^H, X_j^H) \leq (1 - \alpha)dn^2$ for all $ij \in E(R)$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ into G which extends ϕ_0 such that*

(i) *$\phi(X_i^H) = V_i$ for all $i \in [r]_0$ and $H \in \mathcal{H}$;*

(ii) *$|W \cap \bigcap_{j \in [\ell]} \phi(Y_j)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \alpha n$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$;*

(iii) *$\omega(\bigcup_{H \in \mathcal{H}} X_i^H \cap \phi^{-1}(v)) = \omega(\bigcup_{H \in \mathcal{H}} X_i^H)/n \pm \alpha n$ for all $(v, \omega) \in \mathcal{W}_{ver}$.*

4.1.3 Applications

The multipartite framework can be used to obtain results for the non-partite setting. The next theorem applies to graphs G that are (ε, d)-quasirandom; that is, if n is the order of G, then $|N_G(u)| = (d \pm \varepsilon)n$ and $|N_G(u) \cap N_G(v)| = (d^2 \pm \varepsilon)n$ for all distinct $u, v \in V(G)$. In fact, our result extends Theorem 1.4 by including the following test structures to control certain quantities of the packing. Given G and a collection of graphs $\mathcal{H}$ on at most n vertices, we say $(W, Y_1, \ldots, Y_k)$ is an ℓ-*set tester* if $k \leq \ell$ and there exist distinct $H_1, \ldots, H_k \in \mathcal{H}$ such that $W \subseteq V(G)$ and $Y_i \subseteq V(H_i)$ for all $i \in [k]$. We say (v, ω) is an ℓ-*vertex tester* if $v \in V(G)$ and $\omega : \bigcup_{H \in \mathcal{H}} V(H) \to [0, \ell]$.

Theorem 4.3 (Ehard, Joos [32]). *For all $\alpha > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose G is an (ε, d)-quasirandom graph on n vertices and $\mathcal{H}$ is a collection of graphs on at most n vertices with $|\mathcal{H}| \leq \alpha^{-1}n$ and $\sum_{H\in\mathcal{H}} e(H) \leq (1-\alpha)e(G)$ as well as $\Delta(H) \leq \alpha^{-1}$ for all $H \in \mathcal{H}$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ into G such that*

- $|W \cap \bigcap_{i\in[\ell]} \phi(Y_i)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \alpha n$ *for all* $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$;
- $\omega(\bigcup_{H\in\mathcal{H}} V(H)) \cap \phi^{-1}(v)) = \omega(\bigcup_{H\in\mathcal{H}} V(H))/n \pm \alpha n$ *for all* $(v, \omega) \in \mathcal{W}_{ver}$.

In many scenarios when one applies approximate decomposition results, as for example Theorem 4.3, it is important that the graph $G - \phi(\mathcal{H})$ has 'small' maximum degree. Here, this can be easily achieved by utilising vertex testers (v, ω) where ω assigns to all $x \in \bigcup_{H\in\mathcal{H}} V(H)$ the degree of x. We remark that set and vertex testers in our main result are very flexible and capture many desirable properties. For example, Theorem 4.3 implies the approximate decomposition result due to Allen, Böttcher, Clemens and Taraz [5] when restricted to graphs of bounded maximum degree.

We give an example how to apply Theorem 4.3. By exploiting set and vertex testers, we can combine the approximate decomposition result of Theorem 4.3 with Keevash's results on hypergraph decompositions to perfectly decompose pseudorandom graphs into regular spanning graphs as long as only a few graphs contain a few vertices in components of bounded size. This is stronger as some results in [51] where a few graphs with almost all vertices in components of bounded size are required.[2] For this result, we need the stronger pseudorandom notion of typicality as also used by Keevash in [72]. We say a graph G on n vertices is (ε, s, d)-*typical* if $|\bigcap_{u\in U} N_G(u)| = (1\pm\varepsilon)d^{|U|}n$ for all sets $U \subseteq V(G)$ with $|U| \leq s$.

Theorem 4.4 (Ehard, Joos [32]). *For all $\alpha > 0$, there exist $\varepsilon > 0$ and $s, n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose G is a regular (ε, s, d)-typical graph on n vertices and $\mathcal{H}$ is a collection of regular graphs on n vertices with $\sum_{H\in\mathcal{H}} e(H) = e(G)$ as well as $\Delta(H) \leq \alpha^{-1}$ for all $H \in \mathcal{H}$. Suppose there are at least αn graphs $H \in \mathcal{H}$ such that at least αn vertices in H belong to components of order at most α^{-1}. Then there is a decomposition of the edge set of G into $\mathcal{H}$.*

Theorem 4.4 makes progress on a conjecture by Glock, Joos, Kim, Kühn and Osthus who conjecture in [51] that K_n can be decomposed into any collection $\mathcal{H}$ of regular bounded degree graphs with $\sum_{H\in\mathcal{H}} e(H) = \binom{n}{2}$. Without Theorem 4.4, one can show that there is a decomposition of K_n into a collection of r-regular graphs $\mathcal{H}$ whenever at least εn graphs in $\mathcal{H}$ contain only components of size at most ε^{-1} (or Hamilton cycles). Theorem 4.4 implies that $\mathcal{H}$ has to contain only εn graphs with a very small proportion of the vertices in components of size at most ε^{-1}. We prove Theorem 4.4 in Section 4.6.

In Section 4.6 we also show how our proof methods give rise to a blow-up lemma for approximate decompositions for directed graphs, which we derive from a more general setting which was also addressed in [6].

4.2 Proof overview

Before we explain our approach, we briefly sketch the approach of Kim, Kühn, Osthus and Tyomkyn in [79]. Their first step is to stack several graphs $H \in \mathcal{H}$ together to a

[2] The results in [51] consider only 2-regular graphs. However, their proof for the part where they consider collections of graphs $\mathcal{H}$ that contain a few graphs with almost all vertices in components of bounded size carries over verbatim to r-regular graphs for any r if n is large in terms of r.

new graph $\widetilde{H}$ such that $\widetilde{H}[X_i^{\widetilde{H}}, X_j^{\widetilde{H}}]$ is essentially regular for all $ij \in E(R)$.[3] Let $\widetilde{\mathcal{H}}$ be the collection of these graphs $\widetilde{H}$. They prove that such graphs $\widetilde{H}$ can be embedded into G by a probabilistic algorithm in a very uniform way. For some $\gamma \ll \alpha$, they apply this algorithm to γn graphs in $\widetilde{\mathcal{H}}$ in turn. Observe that this may cause edge overlaps in G. Nevertheless, after embedding γn graphs, they remove all 'used' edges from G and repeat. At the end, they eliminate all edge overlaps by unembedding several vertices and complete the packing by utilising a thin edge slice put aside at the beginning.

Our approach is somewhat perpendicular to their approach. We proceed cluster by cluster and find a function ϕ_i which maps almost all vertices in $\bigcup_{H\in\mathcal{H}} X_i^H$ into V_i and which is consistent with our partial packing so far. Our 'Approximate Packing Lemma', stated in Section 4.4, performs one such step using an auxiliary hypergraph where we aim to find a large matching which is pseudorandom with respect to certain weight functions. At the end, we complete the packing by also using a thin edge slice similar to [79]. At the beginning, we partition the clusters of our blow-up instance into many smaller clusters with the only purpose to ensure that $H[X_i^H, X_j^H]$ is a matching (see Section 4.3.2). This preprocessing is comparably simple and first used in [112].

Both the approach in [79] and ours draw on ideas from an alternative proof of the blow-up lemma by Rödl and Ruciński [112]. In spirit, our approach is again closer to the procedure in [112] as they also embed the clusters of H in turn. Many generalizations of the original blow-up lemma build on this alternative proof. We hope that our alternative proof of the blow-up lemma for approximate decompositions paths the way for further developments in the field. In fact, in Chapter 5 we extend our proof methods of the blow-up lemma for approximate decompositions to the setting of decomposing quasirandom hypergraphs.

Some ideas for our proof are taken from the approach of our rainbow blow-up lemma in Chapter 3. In particular, we also employ our main result on pseudorandom hypergraph matchings (Theorem 2.3) from Chapter 2.

Clearly, our proof cannot avoid a certain level of technicalities simply because the statement itself is already somewhat complex. However, we believe that the proof is substantially less complex and technical than the original proof in [79], as well as proofs of related results in the area of graph decompositions and embeddings, as for example [5, 6, 7, 16, 21, 65, 70, 92].

4.3 Preliminaries

4.3.1 The usual blow-up lemma

At the end of our packing algorithm we apply the following version of the blow-up lemma due to Komlós, Sarközy, and Szemerédi.

Theorem 4.5 (Komlós, Sarközy, and Szemerédi [82])**.** *Suppose $1/n \ll \varepsilon \ll 1/\Delta, d$ and $1/n \ll 1/r$. Suppose $(H, G, R, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε, d')-super-regular and Δ-bounded blow-up instance, with $d' \geq d$ as well as $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$ and $(A_i)_{i\in[r]}$ is a collection of graphs such that A_i is bipartite with vertex partition (X_i, V_i) and (ε, d_i)-super-regular for some $d_i \geq d$. Then there is a packing ϕ of H into G such that $\phi(x) \in N_{A_i}(x)$ for all $x \in X_i$ and $i \in [r]$.*

[3] In fact, their main theorem only applies to collections of graphs that are essentially regular and this stacking had to be performed again in [21] and [65] which made the application in both cases technically involved.

4.3.2 Refining partitions

Here, we provide a useful result to refine the vertex partition of a blow-up instance such that every $H \in \mathcal{H}$ only induces a matching between its refined partition classes. While in [112] this procedure was easily obtained by applying the classical Hajnal–Szemerédi Theorem, we perform a random procedure to obtain more control on the mass distribution of a weight function with respect to the refined partition.

The following lemma is stated such that we can also apply it conveniently for refining the vertex set in the non-multipartite setting for the proof of Theorem 4.3, that is, when $r = 1$, which is the reason for the asymmetry in the statements (iv) and (v).

Lemma 4.6 ([32])**.** *Suppose $1/n \ll \beta \ll \alpha$ and $1/n \ll 1/r$. Suppose $\mathcal{H}$ is a collection of at most $\alpha^{-1}n$ graphs, $(X_i^H)_{i\in[r]}$ is a vertex partition of H, and $\Delta(H) \leq \alpha^{-1}$ for every $H \in \mathcal{H}$. Suppose $n/2 \leq |X_i^H| = |X_i^{H'}| \leq 2n$ for all $H, H' \in \mathcal{H}$ and $i \in [r]$. Suppose $\mathcal{W}$ is a set of weight functions $\omega\colon \bigcup_{H\in\mathcal{H},i\in[r]} X_i^H \to [0, \alpha^{-1}]$ with $|\mathcal{W}| \leq \mathrm{e}^{\sqrt{n}}$. Then for all $H \in \mathcal{H}$ and $i \in [r]$, there exists a partition $(X_{i,j}^H)_{j\in[\beta^{-1}]}$ of X_i^H such that for all $H \in \mathcal{H}, \omega \in \mathcal{W}, i, i' \in [r], j, j' \in [\beta^{-1}]$ where $i \neq i'$ or $j \neq j'$, we have that*

- (i) *$X_{i,j}^H$ is independent in H^2;*
- (ii) *$|X_{i,1}^H| \leq \ldots \leq |X_{i,\beta^{-1}}^H| \leq |X_{i,1}^H| + 1$;*
- (iii) *$\omega(X_{i,j}^H) = \beta\omega(X_i^H) \pm \beta^{3/2}n$;*
- (iv) *$\sum_{H\in\mathcal{H}} e_H(X_{i,j}^H, X_{i',j'}^H) = \beta^2 \sum_{H\in\mathcal{H}} e_H(X_i^H, X_{i'}^H) \pm n^{5/3}$ if $i \neq i'$;*
- (v) *$\sum_{H\in\mathcal{H}} e_H(X_{i,j}^H, X_{i,j'}^H) = \binom{\beta^{-1}}{2}^{-1} \sum_{H\in\mathcal{H}} e(H[X_i^H]) \pm n^{5/3}$ if $i = i'$.*

We omit a detailed proof in this thesis and instead refer to [32]. In fact, we will prove a very similar statement in Chapter 5 for hypergraphs (Lemma 5.9). Let us only give an idea of the proof strategy. We first consider every $H \in \mathcal{H}$ in turn and construct a partition that essentially satisfies (i)–(iii) by simply partitioning each cluster randomly into β^{-1} clusters. Then we perform a vertex swapping procedure to resolve some conflicts and obtain a partitioning that precisely satisfies (i)–(iii). In the end, we randomly permute the ordering of these partitions for each $H \in \mathcal{H}$, $i \in [r]$ to also ensure (iv) (respectively (v)).

4.4 Approximate packings

The goal of this section is to provide an 'Approximate Packing Lemma' (Lemma 4.9). Given a blow-up instance $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$, it allows us to embed almost all vertices of $\bigcup_{H\in\mathcal{H}} X_i^H$ into V_i, while maintaining crucial properties for future embedding rounds of other clusters. To describe this setup we define a *packing instance* and collect some more notation.

4.4.1 Packing instances

Given a graph G and a set $\mathcal{E}$, we call $\psi\colon E(G) \to 2^{\mathcal{E}}$ an *edge set labelling* of G. A label $\alpha \in \mathcal{E}$ *appears* on an edge e if $\alpha \in \psi(e)$. We define the *maximum degree $\Delta_\psi(G)$ of ψ* as the maximum number of edges of G on which any fixed label appears. We define the *maximum codegree $\Delta_\psi^c(G)$ of ψ* as the maximum number of edges of G on which any two fixed labels appear together.

Let $r \in \mathbb{N}_0$. We say $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ is a *packing-instance of size r* if

- $\mathcal{H}$ is a collection of graphs, and G and R are graphs, where $V(R) = [r]_0$;
- $\mathcal{A} = \bigcup_{H\in\mathcal{H},i\in[r]_0} A_i^H$ is a union of balanced bipartite graphs A_i^H with vertex partition (X_i^H, V_i);
- $(X_i^H)_{i\in[r]_0}$ is a partition of H into independent sets for every $H \in \mathcal{H}$, and $(V_i)_{i\in[r]_0}$ is a partition of $V(G)$;
- $R = R_A \cup R_B$ is the union of two edge-disjoint graphs with $N_R(0) = [r]$;
- for all $H \in \mathcal{H}$, the graph $H[X_i^H, X_j^H]$ is a matching if $ij \in E(R)$ and empty otherwise;
- $\psi\colon E(\mathcal{A}) \to 2^{\mathcal{E}}$ is an edge set labelling such that $\Delta_\psi(A_i^H) \le 1$ for all $H \in \mathcal{H}, i \in [r]_0$, and for every label $\alpha \in \mathcal{E}$ with $\alpha \in \psi(xv) \cap \psi(x'v')$ and $v, v' \in V(G)$, we have $v = v'$.

In such a case, we write for simplicity $\mathscr{X}_i := \bigcup_{H\in\mathcal{H}} X_i^H$ and $\mathscr{A}_i := \bigcup_{H\in\mathcal{H}} A_i^H$ for each $i \in [r]_0$, and whenever we write $xv \in E(\mathscr{A}_i)$, we tacitly assume that $x \in \mathscr{X}_i, v \in V_i$. The only reason why R is the disjoint union of two graphs lies in the nature of our approach; while R_A represents parts of R as in the statement of our main result (Lemma 4.10, which is very similar to Theorem 4.2), the edges of R_B represent copies of edge slices of G that in the end will be used to complete the approximate packing. We use copies here to obtain a unified setup for the Approximate Packing Lemma, alternatively, we could have used parallel edges in the reduced graph.

The aim of this section is to map almost all vertices of $\mathscr{X}_0$ into V_0 by defining a function $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$ (that is, $x\sigma(x) \in E(\mathscr{A}_0)$) where $\mathscr{X}_0^\sigma \subseteq \mathscr{X}_0$. (Hence, we refer to subgraphs of $\mathscr{A}_i$ as candidacy graphs.) For convenience, we identify such a function σ with its *corresponding edge set* M defined as $M = M(\sigma) := \{xv : x \in \mathscr{X}_0^\sigma, v \in V_0, \sigma(x) = v\}$. We say

$$\text{(4.4.1)} \qquad \begin{array}{l}\sigma\colon \mathscr{X}_0^\sigma \to V_0 \textit{ is a } \text{conflict-free packing} \textit{ if } \sigma|_{\mathscr{X}_0^\sigma\cap X_0^H} \textit{ is injective for all}\\ H\in\mathcal{H} \textit{ and } \psi(e)\cap\psi(f)=\emptyset \textit{ for all distinct } e,f\in M(\sigma).\end{array}$$

The set $\psi(xv)$ will encode the set of edges of G that are used for the embedding when mapping x to v. The property that $\psi(e) \cap \psi(f) = \emptyset$ for all distinct $e, f \in M(\sigma)$ will guarantee that in the proof of our main result (Lemma 4.10) every edge in G is used at most once.

Given a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$, we update the remaining candidacy graphs and their edge set labelling according to the following two definitions. For an illustration, see Figure 4.1.

Definition 4.7 (Updated candidacy graphs)**.** *For a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$ and all $H \in \mathcal{H}, i \in [r]$, let $A_i^H[\sigma]$ be the* updated candidacy graph (with respect to σ) *which is defined by the spanning subgraph of A_i^H that contains precisely those edges $xv \in E(A_i^H)$ for which the following holds: if x has an H-neighbour $x_0 \in \mathscr{X}_0^\sigma$ (which would be unique), then $\sigma(x_0)v \in E(G[V_0, V_i])$.*

Definition 4.8 (Updated labelling)**.** *For a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$, let $\psi[\sigma]$ be the* updated edge set labelling (with respect to σ) *defined as follows: for all $H \in \mathcal{H}, i \in [r]$ and $xv \in E(A_i^H[\sigma])$, if x has an H-neighbour $x_0 \in \mathscr{X}_0^\sigma$, then set $\psi[\sigma](xv) := \psi(xv) \cup \{\sigma(x_0), v\}$, and otherwise set $\psi[\sigma](xv) := \psi(xv)$.*

In order to be able to analyse our packing process in Section 4.5, we carefully maintain quasirandom properties of the candidacy graphs throughout the procedure. To this end, we refer to a packing instance $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ of size r as an $(\varepsilon, \mathbf{d})$*-packing-instance*, where $\mathbf{d} = (d_A, d_B, d_0, \dots, d_r)$, if

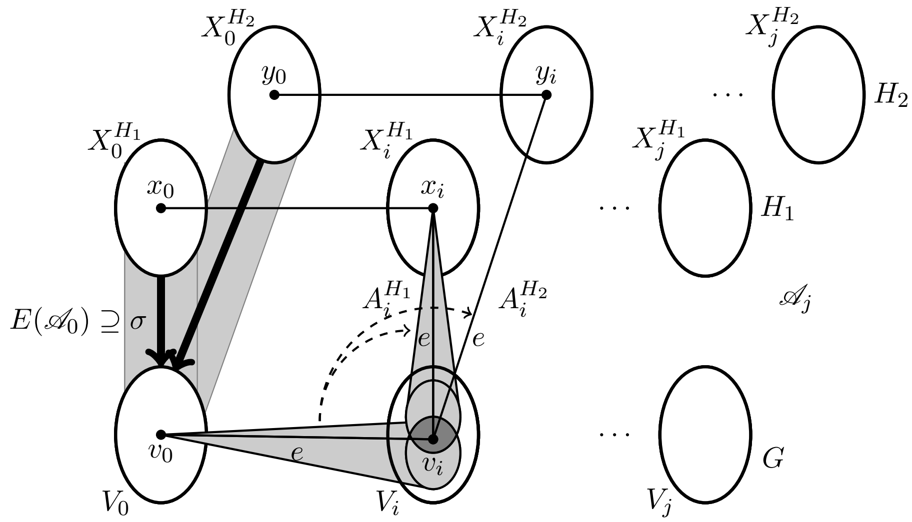


Figure 4.1: We have $\mathcal{H} = \{H_1, H_2\}$ and want to find a conflict-free packing σ in $E(\mathscr{A}_0)$. If σ maps x_0 onto v_0, then only those candidates of x_i remain that are also neighbours of v_0. If σ maps x_0 and y_0 onto v_0, then we add the label e to the edge set label $\psi(x_i v_i)$ and $\psi(y_i v_i)$. This captures the information that if x_i or y_i are mapped onto v_i in $\mathscr{A}_i$, then this embedding uses the edge e.

(P1) $G[V_i, V_j]$ is (ε, d_Z)-super-regular for all $ij \in E(R_Z), Z \in \{A, B\}$;

(P2) A_i^H is (ε, d_i)-super-regular for all $H \in \mathcal{H}, i \in [r]_0$;

(P3) $e_H(N_{A_i^H}(v_i), N_{A_j^H}(v_j)) = (d_i d_j \pm \varepsilon) e_H(X_i^H, X_j^H)$ for all $H \in \mathcal{H}, ij \in E(R_A), v_i v_j \in E(G[V_i, V_j])$;

(P4) $\Delta_\psi(\mathscr{A}_i) \leq (1+\varepsilon) d_i |V_i|$ for all $i \in [r]_0$.

Property (P4) ensures that no edge is a potential candidate for too many graphs in $\mathcal{H}$ and (P3) enables us to maintain this property for future embedding rounds (see Lemma 4.9(IV)$_{\text{L4.9}}$ below). Let $\mathscr{P} = (\mathcal{H}, G, R, \mathcal{A}, \psi)$ be an $(\varepsilon, \mathbf{d})$-packing-instance of size r. Similarly as for a blow-up instance, we say $(W, Y_1, \ldots, Y_k)$ is an *ℓ-set tester* for $\mathscr{P}$ if $k \leq \ell$ and there exist distinct $H_1, \ldots, H_k \in \mathcal{H}$ such that $W \subseteq V_0$ and $Y_j \subseteq X_0^{H_j}$ for all $j \in [k]$. For $i \in N_{R_A}[0]$ and $v \in V_i$, we say $\omega\colon E(\mathscr{A}_i) \to [0, \ell]$ is an *ℓ-edge tester with centre v* for $\mathscr{P}$ if $\omega(x'v') = 0$ for all $x'v' \in E(\mathscr{A}_i)$ with $v' \in V_i, v' \neq v$. We say $\omega\colon E(\mathscr{A}_0) \to [0, \ell]$ is an *ℓ-edge tester with centres in $\mathscr{X}_0$* if there exist vertices $\{x_H\}_{H \in \mathcal{H}}$ with $x_H \in X_0^H$ for each $H \in \mathcal{H}$ such that $\omega(x'v') = 0$ for all $x'v' \in E(\mathscr{A}_0)$ with $x' \notin \{x_H\}_{H \in \mathcal{H}}$. Further, let $\dim(\omega)$ be the *dimension of* ω defined as

$$\dim(\omega) = \begin{cases} 1 & \text{if } \omega(E(\mathscr{A}_i)) = \omega(E(A_i^H)) \text{ for some } H \in \mathcal{H}, \\ 2 & \text{otherwise.} \end{cases} \tag{4.4.2}$$

Moreover, for every $H \in \mathcal{H}$, let H_+ be an auxiliary supergraph of H that is obtained by adding a maximal number of edges between X_0^H and X_i^H for every $i \in [r]$ subject to $H_+[X_0^H, X_i^H]$ being a matching. We call $\mathcal{H}_+ := \bigcup_{H \in \mathcal{H}} H_+$ an *enlarged graph of $\mathcal{H}$*. We say that $\mathscr{P}$ is *nice (with respect to $\mathcal{H}_+$)* if

(N1) $|N_{A_i^H}(x_i) \cap N_G(v_j)| = (d_i d_Z \pm \varepsilon)|V_i|$ for all $x_j v_j \in E(A_j^H)$ whenever $\{x_i\} = N_{H_+}(x_j) \cap X_i^H$, $H \in \mathcal{H}, ij \in E(R_Z), Z \in \{A, B\}$;

(N2) $|N_G(v_i, v_j) \cap V_0| = (d_A^2 \pm \varepsilon)|V_0|$ for all $ij \in E(R_A - \{0\})$ and $v_i v_j \in E(G[V_i, V_j])$.

Using standard regularity methods (see Facts 1.11 and 1.12), it is straightforward to verify the following:

(4.4.3) *For every $(\varepsilon, \mathbf{d})$-packing-instance $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ of size r and every enlarged graph $\mathcal{H}_+$ of $\mathcal{H}$, there exist spanning subgraphs $G' \subseteq G$ and $\mathcal{A}' \subseteq \mathcal{A}$ such that $(\mathcal{H}, G', R, \mathcal{A}', \psi)$ is a nice $(\varepsilon', \mathbf{d})$-packing-instance of size r with respect to $\mathcal{H}_+$ for some ε' with $\varepsilon \ll \varepsilon' \ll 1/r$.*

4.4.2 Approximate Packing Lemma

We now state our Approximate Packing Lemma. Roughly speaking it states that given a packing instance, we can find a conflict-free packing such that the updated candidacy graphs are still super-regular, albeit with a smaller density. Moreover, with respect to certain weight functions on the candidacy graphs, the updated candidacy graphs behave as we would expect this by a random and independent deletion of the edges.

Lemma 4.9 ([32] – Approximate Packing Lemma)**.** *Let $1/n \ll \varepsilon \ll \varepsilon' \ll \mathbf{d}, 1/r, 1/s$. Suppose $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ is an $(\varepsilon, \mathbf{d})$-packing-instance of size r, $\|\psi\| \leq s$, $|\mathcal{H}| \leq sn$, $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]_0$, $\sum_{H \in \mathcal{H}} e_H(X_0^H, X_i^H) \leq d_A n^2$ for all $i \in N_{R_A}(0)$, and $e_H(X_i^H, X_j^H) \geq \varepsilon'^2 n$ for all $H \in \mathcal{H}, ij \in E(R)$. Suppose $\Delta^c_\psi(\mathscr{A}_i) \leq \sqrt{n}$ for all $i \in N_{R_A}[0]$, and suppose $\mathcal{W}_{set}, \mathcal{W}_{edge}$ are sets of s-set testers and s-edge testers of size at most $n^{3\log n}$, respectively.*

Then there is a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$ such that for all $H \in \mathcal{H}$, we have $|\mathscr{X}_0^\sigma \cap X_0^H| \geq (1-\varepsilon')n$ and for all $i \in [r]$ there exists a spanning subgraph $A_i^{H,new}$ of the updated candidacy graph $A_i^H[\sigma]$ (where $\mathscr{A}_i^{new} := \bigcup_{H \in \mathcal{H}} A_i^{H,new}$) with

(I)$_{\text{L4.9}}$ *$A_i^{H,new}$ is $(\varepsilon', d_i d_Z)$-super-regular for all $i \in N_{R_Z}(0), Z \in \{A, B\}$;*

(II)$_{\text{L4.9}}$ *$e_H(N_{A_i^{H,new}}(v_i), N_{A_j^{H,new}}(v_j)) = (d_i d_j d_A^2 \pm \varepsilon')e_H(X_i^H, X_j^H)$ for all $ij \in E(R_A - \{0\})$ and $v_i v_j \in E(G[V_i, V_j])$;*

(III)$_{\text{L4.9}}$ *$\omega(E(\mathscr{A}_i^{new})) = (1 \pm \varepsilon'^2) d_A \omega(E(\mathscr{A}_i)) \pm \varepsilon'^2 n^{\dim(\omega)}$ for all $\omega \in \mathcal{W}_{edge}$ with centre in V_i, $i \in N_{R_A}(0)$;*

(IV)$_{\text{L4.9}}$ *$\Delta_{\psi[\sigma]}(\mathscr{A}_i^{new}) \leq (1+\varepsilon') d_i d_A |V_i|$ for all $i \in N_{R_A}(0)$;*

(V)$_{\text{L4.9}}$ *$\Delta^c_{\psi[\sigma]}(\mathscr{A}_i^{new}) \leq \sqrt{n}$ for all $i \in N_{R_A}(0)$;*

(VI)$_{\text{L4.9}}$ *$|W \cap \bigcap_{j \in [\ell]} \sigma(Y_j \cap \mathscr{X}_0^\sigma)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \varepsilon' n$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$;*

(VII)$_{\text{L4.9}}$ *$\omega(M(\sigma)) = (1 \pm \varepsilon')\omega(E(\mathscr{A}_0))/d_0 n \pm \varepsilon' n$ for all $\omega \in \mathcal{W}_{edge}$ with centre in V_0 or centres in $\mathscr{X}_0$.*

Properties (I)$_{\text{L4.9}}$, (II)$_{\text{L4.9}}$ and (IV)$_{\text{L4.9}}$ ensure that (P2)–(P4) are also satisfied for the updated candidacy graphs $\mathscr{A}_i^{new}$, respectively, and (V)$_{\text{L4.9}}$ ensures that the codegree of the updated labelling $\psi[\sigma]$ is still small on $\mathscr{A}_i^{new}$. Property (III)$_{\text{L4.9}}$ states that the weight of the edge testers on the updated candidacy graphs $\mathscr{A}_i^{new}$ is what we would expect by a random sparsification of the edges in $\mathscr{A}_i$, and (VI)$_{\text{L4.9}}$ and (VII)$_{\text{L4.9}}$ guarantee that σ behaves like a random packing with respect to the set and edge testers.

We split the proof into two steps. In Step 1, we construct an auxiliary hypergraph and apply Theorem 2.3 to obtain the required conflict-free packing σ. By defining suitable weight functions in Step 2, we employ the conclusions of Theorem 2.3 to establish (I)$_{\text{L4.9}}$–(VII)$_{\text{L4.9}}$.

Proof. Let $\mathcal{H}_+$ be an enlarged graph of $\mathcal{H}$, and for $i \in [r]$, let $\mathscr{A}_i^{bad}$ and $\mathscr{A}_i^{good}$ be spanning subgraphs of $\mathscr{A}_i$ such that $\mathscr{A}_i^{bad}$ contains precisely those edges $xv \in E(\mathscr{A}_i)$

where $N_{\mathcal{H}_+}(x) \cap \mathscr{X}_0 = \emptyset$, and $E(\mathscr{A}_i^{good}) := E(\mathscr{A}_i) \setminus E(\mathscr{A}_i^{bad})$. We may assume that $|\mathcal{H}| = sn$ and $\sum_{H\in\mathcal{H}} e_H(X_0^H, X_i^H) \leq (d_A + \varepsilon'^{3/2})n^2$ for all $i \in N_{R_A}(0)$, where the last inequality will be only used in (4.4.31). (Otherwise we artificially add some graphs to $\mathcal{H}$ subject to the condition that still $e_H(X_i^H, X_j^H) \geq \varepsilon'^2 n$ for all $H \in \mathcal{H}, ij \in E(R)$, and accordingly we add some graphs to $\mathcal{A}$ satisfying (P1)–(P4).) We may also assume that $\psi\colon E(\mathcal{A}) \to 2^{\mathcal{E}}$ is such that $|\psi(e)| = s$ for all $e \in E(\mathscr{A}_0)$ (otherwise we add artificial labels that we delete at the end again), and $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ is a nice $(\varepsilon, \mathbf{d})$-packing-instance with respect to $\mathcal{H}_+$ (otherwise we may employ (4.4.3) and replace ε by some $\tilde{\varepsilon}$, where $\varepsilon \ll \tilde{\varepsilon} \ll \varepsilon'$; observe also that this does not cause problems with the weight of the edge testers in $(\mathrm{III})_{\mathrm{L4.9}}$ and $(\mathrm{VII})_{\mathrm{L4.9}}$, as the operation in (4.4.3) only deletes few edges of $\mathcal{A}$ incident to every vertex).

Step 1. Constructing an auxiliary hypergraph

We want to use Theorem 2.3 to find the required conflict-free packing σ in $\mathscr{A}_0$. To this end, let $(V_0^H)_{H\in\mathcal{H}}$ be disjoint copies of V_0, and for $H \in \mathcal{H}$ and $e = x_0v_0 \in E(A_0^H)$, let $e^H := x_0v_0^H$ where v_0^H is the copy of v_0 in V_0^H. Let $f_e := e^H \cup \psi(e)$ for each $e \in E(A_0^H), H \in \mathcal{H}$ and let $\mathcal{H}^{aux}$ be the $(s+2)$-uniform hypergraph with vertex set $\bigcup_{H\in\mathcal{H}}(X_0^H \cup V_0^H) \cup \mathcal{E}$ and edge set $\{f_e \colon e \in E(\mathscr{A}_0)\}$. A key property of the construction of $\mathcal{H}^{aux}$ is a bijection between conflict-free packings σ in $\mathscr{A}_0$ and matchings $\mathcal{M}$ in $\mathcal{H}^{aux}$ by assigning σ to $\mathcal{M} = \{f_e \colon e \in M(\sigma)\}$. (Recall that $M = M(\sigma)$ is the edge set corresponding to σ.)

It is easy to estimate $\Delta(\mathcal{H}^{aux})$ and $\Delta^c(\mathcal{H}^{aux})$ in order to apply Theorem 2.3. Since A_0^H is (ε, d_0)-super-regular for each $H \in \mathcal{H}$, $|X_0^H| = |V_0| = (1 \pm \varepsilon)n$, and $\Delta_\psi(\mathscr{A}_0) \leq (1+\varepsilon)d_0|V_0|$, we conclude that

$$\Delta(\mathcal{H}^{aux}) \leq (d_0 + 3\varepsilon)n =: \Delta. \tag{4.4.4}$$

Note that the codegree in $\mathcal{H}^{aux}$ of two vertices in $\bigcup_{H\in\mathcal{H}}(X_0^H \cup V_0^H)$ is at most 1, and similarly, the codegree in $\mathcal{H}^{aux}$ of a vertex in $\bigcup_{H\in\mathcal{H}}(X_0^H \cup V_0^H)$ and a label in $\mathcal{E}$ is at most 1 because $\Delta_\psi(A_0^H) \leq 1$ for all $H \in \mathcal{H}$. By assumption, $\Delta_\psi^c(\mathscr{A}_0) \leq \sqrt{n}$. Altogether, this implies that

$$\Delta^c(\mathcal{H}^{aux}) \leq \sqrt{n} \leq \Delta^{1-\varepsilon^2}. \tag{4.4.5}$$

Suppose $\mathcal{W} = \bigcup_{\ell\in[s]} \mathcal{W}_\ell$ is a set of size at most $n^{4\log n}$ of given weight functions $\omega \in \mathcal{W}_\ell$ for $\ell \in [s]$ with $\omega\colon \binom{E(\mathscr{A}_0)}{\ell} \to [0, s]$. Note that every weight function $\omega\colon \binom{E(\mathscr{A}_0)}{\ell} \to [0, s]$ naturally corresponds to a weight function $\omega_{\mathcal{H}^{aux}}\colon \binom{E(\mathcal{H}^{aux})}{\ell} \to [0, s]$ by defining $\omega_{\mathcal{H}^{aux}}(\{f_{e_1}, \ldots, f_{e_\ell}\}) := \omega(\{e_1, \ldots, e_\ell\})$. We will explicitly specify $\mathcal{W}$ in Step 2 and it is simple to check that for each $\omega \in \mathcal{W}$ the corresponding weight function $\omega_{\mathcal{H}^{aux}}$ will be clean. Our main idea is to find a hypergraph matching in $\mathcal{H}^{aux}$ that behaves like a typical random matching with respect to $\{\omega_{\mathcal{H}^{aux}} \colon \omega \in \mathcal{W}\}$ in order to establish $(\mathrm{I})_{\mathrm{L4.9}}$–$(\mathrm{VII})_{\mathrm{L4.9}}$.

Suppose $\ell \in [s]$ and $\omega \in \mathcal{W}_\ell$. If $\omega(E(\mathscr{A}_0)) \geq n^{1+\varepsilon/2}$ or $\ell \geq 2$, define $\tilde{\omega} := \omega$. Otherwise, choose $\tilde{\omega}\colon E(\mathscr{A}_0) \to [0, s]$ such that $\omega \leq \tilde{\omega}$ and $\tilde{\omega}(E(\mathscr{A}_0)) = n^{1+\varepsilon/2}$. By (4.4.4) and (4.4.5), we can apply Theorem 2.3 (with $(d_0 + 3\varepsilon)n, \varepsilon^2, s+2, s, \{\tilde{\omega}_{\mathcal{H}^{aux}} \colon \omega \in \mathcal{W}_\ell\}$ playing the roles of $\Delta, \delta, r, L, \mathcal{W}_\ell$) to obtain a matching $\mathcal{M}$ in $\mathcal{H}^{aux}$ that corresponds to a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathscr{A}_0$ with its corresponding edge set $M = M(\sigma)$

that satisfies the following properties (where $\hat{\varepsilon} := \varepsilon^{1/2}$):

(4.4.6)

$$\omega(M) = (1 \pm \hat{\varepsilon})\frac{\omega(E(\mathscr{A}_0))}{(d_0 n)^\ell} \text{ for } \omega \in \mathcal{W}_\ell, \ell \in [s] \text{ where } \omega(E(\mathscr{A}_0)) \geq \|\omega\|_k \Delta^{k+\varepsilon^2} \text{ for all } k \in [\ell];$$

(4.4.7)

$$\omega(M) \leq \max\left\{(1+\hat{\varepsilon})\frac{\omega(E(\mathscr{A}_0))}{d_0 n}, n^\varepsilon\right\} \text{ for all } \omega \in \mathcal{W}_1.$$

One way to exploit (4.4.6) is to control the number of edges in M between sufficiently large sets of vertices. To this end, for subsets $S \subseteq X_0^H$ and $T \subseteq V_0$ for some $H \in \mathcal{H}$ with $|S|, |T| \geq 2\varepsilon n$, we define a weight function $\omega_{S,T} \colon E(A_0^H) \to \{0,1\}$ with

$$\omega_{S,T}(e) := \mathbb{1}_{\{e \in E(A_0^H[S,T])\}}. \tag{4.4.8}$$

That is, $\omega_{S,T}(M)$ counts the number of edges in A_0^H between S and T that lie in M. Since A_0^H is (ε, d_0)-super-regular we have that $e_{A_0^H}(S,T) = (d_0 \pm \varepsilon)|S||T| \geq \varepsilon^3 n^2$ which implies together with (4.4.6) that whenever $\omega_{S,T} \in \mathcal{W}$, then σ is chosen such that

$$|\sigma(S \cap \mathscr{X}_0^\sigma) \cap T| = \omega_{S,T}(M) = (1 \pm 2\hat{\varepsilon})\frac{|S||T|}{n}. \tag{4.4.9}$$

<u>*Step 2.*</u> *Employing weight functions to conclude* (I)$_{\text{L4.9}}$–(VII)$_{\text{L4.9}}$

By Step 1, we may assume that (4.4.6) and (4.4.7) hold for a set of weight functions $\mathcal{W}$ that we will define during this step. We will show that for this choice of $\mathcal{W}$ the conflict-free packing $\sigma \colon \mathscr{X}_0^\sigma \to V_0$ as obtained in Step 1 satisfies (I)$_{\text{L4.9}}$–(VII)$_{\text{L4.9}}$. Similarly as in Definition 4.7 (here, $\mathcal{H}$ is replaced by $\mathcal{H}_+$), we define subgraphs $A_i^{H,*}$ of A_i^H as follows.

(4.4.10) *For all $H \in \mathcal{H}$, $i \in [r]$, let $A_i^{H,*}$ be the spanning subgraph of A_i^H containing precisely those edges $xv \in E(A_i^H)$ for which the following holds: if $\{x_0\} = N_{\mathcal{H}_+}(x) \cap \mathscr{X}_0^\sigma$, then $\sigma(x_0)v \in E(G[V_0, V_i])$.*

Observe that $A_i^{H,*}$ is a spanning subgraph of the updated candidacy graph $A_i^H[\sigma]$ as in Definition 4.7. By taking a suitable subgraph of $A_i^{H,*}$, we will in the end obtain the required candidacy graph $A_i^{H,new}$.

First, we show that $|\mathscr{X}_0^\sigma \cap X_0^H| \geq (1-3\hat{\varepsilon})n$ for each $H \in \mathcal{H}$. Adding $\omega_{X_0^H,V_0}$ as defined in (4.4.8) for every $H \in \mathcal{H}$ to $\mathcal{W}$ and using (4.4.9) yields

$$|\mathscr{X}_0^\sigma \cap X_0^H| = \omega_{X_0^H,V_0}(M) \geq (1-3\hat{\varepsilon})n. \tag{4.4.11}$$

<u>*Step 2.1.*</u> *Checking* (I)$_{\text{L4.9}}$

For all $H \in \mathcal{H}$ and $i \in N_{R_Z}(0), Z \in \{A,B\}$ we proceed as follows. Let $Y_i^H := N_{H_+}(\mathscr{X}_0^\sigma) \cap X_i^H$. We first show that $A_i^{H,*}[Y_i^H, V_i]$ is $(\hat{\varepsilon}^{1/18}, d_i d_Z)$-super-regular (see (4.4.15)). We do so by showing that every vertex in $Y_i^H \cup V_i$ has the appropriate degree, and that the common neighbourhood of most pairs of vertices in V_i have the correct size such that we can employ Theorem 1.13 to guarantee the super-regularity of $A_i^{H,*}[Y_i^H, V_i]$.

Note that $|Y_i^H| \geq |\mathscr{X}_0^\sigma \cap X_0^H| - 2\varepsilon n \geq (1-4\hat{\varepsilon})n$ by (4.4.11). For every vertex $x \in Y_i^H$ with $\{x_0\} = N_{H_+}(x) \cap \mathscr{X}_0^\sigma$, we have $\deg_{A_i^{H,*}}(x) = |N_{A_i^H}(x) \cap N_G(\sigma(x_0))|$.

Since our packing-instance is nice, (N1) implies that $\deg_{A_i^{H,*}}(x) = (d_i d_Z \pm \varepsilon)|V_i|$. For $v \in V_i$, let $N_v := N_{A_i^H}(v)$. Observe that

$$\deg_{A_i^{H,*}[Y_i^H,V_i]}(v) = |\sigma(N_{H_+}(N_v) \cap \mathscr{X}_0^\sigma) \cap N_G(v)|, \tag{4.4.12}$$

and $|N_{H_+}(N_v) \cap X_0^H| = |N_v| \pm 2\varepsilon n = (d_i \pm 5\varepsilon)n$, and $|N_G(v) \cap V_0| = (d_Z \pm 3\varepsilon)n$. Adding for every vertex $v \in V_i$, the weight function $\omega_{S,T}$ as defined in (4.4.8) for $S := N_{H_+}(N_v) \cap X_0^H$ and $T := N_G(v) \cap V_0$ to $\mathcal{W}$, we obtain by (4.4.9) and (4.4.12) that

$$\deg_{A_i^{H,*}[Y_i^H,V_i]}(v) = (1 \pm 2\hat{\varepsilon})|N_{H_+}(N_v) \cap X_0^H||N_G(v) \cap V_0|n^{-1} = (d_i d_Z \pm \hat{\varepsilon}^{1/2})|Y_i^H|. \tag{4.4.13}$$

Next, we use Theorem 1.13 to show that $A_i^{H,*}[Y_i^H, V_i]$ is $(\hat{\varepsilon}^{1/18}, d_i d_Z)$-super-regular. We call a pair of vertices $u, v \in V_i$ *good* if $|N_{A_i^H}(u,v)| = (d_i \pm \varepsilon)^2|X_i^H|$, and $|N_G(u,v) \cap V_0| = (d_Z \pm \varepsilon)^2|V_0|$. By the ε-regularity of A_i^H and $G[V_0, V_i]$, there are at most $2\varepsilon|V_i|^2$ pairs $u, v \in V_i$ which are not good.

For all good pairs $u, v \in V_i$, let $S_{u,v} := N_{H_+}(N_{A_i^H}(u,v)) \cap X_0^H$ and $T_{u,v} := N_G(u,v) \cap V_0$. We add the weight function $\omega_{S_{u,v},T_{u,v}}$ as defined in (4.4.8) to $\mathcal{W}$. Observe that $|S_{u,v}| = |N_{A_i^H}(u,v)| \pm 2\varepsilon n = (d_i \pm \varepsilon^{1/2})^2 n$ and $|T_{u,v}| = (d_Z \pm \varepsilon^{1/2})^2 n$. By (4.4.9), we obtain for all good pairs $u, v \in V_j$ that

$$\begin{aligned} |N_{A_i^{H,*}[Y_i^H,V_i]}(u,v)| &= |\sigma(S_{u,v} \cap \mathscr{X}_0^\sigma) \cap T_{u,v}| = (1 \pm 2\hat{\varepsilon})|S_{u,v}||T_{u,v}|n^{-1} \\ &\le (d_i d_Z + \hat{\varepsilon}^{1/3})^2 |Y_i^H|. \end{aligned} \tag{4.4.14}$$

Now, by (4.4.13) and (4.4.14), we can apply Theorem 1.13, and obtain that

$$A_i^{H,*}[Y_i^H, V_i] \text{ is } (\hat{\varepsilon}^{1/18}, d_i d_Z)\text{-super-regular.} \tag{4.4.15}$$

In order to complete the proof of $(\mathrm{I})_{\mathrm{L4.9}}$, we show that we can find a spanning subgraph $A_i^{H,new}$ of $A_i^{H,*}$ that is $(\varepsilon', d_i d_Z)$-super-regular. Let

$$E(A_i^{H,new}[Y_i^H, V_i]) := E(A_i^{H,*}[Y_i^H, V_i]). \tag{4.4.16}$$

For every vertex $x \in X_i^H \setminus Y_i^H$, we have that $\deg_{A_i^{H,*}}(x) = (d_i \pm \varepsilon)|V_i|$ because A_i^H is (ε, d_i)-super-regular. Suppose $\mathcal{W}^{bad}$ is a collection of at most $n^{4\log n}$ weight functions $\omega^{bad}\colon E(\mathscr{A}_i^{bad}) \to [0, s]$; we will specify $\mathcal{W}^{bad}$ explicitly when we establish $(\mathrm{III})_{\mathrm{L4.9}}$. We claim that we can delete for every vertex $x \in X_i^H \setminus Y_i^H$ some incident edges in $A_i^{H,*}$ and obtain a subgraph $A_i^{H,new}$ such that

$$\deg_{A_i^{H,new}}(x) = (d_i d_Z \pm 2\varepsilon)|V_i| \text{ for every } x \in X_i^H \setminus Y_i^H; \tag{4.4.17}$$

$$\omega^{bad}(E(\mathscr{A}_i^{new})) = (1 \pm \varepsilon) d_Z \omega^{bad}(E(\mathscr{A}_i^{bad})) \pm \varepsilon n \text{ for every } \omega^{bad} \in \mathcal{W}^{bad}. \tag{4.4.18}$$

This can be easily seen by a probabilistic argument: For all $H \in \mathcal{H}$ and $x \in X_i^H \setminus Y_i^H$, we keep each edge incident to x in A_i^H independently at random with probability d_Z. Then, McDiarmid's inequality (Theorem 1.8) together with a union bound yields that (4.4.17) and (4.4.18) hold simultaneously with probability at least, say, 1/2.

Since $|X_i^H| = (1 \pm \varepsilon)n$, we have that $|X_i^H \setminus Y_i^H| \le 4\hat{\varepsilon}n$ by (4.4.11). Hence, (4.4.15) implies together with (4.4.17) that $A_i^{H,new}$ is $(\varepsilon', d_i d_Z)$-super-regular, which establishes $(\mathrm{I})_{\mathrm{L4.9}}$.

Step 2.2. *Checking* $(\mathrm{II})_{\mathrm{L4.9}}$

For all $H \in \mathcal{H}, ij \in E(R_A - \{0\})$, and $v_iv_j \in E(G[V_i, V_j])$ we proceed as follows. Let $\widetilde{E} := E(H[N_{A_i^H}(v_i), N_{A_j^H}(v_j)])$ and

$$\begin{aligned}
S &:= \{\{x_i', x_j'\} \subseteq X_0^H : x_ix_i', x_jx_j' \in E(H_+), x_ix_j \in \widetilde{E}\},\\
S_1 &:= \{S' \in S : |S'| = 1\}, \quad \text{and} \quad S_2 := \{S' \in S : |S'| = 2\},\\
E_1 &:= \{xv \in E(A_0^H) : x \in S_1, v \in N_G(v_i, v_j)\},\\
E_2 &:= \{\{xv, x'v'\} \in \tbinom{E(A_0^H)}{2} : \{x, x'\} \in S_2, v \in N_G(v_i), v' \in N_G(v_j), v \neq v'\}.
\end{aligned}$$

By assumption (see (P3)), we have that $|\widetilde{E}| = (d_id_j \pm \varepsilon)e_H(X_i^H, X_j^H)$. Since $e_H(X_i^H, X_j^H) \geq \varepsilon'^2 n$, we conclude that

$$|S| = |S_1| + |S_2| = (d_id_j \pm \varepsilon)e_H(X_i^H, X_j^H) \pm 4\varepsilon n = (d_id_j \pm \hat{\varepsilon})e_H(X_i^H, X_j^H). \tag{4.4.19}$$

Note that the term of '$\pm 4\varepsilon n$' in (4.4.19) accounts for possible vertices $x_i \in N_{A_i^H}(v_i)$ and $x_j \in N_{A_j^H}(v_j)$ that do not have an H_+-neighbour in X_0^H.

We define the following weight functions $\omega_1 : E(A_0^H) \to \{0, 1\}$ and $\omega_2 : \binom{E(A_0^H)}{2} \to \{0, 1\}$ by setting $\omega_1(e) := \mathbb{1}_{\{e \in E_1\}}$ and $\omega_2(\{e_1, e_2\}) := \mathbb{1}_{\{\{e_1,e_2\} \in E_2\}}$ and add them to $\mathcal{W}$. By the definition of $A_i^{H,new}$ (recall (4.4.10) and (4.4.16)), we crucially observe that

$$e_H(N_{A_i^{H,new}}(v_i), N_{A_j^{H,new}}(v_j)) = \omega_1(M) + \omega_2(M) \pm 5\hat{\varepsilon}n. \tag{4.4.20}$$

Note that the term of '$\pm 5\hat{\varepsilon}n$' in (4.4.20) accounts for possible vertices $x_i \in N_{A_i^H}(v_i)$ and $x_j \in N_{A_j^H}(v_j)$ that do not have an H_+-neighbour in X_0^H (at most $4\varepsilon n$), and possible vertices in S that are left unembedded (at most $4\hat{\varepsilon}n$ by (4.4.11)).

Let us for the moment assume that $|S_1|, |S_2| \geq \varepsilon'^5 n$ (otherwise the claimed estimations in (4.4.23) and (4.4.24) below are trivially true). Since A_0^H is (ε, d_0)-super-regular and $|N_G(v_i, v_j) \cap V_0| = (d_A^2 \pm 3\varepsilon)n$ by (N2), we obtain that

$$\omega_1(E(A_0^H)) = |E_1| = (d_0 \pm \varepsilon)|S_1||N_G(v_i, v_j) \cap V_0| = (d_0d_A^2 \pm \hat{\varepsilon})|S_1|n. \tag{4.4.21}$$

By Fact 1.11, all but at most $6\varepsilon n$ elements $\{x_i', x_j'\} \in S_2$ are such that x_k' has $(d_0 \pm \varepsilon)|N_G(v_k) \cap V_0|$ neighbours in $N_G(v_k) \cap V_0$ for both $k \in \{i, j\}$ because A_0^H is (ε, d_0)-super-regular and $G[V_0, V_k]$ is (ε, d_A)-super-regular. Each of these $6\varepsilon n$ exceptional elements contributes at most $|N_G(v_i) \cap V_0||N_G(v_j) \cap V_0| \leq 3n^2$ to $\omega_2(E(A_0^H))$. This implies that

$$\begin{aligned}
\omega_2(E(A_0^H)) = |E_2| &= (d_0 \pm 2\varepsilon)^2(|S_2| \pm 6\varepsilon n)|N_G(v_i) \cap V_0||N_G(v_j) \cap V_0| \pm 18\varepsilon n^3\\
&= (d_0^2d_A^2 \pm \hat{\varepsilon})|S_2|n^2. \qquad (4.4.22)
\end{aligned}$$

For all $e_1 \in E(A_0^H)$, the number of edges e_2 for which $\{e_1, e_2\} \in E_2$ is at most $2n$, implying $\|\omega_2\|_1\Delta^{1+\varepsilon^2} \leq 2n\Delta^{1+\varepsilon^2} \leq \omega_2(E(A_0^H))$, and clearly, $\|\omega_2\|_2\Delta^{2+\varepsilon^2} \leq \Delta^{2+\varepsilon^2} \leq \omega_2(E(A_0^H))$ and $\|\omega_1\|_1\Delta^{1+\varepsilon^2} \leq \Delta^{1+\varepsilon^2} \leq \omega_1(E(A_0^H))$. (Recall that $\Delta = (d_0 + 3\varepsilon)n$.) Hence, by (4.4.6), we conclude that

$$\omega_1(M) = (1 \pm \hat{\varepsilon})\frac{\omega_1(E(A_0^H))}{d_0n} \overset{(4.4.21)}{=} d_A^2|S_1| \pm \varepsilon'^2 e_H(X_i^H, X_j^H), \tag{4.4.23}$$

$$\omega_2(M) = (1 \pm \hat{\varepsilon})\frac{\omega_2(E(A_0^H))}{(d_0n)^2} \overset{(4.4.22)}{=} d_A^2|S_2| \pm \varepsilon'^2 e_H(X_i^H, X_j^H). \tag{4.4.24}$$

Clearly, the final equalities in (4.4.23) and (4.4.24) are also true if $|S_1|, |S_2| < \varepsilon'^5 n$ because $e_H(X_i^H, X_j^H) \geq \varepsilon'^2 n$. Now, together with (4.4.19) and (4.4.20) this implies that

$$e_H(N_{A_i^{H,new}}(v_i), N_{A_j^{H,new}}(v_j)) = d_A^2|S| \pm 3\varepsilon'^2 e_H(X_i^H, X_j^H) = (d_i d_j d_A^2 \pm \varepsilon')e_H(X_i^H, X_j^H),$$

which establishes $(\mathrm{II})_{\mathrm{L4.9}}$.

Step 2.3. Checking $(\mathrm{III})_{\mathrm{L4.9}}$

We will even show that $(\mathrm{III})_{\mathrm{L4.9}}$ holds for all $\omega \in \mathcal{W}_{edge} \cup \mathcal{W}'_{edge}$ with $\omega\colon E(\mathscr{A}_i) \to [0, s]$ and centre $v \in V_i, i \in N_{R_A}(0)$, where $\mathcal{W}'_{edge}$ is a set of edge testers that we will explicitly specify in Step 2.4 when establishing $(\mathrm{IV})_{\mathrm{L4.9}}$. For all $\omega \in \mathcal{W}_{edge} \cup \mathcal{W}'_{edge}$ with centre $v \in V_i$, $i \in N_{R_A}(0)$ we define a weight function $\omega_0\colon E(\mathscr{A}_0) \to [0, s]$ by

$$\omega_0(x_0v_0) := \begin{cases} \omega(x_iv) & \text{if } \{x_i\} = N_{\mathcal{H}_+}(x_0) \cap \mathscr{X}_i,\ x_iv \in E(\mathscr{A}_i^{good}) \text{ and } v_0v \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

and we add ω_0 to $\mathcal{W}$. (Recall that $\mathscr{A}_i^{good}$ is the spanning subgraph of $\mathscr{A}_i$ containing precisely those edges $x_iv_i \in E(\mathscr{A}_i)$, where $N_{\mathcal{H}_+}(x_i) \cap \mathscr{X}_0 \neq \emptyset$.)

For every edge $x_iv \in E(\mathscr{A}_i^{good})$ with $\{x_0\} = N_{\mathcal{H}_+}(x_i) \cap \mathscr{X}_0$, property (N1) yields that

$$|N_{\mathscr{A}_0}(x_0) \cap N_G(v)| = (d_0 d_A \pm 3\varepsilon)n.$$

Hence, every edge $x_iv \in E(\mathscr{A}_i^{good})$ contributes weight $\omega(x_iv)\cdot(d_0d_A \pm 3\varepsilon)n$ to $\omega_0(E(\mathscr{A}_0))$, and we obtain

$$\omega_0(E(\mathscr{A}_0)) = \omega(E(\mathscr{A}_i^{good}))(d_0 d_A \pm 3\varepsilon)n.$$

By the definition of $\mathscr{A}_i^{new}$ (recall (4.4.10) and (4.4.16)), if $\sigma(x_0) \in N_G(v)$ for $\{x_0\} = N_{\mathcal{H}_+}(x_i) \cap \mathscr{X}_0$, then the edge $x_iv \in E(\mathscr{A}_i^{good})$ is in $E(\mathscr{A}_i^{new})$. Hence, if $x_0v_0 \in M(\sigma) = M$, then this contributes weight $\omega_0(x_0v_0)$ to $\omega(E(\mathscr{A}_i^{new}))$. If $\omega(E(\mathscr{A}_i^{good})) \geq \varepsilon n$, then $\omega_0(E(\mathscr{A}_0)) \geq n^{1+\varepsilon} \geq s\Delta^{1+\varepsilon^2} \geq \|\omega_0\|_1 \Delta^{1+\varepsilon^2}$, and thus (4.4.6) implies that

$$\omega_0(M) = (1 \pm \hat{\varepsilon})\frac{\omega_0(E(\mathscr{A}_0))}{d_0 n} = (1 \pm 2\hat{\varepsilon})d_A\omega(E(\mathscr{A}_i^{good})) \pm \hat{\varepsilon}n. \tag{4.4.25}$$

If $\omega(E(\mathscr{A}_i^{good})) < \varepsilon n$, then (4.4.7) implies that

$$\omega_0(M) \leq \max\left\{(1 + \hat{\varepsilon})\frac{\omega_0(E(\mathscr{A}_0))}{d_0 n}, n^{\varepsilon}\right\} \leq \hat{\varepsilon}n,$$

and hence, (4.4.25) also holds in this case.

We now make a key observation:

$$\omega(E(\mathscr{A}_i^{new})) = \omega_0(M) + \omega(\Lambda) \pm \omega(\Gamma), \tag{4.4.26}$$

for $\Gamma := \{x_iv \in E(\mathscr{A}_i^{good})\colon N_{\mathcal{H}_+}(x_i) \cap \mathscr{X}_0^{\sigma} = \emptyset\}$ and $\Lambda := E(\mathscr{A}_i^{bad}) \cap E(\mathscr{A}_i^{new})$. Next, we want to control $\omega(\Gamma)$ and $\omega(\Lambda)$.

In order to bound $\omega(\Gamma)$, we define a weight function $\omega_\Gamma\colon E(\mathscr{A}_0) \to [0, s]$ by

$$\omega_\Gamma(x_0v_0) := \begin{cases} \omega(x_iv) & \text{if } \{x_i\} = N_{\mathcal{H}_+}(x_0) \cap \mathscr{X}_i,\ x_iv \in E(\mathscr{A}_i^{good}), \\ 0 & \text{otherwise,} \end{cases}$$

and we add ω_Γ to $\mathcal{W}$. Observe that $\omega_\Gamma(M)$ accounts for the ω-weight of edges $x_iv \in E(\mathscr{A}_i^{good})$ such that $N_{\mathcal{H}_+}(x_i) \cap \mathscr{X}_0 \in \mathscr{X}_0^\sigma$ and thus $x_iv \notin \Gamma$. Hence $\omega(\Gamma) = \omega(E(\mathscr{A}_i^{good})) - \omega_\Gamma(M)$. For every vertex $x_0 \in \mathscr{X}_0$, we have $\deg_{\mathscr{A}_0}(x_0) = (d_0 \pm 3\varepsilon)n$. Hence, every edge $x_iv \in E(\mathscr{A}_i^{good})$ contributes weight $\omega(x_iv) \cdot (d_0 \pm 3\varepsilon)n$ to $\omega_\Gamma(E(\mathscr{A}_0))$, and we obtain

$$\omega_\Gamma(E(\mathscr{A}_0)) = \omega(E(\mathscr{A}_i^{good}))(d_0 \pm 3\varepsilon)n.$$

If $\omega(E(\mathscr{A}_i^{good})) \geq \varepsilon n$, then $\omega_\Gamma(E(\mathscr{A}_0)) \geq n^{1+\varepsilon} \geq s\Delta^{1+\varepsilon^2} \geq \|\omega_\Gamma\|_1\Delta^{1+\varepsilon^2}$, and thus (4.4.6) implies that

$$\omega_\Gamma(M) = (1 \pm \hat{\varepsilon})\frac{\omega_\Gamma(E(\mathscr{A}_0))}{d_0 n} = (1 \pm 2\hat{\varepsilon})\omega(E(\mathscr{A}_i^{good})) \pm \hat{\varepsilon}n. \tag{4.4.27}$$

Again, if $\omega(E(\mathscr{A}_i^{good})) < \varepsilon n$, then (4.4.7) implies that (4.4.27) also holds in this case.

Hence, we conclude that

$$\omega(\Gamma) = \omega(E(\mathscr{A}_i^{good})) - \omega_\Gamma(M) \overset{(4.4.27)}{\leq} 2\hat{\varepsilon}\omega(E(\mathscr{A}_i^{good})) + \hat{\varepsilon}n. \tag{4.4.28}$$

In order to bound $\omega(\Lambda)$, we use (4.4.18) and add $\omega|_{E(\mathscr{A}_i^{bad})}$ to $\mathcal{W}^{bad}$. Then (4.4.18) implies that

$$\omega(\Lambda) = (1 \pm \varepsilon)d_A\omega(E(\mathscr{A}_i^{bad})) \pm \varepsilon n. \tag{4.4.29}$$

Finally, equations (4.4.25), (4.4.26), (4.4.28) and (4.4.29) yield that

$$\begin{aligned}\omega(E(\mathscr{A}_i^{new})) &= (1 \pm 2\hat{\varepsilon})d_A\omega(E(\mathscr{A}_i^{good})) + (1 \pm \varepsilon)d_A\omega(E(\mathscr{A}_i^{bad})) \pm 2\hat{\varepsilon}\omega(E(\mathscr{A}_i^{good})) \pm 3\hat{\varepsilon}n\\ &= (1 \pm \varepsilon'^2)d_A\omega(E(\mathscr{A}_i)) \pm \varepsilon'^2 n. \qquad (4.4.30)\end{aligned}$$

This establishes $(\mathrm{III})_{\mathrm{L4.9}}$ for all $\omega \in \mathcal{W}_{edge} \cup \mathcal{W}'_{edge}$.

Step 2.4. *Checking* $(\mathrm{IV})_{\mathrm{L4.9}}$

We show that for the updated edge set labelling $\psi[\sigma]$, we have $\Delta_{\psi[\sigma]}(\mathscr{A}_i^{new}) \leq (1+\varepsilon')d_id_A|V_i|$ for every $i \in N_{R_A}(0)$. Recall that we defined $\psi[\sigma]$ in Definition 4.8 such that for $xv \in E(A_i^{H,new})$, we have $\psi[\sigma](xv) = \psi(xv) \cup \{\sigma(x_0)v\}$, if x has an H-neighbour $x_0 \in \mathscr{X}_0^\sigma$, and otherwise $\psi[\sigma](xv) = \psi(xv)$. We split the proof of $(\mathrm{IV})_{\mathrm{L4.9}}$ into two claims, where Claim 1 bounds the number of edges on which an 'old' label of ψ appears on the updated candidacy graph, and Claim 2 bounds the number of edges on which a 'new' label that we additionally added to $\psi[\sigma]$ appears in the updated candidacy graph. Let $\psi_i \colon E(\mathscr{A}_i) \to 2^{\mathcal{E}_i}$ be the (old) edge set labelling ψ restricted to $\mathscr{A}_i$ and we may assume that $|\mathcal{E}_i| \leq n^4$.

Claim 1. We can add at most n^5 weight functions to $\mathcal{W}'_{edge}$ to ensure that $\Delta_{\psi_i}(\mathscr{A}_i^{new}) \leq (1+\varepsilon')d_id_A|V_i|$ for every $i \in N_{R_A}(0)$.

Proof of claim: For all $i \in N_{R_A}(0)$ and $e \in \mathcal{E}_i$, let $\omega_e \colon E(\mathscr{A}_i) \to \{0,1\}$ be such that $\omega_e(x_iv_i) := \mathbb{1}_{\{e \in \psi_i(x_iv_i)\}}$ and we add ω_e to $\mathcal{W}'_{edge}$. By assumption (see (P4)), we have $\Delta_\psi(\mathscr{A}_i) \leq (1+\varepsilon)d_i|V_i|$, which implies that $\omega_e(E(\mathscr{A}_i)) \leq (1+\varepsilon)d_i|V_i|$. Since (4.4.30) in Step 2.3 is also valid for $\omega_e \in \mathcal{W}'_{edge}$, we conclude that e appears on at most

$$(1+\varepsilon'^2)d_A(1+\varepsilon)d_i|V_i| + \varepsilon'^2 n \leq (1+\varepsilon')d_id_A|V_i|$$

edges of $\mathscr{A}_i^{new}$, which completes the proof of Claim 1. —

Claim 2. *We can add at most n^3 weight functions to $\mathcal{W}$ to ensure that each $e \in E(G[V_0, V_i])$ appears on at most $(1+\varepsilon')d_i d_A|V_i|$ edges of $\mathscr{A}_i^{new}$ for every $i \in N_{R_A}(0)$.*

Proof of claim: For all $i \in N_{R_A}(0)$ and $e = v_0v_i \in E(G[V_0, V_i])$, we proceed as follows. Let $N := N_{\mathscr{A}_0}(v_0) \cap N_{\mathcal{H}}(N_{\mathscr{A}_i}(v_i))$. We define a weight function $\omega_e \colon E(\mathscr{A}_0) \to \{0,1\}$ by $\omega_e(xv) := \mathbb{1}_{\{v=v_0 \text{ and } x\in N\}}$ for every $xv \in E(\mathscr{A}_0)$, and we add ω_e to $\mathcal{W}$. Then, e appears on $\omega_e(M)$ edges of $\mathscr{A}_i^{new}$. Observe that

$$\begin{aligned}\omega_e(E(\mathscr{A}_0)) = |N| \ &= \sum_{H\in\mathcal{H}} e_H(N_{A_0^H}(v_0), N_{A_i^H}(v_i))\\ &\overset{\text{(P3)}}{=} \sum_{H\in\mathcal{H}} (d_0d_i \pm \varepsilon)e_H(X_0^H, X_i^H) \le (d_0d_id_A + 2\varepsilon'^{3/2})n^2,\end{aligned} \tag{4.4.31}$$

where the last inequality holds because $\sum_{H\in\mathcal{H}} e_H(X_0^H, X_i^H) \le (d_A + \varepsilon'^{3/2})n^2$, by assumption. With (4.4.7), we obtain that

$$\omega_e(M) \le \max\left\{(1+\hat{\varepsilon})\frac{\omega_e(E(\mathscr{A}_0))}{d_0n}, n^{\varepsilon}\right\} \overset{(4.4.31)}{\le} (1+\varepsilon')d_id_A|V_i|,$$

which completes the proof of Claim 2. –

Step 2.5. *Checking* $(\mathrm{V})_{\mathrm{L4.9}}$

Recall that $\psi_i \colon E(\mathscr{A}_i) \to 2^{\mathcal{E}_i}$ denotes the edge set labelling ψ restricted to $\mathscr{A}_i$. For each $i \in [r]$, $e = v_0v_i \in E(G[V_0, V_i])$, and $f \in \mathcal{E}_i$, we show that $\{e, f\}$ appears on at most $\sqrt{n}$ edges of $\mathscr{A}_i^{new}$. This will imply $(\mathrm{V})_{\mathrm{L4.9}}$ because any set $\{e', f'\} \in \binom{\mathcal{E}_i}{2}$ appears also on at most $\Delta^c_{\psi}(\mathscr{A}_i) \le \sqrt{n}$ edges of $\mathscr{A}_i^{new}$, and no two edges of $E(G[V_0, V_i])$ appear together as a label on an edge of $\mathscr{A}_i^{new}$. Let

$$\mathscr{X}_0^f := N_{\mathcal{H}_+}(\{x_i \in \mathscr{X}_i \colon x_iv_i \in E(\mathscr{A}_i), f \in \psi_i(x_iv_i)\}) \cap \mathscr{X}_0.$$

We define a weight function $\omega_{e,f} \colon E(\mathscr{A}_0) \to \{0,1\}$ by $\omega_{e,f}(xv) := \mathbb{1}_{\{v=v_0 \text{ and } x\in\mathscr{X}_0^f\}}$ for every $xv \in E(\mathscr{A}_0)$ and add $\omega_{e,f}$ to $\mathcal{W}$. Since $\Delta_{\psi}(\mathscr{A}_i) \le (1+\varepsilon)d_i|V_i|$ by (P4), we obtain that $\omega_{e,f}(E(\mathscr{A}_0)) \le 2n$. Note that $\{e, f\}$ appears on at most $\omega_{e,f}(M)$ edges of $\mathscr{A}_i^{new}$. Now, (4.4.7) implies that $\omega_{e,f}(M) \le n^{\varepsilon}$, which establishes $(\mathrm{V})_{\mathrm{L4.9}}$.

Step 2.6. *Checking* $(\mathrm{VI})_{\mathrm{L4.9}}$

For each $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$ with $H_1, \ldots, H_\ell \in \mathcal{H}$ such that $Y_j \subseteq X_0^{H_j}$, let

$$E_{(W,Y_1,\ldots,Y_\ell)} := \left\{\bigcup\nolimits_{j\in[\ell]}\{xy_j\} \colon xy_j \in E(A_0^{H_j}[W, Y_j]) \text{ for all } j \in [\ell]\right\} \subseteq \binom{E(\mathscr{A}_0)}{\ell},$$

and we define a weight function $\omega_{(W,Y_1,\ldots,Y_\ell)} \colon \binom{E(\mathscr{A}_0)}{\ell} \to \{0,1\}$ by

$$\omega_{(W,Y_1,\ldots,Y_\ell)}(\{e_1, \ldots, e_\ell\}) := \mathbb{1}_{\{\{e_1,\ldots,e_\ell\}\in E_{(W,Y_1,\ldots,Y_\ell)}\}}$$

and add $\omega_{(W,Y_1,\ldots,Y_\ell)}$ to $\mathcal{W}$. Observe that

$$\omega_{(W,Y_1,\ldots,Y_\ell)}(M) = \left|W \cap \bigcap\nolimits_{j\in[\ell]} \sigma(Y_j \cap \mathscr{X}_0^{\sigma})\right|. \tag{4.4.32}$$

In view of the statement, we may assume that $|W|, |Y_j| \ge \varepsilon'^2 n$ for all $j \in [\ell]$. Since $\ell \le s$ and A_0^H is (ε, d_0)-super-regular for every $H \in \mathcal{H}$, we obtain with Fact 1.11 that are at most $\varepsilon^{1/2}n$ vertices in W that do not have $(d_0 \pm \varepsilon)|Y_j|$ many neighbours in Y_j for every $j \in [\ell]$. Hence we obtain that

$$\omega_{(W,Y_1,\dots,Y_\ell)}(E(\mathscr{A}_0)) = |E_{(W,Y_1,\dots,Y_\ell)}| = (d_0^\ell \pm \varepsilon'^2)|W||Y_1|\cdots|Y_\ell|. \tag{4.4.33}$$

For $k \in [\ell]$, any set of k edges $\{e_1,\dots,e_k\}$ is contained in at most $(2n)^{\ell-k}$ ℓ-tuples in $E_{(W,Y_1,\dots,Y_\ell)}$, which implies that

$$\|\omega_{(W,Y_1,\dots,Y_\ell)}\|_k \Delta^{k+\varepsilon^2} \le (2n)^{\ell-k}\Delta^{k+\varepsilon^2} \overset{(4.4.33)}{\le} \omega_{(W,Y_1,\dots,Y_\ell)}(E(\mathscr{A}_0)).$$

Hence, by (4.4.6), we conclude that

$$\omega_{(W,Y_1,\dots,Y_\ell)}(M) = (1 \pm \hat{\varepsilon})\frac{\omega_{(W,Y_1,\dots,Y_\ell)}(E(\mathscr{A}_0))}{(d_0 n)^\ell} \overset{(4.4.33)}{=} \frac{|W||Y_1|\cdots|Y_\ell|}{n^\ell} \pm \varepsilon' n,$$

which establishes $(\mathrm{VI})_{\mathrm{L4.9}}$ by (4.4.32).

Step 2.7. *Checking* $(\mathrm{VII})_{\mathrm{L4.9}}$

We add $\mathcal{W}_{edge}$ to $\mathcal{W}$ and fix some $\omega \in \mathcal{W}_{edge}$. If $\omega(E(\mathscr{A}_0)) \le n^{1+\varepsilon/2}$, then we obtain by (4.4.7) that $\omega(M) \le n^\varepsilon$ and thus, $\omega(M) = (1 \pm \varepsilon')\omega(E(\mathscr{A}_0))/d_0 n \pm \varepsilon' n$. If $\omega(E(\mathscr{A}_0)) \ge n^{1+\varepsilon/2}$, then we obtain by (4.4.6) that $\omega(M) = (1 \pm \hat{\varepsilon})\omega(E(\mathscr{A}_0))/d_0 n$. This establishes $(\mathrm{VII})_{\mathrm{L4.9}}$ and completes the proof of Lemma 4.9. □

4.5 Proof of the main result

The following lemma is very similar to Theorem 4.2. We only require additionally that all graphs in $\mathcal{H}$ only span a matching between two clusters that is either empty or not too small. This reduction has already been used in [112] (and in several other extensions of the blow-up lemma) and it is also not complicated in our framework.

Lemma 4.10 ([32])**.** *Let $1/n \ll \varepsilon \ll \alpha, d$ and $1/n \ll 1/r$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V}, \phi^*)$ is an (ε, d)-super-regular, α^{-1}-bounded and (ε, α)-linked extended blow-up instance, $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$, and $|\mathcal{H}| \le \alpha^{-1}n$. Suppose that $\sum_{H\in\mathcal{H}} e_H(X_i^H, X_j^H) \le (1-\alpha)dn^2$ for all $ij \in E(R)$ and $H[X_i^H, X_j^H]$ is a matching of size at least $\alpha^2 n$ if $ij \in E(R)$ and empty if $ij \in \binom{[r]}{2} \setminus E(R)$ for each $H \in \mathcal{H}$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{2\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ in G which extends ϕ^* such that*

(i) *$\phi(X_i^H) = V_i$ for all $i \in [r]_0$ and $H \in \mathcal{H}$;*

(ii) *$|W \cap \bigcap_{j\in[\ell]} \phi(Y_j)| = |W||Y_1|\cdots|Y_\ell|/n^\ell \pm \alpha n$ for all $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$;*

(iii) *$\omega(\bigcup_{H\in\mathcal{H}} X_i^H \cap \phi^{-1}(v)) = \omega(\bigcup_{H\in\mathcal{H}} X_i^H)/n \pm \alpha n$ for all $(v, \omega) \in \mathcal{W}_{ver}$ and $v \in V_i$.*

We first prove our main result (Theorem 4.2) assuming Lemma 4.10.

Proof of Theorem 4.2. We choose a new constant β such that $\varepsilon \ll \beta \ll \alpha, d$. For each $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$ with $W \subseteq V_i$, $i \in [r]$ and $k \in [\ell]$, let $\omega_{Y_k}\colon \bigcup_{H\in\mathcal{H}} X_i^H \to \{0,1\}$ be such that $\omega_{Y_k}(x) = \mathbb{1}_{\{x\in Y_k\}}$, and let $\mathcal{W}_Y$ be the set containing all those weight functions. We delete from every $H \in \mathcal{H}$ the set X_0^H and apply Lemma 4.6 to this collection of graphs and the set of weight functions $\mathcal{W}^* := \{\omega\colon (v,\omega) \in \mathcal{W}_{ver}\} \cup \mathcal{W}_Y$, which yields a refined partition of $\mathcal{H}$; to be more precise, for all $H \in \mathcal{H}$ and $i \in [r]$, we obtain a partition $(X_{i,j}^H)_{j\in[\beta^{-1}]}$ of X_i^H satisfying (i)–(iv) of Lemma 4.6. Let

$\mathcal{X}'$ be the collection of vertex partitions of the graphs in $\mathcal{H}$ given by $(X_0^H)_{H\in\mathcal{H}}$ and $(X_{i,j}^H)_{H\in\mathcal{H},i\in[r],j\in[\beta^{-1}]}$. In particular, Lemma 4.6(iii) yields that

$$\omega(X_{i,j}^H) = \beta\omega(X_i^H) \pm \beta^{3/2}n, \text{ for all } H \in \mathcal{H}, \omega \in \mathcal{W}^*, i \in [r], j \in [\beta^{-1}]. \tag{4.5.1}$$

Let R' be the graph with vertex set $[r] \times [\beta^{-1}]$ and two vertices $(i,j),(i',j')$ are joined by an edge if $ii' \in E(R)$. Note that $\Delta(R') \leq \alpha^{-1}\beta^{-1}$ because $\Delta(R) \leq \alpha^{-1}$.

According to the refinement $\mathcal{X}'$ of $\mathcal{X}$, we claim that there exists a refined partition $\mathcal{V}'$ of $\mathcal{V}$ consisting of the collection of V_0 together with $(V_{i,j})_{i\in[r],j\in[\beta^{-1}]}$, where $(V_{i,j})_{j\in[\beta^{-1}]}$ is a partition of V_i for every $i \in [r]$ such that

(a) $|W \cap V_{i,j}| = \beta|W| \pm \beta^{3/2}n$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$ and $j \in [\beta^{-1}]$ with $W \subseteq V_i, \ell \in [\alpha^{-1}]$;

(b) $(\mathcal{H}, G, R', \mathcal{X}', \mathcal{V}', \phi_0)$ is an $(\varepsilon^{1/2}, d)$-super-regular, β^{-2}-bounded and $(\varepsilon^{1/2}, \alpha/2)$-linked extended blow-up instance.

Indeed, the existence of $\mathcal{V}'$ follows by a simple probabilistic argument. For each $i \in [r]$, let $\tau_i \colon V_i \to [\beta^{-1}]$ where $\tau_i(v)$ is chosen uniformly at random for every $v \in V_i$, all independently, and let $V_{i,j} := \{v \in V_i \colon \tau_i(v) = j\}$ for every $j \in [\beta^{-1}]$. Chernoff's inequality and a union bound imply that (a) holds simultaneously together with the following properties with probability at least $1 - \mathrm{e}^{-\sqrt{n}}$:

- $G[V_{i,j}, V_{i',j'}]$ is $(\varepsilon^{1/2}, d)$-super-regular for all $ii' \in E(R), j, j' \in [\beta^{-1}]$;
- $|\bigcap_{x_0\in X_0^H\cap N_H(x)} N_G(\phi_0(x_0))\cap V_{i,j}| \geq \alpha/2|V_{i,j}|$ for all $x \in X_{i,j}^H, i \in [r], j \in [\beta^{-1}], H \in \mathcal{H}$.

Standard properties of the multinomial distribution yield that $|V_{i,j}| = |X_{i,j}^H|$ for all $i \in [r]$, $j \in [\beta^{-1}]$, $H \in \mathcal{H}$ with probability at least $\Omega(n^{-r\beta^{-1}})$. To see in (b) that the instance is $(\varepsilon^{1/2}, \alpha/2)$-linked, observe further that the number of vertices in $X_{i,j}^H$ that have a neighbour in X_0^H is at most $\varepsilon|X_i^H| \leq \varepsilon^{1/2}|X_{i,j}^H|$ and $\sum_{H\in\mathcal{H}} |N_H(\phi_0^{-1}(v_0), \phi_0^{-1}(v_0')) \cap X_{i,j}^H| \leq \varepsilon|V_i|^{1/2} \leq \varepsilon^{1/2}|V_{i,j}|^{1/2}$ for all $i \in [r]$, $j \in [\beta^{-1}]$ and distinct $v_0, v_0' \in V_0$. Thus, for every $i \in [r]$, there exists a partition $(V_{i,j})_{j\in[\beta^{-1}]}$ of V_i satisfying (a) and (b). Let $n' := \beta n$.

We show how to lift the vertex and set testers from the original blow-up instance to the just defined blow-up instance. For each $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$ and distinct $H_1, \ldots, H_\ell \in \mathcal{H}$ such that $W \subseteq V_i$ for some $i \in [r]$ and $Y_k \subseteq X_i^{H_k}$ for all $k \in [\ell]$, we define $(W_j, Y_{1,j}, \ldots, Y_{\ell,j})$ by setting $W_j := W \cap V_{i,j}$ and $Y_{k,j} := Y_k \cap X_{i,j}^{H_k}$ for all $j \in [\beta^{-1}], k \in [\ell]$. By (a), we conclude that $|W_j| = \beta|W| \pm \beta^{3/2}n$, and by (4.5.1), we have that $|Y_{k,j}| = \omega_{Y_k}(X_{i,j}^{H_k}) = \beta\omega_{Y_k}(X_i^{H_k}) \pm \beta^{3/2}n = \beta|Y_k| \pm \beta^{3/2}n$. Let $\mathcal{W}'_{set} := \{(W_j, Y_{1,j}, \ldots, Y_{\ell,j}) \colon j \in [\beta^{-1}], (W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}\}$. For each $(v, \omega) \in \mathcal{W}_{ver}$ with $v \in V_{i,j}$, let $\omega' := \omega|_{\bigcup_{H\in\mathcal{H}} X_{i,j}^H}$ and $\mathcal{W}'_{ver} := \{(v, \omega') \colon (v, \omega) \in \mathcal{W}_{ver}\}$.

Next, we add some edges to the graphs in $\mathcal{H}$ ensuring that all matchings between two clusters are either empty or of small linear size. To this end, we add a minimum number of edges to $H[X_{i,j}^H, X_{i',j'}^H]$ for all $\{(i,j),(i',j')\} \in E(R')$ and $H \in \mathcal{H}$ such that the obtained supergraph $H'[X_{i,j}^{H'}, X_{i',j'}^{H'}]$ is a matching of size at least $\beta^4 n$. Note that $\Delta(H') \leq \Delta(R') + \alpha^{-1} \leq \beta^{-2}$. Let $\mathcal{H}'$ be the collection of graphs H' obtained in this manner. Together with Lemma 4.6(iv), we conclude for all $\{(i,j),(i',j')\} \in E(R')$ that

$$\begin{aligned}\sum_{H'\in\mathcal{H}'} e_{H'}(X_{i,j}^{H'}, X_{i',j'}^{H'}) &\leq 2\beta^4 n \cdot \alpha^{-1} n + \sum_{H\in\mathcal{H}} e_H(X_{i,j}^H, X_{i',j'}^H) \\ &\leq \beta^3 n^2 + \beta^2 \sum_{H\in\mathcal{H}} e_H(X_i^H, X_{i'}^H) + n^{5/3} \leq (1 - \alpha/2) d n'^2.\end{aligned}$$

Obviously, it suffices to construct a packing of $\mathcal{H}'$ into G which extends ϕ_0 and satisfies Theorem 4.2(i)–(iii). By (b) and because $\beta \ll \alpha$, also $(\mathcal{H}', G, R', \mathcal{X}', \mathcal{V}', \phi_0)$ is an $(\varepsilon^{1/2}, d)$-super-regular, β^{-2}-bounded and $(\varepsilon^{1/2}, \beta^2)$-linked extended blow-up instance, and we can apply Lemma 4.10 to $(\mathcal{H}', G, R', \mathcal{X}', \mathcal{V}', \phi_0)$ with set testers $\mathcal{W}'_{set}$ and vertex testers $\mathcal{W}'_{ver}$ as follows:

n'	$\varepsilon^{1/2}$	β^2	d	$r\beta^{-1}$
n	ε	α	d	r

Hence, we obtain a packing ϕ of $\mathcal{H}'$ in G which extends ϕ_0 such that for all $i \in [r], j \in [\beta^{-1}]$

(I) $\phi(X^H_{i,j}) \subseteq V_{i,j}$ for all $H \in \mathcal{H}$;

(II) $|W_j \cap \bigcap_{k\in[\ell]} \phi(Y_{k,j})| = |W_j||Y_{1,j}| \cdots |Y_{\ell,j}|/n'^{\ell} \pm \beta^2 n'$ for all $(W_j, Y_{1,j}, \dots, Y_{\ell,j}) \in \mathcal{W}'_{set}$;

(III) $\omega'(\bigcup_{H\in\mathcal{H}} X^H_{i,j} \cap \phi^{-1}(v)) = \omega'(\bigcup_{H\in\mathcal{H}} X^H_{i,j})/n' \pm \beta^2 n'$ for all $(v, \omega') \in \mathcal{W}'_{ver}$ with $v \in V_{i,j}$.

Observe that (I) establishes Theorem 4.2(i).

For $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$, we conclude that

$$\begin{aligned}\left|W \cap \bigcap_{k\in[\ell]} \phi(Y_k)\right| &= \sum_{j\in[\beta^{-1}]} \left|W_j \cap \bigcap_{k\in[\ell]} \phi(Y_{k,j})\right| \\ &\overset{\text{(II)},(4.5.1)}{=} \sum_{j\in[\beta^{-1}]} \left(\frac{\beta^{\ell+1}\left(|W||Y_1|\cdots|Y_\ell| \pm \beta^{1/3} n^{\ell+1}\right)}{(\beta n)^{\ell}} \pm \beta^2 n' \right) \\ &= |W||Y_1|\cdots|Y_\ell|/n^{\ell} \pm \alpha n.\end{aligned}$$

Hence, Theorem 4.2(ii) holds.

For $(v, \omega) \in \mathcal{W}_{ver}$ with $v \in V_{i,j}$ and its corresponding tuple $(v, \omega') \in \mathcal{W}'_{ver}$, we conclude that

$$\begin{aligned}\omega(\textstyle\bigcup_{H\in\mathcal{H}} X^H_i \cap \phi^{-1}(v)) &\overset{\text{(I)}}{=} \omega'(\textstyle\bigcup_{H\in\mathcal{H}} X^H_{i,j} \cap \phi^{-1}(v)) \overset{\text{(III)}}{=} \omega'(\bigcup_{H\in\mathcal{H}} X^H_{i,j})/n' \pm \beta^2 n' \\ &\overset{(4.5.1)}{=} \frac{\beta\omega(\bigcup_{H\in\mathcal{H}} X^H_i) \pm \beta^{4/3} n^2}{\beta n} \pm \beta^2 n' = \omega(\textstyle\bigcup_{H\in\mathcal{H}} X^H_i)/n \pm \alpha n.\end{aligned}$$

This yields Theorem 4.2(iii) and completes the proof. □

Theorem 4.3 can be easily deduced from Theorem 4.2 by randomly partitioning G and applying Lemma 4.6 to $\mathcal{H}$ with $r = 1$. In particular, the proof is very similar to the proof of Theorem 4.2 and therefore omitted. We proceed with the proof of Lemma 4.10.

Proof of Lemma 4.10. We split the proof into four steps. In Step 1, we partition G into two edge-disjoint subgraphs G_A and G_B. In Step 2, we define 'candidacy graphs' that we track for the partial packing in Step 3, where we iteratively apply Lemma 4.9 to consider the clusters in turn. We only use the edges of G_A for the partial packing in Step 3 such that we can complete the packing in Step 4 using the edges of G_B and the ordinary blow-up lemma.

We will proceed cluster by cluster in Step 3 to find a function that packs almost all vertices of $\mathcal{H}$ into G. Since r may be much larger than ε^{-1}, we need to carefully control the growth of the error term. We do so, by considering a proper vertex colouring $c\colon V(R) \to [T]$ of R^3 where $T := \alpha^{-3}$, and choose new constants $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_T, \mu, \gamma$ such that

$$\varepsilon \ll \varepsilon_0 \ll \varepsilon_1 \ll \dots \ll \varepsilon_T \ll \mu \ll \gamma \ll \alpha, d.$$

To obtain the order in which we consider the clusters in turn, we simply relabel the cluster indices such that the colour values are non-decreasing; that is, $c(1) \leq \dots \leq c(r)$. Note that the sets $(c^{-1}(k))_{k\in[T]}$ are independent in R^3. For $i \in [r], t \in [r]_0$, let

$$c_i(t) := \max\{\{0\} \cup \{c(j)\colon j \in N_R[i] \cap [t]\}\}, \quad \text{and} \quad m_i(t) := |N_R(i) \cap [t]|. \tag{4.5.2}$$

That is, if we think of $[t]$ as the indices of clusters that have already been embedded, then $c_i(t)$ denotes the largest colour of an already embedded cluster in the closed neighbourhood of i in R, and $m_i(t)$ denotes the number of neighbours of i in R that have already been embedded.

For $t \in [r]_0$, let

$$\mathscr{X}_t := \bigcup\nolimits_{H\in\mathcal{H}} X_t^H, \qquad \mathcal{X}_t := \bigcup\nolimits_{\ell\in[t]_0} \mathscr{X}_\ell, \qquad \mathcal{V}_t := \bigcup\nolimits_{\ell\in[t]_0} V_\ell.$$

For every vertex tester $(v, \omega) \in \mathcal{W}_{ver}$ with $v \in V_i$ for some $i \in [r]$, we define its *corresponding function* ω_v on $\{x_i v_i\colon x_i \in \mathscr{X}_i, v_i \in V_i\}$ by setting $\omega_v(x_i v_i) := \omega(x_i)\mathbb{1}_{\{v_i=v\}}$. Let

$$\mathcal{W}^i_{edge} := \{\omega_v\colon (v, \omega) \in \mathcal{W}_{ver}, v \in V_i\}. \tag{4.5.3}$$

Step 1. *Partitioning the edges of G*

In order to reserve an exclusive set of edges for the completion in Step 4, we partition the edges of G into two subgraphs G_A and G_B. For each edge e of G independently, we add e to G_B with probability γ and otherwise to G_A. Let $d_A := (1-\gamma)d$, $d_B := \gamma d$, $\alpha_A := (1-\gamma)^{\alpha^{-1}}\alpha/2$ and $\alpha_B := \gamma^{\alpha^{-1}}\alpha/2$. Using Chernoff's inequality, we can easily conclude that with probability at least $1 - 1/n$ we have for all $Z \in \{A, B\}$ that

(4.5.4) *$G_Z[V_i, V_j]$ is $(2\varepsilon, d_Z)$-super-regular for all $ij \in E(R)$,*

(4.5.5) *$|V_i \cap \bigcap_{x_0\in X_0^H\cap N_H(x)} N_{G_Z}(\phi^*(x_0))| \geq \alpha_Z|V_i|$ for all $x \in X_i^H, i \in [r], H \in \mathcal{H}$.*

Hence, we may assume that G is partitioned into G_A and G_B such that (4.5.4) and (4.5.5) hold.

Step 2. *Candidacy graphs*

For $t \in [r]_0$, we call $\phi\colon \bigcup_{H\in\mathcal{H},i\in[t]_0} \widehat{X}_i^H \to \mathcal{V}_t$ a *t-partial packing* if $\widehat{X}_i^H \subseteq X_i^H$, $\phi|_{\widehat{X}_0^H} = \phi^*|_{X_0^H}$, and $\phi(\widehat{X}_i^H) \subseteq V_i$ for all $H \in \mathcal{H}, i \in [t]_0$ such that ϕ is a packing of $(H[\widehat{X}_0^H \cup \ldots \cup \widehat{X}_t^H])_{H\in\mathcal{H}}$ into $G_A[\mathcal{V}_t]$. Note that $\widehat{X}_0^H = X_0^H$, and $\phi|_{\widehat{X}_i^H}$ is injective for all $H \in \mathcal{H}$ and $i \in [t]_0$. For convenience, we often write

$$\mathcal{X}_t^\phi := \bigcup\nolimits_{H\in\mathcal{H},i\in[t]_0} \widehat{X}_i^H.$$

Suppose $t \in [r]_0$ and $\phi_t\colon \mathcal{X}_t^{\phi_t} \to \mathcal{V}_t$ is a t-partial packing. We introduce the notion of *candidates* (with respect to ϕ_t) for future packing rounds and track those relations in two kinds of bipartite auxiliary graphs that we call candidacy graphs: A graph $A_i^H(\phi_t)$ with bipartition (X_i^H, V_i), $i \in [r]$ that will be used to extend the t-partial packing ϕ_t to a $(t+1)$-partial packing ϕ_{t+1} via Lemma 4.9 in Step 3, and a graph $B_i^H(\phi_t)$ that will be used for the completion in Step 4. For convenience, we define $B_i^H(\phi_t)$ on a copy $(X_i^{H,B}, V_i^B)$ of (X_i^H, V_i). That is, for all $H \in \mathcal{H}, i \in [r]$, let $X_i^{H,B}$ and V_i^B be disjoint copies of X_i^H and V_i, respectively. Let π be the bijection that maps a vertex in $\bigcup_{H\in\mathcal{H},i\in[r]}(X_i^H \cup V_i)$ to its copy in $\bigcup_{H\in\mathcal{H},i\in[r]}(X_i^{H,B} \cup V_i^B)$. Let G_+ and H_+ be supergraphs of G_A and $H \in \mathcal{H}$ with vertex partitions $(V_0, \ldots, V_r, V_1^B, \ldots, V_r^B)$ and $(X_0^H, \ldots, X_r^H, X_1^{H,B}, \ldots, X_r^{H,B})$, respectively, and edge sets

$$\begin{aligned} E(G_+) &:= E(G_A) \cup \{u\pi(v) \colon uv \in E(G_B)\}, \\ E(H_+) &:= E(H) \cup \{u\pi(v) \colon uv \in E(H)\}. \end{aligned}$$

Let R_B be the graph on $[r] \cup \{1^B, , \ldots, r^B\}$ with edge set $E(R_B) := \{ij^B \colon ij \in E(R)\}$. By taking copies $(X_i^{H,B}, V_i^B)$ for all (X_i^H, V_i) and defining the candidacy graphs $B_i^H(\phi_t)$ on these copies, and by enlarging G, H and R accordingly to G_+, H_+ and $R \cup R_B$, we will be able to update the candidacy graphs $A_i^H(\phi_t)$ and $B_i^H(\phi_t)$ simultaneously in Step 3 when we apply Lemma 4.9 in order to extend ϕ_t to a $(t+1)$-partial packing ϕ_{t+1}.

We now define $A_i^H(\phi_t)$ and $B_i^H(\phi_t)$. Let $X_i^{H,A} := X_i^H$ and $V_i^A := V_i$ for all $H \in \mathcal{H}$, $i \in [r]$. For $Z \in \{A, B\}$, $H \in \mathcal{H}$ and $i \in [r]$, we say that $v \in V_i^Z$ is a *candidate for* $x \in X_i^{H,Z}$ *given* ϕ_t if

$$\phi_t(N_{H_+}(x) \cap \mathcal{X}_t^{\phi_t}) \subseteq N_{G_+}(v). \tag{4.5.6}$$

For all $Z \in \{A, B\}$, let $Z_i^H(\phi_t)$ be a bipartite graph with vertex partition $(X_i^{H,Z}, V_i^Z)$ and edge set

$$E(Z_i^H(\phi_t)) := \{xv \colon x \in X_i^{H,Z}, v \in V_i^Z, \text{ and } v \text{ is a candidate for } x \text{ given } \phi_t\}. \tag{4.5.7}$$

We call every spanning subgraph of $Z_i^H(\phi_t)$ a *candidacy graph (with respect to ϕ_t)*.

Furthermore, for all $H \in \mathcal{H}$ and $i \in [r]$, we assign to every edge $xv \in E(A_i^H(\phi_t))$ an edge set labelling $\psi_t(xv)$ of size at most α^{-1}. This set encodes the edges between v and $\phi_t(N_H(x) \cap \mathcal{X}_t^{\phi_t})$ in G_A that are covered if we embed x onto v; to be more precise, for all $H \in \mathcal{H}$, $i \in [r]$, and every edge $xv \in E(A_i^H(\phi_t))$, we set

$$\psi_t(xv) := E\big(G_A\big[\phi_t(N_H(x) \cap \mathcal{X}_t^{\phi_t}), \{v\}\big]\big). \tag{4.5.8}$$

Tracking this set enables us to extend a t-partial packing ϕ_t to a $(t+1)$-partial packing ϕ_{t+1} by finding a conflict-free embedding (see definition in (4.4.1)) in $\bigcup_{H\in\mathcal{H}} A_{t+1}^H(\phi_t)$ via Lemma 4.9. Since $|N_H(x) \cap \mathcal{X}_t^{\phi_t}| \leq \alpha^{-1}$, we have $|\psi_t(xv)| \leq \alpha^{-1}$.

Before we proceed to Step 3 and extend ϕ_t to ϕ_{t+1}, we consider the candidacy graphs and their edge set labelling with respect to ϕ^*.

Claim 1. For all $H \in \mathcal{H}, i \in [r], Z \in \{A, B\}$, there exists a candidacy graph $Z_i^H \subseteq Z_i^H(\phi^*)$ with respect to ϕ^* (where $\mathscr{A}_i := \bigcup_{H\in\mathcal{H}} A_i^H$) such that

(C1.1) *Z_i^H is $(\varepsilon_0, \alpha_Z)$-super-regular;*

(C1.2) *$\Delta_{\psi_0}(\mathscr{A}_i) \leq \varepsilon_0 n$;*

(C1.3) *$\Delta_{\psi_0}^c(\mathscr{A}_i) \leq \sqrt{n}$;*

(C1.4) *$e_H(N_{A_i^H}(v_i), N_{A_j^H}(v_j)) = (\alpha_A^2 \pm \varepsilon_0) e_H(X_i^H, X_j^H)$ for all $v_iv_j \in E(G_A[V_i, V_j])$, $ij \in E(R)$;*

(C1.5) *$\omega_v(E(\mathscr{A}_i)) = \alpha_A \omega(\mathscr{X}_i) \pm \varepsilon_0 n^2$ for all $\omega_v \in \mathcal{W}_{edge}^i$.*

Proof of claim: We fix $H \in \mathcal{H}, i \in [r], ij \in E(R), Z \in \{A,B\}$, $v_iv_j \in E(G_A[V_i,V_j])$ and $\omega_v \in \mathcal{W}^i_{edge}$ as defined in (4.5.3). For each $k \in \{i,j\}$, let $\widetilde{X}^H_k$ be the set of vertices in X^H_k that have a neighbour in X^H_0. Observe that $Z^H_k(\phi^*)[X^H_k \setminus \widetilde{X}^H_k, V_k]$ is a complete bipartite graph for $k \in \{i,j\}$, and $e_H(N_{Z^H_i(\phi^*)[X^H_i \setminus \widetilde{X}^H_i, V_i]}(v_i), N_{Z^H_j(\phi^*)[X^H_j \setminus \widetilde{X}^H_j, V_j]}(v_j)) = e_H(X^H_i, X^H_j) \pm 4\varepsilon n$ because $|\widetilde{X}^H_k| \le \varepsilon |X^H_k| \le 2\varepsilon n$. By (4.5.5) and the definition of candidates in (4.5.6), we obtain that $\deg_{Z^H_i(\phi^*)}(x_i) \ge \alpha_Z |V_i|$ for all $x_i \in \widetilde{X}^H_i$. Note further that $\omega_v(E(\bigcup_{H\in\mathcal{H}} A^H_i[X^H_i \setminus \widetilde{X}^H_i, V_i])) = \omega(\mathscr{X}_i) \pm \varepsilon^{1/2}n^2$. Hence, there exists a subgraph $Z^H_i \subseteq Z^H_i(\phi^*)$ that satisfies (C1.1), (C1.4) and (C1.5), which can be seen by keeping each edge in $Z^H_i(\phi^*)[X^H_i \setminus \widetilde{X}^H_i, V_i]$ independently at random with probability α_Z and by possibly removing some edges incident to $x_i \in \widetilde{X}^H_i$ in $Z^H_i(\phi^*)$ deterministically. In order to see (C1.2), note that $|\psi_0^{-1}(v_0v_i)| \le |N_{\mathcal{H}}(\phi_0^{-1}(v_0)) \cap \mathscr{X}_i| \le \alpha^{-1}|\phi_0^{-1}(v_0)| \le \alpha^{-1}\varepsilon|\mathcal{H}| \le \varepsilon_0 n$. Since the blow-up instance is (ε,α)-linked, we have $\sum_{H\in\mathcal{H}} |N_H(\phi_0^{-1}(v_0), \phi_0^{-1}(v_0')) \cap X^H_i| \le \varepsilon|V_i|^{1/2}$ for all distinct $v_0v_i, v_0'v_i \in E(G[V_0,V_i])$, $i \in [r]$, which implies (C1.3). This completes the proof of the claim. $-$

Step 3. Induction

We inductively prove that the following statement $\mathbf{S}(t)$ holds for all $t \in [r]_0$, which will provide a partial packing of $\mathcal{H}$ into G_A.

$\mathbf{S}(t)$. For all $H \in \mathcal{H}$ and $Z \in \{A,B\}$, there exists a t-partial packing $\phi_t \colon \mathcal{X}^{\phi_t}_t \to \mathcal{V}_t$ with $|\mathcal{X}^{\phi_t}_t \cap X^H_i| \ge (1-\varepsilon_{c_i(t)})n$ for all $i \in [t]$, and there exists a candidacy graph $Z^H_i \subseteq Z^H_i(\phi_t)$ (where $\mathscr{A}_i := \bigcup_{H\in\mathcal{H}} A^H_i$) such that

(a) Z^H_i is $(\varepsilon_{c_i(t)}, \alpha_Z d_Z^{m_i(t)})$-super-regular for all $i \in [r]\setminus[t]$ if $Z = A$ and for all $i \in [r]$ if $Z = B$;

(b) $\Delta_{\psi_t}(\mathscr{A}_i) \le (1+\varepsilon_{c_i(t)})\alpha_A d_A^{m_i(t)}|V_i|$ for all $i \in [r]\setminus[t]$;

(c) $\Delta^c_{\psi_t}(\mathscr{A}_i) \le \sqrt{n}$ for all $i \in [r]\setminus[t]$;

(d) $e_H(N_{A^H_i}(v_i), N_{A^H_j}(v_j)) = (\alpha_A^2 d_A^{m_i(t)+m_j(t)} \pm \varepsilon_{\max\{c_i(t),c_j(t)\}})e_H(X^H_i, X^H_j)$ for all $H \in \mathcal{H}$, $ij \in E(R-[t])$ and $v_iv_j \in E(G_A[V_i,V_j])$;

(e) $|\phi_t^{-1}(v)| \ge (1-\varepsilon^{1/2}_{c_i(t)})|\mathcal{H}| - \varepsilon_{c_i(t)}n$ and $|\phi_t^{-1}(v) \cap N_{\mathcal{H}}(\mathcal{X}_t \setminus \mathcal{X}^{\phi_t}_t)| \le \varepsilon^{1/2}_{c_i(t)}n$ for all $v \in V_i$, $i \in [t]$;

(f) $\omega_v(E(\mathscr{A}_i)) = \alpha_A d_A^{m_i(t)}\omega(\mathscr{X}_i) \pm \varepsilon_{c_i(t)}n^2$ for all $\omega_v \in \mathcal{W}^i_{edge}$ and $i \in [r]\setminus[t]$;

(g) $|W \cap \bigcap_{j\in[\ell]} \phi_t(Y_j \cap \mathcal{X}^{\phi_t}_t)| = |W||Y_1|\cdots|Y_\ell|/n^\ell \pm \alpha n/2$ for all $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$ with $W \subseteq V_i$, $i \in [t]$;

(h) $\omega(\mathscr{X}_i \cap \phi_t^{-1}(v)) = \omega(\mathscr{X}_i)/n \pm \alpha n/2$ for all $(v,\omega) \in \mathcal{W}_{ver}$ with $v \in V_i$, $i \in [t]$.

Properties $\mathbf{S}(t)$(a)–(d) will be used particularly to establish $\mathbf{S}(t+1)$ by applying Lemma 4.9. Property (f) enables us to establish (h), which together with (g) basically implies Lemma 4.10(ii) and (iii) as we merely modify the r-partial packing ϕ_r for the completion in Step 4 where we exploit (a) (for $Z = B$) and (e).

The statement $\mathbf{S}(0)$ holds for $\phi_0 = \phi^*$ by Claim 1. Hence, we assume the truth of $\mathbf{S}(t)$ for some $t \in [r-1]_0$ and let $\phi_t \colon \mathcal{X}^{\phi_t}_t \to \mathcal{V}_t$ and A^H_i and B^H_i be as in $\mathbf{S}(t)$; we set $\mathscr{A}_i := \bigcup_{H\in\mathcal{H}} A^H_i$ and $\mathscr{B}_i := \bigcup_{H\in\mathcal{H}} B^H_i$. We will extend ϕ_t such that $\mathbf{S}(t+1)$ holds. Any function $\sigma \colon \mathscr{X}^\sigma_{t+1} \to V_{t+1}$ with $\mathscr{X}^\sigma_{t+1} \subseteq \mathscr{X}_{t+1}$ extends ϕ_t to a function $\phi_{t+1} \colon \mathcal{X}^{\phi_t}_t \cup \mathscr{X}^\sigma_{t+1} \to \mathcal{V}_{t+1}$ as follows:

$$\phi_{t+1}(x) := \begin{cases} \phi_t(x) & \text{if } x \in \mathcal{X}^{\phi_t}_t, \\ \sigma(x) & \text{if } x \in \mathscr{X}^\sigma_{t+1}. \end{cases} \tag{4.5.9}$$

We now make a key observation: By definition of the candidacy graphs $\mathscr{A}_{t+1}$ and their edge set labellings as in (4.5.8), if σ is a conflict-free packing in $\mathscr{A}_{t+1}$ as defined in (4.4.1), then ϕ_{t+1} is a $(t+1)$-partial packing.

We aim to apply Lemma 4.9 in order to obtain a conflict-free packing σ in $\mathscr{A}_{t+1}$. Let

$$\begin{aligned}
\mathcal{H}_{t+1} &:= \bigcup_{H\in\mathcal{H}} H_+\left[\textstyle\bigcup_{i\in N_R[t+1]\setminus[t]} X_i^H \cup \bigcup_{i\in N_R(t+1)} X_i^{H,B}\right],\\
G_{t+1} &:= G_+\left[\textstyle\bigcup_{i\in N_R[t+1]\setminus[t]} V_i \cup \bigcup_{i\in N_R(t+1)} V_i^B\right],\\
\mathcal{A}_{t+1} &:= \textstyle\bigcup_{i\in N_R[t+1]\setminus[t]} \mathscr{A}_i \cup \bigcup_{i\in N_R(t+1)} \mathscr{B}_i,\\
R_{t+1} &:= R[N_R[t+1]\setminus[t]] \cup R_B[N_{R_B}[t+1]].
\end{aligned}$$

Note that $\mathscr{P} := (\mathcal{H}_{t+1}, G_{t+1}, R_{t+1}, \mathcal{A}_{t+1}, \psi_t|_{E(\mathcal{A}_{t+1})})$ is a packing instance of size $\deg_{R_{t+1}}(t+1)$ with $t+1$ playing the role of 0, and we claim that $\mathscr{P}$ is indeed an $(\varepsilon_{c(t+1)-1}, \mathbf{d})$-packing instance, where $\mathbf{d} = \big(d_A, d_B, (\alpha_A d_A^{m_i(t)})_{i\in N_R[t+1]\setminus[t]}, (\alpha_B d_B^{m_i(t)})_{i\in N_R(t+1)}\big)$. Observe that by definition of $c_i(t)$ and $m_i(t)$ in (4.5.2), we have:

(4.5.10)
If $i \in N_R(t+1)$, then $m_i(t+1) = m_i(t)+1$, and $c(t+1) = c_i(t+1) > \max\{c_i(t), c_j(t)\}$ for all $j \in N_R(i)$. If $i \in [r]\setminus N_R(t+1)$, then $m_i(t+1) = m_i(t)$.

Note that for the inequality in (4.5.10) we used that no pair of adjacent vertices in R has two neighbours in R that are coloured alike as we have chosen the vertex colouring as a colouring in R^3. In particular, we infer from (4.5.10) that $\varepsilon_{c(t+1)-1} = \varepsilon_{c_i(t+1)-1} \geq \varepsilon_{c_i(t)}$ for all $i \in N_R(t+1)$. Therefore, (P1) follows from (4.5.4), property (P2) follows from $\mathbf{S}(t)$(a), property (P3) follows from $\mathbf{S}(t)$(d) with $R[N_R[t+1]\setminus[t]]$ playing the role of R_A, and (P4) follows from $\mathbf{S}(t)$(b).

Observe further that

- ψ_t as defined in (4.5.8) satisfies $\|\psi_t\| \leq \alpha^{-1}$;
- $\sum_{H\in\mathcal{H}} e_H(X_{t+1}^H, X_i^H) \leq (1-\alpha)dn^2 \leq d_A n^2$ for all $i \in N_R(t+1)\setminus[t]$;
- $\Delta_{\psi_t}^c(\mathscr{A}_i) \leq \sqrt{n}$ for all $i \in N_R[t+1]\setminus[t]$ by $\mathbf{S}(t)$(c).

Hence, we can apply Lemma 4.9 to $\mathscr{P}$ with

n	$\varepsilon_{c(t+1)-1}$	$\varepsilon_{c(t+1)}$	α^{-1}	$\deg_{R_{t+1}}(t+1)$	$R[N_R[t+1]\setminus[t]]$
n	ε	ε'	s	r	R_A

and with set testers $\mathcal{W}_{set}^{t+1}$ where we denote by $\mathcal{W}_{set}^{t+1} \subseteq \mathcal{W}_{set}$ the set of set testers $(W, Y_1, \ldots, Y_\ell)$ with $W \subseteq V_{t+1}$, and with edge testers $\mathcal{W}_{edge}^{t+1} \cup \mathcal{W}_{edge}^*$ where we will define the set $\mathcal{W}_{edge}^*$ when proving $\mathbf{S}(t+1)$(d) and (e) in Steps 3.3 and 3.4.

Let $\sigma\colon \mathscr{X}_{t+1}^\sigma \to V_{t+1}^\sigma$ be the conflict-free packing in $\mathscr{A}_{t+1}$ obtained from Lemma 4.9 with $|\mathscr{X}_{t+1}^\sigma \cap X_{t+1}^H| \geq (1-\varepsilon_{c_i(t+1)})n$ for all $H \in \mathcal{H}$, which extends ϕ_t to ϕ_{t+1} as defined in (4.5.9). Fix some $H \in \mathcal{H}$. By Definition 4.7, the updated candidacy graphs with respect to σ obtained from Lemma 4.9 are also updated candidacy graphs with respect to ϕ_{t+1} as defined in (4.5.7) in Step 2. Hence, the graphs $A_i^{H,new}$ in Lemma 4.9 correspond to subgraphs $\widetilde{A}_i^H \subseteq A_i^H(\phi_{t+1})$ for all $i \in N_R(t+1)\setminus[t]$, and $\widetilde{B}_i^H \subseteq B_i^H(\phi_{t+1})$ for all $i \in N_R(t+1)$ that satisfy $(\mathrm{I})_{\mathrm{L4.9}}$–$(\mathrm{VII})_{\mathrm{L4.9}}$.

<u>*Step 3.1.*</u> *Checking* $\mathbf{S}(t+1)$(a)

By $(\mathrm{I})_{\mathrm{L4.9}}$ and (4.5.10), we obtain that $\widetilde{A}_i^H$ is $(\varepsilon_{c_i(t+1)}, \alpha_A d_A^{m_i(t+1)})$-super-regular for all $i \in N_R(t+1)\setminus[t]$, and $\widetilde{B}_i^H$ is $(\varepsilon_{c_i(t+1)}, \alpha_B d_B^{m_i(t+1)})$-super-regular for all $i \in N_R(t+1)$. Note that for each $i \in [r]\setminus N_R(t+1)$, we have $m_i(t) = m_i(t+1)$ and $A_i^H(\phi_t) = A_i^H(\phi_{t+1})$ and $B_i^H(\phi_t) = B_i^H(\phi_{t+1})$. For $i \in [r] \setminus [t+1], i' \in [r]$, let

$$\widehat{A}_i^H := \begin{cases} \widetilde{A}_i^H & \text{if } i \in N_R(t+1) \setminus [t], \\ A_i^H & \text{otherwise.} \end{cases} \qquad \widehat{B}_{i'}^H := \begin{cases} \widetilde{B}_{i'}^H & \text{if } i' \in N_R(t+1), \\ B_{i'}^H & \text{otherwise.} \end{cases} \tag{4.5.11}$$

Then the graphs $\widehat{A}_i^H$ and $\widehat{B}_{i'}^H$ are candidacy graphs satisfying $\mathbf{S}(t+1)$(a).

Step 3.2. *Checking* $\mathbf{S}(t+1)$(b) *and* $\mathbf{S}(t+1)$(c)

The new edge set labelling ψ_{t+1} as defined in (4.5.8) corresponds to the updated edge set labelling as in Definition 4.8. By $(\mathrm{IV})_{\mathrm{L4.9}}$, $\mathbf{S}(t)$(b) and (4.5.10), we obtain for every $i \in [r] \setminus [t+1]$ that $\Delta_{\psi_{t+1}}(\bigcup_{H\in\mathcal{H}} \widehat{A}_i^H) \le (1+\varepsilon_{c_i(t+1)})\alpha_A d_A^{m_i(t+1)}|V_i|$. This establishes $\mathbf{S}(t+1)$(b). Similarly, by $(\mathrm{V})_{\mathrm{L4.9}}$ with $R[N_R[t+1]\setminus[t]]$ playing the role of R_A and by $\mathbf{S}(t)$(c), we obtain for every $i \in [r] \setminus [t+1]$ that $\Delta^c_{\psi_{t+1}}(\bigcup_{H\in\mathcal{H}} \widehat{A}_i^H) \le \sqrt{n}$, which establishes $\mathbf{S}(t+1)$(c).

Step 3.3. *Checking* $\mathbf{S}(t+1)$(d)

In order to show $\mathbf{S}(t+1)$(d), fix $H \in \mathcal{H}$, $ij \in E(R-[t+1])$, and $v_iv_j \in E(G[V_i,V_j])$. Observe, that $|\{i,j\} \cap N_R(t+1)| \in \{0,1,2\}$.

If $|\{i,j\} \cap N_R(t+1)| = 2$, then this implies together with (4.5.10) that $c_i(t+1) = c_j(t+1) = c(t+1) > \max\{c_i(t), c_j(t)\}$, and $m_i(t)+1 = m_i(t+1)$ as well as $m_j(t)+1 = m_j(t+1)$. Hence we obtain by $(\mathrm{II})_{\mathrm{L4.9}}$ (with $R[N_R[t+1]\setminus[t]]$ playing the role of R_A) and $\mathbf{S}(t)$(d) that

$$e_H(N_{\widehat{A}_i^H}(v_i), N_{\widehat{A}_j^H}(v_j)) = \left(\alpha_A^2 d_A^{m_i(t+1)+m_j(t+1)} \pm \varepsilon_{\max\{c_i(t+1),c_j(t+1)\}}\right) e_H(X_i^H, X_j^H). \tag{4.5.12}$$

If $|\{i,j\}\cap N_R(t+1)| = 1$, say $i \in N_R(t+1)$, then this implies together with (4.5.10) that

$$c_i(t+1) = \max\{c_i(t+1), c_j(t+1)\} = c(t+1) > \max\{c_i(t), c_j(t)\}, \text{ and } m_i(t)+1 = m_i(t+1),\ m_j(t) = m_j(t+1). \tag{4.5.13}$$

By (4.5.11), we have that $\widehat{A}_j^H = A_j^H$ because $j \notin N_R(t+1)$. Let $N := N_{A_i^H}(v_i) \cap N_H(N_{A_j^H}(v_j))$ and we define a weight function $\omega_N \colon E(\mathscr{A}_i) \to \{0,1\}$ by $\omega_N(xv) := \mathbb{1}_{\{v=v_i\}}\mathbb{1}_{\{x\in N\}}$ and add ω_N to $\mathcal{W}^*_{edge}$. Note that $\dim(\omega_N) = 1$ (with $\dim(\omega_N)$ defined as in (4.4.2)) and that $\omega_N(E(\mathscr{A}_i)) = |N| = (\alpha_A^2 d_A^{m_i(t)} d_A^{m_j(t)} \pm \varepsilon_{\max\{c_i(t),c_j(t)\}}) e_H(X_i^H, X_j^H)$ by $\mathbf{S}(t)$(d). This implies that

$$\begin{aligned} e_H(N_{\widehat{A}_i^H}(v_i), N_{\widehat{A}_j^H}(v_j)) &= |N_{\widehat{A}_i^H}(v_i) \cap N_H(N_{A_j^H}(v_j))| = |N_{\widehat{A}_i^H}(v_i) \cap N| = \omega_N(E(\widehat{A}_i^H)) \\ &\overset{(\mathrm{III})_{\mathrm{L4.9}}}{=} (1 \pm \varepsilon_{c(t+1)}^2) d_A \omega_N(E(\mathscr{A}_i)) \pm \varepsilon_{c(t+1)}^2 n \\ &\overset{(4.5.13)}{=} \left(\alpha_A^2 d_A^{m_i(t+1)+m_j(t+1)} \pm \varepsilon_{\max\{c_i(t+1),c_j(t+1)\}}\right) e_H(X_i^H, X_j^H). \end{aligned}$$

If $|\{i,j\}\cap N_R(t+1)| = 0$, then this implies together with (4.5.10) and (4.5.11), that $m_i(t) = m_i(t+1)$, $m_j(t) = m_j(t+1)$, and $\widehat{A}_i^H = A_i^H$. Consequently, (4.5.12) holds which establishes $\mathbf{S}(t+1)$(d).

Step 3.4. *Checking* $\mathbf{S}(t+1)$(e)

In order to establish $\mathbf{S}(t+1)$(e), we first consider $v_i \in V_i$ for $i \in N_R(t+1) \cap [t]$. We define a weight function $\omega_i^*\colon E(\mathscr{A}_{t+1}) \to \{0,1\}$ by $\omega_{v_i}^*(xv) := \mathbb{1}_{\{x \in \mathscr{X}_i^*\}}$ for $\mathscr{X}_i^* := N_{\mathcal{H}}(\phi_t^{-1}(v_i)) \cap \mathscr{X}_{t+1}$ and every $xv \in E(\mathscr{A}_{t+1})$, and we add ω_{v_i} to $\mathcal{W}_{edge}^*$. By $\mathbf{S}(t)$(a), we have

$$\omega_{v_i}(E(\mathscr{A}_{t+1})) = (\alpha_A d_A^{m_{t+1}(t)} \pm 3\varepsilon_{c_{t+1}(t)})|\mathscr{X}_i^*|n, \tag{4.5.14}$$

and by $\mathbf{S}(t)$(e), we have

$$|\phi_{t+1}^{-1}(v_i) \cap N_{\mathcal{H}}(\mathcal{X}_{t+1} \setminus \mathcal{X}_{t+1}^{\phi_{t+1}})| \leq \varepsilon_{c_i(t)}^{1/2} n + |\mathscr{X}_i^*| - \omega_{v_i}(M) \tag{4.5.15}$$

with $M = M(\sigma)$ being the corresponding edge set to σ. By $(\mathrm{VII})_{\mathrm{L4.9}}$, we obtain that

$$\omega_{v_i}(M) \overset{(\mathrm{VII})_{\mathrm{L4.9}}}{=} (1 \pm \varepsilon_{c(t+1)}) \frac{\omega_{v_i}(E(\mathscr{A}_{t+1}))}{\alpha_A d_A^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n \overset{(4.5.14)}{\geq} (1 - \varepsilon_{c(t+1)}^{3/4})|\mathscr{X}_i^*| - \varepsilon_{c(t+1)} n.$$

Together with (4.5.15), this implies that $|\phi_{t+1}^{-1}(v_i) \cap N_{\mathcal{H}}(\mathcal{X}_{t+1} \setminus \mathcal{X}_{t+1}^{\phi_{t+1}})| \leq \varepsilon_{c(t+1)}^{1/2} n$.

Hence, it now suffices to establish $\mathbf{S}(t+1)$(e) for all $v_{t+1} \in V_{t+1}$ by $\mathbf{S}(t)$(e). We define weight functions $\omega_{v_{t+1}}, \omega_{v_{t+1}}^*\colon E(\mathscr{A}_{t+1}) \to \{0,1\}$ by $\omega_{v_{t+1}}(xv) := \mathbb{1}_{\{v = v_{t+1}\}}$ and $\omega_{v_{t+1}}^*(xv) := \mathbb{1}_{\{v = v_{t+1} \text{ and } x \in \mathscr{X}_{t+1}^*\}}$ for $\mathscr{X}_{t+1}^* := N_{\mathcal{H}}(\mathcal{X}_t \setminus \mathcal{X}_t^{\phi_t}) \cap \mathscr{X}_{t+1}$ and every $xv \in E(\mathscr{A}_{t+1})$, and we add $\omega_{v_{t+1}}$ and $\omega_{v_{t+1}}^*$ to $\mathcal{W}_{edge}^*$. Observe that $\mathbf{S}(t)$ implies that

$$\omega_{v_{t+1}}(E(\mathscr{A}_{t+1})) = (\alpha_A d_A^{m_{t+1}(t)} \pm 3\varepsilon_{c_{t+1}(t)})|\mathcal{H}|n, \tag{4.5.16}$$

$$\omega_{v_{t+1}}^*(E(\mathscr{A}_{t+1})) \leq |\mathscr{X}_{t+1}^*| \leq \varepsilon_{c_{t+1}(t)}^{1/2} |\mathcal{H}|n, \tag{4.5.17}$$

and we have that $|\phi_{t+1}^{-1}(v_{t+1})| = \omega_{v_{t+1}}(M)$ and $|\phi_{t+1}^{-1}(v_{t+1}) \cap \mathscr{X}_{t+1}^*| \leq \omega_{v_{t+1}}^*(M)$. By $(\mathrm{VII})_{\mathrm{L4.9}}$, we obtain that

$$\omega_{v_{t+1}}(M) \overset{(\mathrm{VII})_{\mathrm{L4.9}}}{=} (1 \pm \varepsilon_{c(t+1)}) \frac{\omega_{v_{t+1}}(E(\mathscr{A}_{t+1}))}{\alpha_A d_A^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n \overset{(4.5.16)}{\geq} (1 - \varepsilon_{c(t+1)}^{1/2})|\mathcal{H}| - \varepsilon_{c(t+1)} n;$$

$$\omega_{v_{t+1}}^*(M) \overset{(\mathrm{VII})_{\mathrm{L4.9}}}{=} (1 \pm \varepsilon_{c(t+1)}) \frac{\omega_{v_{t+1}}^*(E(\mathscr{A}_{t+1}))}{\alpha_A d_A^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n \overset{(4.5.17)}{\leq} \varepsilon_{c(t+1)}^{1/2} n.$$

Note that $\varepsilon_{c(t+1)} = \varepsilon_{c_{t+1}(t+1)}$. Altogether, this establishes $\mathbf{S}(t+1)$(e).

Step 3.5. Checking $\mathbf{S}(t+1)$(f)–(h)

In order to establish $\mathbf{S}(t+1)$(f), consider $\omega_v \in \mathcal{W}_{edge}^i$ for $i \in N_R(t+1) \setminus [t+1]$. By (4.5.10), it holds that $c(t+1) = c_i(t+1)$. With $(\mathrm{III})_{\mathrm{L4.9}}$ we obtain that

$$\begin{aligned} \omega_v(E(\textstyle\bigcup_{H \in \mathcal{H}} \widehat{A}_i^H)) &= (1 \pm \varepsilon_{c(t+1)}^2) d_A \omega_v(E(\mathscr{A}_i)) \pm \varepsilon_{c(t+1)}^2 n^2 \\ &\overset{\mathbf{S}(t)(\mathrm{f})}{=} \alpha_A d_A^{m_i(t+1)} \omega(\mathscr{X}_i) \pm \varepsilon_{c_i(t+1)} n^2, \end{aligned}$$

which together with $\mathbf{S}(t)$(f) establishes $\mathbf{S}(t+1)$(f).

Next we verify $\mathbf{S}(t+1)$(g). Note that $(\mathrm{VI})_{\mathrm{L4.9}}$ implies that $|W \cap \bigcap_{j \in [\ell]} \sigma(Y_j \cap \mathscr{X}_{t+1}^\sigma)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \varepsilon_{c(t+1)} n$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}^{t+1}$, which together with $\mathbf{S}(t)$(g) yields $\mathbf{S}(t+1)$(g).

In order to establish $\mathbf{S}(t+1)$(h), let $\mathcal{W}_{ver}^{t+1} \subseteq \mathcal{W}_{ver}$ be the set of vertex testers (v, ω) with $v \in V_{t+1}$. Hence, for all $(v, \omega) \in \mathcal{W}_{ver}^{t+1}$ and and its corresponding edge tester

$\omega_v \in \mathcal{W}^{t+1}_{edge}$ as defined in (4.5.3), property $(\mathrm{VII})_{\mathrm{L4.9}}$ implies that

$$\begin{aligned}\omega(\mathscr{X}_{t+1} \cap \sigma^{-1}(v)) = \omega_v(M) &\overset{(\mathrm{VII})_{\mathrm{L4.9}}}{=} (1 \pm \varepsilon_{c(t+1)})\frac{\omega_v(E(\mathscr{A}_{t+1}))}{\alpha_A d_A^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n \\ &\overset{\mathbf{S}(t)(\mathrm{f})}{=} (1 \pm \varepsilon_{c(t+1)})\frac{\alpha_A d_A^{m_{t+1}(t)}\omega(\mathscr{X}_{t+1}) \pm \varepsilon_{c_{t+1}(t)} n^2}{\alpha_A d_A^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n \\ &= \frac{\omega(\mathscr{X}_{t+1})}{n} \pm \alpha n/2.\end{aligned}$$

Together with $\mathbf{S}(t)$(h), this yields $\mathbf{S}(t+1)$(h).

Step 4. Completion

Let $\phi_r\colon \bigcup_{H\in\mathcal{H}, i\in[r]_0} \widehat{X}_i^H \to \mathcal{V}_r$ be an r-partial packing satisfying $\mathbf{S}(r)$ with (ε_T, d_i)-super-regular candidacy graphs $B_i^H \subseteq B_i^H(\phi_r)$ where $d_i := \alpha_B d_B^{\deg_R(i)}$ for all $i \in [r]$. We aim to apply iteratively the ordinary blow-up lemma in order to complete the partial packing ϕ_r using the edges in G_B. Recall that $\varepsilon_T \ll \mu \ll \gamma \ll \alpha, d$. Our general strategy is as follows. For every $H \in \mathcal{H}$ in turn, we choose a set $X_i \subseteq X_i^H$ for all $i \in [r]$ of size roughly μn by selecting every vertex uniformly at random with the appropriate probability and adding $X_i^H \setminus \widehat{X}_i^H$ deterministically. Afterwards, we apply the blow-up lemma to embed $H[X_1, \dots, X_r]$ into G_B, which together with ϕ_r yields a complete embedding of H into $G_A \cup G_B$. Before we proceed with the details of our procedure (see Claim 3), we verify in Claim 2 that we can indeed apply the blow-up lemma to a subgraph of $H \in \mathcal{H}$ provided some easily verifiable conditions are satisfied.

Recall that we have defined the candidacy graph B_i^H on a copy $(X_i^{H,B}, V_i^B)$ via the bijection π only to conveniently apply Lemma 4.9 in Step 3. That is, for all $H \in \mathcal{H}, i \in [r]$, we can identify B_i^H with an isomorphic bipartite graph on (X_i^H, V_i) and edge set $\{xv\colon \pi(x)\pi(v) \in E(B_i^H)\}$. Let $\mathcal{B}$ be the union over all $H \in \mathcal{H}, i \in [r]$ of these graphs. For $H \in \mathcal{H}$ and a subgraph G° of G_B, we write $\mathcal{B}_{G^\circ}[X_i^H, V_i]$ for the graph that arises from $\mathcal{B}[X_i^H, V_i]$ by deleting every edge xv with $x \in X_i^H, v \in V_i$ for which there exists $x'x \in E(H)$ such that $\phi_r(x')v \in E(G^\circ)$. (We may think of $E(G^\circ)$ as the edge set in G_B that we have already used in our completion step for packing some other graphs of $\mathcal{H}$ into $G_A \cup G_B$.) For future reference, we observe that

$$\Delta(\mathcal{B}[X_i^H, V_i] - \mathcal{B}_{G^\circ}[X_i^H, V_i]) \leq \alpha^{-1}\Delta(G^\circ) \text{ for all } i \in [r]. \tag{4.5.18}$$

Claim 2. Suppose $H \in \mathcal{H}$, $G^\circ \subseteq G_B$ and $W_i \subseteq V_i$ for all $i \in [r]$ such that the following hold:

(a) *$V_i \setminus \phi_r(\widehat{X}_i^H) \subseteq W_i \subseteq V_i$ and $|W_i| = (\mu \pm \varepsilon_T^{1/2})n$ for all $i \in [r]$;*

(b) *$\mathcal{B}_{G^\circ}[X_i, W_i]$ is $(\mu^{1/31}, d_i)$-super-regular where $X_i := (X_i^H \cap \phi_r^{-1}(W_i \cap \phi_r(\widehat{X}_i^H))) \cup (X_i^H \setminus \widehat{X}_i^H)$;*

(c) *$G_B[W_i, W_j]$ is $(\varepsilon_T^{1/3}, d_B)$-super-regular for all $ij \in E(R)$;*

(d) *$|N_{G^\circ}(v) \cap W_i| \leq \mu^{3/2} n$ for all $v \in V_j$ and $ij \in E(R)$.*

Then there exists an embedding $\phi^{\widetilde{H}}$ of $\widetilde{H} := H[X_1, \dots, X_r]$ into $\widetilde{G} := G_B[W_1, \dots, W_r] - G^\circ$ such that $\phi' := \phi^{\widetilde{H}} \cup \phi_r|_{V(H)\setminus V(\widetilde{H})}$ is an embedding of H into G where $\phi'(X_i^H) = V_i$

for all $i \in [r]$ and all edges incident to a vertex in $\bigcup_{i\in[r]} X_i$ are embedded on an edge in $G_B - G^\circ$.

Proof of claim: Note that $|X_i| = |W_i| = (\mu \pm \varepsilon_T^{1/2})n$ (by (a) and (b)) and $\widetilde{H}[X_i, X_j]$ is empty whenever $ij \notin E(R)$ and a matching otherwise. Moreover, by (c) and (d), Fact 1.12 yields that $\widetilde{G}[W_i, W_j]$ is $(\mu^{1/31}, d_B)$-super-regular for all $ij \in E(R)$ (with room to spare). This shows that $(\widetilde{H}, \widetilde{G}, R, (X_i)_{i\in[r]}, (W_i)_{i\in[r]})$ is a $(\mu^{1/31}, d_B)$-super-regular blow-up instance. We apply Theorem 4.5 to this blow-up instance (with $(\mathcal{B}_{G^\circ}[X_i, W_i])_{i\in[r]}$ playing the role of $(A_i)_{i\in[r]}$) and obtain an embedding $\phi^{\widetilde{H}}$ of $\widetilde{H}$ into $\widetilde{G}$. Recall that $\phi_r|_{V(H)\setminus V(\widetilde{H})}$ is an embedding of $H - V(\widetilde{H})$ into G_A. For $x \in V(\widetilde{H})$, $x' \in V(H) \setminus V(\widetilde{H})$ and $xx' \in E(H)$, we conclude that $\phi^{\widetilde{H}}(x)\phi_r(x') \in E(G_B - G^\circ)$ by the definition of the candidacy graphs in (4.5.7) and the definition of $\mathcal{B}_{G^\circ}$. $-$

Claim 3. For all $H \in \mathcal{H}$ and $i \in [r]$, there exist sets $\overline{X}_i^H \subseteq \widehat{X}_i^H$ with $|\overline{X}_i^H| \geq (1-2\mu)n$ and a packing ϕ of $\mathcal{H}$ into G that extends ϕ^* such that $\phi|_{\overline{X}_i^H} = \phi_r|_{\overline{X}_i^H}$ as well as $\phi(X_i^H) = V_i$.

Proof of claim: We write $\mathcal{H} = \{H_1, \dots, H_{|\mathcal{H}|}\}$ and let $\mathcal{H}_h := \{H_1, \dots, H_h\}$ for all $h \in [|\mathcal{H}|]_0$. For all $v \in V_i$, $i \in [r]$, let $\tau^h(v) := |\{H \in \mathcal{H}_h : v \in V_i \setminus \phi_r(\widehat{X}_i^H)\}|$ and $\sigma^h(v) := |\{\phi_r^{-1}(v) \cap \bigcup_{H\in\mathcal{H}_h, j\in N_R(i)} N_H(X_j^H \setminus \widehat{X}_j^H)\}|$. We inductively prove that the following statement $\mathbf{C}(h)$ holds for all $h \in [|\mathcal{H}|]_0$.

C(h). There exists a packing ϕ^h of $\mathcal{H}_h$ into G that extends ϕ^* such that for $G_h^\circ := G_B \cap \phi^h(\mathcal{H}_h)$ we have

(A) $\deg_{G_h^\circ}(v) \leq \alpha^{-1}(\tau^h(v) + \sigma^h(v)) + \mu^{2/3}n$ for all $v \in V(G)$;

(B) for all $H \in \mathcal{H}_h$ and $i \in [r]$, there exist sets $\overline{X}_i^H \subseteq \widehat{X}_i^H$ with $|\overline{X}_i^H| \geq (1-2\mu)n$ such that $\phi^h|_{\overline{X}_i^H} = \phi_r|_{\overline{X}_i^H}$ as well as $\phi^h(X_i^H) = V_i$.

Let G_0° be the edgeless graph on $V(G)$ and ϕ^0 be the empty function; then $\mathbf{C}(0)$ holds. Hence, we may assume the truth of $\mathbf{C}(h)$ for some $h \in [|\mathcal{H}|-1]_0$ and let ϕ^h and G_h° be as in $\mathbf{C}(h)$.

By $\mathbf{C}(h)$(B), there are at most $\sum_{H\in\mathcal{H}_h, j\in N_R[i]} \alpha^{-1}|X_j^H \setminus \overline{X}_j^H| \leq 5\alpha^{-3}\mu n^2$ edges of G_h° incident to a vertex in V_i for each $i \in [r]$. Hence, there are at most $10\alpha^{-3}\mu^{1/3}n$ vertices in V_i of degree at least $\mu^{2/3}n/2$ in G_h°. Let $V_i^{high} \subseteq V_i$ be a set of size $\mu^{1/4}n$ that contains all vertices of degree at least $\mu^{2/3}n/2$ in G_h°. For all $i \in [r]$, we select every vertex in $\phi_r(\widehat{X}_i^{H_{h+1}}) \setminus V_i^{high}$ independently with probability $\mu(1-\mu^{1/4})^{-1}$ and denote by W_i their union together with $V_i \setminus \phi_r(\widehat{X}_i^{H_{h+1}})$; we define $X_i := (X_i^{H_{h+1}} \cap \phi_r^{-1}(W_i \cap \phi_r(\widehat{X}_i^{H_{h+1}}))) \cup (X_i^{H_{h+1}} \setminus \widehat{X}_i^{H_{h+1}})$. Note that $\mathbf{S}(r)$ implies that

$$\left|X_i^{H_{h+1}} \setminus \widehat{X}_i^{H_{h+1}}\right| = \left|V_i \setminus \phi_r(\widehat{X}_i^{H_{h+1}})\right| \leq 2\varepsilon_T n \text{ for all } i \in [r]. \tag{4.5.19}$$

In the following we will show that the assumptions of Claim 2 are satisfied with probability at least $1/2$, say, for H_{h+1} and G_h° playing the roles of H and G°, respectively. In particular, there is a choice for W_i such that the assumptions of Claim 2 hold.

To obtain (a), we apply Chernoff's inequality to the sum of indicator variables which indicate whether a vertex in V_i is randomly selected. Together with (4.5.19), this shows that (a) holds with probability at least $1 - 1/n^3$, say.

By $\mathbf{C}(h)$(A) and $\mathbf{S}(r)$(e), we obtain that $\Delta(G_h^\circ) \leq 2\mu^{2/3}n$. We exploit (4.5.18) and conclude that $\Delta(\mathcal{B}[X_i^{H_{h+1}}, V_i] - \mathcal{B}_{G_h^\circ}[X_i^{H_{h+1}}, V_i]) \leq 2\alpha^{-1}\mu^{2/3}n$ for all $i \in [r]$. Thus Fact 1.12 implies that $\mathcal{B}_{G_h^\circ}[X_i^{H_{h+1}}, V_i]$ is $(\mu^{1/5}, d_i)$-super-regular for all $i \in [r]$. For all $i \in [r]$ and $x \in X_i^{H_{h+1}}$, Chernoff's inequality implies that $|N_{\mathcal{B}_{G_h^\circ}}(x) \cap W_i| = (d_i \pm 2\mu^{1/5})|W_i|$ and similarly, for all $v \in V_i$, we have $|N_{\mathcal{B}_{G_h^\circ}}(v) \cap X_i| = (d_i \pm 2\mu^{1/5})|X_i|$. Moreover, for all distinct v, v' with $|N_{\mathcal{B}_{G_h^\circ}}(v, v')| = (d_i \pm 2\mu^{1/5})^2 n$ (which we call *good*, and there are at least $(1 - 2\mu^{1/5})\binom{n}{2}$ good pairs), we also obtain $|N_{\mathcal{B}_{G_h^\circ}}(v, v') \cap X_i| = (d_i \pm 3\mu^{1/5})^2|X_i|$, all with probability at least $1 - 1/n^3$. Observe that Theorem 1.8 implies that there at least $(1 - 3\mu^{1/5})\binom{\mu n}{2}$ good pairs in W_i also with probability at least $1 - 1/n^3$. Therefore, we may apply Theorem 1.13 and obtain that $\mathcal{B}_{G_h^\circ}[X_i, W_i]$ is $(\mu^{1/31}, d_i)$-super-regular for all $i \in [r]$ which yields (b) with probability at least $1 - 1/n^2$, say.

To obtain (c), for each $ij \in E(R)$, we proceed as follows. Observe first that $G_B[V_i, V_j]$ is $(2\varepsilon, d_B)$-super-regular by (4.5.4). Hence $G_B[W_i, W_j]$ is clearly $\varepsilon_T^{1/3}$-regular as $|W_i|, |W_j| \geq \mu n/2$ by (a). Therefore, we only need to control the degrees of the vertices in $G_B[W_i, W_j]$ which follows directly by Chernoff's inequality with probability at least $1 - 1/n^3$ and because of (4.5.19).

Since $\Delta(G_h^\circ) \leq 2\mu^{2/3}n$ and because of (4.5.19), we conclude by Chernoff's inequality that (d) holds with probability at least $1 - 1/n^3$.

Therefore, the assumptions of Claim 2 are achieved by our construction with probability at least 1/2. Fix such a choice for $W_1, \dots, W_r$ and apply Claim 2 that returns an embedding $\phi^{\widetilde{H}_{h+1}}$ of $\widetilde{H}_{h+1} := H_{h+1}[X_1, \dots, X_r]$ into $G_B[W_1, \dots, W_r] - G_h^\circ$ such that $\phi' := \phi^{\widetilde{H}_{h+1}} \cup \phi_r|_{V(H_{h+1}) \setminus V(\widetilde{H}_{h+1})}$ is an embedding of H_{h+1} into G where $\phi'(X_i^H) = V_i$ and all edges incident to a vertex in $\bigcup_{i \in [r]} X_i$ are embedded on an edge in $G_B - G_h^\circ$. We define $\phi^{h+1} := \phi^h \cup \phi'$ and obtain $\mathbf{C}(h+1)$(B) with $\overline{X}_i^{H_{h+1}} := X_i^{H_{h+1}} \setminus X_i$. It is straightforward to check that by our construction also $\mathbf{C}(h+1)$(A) holds. −

Let ϕ be as in Claim 3. This directly implies conclusion (i) of Lemma 4.10. Conclusion (ii) of Lemma 4.10 follows from Claim 3 together with $\mathbf{S}(r)$(g) as we merely modified ϕ_r to obtain ϕ. For a similar reason, Lemma 4.10 (iii) follows from Claim 3 and $\mathbf{S}(r)$(h). This completes the proof. □

4.6 Applications

4.6.1 Proof of Theorem 4.4

In this section we prove Theorem 4.4 which provides an illustration for an application of vertex and set testers so that the leftover is suitably well-behaved. For a graph H, let H^+ arise from H by adding a labelled vertex x and joining x to all vertices of H. We call x the *apex vertex* of H^+.

Theorem 4.11 (Keevash [72, cf. Theorem 7.8], cf. [51, Theorem 3.6 and Corollary 3.7]). *Suppose $1/n \ll \varepsilon \ll 1/s \ll d_0, 1/m$. Suppose H is an r-regular graph on m vertices. Let G be a graph with vertex partition (V, W) such that W is an independent set, $d_0 n \leq |W| \leq |V| = n$ and $|\bigcap_{x \in V' \cup W'} N_G(x)| = (1 \pm \varepsilon)d_V^{|V'|}d_W^{|W'|}n$ for all $V' \subseteq V, W' \subseteq W$ with $1 \leq |V'| + |W'| \leq s$ where $d_V = rd|W|/n$ and $d_W = d$ for some $d \geq d_0$. Suppose that $|N_G(v) \cap V| = r|N_G(v) \cap W|$ for all $v \in V$ and m divides $\deg_G(w)$ for all $w \in W$. Then there is a decomposition of the edge set of G into copies of H^+ where the apex vertices are contained in W.*

Proof of Theorem 4.4. Choose $\varepsilon \ll \delta \ll 1/s \ll \beta \ll \alpha$. Among the αn graphs in $\mathcal{H}$ that contain at least αn vertices in components of size at most α^{-1}, there is a collection $\mathcal{H}'$ of βn r-regular graphs that contain each at least βn vertices in components all isomorphic to some graph J where $|V(J)| \leq \alpha^{-1}$ and $r \in [\alpha^{-1}]$.

For all $H \in \mathcal{H}'$, let H^- arise from H by deleting $\beta n/|V(J)|$ components isomorphic to J. We denote by I^H a set of βn isolated vertices disjoint from $V(H^-)$. Let $\widetilde{\mathcal{H}} := (\mathcal{H} \setminus \mathcal{H}') \cup \bigcup_{H \in \mathcal{H}'} (H^- \cup I^H)$. Let G_1 be a $(2\varepsilon, s, d_1)$-typical subgraph of G where $d_1 := (1+\delta)(d-\beta^2 r)$ such that $G-G_1$ is $(2\varepsilon, s, d-d_1)$-typical; that is, $e(\widetilde{\mathcal{H}}) \leq (1-\delta/2)e(G_1)$. Clearly, G_1 exists by considering a random subgraph and then applying Chernoff's inequality. Now we apply Theorem 4.3 to obtain a packing ϕ of $\widetilde{\mathcal{H}}$ in G_1 with δ^2 playing the role of α and sets of set and vertex testers $\mathcal{W}_{set}$, $\mathcal{W}_{ver}$ defined as follows. For all $H_1, \ldots, H_{\ell_1} \in \mathcal{H}'$ and $v_1, \ldots, v_{\ell_2} \in V(G)$ with $1 \leq \ell_1$ and $\ell_1 + \ell_2 \leq s$, we add the set tester $(V', I^{H_1}, \ldots, I^{H_{\ell_1}})$ to $\mathcal{W}_{set}$ where $V' := V(G) \cap \bigcap_{i \in [\ell_2]} N_{G-G_1}(v_i)$. Then Theorem 4.3 implies that (where $d_2 := d - d_1$)

$$\left| \bigcap_{i \in [\ell_1]} \phi(I^{H_i}) \cap \bigcap_{i \in [\ell_2]} N_{G-G_1}(v_i) \right| = (\beta^{\ell_1} d_2^{\ell_2} \pm 2\delta^2)n. \tag{4.6.1}$$

For each $v \in V(G)$, we define a vertex tester (v, ω) where ω assigns every vertex in $V(H)$ its degree for all $H \in \widetilde{\mathcal{H}}$ and add (v, ω) to $\mathcal{W}_{ver}$. Then Theorem 4.3 implies that $\Delta(G_1 - \phi(\widetilde{\mathcal{H}})) \leq 2\delta n$.

Let $G_2 := G - \phi(\widetilde{\mathcal{H}})$. Next, we add βn vertices $W := \{v_H\}_{H \in \mathcal{H}'}$ to G_2 and join v_H to all vertices in $\phi(I^H)$ and denote this new graph by G_3. Let $d_V := \beta^2 r$ and $d_W := \beta$. Hence (4.6.1), the typicality of $G - G_1$, and $\Delta(G_1 - \phi(\widetilde{\mathcal{H}})) \leq 2\delta n$ imply that for all $w_1, \ldots, w_{\ell_1} \in W$ and $v_1, \ldots, v_{\ell_2} \in V(G)$

$$\bigcap_{x \in \{w_1, \ldots, w_{\ell_1}, v_1, \ldots, v_{\ell_2}\}} N_{G_3}(x) = (1 \pm \sqrt{\delta})\beta^{\ell_1} \cdot (\beta^2 r)^{\ell_2} n = (1 \pm \sqrt{\delta}) d_W^{\ell_1} d_V^{\ell_2} n$$

whenever $1 \leq \ell_1 + \ell_2 \leq s$. We apply Theorem 4.11 to G_3 to obtain a decomposition of G_3 into copies of J^+ where the apex vertices are contained in W. Observe that this yields the desired decomposition of G into $\mathcal{H}$. Indeed, for $H \in \mathcal{H}'$, let $\mathcal{J}_H$ be the set of all copies of J^+ in G_3 whose apex vertex is v_H; hence $|\mathcal{J}_H| = |I^H|/|V(J)| = \beta n/|V(J)|$. We define a packing ϕ' of $\mathcal{H}$ in G as follows. For all $H \in \mathcal{H} \setminus \mathcal{H}'$, let $\phi' |_{V(H)} := \phi |_{V(H)}$. For all $H \in \mathcal{H}'$, let $\phi' |_{V(H^-)} := \phi |_{V(H^-)}$ and each component of $H - V(H^-)$ (which is isomorphic to J) is mapped to $C - v_H$ for some $C \in \mathcal{J}_H$ such that every $C - v_H$ is the image of exactly one component isomorphic to J of $H - V(H^-)$. $\square$

4.6.2 Decomposing directed graphs

In this section we discuss how some modifications of our proof method also yield a blow-up lemma for approximate decomposition of digraphs. In fact, we can derive such a result from a more general statement where we allow the graphs in $H \in \mathcal{H}$ to be edge-coloured, say with k colours, and we are given k host graphs $G_1, \ldots, G_k$ on the same vertex set (whose union may be a multigraph), each with the same partite structure as the graphs in $\mathcal{H}$. Given that for each colour $\ell \in [k]$ and each bipartite pair i, j of the partition, the sum over all $H \in \mathcal{H}$ of ℓ-coloured edges in $E(H[X_i^H, X_j^H])$ is slightly less than the number of edges in $E(G_\ell[V_i, V_j])$, then we can pack $\mathcal{H}$ into $G_1, \ldots, G_k$ such that for all colours $\ell \in [k]$, each ℓ-coloured $\mathcal{H}$-edge is mapped onto G_ℓ. From this general statement one can easily obtain a corresponding approximate decomposition result for digraphs by taking $k = 2$ and colouring all arcs of the same direction with the same colour. Such a setting has already been discussed in [6, Remark 7.4] in the context of embedding a single graph H into sparse host graphs.

Let us state this general result more precisely. For a collection of graphs $\mathcal{H}$, an edge-colouring $\xi\colon E(\mathcal{H}) \to [k]$ and a collection of graphs $\mathcal{G} = \{G_1, \dots, G_k\}$, we call $\phi\colon \bigcup_{H\in\mathcal{H}} V(H) \to \bigcup_{\ell\in[k]} V(G_\ell)$ a *ξ-respecting packing of $\mathcal{H}$ into $\mathcal{G}$* if $\phi|_{V(H)}$ is injective for all $H \in \mathcal{H}$ and for each $\ell \in [k]$, the function ϕ injectively maps ℓ-coloured edges in $\mathcal{H}$ onto edges in G_ℓ. Further, we extend the definition of a blow-up instance to this k-coloured version. For each colour $\ell \in [k]$ and $H \in \mathcal{H}$, let $H[\xi^{-1}(\ell)]$ be the spanning subgraph of H that contains all ℓ-coloured edges, and let $\mathcal{H}[\xi^{-1}(\ell)] := \{H[\xi^{-1}(\ell)]\}_{H\in\mathcal{H}}$. We call $\mathscr{B} = (\mathcal{H}, \mathcal{G}, R, \mathcal{X}, \mathcal{V}, \phi_0, \xi)$ an *(ε, d)-super-regular, α^{-1}-bounded, k-coloured extended blow-up instance* if

- $V(G_\ell) = \mathcal{V}$ for all $\ell \in [k]$;
- $(\mathcal{H}[\xi^{-1}(\ell)], G_\ell, R, \mathcal{X}, \mathcal{V}, \phi_0)$ is an (ε, d)-super-regular, α^{-1}-bounded, extended blow-up instance which is (ε, α)-linked for every $\ell \in [k]$.

Similarly as for an extended blow-up instance, we say that $\mathscr{B}$ is (ε, α)-linked where we replace the second condition by $|V_i \cap \bigcap_{\ell\in[k]\colon x_0\in X_0^H\cap N_{H[\xi^{-1}(\ell)]}(x)} N_{G_\ell}(\phi_0(x_0))| \geq \alpha|V_i|$ for all $x \in X_i^H, i \in [r], H \in \mathcal{H}$. The notion of set and vertex testers as defined in Section 4.1.2 extends analogously to $\mathscr{B}$.

Theorem 4.12 (Ehard, Joos [32]). *Let $1/n \ll \varepsilon \ll \alpha, d, 1/k$ and $1/n \ll 1/r$. Suppose $(\mathcal{H}, \mathcal{G}, R, \mathcal{X}, \mathcal{V}, \phi_0, \xi)$ is an (ε, d)-super-regular, α^{-1}-bounded, k-coloured extended blow-up instance which is (ε, α)-linked, $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$, $|\mathcal{H}| \leq \alpha^{-1}n$, and $\sum_{H\in\mathcal{H}} e_{H[\xi^{-1}(\ell)]}(X_i^H, X_j^H) \leq (1-\alpha)dn^2$ for all $\ell \in [k]$ and $ij \in E(R)$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{\log n}$, respectively. Then there is a ξ-respecting packing ϕ of $\mathcal{H}$ into $\mathcal{G}$ which extends ϕ_0 such that*

(i) $\phi(X_i^H) = V_i$ *for all* $i \in [r]_0$ *and* $H \in \mathcal{H}$*;*

(ii) $|W \cap \bigcap_{j\in[\ell]} \phi(Y_j)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \alpha n$ *for all* $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$*;*

(iii) $\omega(\bigcup_{H\in\mathcal{H}} X_i^H \cap \phi^{-1}(v)) = \omega(\bigcup_{H\in\mathcal{H}} X_i^H)/n \pm \alpha n$ *for all* $(v, \omega) \in \mathcal{W}_{ver}$*.*

In the following, let us discuss how modifications of our proof method also imply Theorem 4.12. Even though Theorem 4.2 can be derived from Theorem 4.12, for the sake of a cleaner presentation of the proof of Theorem 4.2, we only point out the minor modifications that have to be made to turn it into a proof of Theorem 4.12.

Proof sketch of Theorem 4.12

We first observe that the refinement of the partitions via Lemma 4.6 can be performed essentially independently for each $G_1, \dots, G_k \in \mathcal{G}$. Hence, we can proceed similarly as in the proof of Lemma 4.10 and assume that all graphs in $\mathcal{H}$ only span a matching between two clusters respecting the reduced graph R. The two main ingredients for proving Lemma 4.10 are a partial packing result that packs almost all vertices of $\mathcal{H}$ into the host graph, and the completion step that turns this partial packing into a complete one.

We first describe how we can obtain such a partial packing result. To that end, we can proceed analogously as in the inductive Step 3 of the proof of Lemma 4.10 but we have to modify the definition of our candidacy graphs to obtain a partial ξ-respecting packing. That is, $v \in V_i$ should only be a suitable candidate for $x \in X_i^H$ if for each incident ℓ-coloured edge incident to x, there exists also corresponding edge incident to v in G_ℓ. Hence, in more detail, we modify (4.5.6) as follows. Given a t-partial ξ-respecting packing $\phi_t\colon \mathcal{X}_t^{\phi_t} \to \mathcal{V}_t$, and $x \in X_i^H$, $v \in V_i$ for $H \in \mathcal{H}$, $i \in [r]$, we say v *is a ξ-respecting candidate for x given ϕ_t* if

$$\phi_t(N_{H[\xi^{-1}(\ell)]}(x) \cap \mathcal{X}_t^{\phi_t}) \subseteq N_{G_\ell}(v) \text{ for each } \ell \in [k].$$

(Here we ignored the notion of H_+ and G_+ as in (4.5.6) which can be defined analogously.) By adapting the definition of updated candidacy graphs and the updated labelling of the candidacy graphs in Definitions 4.7 and 4.8 for Lemma 4.9, accordingly, one can see that we can inductively apply Lemma 4.9 to obtain a partial ξ-respecting packing similar as in Step 3.

Next, let us describe how we can turn such a partial ξ-respecting packing $\phi_r\colon \bigcup_{H\in\mathcal{H}, i\in[r]_0} \widehat{X}_i^H \to \mathcal{V}_r$ into a complete one. In Step 4 of Lemma 4.10 we iteratively applied the usual blow-up lemma in order to complete the partial packing using the edge slice G_B of the host graph G that we set aside in the beginning. Before obtaining our partial ξ-respecting packing, we also partition the edges of each graph G_ℓ for $\ell \in [k]$ into graphs $G_{\ell,A}$ and $G_{\ell,B}$, analogously as in Step 1 of the proof of Lemma 4.10. We use the graphs $G_{\ell,A}$ to obtain our partial ξ-respecting packing and we utilise $G_{\ell,B}$ to complete this embedding. Similar as in the proof of Lemma 4.10, we also track candidacy graphs B_i^H with bipartition (X_i^H, V_i) with respect to the collection of graphs $\{G_{\ell,B}\}_{\ell\in[k]}$ during our partial packing procedure that incorporate which candidates can be used for the completion. In order to proceed analogously as in Step 4, we would need a coloured version of the usual blow-up lemma that yields a ξ-respecting embedding of one single ξ-coloured graph H into a collection of graphs $\mathcal{G}$. Unfortunately, we are not aware of such a result. However, instead of the usual blow-up lemma we can also apply a simple matching argument for the completion. We proceed as follows. As in Step 4, for every $H \in \mathcal{H}$ in turn, we choose a small set of vertices $X_i \subseteq X_i^H$ for each $i \in [r]$ by selecting every vertex uniformly at random with probability roughly μ and adding $X_i^H \setminus \widehat{X}_i^H$ deterministically. Afterwards, we use a matching argument to embed $H[X_1, \dots, X_r]$ into $\{G_{\ell,B}\}_{\ell\in[k]}$ where we respect the ξ-colouring of the edges in H. In particular, it suffices to replace the application of the usual blow-up lemma in Claim 2 of Step 4 with the new matching argument. Proceeding similarly as in Claim 3 of Step 4 will then finish the completion.

The crucial observation is as follows. When we select vertices in X_i^H independently with probability roughly μ and consider the subgraph of H induced by these vertices, then only at most $2\mu^2 n$ vertices in each X_i are incident to an edge in this subgraph (even after adding $X_i^H \setminus \widehat{X}_i^H$ deterministically). Therefore, we may simply embed greedily all such vertices one after another to a potential candidate in V_i and afterwards are left with the task to embed only vertices where all neighbours have already been embedded. However, this is easy; simply choose in each candidacy graph (which will still be sufficiently super-regular) any perfect matching and this completes the embedding.

Chapter 5

A hypergraph blow-up lemma for approximate decompositions

The content of this chapter is based on the preprint [31] with Felix Joos.

5.1 Introduction to hypergraph decompositions

Although, there are numerous (approximate) decomposition results for spanning structures in graphs, the situation for hypergraphs is notably different and there are only few results concerning types of Hamilton cycles and H-factors (as already mentioned Section 1.3.3 in the introduction). In this chapter we provide a versatile result for approximate decompositions of quasirandom hypergraphs into families of spanning bounded degree hypergraphs.

One key feature of our results is their applicability to hypergraphs with vanishing density, which answers Question 1.5 of Kim, Kühn, Osthus and Tyomkyn [79].

In particular, for further applications it turned out that (approximate) decompositions of multipartite hypergraphs are highly desirable (to name only two examples, see Kim, Kühn, Osthus and Tyomkyn [79] and Keevash [72] which are used in [51, 65, 75, 76]). In view of this, we provide all our tools also for the multipartite setting. Our results answer and address questions of Keevash [73] and Kim, Kühn, Osthus and Tyomkyn [79].

Whether we have a strong control over the actual (approximate) decomposition, locally and globally, is another decisive factor if such a result is a powerful tool for further applications. Therefore, we make a considerable effort to implement two types of versatile test functions with respect to which the decomposition behaves random-like (for more details we refer the reader to our more technical results in Section 5.1.1).

The simplified version in Theorem 1.6 is a direct consequence of our main result. Let us recall the following definitions. For a k-graph G, we define the neighbourhood $N_G(S)$ of a $(k-1)$-set S of vertices as the set of vertices that form an edge together with S. Let $\varepsilon > 0$, $t \in \mathbb{N}$, $d \in (0, 1]$ and suppose G has n vertices. We say an n-vertex k-graph G is (ε, t, d)-typical if $|\bigcap_{S\in\mathcal{S}} N_G(S)| = (1 \pm \varepsilon)d^{|\mathcal{S}|}n$ for all sets $\mathcal{S}$ of $(k-1)$-sets of $V(G)$ with $|\mathcal{S}| \leq t$. Observe that the binomial random hypergraph is with high probability (ε, t, d)-typical whenever ε, t are fixed and $d \geq n^{-\varepsilon}$ and consequently with high probability these k-graphs can be approximately decomposed into any list of bounded degree hypergraphs with almost as many edges.

5.1.1 Multipartite graphs and the main result

Let us now turn to the statement of our main result. In fact, it applies to multipartite hypergraphs, but it easily implies a similar statement for non-partite graphs.

We say that a family/multiset of k-graphs $\mathcal{H} = \{H_1, \dots, H_s\}$ *packs* into a k-graph G if there is a function $\phi : \bigcup_{H \in \mathcal{H}} V(H) \to V(G)$ such that $\phi|_{V(H)}$ is injective and ϕ injectively maps edges onto edges. In such a case, we call ϕ a *packing of $\mathcal{H}$ into G*. Our general aim is to pack a collection $\mathcal{H}$ of multipartite k-graphs into a quasirandom host k-graph G having the same multipartite structure which is captured by a so-called 'reduced graph' R. We say $(H, G, R, \mathcal{X}, \mathcal{V})$ is a *blow-up instance* of size (n, k, r) if

- H, G, R are k-graphs where $V(R) = [r]$;
- $\mathcal{X} = (X_i)_{i \in [r]}$ is a vertex partition of H such that $|e \cap X_i| \leq 1$ for all $e \in E(H), i \in [r]$;
- $\mathcal{V} = (V_i)_{i \in [r]}$ is a vertex partition of G such that $|V_i| = |X_i| = (1 \pm 1/2)n$ for all $i \in [r]$;
- $H[X_{i_1}, \dots, X_{i_k}]$ is empty whenever $\{i_1, \dots, i_k\} \notin E(R)$.

We also refer to $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ as a *blow-up instance* if $\mathcal{H}$ is a collection of k-graphs and $\mathcal{X}$ is a collection of vertex partitions $(X_i^H)_{i \in [r], H \in \mathcal{H}}$ so that $(H, G, R, (X_i^H)_{i \in [r]}, \mathcal{V})$ is a blow-up instance for every $H \in \mathcal{H}$.

Given a blow-up instance $\mathscr{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ of size (n, k, r), we generalize the notion of typicality to the multipartite setting given by the reduced graph R. For finite sets $A_1, \dots, A_\ell$, we write $\bigsqcup_{i \in [\ell]} A_i := \{\{a_1, \dots, a_\ell\} \colon a_i \in A_i \text{ for all } i \in [\ell]\}$. We say G is (ε, t, d)-*typical with respect to* R if for all $i \in [r]$ and all sets $\mathcal{S} \subseteq \bigcup_{\mathscr{r} \in E(R) \colon i \in \mathscr{r}} V_{\sqcup \mathscr{r} \setminus \{i\}}$ with $|\mathcal{S}| \leq t$, we have $|V_i \cap \bigcap_{S \in \mathcal{S}} N_G(S)| = (1 \pm \varepsilon) d^{|\mathcal{S}|} |V_i|$. We say the blow-up instance $\mathscr{B}$ is (ε, t, d)-*typical* if G is (ε, t, d)-typical with respect to R and $|V_i| = |X_i^H| = (1 \pm \varepsilon)n$ for all $H \in \mathcal{H}, i \in [r]$. We denote the maximum vertex degree of a k-graph H by $\Delta(H)$. We say $\mathscr{B}$ is Δ-*bounded* if $\Delta(R), \Delta(H) \leq \Delta$ for each $H \in \mathcal{H}$.

We first state a simplified version of our main result for multipartite graphs.

Theorem 5.1 (Ehard, Joos [31])**.** *For all $\alpha \in (0, 1]$, there exist $n_0, t \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ is an (ε, t, d)-typical and α^{-1}-bounded blow-up instance of size (n, k, r) with $k \leq \alpha^{-1}$, $r \leq n^{\log n}$, $d \geq n^{-\varepsilon}$, $|\mathcal{H}| \leq n^{k+1}$, and $\sum_{H \in \mathcal{H}} e_H(X_{i_1}^H, \dots, X_{i_k}^H) \leq (1 - \alpha) d n^k$ for all $\{i_1, \dots, i_k\} \in E(R)$. Then there is a packing ϕ of $\mathcal{H}$ into G such that $\phi(X_i^H) = V_i$ for all $i \in [r]$ and $H \in \mathcal{H}$.*

In numerous applications of the original blow-up lemma for graphs, it has been essential that it provides additional features. In view of this, we make a substantial effort to also include further tools in our results that allow us to control the structure of the packings and will be very useful for future applications. We achieve this with two different types of what we call *testers*. The first tester is a so-called *set tester*; with the setting as in Theorem 5.1, we can fix a set $Y \subseteq X_i^H$ and $W \subseteq V_i$ for some $i \in [r]$ and $H \in \mathcal{H}$. Then we find a packing ϕ such that $|W \cap \phi(Y)| = |W||Y|/n \pm \alpha n$. Moreover, we can even fix several sets Y_j as above in multiple k-graphs H_j in $\mathcal{H}$ and the size of their common intersection with W is as large as we would expect it to be in an idealized random packing.

The second type of tester is a so-called *vertex tester*. In the simplest form, we fix a vertex $c \in V_i$ with $i \in [r]$ and define a weight function on $\bigcup_{H \in \mathcal{H}} X_i^H$. Then we find a packing such that the weight of the vertices embedded onto c is roughly the total weight divided by n. However, for many applications it is not enough to control single vertices – this is one reason why it is difficult to apply the hypergraph blow-up lemma due to Keevash [70]. Here, we make a considerable effort to provide a tool to deal with larger sets. To this end, for a vertex tester, we can also fix $c_i \in V_i$ for $i \in I$ for some $(k-1)$-set $I \subseteq \mathscr{r} \in E(R)$ and define a weight function on the $(k-1)$-sets that could be potentially embedded onto $\{c_i\}_{i\in I}$. Then our main result yields an embedding where the weight of the actually embedded $(k-1)$-sets onto $\{c_i\}_{i\in I}$ is the appropriate proportion of the total weight assigned.

Let us now formally define these two types of testers. Suppose $\mathscr{B}$ is a blow-up instance as above. We say $(W, Y_1, \ldots, Y_m)$ is an *ℓ-set tester* for $\mathscr{B}$ if $m \leq \ell$ and there exist $i \in [r]$ and distinct $H_1, \ldots, H_m \in \mathcal{H}$ such that $W \subseteq V_i$ and $Y_j \subseteq X_i^{H_j}$ for all $j \in [m]$. We say $(\omega, \mathscr{c})$ is an *ℓ-vertex tester* for $\mathscr{B}$ with centres $\mathscr{c} = \{c_i\}_{i\in I}$ in $I \subseteq [r]$, if

- $|I| \leq k-1$ and $I \subseteq \mathscr{r}$ for some $\mathscr{r} \in E(R)$, and $c_i \in V_i$ for each $i \in I$, and
- ω is a weight function on the $|I|$-tuples $\mathscr{X}_{\sqcup I} := \bigcup_{H\in\mathcal{H}}(\bigsqcup_{i\in I} X_i^H)$ with $\omega\colon \mathscr{X}_{\sqcup I} \to [0,\ell]$ and whenever $|I| \geq 2$, we have $\mathrm{supp}(\omega) = \omega^{-1}((0,\ell]) \subseteq \{\mathscr{x} \in \mathscr{X}_{\sqcup I} \colon \mathscr{x} = \mathscr{e} \cap \mathscr{X}_{\sqcup I}$ for some $\mathscr{e} \in \mathcal{H}\}$.

For an ℓ-tuple function $\omega\colon \binom{X}{\ell} \to \mathbb{R}_{\geq 0}$ on a finite set X, we define $\omega(X') := \sum_{S\in\binom{X'}{\ell}} \omega(S)$ for any $X' \subseteq X$. The following theorem is our main result.

Theorem 5.2 (Ehard, Joos [31])**.** *Suppose the assumptions of Theorem 5.1 hold and suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{2\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ into G in Theorem 5.1 such that*

(i) $|W \cap \bigcap_{j\in[m]} \phi(Y_j)| = |W||Y_1|\cdots|Y_m|/n^m \pm \alpha n$ *for all* $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$;

(ii) $\omega(\phi^{-1}(\mathscr{c})) = (1 \pm \alpha)\omega(\mathscr{X}_{\sqcup I})/n^{|I|} \pm n^\alpha$ *for all* $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ *with centres* $\mathscr{c}$ *in* I.

In the same line, we also provide these two types of testers if G is a (non-multipartite) quasirandom k-graph and the result in fact follows from Theorem 5.2. The definition is adapted in the obvious way. Suppose the vertex set of G is V and we aim to pack $\mathcal{H}$ into G where the vertex set of $H \in \mathcal{H}$ is denoted by X^H. For set testers $(W, Y_1, \ldots, Y_m)$, we proceed as above but select $W \subseteq V$ and $Y_j \subseteq X^{H_j}$. For vertex testers $(\omega, \mathscr{c})$ with $\{c_i\}_{i\in I}$ and $|I| \leq k-1$, we also proceed as above but require that $\{c_i\}_{i\in I}$ is an $|I|$-set in V and ω is a function from the union over all $H \in \mathcal{H}$ of the ordered $|I|$-sets of X^H into $[0,\ell]$, where $\mathrm{supp}(\omega)$ contains only $|I|$-tuples that are contained in an edge of H if $|I| \geq 2$. With this we obtain the following result.

Theorem 5.3 (Ehard, Joos [31])**.** *For all $\alpha \in (0,1]$, there exist $n_0, t \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose G is an (ε, t, d)-typical k-graph on n vertices with $k \leq \alpha^{-1}$, $d \geq n^{-\varepsilon}$ and $\mathcal{H}$ is a family of k-graphs on n vertices with $\Delta(H) \leq \alpha^{-1}$ for all $H \in \mathcal{H}$ and $|\mathcal{H}| \leq n^k$ such that $\sum_{H\in\mathcal{H}} e(H) \leq (1-\alpha)e(G)$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ into G such that*

(i) $|W \cap \bigcap_{\ell\in[m]} \phi(Y_\ell)| = |W||Y_1|\cdots|Y_m|/n^m \pm \alpha n$ *for all* $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$;

(ii) $\omega(\phi^{-1}(\mathscr{c})) = (1 \pm \alpha)\omega(\bigcup_{H\in\mathcal{H}} V(H))/n^{|I|} \pm n^\alpha$ *for all* $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ *with centres* $\mathscr{c}$ *in* I.

We illustrate applications of our main results to the setting of hypergraph decompositions as well as decompositions of simplicial complexes in Section 5.8.

5.2 Proof overview

In the following we outline our highlevel approach for the proof of our main result in the multipartite setting, that is, assuming we aim to pack k-graphs $H \in \mathcal{H}$ with vertex partition $(X_1^H, \dots, X_r^H)$ into G with vertex partition $(V_1, \dots, V_r)$. Our general approach is to consider each cluster V_i in turn and embed simultaneously almost all vertices of $\bigcup_{H\in\mathcal{H}} X_i^H$ onto V_i. Afterwards we complete the embedding with another procedure. We draw and extend several ideas from our proof in Chapter 4 of the blow-up lemma for approximate decompositions.

Let us turn to a more detailed description. At the very beginning, we will use a simple reduction to the case where each $H[X_{i_1}^H, \dots, X_{i_k}^H]$ induces a matching for all $\{i_1, \dots, i_k\} \in E(R)$ and $H \in \mathcal{H}$ by simply splitting the clusters into smaller clusters (see Lemma 5.9). This makes the analysis simpler and cleaner, and we assume this setting from now on.

The proof of the main result consists of two central parts. The first part is an iterative procedure that considers each cluster V_i of G in turn and provides a partial packing that embeds almost all vertices of the graphs in $\mathcal{H}$ onto vertices in G (at the beginning we fix some suitable ordering as r may be even larger than n, in particular, r is not bounded by a function in terms of ε). In the i-th step, we embed almost all vertices of $\bigcup_{H\in\mathcal{H}} X_i^H$ onto V_i by finding simultaneously for each $H \in \mathcal{H}$ an almost perfect matching within a 'candidacy graph' A_i^H, which is an auxiliary bipartite graph between X_i^H and V_i such that $xv \in E(A_i^H)$ only if v is still a suitable image for x with respect to the embedding obtained in previous steps. Of course, we have to guarantee that this indeed yields an edge-disjoint packing; that is, when we map $x_1 \in X_i^{H_1}$ and $x_2 \in X_i^{H_2}$ on the same vertex $v \in V_i$, then we have to ensure that for all $\mathscr{e}_j \in E(H_j)$ with $x_j \in \mathscr{e}_j$, $j \in [2]$, the $(k-1)$-sets $\mathscr{e}_1 \setminus \{x_1\}$ and $\mathscr{e}_2 \setminus \{x_2\}$ are not embedded onto the same $(k-1)$-set (provided they are already embedded). We achieve this by defining an auxiliary hypergraph $\mathcal{H}_{aux}$ with respect to the candidacy graphs to which we apply Theorem 2.3. There will be a bijection between matchings in $\mathcal{H}_{aux}$ and valid embeddings of $\bigcup_{H\in\mathcal{H}} X_i^H$ into V_i. This is one of the main ingredients in the first part of our proof.

Let us give more details for the construction of $\mathcal{H}_{aux}$. Assume we are in the i-th step of the partial packing procedure and already found a partial packing $\phi_\circ$ of the initial $i-1$ clusters. We define a labelling ψ on the edges of the candidacy 2-graphs such that for every edge $xv \in E(A_i^H)$ and $H \in \mathcal{H}$, the labelling $\psi(xv)$ contains the set of G-edges that are used when we extend $\phi_\circ$ by embedding x onto v. Let us for simplicity assume that only one G-edge would be covered when we embed x onto v, say, $\psi(xv) = \mathscr{g}_{xv} \in E(G)$. Since the packing will map multiple vertices of $\bigcup_{H\in\mathcal{H}} X_i^H$ onto the same vertex v in V_i, we consider disjoint copies $(V_i^H)_{H\in\mathcal{H}}$ of V_i where the vertex v^H is the copy of v. For every $xv \in E(A_i^H)$ and $H \in \mathcal{H}$, we define the 3-set $\mathscr{h}_{xv} := \{x, v^H, \psi(xv)\}$ and let $\mathcal{H}_{aux}$ be the 3-graph with vertex set $\bigcup_{H\in\mathcal{H}}(X_i^H \cup V_i^H) \cup E(G)$ and edge set $\{\mathscr{h}_{xv} : xv \in E(A_i^H) \text{ for some } H \in \mathcal{H}\}$. It is easy to see that there is a one-to-one correspondence between matchings in $\mathcal{H}_{aux}$ and valid embeddings of $\bigcup_{H\in\mathcal{H}} X_i^H$ into V_i such that no G-edge is used more than once. This hypergraph construction is similar as in Chapter 4.

Of course, whether we can iteratively apply this procedure depends on the choice of the partial packing in each step. Hence, with the aim of avoiding a future failure of the process, we have to maintain several pseudorandom properties throughout the entire process. For instance, we have to ensure that there are many candidates available in each step; in more detail, we guarantee that the updated candidacy graphs after each step remain super-regular even though they naturally become sparser after each embedding step.

Unfortunately, it is not enough to consider only candidacy graphs between pairs of clusters. For clusters indexed by elements in I where $|I| \leq k-1$, we consider candidacy graphs A_I^H on the clusters $\bigcup_{i\in I}(X_i^H \cup V_i)$ with edges of size $2|I|$. An edge $\mathscr{a}$ in A_I^H will then indicate whether the entire set of $|I|$ vertices in $\mathscr{a} \cap \bigcup_{i\in I} X_i^H$ can (still) be mapped onto $\mathscr{a} \cap \bigcup_{i\in I} V_i$. We further discuss the purpose of these candidacy graphs in Section 5.4.1, where we define them precisely.

The main source yielding a smooth trajectory of our partial packing procedure is the aforementioned Theorem 2.3 from [29], which provides a tool that gives rise to a pseudorandom matching in $\mathcal{H}_{aux}$ with respect to tuple-weight functions. One difficulty of the first stage of our proof is the careful definition of these tuple-weight functions. For instance, we have to ensure that we can indeed iteratively apply Theorem 2.3. Moreover, we have to guarantee that we can turn the partial packing into a complete one in the second part of our proof. To that end, we define very flexible but complex weight functions on tuples of (hyper)edges of the candidacy graphs that we call *edge testers*. Dealing with hypergraphs, and especially with hypergraphs with vanishing density, makes it significantly more complex to control the weight of these edge testers during our partial packing procedure than for a similar approach for simple graphs.

One single embedding step is performed by our so-called 'Approximate Packing Lemma' (see Section 5.5). The process where we iteratively apply our Approximate Packing Lemma is described in Section 5.6 and will provide a partial packing that maps almost all vertices of the graphs in $\mathcal{H}$ onto vertices in G.

The second part of the proof deals with embedding the remaining vertices and turning the obtained partial packing into a complete one. Our general strategy is to apply a randomized procedure where we unembed several vertices that we already embedded in the first part and find the desired packing by using a small edge-slice G_B of G put aside at the beginning (that is, we did not use the edges of G_B for the partial packing in the first stage). Of course, we again have to track which vertices are still suitable images during the completion and respect the partial packing of the first part. To that end, for each $H \in \mathcal{H}$ and $i \in [r]$, we track a second type of candidacy graphs B_i^H between X_i^H and V_i with respect to G_B, where $xv \in E(B_i^H)$ only if we could map x onto v during the completion. In fact, we track these candidacy graphs already during the partial packing procedure and carefully control several quantities using our edge testers. In the completion step, we can then apply a randomized matching procedure within the candidacy graphs B_i^H to turn the partial packing into a complete one.

As in many other results that were originally proven for graphs and later lifted to k-graphs for $k \geq 3$, we have to overcome numerous difficulties that are specific to hypergraphs. In our case this includes for example the much more complex intersection structure among hyperedges, which in turn complicates the analysis of our partial packing procedure considerably. To this end, several novel ideas are needed.

Let us highlight one obstacle. Suppose we are in the i-th step of the iteration where we aim to embed essentially all vertices $\bigcup_{H\in\mathcal{H}} X_i^H$ onto V_i, then all $x \in \bigcup_{H\in\mathcal{H}} X_i^H$ have to be grouped according to the edge intersection pattern of all edges that contain x and the edges that intersect these edges with respect to the clusters that have been considered earlier. In this context, we will define the patterns of edges in H in Section 5.4.2.

As we alluded to earlier, a strong control over the actual packing is of importance for further applications when an entire decomposition is sought. In particular, it is often not enough to control how many vertices of a certain set are mapped to a particular vertex, but how many $(k-1)$-sets are embedded to a particular $(k-1)$-set. To illustrate this, it is significantly stronger to claim that $\Delta_{k-1}(G - \phi(\mathcal{H})) \leq \alpha n$

than $\Delta(G-\phi(\mathcal{H})) \leq \alpha n^{k-1}$ (where $\Delta_{k-1}(G)$ refers to the maximum number of edges containing a particular $(k-1)$-set in G). It is already complicated enough to control such quantities for the multipartite setting, but in order to transfer this ability to general quasirandom k-graphs, it is necessary to allow vanishing densities of magnitude $o(\log^{-k} n)$. This is due to the following observation. Suppose $\mathcal{P}$ is a partition of $[n]$ and say a set S is *crossing* (with respect to $\mathcal{P}$) if each part of $\mathcal{P}$ contains at most one element of S. For $k \geq 3$, we need at least polylog n partitions of $[n]$, which are non-trivial but have only a constant number of parts, such that all $(k-1)$-sets of $[n]$ are (roughly) equally often crossing for the partitions, whereas for $k=2$ one partition suffices (because 1-element sets are always crossing).

Unfortunately, considering sparse k-graphs adds another complexity level to the problem. To be more precise, $o(n)$ and $o(dn)$ no longer mean the same where $d \geq n^{-\varepsilon}$ refers to the density, and thus, terms of size $o(n)$ can no longer be ignored. Essentially at all stages of the proof a substantially more careful analysis is needed to make sure that several quantities are not only $o(n)$ but $o(d^m n)$ because in many natural auxiliary (hyper)graphs considered in our proof vertices have typically $d^m n$ neighbours, where $m \in \mathbb{N}$ grows as we proceed in our procedure.

5.3 Preliminaries

In this section we collect some important tools for Chapter 5.

5.3.1 Notation

Let us introduce some general notation and (hyper)graph terminology, some of which we already introduced in Section 1.5.1, but we recall it here for completeness.

For $\ell \in \mathbb{N}$, we write $[\ell]_0 := [\ell] \cup \{0\} = \{0,1,\ldots,\ell\}$ and $-[\ell] := \{-\ell,\ldots,-1\}$, where $[0] := \emptyset$. We refer to a set of cardinality ℓ as an ℓ-set. For sets $A_1,\ldots,A_\ell$ and $I \subseteq [\ell]$, we write $A_{\cup I} := \bigcup_{i\in I} A_i$. For a tuple $\mathbf{a} = (a_1,\ldots,a_\ell) \in \mathbb{R}^\ell$ and $I \subseteq [\ell]$, we write $\mathbf{a}_I := (a_i)_{i\in I}$ and $\|\mathbf{a}\| := \sum_{i\in[\ell]} a_i$. For a finite set A and $\ell \in \mathbb{N}$, we write 2^A for the powerset of A and $\binom{A}{\ell}$ for the set of all ℓ-subsets of A. For a graph G, let $\binom{E(G)}{\ell}^{=} \subseteq \binom{E(G)}{\ell}$ be the set of all matchings of size ℓ in G. For finite sets $A_1,\ldots,A_\ell$, $\ell \in \mathbb{N}$, we write $\bigsqcup_{i\in[\ell]} A_i := \{\{a_1,\ldots,a_\ell\}\colon a_i \in A_i \text{ for all } i \in [\ell]\}$, and conversely, whenever we write $\{a_1,\ldots,a_\ell\} \in \bigsqcup_{i\in[\ell]} A_i$, we tacitly assume that $a_i \in A_i$ for all $i \in [\ell]$. For $I \subseteq [\ell]$, we write $A_{\sqcup I} := \bigsqcup_{i\in I} A_i$. Whenever we consider an index set $\{i_1,\ldots,i_\ell\} \subseteq \mathbb{Z}$, we tacitly assume that $i_1 \leq i_2 \leq \ldots \leq i_\ell$. For a real-valued function $f\colon A \to \mathbb{R}_{\geq 0}$, let $\mathrm{supp}(f) := \{a \in A\colon f(a) > 0\}$ be its support. For a function $g\colon A \to B$, let $g(A') := \bigcup_{a\in A'\cap A} g(a)$.

For a k-graph G, let $V(G)$ ad $E(G)$ denote the vertex set and edge set, respectively. For pairwise disjoint subsets $V_1,\ldots,V_k \subseteq V(G)$, let $G[V_1,\ldots,V_k]$ be the k-partite subgraph of G induced between $V_1,\ldots,V_k$. Let $e(G)$ denote the number of edges in G and let $e_G(V_1,\ldots,V_k) := e(G[V_1,\ldots,V_k])$. For set S of at most $k-1$ vertices of G, we define the neighbourhood of S in G by $N_G(S) := \{g \setminus S\colon g \in E(G), g \cap S = S\}$ and let $\deg_G(S) := |N_G(S)|$. Note that $N_G(S)$ is a set of $(k-|S|)$-tuples. For $m \in [k-1]$, let $\Delta_m(G) := \max_{S\in\binom{V(G)}{m}} \deg_G(S)$ denote the maximum m-degree of G. We usually write $\Delta(G)$ instead of $\Delta_1(G)$ and call $\Delta_2(G) = \Delta^c(G)$ the maximum codegree of G. To refer to the vertices contained in the tuples of $N_G(S)$, let $N_G^{\cup}(S) := \bigcup N_G(S)$. Further, we say that u is a G-vertex if $u \in V(G)$, and u is a G-neighbour of $v \in V(G)$ if $u \in N_G^{\cup}(v)$. We simplify the notation for a 2-graph G as follows. We usually write $N_G(v)$ instead of $N_G^{\cup}(v)$ for the neighbourhood of a vertex v and let $N_G[v] := N_G(v) \cup \{v\}$. For vertices u and v of G, let $N_G(u \wedge v) := N_G(u) \cap N_G(v)$,[1] and for a subset $S \subseteq V(G)$, let $N_G(S) := (\bigcup_{v \in S} N_G(v)) \setminus S$. We frequently treat collections of (hyper)graphs as the (hyper)graph obtained by taking the disjoint union of all members.

We say a k-graph G on n vertices is *(ε, t, d)-typical* if for all sets $\mathcal{S} \subseteq \binom{V(G)}{k-1}$ with $|\mathcal{S}| \leq t$, we have $|\bigcap_{S \in \mathcal{S}} N_G(S)| = (1 \pm \varepsilon) d^{|\mathcal{S}|} n$. We often write $N_G(\mathcal{S}) := \bigcap_{S \in \mathcal{S}} N_G(S)$. Throughout Chapter 5, we usually denote a $(k-1)$-set with the letter S, and a set of $(k-1)$-sets with the letter $\mathcal{S}$.

For a k-graph G, we denote by G_* the 2-graph with vertex set $V(G)$ and edge set $\bigcup_{g \in E(G)} \binom{g}{2}$. That is, G_* arises from G by replacing each hyperedge in G with a clique of size k.

As in Chapter 4 (see Section 4.4.1), for a graph G and a finite set $\mathcal{E}$, we call $\psi \colon E(G) \to 2^{\mathcal{E}}$ an *edge set labelling* of G. A label $\alpha \in \mathcal{E}$ *appears* on an edge e if $\alpha \in \psi(e)$. Let $\|\psi\|$ be the maximum number of labels that appear on any edge of G. We define the *maximum degree* $\Delta_\psi(G)$ *of* ψ as the maximum number of edges of G on which any fixed label appears, and the *maximum codegree* $\Delta^c_\psi(G)$ *of* ψ as the maximum number of edges of G on which any two fixed labels appear together.

5.3.2 Sparse graph regularity

In this section we introduce the quasirandom notion of (sparse) ε-regularity for 2-graphs. Even though we already introduced similar notation and results in Section 1.5.3, we carefully state the analogue results in this section as we allow that the density d of our host graph tends to 0 with the number of vertices.

For a bipartite graph G with vertex partition (V_1, V_2), we define the *density* of W_1, W_2 with $W_i \subseteq V_i$ by $d_G(W_1, W_2) := e_G(W_1, W_2)/|W_1||W_2|$. We say G is *(ε, d)-regular* if $d_G(W_1, W_2) = (1 \pm \varepsilon)d$ for all $W_i \subseteq V_i$ with $|W_i| \geq \varepsilon |V_i|$, and G is *(ε, d)-super-regular* if in addition $|N_G(v) \cap V_{3-i}| = (1 \pm \varepsilon) d |V_{3-i}|$ for each $i \in [2]$ and $v \in V_i$.

We collect several results for (sparse) ε-regular graphs. The following two standard results concern the robustness of ε-regular graphs.

Fact 5.4. *Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B) and $Y \subseteq B$ with $|Y| \geq \varepsilon |B|$. Then all but at most $2\varepsilon |A|$ vertices of A have $(1 \pm \varepsilon) d |Y|$ neighbours in Y.*

Fact 5.5. *Suppose $1/n \ll \varepsilon$ and $d > 0$. Suppose G is an (ε, d)-super-regular bipartite graph with vertex partition (A, B), where $\varepsilon^{1/6} n \leq |A|, |B| \leq n$. If $\Delta(H) \leq \varepsilon dn$ and $X \subseteq A \cup B$ with $|X| \leq \varepsilon dn$, then $G[A \setminus X, B \setminus X] - H$ is $(\varepsilon^{1/3}, d)$-super-regular.*

The following result from [8] is useful to establish ε-regularity for sparse graphs and is an analogue statement for the sparse setting of Theorem 1.13. For a graph G, let $C_4(G)$ be the number of copies of a cycle on four vertices in G.

Theorem 5.6 ([8, Lemma 13]). *Suppose $1/n \ll \varepsilon$ and $d \geq n^{-\varepsilon}$. Suppose G is a bipartite graph with vertex partition (A, B), $|A|, |B| = (1 \pm \varepsilon)n$, density $d_G(A, B) = (1 \pm \varepsilon)d$, and $C_4(G) < (1 + \varepsilon) d^4 |A|^2 |B|^2 / 4$. Then G is $(\varepsilon^{1/13}, d)$-regular.*

Further, if the common neighbourhood of most pairs in an (ε, d)-super-regular graph has the appropriate size, we can establish the following useful bounds on the number of edges between subsets of vertices that we allow to be very small.

[1] Note that in the previous chapters we used the notation $N_G(u, v)$ to denote $N_G(u) \cap N_G(v)$. However, this might be confusing since we also use the notation $N_G(\{u, v\})$ if G is a hypergraph, and thus, from now on we use $N_G(u \wedge v)$.

Lemma 5.7 ([31]). *Suppose $1/n \ll \varepsilon$ and $d \geq n^{-\varepsilon}$. Suppose G is an (ε, d)-super-regular graph with bipartition (A, B) and $|A| = |B| = n$, and for all but at most $n^{3/2}$ pairs $\{a, a'\}$ in A, we have $|N_G(a \wedge a')| \leq (1 + \varepsilon)d^2 n$. If $X \subseteq A$ and $Y \subseteq B$ with $n^{3/4+3\varepsilon} \leq |X|, |Y| \leq \varepsilon n$, then $e_G(X, Y) \leq \varepsilon^{1/3} dn \max\{|X|, |Y|\}$.*

Proof. We have that

$$\frac{1}{2}\sum_{b \in Y} |N_G(b) \cap X|^2 = \sum_{\{a,a'\} \in \binom{X}{2}} |N_G(a \wedge a') \cap Y| \leq \frac{1}{2}(1 + 2\varepsilon)|X|^2 d^2 n.$$

Combining this with

$$\frac{1}{2}\sum_{b \in Y} |N_G(b) \cap X|^2 \geq \frac{e_G(X, Y)^2}{2|Y|}.$$

yields that

$$e_G(X, Y)^2 \leq (1 + 2\varepsilon)d^2 |X|^2 |Y| n. \tag{5.3.1}$$

Suppose first that $|Y| \leq |X|$, and suppose for a contradiction that $e_G(X, Y) \geq \varepsilon^{1/3} dn|X|$. Together with (5.3.1) this implies that $|Y| \geq \varepsilon^{2/3} n/4$, which is a contradiction to $|Y| \leq |X| \leq \varepsilon n$.

Next, suppose $|X| \leq |Y|$, and suppose for a contradiction that $e_G(X, Y) \geq \varepsilon^{1/3} dn|Y|$. Together with (5.3.1) this implies that $|X| \geq \varepsilon^{1/3}(|Y|n)^{1/2}/2$. Since $|Y| \geq |X|$, this yields that $|Y| \geq \varepsilon^{2/3} n/4$, which is a contradiction to $|Y| \leq \varepsilon n$. This completes the proof. □

We will also need the following result that is similar to [9, Lemma 2] and guarantees that a (sparse) (ε, d)-super-regular balanced bipartite graph of order $2n$ contains a spanning m-regular subgraph (an m-factor) for $m = (1 - 2\varepsilon^{1/3})dn$ provided that most pairs of vertices have the appropriate number of common neighbours. It can be proved along the same lines as in [9] by employing Lemma 5.7.

Lemma 5.8 ([31]). *Suppose $1/n \ll \varepsilon$ and $d \geq n^{-\varepsilon}$. Suppose G is an (ε, d)-super-regular bipartite graph with vertex partition (A, B) where $|A| = |B| = n$, and suppose that all but at most $n^{3/2}$ pairs $\{a, a'\}$ in A satisfy that $|N_G(a \wedge a')| \leq (1 + \varepsilon)d^2 n$. Then G contains an m-factor for $m = (1 - 2\varepsilon^{1/3})dn$.*

5.3.3 Refining partitions

In this section we provide a useful result to refine the vertex partition of a collection $\mathcal{H}$ of k-graphs of bounded degree such that every $H \in \mathcal{H}$ only induces a matching between any k-set of the refined partition. The results in this section are very similar to the analogous result for 2-graphs (Lemma 4.6). For 2-graphs, a similar approach was already used in [112] by simply applying the classical Hajnal–Szemerédi Theorem. Our result is based on a random procedure which enables us to sufficiently control the weight distribution of a weight function with respect to the refined partition.

Lemma 5.9 ([31]). *Suppose $1/n \ll \varepsilon \ll \beta \ll \alpha, 1/k$ and $r \leq n^{\log n}$. Suppose $\mathcal{H}$ is a collection of at most n^{2k} k-graphs, $(X_i^H)_{i \in [r]}$ is a vertex partition of H, and $\Delta(H) \leq \alpha^{-1}$ for every $H \in \mathcal{H}$. Suppose $n/2 \leq |X_i^H| = |X_i^{H'}| \leq 2n$ for all $H, H' \in \mathcal{H}$ and $i \in [r]$. Suppose $\mathcal{W}$ is a set of weight functions $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$ with $I \subseteq [r]$, $|I| \leq k$, $\mathrm{supp}(\omega) \subseteq \{\boldsymbol{x} \in \mathscr{X}_{\sqcup I} : \boldsymbol{x} \subseteq \boldsymbol{e}$ for some $\boldsymbol{e} \in E(\mathcal{H})\}$, and $|\mathcal{W}| \leq n^{5\log n}$. Then for all $H \in \mathcal{H}$ and $i \in [r]$, there exists a partition $(X_{i,j}^H)_{j \in [\beta^{-1}]}$ of X_i^H such that*

(i) $X^H_{i,j}$ *is independent in* H^2_* *for all* $H \in \mathcal{H}$, $i \in [r]$, $j \in [\beta^{-1}]$;

(ii) $|X^H_{i,1}| \leq \ldots \leq |X^H_{i,\beta^{-1}}| \leq |X^H_{i,1}| + 1$ *for all* $H \in \mathcal{H}$, $i \in [r]$;

(iii) $\omega(X^H_{i,j}) = \beta\omega(X^H_i) \pm \beta^{3/2}n$ *for all* $H \in \mathcal{H}$, $i \in [r]$, $j \in [\beta^{-1}]$, *and* $\omega \in \mathcal{W}$ *with* $\omega\colon \mathscr{X}_i \to [0, \alpha^{-1}]$;

(iv) $\omega(\bigcup_{H\in\mathcal{H}}(\bigsqcup_{\ell\in[|I|]} X^H_{i_\ell,j_\ell})) = (1 \pm \varepsilon)\beta^{|I|}\omega(\mathscr{X}_{\sqcup I})$ *for all* $\omega \in \mathcal{W}$ *with* $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$, $\omega(\mathscr{X}_{\sqcup I}) \geq n^{1+\varepsilon}$, $I = \{i_1, \ldots, i_{|I|}\} \subseteq [r]$, $|I| \leq k$, *and* $j_1, \ldots, j_{|I|} \in [\beta^{-1}]$.

We give an analogous statement for the non-multipartite setting, that is, when $r = 1$, where the given weight functions assign weight on vertex tuples. It can be proved along the lines of the proof of Lemma 5.9. For a finite set A and $\ell \in \mathbb{N}$, let $\binom{A}{\ell}_\prec$ be the set of all ℓ-tuples with non-repetitive entries of A.

Lemma 5.10 ([31]). *Suppose* $1/n \ll \varepsilon \ll \beta \ll \alpha, 1/k$. *Suppose* $\mathcal{H}$ *is a collection of at most* n^{2k} k*-graphs on* n *vertices with* $\Delta(\mathcal{H}) \leq \alpha^{-1}$. *Suppose* $\mathcal{W}$ *is a set of at most* $n^{5\log n}$ *weight functions* $\omega\colon \bigcup_{H\in\mathcal{H}} \binom{V(H)}{m}_\prec \to [0, \alpha^{-1}]$ *with* $m \in [k-1]$, *and whenever* $m \geq 2$, *we have* $\mathrm{supp}(\omega) \subseteq \bigcup_{H\in\mathcal{H}}\{\boldsymbol{x} \in \binom{V(H)}{m}_\prec : \boldsymbol{x} \subseteq \boldsymbol{e}$ *for some* $\boldsymbol{e} \in E(H)\}$. *Then for all* $H \in \mathcal{H}$, *there exists a partition* $(X^H_j)_{j\in[\beta^{-1}]}$ *of* $V(H)$ *such that*

(i) X^H_j *is an independent set in* H^2_* *for all* $H \in \mathcal{H}$, $j \in [\beta^{-1}]$;

(ii) $|X^H_1| \leq \ldots \leq |X^H_{\beta^{-1}}| \leq |X^H_1| + 1$ *for all* $H \in \mathcal{H}$;

(iii) $\omega(X^H_j) = \beta\omega(V(H)) \pm \beta^{3/2}n$ *for all* $H \in \mathcal{H}$, $j \in [\beta^{-1}]$, *and* $\omega \in \mathcal{W}$ *with* $\omega\colon \bigcup_{H\in\mathcal{H}} V(H) \to [0, \alpha^{-1}]$;

(iv) $\omega(\bigcup_{H\in\mathcal{H}}(X^H_{j_1} \times \cdots \times X^H_{j_m})) = (1 \pm \beta^{1/2})\beta^m\omega(V(\mathcal{H}))$ *for all* $\omega \in \mathcal{W}$ *with* $\omega\colon \bigcup_{H\in\mathcal{H}} \binom{V(H)}{m}_\prec \to [0, \alpha^{-1}]$, $\omega(V(\mathcal{H})) \geq n^{1+\varepsilon}$, *and* $\{j_1, \ldots, j_m\} \in \binom{[\beta^{-1}]}{m}$.

Proof of Lemma 5.9. Our general approach is as follows. We first consider every $H \in \mathcal{H}$ in turn and construct a partition $(Y^H_{i,j})_{j\in[\beta^{-1}]}$ that essentially satisfies (i) and (ii) with $Y^H_{i,j}$ playing the role of $X^H_{i,j}$. Then we perform a vertex swapping procedure to resolve some conflicts in $Y^H_{i,j}$ and obtain $Z^H_{i,j}$. In the end, we randomly permute the ordering of $(Z^H_{i,j})_{j\in[\beta^{-1}]}$ for each $H \in \mathcal{H}, i \in [r]$ to also ensure (iv).

To simplify notation, we assume from now on that $|X^H_i|$ is divisible by β^{-1} for all $H \in \mathcal{H}$, $i \in [r]$, and at the end we explain how very minor modifications yield the general case. Recall that H_* is the 2-graph on $V(H)$ that arises from H by replacing each hyperedge with a clique of size k. For future reference, we also recall that for every $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$ with multiset $I \subseteq [r]$, $|I| \leq k$, we have by assumption that

$$\mathrm{supp}(\omega) \subseteq \{\boldsymbol{x} \in \mathscr{X}_{\sqcup I} : \boldsymbol{x} \subseteq \boldsymbol{e} \text{ for some } \boldsymbol{e} \in \mathcal{H}\}. \tag{5.3.2}$$

Note in particular that (5.3.2) implies that $\omega(X^H_{\sqcup I}) \leq 2k\alpha^{-2}n$ for every $H \in \mathcal{H}$.

Let $H \in \mathcal{H}$ be fixed. We claim that there exist partitions $(Y^H_{i,j})_{j\in[\beta^{-1}]}$ of X^H_i for each $i \in [r]$ such that for all $i \in [r], j \in [\beta^{-1}]$

(a) $|Y^H_{i,j}| = \beta|X^H_i| \pm \beta^2 n$;

(b) at most $\beta^{9/5}n$ pairs of vertices in $Y^H_{i,j}$ are adjacent in H^2_*;

(c) $\omega(Y^H_{i,j}) = \beta\omega(X^H_i) \pm \beta^2 n$ for all $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_i \to [0, \alpha^{-1}]$.

Indeed, the existence of such partitions can be seen by assigning every vertex in X_i^H uniformly at random to some $Y_{i,j}^H$ for $j \in [\beta^{-1}]$. Together with a union bound and Lemma 1.8, we conclude that (a)–(c) hold simultaneously with positive probability (where we employ (5.3.2) for (c)).

Next, we slightly modify these partitions $Y_{i,j}^H$ to obtain a new collection of partitions $Z_{i,j}^H$. For all $i \in [r]$, $j \in [\beta^{-1}]$, let $W_{i,j}^H \subseteq Y_{i,j}^H$ be such that $|Y_{i,j}^H \setminus W_{i,j}^H| = \beta|X_i^H| - \beta^{5/3}n$ and $W_{i,j}^H$ contains all vertices in $Y_{i,j}^H$ that contain an H_*^2-neighbour in $Y_{i,j}^H$ (the sets $W_{i,j}^H$ clearly exist by (a) and (b)). For all $i \in [r]$, $j \in [\beta^{-1}]$, let $\{w_1^i, \dots, w_s^i\} = W_i^H := \bigcup_{j\in[\beta^{-1}]} W_{i,j}^H$ and observe that $s = |X_i^H| - \sum_{j\in[\beta^{-1}]} |Y_{i,j}^H \setminus W_{i,j}^H| = \beta^{2/3}n$.

Now for every $i \in [r]$, arbitrarily assign labels in $[\beta^{-1}]$ to the vertices in W_i such that each label is used exactly $\beta^{5/3}n$ times. Let $Z_{i,j}^H(0) := Y_{i,j}^H \setminus W_{i,j}^H$ for all $i \in [r]$, $j \in [\beta^{-1}]$. To obtain the desired partitions we perform the following swap procedure for every $i \in [r]$. For every $t \in [s]$ in turn we do the following. Say $w_t^i \in W_{i,j}^H$ and w_t^i received label j'. We select $j'' \in [\beta^{-1}] \setminus \{j, j'\}$ such that w_t^i has no H_*^2-neighbour in $Z_{i,j''}^H(t-1)$ and such that $Z_{i,j''}^H(t-1)$ contains a vertex w that has no H_*^2-neighbour in $Z_{i,j'}^H(t-1)$. In such a case we say that j'' is *selected* in step t. Then we define $Z_{i,j''}^H(t) := (Z_{i,j''}^H(t-1) \cup \{w_t^i\}) \setminus \{w\}$, $Z_{i,j'}^H(t) := Z_{i,j'}^H(t) \cup \{w\}$ and $Z_{i,\ell}^H(t) := Z_{i,\ell}^H(t-1)$ for all $\ell \in [\beta^{-1}] \setminus \{j', j''\}$. Note that $\beta|X_i^H| - \beta^{5/3}n \le |Z_{i,\ell}(t)| \le \beta|X_i^H|$ for all $t \in [s]$, $\ell \in [\beta^{-1}]$. Observe also that we have always at least $\beta^{-1}/2$ choices to select j'' in step t. As $s = \beta^{2/3}n$, we can ensure that each $j'' \in [\beta^{-1}]$ is selected, say, at most $10\beta^{5/3}n$ times. We write $Z_{i,j}^H := Z_{i,j}^H(s)$ and it is plain to verify that for all $j \in [\beta^{-1}]$ we have

(a′) $|Z_{i,j}^H| = \beta|X_i^H|$;

(b′) $Z_{i,j}^H$ is independent in H_*^2;

(c′) $\omega(Z_{i,j}^H) = \beta\omega(X_i^H) \pm \beta^{3/2}n/2$ for all $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_i \to [0, \alpha^{-1}]$.

As $H \in \mathcal{H}$ is chosen arbitrarily, the statements (a′)–(c′) hold for all $H \in \mathcal{H}$. Note that (c′) implies (iii).

It remains to show how to find permutations $\{\pi_i^H\}_{H\in\mathcal{H}, i\in[r]}$ such that (iv) also holds for $\omega \in \mathcal{W}$ with $\omega(\mathscr{X}_{\sqcup I}) \ge n^{1+\varepsilon}$ where $X_{i,j}^H := Z_{i,\pi_i^H(j)}^H$. This can be easily achieved by considering random permutations. To see this, fix $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$, $I = \{i_1, \dots, i_{|I|}\} \subseteq [r]$, $2 \le |I| \le k$, and $j_1, \dots, j_{|I|} \in [\beta^{-1}]$. Note that we would expect that $\omega(\bigcup_{H\in\mathcal{H}}(\bigsqcup_{\ell\in[|I|]} X_{i_\ell,j_\ell}^H)) = \beta^{|I|}\omega(\mathscr{X}_{\sqcup I}) \pm \beta^{4/3}n$. Hence, by Freedman's inequality (Lemma 1.9) and a union bound we obtain with probability, say, at least 1/2, that $\omega(\bigcup_{H\in\mathcal{H}}(\bigsqcup_{\ell\in[|I|]} X_{i_\ell,j_\ell}^H)) = (1 \pm \varepsilon/2)\beta^{|I|}\omega(\mathscr{X}_{\sqcup I})$ for all $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$, $I = \{i_1, \dots, i_{|I|}\} \subseteq [r]$, $|I| \le k$, and $j_1, \dots, j_{|I|} \in [\beta^{-1}]$. This establishes (iv).

In the beginning we made the assumption that β^{-1} divides $|X_i^H|$. To avoid this assumption, we simply remove a set $\widetilde{X}_i^H$ of size at most $\beta^{-1} - 1$ from X_i^H such that β^{-1} divides $X_i^H \setminus \widetilde{X}_i^H$ and perform the entire procedure with $X_i^H \setminus \widetilde{X}_i^H$ instead of X_i^H. To that end, for all $H \in \mathcal{H}$ and $i \in [r]$, let $r_i^H \in [\beta^{-1} - 1]_0$ be such that $r_i^H = |X_i^H| \mod \beta^{-1}$, and let $\widetilde{X}_i^H \subseteq X_i^H$ be such that

- $|\widetilde{X}_i^H| = r_i^H$;
- $\omega(\bigcup_{H\in\mathcal{H}} \widetilde{X}_{\sqcup I}^H) \le \varepsilon^2\omega(\mathscr{X}_{\sqcup I})$ for all $\omega \in \mathcal{W}$ with $\omega\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$.

The existence of such sets $\widetilde{X}_i^H$ can be easily seen by taking each $\widetilde{X}_i^H$ as a subset of X_i^H of size exactly r_i^H uniformly and independently at random for all $i \in [r]$ and $H \in \mathcal{H}$. We now perform the procedure with $X_i^H \setminus \widetilde{X}_i^H$ instead of X_i^H. That is, for all $H \in \mathcal{H}$

and $i \in [r]$, we obtain a partition $(\widetilde{X}^H_{i,j})_{j\in[\beta^{-1}]}$ of $X^H_i \setminus \widetilde{X}^H_i$ satisfying (i)–(iv), where $|\widetilde{X}^H_{i,j}| = |\widetilde{X}^H_{i,j'}|$ for all $i \in [r], j, j' \in [\beta^{-1}]$, and (iii) and (iv) hold with error term '$\pm\beta^{3/2}n/2$' and '$(1 \pm \varepsilon/2)$', respectively. At the very end we add the vertices in $\widetilde{X}^H_i$ to the partition $(\widetilde{X}^H_{i,j})_{j\in[\beta^{-1}]}$ while preserving (i) and (ii). We may do so by performing a swap argument as before. Together with (5.3.2), observe that the error bounds give us enough room to spare. □

5.4 Blow-up instances and candidacy graphs

In this section we introduce more notation concerning blow-up instances, which will be useful throughout our packing procedure (in Sections 5.5 and 5.6). Let $\mathscr{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ be a blow-up instance of size (n, k, r) that is fixed throughout Section 5.4. Note that the reduced graph R with vertex set $[r]$ gives us a natural ordering of the clusters and we assume this ordering to be fixed (for a different ordering, just relabel the cluster indices). For simplicity, we often write $\mathscr{X}_i := \bigcup_{H\in\mathcal{H}} X^H_i$ for $i \in [r]$ and $\mathscr{X}_{\sqcup I} := \bigcup_{H\in\mathcal{H}}(\bigsqcup_{i\in I} X^H_i)$ for $I \subseteq [r]$. Further, we call $I \subseteq [r]$ an *index set (of $\mathscr{B}$)*, if $I \subseteq \mathscr{r}$ for some $\mathscr{r} \in E(R)$.

We will introduce some important quantities that we control during our packing procedure. For instance, we track for each edge $\mathscr{g} \in E(G)$ (and for each subset of $\mathscr{g}$), the set of $\mathcal{H}$-edges that still could be mapped onto $\mathscr{g}$ given a function ϕ that already maps vertices of some clusters in $\mathcal{H}$ onto vertices in G (see Definition 5.16 in Section 5.4.4). Similarly, we track for distinct edges $\mathscr{g}, \mathscr{h} \in E(G)$, the set of tuples of $\mathcal{H}$-edges that still could be mapped together onto $\mathscr{g}$ and $\mathscr{h}$, respectively, with respect to ϕ (see Definition 5.17). To track these quantities, we define *edge testers* (see Definitions 5.14–5.15) on *candidacy graphs* (see Definition 5.11) in Sections 5.4.3 and 5.4.1, respectively. The definition of these edge testers depends on how the edges in $\mathcal{H}$ intersect, and to that end, we define *patterns* (see Definitions 5.12–5.13) in Section 5.4.2.

5.4.1 Candidacy graphs

For the purpose of tracking which sets of vertices in $\mathcal{H}$ are still suitable images for sets of vertices in G, we will consider auxiliary *candidacy graphs.* To that end, assume we are given $r_\circ \le r$ and a mapping $\phi\colon \bigcup_{H\in\mathcal{H}} \widehat{X}^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$ with $\widehat{X}^H_q \subseteq X^H_q$ for $q \in [r_\circ]$ and $\phi|_{V(H)}$ is injective for each $H \in \mathcal{H}$; that is, ϕ already embeds some $\mathcal{H}$-vertices onto G-vertices. We assume $r_\circ$ and ϕ to be fixed throughout the entire Section 5.4. We define candidacy (hyper)graphs with respect to ϕ and the blow-up instance $\mathscr{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ such that a (hyper)edge $\mathscr{a}$ of the candidacy graph incorporates the property that the set of $\mathcal{H}$-vertices in $\mathscr{a}$ can still be mapped onto the set of G-vertices in $\mathscr{a}$ with respect to potential $\mathcal{H}$-vertices that are already mapped onto G-vertices by ϕ.

Definition 5.11 (Candidacy graphs $A^H_I(\phi)$)**.** *For all $H \in \mathcal{H}$ and every index set $I \subseteq [r]$, let $A^H_I(\phi)$ be the $2|I|$-graph with vertex set $X^H_{\cup I} \cup V_{\cup I}$ and $\bigcup_{i\in I}\{x_i, v_i\} \in E(A^H_I(\phi))$ for $\{x_i, v_i\} \in X^H_i \sqcup V_i$ if all $\mathscr{e} = \mathscr{e}_\circ \cup \mathscr{e}_m \in E(H[\widehat{X}^H_{\cup([r_\circ]\setminus I)}, X^H_{\cup I_m}])$ with $m \in [|I|]$, $I_m \in \binom{I}{m}$, $\mathscr{e}_\circ \subseteq \binom{\widehat{X}^H_{\cup([r_\circ]\setminus I)}}{k-m}$, and $\mathscr{e}_m = \{x_i\}_{i\in I_m}$ satisfy*

$$\phi(\mathscr{e}_\circ) \cup \{v_i\}_{i\in I_m} \in E(G[V_{\cup([r_\circ]\setminus I)}, V_{\cup I_m}]). \tag{5.4.1}$$

We call $A^H_I(\phi)$ the candidacy graph with respect to ϕ and G.

Let us describe Definition 5.11 in words; for an illustration, see Figure 5.1. Suppose first that we are given an index set $I \subseteq [r] \setminus [r_\circ]$. Then the set I contains the indices of clusters whose vertices are not yet embedded by $\phi\colon \bigcup_{H\in\mathcal{H}} \widehat{X}^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$. If the set of vertices $\{x_i\}_{i\in I} \in X^H_{\sqcup I}$ can still be mapped onto $\{v_i\}_{i\in I} \in V_{\sqcup I}$, then $\{v_i\}_{i\in I}$ are still suitable candidates for $\{x_i\}_{i\in I}$ and we store this information in the candidacy graph $A^H_I(\phi)$ by adding the edge $\bigcup_{i\in I}\{x_i, v_i\} \in E(A^H_I(\phi))$. Let us spell out what it means that $\{x_i\}_{i\in I}$ can still be mapped onto $\{v_i\}_{i\in I}$. It means that all H-edges e

- that intersect $\{x_i\}_{i\in I}$, say in a set of m vertices e_m which lie in the clusters with indices $\{i_1, \dots, i_m\}$,
- and whose other $k-m$ vertices $e_\circ = e \setminus e_m$ are embedded by ϕ,

satisfy (5.4.1), that is

- the embedding $\phi_\circ(e_\circ)$ together with the vertices of $\{v_i\}_{i\in I}$ in the clusters with $\{i_1, \dots, i_m\}$, that is $\phi(e_\circ) \cup \{v_{i_1}, \dots, v_{i_m}\}$, forms an edge in G.

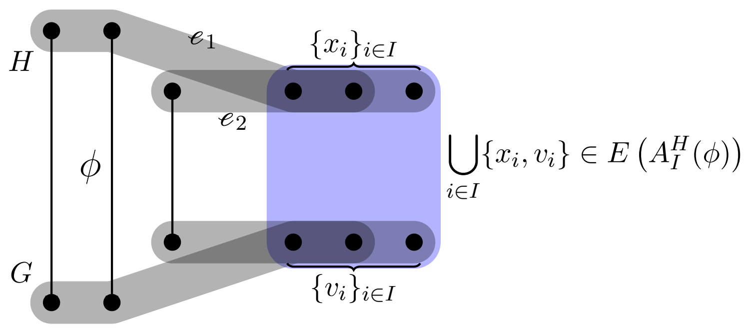


Figure 5.1: We have $k = 4$ in this figure. The set of vertices $\{x_i\}_{i\in I}$ can be mapped onto $\{v_i\}_{i\in I}$, and thus $\bigcup_{i\in I}\{x_i, v_i\}$ is an edge in the candidacy graph $A^H_I(\phi)$ if the embedded parts of the H-edges e_1 and e_2 together with the vertices in $\{v_i\}_{i\in I}$ also form corresponding edges in G.

We note that we allow $I \subseteq [r]$ in Definition 5.11 (and not only $I \subseteq [r]\setminus[r_\circ]$) because will use this for $I \subseteq [r]$, $|I| = 1$, to track a second type of candidacy graphs between already embedded clusters. At the end of our procedure we will use this second type of candidacy graphs to turn an approximate packing into a complete one.

Let us continue with another comment. Clearly, for $I = \{i\}$, the candidacy graph $A^H_i(\phi)$ is a bipartite 2-graph.[2] It is worth pointing out a crucial difference between $\bigcup_{i\in I} A^H_i(\phi)$ and $A^H_I(\phi)$: If $\{x_i v_i\}_{i\in I} \in \bigsqcup_{i\in I} E(A^H_i(\phi))$, then each vertex x_i on its own can still be mapped onto v_i, whereas if $\bigcup_{i\in I}\{x_i, v_i\} \in E(A^H_I(\phi))$, then the entire set $\{x_i\}_{i\in I}$ can still be mapped onto $\{v_i\}_{i\in I}$.

Let $\mathcal{A}_I(\phi) := \bigcup_{H\in\mathcal{H}} A^H_I(\phi)$ and let $\mathcal{A}(\phi)$ be the collection of all $\mathcal{A}_I(\phi)$ for all index sets $I \subseteq [r] \setminus [r_\circ]$. We also refer to subgraphs of $A^H_I(\phi)$ as candidacy graphs.

To suitably control the candidacy graphs during our approximate packing procedure, it will be important that the neighbourhood $N_{A^H_i}(x)$ in the candidacy graph A^H_i of an H-vertex x is the intersection of neighbourhoods of $(k-1)$-sets in G. To that

[2]For the sake of readability, we write A^H_i instead of $A^H_{\{i\}}$.

end, for $\varepsilon > 0$, $q \in \mathbb{N}$, $H \in \mathcal{H}$, $i \in [r]$ and a candidacy graph $A_i^H \subseteq A_i^H(\phi)$, we say

(5.4.2) *A_i^H is (ε, q)-well-intersecting with respect to G, if for every $x \in X_i^H$, we can find $\mathcal{S}_x \subseteq \binom{V(G)}{k-1}$ with $|\mathcal{S}_x| \leq q$ such that $N_{A_i^H}(x) = V_i \cap N_G(\mathcal{S}_x)$, and every $x \in X_i^H$ is contained in at most $n^{1/4+\varepsilon}$ pairs $\{x, x'\} \in \binom{X_i^H}{2}$ such that $\mathcal{S}_x \cap \mathcal{S}_{x'} \neq \emptyset$.*

We note that during our packing procedure the sets $\mathcal{S}_x$ will be uniquely determined and thus, (5.4.2) is indeed well-defined.

5.4.2 Patterns

The behaviour of several parameters in our packing procedure depends on the intersection pattern of the edges in $\mathcal{H}$, that is, how edges in $\mathcal{H}$ intersect and overlap. To this end, we associate two vectors in $\mathbb{N}_0^r$ with certain sets of vertices in $H \in \mathcal{H}$ that we call 1^{st}-*pattern* and 2^{nd}-*pattern*. Even though we need the precise definitions of these patterns at certain points throughout the paper, it mostly suffices to remember that every set of vertices and every edge in $\mathcal{H}$ has a unique 1^{st}- and 2^{nd}-pattern. This allows us to track certain quantities with respect to their patterns. We proceed to the precise definitions of 1^{st}- and 2^{nd}-patterns.

In order to conveniently define these vectors for a given set of vertices in $H \in \mathcal{H}$, we consider supergraphs of H (namely, for $Z = B$ in Definition 5.12 below). We do this because we define candidacy graphs in Section 5.4.1, and we will in fact consider two collections $\mathcal{A}$ and $\mathcal{B}$ of candidacy graphs (as mentioned in the proof overview), and thus, we also have to distinguish two types of 1^{st}-patterns and 2^{nd}-patterns for both $Z \in \{A, B\}$. To that end, it is more convenient to imagine that the clusters associated with the candidacy graphs in $\mathcal{B}$ are copies of the original cluster. For all $H \in \mathcal{H}$, $J \subseteq [r]$, and $j \in J$, let $X_j^{H,B}$ be a disjoint copy of X_j^H, and let π be the bijection that maps a vertex in X_j^H to its copy in $X_j^{H,B}$. Let H_J be the supergraph of H with vertex set $V(H_J) := V(H) \cup X_{\cup J}^{H,B}$ and edge set $E(H_J) := E(H) \cup \{\pi(x) \cup (\mathscr{e} \setminus \{x\}) \colon \mathscr{e} \in E(H), x \in \mathscr{e} \cap X_{\cup J}^H\}$.

We now define 1^{st}-patterns and 2^{nd}-patterns and give an illustration in Figure 5.2. Note that $H_\emptyset - H$ is the empty graph.

Definition 5.12 (Patterns). *For all $Z \in \{A, B\}$, index sets $I \subseteq [r]$, $J \subseteq I$, and all $\mathscr{x} = \{x_i\}_{i \in I} \in X_{\cup I}^H$ for some $H \in \mathcal{H}$, let $\mathscr{x}' := \{x_i\}_{i \in I \setminus J} \cup \{\pi(x_j)\}_{j \in J}$, $H_A := H$ and $H_B := H_J - H$. We define the* 1^{st}-pattern $\mathbf{p}^Z(\mathscr{x}, J) \in \mathbb{N}_0^r$ *and the* 2^{nd}-pattern $\mathbf{p}^{Z,2nd}(\mathscr{x}, J) \in \mathbb{N}_0^r$ *as r-tuples where their ℓ-th entry $\mathbf{p}^Z(\mathscr{x}, J)_\ell$ and $\mathbf{p}^{Z,2nd}(\mathscr{x}, J)_\ell$ for $\ell \in [r]$ is given by*

(5.4.3)
$$\mathbf{p}^Z(\mathscr{x}, J)_\ell := \left|\left\{\mathscr{f} \in E(H_Z) : (\mathscr{f} \cap X_\ell^H) \setminus \{x_i\}_{i \in I \setminus J} \neq \emptyset, \mathscr{f} \setminus X_{\cup[\ell]}^H \subseteq \mathscr{x}', |\mathscr{f} \setminus X_{\cup[\ell]}^H| \geq 2\right\}\right|;$$

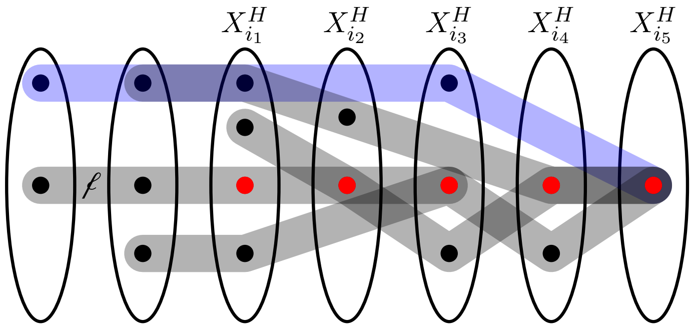
(5.4.4)
$$\mathbf{p}^{Z,2nd}(\mathscr{x}, J)_\ell := \left|\left\{\mathscr{f} \in E(H_Z) : \mathscr{f} \cap X_\ell^H \in \{x_i\}_{i \in I \setminus J}, \mathscr{f} \setminus X_{\cup[\ell]}^H \subseteq \mathscr{x}', |\mathscr{f} \setminus X_{\cup[\ell]}^H| = 1\right\}\right|.$$

Let us describe Definition 5.12 in words. Consider some fixed entry for $\ell \in [r]$. Depending on $Z \in \{A, B\}$, we either count edges in H_A or in H_B; note that the edges in $H_B = H_J - H$ always contain exactly one copied vertex in $X_{\cup J}^{H,B}$. Further, $\mathscr{x} \in X_{\cup I}^H$ determines the set I and hence the definitions in (5.4.3) and (5.4.4) display no additional dependence on I.

We first describe the 1^{st}-pattern entry $\mathbf{p}^Z(\mathscr{x}, J)_\ell$ as defined in (5.4.3). The condition '$(\mathscr{f} \cap X_\ell^H) \setminus \{x_i\}_{i \in I \setminus J} \neq \emptyset$' means that $\mathscr{f}$ has a non-empty intersection with

X_ℓ^H which does not lie in $\{x_i\}_{i\in I\setminus J}$; in particular $\mathscr{x}, \mathscr{x}' \neq \mathscr{e}$. The last two conditions in (5.4.3) mean that all vertices of $\mathscr{e} \setminus X_{\cup[\ell]}^H$ lie in $\mathscr{x}'$ and these are at least two vertices. In that sense, $\mathscr{e} \cap X_\ell^H$ is the 'last' vertex of $\mathscr{e}$ not contained in $\mathscr{x}'$.

We now describe the 2^{nd}-pattern entry $\mathbf{p}^{Z,2nd}(\mathscr{x}, J)_\ell$ as defined in (5.4.4). The condition '$\mathscr{e} \cap X_\ell^H \in \{x_i\}_{i\in I\setminus J}$' means that $\mathscr{e}$ has a non-empty intersection with X_ℓ^H that lies in $\{x_i\}_{i\in I\setminus J}$; note that we may also count edges $\mathscr{e} \in E(H_Z)$ with $\mathscr{e} = \mathscr{x}'$ if $\mathscr{x}' \in E(H_Z)$. The last two conditions in (5.4.4) mean that $k-1$ vertices of $\mathscr{e}$ lie in $X_{\cup[\ell]}^H$ and the other vertex of $\mathscr{e}$ lies in $\mathscr{x}'$; note that the last two conditions in (5.4.4) imply that this must be the copied vertex $\mathscr{e} \cap X_{\cup J}^{H,B}$ if $Z = B$.



$$\mathbf{p}^{A}(\mathscr{x}, \emptyset) = (\ 0\ ,\ 1\ ,\ 0\ ,\ 1\ ,\ 1\ ,\ 0\ ,\ 0\ )$$
$$\mathbf{p}^{A,2nd}(\mathscr{x}, \emptyset) = (\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ 2\ ,\ 0\ )$$
$$\mathbf{p}^{B}(\mathscr{x}, \{i_1, i_2\}) = (\ 0\ ,\ 0\ ,\ \boxed{1}\ ,\ 1\ ,\ 1\ ,\ 0\ ,\ 0\ )$$
$$\mathbf{p}^{B,2nd}(\mathscr{x}, \{i_1, i_2\}) = (\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ \boxed{2}\ ,\ 0\ ,\ 1\ )$$
$$\mathbf{p}^{B}(\mathscr{x}, \{i_1, \dots, i_4\}) = (\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 1\ ,\ 2\ ,\ 0\ )$$
$$\mathbf{p}^{B,2nd}(\mathscr{x}, \{i_1, \dots, i_4\}) = (\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 4\ )$$

Figure 5.2: We have $k = 5$, $I = \{i_1, \dots, i_5\}$ in this figure and the set $\mathscr{x} = \{x_{i_1}, \dots, x_{i_5}\} \in X_{\cup I}^H$, $\mathscr{x} \notin E(H)$, consists of the five red vertices. Note that the blue hyperedge does not play a role for any of the displayed 1^{st}-patterns or 2^{nd}-patterns. Let us explain the marked entries; all others can be checked similarly. For $\mathbf{p}^B(\mathscr{x}, \{i_1, i_2\})$ we consider (5.4.3) for $Z = B$ and the displayed edge $\mathscr{e} \in E(H)$. For $J = \{i_1, i_2\}$ we will obtain two copied edges $\mathscr{e}_{i_1} := (\mathscr{e} \setminus \{x_{i_1}\}) \cup \{x'_{i_1}\}$ and $\mathscr{e}_{i_2} := (\mathscr{e} \setminus \{x_{i_2}\}) \cup \{x'_{i_2}\}$ of $\mathscr{e}$ in $E(H_B) = E(H_J) \setminus E(H)$. By checking the conditions in (5.4.3), we note that $\mathscr{e}_{i_2}$ accounts for the marked entry $\boxed{1}$ and $\mathscr{e}_{i_1}$ accounts for 1 of $\mathbf{p}^B(\mathscr{x}, \{i_1, i_2\})$. Similarly, by checking the conditions in (5.4.4), both edges $\mathscr{e}_{i_1}$ and $\mathscr{e}_{i_2}$ account for the marked entry $\boxed{2}$ of $\mathbf{p}^{B,2nd}(\mathscr{x}, \{i_1, i_2\})$.

We make the following important observation concerning Definition 5.12. We claim that

$$\|\mathbf{p}^Z(\mathscr{x}, J)\| = \|\mathbf{p}^{Z,2nd}(\mathscr{x}, J)\| - \mathbb{1}\{\mathscr{x}' \in E(H_Z)\}. \tag{5.4.5}$$

To see that this is true, let us first assume that $\mathscr{x}' \notin E(H_Z)$. Note that every $\mathscr{e}$ that contributes to $\|\mathbf{p}^Z(\mathscr{x}, J)\|$, has a 'penultimate' vertex that lies in $\{x_i\}_{i\in I\setminus J}$, that is, there exists an index ℓ such that the conditions in (5.4.4) are satisfied, and thus $\mathscr{e}$ also contributes to $\|\mathbf{p}^{Z,2nd}(\mathscr{x}, J)\|$. Conversely, every $\mathscr{e}$ that contributes to $\|\mathbf{p}^{Z,2nd}(\mathscr{x}, J)\|$ has a 'last' vertex not contained in $\{x_i\}_{i\in I\setminus J}$ if $\mathscr{x}' \notin E(H_Z)$, and thus $\mathscr{e}$ also contributes to $\|\mathbf{p}^Z(\mathscr{x}, J)\|$. Hence, $\|\mathbf{p}^Z(\mathscr{x}, J)\| = \|\mathbf{p}^{Z,2nd}(\mathscr{x}, J)\|$ if $\mathscr{x}' \notin E(H_Z)$. If $\mathscr{x}' \in E(H_Z)$ (and thus $|I| = k$), then $\mathscr{e} = \mathscr{x}'$ additionally contributes to $\|\mathbf{p}^{Z,2nd}(\mathscr{x}, J)\|$ but not to $\|\mathbf{p}^Z(\mathscr{x}, J)\|$ because of the condition '$(\mathscr{e} \cap X_\ell^H) \setminus \{x_i\}_{i\in I\setminus J} \neq \emptyset$' in (5.4.3). This implies (5.4.5). Further, note that $\mathscr{x}' \in E(H_A)$ only if $J = \emptyset$, and $\mathscr{x}' \in E(H_B)$ only if $|J| = 1$.

We will also consider a set of vertex tuples that lie in an edge in $\mathcal{H}$ and have certain patterns.

Definition 5.13 ($E_{\mathcal{H}}(\boldsymbol{p}, I, J)$)**.** *For all index sets $I \subseteq [r]$, and $\mathbf{p}, \mathbf{p}^{2nd} \in \mathbb{N}_0^r$, let*

$$E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, I) := \big\{ x \in \mathscr{X}_{\sqcup I} : \mathbf{p}^A(x, \emptyset) = \mathbf{p}, \mathbf{p}^{A,2nd}(x, \emptyset) = \mathbf{p}^{2nd}, \\ x = e \cap \mathscr{X}_{\sqcup I} \text{ for some } e \in E(\mathcal{H}) \big\}.$$

More generally, we allow to specify whether some vertices of $x \in \mathscr{X}_{\sqcup I}$ lie in clusters with indices in $J \subseteq I$. To this end, for all index sets $I \subseteq [r]$, all $J \subseteq I$, and $\boldsymbol{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd}) \in \mathbb{N}_0^r \times \mathbb{N}_0^r \times \mathbb{N}_0^r \times \mathbb{N}_0^r = (\mathbb{N}_0^r)^4$, let

$$E_{\mathcal{H}}(\boldsymbol{p}, I, J) := \big\{ x \in \mathscr{X}_{\sqcup I} : \mathbf{p}^Z(x, J) = \mathbf{p}^Z, \mathbf{p}^{Z,2nd}(x, J) = \mathbf{p}^{Z,2nd} \text{ for both } Z \in \{A, B\}, \\ x = e \cap \mathscr{X}_{\sqcup I} \text{ for some } e \in E(\mathcal{H}) \big\}.$$

5.4.3 Edge testers

Recall that we consider some fixed $r_\circ \leq r$ and $\phi \colon \bigcup_{H \in \mathcal{H}} \widehat{X}^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$. Let $\mathcal{A}$ be a collection of candidacy graphs $\mathcal{A}_I = \bigcup_{H \in \mathcal{H}} A^H_I \subseteq \mathcal{A}_I(\phi)$ for all index sets $I \subseteq [r] \setminus [r_\circ]$. We will use weight functions on the edges of the candidacy graphs, which we also call edge testers, in order to track important quantities during our packing procedure.

We start with the definition of *simple edge testers* (Definition 5.14). In Definition 5.15 we introduce more complex edge testers that include simple edge testers. However, because we will frequently use weight functions on the candidacy graphs in form of simple edge testers, we include both definitions for the readers' convenience.

To that end, given an index set $I \subseteq [r]$, a 1^{st}-pattern vector $\mathbf{p}$, a 2^{nd}-pattern vector $\mathbf{p}^{2nd}$, vertices $c \in V_{\sqcup I}$ that we call *centres*, and an *(initial) weight function* $\omega_\iota \colon \mathscr{X}_{\sqcup I} \to \mathbb{R}_{\geq 0}$ with $\operatorname{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, I)$, we define a *simple edge tester* $(\omega, \omega_\iota, c, \mathbf{p}, \mathbf{p}^{2nd})$ *with respect to ϕ and the candidacy graphs in $\mathcal{A}$* in the following Definition 5.14. Ultimately, our aim is to track the ω_ι-weight of tuples in $\mathscr{X}_{\sqcup I}$ that are mapped onto the centres c. We can think of $\omega \colon E(\mathcal{A}_{I \setminus [r_\circ]}) \to \mathbb{R}_{\geq 0}$ as an updated weight function that restricts the ω_ι-weight that still can be mapped onto the centres c with respect to $\phi \colon \bigcup_{H \in \mathcal{H}} \widehat{X}^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$ and the candidacy graphs in $\mathcal{A}$, see also Figure 5.3. Since $\operatorname{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, I)$, we specify the 1^{st}-pattern and 2^{nd}-pattern of the tuples in $\operatorname{supp}(\omega_\iota)$ and thus we know exactly how those tuples intersect with edges in $\mathcal{H}$. This will allow us to precisely control the weight of an edge tester during our packing procedure.

Definition 5.14 (Simple edge tester $(\omega, \omega_\iota, c, \mathbf{p}, \mathbf{p}^{2nd})$)**.** *For an index set $I \subseteq [r]$, $\omega_\iota \colon \mathscr{X}_{\sqcup I} \to [0, s]$ for some $s \in \mathbb{R}_{>0}$ with $\operatorname{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, I)$ for given $\mathbf{p}, \mathbf{p}^{2nd} \in \mathbb{N}_0^r$, and $c = \{c_i\}_{i \in I} \in V_{\sqcup I}$, let $\omega \colon E(\mathcal{A}_{I \setminus [r_\circ]}) \to [0, s]$ be defined by*

$$\omega(a) := \mathbb{1}\{a \cap V_{I \setminus [r_\circ]} = \{c_i\}_{i \in I \setminus [r_\circ]}\} \cdot \omega_\iota(x) \tag{5.4.6}$$

for all $a \in E(A^H_{I \setminus [r_\circ]})$ and $H \in \mathcal{H}$ where $x = (\phi|_{V(H)})^{-1}(\{c_i\}_{i \in I \cap [r_\circ]}) \cup (a \cap X^H_{\cup(I \setminus [r_\circ])}) \in X^H_{\sqcup I}$. If no such x exists, we set $\omega(a) := 0$. We say $(\omega, \omega_\iota, c, \mathbf{p}, \mathbf{p}^{2nd})$ is a simple s-edge tester with respect to $(\omega_\iota, c, \mathbf{p}, \mathbf{p}^{2nd})$, ϕ and $\mathcal{A}$.

For the readers' convenience, let us discuss Definition 5.14 in detail. Suppose we are given an (initial) weight function $\omega_\iota \colon \mathscr{X}_{\sqcup I} \to [0, s]$ with centres $c = \{c_i\}_{i \in I} \in V_{\sqcup I}$ for an index set $I \subseteq [r]$ and $\operatorname{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, I)$ with 1^{st}-pattern $\mathbf{p}$ and 2^{nd}-pattern $\mathbf{p}^{2nd}$. Recall that our aim is to control the ω_ι-weight of tuples in $\mathscr{X}_{\sqcup I}$ that are mapped onto the centres c. Therefore, for an edge $a \in E(\mathcal{A}_{I \setminus [r_\circ]})$, we put the weight $\omega_\iota(x)$ onto a only if the following are satisfied:

- $\mathscr{a}$ contains the centres $\{c_i\}_{i\in I\setminus[r_\circ]}$ of the not yet embedded clusters (which is incorporated by the indicator function in (5.4.6)), and

- $\mathscr{x}$ is such that the vertices of $\mathscr{x}$ in $X^H_{\cup(I\setminus[r_\circ])}$ are contained in $\mathscr{a}$ and ϕ maps the vertices in $\mathscr{x}\cap X^H_{\cup(I\cap[r_\circ])}$ onto $\{c_i\}_{i\in I\cap[r_\circ]}$.

For an illustration, see Figure 5.3, where we also demonstrate in part (B) how we update these edge testers when we enlarge the embedding ϕ by embedding further clusters of $\mathcal{H}$ into G. This is one of the main purposes of our Approximate Packing Lemma (Lemma 5.18) in Section 5.5.2.

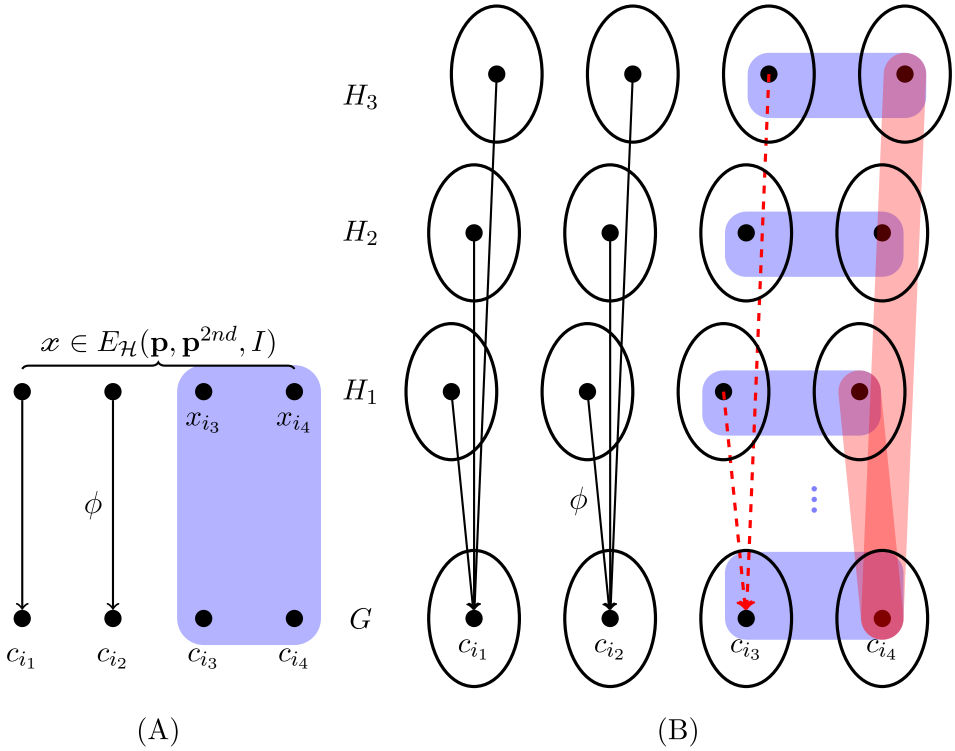


Figure 5.3: (A) illustrates one tuple $\mathscr{x}\in E_{\mathcal{H}}(\mathbf{p},\mathbf{p}^{2nd},I)$ for $I:=\{i_1,\dots,i_4\}$ and $\{i_1,i_2\}\subsetneq[r_\circ]$ whose initial weight $\omega_\iota(\mathscr{x})$ accounts for the weight $\omega(\mathscr{a})$ of $\mathscr{a}=\{x_{i_3},x_{i_4},c_{i_3},c_{i_4}\}\in E(A^H_{I\setminus[r_\circ]})$ in (5.4.6) for a simple edge tester $(\omega,\omega_\iota,\mathscr{c},\mathbf{p},\mathbf{p}^{2nd})$.
(B) illustrates several such tuples and the corresponding edges in $A^{H_\ell}_{I\setminus[r_\circ]}$ for $\ell\in[3]$. Further, assume that we enlarge ϕ to some function ϕ' by embedding the vertices of the third cluster of H_1,H_2,H_3, respectively, onto the third cluster of G which is illustrated by the red dashed line. Consequently, the size of the hyperedges of the updated candidacy graphs with respect to ϕ' will be reduced by 2. In (B) only the vertices of H_1 and H_3 in the third cluster that are contained in the corresponding edges of $A^{H_1}_{I\setminus[r_\circ]}$ and $A^{H_3}_{I\setminus[r_\circ]}$ are mapped onto the centre c_{i_3}, and thus by Definition 5.14 only the weight of the tuples $\mathscr{x}\in E_{\mathcal{H}}(\mathbf{p},\mathbf{p}^{2nd},I)$ of H_1 and H_3 will account for the weight of the two red edges of the updated candidacy graphs.

Let us now comment on the purpose of more complex edge testers as defined in Definition 5.15. Our partial packing procedure will only provide a packing $\phi\colon\bigcup_{H\in\mathcal{H}}\widehat{X}^H_{\cup[r_\circ]}\to V_{\cup[r_\circ]}$ for $[r_\circ]\subseteq[r]$ that maps almost all vertices in $\bigcup_{H\in\mathcal{H}}X^H_{\cup[r_\circ]}$ onto vertices in $V_{\cup[r_\circ]}$ and leaves the vertices $\bigcup_{H\in\mathcal{H}}(X^H_{\cup[r_\circ]}\setminus\widehat{X}^H_{\cup[r_\circ]})$ unembedded. We will often call the vertices $\bigcup_{H\in\mathcal{H}}(X^H_{\cup[r_\circ]}\setminus\widehat{X}^H_{\cup[r_\circ]})$ *unembedded by* ϕ or simply the *leftover* of the partial packing ϕ. In the end, we will have to turn such a partial packing into a complete one. Therefore, we will utilize a second collection of candidacy graphs $\mathcal{B}$

during the partial packing that tracks candidates that correspond to edges in G that we reserved in the beginning for the completion step.[3] In order for this to work, we have to take care that the leftover is well-behaved with respect to the candidacy graphs in $\mathcal{B}$. We achieve this by using weight functions on 2-tuples consisting of one hyperedge of an $\mathcal{A}$-candidacy graph and of a collection of edges within the $\mathcal{B}$-candidacy graphs. That is, assume we are initially given a weight function $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to [0,s]$ (that we therefore often call initial weight function) with centres $\mathscr{c} = \{c_i\}_{i\in I} \in V_{\sqcup I}$ for an index set $I \subseteq [r]$ and recall that our overall aim is to track the ω_ι-weight of tuples in $\mathscr{X}_{\sqcup I}$ that can be mapped onto the centres in $\mathscr{c}$. Now, in Definition 5.15 of our *general edge testers* we allow to specify a set $J \subseteq I$ of indices where we track the ω_ι-weight of tuples $\mathscr{x} = \{x_i\}_{i\in I} \in X^H_{\sqcup I}$ for all $H \in \mathcal{H}$ such that for each $j \in J$, the vertex x_j can be mapped onto c_j within the candidacy graph $B^H_j \in \mathcal{B}$. That is, if the vertices $\{x_j\}_{j\in J}$ are left unembedded, then they can potentially still be mapped onto $\{c_j\}_{j\in J}$ during the completion process using the candidacy graphs $\mathcal{B}$. Further, we even allow to specify disjoint subsets J_X and J_V of J, where J_X encodes that exactly the vertices $\{x_j\}_{j\in J_X}$ of $\mathscr{x}$ are left unembedded, and J_V encodes whether the tuple $\mathscr{x} \in X^H_{\sqcup I}$ lies in a graph H such that $\phi|_{V(H)}$ leaves the centres $\{c_j\}_{j\in J_V}$ uncovered.

Assume $\mathcal{B}$ is a fixed collection of candidacy graphs $\mathcal{B}_j = \bigcup_{H\in\mathcal{H}} B^H_j \subseteq \mathcal{B}_j(\phi)$ for all $j \in [r]$, defined as in Definition 5.11. To make our partial packing procedure more uniform, we will sometimes also treat vertices that are left unembedded by ϕ as embedded by some extension $\phi^+\colon \bigcup_{H\in\mathcal{H}} X^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$ of ϕ (which only serves as a dummy extension and is not necessarily a packing).

Definition 5.15 ((General) edge tester $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$). *For an index set $I \subseteq [r]$, $J \subseteq I$, disjoint sets $J_X, J_V \subseteq J$, $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to [0,s]$ for some $s \in \mathbb{R}_{>0}$ with* $\operatorname{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathscr{p}, I, J)$ *for given $\mathscr{p} \in (\mathbb{N}_0^r)^4$, and $\mathscr{c} = \{c_i\}_{i\in I} \in V_{\sqcup I}$, let $\omega\colon \bigcup_{H\in\mathcal{H}} \big(E(A^H_{I\setminus([r_\circ]\cup J)}) \sqcup \big(\bigsqcup_{j\in J} E(B^H_j)\big)\big) \to [0,s]$ be defined by*

(5.4.7)
$$\omega(\{\mathscr{a}, \mathscr{b}\}) := \mathbb{1}\{(\mathscr{a} \cup b_{\cup J}) \cap V_{\cup((I\setminus[r_\circ])\cup J)} = \{c_i\}_{i\in(I\setminus[r_\circ])\cup J}\} \sum_{\mathscr{x} \text{ is } \{\mathscr{a},\mathscr{b}\}\text{-suitable}} \omega_\iota(\mathscr{x})$$

*for all $\mathscr{a} \in E(A^H_{I\setminus([r_\circ]\cup J)})$, $\mathscr{b} = \{b_j\}_{j\in J} \in \bigsqcup_{j\in J} E(B^H_j)$ and $H \in \mathcal{H}$, where we say that $\mathscr{x} \in X^H_{\sqcup I}$ is $\{\mathscr{a}, \mathscr{b}\}$-*suitable *if*

(i)$_{\text{D5.15}}$ $(\mathscr{a} \cup b_{\cup J}) \cap X^H_{\cup((I\setminus[r_\circ])\cup J)} = \mathscr{x} \cap X^H_{\cup((I\setminus[r_\circ])\cup J)}$;

(ii)$_{\text{D5.15}}$ $\{c_i\}_{i\in(I\cap[r_\circ])\setminus J} \subseteq \phi(\mathscr{x} \cap \widehat{X}^H_{\cup[r_\circ]})$;

(iii)$_{\text{D5.15}}$ $c_j \notin \phi(\widehat{X}^H_j)$ *for all* $j \in J_V \cap [r_\circ]$;

(iv)$_{\text{D5.15}}$ $\mathscr{x} \cap (X^H_{\cup[r_\circ]} \setminus \widehat{X}^H_{\cup[r_\circ]}) = \mathscr{x} \cap X^H_{\cup(J_X\cap[r_\circ])}$;

(v)$_{\text{D5.15}}$ $\phi^+(\mathscr{x} \cap X^H_{\cup(J\cap[r_\circ])}) \cap \mathscr{c} = \emptyset$.

Note that for each $\{\mathscr{a}, \mathscr{b}\}$, there is at most one $\{\mathscr{a}, \mathscr{b}\}$-suitable tuple $\mathscr{x}$. If no $\{\mathscr{a}, \mathscr{b}\}$-suitable tuple $\mathscr{x}$ exists, we set $\omega(\{\mathscr{a}, \mathscr{b}\}) := 0$. We say $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ is an s-edge tester with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, (ϕ, ϕ^+), $\mathcal{A}$ and $\mathcal{B}$.

[3] In fact, we partition the edge set of the host graph G into two k-graphs G_A and G_B, and $\mathcal{A}$ will be a collection of candidacy graphs with respect to ϕ and G_A, and $\mathcal{B}$ will be a collection of candidacy graphs with respect to ϕ and G_B.

We often write J_{XV} for $J_X \cup J_V$ if J_X and J_V are fixed. Let us comment on Definition 5.15. Suppose we are given an initial weight function $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to [0,s]$ with centres $\mathscr{c} = \{c_i\}_{i\in I} \in V_{\sqcup I}$ for an index set $I \subseteq [r]$ and $J \subseteq I$, and $\mathrm{supp}(\omega_\iota) \subseteq E_{\mathcal{H}}(\mathscr{p}, I, J)$. As in the case for simple edge testers, our aim is to control the ω_ι-weight of tuples $\mathscr{x}$ in $\mathscr{X}_{\sqcup I}$ that are mapped onto the centres $\mathscr{c}$. The set $J \subseteq I$ allows us to specify some vertices $\mathscr{x} \cap X^H_{\cup J}$ of such tuples $\mathscr{x}$ that are not yet embedded by ϕ^+ onto their centers $\{c_i\}_{i\in J}$ and which will potentially be embedded onto those during the completion process. For the completion, we will use the candidacy graphs B^H_j in $\mathcal{B}$ and therefore, $(\mathscr{x} \cap X^H_j) \cup \{c_j\} = b_j \in E(B^H_j)$ for each $j \in J$. Furthermore, the sets $J_X, J_V \subseteq J$ encode the situations that

- only the vertices $\mathscr{x} \cap X^H_{\cup(J_X\cap[r_\circ])}$ of $\mathscr{x}$ are left unembedded by ϕ (see (iv)$_{\mathrm{D5.15}}$), or
- the centres $\{c_j\}_{j\in J_V\cap[r_\circ]}$ are uncovered by $\phi|_{V(H)}$ (see (iii)$_{\mathrm{D5.15}}$).

Note that a (general) edge tester $(\omega, \omega_\iota, J = \emptyset, J_X = \emptyset, J_V = \emptyset, \mathscr{c}, (\mathbf{p}, \mathbf{p}^{2nd}, \mathbf{0}, \mathbf{0}))$ is equivalent to a simple edge tester $(\omega, \omega_\iota, \mathscr{c}, \mathbf{p}, \mathbf{p}^{2nd})$ with respect to $(\omega_\iota, \mathscr{c}, \mathbf{p}, \mathbf{p}^{2nd})$, ϕ and $\mathcal{A}$.

5.4.4 Sets of suitable $\mathcal{H}$-edges

Next, we define (sub)sets of $\mathcal{H}$-edges that we track during our packing procedure. Recall that we consider some fixed $r_\circ \le r$ and $\phi\colon \bigcup_{H\in\mathcal{H}} \widehat{X}^H_{\cup[r_\circ]} \to V_{\cup[r_\circ]}$. In Definition 5.16, we define a set $\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})$ for every edge $\mathscr{g} \in E(G)$ that contains the sets of vertices that are contained in an $\mathcal{H}$-edge with 1^{st}-pattern $\mathbf{p}$ and 2^{nd}-pattern $\mathbf{p}^{2nd}$, and that still could be mapped together onto $\mathscr{g}$ with respect to ϕ and the candidacy graphs in $\mathcal{A}$. We can track the size of this set $\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})$ by using simple edge testers.

Definition 5.16 ($\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})$). *For all $\mathscr{g} = \mathscr{g}_\circ \cup \mathscr{g}_m \in E(G[V_{\cup\mathscr{r}}])$ for some $\mathscr{r} \in E(R)$ with $\mathscr{g}_\circ \in \binom{V_{\cup[r_\circ]}}{k-m}$, $m \in [k]$, and with $I := \mathscr{r} \setminus [r_\circ]$, $|I| = m$, $\mathscr{g}_m \in V_{\sqcup I}$, and for all $\mathbf{p}, \mathbf{p}^{2nd} \in \mathbb{N}_0^r$, let*

$$\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A}) := \bigcup_{H\in\mathcal{H}} \Big\{ \mathscr{x}_m \in X^H_{\sqcup I}\colon$$

$$\mathscr{x}_m \cup \mathscr{g}_m \in E(A^H_I), \tag{5.4.8}$$

$$(\phi|_{V(H)})^{-1}(\mathscr{g}_\circ) \cup \mathscr{x}_m \in E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r}) \Big\}. \tag{5.4.9}$$

Further, for $\omega_\iota\colon \mathscr{X}_{\sqcup\mathscr{r}} \to \{0,1\}$ with $\omega_\iota(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})\}$, we call the simple 1-edge tester $(\omega, \omega_\iota, \mathscr{g}, \mathbf{p}, \mathbf{p}^{2nd})$ with respect to $(\omega_\iota, \mathscr{g}, \mathbf{p}, \mathbf{p}^{2nd})$, ϕ and $\mathcal{A}$ (as defined in Definition 5.14), the edge tester for $\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})$.

Let us describe Definition 5.16 in words. Suppose we are given an edge $\mathscr{g} = \mathscr{g}_\circ \cup \mathscr{g}_m \in E(G[V_{\cup\mathscr{r}}])$, where $\mathscr{g}_\circ$ contains the vertices of $\mathscr{g}$ that lie in clusters that are already embedded by ϕ and $\mathscr{g}_m$ contains the remaining m vertices of $\mathscr{g}$ in the not yet embedded clusters with indices $I = \mathscr{r} \setminus [r_\circ]$. For each $H \in \mathcal{H}$, we track the set of vertices $\mathscr{x}_m \in X^H_{\sqcup I}$ where

- $\mathscr{x}_m$ still could be mapped onto $\mathscr{g}_m$ (that is, $\mathscr{x}_m \cup \mathscr{g}_m \in E(A^H_I)$ in (5.4.8)), and
- $\mathscr{x}_m$ lies in an H-edge $\mathscr{e}$ with 1^{st}-pattern $\mathbf{p}$ and 2^{nd}-pattern $\mathbf{p}^{2nd}$ such that if we map $\mathscr{x}_m$ onto $\mathscr{g}_m$, then $\mathscr{e}$ is mapped onto $\mathscr{g}$ (that is, $(\phi|_{V(H)})^{-1}(\mathscr{g}_\circ) \cup \mathscr{x}_m \in E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})$ in (5.4.9)).

Further, note that for a simple edge tester $(\omega, \omega_\iota, g, \mathbf{p}, \mathbf{p}^{2nd})$ for $\mathcal{X}_{g,\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})$ as in Definition 5.16, we have that $\omega(E(\mathcal{A}_I)) = |\mathcal{X}_{g,\mathbf{p},\mathbf{p}^{2nd},\phi}(\mathcal{A})|$ for $I = r \setminus [r_\circ]$ because the indicator function in (5.4.6) corresponds to (5.4.8), and the choice of x in (5.4.6) corresponds to (5.4.9) by the definition of ω_ι.

Next, we define in Definition 5.17 a set $E_{g,h,\phi}(\mathcal{A})$ for all distinct G-edges g, h with identical G-vertex in the last cluster such that $E_{g,h,\phi}(\mathcal{A})$ contains the tuples of $\mathcal{H}$-edges (e, f) with identical $\mathcal{H}$-vertex in the last cluster, and e and f can still be mapped onto g and h with respect to ϕ and the candidacy graphs in $\mathcal{A}$. In this case, we ignore the patterns as we only aim for an upper bound on the number of these edges and have some room to spare.

Definition 5.17 ($E_{g,h,\phi}(\mathcal{A})$)**.** *For edges $g = \{v_{i_1}, \dots, v_{i_k}\}, h = \{w_{j_1}, \dots, w_{j_k}\} \in E(G)$ with $v_{i_k} = w_{j_k}$ and both $z \in \{g, h\}$, let*

$$E_{z,\phi}(\mathcal{A}) := E\left(\mathcal{H}\left[\phi^{-1}(z \cap V_{\cup[r_\circ]}) \cup \bigcup\nolimits_{i\in[r]\setminus[r_\circ]} N_{\mathcal{A}_i}(z \cap V_i)\right]\right);$$
$$E_{g,h,\phi}(\mathcal{A}) := \left\{(e, f) \in E_{g,\phi}(\mathcal{A}) \times E_{h,\phi}(\mathcal{A}) \colon e \cap f \cap \mathscr{X}_{i_k} \neq \emptyset\right\}.$$

5.5 Approximate Packing Lemma

In this section we provide our 'Approximate Packing Lemma' (Lemma 5.18). Given a blow-up instance $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$, it allows us for one cluster to embed almost all vertices of $\bigcup_{H\in\mathcal{H}} X_i^H$ into V_i, while maintaining crucial properties for future embedding rounds of other clusters. To describe this setup we define a *packing instance* and collect some more notation.

5.5.1 Packing instances

Our general understanding of a packing instance is as follows. Recall that we will consider the clusters of a blow-up instance one after another. A packing instance arises from a blow-up instance where we have already embedded vertices of some clusters (which is given by a function $\phi_\circ$) and focuses only on one particular cluster (denoted by $\bigcup_{H\in\mathcal{H}} X_0^H$ and V_0) and all clusters that are close to the considered cluster (measured in the reduced graph R). We track *candidacy graphs* as defined in Definition 5.11 and consider a collection of candidacy graphs $\mathcal{A}$ between the clusters in $\mathcal{H}$ and G that will be used for future embedding rounds. In order to be able to turn a partial packing into a complete one in the end, we do not only track the collection of candidacy graphs in $\mathcal{A}$ but also a second collection of candidacy graphs $\mathcal{B}$, where the candidacy graphs in $\mathcal{B}$ will be used for the completion step. To that end, we also assume that the edges of G are partitioned into two parts G_A and G_B such that the edges in G_A are used for the approximate packing and the edges in G_B are reserved for the completion step. That is, we will think of the graphs in $\mathcal{A}$ as candidacy graphs with respect to $\phi_\circ$ and G_A, and of the graphs in $\mathcal{B}$ as candidacy graphs with respect to $\phi_\circ$ and G_B.

We make this more precise. Let $n, k, r, r_\circ \in \mathbb{N}_0$. We say $\mathscr{P} = (\mathcal{H}, G_A, G_B, R, \mathcal{A}, \mathcal{B}, \phi_\circ^-, \phi_\circ)$ is a *packing instance of size* $(n, k, r, r_\circ)$ if

- $\mathcal{H}$ is a collection of k-graphs, G_A and G_B are edge-disjoint k-graphs on the same vertex set, and R is a k-graph where $V(R) = -[r_\circ] \cup [r]_0$;
- $\{X_i^H\}_{i\in V(R)}$ is a vertex partition of $H \in \mathcal{H}$ such that $|e \cap X_i^H| \leq 1$ for all $e \in E(H)$, $H \in \mathcal{H}$;
- $\{V_i\}_{i\in V(R)}$ is a vertex partition of G_A as well as G_B;

- $|X_i^H| = |V_i| = (1 \pm 1/2)n$ for each $i \in V(R)$;
- for all $H \in \mathcal{H}$, the hypergraph $H[X^H_{\cup r}]$ is a matching if $r \in E(R)$ and empty if $r \in \binom{V(R)}{k} \setminus E(R)$;
- $\mathcal{A} = \bigcup_{H\in\mathcal{H}, I\subseteq[r]_0\colon I \text{ is an index set}} A_I^H$ is a union of candidacy graphs with respect to $\phi_\circ$ and G_A; in particular, A_I^H is $2|I|$-uniform, and A_i^H is a balanced bipartite 2-graph with vertex partition (X_i^H, V_i) for each $i \in [r]_0$;
- $\mathcal{B} = \bigcup_{H\in\mathcal{H}, j\in V(R)} B_j^H$ is a union of candidacy graphs with respect to $\phi_\circ$ and G_B; in particular, B_j^H is a balanced bipartite 2-graph with vertex partition (X_j^H, V_j) for each $j \in V(R)$;
- $\phi_\circ^-\colon \bigcup_{H\in\mathcal{H}} X^{H,-}_{\cup-[r_\circ]} \to V_{\cup-[r_\circ]}$ with $X_i^{H,-} \subseteq X_i^H$, $\phi_\circ^-(X_i^{H,-}) \subseteq V_i$ and $\phi_\circ^-|_{X_i^{H,-}}$ is injective for all $H \in \mathcal{H}, i \in -[r_\circ]$, and $\phi_\circ\colon \bigcup_{H\in\mathcal{H}} X^H_{\cup-[r_\circ]} \to V_{\cup-[r_\circ]}$ is an extension of $\phi_\circ^-$ with $\phi_\circ(X_i^H) = V_i$ and $\phi_\circ|_{X_i^H}$ is bijective for all $H \in \mathcal{H}, i \in -[r_\circ]$.

For simplicity, we often write $G := G_A \cup G_B$, $\mathscr{X}_i := \bigcup_{H\in\mathcal{H}} X_i^H$, $\mathscr{X}_i^- := \bigcup_{H\in\mathcal{H}} X_i^{H,-}$, $\mathcal{A}_I := \bigcup_{H\in\mathcal{H}} A_I^H$, $\mathcal{B}_i := \bigcup_{H\in\mathcal{H}} B_i^H$, $\mathcal{X}_\circ := \mathscr{X}_{\cup-[r_\circ]}$, $\mathcal{X}_\circ^- := \mathscr{X}^-_{\cup-[r_\circ]}$ and $\mathcal{V}_\circ := V_{\cup-[r_\circ]}$ for all $i \in V(R)$ and index sets $I \subseteq [r]_0$. Note that the packing instance $\mathscr{P}$ naturally corresponds to a blow-up instance

$$(\mathcal{H}, G, R, \{X_i^H\}_{i\in V(R), H\in\mathcal{H}}, \{V_i\}_{i\in V(R)})$$

of size $(n, k, r + r_\circ + 1)$. In particular, we also use the notation of Section 5.4. For the sake of a better readability, we stick to some conventions:

We will often use the letters $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$ as many arguments for candidacy graphs $\mathcal{A}_I$ with respect to G_A, and candidacy graphs $\mathcal{B}_i$ with respect to G_B are the same. Whenever we write $xv \in E(\mathcal{Z}_i)$ for some $i \in [r]$, we tacitly assume that $x \in \mathscr{X}_i, v \in V_i$. We usually denote edges in $\mathcal{Z}_i$ by (non-calligraphic) letters e, f, and hyperedges in $\mathcal{A}_I$ by a and a collection of edges from $\bigsqcup_{j\in J} E(\mathcal{B}_j)$ by $b = \{b_j\}_{j\in J}$, where we allow to slightly abuse the notation and often treat b as $b_{\cup J}$. Whenever we write $\{v_{i_1}, \dots, v_{i_k}\} \in E(G_Z)$, we tacitly assume that $v_{i_\ell} \in V_{i_\ell}$ for all $\ell \in [k]$; analogously for $\{x_{i_1}, \dots, x_{i_k}\} \in E(H)$. We usually refer to hyperedges in G_Z with letters g, h, hyperedges in $\mathcal{H}$ with letters e, f, and hyperedges in R with r.

The aim of this section is to map almost all vertices of $\mathscr{X}_0$ into V_0 by defining a function $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathcal{A}_0$ (that is, $x\sigma(x) \in E(\mathcal{A}_0)$) where $\mathscr{X}_0^\sigma \subseteq \mathscr{X}_0$, while maintaining several properties for the other candidacy graphs. We identify such a function σ with its *corresponding edge set* $M(\sigma)$ defined as

$$M(\sigma) := \{xv\colon x \in \mathscr{X}_0^\sigma, v \in V_0, \sigma(x) = v\}. \tag{5.5.1}$$

To incorporate that σ has to be chosen such that each edge in G_A is used at most once, we define an edge set labelling ψ *with respect to* $\mathscr{P}$ on $\mathcal{A}_0$ as follows. For every edge $xv \in E(\mathcal{A}_0)$, we set

$$\psi(xv) := \left\{\phi_\circ^-(e \setminus \{x\}) \cup \{v\}\colon x \in e \in E(\mathcal{H}) \text{ with } e \setminus \{x\} \subseteq \mathcal{X}_\circ^-\right\}. \tag{5.5.2}$$

We defined the candidacy graphs $\mathcal{A}_0$ in Definition 5.11 such that $xv \in E(\mathcal{A}_0)$ only if $\phi_\circ^-(e \setminus \{x\}) \cup \{v\} \in E(G_A)$ for each such edge e as in (5.5.2). That is, $\psi(xv)$ encodes the set of edges in G_A that are used for the packing when mapping x onto v. We say

(5.5.3) *$\sigma\colon \mathscr{X}_0^\sigma \to V_0$ is a* conflict-free packing *if $\sigma|_{\mathscr{X}_0^\sigma \cap X_0^H}$ is injective for all $H \in \mathcal{H}$ and $\psi(e) \cap \psi(f) = \emptyset$ for all distinct $e, f \in M(\sigma)$.*

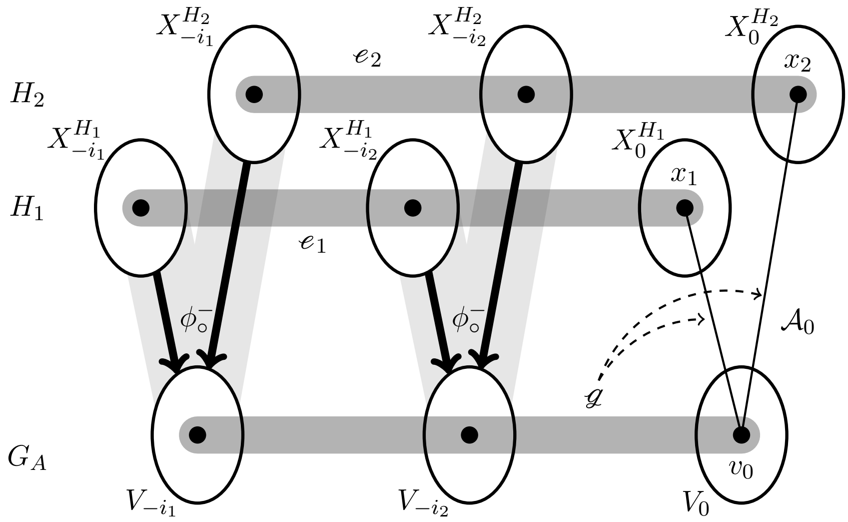


Figure 5.4: We have $\mathcal{H} = \{H_1, H_2\}$ and $k = 3$. If $\phi_\circ^-(e_1 \setminus \{x_1\}) = \phi_\circ^-(e_2 \setminus \{x_2\}) = g \setminus \{v_0\}$, then we add the label g to the edge set label $\psi(x_1v_0)$ and $\psi(x_2v_0)$ of the edges x_1v_0 and x_2v_0 of the candidacy graph $\mathcal{A}_0$. This captures the information that if x_1 or x_2 are mapped onto v_0 by σ in $\mathcal{A}_0$, then this embedding uses the edge $g \in E(G_A)$.

Crucially note that the property that $\psi(e) \cap \psi(f) = \emptyset$ for all distinct $e, f \in M(\sigma)$ will guarantee that every edge in G_A is used at most once. For an illustration, see Figure 5.4.

Given a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathcal{A}_0$, we update the remaining candidacy graphs with respect to σ. To account for the vertices in $\mathscr{X}_0 \setminus \mathscr{X}_0^\sigma$ that are left unembedded by σ, we will consider an extension σ^+ of σ such that σ^+ also maps every vertex $x_0 \in \mathscr{X}_0 \setminus \mathscr{X}_0^\sigma$ to V_0 and $\sigma^+|_{X_0^H}$ is injective for all $H \in \mathcal{H}$ (and hence bijective). We call such a σ^+ a *cluster-injective extension* of σ. The purpose of σ^+ is that $\sigma^+|_{\mathscr{X}_0 \setminus \mathscr{X}_0^\sigma}$ will serve as a 'dummy' extension resulting in an easier analysis of the packing process as σ^+ will impose further restriction that culminate in more consistent candidacy graphs. Using Definition 5.11, we will consider the *(updated) candidacy graphs* $A_I^H(\phi_\circ \cup \sigma^+)$ with respect to $\phi_\circ \cup \sigma^+$ and G_A for index sets $I \subseteq [r]$, as well as the *(updated) candidacy graphs* $B_j^H(\phi_\circ \cup \sigma^+)$ with respect to $\phi_\circ \cup \sigma^+$ and G_B for $j \in V(R)$.

To track our packing process, we carefully maintain quasirandom properties of the candidacy graphs throughout the entire procedure. Our Approximate Packing Lemma will guarantee that we can find a conflict-free packing that behaves like an idealized random packing with respect to given sets of *edge testers* (as defined in Definition 5.15), and with respect to weight functions $\omega\colon \binom{E(\mathcal{A}_0)}{\ell}^= \to [0, s]$ for $\ell \leq s$ that we will call *local testers*.

To that end, assume we are given a packing instance $\mathscr{P} = (\mathcal{H}, G_A, G_B, R, \mathcal{A}, \mathcal{B}, \phi_\circ^-, \phi_\circ)$ of size $(n, k, r, r_\circ)$, and $\mathbf{d} = (d_A, d_B, (d_{i,A})_{i \in [r]_0}, (d_{i,B})_{i \in V(R)})$, and $q, t \in \mathbb{N}$, as well as a set $\mathcal{W}_{edge}$ of edge testers. We say $\mathscr{P}$ is an $(\varepsilon, q, t, \mathbf{d})$*-packing instance with suitable edge testers* $\mathcal{W}_{edge}$ if $|X_i^H| = |V_i| = (1 \pm \varepsilon)n$ for all $i \in V(R)$, and the following properties are satisfied (recall (5.4.2) and Definitions 5.15–5.17 for (P2)–(P5), respectively, and that we write J_{XV} for $J_X \cup J_V$):

(P1) for all $i \in V(R)$ and all pairs of disjoint sets $\mathcal{S}_A, \mathcal{S}_B \subseteq \bigcup_{r \in E(R)\colon i \in r} V_{\sqcup r \setminus \{i\}}$ with $|\mathcal{S}_A \cup \mathcal{S}_B| \leq t$, we have $\big|V_i \cap N_{G_A}(\mathcal{S}_A) \cap N_{G_B}(\mathcal{S}_B)\big| = (1 \pm \varepsilon) d_A^{|\mathcal{S}_A|} d_B^{|\mathcal{S}_B|} n$;

(P2) for all $H \in \mathcal{H}, i \in [r]_0, j \in V(R)$, we have that A_i^H is $(\varepsilon, d_{i,A})$-super-regular and (ε, q)-well-intersecting with respect to G_A, and B_j^H is $(\varepsilon, d_{j,B})$-super-regular and (ε, q)-well-intersecting with respect to G_B;

(P3) for every edge tester $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}_{edge}$ with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, $(\phi_\circ^-, \phi_\circ)$, $\mathcal{A}$ and $\mathcal{B}$, with centres $\mathscr{c} \in V_{\sqcup I}$ for $I \subseteq V(R)$, $I_{r_0} := (I \cap [r]_0) \setminus J$, and patterns $\mathscr{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd}) \in (\mathbb{N}_0^{r_\circ+r+1})^4$, we have that

$$\omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)\right) = \Big(\mathbb{1}\{J_{XV} \cap -[r_\circ] = \emptyset\} \pm \varepsilon\Big) \prod_{Z\in\{A,B\}} d_Z^{\|\mathbf{p}^Z_{-[r_\circ]_0}\| - \|\mathbf{p}^{Z,2nd}_{-[r_\circ]_0}\|} \prod_{i\in I_{r_0}} d_{i,A} \prod_{j\in J} d_{j,B} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ])\setminus J|}} \pm n^{\varepsilon};$$

(P4) for all $\mathscr{g} \in E(G_A[V_{\cup \mathscr{r}}])$ for some $\mathscr{r} \in E(R)$ with $\mathscr{r} \cap [r]_0 \neq \emptyset$, and all $\mathbf{p}, \mathbf{p}^{2nd} \in \mathbb{N}_0^{r_\circ+r+1}$, the set $\mathcal{W}_{edge}$ contains the edge tester for $\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})$;

(P5) for all $\mathscr{g} = \{v_{i_1}, \dots, v_{i_k}\}, \mathscr{h} = \{w_{j_1}, \dots, w_{j_k}\} \in E(G_A)$ with $v_{i_k} = w_{j_k}$ and $I := \{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\} =: J$, we have that

$$\left|E_{\mathscr{g},\mathscr{h},\phi_\circ^-}(\mathcal{A})\right| \leq \max\left\{n^{k-|(I\cup J)\cap -[r_\circ]|+\varepsilon}, n^{\varepsilon}\right\}.$$

Note that (P1) also implies that G_Z is $(3\varepsilon, t, d_Z)$-typical with respect to R for each $Z \in \{A, B\}$.

Furthermore, we call a function $\omega\colon \binom{E(\mathcal{A}_0)}{\ell}^{=} \to [0, s]$ for $\ell \leq s$ a *local s-tester (for the $(\varepsilon, q, t, \mathbf{d})$-packing instance $\mathscr{P}$)* if $\|\omega\|_{\ell'} \leq n^{\ell-\ell'+\varepsilon^2}$ for every $\ell' \in [\ell]$. We introduce some more notation:

(5.5.4) *Let $E_i^R := \{\mathscr{r} \in E(R)\colon \{0, i\} = \mathscr{r} \cap ([r]_0 \cup \{i\})\}$ and $b_i := |E_i^R| = \deg_{R[-[r_\circ]\cup\{0,i\}]}(\{0, i\})$ for each $i \in V(R) \setminus \{0\}$. For $I \subseteq V(R)$, let $b_I := \sum_{i\in I\setminus\{0\}} b_i$. For $i \in [r], j \in V(R) \setminus \{0\}$, let $d_{i,A}^{new} := d_{i,A} d_A^{b_i}$ and $d_{j,B}^{new} := d_{j,B} d_B^{b_j}$.*

Note that for $i \in V(R) \setminus N_{R_*}[0]$, we have $b_i = 0$ and thus $d_{i,Z}^{new} = d_{i,Z}$ for $Z \in \{A, B\}$.

5.5.2 Approximate Packing Lemma

We now state the Approximate Packing Lemma, which is the key tool for the proof of our main result.

Lemma 5.18 ([31] — Approximate Packing Lemma)**.** *Let $1/n \ll \varepsilon \ll \varepsilon' \ll 1/t \ll 1/k, 1/q, 1/r, 1/(r_\circ+1), 1/s$. Suppose $(\mathcal{H}, G_A, G_B, R, \mathcal{A}, \mathcal{B}, \phi_\circ^-, \phi_\circ)$ is an $(\varepsilon, q, t, \mathbf{d})$-packing instance of size $(n, k, r, r_\circ)$ with $\mathbf{d} \geq n^{-\varepsilon}$ and suitable s-edge-testers $\mathcal{W}_{edge}$. Suppose further that $\mathcal{W}_{local}$ is a set of local s-testers, $\mathcal{W}_0$ is a set of tuples (ω, c) with $\omega\colon \mathscr{X}_0 \to [0, s]$, $c \in V_0$, and $|\mathcal{W}_{edge}|, |\mathcal{W}_{local}|, |\mathcal{W}_0| \leq n^{4\log n}$, $|\mathcal{H}| \leq n^{2k}$, as well as $e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) \leq d_A n^k$ for all $\mathscr{r} \in E(R)$.*

Then there is a conflict-free packing $\sigma\colon \mathscr{X}_0^\sigma \to V_0$ in $\mathcal{A}_0$ and a cluster-injective extension σ^+ of σ such that for all $H \in \mathcal{H}$, we have $|\mathscr{X}_0^\sigma \cap X_0^H| \geq (1 - \varepsilon')n$, and for all index sets $I_A \subseteq [r]$, $I_B \in V(R)$ and $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$, there exist spanning subgraphs $Z_{I_Z}^{H,new}$ of the candidacy graphs $Z_{I_Z}^H(\phi_\circ \cup \sigma^+)$ with respect to $\phi_\circ \cup \sigma^+$ and G_Z (where $\mathcal{Z}_{I_Z}^{new} := \bigcup_{H\in\mathcal{H}} Z_{I_Z}^{H,new}$ and $\mathcal{Z}^{new}$ is the collection of all $\mathcal{Z}_{I_Z}^{new}$) such that

(I)$_{\text{L5.18}}$ *$Z_i^{H,new}$ is $(\varepsilon', d_{i,Z}^{new})$-super-regular and $(\varepsilon', q+\Delta(R))$-well-intersecting with respect to G_Z for all $H \in \mathcal{H}$, and all $i \in [r]$ if $Z = A$, and all $i \in V(R) \setminus \{0\}$ if $Z = B$;*

(II)$_{\text{L5.18}}$ *for every (general) s-edge tester $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}_{edge}$ with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, $(\phi_\circ^-, \phi_\circ)$, $\mathcal{A}$ and $\mathcal{B}$, with centres $\mathscr{c} \in V_{\sqcup I}$ for $I \subseteq V(R)$, $I_r := (I \cap [r]) \setminus J$, $I_r \cup J \neq \emptyset$, and patterns $\mathscr{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd}) \in (\mathbb{N}_0^{r_\circ+r+1})^4$, the s-edge tester $(\omega^{new}, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ defined as in Definition 5.15 with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, $(\phi_\circ^- \cup \sigma, \phi_\circ \cup \sigma^+)$, $\mathcal{A}^{new}$ and $\mathcal{B}^{new}$ satisfies*

$$\omega^{new}\Big(E(\mathcal{A}_{I_r}^{new}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j^{new})\Big) = \big(\mathbb{1}\{J_{XV} \cap -[r_\circ]_0 = \emptyset\} \pm \varepsilon'^2\big) \prod_{Z\in\{A,B\}} d_Z^{\|\mathbf{p}^Z_{-[r_\circ]_0}\| - \|\mathbf{p}^{Z,2nd}_{-[r_\circ]_0}\|} \prod_{i\in I_r} d_{i,A}^{new} \prod_{j\in J} d_{j,B}^{new} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ]_0)\setminus J|}} \pm n^{\varepsilon'};$$

(III)$_{\text{L5.18}}$ *$\omega(M(\sigma)) = (1 \pm \varepsilon'^2)\omega(E(\mathcal{A}_0))/(d_{0,A}n)^\ell \pm n^\varepsilon$ for every local s-tester $\omega \in \mathcal{W}_{local}$ with $\omega\colon \binom{E(\mathcal{A}_0)}{\ell}^= \to [0,s]$;*

(IV)$_{\text{L5.18}}$ *$\omega(\{x \in \mathscr{X}_0 \setminus \mathscr{X}_0^\sigma \colon \sigma^+(x) = c\}) \le \omega(\mathscr{X}_0)/n^{1-\varepsilon} + n^\varepsilon$ for every $(\omega, c) \in \mathcal{W}_0$.*

Properties (I)$_{\text{L5.18}}$ and (II)$_{\text{L5.18}}$ ensure that (P2) and (P3) are also satisfied for the updated candidacy graphs $\mathcal{A}^{new}$ and $\mathcal{B}^{new}$, respectively. Property (III)$_{\text{L5.18}}$ states that σ behaves like a random packing with respect to the local testers, which for instance can be used to establish (P5) for future packing rounds. Property (IV)$_{\text{L5.18}}$ allows to control the weight on vertices that are not embedded by σ but are nevertheless mapped onto a specific vertex c by the extension σ^+.

Proof. We split the proof into three parts. In Part A we construct an auxiliary supergraph H_+ of every $H \in \mathcal{H}$ by adding some hyperedges to $H[X_{\cup\mathscr{r}}^H]$ for every $\mathscr{r} \in E(R)$ in order to make the packing procedure more uniform. In Part B we construct an auxiliary hypergraph $\mathcal{H}_{aux}$ for $\mathcal{A}_0$ such that we can use Theorem 2.3 to find a conflict-free packing in $\mathcal{A}_0$. In order to be able to apply Theorem 2.3, we exploit (P5) as well as (P2) together with (P3) to control $\Delta_2(\mathcal{H}_{aux})$ and $\Delta(\mathcal{H}_{aux})$, respectively. In Part C we define weight functions and employ the conclusion of Theorem 2.3 to establish (I)$_{\text{L5.18}}$–(III)$_{\text{L5.18}}$.

Let $\Delta_R := 2\binom{r+r_\circ}{k-1}$. Note that $\Delta(R), \Delta(\mathcal{H}), b_i \le \Delta_R$ for all $i \in [r]$. For simplicity we write $d_0 := d_{0,A}$ and $\hat\varepsilon := \varepsilon^{1/2}$. Further, we choose a new constant Δ such that $\varepsilon' \ll 1/\Delta \ll 1/t$.

Part A. Construction of H_+

We construct an auxiliary supergraph H_+ of H by artificially adding some edges to H for every $\mathscr{r} \in E_i^R$ and $i \in V(R) \setminus \{0\}$. (Recall (5.5.4) for the definition of E_i^R and note that it can be that $\mathscr{r} \in E_i^R \cap E_j^R$ for $i, j \in -[r_\circ]$.) For every $H \in \mathcal{H}$, we proceed as follows. We obtain H_+ from H by adding a minimal number of hyperedges of size k subject to the conditions that for all $i \in V(R) \setminus \{0\}$ and $\mathscr{r} \in E_i^R$, an H_+-edge meets every cluster in $H[X_{\cup\mathscr{r}}^H]$ exactly once, and

(a) $\deg_{H_+[X_{\cup\mathscr{r}}^H]}(x_i) \in \{1, 2\}$ for all $x_i \in X_i^H$, and $\deg_{H_+[X_{\cup\mathscr{r}}^H]}(x_0) \le 2$ for all $x_0 \in X_0^H$;

(b) for all $\mathscr{e} \in E(H_+[X_{\cup\mathscr{r}}^H])$, we have $|\{x \in \mathscr{e} \colon \deg_{H_+[X_{\cup\mathscr{r}}^H]}(x) = 2\}| \le 1$;

(c) for all $\{x_0, x_i\} \in X_0^H \sqcup X_i^H$, if $\{x_0, x_i\} \subseteq e$ for some $e \in E(H_+) \setminus E(H)$, then $\{x_0, x_i\} \nsubseteq f$ for all $f \in E(H)$.

Note that (a)–(c) can be met because $|X_i^H| = (1 \pm \varepsilon)n$ for all $i \in V(R)$ and $\Delta(R) \leq \Delta_R$. For all $i \in V(R) \setminus \{0\}$, let H_+^i be an arbitrary but fixed k-graph $H \subseteq H_+^i \subseteq H_+$ such that for all $r \in E_i^R$, we have $\deg_{H_+^i[X_{\cup r}^H]}(x_i) = 1$. Observe that by the construction of H_+, we have $\deg_{H_+[X_{\cup r}^H]}(x_i) = 1$ for all $x_i \in X_i^H$, $i \in [r]$, $r \in E_i^R$, and thus $H_+^i = H_+$ for all $i \in [r]$. We make some observations.

(5.5.5)

For all $x \in X_i^H, i \in V(R) \setminus \{0\}$, *we have* $\sum_{r \in E_i^R} \deg_{H_+^i[X_{\cup r}^H]}(x) = b_i$.

(5.5.6)
By (a), *for every* $x \in X_i^H, i \in V(R) \setminus \{0\}$, *there are at most* Δ_R *vertices* $x' \in X_i^H \setminus \{x\}$ *such that* x *and* x' *have a common neighbour in* X_0^H *in* $H_+[\mathcal{X}_\circ, X_0^H, X_i^H]$, *that is,* $e_x \cap e_{x'} \cap X_0^H \neq \emptyset$ *with* $e_y \in E(H_+[\mathcal{X}_\circ, X_0^H, X_i^H])$ *and* $y \in e_y$ *for both* $y \in \{x, x'\}$.

(5.5.7)
If $\{x_0, x_i\}$ *lies in an edge of* $H_+ - H$, *then* $\{x_0, x_i\}$ *does not lie in an edge of* H.

(5.5.8)
If $|e \cap X_{\cup[r]}^H| \geq 2$, *then* $e \in E(H)$ *for all* $e \in E(H_+)$.

We introduce some simpler notation how to denote edges in H_+ (respectively H_+^i) that contain a vertex $x \in X_0^H$. Let $\mathcal{H}_+ := \bigcup_{H \in \mathcal{H}} H_+$, and for $i \in V(R) \setminus \{0\}$, let $\mathcal{H}_+^i := \bigcup_{H \in \mathcal{H}} H_+^i$. For all $x \in \mathscr{X}_0$ and $\mathbf{y} \in \mathscr{X}_{\cup I}$ for some $I \subseteq V(R)$, let

(5.5.9)
$$E_{x,\mathbf{y}} := \big\{e \in E(\mathcal{H}_+[\mathcal{X}_\circ, \mathscr{X}_0, \mathscr{X}_{\cup I}]) \colon x \in e, e \cap \mathscr{X}_{\cup I} \subseteq \{x\} \cup \mathbf{y} \cup \mathcal{X}_\circ, e \cap (\mathbf{y} \setminus \mathscr{X}_0) \neq \emptyset,$$
$$\text{if } e \cap (\mathbf{y} \setminus \mathscr{X}_0) \in \mathscr{X}_i \text{ for some } i \in V(R) \setminus \{0\}, \text{ then } e \in E(\mathcal{H}_+^i)\big\}.$$

That is, $E_{x,\mathbf{y}}$ contains essentially all $\mathcal{H}_+$-edges that contain $x \in \mathscr{X}_0$ and a non-empty subset of $\mathbf{y}$ and whose remaining vertices are already embedded and lie in $\mathcal{X}_\circ$. In particular, if $\mathbf{y} = y$ is a single vertex $y \in \mathscr{X}_i$, then $E_{x,y}$ contains all $\mathcal{H}_+^i$-edges that contain x and y and whose remaining vertices lie in $\mathcal{X}_\circ$. Hence, note that by definition of $\mathcal{H}_+$ and as observed in (5.5.5), we have that

(5.5.10)
$$\Big|\bigcup_{x \in \mathscr{X}_0} E_{x,y}\Big| = b_i \text{ for all } y \in \mathscr{X}_i, i \in V(R) \setminus \{0\}.$$

Part B. Applying Theorem 2.3

Our strategy is to utilize Theorem 2.3 to find the required conflict-free packing σ in $\mathcal{A}_0$. To that end, we will define an auxiliary hypergraph $\mathcal{H}_{aux}$ for $\mathcal{A}_0$. Let $\psi \colon E(\mathcal{A}_0) \to 2^{\mathcal{E}}$ be the edge set labelling with respect to the packing instance as defined in (5.5.2). For all $H \in \mathcal{H}$, the hypergraph $H[X_{\cup r}^H]$ is a matching if $r \in E(R)$ and empty otherwise, and thus we have that $\|\psi\| \leq \binom{r_\circ}{k-1} \leq \Delta_R$. In the following, we may assume that $|\psi(e)| = \Delta_R$ for all $e \in E(\mathcal{A}_0)$ as we may simply add distinct artificial dummy labels that we ignore afterwards again.

Further, let $(V_0^H)_{H \in \mathcal{H}}$ be disjoint copies of V_0, and for all $H \in \mathcal{H}$ and $e = x_0 v_0 \in E(A_0^H)$, let $e^H := x_0 v_0^H$ where v_0^H is the copy of v_0 in V_0^H. Let $\hbar_e := e^H \cup \psi(e)$ for each $e \in E(A_0^H)$, $H \in \mathcal{H}$ and let $\mathcal{H}_{aux}$ be the $(\Delta_R + 2)$-graph with vertex set $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H) \cup \mathcal{E}$ and edge set $\{\hbar_e \colon e \in E(\mathcal{A}_0)\}$. A key property of the construction of $\mathcal{H}_{aux}$ is a bijection between conflict-free packings σ in $\mathcal{A}_0$ and matchings $\mathcal{M}$ in $\mathcal{H}_{aux}$ by assigning σ to $\mathcal{M} = \{\hbar_e : e \in M(\sigma)\}$. (Recall that $M(\sigma)$ is the edge set corresponding to σ as defined in (5.5.1).)

Step 1. Estimating $\Delta(\mathcal{H}_{aux})$ and $\Delta_2(\mathcal{H}_{aux})$

In order to apply Theorem 2.3 to $\mathcal{H}_{aux}$, we estimate $\Delta(\mathcal{H}_{aux})$ and $\Delta_2(\mathcal{H}_{aux})$. We first claim that

$$\Delta(\mathcal{H}_{aux}) \leq (1+\varepsilon^{2/3})d_0 n =: \Delta_{aux}. \tag{5.5.11}$$

Since A_0^H is (ε, d_0)-super-regular and $|X_0^H| = |V_0| = (1 \pm \varepsilon)n$ for each $H \in \mathcal{H}$, we have that an appropriate upper bound on $\Delta_\psi(\mathcal{A}_0)$ immediately establishes (5.5.11). In the following we derive such an upper bound on $\Delta_\psi(\mathcal{A}_0)$ by employing property (P3). For all $\mathscr{r} \in E(R)$ with $0 \in \mathscr{r}$, $|\mathscr{r} \cap -[r_\circ]| = k-1$, and all $\mathscr{g} \in E(G_A[V_{\cup \mathscr{r}}])$ with $\mathscr{g} \setminus \mathcal{V}_\circ = \{v_0\}$, note that $\bigcup_{\mathbf{p},\mathbf{p}^{2nd}} \mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})$ contains by Definition 5.16 all vertices $x_0 \in N_{\mathcal{A}_0}(v_0)$ (compare with (5.4.8)) that are contained in an $\mathcal{H}$-edge that could be mapped onto $\mathscr{g}$ with respect to $\phi_\circ^-$ and $\mathcal{A}$ (compare with (5.4.9)). Hence, by the definition of the edge set labelling ψ in (5.5.2), $\mathscr{g}$ appears as a label of ψ on at most $\sum_{\mathbf{p},\mathbf{p}^{2nd}} |\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})|$ edges of $\mathcal{A}_0$. Note that by Definition 5.13 of $E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})$, we obtain

$$\sum_{\mathbf{p},\mathbf{p}^{2nd}} |E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})| = e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) \leq d_A n^k, \tag{5.5.12}$$

where the last inequality holds by assumption of Lemma 5.18. For all $x_0 \in N_{A_0^H}(v_0)$ for some $H \in \mathcal{H}$, let $\mathscr{g}_{x_0}^{-1} := (\phi_\circ^-|_{V(H)})^{-1}(\mathscr{g} \setminus \{v_0\}) \cup \{x_0\}$. If $\mathscr{g}_{x_0}^{-1} \in \mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})$ for $\mathbf{p}, \mathbf{p}^{2nd} \in \mathbb{N}_0^r$, then we have by (5.4.9) that $\mathscr{g}_{x_0}^{-1} \in E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})$, and thus

$$\mathbf{p}^A\left(\mathscr{g}_{x_0}^{-1}, \emptyset\right) = \mathbf{p}, \quad \mathbf{p}^{A,2nd}\left(\mathscr{g}_{x_0}^{-1}, \emptyset\right) = \mathbf{p}^{2nd}, \quad \text{and} \quad \|\mathbf{p}\| - \|\mathbf{p}^{2nd}\| \overset{(5.4.5)}{=} -1.$$

By property (P4), the set $\mathcal{W}_{edge}$ contains the (simple) edge tester $(\omega, \omega_\iota, \mathscr{g}, \mathbf{p}, \mathbf{p}^{2nd})$ for $\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})$ (as defined in Definition 5.16) with $\omega(E(\mathcal{A}_0)) = |\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})|$ and $\omega_\iota(\mathscr{X}_{\sqcup \mathscr{r}}) = |E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})|$. Hence, by (P3) (with $I = \mathscr{r}$, $J = J_X = J_V = \emptyset$, $\mathbf{p}^A = \mathbf{p}$, $\mathbf{p}^{A,2nd} = \mathbf{p}^{2nd}$, $\mathbf{p}^B = \mathbf{p}^{B,2nd} = \mathbf{0}$), we obtain

$$\begin{aligned}
\Delta_\psi(\mathcal{A}_0) &\leq \sum_{\mathbf{p},\mathbf{p}^{2nd}} |\mathcal{X}_{\mathscr{g},\mathbf{p},\mathbf{p}^{2nd},\phi_\circ^-}(\mathcal{A})| \\
&\overset{(P3)}{\leq} \sum_{\mathbf{p},\mathbf{p}^{2nd}} \left((1+\varepsilon)d_A^{-1} d_0 \frac{|E_{\mathcal{H}}(\mathbf{p}, \mathbf{p}^{2nd}, \mathscr{r})|}{n^{k-1}} + n^\varepsilon\right) \overset{(5.5.12)}{\leq} (1+\varepsilon^{2/3})d_0 n.
\end{aligned}$$

This establishes (5.5.11).

Next, we claim that

$$\Delta_2(\mathcal{H}_{aux}) \leq n^\varepsilon \leq \Delta_{aux}^{1-\varepsilon^2}. \tag{5.5.13}$$

Note that the codegree in $\mathcal{H}_{aux}$ of two vertices in $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H)$ is at most 1, and similarly, the codegree in $\mathcal{H}_{aux}$ of a vertex in $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H)$ and a label in $\mathcal{E}$ is at most 1 because $\Delta_\psi(A_0^H) \leq 1$ for all $H \in \mathcal{H}$. Hence, an appropriate upper bound on $\Delta_\psi^c(\mathcal{A}_0)$ establishes (5.5.13). In the following we derive such an upper bound on $\Delta_\psi^c(\mathcal{A}_0)$ by employing (P5). For all $\mathscr{g} = \{v_{i_1}, \ldots, v_{i_k}\}, \hbar = \{w_{j_1}, \ldots, w_{j_k}\} \in E(G_A)$ with $v_{i_k} = w_{j_k} \in V_0$, $(\mathscr{g} \cup \hbar) \setminus \{v_{i_k}\} \subseteq \mathcal{V}_\circ$, and $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$, note that $\mathscr{g}$ and $\hbar$ appear together as labels of ψ on at most $|E_{\mathscr{g},\hbar,\phi_\circ^-}(\mathcal{A})|$ edges of $\mathcal{A}_0$. This follows immediately from Definition 5.17 of $E_{\mathscr{g},\hbar,\phi_\circ^-}(\mathcal{A})$. Note further that $|\{i_1,\dots,i_k,j_1,\dots,j_k\}\cap -[r_\circ]| \geq k$ because $(\mathscr{g}\cup\hbar)\setminus\{v_{i_k}\}\subseteq\mathcal{V}_\circ$, and $\{i_1,\dots,i_k\}\neq\{j_1,\dots,j_k\}$. Hence, by (P5), we have

$$\left|E_{\mathscr{g},\hbar,\phi_\circ^-}(\mathcal{A})\right| \leq \max\left\{n^{k-|\{i_1,\dots,i_k,j_1,\dots,j_k\}\cap-[r_\circ]|+\varepsilon}, n^\varepsilon\right\} = n^\varepsilon,$$

and thus, $\Delta_\psi^c(\mathcal{A}_0)\leq n^\varepsilon$, which establishes (5.5.13).

Step 2. Applying Theorem 2.3 to $\mathcal{H}_{aux}$

Suppose $\mathcal{W}=\bigcup_{\ell\in[\Delta]}\mathcal{W}_\ell$ is a set of size at most $n^{4\log n}$ of given weight functions $\omega\in\mathcal{W}_\ell$ for $\ell\in[\Delta]$ with $\omega\colon\binom{E(\mathcal{A}_0)}{\ell}\to[0,\Delta]$ and

$$\|\omega\|_{\ell'}\leq n^{\ell-\ell'+\varepsilon^2} \text{ for every } \ell'\in[\ell]. \tag{5.5.14}$$

Note that every weight function $\omega\colon\binom{E(\mathcal{A}_0)}{\ell}\to[0,\Delta]$ naturally corresponds to a weight function $\omega_{\mathcal{H}_{aux}}\colon\binom{E(\mathcal{H}_{aux})}{\ell}\to[0,\Delta]$ by defining $\omega_{\mathcal{H}_{aux}}(\{\hbar_{e_1},\dots,\hbar_{e_\ell}\}):=\omega(\{e_1,\dots,e_\ell\})$. We will explicitly specify $\mathcal{W}$ in Part C, where every weight function $\omega\colon\binom{E(\mathcal{A}_0)}{\ell}\to[0,\Delta]$ in $\mathcal{W}_\ell$ for $\ell\in[\Delta]$ will be defined such that $\operatorname{supp}(\omega)\subseteq\bigcup_{H\in\mathcal{H}}\binom{E(A_0^H)}{\ell}^{=}$ and in particular, such that the corresponding weight function $\omega_{\mathcal{H}_{aux}}$ will also be clean, that is $\operatorname{supp}(\omega_{\mathcal{H}_{aux}})\subseteq\binom{E(\mathcal{H}_{aux})}{\ell}^{=}$. Our main idea is to find a hypergraph matching in $\mathcal{H}_{aux}$ that behaves like a typical random matching with respect to $\{\omega_{\mathcal{H}_{aux}}\colon\omega\in\mathcal{W}\}$ in order to establish (I)$_{\text{L5.18}}$–(III)$_{\text{L5.18}}$.

Suppose $\ell\in[\Delta]$ and $\omega\in\mathcal{W}_\ell$. If $\omega(E(\mathcal{A}_0))\geq n^{\ell+\varepsilon/2}$, define $\tilde\omega:=\omega$. Otherwise, arbitrarily choose $\tilde\omega\colon\binom{E(\mathcal{A}_0)}{\ell}\to[0,\Delta]$ such that $\omega\leq\tilde\omega$, $\tilde\omega$ satisfies that $\operatorname{supp}(\tilde\omega)\subseteq\bigcup_{H\in\mathcal{H}}\binom{E(A_0^H)}{\ell}^{=}$ and in particular $\operatorname{supp}(\tilde\omega_{\mathcal{H}_{aux}})\subseteq\binom{E(\mathcal{H}_{aux})}{\ell}^{=}$, $\tilde\omega(E(\mathcal{A}_0))=n^{\ell+\varepsilon/2}$, and $\|\tilde\omega\|_{\ell'}\leq n^{\ell-\ell'+2\varepsilon^2}$ for all $\ell'\in[\ell]$. By (5.5.11) and (5.5.13), we can apply Theorem 2.3 (with $\Delta_{aux},\varepsilon^2,\Delta_R+2,\Delta,\{\tilde\omega_{\mathcal{H}_{aux}}\colon\omega\in\mathcal{W}_\ell\}$ playing the roles of the parameters $\Delta,\delta,r,L,\mathcal{W}_\ell$ of Theorem 2.3, respectively) to obtain a matching $\mathcal{M}$ in $\mathcal{H}_{aux}$ that corresponds to a conflict-free packing $\sigma\colon\mathscr{X}_0^\sigma\to V_0$ in $\mathcal{A}_0$ with its corresponding edge set $M(\sigma)$ that satisfies the following properties (where $\hat\varepsilon=\varepsilon^{1/2}$):

$$\omega(M(\sigma))=(1\pm\varepsilon)(1-\ell\varepsilon^{2/3})\frac{\omega(E(\mathcal{A}_0))}{(d_0n)^\ell}\pm n^\varepsilon \tag{5.5.15}$$

$$=(1\pm\hat\varepsilon)\frac{\omega(E(\mathcal{A}_0))}{(d_0n)^\ell}\pm n^\varepsilon \text{ for all } \omega\in\mathcal{W}_\ell,\ell\in[\Delta]. \tag{5.5.16}$$

Part C. Employing weight functions to conclude (I)$_{\text{L5.18}}$–(III)$_{\text{L5.18}}$

Let $\sigma\colon\mathscr{X}_0^\sigma\to V_0$ be the conflict-free packing in $\mathcal{A}_0$ as obtained in Part B and let σ^+ be a cluster-injective extension of σ chosen uniformly and independently at random. We will show that the random σ^+ satisfies with high probability the conclusions of the lemma and thus, there exists a suitable cluster-injective extension σ^+ by picking one such extension deterministically. We may assume that (5.5.16) holds for a set of weight functions $\mathcal{W}$. Each of these weight functions will only depend on our input parameters. Hence, we could define them right away but for the sake of a cleaner presentation we postpone their definitions to the specific situations when we employ those weight functions to establish (I)$_{\text{L5.18}}$–(III)$_{\text{L5.18}}$. We now define the candidacy graphs $Z_{I_Z}^{H,new}\subseteq Z_{I_Z}^H(\phi_\circ\cup\sigma^+)$ for $Z\in\{A,B\}$ and all index sets $I=I_A\subseteq[r]$, and $i=I_B\in V(R)$. If $I_Z\cap\mathscr{r}=\emptyset$ for all $\mathscr{r}\in E(R)$ with $0\in\mathscr{r}$, then we set $Z_{I_Z}^{H,new}:=Z_{I_Z}^H$. Otherwise, let

$$A_I^{H,new}:=A_I^{H+}(\phi_\circ\cup\sigma^+),\qquad\text{and}\qquad B_i^{H,new}:=B_i^{H_+^i}(\phi_\circ\cup\sigma^+), \tag{5.5.17}$$

with $A_I^{H_+}(\phi_\circ \cup \sigma^+)$ defined as in Definition 5.11 with respect to H_+, $\phi_\circ \cup \sigma^+$ and G_A, as well as $B_i^{H_+^i}(\phi_\circ \cup \sigma^+)$ defined with respect to H_+^i, $\phi_\circ \cup \sigma^+$ and G_B.

Before we establish (I)$_{\text{L5.18}}$–(III)$_{\text{L5.18}}$, we first estimate $|\mathscr{X}_0^\sigma \cap X_0^H|$ for each $H \in \mathcal{H}$. We define a weight function $\omega_H\colon E(A_0^H) \to \{0,1\}$ for each $H \in \mathcal{H}$ by $\omega_H(e) := \mathbb{1}\{e \in E(A_0^H)\}$, and add ω_H to $\mathcal{W}$. Note that $\omega_H(E(A_0^H)) = (1 \pm 3\varepsilon)d_0 n^2$ because A_0^H is (ε, d_0)-super-regular by (P2). By (5.5.15), we obtain

$$|\mathscr{X}_0^\sigma \cap X_0^H| = \omega_H(M(\sigma)) = (1 \pm 5\varepsilon)(1 - \varepsilon^{2/3})\frac{d_0 n^2}{d_0 n} \pm n^\varepsilon,$$

and thus,

$$(1 - \varepsilon^{2/3}/2)n \geq |\mathscr{X}_0^\sigma \cap X_0^H| \geq (1 - \hat{\varepsilon})n. \tag{5.5.18}$$

We first prove (II)$_{\text{L5.18}}$, as we can use this for establishing (I)$_{\text{L5.18}}$.

<u>*Step 3.*</u> *Preparation for checking* (II)$_{\text{L5.18}}$

We will even show that (II)$_{\text{L5.18}}$ holds for edge testers in $\mathcal{W}_{edge} \cup \mathcal{W}'_{edge}$, where $\mathcal{W}'_{edge}$ is a set of suitable edge testers satisfying (P3) that we will explicitly specify in Step 11 when establishing (I)$_{\text{L5.18}}$. Throughout Steps 3–10 let $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}_{edge} \cup \mathcal{W}'_{edge}$ be fixed. That is, we fix an index set $I \subseteq V(R)$, $J \subseteq I$, disjoint sets $J_X, J_V \subseteq J$, and let $I_{r_0} := (I \cap [r]_0) \setminus J$, $I_r := (I \cap [r]) \setminus J$, $J_{XV} := J_X \cup J_V$, and we fix $\omega\colon E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j) \to [0, s]$, $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to [0, s]$, $\mathscr{c} = \{c_i\}_{i \in I} \in V_{\sqcup I}$, and $\mathscr{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd}) \in (\mathbb{N}_0^{r_\circ + r + 1})^4$ with $\mathbf{p}^Z = (p_i^Z)_{i \in V(R)}, \mathbf{p}^{Z,2nd} = (p_i^{Z,2nd})_{i \in V(R)} \in \mathbb{N}_0^{r_\circ + r + 1}$ for $Z \in \{A, B\}$, and we may assume that $I \cap \mathscr{r} \neq \emptyset$ for some $\mathscr{r} \in E(R)$ with $0 \in \mathscr{r}$ because otherwise we do not update the weight of the edge tester. Recall that the statement (II)$_{\text{L5.18}}$ concerns the weight of the edge tester $(\omega^{new}, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ defined as in Definition 5.15 with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, $(\phi_\circ^- \cup \sigma, \phi_\circ \cup \sigma^+)$, $\mathcal{A}^{new}$ and $\mathcal{B}^{new}$.

We will consider three different cases depending on whether $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, $0 \in I \setminus J$, and $0 \in J_{XV}$. Even though we proceed similarly in each of these cases, the effects on (II)$_{\text{L5.18}}$ are quite different in each scenario as we try to illude in the following. Recall that (II)$_{\text{L5.18}}$ ensures that (P3) is also satisfied for the updated candidacy graphs. If $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, we have to update the density factors whereas the magnitude of $\omega_\iota(\mathscr{X}_{\sqcup I})/n^{|(I \cap -[r_\circ]) \setminus J|}$ in (P3) equals the magnitude of $\omega_\iota(\mathscr{X}_{\sqcup I})/n^{|(I \cap -[r_\circ]_0) \setminus J|}$ in (II)$_{\text{L5.18}}$. In contrast, if $0 \in I \setminus J$, we additionally have to ensure that the magnitude of $\omega_\iota(\mathscr{X}_{\sqcup I})/n^{|(I \cap -[r_\circ]) \setminus J|}$ in (P3) will be updated by a factor of n^{-1} to obtain the the magnitude of $\omega_\iota(\mathscr{X}_{\sqcup I})/n^{|(I \cap -[r_\circ]_0) \setminus J|}$ in (II)$_{\text{L5.18}}$. If $0 \in J_{XV}$ the magnitudes are again equal, but besides updating the densities we additionally have to consider that $0 \in J_{XV}$ and thus (P3) will potentially be updated by the factor $(\mathbb{1}\{J_{XV} \cap -[r_\circ]_0 = \emptyset\} \pm \varepsilon'^2) = 0 \pm \varepsilon'^2$.

We collect some common notation that will be used to establish (II)$_{\text{L5.18}}$. Recall that the centres $\mathscr{c}$ are fixed and for all $H \in \mathcal{H}$ and $\mathscr{a}\mathscr{b} = \{\mathscr{a}, \mathscr{b}\} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j)$ with $\mathscr{b} = \{b_j\}_{j \in J}$, we have $\omega(\mathscr{a}\mathscr{b}) > 0$ only if $\{c_i\}_{i \in I_{r_0} \cup J} = (\mathscr{a} \cup \mathscr{b}) \cap V_{\cup(I_{r_0} \cup J)}$ by Definition 5.15 of an edge tester in (5.4.7). (Recall that we allow to treat $\mathscr{b} = \{b_j\}_{j \in J} \in \bigsqcup_{j \in J} E(\mathcal{B}_j)$ as $b_{\cup J}$.) To that end, for all $H \in \mathcal{H}$ and $\mathscr{a}\mathscr{b} \in E(A_{I_{r_0}}^H) \sqcup \bigsqcup_{j \in J} E(B_j^H)$ with $\{c_i\}_{i \in I_{r_0} \cup J} = (\mathscr{a} \cup \mathscr{b}) \cap V_{\cup(I_{r_0} \cup J)}$, let $\mathbf{y}_{\mathscr{a}\mathscr{b}} = \{y_i\}_{i \in I_{r_0} \cup J} := (\mathscr{a} \cup \mathscr{b}) \cap X_{\cup(I_{r_0} \cup J)}^H$. Our overall strategy in all three cases is to define for vertices x in some set $\mathscr{X}_0^{\mathscr{a}\mathscr{b}} \subseteq X_0^H$ a target set $T_{x,\mathscr{a}\mathscr{b}}$ of suitable images for x such that if all $x \in \mathscr{X}_0^{\mathscr{a}\mathscr{b}}$ are embedded into $T_{x,\mathscr{a}\mathscr{b}}$, then $\mathscr{a}\mathscr{b}$ (or $\mathscr{a}\mathscr{b} \setminus \{c_0, y_0\}\}$ if $0 \in I \setminus J$) is an element in $E(A_{I_r}^{H,new}) \sqcup \bigsqcup_{j \in J} E(B_j^{H,new})$. Hence for all $H \in \mathcal{H}$, $x \in X_0^H$ and $\mathscr{a}\mathscr{b} \in E(A_{I_{r_0}}^H) \sqcup \bigsqcup_{j \in J} E(B_j^H)$ with

$\omega(\mathscr{ab}) > 0$, we define the following sets. We give more motivation for these definitions in the subsequent paragraph. For an edge $\mathscr{e} \in E(H)$, let $\mathscr{r}_{\mathscr{e}} \in E(R)$ be such that $\mathscr{e} \in X^H_{\cup \mathscr{r}_{\mathscr{e}}}$.

$$
\begin{aligned}
\mathcal{S}_{x,\mathscr{ab},\bar{J}} &:= \left\{\phi_\circ(\mathscr{e}) \cup \{c_i\}_{i\in \mathscr{r}_{\mathscr{e}} \cap I_r} \colon \mathscr{e} \in E_{x,\{y_i\}_{i\in I_r}}\right\};\\
\mathcal{S}_{x,\mathscr{ab},J} &:= \left\{S = \phi_\circ(\mathscr{e} \setminus \{y_j\}) \cup \{c_i\}_{i\in \mathscr{r}_{\mathscr{e}} \cap (I_r \cup \{j\})} \colon |S| = k-1, \mathscr{e} \in E_{x,\mathbf{y}_{\mathscr{ab}}}, j \in J \setminus \{0\}, y_j \in \mathscr{e}\right\};\\
V_{x,\mathscr{ab}} &:= V_0 \cap N_{G_A}(\mathcal{S}_{x,\mathscr{ab},\bar{J}}) \cap N_{G_B}(\mathcal{S}_{x,\mathscr{ab},J});\\
T_{x,\mathscr{ab}} &:= V_{x,\mathscr{ab}} \cap N_{A_0^H}(x).
\end{aligned}
$$

That is, $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ and $\mathcal{S}_{x,\mathscr{ab},J}$ are sets of $(k-1)$-sets. In general, these $(k-1)$-sets consist of the image $\phi_\circ(\mathscr{e}) = \phi_\circ(\mathscr{e} \cap \mathcal{X}_\circ)$ of an H_+-edge $\mathscr{e} \in E_{x,\mathbf{y}_{\mathscr{ab}}}$ together with the centres corresponding to the clusters that $\mathscr{e}$ intersects. The set $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ contains all $(k-1)$-sets that only intersect with clusters of I_r, whereas $\mathcal{S}_{x,\mathscr{ab},J}$ contains $(k-1)$-sets that intersect with a cluster of $J \setminus \{0\}$. Consequently, $T_{x,\mathscr{ab}}$ is the intersection of the A_0^H-neighbourhood of x in V_0 with the common neighbourhood $V_{x,\mathscr{ab}}$ in G_A and G_B of all these $(k-1)$-sets in $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ and $\mathcal{S}_{x,\mathscr{ab},J}$ (see also Figures 5.5 and 5.6). Note that $\mathcal{S}_{x,\mathscr{ab},\bar{J}} \cup \mathcal{S}_{x,\mathscr{ab},J} = \emptyset$ if $E_{x,\mathbf{y}_{\mathscr{ab}}} = \emptyset$. Further, since $\omega(\mathscr{ab}) > 0$ and it is required for the edge tester ω that $\phi_\circ$ does not map any vertices $\{y_j\}_{j\in J\cap -[r_\circ]}$ onto its centres by $(\mathrm{v})_{\mathrm{D5.15}}$, we have that $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ and $\mathcal{S}_{x,\mathscr{ab},J}$ are disjoint sets. We estimate the sizes of $T_{x,\mathscr{ab}}$ and $V_{x,\mathscr{ab}}$ in Step 3.2. Further, for $\mathscr{ab} = \{\mathscr{a}, \mathscr{b}\} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)$, let $x_{\mathscr{a}} := \mathscr{a} \cap X_0^H$, $x_{\mathscr{b}} := \mathscr{b} \cap X_0^H$ (note that $x_{\mathscr{a}}$ or $x_{\mathscr{b}}$ might be empty), and

$$
\mathscr{X}_0^{\mathscr{ab}} := \left\{x \in X_0^H \setminus \{x_{\mathscr{a}}\} \colon |\mathcal{S}_{x,\mathscr{ab},\bar{J}} \cup \mathcal{S}_{x,\mathscr{ab},J}| \geq 1\right\}. \tag{5.5.19}
$$

<u>*Step 3.1.*</u> *Weight functions to establish* $(\mathrm{II})_{\mathrm{L5.18}}$

We emphasize again that the general strategy for establishing $(\mathrm{II})_{\mathrm{L5.18}}$ is to define tuple weight functions for the edges between the vertices $x \in \mathscr{X}_0^{\mathscr{ab}}$ and their corresponding target sets $T_{x,\mathscr{ab}}$, which we will do in this step depending on the three cases whether $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, $0 \in I \setminus J$, and $0 \in J_{XV}$.

For all $H \in \mathcal{H}$, $\mathscr{ab} = \{\mathscr{a}, \mathscr{b}\} \in E(A^H_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(B^H_j)$ with $\omega(\mathscr{ab}) > 0$, and $\mathscr{X}_0^{\mathscr{ab}}$ as defined in (5.5.19), we make the following definition. (For notational convenience, we treat $\{\emptyset\}$ as $\emptyset$ in the following definition.)

$$
\begin{aligned}
E_{\mathscr{ab}} := \Bigg\{&\{\mathscr{a} \cap (X_0^H \cup V_0)\} \cup \{e_x\}_{x\in \mathscr{X}_0^{\mathscr{ab}}} \in \binom{E(A_0^H)}{\mathbb{1}\{0 \in I \setminus J\} + |\mathscr{X}_0^{\mathscr{ab}}|}^{=} :\\
&e_x \in E\left(A_0^H[\{x\}, T_{x,\mathscr{ab}}]\right) \text{ for all } x \in \mathscr{X}_0^{\mathscr{ab}}\Bigg\}.
\end{aligned} \tag{5.5.20}
$$

Let us explain the definition of $E_{\mathscr{ab}}$. If $0 \in (V(R)\setminus I)\cup(J\setminus J_{XV})$, then $\mathscr{a}\cap(X_0^H \cup V_0) = \emptyset$. Thus, $E_{\mathscr{ab}}$ is the set of clean $|\mathscr{X}_0^{\mathscr{ab}}|$-tuples of edges in $E(A_0^H)$ between a vertex $x \in \mathscr{X}_0^{\mathscr{ab}}$ and its target set $T_{x,\mathscr{ab}}$. If $0 \in I \setminus J$, then $\mathscr{a} \cap (X_0^H \cup V_0) = \{x_{\mathscr{a}}, c_0\}$. (Recall that $x_{\mathscr{a}} = \mathscr{a} \cap X_0^H$.) Thus, $E_{\mathscr{ab}}$ is the set of clean $(1+|\mathscr{X}_0^{\mathscr{ab}}|)$-tuples of edges in $E(A_0^H)$ where we additionally require that the tuple contains the edge $x_{\mathscr{a}}c_0$. If $0 \in J_{XV}$, we will not make use of the definition of $E_{\mathscr{ab}}$.

With $E_{\mathscr{ab}}$ we can define the following weight function $\omega_{\mathscr{ab}} \colon \binom{E(\mathcal{A}_0)}{\mathbb{1}\{0\in I\setminus J\}+|\mathscr{X}_0^{\mathscr{ab}}|} \to [0, s]$ by

$$
\omega_{\mathscr{ab}}(\mathbf{e}) := \omega(\mathscr{ab}) \cdot \mathbb{1}\{\mathbf{e} \in E_{\mathscr{ab}}\}.
$$

The motivation behind this is the following observation for the two cases $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, and $0 \in I \setminus J$. We claim that for $\mathscr{ab} = \{\mathscr{a}, \mathscr{b}\} \in E(A^H_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(B^H_j)$

with $\omega(\mathscr{ab}) > 0$, if $\omega_{\mathscr{ab}}(M(\sigma)) > 0$ and $\sigma^+(x_{\mathscr{b}}) \neq c_0$ if $0 \in J$, then $\mathscr{ab}^{new} := \{\mathscr{a} \setminus (\mathscr{a} \cap (X_0^H \cup V_0)), \mathscr{b}\} \in E(A_{I_r}^{H,new}) \sqcup \bigsqcup_{j\in J} E(B_j^{H,new})$. To see that this is true, note that if $\omega_{\mathscr{ab}}(M(\sigma)) > 0$, then the definition of the target sets $T_{x,\mathscr{ab}}$ implies that (5.4.1) of Definition 5.11 for the updated candidacy graphs $A_{I_r}^{H,new} = A_{I_r}^{H_+}(\phi_\circ \cup \sigma^+)$ and $B_j^{H,new} = B_j^{H_+^j}(\phi_\circ \cup \sigma^+)$ is satisfied. Hence in this case, for the edge tester ω^{new} as defined in the statement of (II)$_{\text{L5.18}}$, we obtain by (5.4.7) of Definition 5.15 of ω^{new} that $\omega^{new}(\mathscr{ab}^{new}) = \omega(\mathscr{ab})$ requiring that $\sigma^+(x_{\mathscr{b}}) \neq c_0$ if $0 \in J$ so that (v)$_{\text{D5.15}}$ of Definition 5.15 is satisfied. Note that $\mathscr{ab}^{new} = \mathscr{ab}$ if $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$.

In order to ensure that $\sigma^+(x_{\mathscr{b}}) \neq c_0$ if $0 \in J$, we apply an inclusion-exclusion principle and introduce another weight function $\omega^-_{\mathscr{ab}}$ that accounts for the weight in the case that $0 \in J$ and $\sigma^+(x_{\mathscr{b}}) = c_0$. To that end, similarly as in (5.5.20), for all $H \in \mathcal{H}$, $\mathscr{ab} = \{\mathscr{a}, \mathscr{b}\} \in E(A_{I_{r_0}}^H) \sqcup \bigsqcup_{j\in J} E(B_j^H)$ with $\omega(\mathscr{ab}) > 0$, and $\mathscr{X}_0^{\mathscr{ab}}$ as defined in (5.5.19), we define the following set of edge tuples

$$\begin{aligned} E^-_{\mathscr{ab}} := \Bigg\{ &\{\mathscr{b} \cap (X_0^H \cup V_0)\} \cup \{e_x\}_{x \in \mathscr{X}_0^{\mathscr{ab}} \setminus \{x_{\mathscr{b}}\}} \in \binom{E(A_0^H)}{\mathbb{1}\{0 \in J\} + |\mathscr{X}_0^{\mathscr{ab}} \setminus \{x_{\mathscr{b}}\}|}^{=} : \\ &e_x \in E\left(A_0^H[\{x\}, T_{x,\mathscr{ab}}]\right) \text{ for all } x \in \mathscr{X}_0^{\mathscr{ab}} \setminus \{x_{\mathscr{b}}\} \Bigg\}. \end{aligned}$$

Analogously to $\omega_{\mathscr{ab}}$, we define the weight function $\omega^-_{\mathscr{ab}} \colon \binom{E(\mathcal{A}_0)}{\mathbb{1}\{0\in J\} + |\mathscr{X}_0^{\mathscr{ab}} \setminus \{x_{\mathscr{b}}\}|} \to [0, s]$ by $\omega^-_{\mathscr{ab}}(\mathbf{e}) := \omega(\mathscr{ab}) \cdot \mathbb{1}\{\mathbf{e} \in E^-_{\mathscr{ab}}\}$.

The size of the tuple weight functions depends on the cardinality of $\mathscr{X}_0^{\mathscr{ab}}$. To that end, let

$$b_{\max} := \max\{|\mathscr{X}_0^{\mathscr{ab}}| \colon \mathscr{ab} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j), \omega(\mathscr{ab}) > 0\}, \tag{5.5.21}$$

and we will group the tuple functions for all $\mathscr{ab}$ with $\omega(\mathscr{ab}) > 0$ into all possible $(\mathbb{1}\{0 \in I \setminus J\} + b)$-tuple weight functions for $b \in [b_{\max}]_0$. To this end, for each $b \in [b_{\max}]_0$, we set

$$\begin{aligned} \Omega_b &:= \left\{ \mathscr{ab} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j) \colon \omega(\mathscr{ab}) > 0, |\mathscr{X}_0^{\mathscr{ab}}| = b \right\}, \\ \Omega_b^- &:= \left\{ \mathscr{ab} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j) \colon \omega(\mathscr{ab}) > 0, |\mathscr{X}_0^{\mathscr{ab}} \setminus \{x_{\mathscr{b}}\}| = b \right\}, \end{aligned} \tag{5.5.22}$$

as well as

$$\omega_b := \sum_{\mathscr{ab} \in \Omega_b} \omega_{\mathscr{ab}}, \quad \text{and} \quad \omega_b^- := \sum_{\mathscr{ab} \in \Omega_b^-} \omega^-_{\mathscr{ab}}.$$

In the two cases when $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, and $0 \in I \setminus J$, we will see that $\sum_{b \in [b_{\max}]_0} \omega_b(M(\sigma))$ is the major contribution to $\omega^{new}\big(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j^{new})\big)$. However, since $\mathscr{X}_0^\sigma$ is a proper subset of $\mathscr{X}_0$, we additionally need to consider those $\mathscr{ab} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)$ for which a relevant vertex in $\mathscr{X}_0^{\mathscr{ab}}$ has not been embedded by σ as this might also contribute to the weight of ω^{new}. This is also the case when $0 \in J_{XV}$, because then we require that either $x_{\mathscr{b}} = \mathscr{b} \cap X_0^H$ is not embedded or no H-vertex is mapped onto the centre c_0.

We collect some notation. For all $\mathscr{ab} \in E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)$ with $\omega(\mathscr{ab}) > 0$, let $H_{\mathscr{ab}} \in \mathcal{H}$ be such that $\mathscr{ab} \in E(A_{I_{r_0}}^{H_{\mathscr{ab}}}) \sqcup \bigsqcup_{j\in J} E(B_j^{H_{\mathscr{ab}}})$, and let $\mathscr{X}_0^{\mathscr{ab},\bar{\sigma}} := \mathscr{X}_0^{\mathscr{ab}} \setminus \mathscr{X}_0^\sigma$. Recall that $J_X, J_V \subseteq J$ are disjoint sets and $J_{XV} = J_X \cup J_V$. Let $z_{\mathscr{b}} := x_{\mathscr{b}}$ if $0 \in J_X$, and let $z_{\mathscr{b}} := c_0$ if $0 \in J_V$. We can now define the following set of edge tuples that we describe in detail below. For $b \in [b_{\max}]_0$, $\ell \in [b]_0$, $m_A, m_B \in [\Delta]_0$, let

(5.5.23)
$$\Gamma_b(\ell, m_A, m_B) := \Big\{ \mathscr{ab} \in \Omega_b \colon |\mathscr{X}_0^{\mathscr{ab},\bar\sigma}| = \ell, \sum_{x \in \mathscr{X}_0^{\mathscr{ab},\bar\sigma}} |\mathcal{S}_{x,\mathscr{ab},\bar J}| = m_A, \sum_{x \in \mathscr{X}_0^{\mathscr{ab},\bar\sigma}} |\mathcal{S}_{x,\mathscr{ab},J}| = m_B,$$
$$\sigma(x) \in T_{x,\mathscr{ab}} \text{ for all } x \in \mathscr{X}_0^{\mathscr{ab}} \cap \mathscr{X}_0^{\sigma}, \text{ if } 0 \in I \setminus J \text{ then } \sigma(x_{\mathscr{a}}) = \{c_0\},$$
$$\text{if } 0 \in J_{XV} \text{ then } z_{\mathscr{b}} \in (\mathscr{X}_0 \setminus \mathscr{X}_0^{\sigma}) \cup (V_0 \setminus \sigma(X_0^{H_{\mathscr{ab}}} \cap \mathscr{X}_0^{\sigma})) \Big\}.$$

That is, $\Gamma_b(\ell, m_A, m_B)$ is the set of edges $\mathscr{ab} \in \Omega_b$ such that there exists an ℓ-set $\mathscr{X}_0^{\mathscr{ab},\bar\sigma}$ of vertices in $\mathscr{X}_0^{\mathscr{ab}}$ which are not embedded by σ, and these ℓ vertices in $\mathscr{X}_0^{\mathscr{ab},\bar\sigma}$ contribute m_A and m_B many $(k-1)$-sets, and all remaining $b-\ell$ vertices $x \in \mathscr{X}_0^{\mathscr{ab}} \cap \mathscr{X}_0^{\sigma}$ are embedded onto their target set $T_{x,\mathscr{ab}}$. Additionally, if $0 \in I \setminus J$, then we require that $x_{\mathscr{a}}$ is mapped onto c_0 by σ, and if $0 \in J_X$, then we require that $x_{\mathscr{b}}$ is not embedded by σ, and if $0 \in J_V$, then we require that no vertex of $X_0^{H_{\mathscr{ab}}}$ is mapped onto c_0 by σ. Further, let

(5.5.24)
$$\Gamma_b^{hit}(\ell, m_A, m_B) := \Big\{ \mathscr{ab} \in \Gamma_b(\ell, m_A, m_B) \colon \sigma^+(x) \in V_{x,\mathscr{ab}} \text{ for all } x \in \mathscr{X}_0^{\mathscr{ab},\bar\sigma},$$
$$\text{if } x_{\mathscr{b}} \in \mathscr{X}_0^{\mathscr{ab},\bar\sigma} \text{ then } \sigma^+(x_{\mathscr{b}}) \neq c_0 \Big\}.$$

That is, $\Gamma_b^{hit}(\ell, m_A, m_B) \subseteq \Gamma_b(\ell, m_A, m_B)$ contains those edges $\mathscr{ab} \in \Omega_b$, where the not embedded vertices in $\mathscr{X}_0^{\mathscr{ab}}$ are nevertheless mapped onto their target set $V_{x,\mathscr{ab}}$ by the random cluster-injective extension σ^+ of σ, and σ^+ does not map $x_{\mathscr{b}}$ onto c_0. Thus, in such a case the weight of $\mathscr{ab}$ will 'accidentally' be taken into account in addition to the 'real' contribution given by σ (compare also with (5.5.25) below).

Crucially note that in the two cases when $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$ and $0 \in I \setminus J$, we know for $\mathscr{ab} = \{\mathscr{a}, \mathscr{b}\} \in \Omega_b$ that $\mathscr{ab}^{new} = \{\mathscr{a} \setminus (\mathscr{a} \cap (X_0^H \cup V_0)), \mathscr{b}\} \in E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j^{new})$ only if $\sigma^+(x_{\mathscr{b}}) \neq c_0$ if $0 \in J$ and either $\omega_{\mathscr{ab}}(M(\sigma)) = \omega(\mathscr{ab}) > 0$ or $\mathscr{ab} \in \Gamma_b^{hit}(\ell, m_A, m_B)$ for some $\ell \in [b]$, $m_A, m_B \in [\Delta]_0$. (Recall that $\mathscr{ab}^{new} = \mathscr{ab}$ if $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$.) This holds by (5.4.1) of Definition 5.11 and since we defined $\mathcal{A}_{I_r}^{new}$ and $\mathcal{B}_j^{new}$ in (5.5.17) as updated candidacy graphs with respect to $\phi_\circ \cup \sigma^+$. (Note that since we choose σ^+ as a 'dummy' enlargement, we do not require that $\sigma^+(x) \in N_{\mathcal{A}_0}(x)$, which is the reason why $\sigma^+(x) \in V_{x,\mathscr{ab}}$ in (5.5.24) instead of $\sigma^+(x) \in T_{x,\mathscr{ab}}$.) Hence, for the two cases when $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$ and $0 \in I \setminus J$, we make the following key observation:

$$\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \textstyle\bigsqcup_{j \in J} E(\mathcal{B}_j^{new})\right)$$
(5.5.25)
$$= \sum_{b \in [b_{\max}]_0} \left(\omega_b(M(\sigma)) - \mathbb{1}\{0 \in J \setminus J_{XV}\}\omega_b^-(M(\sigma))\right) + \sum_{\substack{b \in [b_{\max}], \ell \in [b], \\ m_A, m_B \in [\Delta]_0}} \omega\left(\Gamma_b^{hit}(\ell, m_A, m_B)\right).$$

In the case that $0 \in J_{XV}$, it suffices in view of the statement to establish an upper bound for $\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j^{new})\right)$. Similarly as in (5.5.25), we have in this case for $\mathscr{ab} \in \Omega_b$ and $b \in [b_{\max}]_0$ that $\mathscr{ab} \in E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j^{new})$ only if $\mathscr{ab} \in \Gamma_b^{hit}(\ell, m_A, m_B)$ for some $\ell \in [b]_0$, $m_A, m_B \in [\Delta]_0$. This holds by (5.4.1) of Definition 5.11 and since we defined $\mathcal{A}_{I_r}^{new}$ and $\mathcal{B}_j^{new}$ in (5.5.17) as updated candidacy graphs with respect to $\phi_\circ \cup \sigma^+$. Hence, for the case that $0 \in J_{XV}$, we make the following key observation:

$$\omega^{new}\left(E(\mathcal{A}^{new}_{I_r}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}^{new}_j)\right) \le \sum_{\substack{b\in[b_{\max}]_0,\ell\in[b]_0,\\ m_A,m_B\in[\Delta]_0}} \omega\left(\Gamma^{hit}_b(\ell, m_A, m_B)\right). \tag{5.5.26}$$

In Steps 4–7, we will estimate the weight of the contributing terms in (5.5.25) and (5.5.26). To do so, we first determine the sizes of the target sets $T_{x,\mathscr{ab}}$ and $V_{x,\mathscr{ab}}$ in the next step.

Step 3.2. *Size of the target sets $T_{x,\mathscr{ab}}$ and $V_{x,\mathscr{ab}}$*

Let $b \in [b_{\max}]$ and $\mathscr{ab} \in \Omega_b$ be fixed. We observe that

(5.5.27)
$$|V_{x,\mathscr{ab}}| = (1\pm 3\varepsilon) d_A^{|\mathcal{S}_{x,\mathscr{ab},\bar{J}}|} d_B^{|\mathcal{S}_{x,\mathscr{ab},J}|} n, \quad \text{and} \quad |T_{x,\mathscr{ab}}| = (1\pm 3\varepsilon) d_A^{|\mathcal{S}_{x,\mathscr{ab},\bar{J}}|} d_B^{|\mathcal{S}_{x,\mathscr{ab},J}|} d_0 n,$$

for $x \in \mathscr{X}_0^{\mathscr{ab}}$, where we used (P1) and that A_0^H is (ε, d_0)-super-regular and (ε, q)-well-intersecting for each $H \in \mathcal{H}$. For an illustration of the sets $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ and $T_{x,\mathscr{ab}}$ in the case that $0 \notin I$ and $J = \emptyset$, see Figure 5.5.

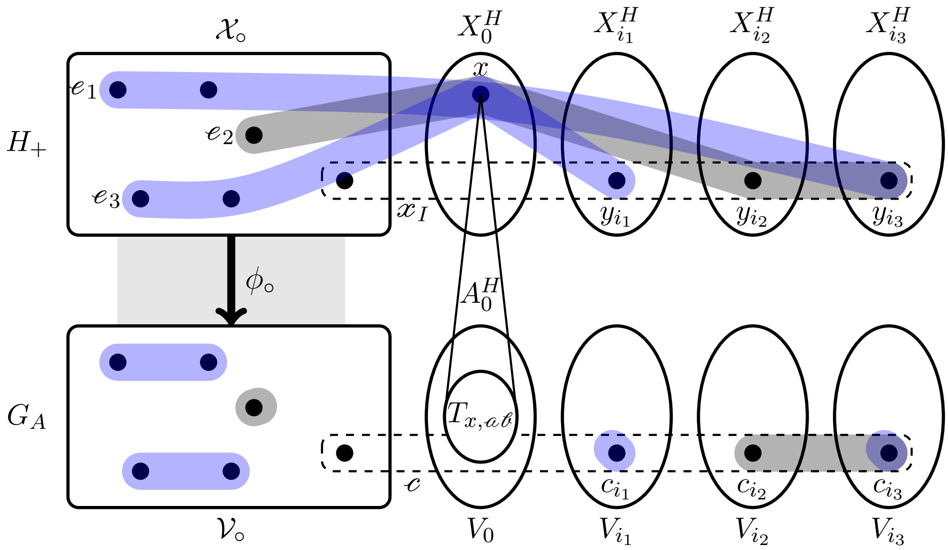


Figure 5.5: This illustrates the case that $0 \notin I$ and for simplicity $J = \emptyset$. That is, we consider the edge $\mathscr{ab} = \{c_{i_1}, c_{i_2}, c_{i_3}, y_{i_1}, y_{i_2}, y_{i_3}\} \in E(A^H_{I_{r_0}})$ with $\mathbf{y}_{\mathscr{ab}} = \{y_{i_1}, y_{i_2}, y_{i_3}\} \subseteq x_I$. Note that the edges e_1, e_2, e_3 in H_+ belong to $E_{x,\mathbf{y}_{\mathscr{ab}}}$. For the set $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$, we have $\mathcal{S}_{x,\mathscr{ab},\bar{J}} = \{\phi_\circ(e_1) \cup \{c_{i_1}\}, \phi_\circ(e_2) \cup \{c_{i_2}, c_{i_3}\}, \phi_\circ(e_3) \cup \{c_{i_3}\}\}$. Accordingly, $T_{x,\mathscr{ab}}$ is the intersection in V_0 of the G_A-neighbourhoods of these $(k-1)$-sets in $\mathcal{S}_{x,\mathscr{ab},\bar{J}}$ and the neighbourhood of x in A_0^H. Note that the blue edges e_1 and e_3 in H_+ satisfy that $|e_1 \cap \mathbf{y}_{\mathscr{ab}}| = |e_3 \cap \mathbf{y}_{\mathscr{ab}}| = 1$, and thus they do not account for the 1^{st}-pattern $\mathbf{p}^A(x_I, \emptyset)$ of x_I by (5.4.3) of Definition 5.12. By considering all possible $x \in X_0^H$, there are in total $b_{I_r} = \sum_{i\in I_r} b_i$ many such blue edges in H_+. Further, note that the grey edge e_2 satisfies that $|e_2 \cap \mathbf{y}_{\mathscr{ab}}| = 2$ and thus, e_2 belongs to H by (5.5.8) and accounts for the 1^{st}-pattern $\mathbf{p}^A(x_I, \emptyset)$ of x_I. Again, by considering all possible $x \in X_0^H$, there are in total $\mathbf{p}^A(x_I, \emptyset)_0$ many such grey edges in H_+.

For $b_{I_r} = \sum_{i\in I_r} b_i$ and $b_J = \sum_{j\in J} b_j$ as defined in (5.5.4), we claim that

(5.5.28)
$$\sum_{x\in\mathscr{X}_0^{\mathscr{ab}}} |\mathcal{S}_{x,\mathscr{ab},\bar{J}}| = b_{I_r} + p_0^A - p_0^{A,2nd}, \quad \text{and} \quad \sum_{x\in\mathscr{X}_0^{\mathscr{ab}}} |\mathcal{S}_{x,\mathscr{ab},J}| = b_J + p_0^B - p_0^{B,2nd}.$$

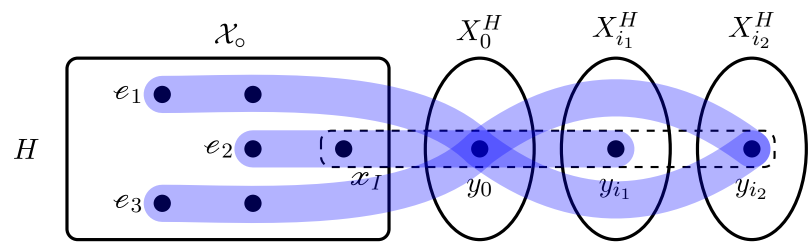


Figure 5.6: This illustrates the case that $0 \in I$ and $\mathbf{y}_{ab} = \{y_0, y_{i_1}, y_{i_2}\} \subseteq x_I$, and we assume that $I_{r_0} = \{0, i_1, i_2\}$ and $J = \emptyset$. Note that $y_0 = x_a$. By Definition 5.14 of an edge tester we require that x_I lies in an H-edge if we assigned positive weight to the tuple x_I. Hence, by the construction of H_+ (see (5.5.7)), we have that e_1, e_2, e_3 are edges in H. Note that the edges e_1, e_2, e_3 in H belong to $E_{y_0,y_{i_1}} \cup E_{y_0,y_{i_2}}$. By definition, $y_0 = x_a \notin \mathcal{X}_0^{ab}$ and we thus do not consider the possible target set $T_{y_0,ab}$ because y_0 has to be mapped onto the centre c_0 in G. Hence, the edges e_1, e_2, e_3 do not account for $\sum_{x \in \mathcal{X}_0^{ab}} |\mathcal{S}_{x,ab,\bar{J}}|$. Since $\mathbf{y}_{ab} \subseteq x_I$, there are $\mathbf{p}^{A,2nd}(x_I, \emptyset)_0$ many such blue edges by the definition of the 2^{nd}-pattern $\mathbf{p}^{A,2nd}(x_I, \emptyset)$ in Definition 5.12.

We establish the first equation in (5.5.28); the second one then follows similarly. At this part of the proof it is crucial to refresh Definition 5.12 because we make use of all the details of the pattern definitions. Further, recall that $\mathbf{y}_{ab} = \{y_i\}_{i \in I_{r_0} \cup J} := (a \cup b) \cap X^H_{\cup(I_{r_0} \cup J)}$ and thus $x_a = y_0$ if $0 \in I_{r_0}$; otherwise, $x_a = \emptyset$. Note that in order to compute $|\mathcal{S}_{x,ab,\bar{J}}|$ it is equivalent to count $|E_{x,\{y_i\}_{i \in I_r}}|$. By definition of $\mathcal{H}^i_+$, we have that $|\bigcup_{x \in \mathcal{X}_0} E_{x,y_i}| = b_i$ for all $i \in (I_r \cup J) \setminus \{0\}$ (see (5.5.10)). That is, there are $b_{I_r} = \sum_{i \in I_r} b_i$ edges $e \in \bigcup_{x \in \mathcal{X}_0} E_{x,\{y_i\}_{i \in I_r}}$ with $|e \cap \{y_i\}_{i \in I_r}| = 1$. Out of these b_{I_r} edges, we claim that $p_0^{A,2nd}$ edges e satisfy that $x_a \in e$, that is, $\sum_{i \in I_r} |E_{x_a,y_i}| = p_0^{A,2nd}$. For an illustration of these edges in $\bigcup_{i \in I_r} E_{x_a,y_i}$, see Figure 5.6. Indeed, since $\omega(ab) > 0$, we have by Definition 5.14 of an edge tester that $\mathbf{y}_{ab} \subseteq x_I$ for some $x_I \in \mathcal{X}_{\sqcup I}$ with $x_I \in E_{\mathcal{H}}(p, I, J)$. Thus, by Definition 5.13 of $E_{\mathcal{H}}(p, I, J)$, we have that $\mathbf{p}^A(x_I, J) = \mathbf{p}^A = (p_i^A)_{i \in V(R)}$ and $\mathbf{p}^{A,2nd}(x_I, J) = \mathbf{p}^{A,2nd} = (p_i^{A,2nd})_{i \in V(R)}$. Hence, by the definition of a 2^{nd}-pattern in Definition 5.12 and because $\mathcal{H}_+$ is constructed such that each subset $\{x_a, y_i\}$ only lies in proper edges of $\mathcal{H}$ due to (c) and (5.5.7), we have $p_0^{A,2nd} = \sum_{i \in I_r} |E_{x_a,y_i}|$. By the definition of $\mathcal{X}_0^{ab}$ in (5.5.19) which excludes x_a, this accounts for the term '$b_{I_r} - p_0^{A,2nd}$' in (5.5.28). It is worth pointing out that $p_0^{A,2nd} = p_0^{B,2nd} = 0$ if $0 \notin I_{r_0}$ because then $x_a = \emptyset$.

Further, we claim that there are p_0^A edges $e \in \bigcup_{x \in \mathcal{X}_0} E_{x,\{y_i\}_{i \in I_r}}$ with $|e \cap \{y_i\}_{i \in I_r}| \geq 2$ but $x_a \notin e$. Since $\mathbf{y}_{ab} \subseteq x_I$ for some $x_I \in \mathcal{X}_{\sqcup I}$ with $x_I \in E_{\mathcal{H}}(p, I, J)$, we have that $\mathbf{p}^A(x_I, J) = \mathbf{p}^A = (p_i^A)_{i \in V(R)}$. Hence, by the definition of a 1^{st}-pattern in Definition 5.12 and by (5.5.8), there are p_0^A edges $e \in \bigcup_{x \in \mathcal{X}_0} E_{x,\{y_i\}_{i \in I_r}}$ with $|e \cap \{y_i\}_{i \in I_r}| \geq 2$ because of the last two conditions in (5.4.3), and all of these edges e satisfy that $x_a \notin e$ due to the condition '$(f \cap X_\ell^H) \setminus \{x_i\}_{i \in I \setminus J} \neq \emptyset$' in (5.4.3). Altogether, this implies (5.5.28).

Hence, for $ab \in \Omega_b$, we have by (5.5.27) and (5.5.28) that

$$\prod_{x \in \mathcal{X}_0^{ab}} |T_{x,ab}| = (1 \pm \hat{\varepsilon}) d_A^{b_{I_r} + p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} d_0^b n^b. \tag{5.5.29}$$

Step 4. Estimating $\sum_{b \in [b_{\max}]_0} \omega_b(M(\sigma))$ in (5.5.25)

In this step we estimate the contribution of the first term $\sum_{b \in [b_{\max}]_0} \omega_b(M(\sigma))$ in (5.5.25). Throughout this step, let us consider the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$. At the end of the step we explain how the estimate of $\sum_{b \in [b_{\max}]_0} \omega_b(M(\sigma))$ changes if

$0 \in I \setminus J$. Note, if $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, then ω_0 is the empty function and thus, $\sum_{b\in[b_{\max}]_0} \omega_b(M(\sigma)) = \sum_{b\in[b_{\max}]} \omega_b(M(\sigma))$. (That is, $b = 0$ is only relevant if $0 \in I \setminus J$.)

We first consider $\omega_b(M(\sigma))$ for some $b \in [b_{\max}]$. By (5.5.29) and the definition of ω_b, we have

$$\omega_b(E(\mathcal{A}_0)) = (1 \pm 2\hat{\varepsilon}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} d_0^b n^b \sum_{\mathscr{a}\mathscr{b}\in\Omega_b} \omega(\mathscr{a}\mathscr{b}). \tag{5.5.30}$$

We verify that ω_b satisfies (5.5.14). For all $\{e_1, \ldots, e_{b'}\} \in \binom{E(\mathcal{A}_0)}{b'}, b' \in [b]$, the number of edges $\{e_{b'+1}, \ldots, e_b\}$ such that $\mathbf{e} = \{e_1, \ldots, e_b\} \in \binom{E(\mathcal{A}_0)}{b}$ with $\omega_b(\mathbf{e}) > 0$ is at most $\Delta n^{b-b'}$ (recall that we have chosen Δ such that $\varepsilon' \ll 1/\Delta \ll 1/t \ll 1/k, 1/r, 1/r_\circ, 1/s$), implying that $\|\omega_b\|_{b'} \leq \Delta^2 n^{b-b'} \leq n^{b-b'+\varepsilon^2}$. Hence, by adding ω_b to $\mathcal{W}$, (5.5.16) implies that

$$\begin{aligned} \omega_b(M(\sigma)) &= (1 \pm \hat{\varepsilon}) \frac{\omega_b(E(\mathcal{A}_0))}{(d_0 n)^b} \pm n^{\varepsilon} \\ &\overset{(5.5.30)}{=} (1 \pm \hat{\varepsilon}^{1/2}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} \sum_{\mathscr{a}\mathscr{b}\in\Omega_b} \omega(\mathscr{a}\mathscr{b}) \pm n^{\varepsilon}. \end{aligned} \tag{5.5.31}$$

Finally, observe that

$$\sum_{b\in[b_{\max}]} \sum_{\mathscr{a}\mathscr{b}\in\Omega_b} \omega(\mathscr{a}\mathscr{b}) = \sum_{\mathscr{a}\mathscr{b}\in E(\mathcal{A}_{I_{r_0}})\sqcup\bigsqcup_{j\in J} E(\mathcal{B}_j)} \omega(\mathscr{a}\mathscr{b}) = \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j)\right),$$

and thus (5.5.31) implies that

$$\sum_{b\in[b_{\max}]} \omega_b(M(\sigma)) = (1 \pm \hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j)\right) \pm n^{2\varepsilon}, \tag{5.5.32}$$

which is the desired estimate of $\sum_{b\in[b_{\max}]} \omega_b(M(\sigma)) = \sum_{b\in[b_{\max}]_0} \omega_b(M(\sigma))$ in the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$.

Let us now assume that $0 \in I \setminus J$ and we explain how the estimate of $\sum_{b\in[b_{\max}]_0} \omega_b(M(\sigma))$ changes. (Note that we allow for $b = 0$.) The intuition is that we additionally require that $x_{\mathscr{a}}$ is mapped onto c_0 which we would expect to happen in an idealized random setting with probability roughly $(d_0 n)^{-1}$. In fact, (5.5.30) is still true in the case that $0 \in I \setminus J$, but note that ω_b is now a $(1 + b)$-tuple weight function which yields an additional factor of $(d_0 n)^{-1}$ in (5.5.31) and thus also in (5.5.32). Hence, we obtain (5.5.32) with an additional factor of $(d_0 n)^{-1}$ as the desired estimate in the case that $0 \in I \setminus J$.

Step 5. Estimating $\sum_{b\in[b_{\max}]_0} \omega_b^-(M(\sigma))$ in (5.5.25) if $0 \in J \setminus J_{XV}$

In this step we establish an upper bound on the contribution of the minuend $\sum_{b\in[b_{\max}]_0} \omega_b^-(M(\sigma))$ in (5.5.25) and therefore, suppose that $0 \in J \setminus J_{XV}$.

We first consider $\omega_b^-(M(\sigma))$ for some $b \in [b_{\max}]_0$. It suffices to establish only a rough upper bound which follows directly by the definition of ω_b^-:

$$\omega_b^-(E(\mathcal{A}_0)) \leq 2n^b \sum_{\mathscr{a}\mathscr{b}\in\Omega_b^-} \omega(\mathscr{a}\mathscr{b}). \tag{5.5.33}$$

It is easy to verify that ω_b^- satisfies (5.5.14). Further, note that ω_b^- is a $(b+1)$-tuple weight function- Hence, by adding ω_b to $\mathcal{W}$, (5.5.16) implies that

$$\omega_b^-(M(\sigma)) \leq (1+\hat{\varepsilon})\frac{\omega_b^-(E(\mathcal{A}_0))}{(d_0 n)^{b+1}} + n^{\varepsilon} \overset{(5.5.33)}{\leq} n^{\hat{\varepsilon}-1} \sum_{\mathscr{a}\mathscr{b} \in \Omega_b^-} \omega(\mathscr{a}\mathscr{b}) + n^{\varepsilon}. \tag{5.5.34}$$

Finally, observe that

$$\sum_{b \in [b_{\max}]_0} \sum_{\mathscr{a}\mathscr{b} \in \Omega_b^-} \omega(\mathscr{a}\mathscr{b}) \leq \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j)\right),$$

and thus (5.5.34) implies that

$$\sum_{b \in [b_{\max}]_0} \omega_b^-(M(\sigma)) \leq n^{\hat{\varepsilon}-1}\omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j)\right) + n^{2\varepsilon}. \tag{5.5.35}$$

Hence, combining (5.5.32) and (5.5.35) in the case that $0 \in J \setminus J_{XV}$, yields that

$$\begin{aligned}\sum_{b \in [b_{\max}]_0} \left(\omega_b(M(\sigma)) - \omega_b^-(M(\sigma))\right) =& (1 \pm 2\hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} \\ & \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j \in J} E(\mathcal{B}_j)\right) \pm 2n^{2\varepsilon}.\end{aligned} \tag{5.5.36}$$

<u>*Step 6.*</u> *Estimating $\omega(\Gamma_b(\ell, m_A, m_B))$*

In this step we derive an upper bound for $\omega(\Gamma_b(\ell, m_A, m_B))$ for fixed $b \in [b_{\max}]_0$, $\ell \in [b]_0$, $m_A, m_B \in [\Delta]_0$, and $\Gamma_b(\ell, m_A, m_B) \supseteq \Gamma_b^{hit}(\ell, m_A, m_B)$ as defined in (5.5.23). We will use this bound in the subsequent Step 7 to derive an upper bound for $\omega(\Gamma_b^{hit}(\ell, m_A, m_B))$ as in (5.5.25) and (5.5.26). Throughout this step, let us again consider the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, and thus $b, \ell > 0$. At the end of the step, we explain how the estimate changes if $0 \in I \setminus J$ or $0 \in J_{XV}$.

Our general strategy is based on the following inclusion-exclusion principle. For every $\mathscr{a}\mathscr{b} \in \Omega_b$, we estimate the

(5.5.37) *ω-weight $\omega(\mathscr{a}\mathscr{b})$ with tuples $x \in \binom{\mathscr{X}_0^{\mathscr{a}\mathscr{b}}}{b}$ such that $b-\ell$ vertices x of x are mapped onto their target set $T_{x,\mathscr{a}\mathscr{b}}$ and the remaining ℓ vertices $x_1, \ldots, x_\ell$ of x satisfy*

$$(*) \qquad \sum_{i \in [\ell]} |\mathcal{S}_{x_i, \mathscr{a}\mathscr{b}, \bar{J}}| = m_A, \qquad \sum_{i \in [\ell]} |\mathcal{S}_{x_i, \mathscr{a}\mathscr{b}, J}| = m_B,$$

(5.5.38) *ω-weight $\omega(\mathscr{a}\mathscr{b})$ as in* (5.5.37) *with tuples x where we additionally require that the remaining ℓ vertices vertices $x_1, \ldots, x_\ell$ of x are embedded by σ and satisfy $(*)$.*

Now, subtracting (5.5.38) from (5.5.37) yields the

(5.5.39) *ω-weight $\omega(\mathscr{a}\mathscr{b})$ as in* (5.5.37) *with tuples x where at least one of the remaining ℓ vertices $x_1, \ldots, x_\ell$ of x is not embedded by σ, and $x_1, \ldots, x_\ell$ satisfy $(*)$.*

Hence, summing over the ω-weight as in (5.5.39) for all $\mathscr{a}\mathscr{b} \in \Omega_b$ yields an upper bound for
$\omega(\Gamma_b(\ell, m_A, m_B))$ as defined in (5.5.23) when $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$.

First, in order to estimate (5.5.37) and (5.5.38), we define the following sets of tuples of edges in $\mathcal{A}_0$. For all $H \in \mathcal{H}$ and $ab \in E(A^H_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(B^H_j)$ with $ab \in \Omega_b$, let

$$\mathscr{X}_0^{ab,\bar{\sigma},m_A,m_B} := \left\{ \mathscr{X} \in \binom{\mathscr{X}_0^{ab}}{\ell} : \sum_{x\in\mathscr{X}} |\mathcal{S}_{x,ab,\bar{J}}| = m_A, \sum_{x\in\mathscr{X}} |\mathcal{S}_{x,ab,J}| = m_B \right\};$$

(5.5.40)

$$E^{(5.5.37)}_{ab} := \bigcup_{\mathscr{X}\in\mathscr{X}_0^{ab,\bar{\sigma},m_A,m_B}} \left\{ \{e_x\}_{x\in\mathscr{X}_0^{ab}\setminus\mathscr{X}} \in \binom{E(A^H_0)}{b-\ell}^= : e_x \in E(A^H_0[\{x\},T_{x,ab}]) \text{ for all } x \in \mathscr{X}_0^{ab}\setminus\mathscr{X} \right\};$$

(5.5.41)

$$E^{(5.5.38)}_{ab} := \bigcup_{\mathscr{X}\in\mathscr{X}_0^{ab,\bar{\sigma},m_A,m_B}} \left\{ \{e_x\}_{x\in\mathscr{X}_0^{ab}} \in \binom{E(A^H_0)}{b}^= : x \in e_x \text{ for all } x\in\mathscr{X}_0^{ab},\ e_x \in E(A^H_0[\{x\},T_{x,ab}]) \text{ for all } x \in \mathscr{X}_0^{ab}\setminus\mathscr{X} \right\}.$$

Note that the edges in $E^{(5.5.37)}_{ab}$ and $E^{(5.5.38)}_{ab}$ correspond to the described situations in (5.5.37) and (5.5.38), respectively. We define weight functions $\omega^{(5.5.37)}_{ab} : \binom{E(A^H_0)}{b-\ell} \to [0,s]$ and $\omega^{(5.5.38)}_{ab} : \binom{E(A^H_0)}{b} \to [0,s]$ by

$$\omega^{(5.5.37)}_{ab}(\mathbf{e}) := \mathbb{1}\{\mathbf{e}\in E^{(5.5.37)}_{ab}\}\cdot\omega(ab), \quad \text{and} \quad \omega^{(5.5.38)}_{ab}(\mathbf{e}) := \mathbb{1}\{\mathbf{e}\in E^{(5.5.38)}_{ab}\}\cdot\omega(ab).$$

Let

$$\omega^{(5.5.37)}_{\Gamma}(\mathbf{e}) := \sum_{ab\in\Omega_b} \omega^{(5.5.37)}_{ab}(\mathbf{e}), \quad \text{and} \quad \omega^{(5.5.38)}_{\Gamma}(\mathbf{e}) := \sum_{ab\in\Omega_b} \omega^{(5.5.38)}_{ab}(\mathbf{e}).$$

We estimate $\omega^{(5.5.37)}_{\Gamma}(E(\mathcal{A}_0))$ and $\omega^{(5.5.38)}_{\Gamma}(E(\mathcal{A}_0))$. By (5.5.29) and the definition of $E^{(5.5.37)}_{ab}$ and $E^{(5.5.38)}_{ab}$ in (5.5.40) and (5.5.41), respectively, we obtain

(5.5.42)

$$\omega^{(5.5.37)}_{\Gamma}(E(\mathcal{A}_0)) = (1\pm 2\hat{\varepsilon}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}-m_A} d_B^{b_J+p_0^B-p_0^{B,2nd}-m_B} (d_0 n)^{b-\ell} \sum_{ab\in\Omega_b}\omega(ab);$$

(5.5.43)

$$\omega^{(5.5.38)}_{\Gamma}(E(\mathcal{A}_0)) = (1\pm 2\hat{\varepsilon}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}-m_A} d_B^{b_J+p_0^B-p_0^{B,2nd}-m_B} (d_0 n)^{b} \sum_{ab\in\Omega_b}\omega(ab).$$

Again, we can add $\omega^{(5.5.37)}_{\Gamma}$ and $\omega^{(5.5.38)}_{\Gamma}$ to $\mathcal{W}$ and employ property (5.5.16). This yields that

$$\begin{aligned} \omega^{(5.5.37)}_{\Gamma}(M(\sigma)) &= (1\pm\hat{\varepsilon})\frac{\omega^{(5.5.37)}_{\Gamma}(E(\mathcal{A}_0))}{(d_0 n)^{b-\ell}} \pm n^{\varepsilon} \\ &\overset{(5.5.42)}{=} (1\pm 4\hat{\varepsilon}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}-m_A} d_B^{b_J+p_0^B-p_0^{B,2nd}-m_B} \sum_{ab\in\Omega_b}\omega(ab) \pm n^{\varepsilon} \end{aligned} \tag{5.5.44}$$

and

$$\begin{aligned}\omega_\Gamma^{(5.5.38)}(M(\sigma)) &= (1\pm\hat\varepsilon)\frac{\omega_\Gamma^{(5.5.38)}(E(\mathcal{A}_0))}{(d_0n)^b} \pm n^\varepsilon \\ &\overset{(5.5.43)}{=} (1\pm 4\hat\varepsilon) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}-m_A} d_B^{b_J+p_0^B-p_0^{B,2nd}-m_B} \sum_{\mathcal{ab}\in\Omega_b} \omega(\mathcal{ab}) \pm n^\varepsilon. \end{aligned} \tag{5.5.45}$$

Finally, as observed in (5.5.39), subtracting (5.5.45) from (5.5.44) gives us an upper bound on $\omega(\Gamma_b(\ell, m_A, m_B))$. We obtain

$$\begin{aligned}\omega(\Gamma_b(\ell,m_A,m_B)) &\le \omega_\Gamma^{(5.5.37)}(M(\sigma)) - \omega_\Gamma^{(5.5.38)}(M(\sigma)) \\ &\le 8\hat\varepsilon d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}-m_A} d_B^{b_J+p_0^B-p_0^{B,2nd}-m_B} \sum_{\mathcal{ab}\in\Omega_b} \omega(\mathcal{ab}) + 2n^\varepsilon. \end{aligned} \tag{5.5.46}$$

Let us now first assume that $0 \in I \setminus J$ and we explain how the estimate on $\omega(\Gamma_b(\ell, m_A, m_B))$ changes. If $0 \in I \setminus J$, then by the definition of $\Gamma_b(\ell, m_A, m_B)$ in (5.5.23), we additionally require that $x_{\mathcal{a}}$ is mapped onto c_0 by σ which we would again expect to happen with probability roughly $(d_0n)^{-1}$. That is, we have to modify the definitions in (5.5.40) and (5.5.41) by additionally adding the edge $x_{\mathcal{a}}c_0$ to the tuples. Again, the estimates for the total weights in (5.5.42) and (5.5.43) are still true but we obtain an additional factor of $(d_0n)^{-1}$ in (5.5.44) and (5.5.45) as the sizes of the tuple functions increased by 1. Thus, we also obtain (5.5.46) with an additional factor of $(d_0n)^{-1}$ which will be our desired estimate in the case that $0 \in I \setminus J$.

Finally, let us assume that $0 \in J_{XV}$ and we explain how the estimate of the weight $\omega(\Gamma_b(\ell, m_A, m_B))$ changes. If $0 \in J_{XV}$, then by the definition of $\Gamma_b(\ell, m_A, m_B)$ in (5.5.23), we additionally require that either $x_{\mathcal{b}}$ is left unembedded by σ, or no $H^{\mathcal{ab}}$-vertex is mapped onto c_0. This can be achieved by modifying the definition in (5.5.41) such that the edge tuples are increased by adding the A_0^H-edges $e_{z_{\mathcal{b}}}$ such that $z_{\mathcal{b}} \in e_{z_{\mathcal{b}}}$. This ensures that for $z_{\mathcal{b}} \in \{x_{\mathcal{b}}, c_0\}$, we either have $x_{\mathcal{b}}$ is left unembedded by σ, or no $H^{\mathcal{ab}}$-vertex is mapped onto c_0. The modification adds another factor of d_0n to the total weight in (5.5.43) but also another factor of $(d_0n)^{-1}$ to (5.5.45). Thus, (5.5.46) will also be our desired estimate in the case that $0 \in J_{XV}$.

<u>*Step 7.*</u> *Estimating $\omega(\Gamma_b^{hit}(\ell, m_A, m_B))$ in* (5.5.25) *and* (5.5.26)

We will use the bounds on $\omega(\Gamma_b(\ell, m_A, m_B))$ of Step 6 to derive an upper bound for the ω-weight $\omega(\Gamma_b^{hit}(\ell, m_A, m_B))$. Let us first assume the two cases that $0 \in (V(R)\setminus I) \cup (J \setminus J_{XV})$, and $0 \in J_{XV}$, since both cases yield the same bound on $\omega(\Gamma_b(\ell, m_A, m_B))$ in (5.5.46). The general idea is that we will obtain additional factors '$d_A^{m_A}$' and '$d_B^{m_B}$' in (5.5.46) when we extend σ to σ^+, that is, when we embed the ℓ unembedded $\mathcal{H}_+$-neighbours of each $\mathcal{ab}$ contributing to $\omega(\Gamma_b(\ell, m_A, m_B))$. Only if these ℓ vertices are mapped onto their target set $V_{x,\mathcal{ab}}$, then $\mathcal{ab}$ also contributes to $\omega(\Gamma_b^{hit}(\ell, m_A, m_B))$; that is, if $\sigma^+(x) \in V_{x,\mathcal{ab}}$ for all $x \in \mathscr{X}_0^{\mathcal{ab}} \setminus \mathscr{X}_0^\sigma$. (Recall the definition of $\Gamma_b^{hit}(\ell, m_A, m_B)$ in (5.5.24).) This happens roughly with probability $d_A^{m_A} d_B^{m_B}$. We proceed with the details.

Note that $\Gamma_b^{hit}(\ell, m_A, m_B) = \Gamma_b(\ell, m_A, m_B)$ for $\ell = 0$, and thus we may consider fixed $b \in [b_{\max}]$, $\ell \in [b]$, $m_A, m_B \in [\Delta]_0$ with $m_A + m_B > 0$. Note that we extend $\sigma|_{V(H)}$ to $\sigma^+|_{V(H)}$ for every $H \in \mathcal{H}$ by choosing a bijective mapping of $X_0^H \setminus \mathscr{X}_0^\sigma$ into $V_0 \setminus \sigma(X_0^H \cap \mathscr{X}_0^\sigma)$ uniformly and independently at random. To that end, for $\mathcal{ab} \in E(A_{I_{r_0}}^H) \sqcup \bigsqcup_{j\in J} E(B_j^H)$, $H \in \mathcal{H}$, and $x \in \mathscr{X}_0^{\mathcal{ab}}$, let $V_{x,\mathcal{ab},\bar\sigma} := V_{x,\mathcal{ab}} \setminus \sigma(X_0^H \cap \mathscr{X}_0^\sigma)$. We first estimate $|V_{x,\mathcal{ab},\bar\sigma}|$. To that end, let $\mathcal{ab} \in E(A_{I_{r_0}}^H) \sqcup \bigsqcup_{j\in J} E(B_j^H)$, $H \in \mathcal{H}$, and

$x \in \mathscr{X}_0^{ab}$ be fixed. For every $v \in V_{x,ab}$, we define a weight function $\omega_v \colon E(A_0^H) \to \{0,1\}$ by $\omega_v(e) := \mathbb{1}\{v \in e\}$ and let $\omega_{V_{x,ab}} := \sum_{v \in V_{x,ab}} \omega_v$. Observe that $\omega_{V_{x,ab}}(M(\sigma))$ counts the vertices $v \in V_{x,ab} \setminus V_{x,ab,\bar{\sigma}}$. Hence,

$$|V_{x,ab,\bar{\sigma}}| = |V_{x,ab}| - \omega_{V_{x,ab}}(M(\sigma)). \tag{5.5.47}$$

Since A_0^H is (ε, d_0) super-regular, we have

$$\omega_{V_{x,ab}}(E(A_0^H)) = (1 \pm 3\varepsilon) d_0 n |V_{x,ab}|.$$

Adding $\omega_{V_{x,ab}}$ to $\mathcal{W}$ and employing (5.5.15) yields that

$$\omega_{V_{x,ab}}(M(\sigma)) = (1 \pm \varepsilon)(1 - \varepsilon^{2/3}) \frac{\omega_{V_{x,ab}}(E(A_0^H))}{d_0 n} \pm n^{\varepsilon} = (1 \pm 5\varepsilon)(1 - \varepsilon^{2/3}) |V_{x,ab}|.$$

We conclude that

$$|V_{x,ab,\bar{\sigma}}| \overset{(5.5.47)}{\leq} 2\varepsilon^{2/3} |V_{x,ab}| \overset{(5.5.27)}{\leq} 3\varepsilon^{2/3} d_A^{|\mathcal{S}_{x,ab,\bar{J}}|} d_B^{|\mathcal{S}_{x,ab,J}|} n. \tag{5.5.48}$$

For $ab \in \Gamma_b(\ell, m_A, m_B)$, let $\mathscr{X}_0^{ab,\bar{\sigma}} := \mathscr{X}_0^{ab} \backslash \mathscr{X}_0^{\sigma}$. By the definition of $\Gamma_b(\ell, m_A, m_B)$ in (5.5.23), we have

$$\sum_{x \in \mathscr{X}_0^{ab,\bar{\sigma}}} |\mathcal{S}_{x,ab,\bar{J}}| = m_A, \qquad \sum_{x \in \mathscr{X}_0^{ab,\bar{\sigma}}} |\mathcal{S}_{x,ab,J}| = m_B.$$

Hence, by (5.5.48) and because $|\mathscr{X}_0^{ab,\bar{\sigma}}| = \ell$, we obtain

$$\prod_{x \in \mathscr{X}_0^{ab,\bar{\sigma}}} |V_{x,ab,\bar{\sigma}}| \leq d_A^{m_A} d_B^{m_B} (3\varepsilon^{2/3} n)^{\ell}. \tag{5.5.49}$$

By (5.5.18), we have that $V_0 \setminus \sigma(X_0^H \cap \mathscr{X}_0^{\sigma}) \geq \varepsilon^{2/3} n/3$. Now, with (5.5.49) we obtain that the probability that all ℓ vertices $x \in \mathscr{X}_0^{ab,\bar{\sigma}}$ are mapped onto their target set $V_{x,ab,\bar{\sigma}}$ and if $x_b \in \mathscr{X}_0^{ab,\bar{\sigma}}$ then it is not mapped onto c_0 — that is, the probability that $ab \in \Gamma_b(\ell, m_A, m_B)$ is also contained in $\Gamma_b^{hit}(\ell, m_A, m_B)$ — is at most

$$\frac{d_A^{m_A} d_B^{m_B} (3\varepsilon^{2/3} n)^{\ell}}{\varepsilon^{2/3} n \cdot (\varepsilon^{2/3} n - 1) \cdots (\varepsilon^{2/3} n - \ell + 1)/3^{\ell}} \leq 10^{\ell} d_A^{m_A} d_B^{m_B}.$$

Finally, we can derive an upper bound for the expected value of $\omega(\Gamma_b^{hit}(\ell, m_A, m_B))$.

$$\begin{aligned} \mathbb{E}\left[\omega\left(\Gamma_b^{hit}(\ell, m_A, m_B)\right)\right] &\leq \omega\left(\Gamma_b(\ell, m_A, m_B)\right) 10^{\ell} d_A^{m_A} d_B^{m_B} \\ &\overset{(5.5.46)}{\leq} \varepsilon^{1/4} d_A^{b_{I_r} + p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} \sum_{ab \in \Omega_b} \omega(ab) + n^{2\varepsilon}. \end{aligned}$$

By using Theorem 1.8 and a union bound, we can establish concentration with probability, say, at least $1 - \mathrm{e}^{-n^{\varepsilon}}$. Thus, we conclude

$$\omega\left(\Gamma_b^{hit}(\ell, m_A, m_B)\right) \leq 2\varepsilon^{1/4} d_A^{b_{I_r} + p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} \sum_{ab \in \Omega_b} \omega(ab) + 2n^{2\varepsilon} \tag{5.5.50}$$

for all $b \in [b_{\max}]$, $\ell \in [b]$, $m_A, m_B \in [\Delta]_0$ with $m_A + m_B > 0$. By summing over all values of b, ℓ, m_A, m_B, we obtain

$$\sum_{\substack{b\in[b_{\max}],\ell\in[b],\\ m_A,m_B\in[\Delta]_0}} \omega\left(\Gamma_b^{hit}(\ell, m_A, m_B)\right)$$

$$\overset{(5.5.50)}{\leq} \varepsilon^{1/5} d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)\right) + n^{3\varepsilon}, \tag{5.5.51}$$

which is the desired estimate for the second term in (5.5.25) for the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, and it is the desired estimate for the right hand side in (5.5.26) for the case that $0 \in J_{XV}$ (where we also allow $b, \ell = 0$ in the summation).

In the case that $0 \in I \setminus J$, we obtain (5.5.51) with an additional factor of $(d_0 n)^{-1}$ since the estimate on $\omega\left(\Gamma_b(\ell, m_A, m_B)\right)$ from the previous step yields an additional factor of $(d_0 n)^{-1}$.

Step 8. *Concluding* $(\mathrm{II})_{\mathrm{L5.18}}$ *if* $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$

Finally, we can establish $(\mathrm{II})_{\mathrm{L5.18}}$ if $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$ by the estimates derived in the Steps 4 and 7. So, let us assume that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$. Using (5.5.32) (respectively (5.5.36) if $0 \in J \setminus J_{XV}$) and (5.5.51) in our key observation (5.5.25) yields that

$$\begin{aligned}\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j^{new})\right) =&(1 \pm 3\hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} \\ &\omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)\right) \pm 2n^{3\varepsilon}.\end{aligned} \tag{5.5.52}$$

We will use (P3) for $\omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)\right)$ in order to obtain $(\mathrm{II})_{\mathrm{L5.18}}$ from (5.5.52). For $0 \notin I$ or $0 \in J \setminus J_{XV}$, we have that

$$d_A^{b_{I_r}} \prod_{i\in I_{r_0}} d_{i,A} = \prod_{i\in I_r} d_A^{b_i} d_{i,A} = \prod_{i\in I_r} d_{i,A}^{new} \tag{5.5.53}$$

for $d_{i,A}^{new}$ defined as in (5.5.4) because $I_{r_0} = I_r$, and similarly

$$d_B^{b_J} \prod_{j\in J} d_{j,B} = \prod_{j\in J} d_B^{b_j} d_{j,B} = \prod_{j\in J} d_{j,B}^{new}, \tag{5.5.54}$$

because by the definition in (5.5.4), we have $b_0 = 0$ and $d_B^{b_0} = d_B^0 = 1$ and $d_{0,B} = d_{0,B}^{new}$. Hence, altogether using (P3) together with (5.5.52), we obtain

$$\begin{aligned}
&\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j^{new})\right) \\
&= (1 \pm 3\hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}} \\
&\quad \Bigg(\left(\mathbb{1}\{J_{XV} \cap -[r_\circ] = \emptyset\} \pm \varepsilon\right) d_A^{\|\mathbf{p}^A_{-[r_\circ]}\| - \|\mathbf{p}^{A,2nd}_{-[r_\circ]}\|} d_B^{\|\mathbf{p}^B_{-[r_\circ]}\| - \|\mathbf{p}^{B,2nd}_{-[r_\circ]}\|} \\
&\quad \prod_{i\in I_{r_0}} d_{i,A} \prod_{j\in J} d_{j,B} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ])\setminus J|}} \pm n^\varepsilon\Bigg) \pm 2n^{3\varepsilon} \\
&\overset{(5.5.53),(5.5.54)}{=} \left(\mathbb{1}\{J_{XV} \cap -[r_\circ]_0 = \emptyset\} \pm \varepsilon'^2\right) d_A^{\|\mathbf{p}^A_{-[r_\circ]_0}\| - \|\mathbf{p}^{A,2nd}_{-[r_\circ]_0}\|} d_B^{\|\mathbf{p}^B_{-[r_\circ]_0}\| - \|\mathbf{p}^{B,2nd}_{-[r_\circ]_0}\|} \\
&\quad \prod_{i\in I_r} d_{i,A}^{new} \prod_{j\in J} d_{j,B}^{new} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ]_0)\setminus J|}} \pm n^{\varepsilon'},
\end{aligned}$$

which establishes (II)$_{\text{L5.18}}$ in the case of $0 \notin I$ or $0 \in J \setminus J_{XV}$.

Step 9. Concluding (II)$_{\text{L5.18}}$ *if* $0 \in I \setminus J$

Similar as in Step 8, we can establish (II)$_{\text{L5.18}}$ if $0 \in I \setminus J$ by the estimates derived in the Steps 4 and 7. So, let us assume that $0 \in I \setminus J$, and recall that we obtained an additional factor of $(d_0 n)^{-1}$ in (5.5.32) and (5.5.51). Together with our key observation (5.5.25) this yields that

$$\begin{aligned}&\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup\nolimits_{j\in J} E(\mathcal{B}_j^{new})\right)\\ &= (1 \pm 2\hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} (d_0 n)^{-1} \omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup\nolimits_{j\in J} E(\mathcal{B}_j)\right) \pm 2n^{2\varepsilon}.\end{aligned} \tag{5.5.55}$$

We will use (P3) for $\omega\left(E(\mathcal{A}_{I_{r_0}}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)\right)$ in order to obtain (II)$_{\text{L5.18}}$ from (5.5.55). For $0 \in I \setminus J$, we have that

$$d_0^{-1} d_A^{b_{I_r}} \prod_{i\in I_{r_0}} d_{i,A} = \prod_{i\in I_r} d_A^{b_i} d_{i,A} = \prod_{i\in I_r} d_{i,A}^{new} \tag{5.5.56}$$

for $d_{i,A}^{new}$ defined as in (5.5.4) because $d_0 = d_{0,A}$, and similarly

$$d_B^{b_J} \prod_{j\in J} d_{j,B} = \prod_{j\in J} d_{j,B}^{new}. \tag{5.5.57}$$

Hence, altogether using (P3) together with (5.5.55), we obtain

$$\begin{aligned}\omega^{new}&\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup\nolimits_{j\in J} E(\mathcal{B}_j^{new})\right)\\ =\ & (1 \pm \hat{\varepsilon}^{1/3}) d_A^{b_{I_r}+p_0^A - p_0^{A,2nd}} d_B^{b_J + p_0^B - p_0^{B,2nd}} (d_0 n)^{-1}\\ &\Bigg(\Big(\mathbb{1}\{J_{XV} \cap -[r_\circ] = \emptyset\} \pm \varepsilon\Big) d_A^{\|\mathbf{p}_{-[r_\circ]}^A\| - \|\mathbf{p}_{-[r_\circ]}^{A,2nd}\|} d_B^{\|\mathbf{p}_{-[r_\circ]}^B\| - \|\mathbf{p}_{-[r_\circ]}^{B,2nd}\|}\\ &\prod_{i\in I_{r_0}} d_{i,A} \prod_{j\in J} d_{j,B} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ])\setminus J|}} \pm n^{\varepsilon}\Bigg) \pm 2n^{2\varepsilon}\\ \overset{(5.5.56),(5.5.57)}{=}\ & \Big(\mathbb{1}\{J_{XV} \cap -[r_\circ]_0 = \emptyset\} \pm \varepsilon'^2\Big) d_A^{\|\mathbf{p}_{-[r_\circ]_0}^A\| - \|\mathbf{p}_{-[r_\circ]_0}^{A,2nd}\|} d_B^{\|\mathbf{p}_{-[r_\circ]_0}^B\| - \|\mathbf{p}_{-[r_\circ]_0}^{B,2nd}\|}\\ &\prod_{i\in I_r} d_{i,A}^{new} \prod_{j\in J} d_{j,B}^{new} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ]_0)\setminus J|}} \pm n^{\varepsilon'},\end{aligned}$$

which establishes (II)$_{\text{L5.18}}$ in the case of $0 \in I \setminus J$.

Step 10. Concluding (II)$_{\text{L5.18}}$ *if* $0 \in J_{XV}$

For the last case that $0 \in J_{XV}$, we employ the estimate derived in Step 7 in our key observation 5.5.26. For $0 \in J_{XV}$, we have that

$$d_A^{b_{I_r}} \prod_{i\in I_{r_0}} d_{i,A} = \prod_{i\in I_r} d_A^{b_i} d_{i,A} = \prod_{i\in I_r} d_{i,A}^{new} \tag{5.5.58}$$

for $d_{i,A}^{new}$ defined as in (5.5.4) because $I_{r_0} = I_r$, and

$$d_B^{b_J} \prod_{j\in J} d_{j,B} = \prod_{j\in J} d_B^{b_j} d_{j,B} = \prod_{j\in J} d_{j,B}^{new}, \tag{5.5.59}$$

because by the definition in (5.5.4), we have $b_0 = 0$ and $d_B^{b_0} = d_B^0 = 1$ and $d_{0,B} = d_{0,B}^{new}$. Hence, altogether using (P3) together with (5.5.51) in our key observation (5.5.26), we obtain

$$\begin{aligned}
&\omega^{new}\left(E(\mathcal{A}_{I_r}^{new}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j^{new})\right)\\
&\leq \quad \varepsilon^{1/5} d_A^{b_{I_r}+p_0^A-p_0^{A,2nd}} d_B^{b_J+p_0^B-p_0^{B,2nd}}\\
&\qquad \left(\left(\mathbb{1}\{J_{XV}\cap -[r_\circ]=\emptyset\}+\varepsilon\right) d_A^{\|\mathbf{p}^A_{-[r_\circ]}\|-\|\mathbf{p}^{A,2nd}_{-[r_\circ]}\|} d_B^{\|\mathbf{p}^B_{-[r_\circ]}\|-\|\mathbf{p}^{B,2nd}_{-[r_\circ]}\|}\right.\\
&\qquad \left.\prod_{i\in I_{r_0}} d_{i,A}\prod_{j\in J} d_{j,B}\frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ])\setminus J|}}+n^{\varepsilon}\right)+2n^{2\varepsilon}\\
&\overset{(5.5.58),(5.5.59)}{\leq} \varepsilon'^2 \prod_{Z\in\{A,B\}} d_Z^{\|\mathbf{p}^Z_{-[r_\circ]_0}\|-\|\mathbf{p}^{Z,2nd}_{-[r_\circ]_0}\|}\prod_{i\in I_r} d_{i,A}^{new}\prod_{j\in J} d_{j,B}^{new}\frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap -[r_\circ]_0)\setminus J|}}+n^{\varepsilon'},
\end{aligned}$$

which establishes $(\mathrm{II})_{\mathrm{L5.18}}$ in the case of $0 \in J_{XV}$ and concludes the proof of $(\mathrm{II})_{\mathrm{L5.18}}$.

Step 11. Checking $(\mathrm{I})_{\mathrm{L5.18}}$

In order to establish $(\mathrm{I})_{\mathrm{L5.18}}$, we fix $H \in \mathcal{H}$, $Z \in \{A, B\}$ and $i \in [r]$ if $Z = A$, $i \in V(R) \setminus \{0\}$ if $Z = B$, and we may assume that $i \in N_{R_*}(0)$ otherwise $Z_i^{H,new} = Z_i^H$. We will first show that $Z_i^{H,new}$ as defined in (5.5.17) is $(\varepsilon', d_i^{new})$-regular by employing Theorem 5.6. In order to do so, we verify that every vertex in X_i^H has the appropriate degree in $Z_i^{H,new}$ and that most pairs of vertices in X_i^H have the appropriate common neighbourhood in $Z_i^{H,new}$. These properties follow easily due to the typicality of G_Z and because Z_i^H is (ε, q)-well-intersecting. Finally, we show that also each vertex in V_i has the correct degree in $Z_i^{H,new}$ by employing $(\mathrm{II})_{\mathrm{L5.18}}$. Altogether this will imply that $Z_i^{H,new}$ is $(\varepsilon', d_i^{new})$-super-regular. Since we basically obtain $Z_i^{H,new}$ from Z_i^H by restricting the neighbourhood of every vertex by $b_i \leq \Delta(R)$ additional $(k-1)$-sets in G_Z (see (5.5.60) and (5.5.61) below), we will obtain directly from (P1) that $Z_i^{H,new}$ is $(\varepsilon', q + \Delta(R))$-well-intersecting. We proceed with the details.

For every vertex $y \in X_i^H$, let

$$\mathcal{S}_y := \{\phi_\circ(\mathscr{e}) \cup \sigma^+(x) \colon \mathscr{e} \in E_{x,y}, x \in \mathscr{X}_0\} \tag{5.5.60}$$

with $E_{x,y}$ defined as in (5.5.9). Note that $\mathcal{S}_y \subseteq \bigcup_{\mathscr{r}\in E(R)\colon i\in \mathscr{r}} V_{\sqcup \mathscr{r}\setminus\{i\}}$ and $|\mathcal{S}_y| = b_i$. Since G_Z is (ε, t, d_Z) typical with respect to R by (P1), and since Z_i^H is $(\varepsilon, d_{i,Z})$-super-regular and (ε, q)-well-intersecting by (P2), we conclude by the definition of $Z_i^{H,new}$ as in (5.5.17) that for all $y \in X_i^H$, we have

$$\deg_{Z_i^{H,new}}(y) = |N_{Z_i^H}(y) \cap N_{G_Z}(\mathcal{S}_y)| = (1 \pm \hat{\varepsilon}) d_{i,Z} d_Z^{b_i} |V_i|. \tag{5.5.61}$$

Note that (5.5.61) implies in particular that the density of X_i^H and V_i in $Z_i^{H,new}$ is $d_{Z_i^{H,new}}(X_i^H, V_i) = (1 \pm \hat{\varepsilon}) d_{i,Z} d_Z^{b_i}$.

We can proceed similarly as for the conclusion (5.5.61) and obtain that all but at most $n^{3/2}$ pairs $\{y, y'\} \in \binom{X_i^H}{2}$ satisfy

$$|N_{Z_i^{H,new}}(y \wedge y')| = (1 \pm \hat{\varepsilon})(d_{i,Z} d_Z^{b_i})^2 |V_i|. \tag{5.5.62}$$

To see (5.5.62), note that

$$N_{Z_i^{H,new}}(y \wedge y') = N_{Z_i^H}(y \wedge y') \cap N_{G_Z}(\mathcal{S}_y) \cap N_{G_Z}(\mathcal{S}_{y'}),$$

and for all but at most $2\Delta_R n$ pairs $\{y, y'\} \in \binom{X_i^H}{2}$, we have $|\mathcal{S}_y \cup \mathcal{S}_{y'}| = 2b_i$ by (5.5.6). By employing again (P1) and (P2), we obtain (5.5.62), where we used (5.4.2) that for all but at most $2n \cdot n^{1/4+\varepsilon}$ pairs $\{y, y'\} \in \binom{X_i^H}{2}$ the sets of $(k-1)$-sets corresponding to $N_{Z_i^H}(y)$ and $N_{Z_i^H}(y')$ are disjoint. Hence, all but at most $2n^{5/4+\varepsilon} + 2\Delta_R n \leq n^{3/2}$ pairs $\{y, y'\} \in \binom{X_i^H}{2}$ satisfy (5.5.62).

We can now easily derive an upper bound for the number of 4-cycles in $Z_i^{H,new}$ by (5.5.62). To that end, note that every pair $\{y, y'\} \in \binom{X_i^H}{2}$ together with a pair of common neighbours in $N_{Z_i^{H,new}}(y \wedge y')$ forms a 4-cycle in $Z_i^{H,new}$. Hence, by (5.5.62), the number of 4-cycles in $Z_i^{H,new}$ is at most

$$\begin{aligned} C_4(Z_i^{H,new}) &\leq \frac{|X_i^H|^2}{2} \cdot (1+2\hat{\varepsilon}) \frac{(d_{i,Z} d_Z^{b_i})^4 |V_i|^2}{2} + n^{3/2} \cdot n^2 \\ &\leq (1+3\hat{\varepsilon}) \frac{(d_{i,Z} d_Z^{b_i})^4 |X_i^H|^2 |V_i|^2}{4}. \end{aligned}$$

Thus, we can apply Theorem 5.6 and obtain that

$$Z_i^{H,new} \text{ is } (\varepsilon', d_{i,Z} d_Z^{b_i})\text{-regular.} \tag{5.5.63}$$

For every $v \in V_i$, in order to control the degree of v in $Z_i^{H,new}$, we define weight functions $\omega_v \colon E(Z_i^H) \to \{0,1\}$ by $\omega_v(x_i v_i) := \mathbb{1}\{v_i = v\}$, and $\omega_\iota \colon \mathscr{X}_i \to \{0,1\}$ by $\omega_\iota := \mathbb{1}\{x \in X_i^H\}$, and add the 1-edge tester

$$(\omega_v, \omega_\iota, J = J_Z, J_X = \emptyset, J_V = \emptyset, \mathcal{c} = \{v\}, \mathcal{p} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}))$$

to $\mathcal{W}'_{edge}$ for $J_A := \emptyset$ and $J_B := \{i\}$. This is indeed a (general) 1-edge tester satisfying Definition 5.14. In particular, $\mathbf{p}^A(x) = \mathbf{p}^{A,2nd}(x) = \mathbf{p}^B(x) = \mathbf{p}^{B,2nd}(x) = \mathbf{0}$ for every $x \in X_i^H$ by Definition 5.12 because the 1^{st}-pattern and 2^{nd}-pattern of a single vertex is always $\mathbf{0}$. Since Z_i^H is $(\varepsilon, d_{i,Z})$-super-regular, we have that

$$\omega_v(E(Z_i^H)) = (1 \pm \varepsilon) d_{i,Z} |X_i^H|.$$

Hence in particular, this general edge tester satisfies (P3). By $(\mathrm{II})_{\mathrm{L}5.18}$, we obtain

$$\deg_{Z_i^{H,new}}(v) = \omega_v^{new}(E(Z_i^{H,new})) \overset{(\mathrm{II})_{\mathrm{L}5.18}}{=} (1 \pm \varepsilon'^2) d_{i,Z}^{new} |X_i^H| \pm n^{\varepsilon'}.$$

Together with (5.5.61) and (5.5.63), this implies that $Z_i^{H,new}$ is $(\varepsilon', d_{i,Z}^{new})$-super-regular because $d_{i,Z}^{new} = d_{i,Z} d_Z^{b_i}$ (see (5.5.4)). Further, since the neighbourhood of every vertex $y \in X_i^H$ in $Z_i^{H,new}$ is the intersection of a set $\mathcal{S}_y^{new}$ of $(k-1)$-sets in G_Z (see (5.5.61)), and every $y \in X_i^H$ is contained in at most $n^{1/4+\varepsilon} + \Delta_R$ pairs $\{y, y'\} \in \binom{X_i^H}{2}$ such that $\mathcal{S}_y^{new} \cap \mathcal{S}_{y'}^{new} \neq \emptyset$, we also obtain that $Z_i^{H,new}$ is $(\varepsilon', q + \Delta(R))$-well-intersecting as defined in (5.4.2). This establishes $(\mathrm{I})_{\mathrm{L}5.18}$.

Step 12. *Checking* $(\mathrm{III})_{\mathrm{L}5.18}$

For every $\omega \in \mathcal{W}_{local}$ with $\omega \colon \binom{E(\mathcal{A}_0)}{\ell}^{=} \to [0, s]$, we add ω to $\mathcal{W}$. Hence, (5.5.16) yields $(\mathrm{III})_{\mathrm{L}5.18}$.

Step 13. *Checking* $(\mathrm{IV})_{\mathrm{L}5.18}$

In order to establish $(\mathrm{IV})_{\mathrm{L}5.18}$, we fix $(\omega, c) \in \mathcal{W}_0$ with $\omega \colon \mathscr{X}_0 \to [0, s]$ and $c \in V_0$. By (5.5.18), we have that $V_0 \setminus \sigma(X_0^H \cap \mathscr{X}_0^\sigma) \geq \varepsilon^{2/3} n/3$ for every $H \in \mathcal{H}$, and thus, the probability for a vertex $x \in \mathscr{X}_0 \setminus \mathscr{X}_0^\sigma$ to be mapped onto c is at most $3/\varepsilon^{2/3} n$. We therefore expect that $\omega(\{x \in \mathscr{X}_0 \setminus \mathscr{X}_0^\sigma \colon \sigma^+(x) = c\}) \leq \omega(\mathscr{X}_0)/n^{1-2\varepsilon}$. By an application of Theorem 1.8 and a union bound, we can establish concentration with probability, say at least $1 - \mathrm{e}^{-n^{\varepsilon}}$. This establishes $(\mathrm{IV})_{\mathrm{L}5.18}$ and completes the proof of Lemma 5.18. $\square$

5.6 Iterative packing

In this section we essentially prove our main result, Theorem 5.2. We prove the following lemma whose statement is very similar because we only require additionally that for every graph $H \in \mathcal{H}$ and every reduced edge $\mathscr{r} \in E(R)$, the graph $H[X^H_{\cup \mathscr{r}}]$ is a matching. This reduction can be achieved by an application of Lemma 5.9 and simplifies several arguments; it is presented in the proof of Theorem 5.2 in Section 5.7.

Lemma 5.19 ([31])**.** *Let $1/n \ll \varepsilon \ll 1/t \ll \alpha, 1/k$ and $r \leq n^{2\log n}$ as well as $d \geq n^{-\varepsilon}$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ is an (ε, t, d)-typical and α^{-1}-bounded blow-up instance of size (n, k, r) and $|\mathcal{H}| \leq n^{2k}$. Suppose that $e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) \leq (1-\alpha)dn^k$ for all $\mathscr{r} \in E(R)$, and $H[X^H_{\cup \mathscr{r}}]$ is a matching if $\mathscr{r} \in E(R)$ and empty if $\mathscr{r} \in \binom{[r]}{k} \setminus E(R)$ for each $H \in \mathcal{H}$. Suppose $\mathcal{W}_{set}, \mathcal{W}_{ver}$ are sets of α^{-1}-set testers and α^{-1}-vertex testers of size at most $n^{3\log n}$, respectively. Then there is a packing ϕ of $\mathcal{H}$ into G such that*

(i) $\phi(X^H_i) = V_i$ *for all* $i \in [r]$ *and* $H \in \mathcal{H}$*;*

(ii) $|W \cap \bigcap_{j \in [\ell]} \phi(Y_j)| = |W||Y_1| \cdots |Y_\ell|/n^\ell \pm \alpha n$ *for all* $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$*;*

(iii) $\omega(\phi^{-1}(\mathscr{c})) = (1 \pm \alpha)\omega(\mathscr{X}_{\sqcup I})/n^{|I|} \pm n^\alpha$ *for all* $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ *with centres* $\mathscr{c}$ *in* I*.*

Proof of Lemma 5.19. We split the proof into five steps. In Step 1, we define a vertex colouring of the reduced graph which will incorporate in which order we consider the clusters in turn. In Step 2, we partition G into two edge-disjoint subgraphs G_A and G_B. In Step 3, we introduce candidacy graphs and edge testers that we track for the partial packing in Step 4, where we iteratively apply Lemma 5.18 and consider the clusters in turn with respect to the ordering of the clusters given by the colouring obtained in Step 1. We only use the edges of G_A for the partial packing in Step 4 such that we can complete the packing in Step 5 using the edges of G_B.

Step 1. *Notation and colouring of the reduced graph*

We will proceed cluster by cluster in Step 4 to find a function that packs almost all vertices of $\mathcal{H}$ into G. Since we allow r to grow with n and only require that $r \leq n^{\log n}$, we need to carefully control the growth of the error term. Recall that R_* is the 2-graph with vertex set $V(R)$ and edge set $\bigcup_{\mathscr{r} \in E(R)} \binom{\mathscr{r}}{2}$. Let $c\colon V(R) \to [T]$ be a proper vertex colouring of R^3_* where $T := k^3\alpha^{-3}$. The colouring naturally yields an order in which we consider the clusters in turn. To this end, we simply relabel the cluster indices such that the colour values are non-decreasing; that is, $c(1) \leq \cdots \leq c(r)$. Note that the sets $(c^{-1}(j))_{j \in [T]}$ are independent in R^3_*. We choose new constants $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_T, \mu, \gamma$ such that

$$\varepsilon \ll \varepsilon_0 \ll \varepsilon_1 \ll \cdots \ll \varepsilon_T \ll \mu \ll \gamma \ll 1/t \ll \alpha, 1/k.$$

For $i, q \in [r]$ and $I \subseteq [r]$, we define counters $c_i(q), c_I(q), m_i(q)$ (see (5.6.1)–(5.6.3) below). Our intuition is the following: If we think of $[q]$ as the indices of clusters that have already been embedded, then $c_i(q)$ is the largest colour of an already embedded cluster in the closed neighbourhood of i in R_*. That is, $c_i(q)$ is the largest colour that is relevant to i after embedding the first q clusters, and $c_i(q)$ will incorporate how to update the error term.

To be more precise, for $i, q \in [r]$ and an index set $I \subseteq [r]$ (that is, $I \subseteq \mathscr{r} \in E(R)$), let

$$c_i(q) := \max\left\{\{0\} \cup \{c(j)\colon j \in N_{R_*}[i] \cap [q]\}\right\}; \tag{5.6.1}$$

$$c_I(q) := \max_{i \in I}\left\{0, c_i(q)\right\}. \tag{5.6.2}$$

Similarly as $c_i(q)$, we define $m_i(q)$ as the number of edges in R that contain i and where $k-1$ clusters excluding i have already been embedded. That is, for $i, q \in [r]$, let

$$m_i(q) := |\{\mathscr{r} \in E(R) \colon i \in \mathscr{r}, |\mathscr{r} \cap [q] \setminus \{i\}| = k-1\}|\,. \tag{5.6.3}$$

Further, for all $i \in [r]$ and index sets $I \subseteq [r]$, we set $c_i(0) = c_I(0) = m_i(0) := 0$.

For $q \in [r]$, recall that $\mathscr{X}_q = \bigcup_{H \in \mathcal{H}} X_q^H$, and we set

$$\mathcal{X}_q := \bigcup\nolimits_{\ell \in [q]} \mathscr{X}_\ell, \qquad\qquad \mathcal{V}_q := \bigcup\nolimits_{\ell \in [q]} V_\ell.$$

Step 2. Partitioning the edges of G

In order to reserve an exclusive set of edges for the completion in Step 5, we partition the edges of G into two subgraphs G_A and G_B. For each edge $\mathscr{g}$ of G independently, we add $\mathscr{g}$ to G_B with probability γ and otherwise to G_A. Let $d_A := (1-\gamma)d$ and $d_B := \gamma d$. Using Chernoff's inequality and a union bound, we can easily conclude that with probability at least $1 - 1/n$ it holds that

(5.6.4)
for all $i \in [r]$ and all pairs of disjoint sets $\mathcal{S}_A, \mathcal{S}_B \subseteq \bigcup_{\mathscr{r} \in E(R) \colon i \in \mathscr{r}} V_{\sqcup \mathscr{r} \setminus \{i\}}$ with $|\mathcal{S}_A \cup \mathcal{S}_B| \leq t$, we have $\left|V_i \cap N_{G_A}(\mathcal{S}_A) \cap N_{G_B}(\mathcal{S}_B)\right| = (1 \pm \varepsilon_0) d_A^{|\mathcal{S}_A|} d_B^{|\mathcal{S}_B|} n.$

Hence, we may assume that G is partitioned into G_A and G_B such that (5.6.4) holds. In particular, (5.6.4) implies that G_Z is $(2\varepsilon_0, t, d_Z)$-typical with respect to R for both $Z \in \{A, B\}$.

Step 3. Partial packings, candidacy graphs and edge testers

For $q \in [r]$, we call $\phi \colon \bigcup_{H \in \mathcal{H}, i \in [q]} \widehat{X}_i^H \to \mathcal{V}_q$ a *q-partial packing* if $\widehat{X}_i^H \subseteq X_i^H$ and $\phi(\widehat{X}_i^H) \subseteq V_i$ for all $H \in \mathcal{H}, i \in [q]$ such that ϕ is a packing of $(H[\widehat{X}_1^H \cup \ldots \cup \widehat{X}_q^H])_{H \in \mathcal{H}}$ into $G_A[\mathcal{V}_q]$. Note that $\phi|_{\widehat{X}_i^H}$ is injective for all $H \in \mathcal{H}$ and $i \in [q]$. For convenience, we often write

$$\mathcal{X}_q^\phi := \bigcup\nolimits_{H \in \mathcal{H}, i \in [q]} \widehat{X}_i^H.$$

Further, we call $\phi^+ \colon \mathcal{X}_q \to \mathcal{V}_q$ a *cluster-injective extension* of ϕ if ϕ^+ is an extension of ϕ such that $\phi^+|_{X_i^H}$ is injective (and thus bijective) and $\phi^+(X_i^H) = V_i$ for all $H \in \mathcal{H}, i \in [q]$. Note that we do not even require that ϕ^+ is an embedding of $H[X_1^H \cup \ldots \cup X_q^H]$ for $H \in \mathcal{H}$.

Suppose $q \in [r]$ and $\phi_q \colon \mathcal{X}_q^{\phi_q} \to \mathcal{V}_q$ is a q-partial packing with a cluster-injective extension ϕ_q^+. We consider two kinds of candidacy graphs as in Definition 5.11: Candidacy graphs $A_I^H(\phi_q^+)$ with respect to ϕ_q^+ and G_A for all index sets $I \subseteq [r] \setminus [q]$, and candidacy graphs $B_j^H(\phi_q^+)$ with respect to ϕ_q^+ and G_B for all $j \in [r]$. The candidacy graphs $A_I^H(\phi_q^+)$ will be used to extend the q-partial packing ϕ_q to a $(q+1)$-partial packing ϕ_{q+1} via Lemma 5.18 in Step 4, whereas the candidacy graphs $B_j^H(\phi_r^+)$ will be used for the completion in Step 5.

Given ϕ_q, ϕ_q^+ and a collection $\mathcal{A}^q$ and $\mathcal{B}^q$ of candidacy graphs $A_I^{H,q} \subseteq A_I^H(\phi_q^+)$ and $B_j^{H,q} \subseteq B_j^H(\phi_q^+)$ for all index sets $I \subseteq [r] \setminus [q]$ and $j \in [r]$, we introduce (general) edge testers as in Definition 5.15 to track several quantities during our packing procedure.

To that end, we first define a set $\mathcal{W}_{initial}$ of tuples $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$. We also define a superset $\mathcal{W}_{hit}$ of $\mathcal{W}_{ver}$ containing tuples $(\omega, \mathscr{c})$. For every vertex tester $(\omega_{ver}, \mathscr{c}) \in \mathcal{W}_{ver}$ as in the assumptions of Lemma 5.19 with centres $\mathscr{c} \in V_{\sqcup I}$ for an index set $I \subseteq [r]$, and for all $J \subseteq I$ and pairs of disjoint sets $J_X, J_V \subseteq J$, all $\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd} \in [k\alpha^{-1}]_0^r$, we define a tuple $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ with $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to [0, \alpha^{-1}]$, $\mathscr{p} := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$, by

$$\omega_\iota(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E_{\mathcal{H}}(\mathscr{p}, I, J)\} \cdot \omega_{ver}(\mathscr{x}), \tag{5.6.5}$$

and we add this tuple to $\mathcal{W}_{initial}$. (Recall Definition 5.13 for $E_{\mathcal{H}}(\mathscr{p}, I, J)$.) We also add $(\omega_{ver}, \mathscr{c})$ to $\mathcal{W}_{hit}$.

Similarly, for every $\mathscr{r} \in E(R)$, all $J \subseteq \mathscr{r}$ and pairs of disjoint sets $J_X, J_V \subseteq J$, $\mathscr{g} \in V_{\sqcup \mathscr{r}}$, and $\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd} \in [k\alpha^{-1}]_0^r$, we define a tuple $(\omega_\iota, J, J_X, J_V, \mathscr{g}, \mathscr{p})$ with $\omega_\iota\colon \mathscr{X}_{\sqcup \mathscr{r}} \to \{0,1\}$, $\mathscr{p} := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$, by

$$\omega_\iota(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E_{\mathcal{H}}(\mathscr{p}, \mathscr{r}, J)\}, \tag{5.6.6}$$

and we add this tuple to $\mathcal{W}_{initial}$, and we define a tuple $(\omega, \mathscr{g})$ with $\omega\colon \mathscr{X}_{\sqcup \mathscr{r}} \to \{0,1\}$ by $\omega(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E(\mathcal{H})\}$ and add $(\omega, \mathscr{g})$ to $\mathcal{W}_{hit}$.

To control the number of unembedded H-vertices in one graph H that could potentially be mapped onto a fixed vertex v during the completion, we define for all $j \in [r]$, $H \in \mathcal{H}$, and $v \in V_j$, a tuple $(\omega_{\iota,H}, J = \{j\}, J_X = \{j\}, J_V = \emptyset, \mathscr{c} = \{v\}, (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}))$ with $\omega_{\iota,H}\colon \mathscr{X}_j \to \{0,1\}$ by

$$\omega_{\iota,H}(x) := \mathbb{1}\{x \in X_j^H\}, \tag{5.6.7}$$

and we add this tuple to $\mathcal{W}_{initial}$.

For one single graph $H \in \mathcal{H}$, we also consider tuples with only two centres. That is, for all $H \in \mathcal{H}$, $\mathscr{r} \in E(R)$, distinct $j, j_X \in \mathscr{r}$, $v \in V_j$, $w \in V_{j_X}$, and $\mathbf{p}^A = \mathbf{p}^{A,2nd} = \mathbf{0}, \mathbf{p}^B, \mathbf{p}^{B,2nd} \in [k\alpha^{-1}]_0^r$, we define $I = J := \{j, j_X\}$, $J_X := \{j_X\}$, $J_V := \emptyset$ and a tuple $(\omega_\iota, J, J_X, J_V, \{v,w\}, \mathscr{p})$ with $\omega_\iota\colon \mathscr{X}_{\sqcup I} \to \{0,1\}$, $\mathscr{p} := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$, by

$$\omega_\iota(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E_{\mathcal{H}}(\mathscr{p}, I, J), \mathscr{x} \subseteq V(H)\}, \tag{5.6.8}$$

and we add this tuple to $\mathcal{W}_{initial}$.

We now define a set $\mathcal{W}_{edge}^q = \mathcal{W}_{edge}^q(\phi_q, \phi_q^+, \mathcal{A}^q, \mathcal{B}^q)$ of edge testers with respect to the elements in $\mathcal{W}_{initial}$. That is, for every $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}_{initial}$, let $(\omega_q, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ be the edge tester with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$, (ϕ_q, ϕ_q^+), $\mathcal{A}^q$ and $\mathcal{B}^q$ as in Definition 5.15, and we add $(\omega_q, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p})$ to $\mathcal{W}_{edge}^q$.

Step 4. Induction

We inductively prove that the following statement $\mathbf{S}(q)$ holds for all $q \in [r]_0$, which will provide a partial packing of $\mathcal{H}$ into G_A.

$\mathbf{S}(q)$. For all $H \in \mathcal{H}$, there exists a q-partial packing $\phi_q\colon \mathcal{X}_q^{\phi_q} \to \mathcal{V}_q$ with $|\mathcal{X}_q^{\phi_q} \cap X_i^H| \geq (1 - \varepsilon_{c_i(q)})n$ for all $i \in [q]$, and with a cluster-injective extension ϕ_q^+ of ϕ_q, and for all index sets $I_A \subseteq [r] \setminus [q]$, $I_B \in [r]$, and $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$, there exist subgraphs $Z_{I_Z}^{H,q}$ of the candidacy graphs $Z_{I_Z}^H(\phi_q^+)$ with respect to ϕ_q^+ and G_Z (where $\mathcal{Z}_{I_Z}^q := \bigcup_{H \in \mathcal{H}} Z_{I_Z}^{H,q}$ and $\mathcal{Z}^q$ is the collection of all $\mathcal{Z}_{I_Z}^q$) such that

(a) $Z_{i_Z}^{H,q}$ is $(\varepsilon_{c_{i_Z}(q)}, d_Z^{m_{i_Z}(q)})$-super-regular and $(\varepsilon_{c_{i_Z}(q)}, c_{i_Z}(q)t^{1/2})$-well-intersecting with respect to G_Z for all $i_A \in [r] \setminus [q]$, $i_B \in [r]$ and $Z \in \{A, B\}$;

(b) for every edge tester $(\omega_q, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}_{edge}^q(\phi_q, \mathcal{A}^q, \mathcal{B}^q)$ with centres $\mathscr{c} \in V_{\sqcup I}$ for $I \subseteq [r]$, non-empty $I_q := (I \setminus [q]) \cup J$, patterns $\mathscr{p} =$

$(\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$, and with $J_{XV} := J_X \cup J_V$, we have that

$$\omega_q\left(E(\mathcal{A}^q_{I_q\setminus J}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}^q_j)\right) = \left(\mathbb{1}\{J_{XV}\cap[q]=\emptyset\} \pm \varepsilon_{c_{I_q}(q)}\right) \prod_{Z\in\{A,B\}} d_Z^{\|\mathbf{p}^Z_{[q]}\|-\|\mathbf{p}^{Z,2nd}_{[q]}\|}$$
$$\prod_{i\in I_q\setminus J} d_A^{m_i(q)} \prod_{j\in J} d_B^{m_j(q)} \frac{\omega_\iota(\mathscr{X}_{\sqcup I})}{n^{|(I\cap[q])\setminus J|}} \pm n^{\varepsilon_{c_{I_q}(q)}};$$

(c) for all $\mathscr{g} = \{v_{i_1},\dots,v_{i_k}\}, \mathscr{h} = \{w_{j_1},\dots,w_{j_k}\} \in E(G_A)$ with $v_{i_k} = w_{j_k}$, $I := \{i_1,\dots,i_k\} \neq \{j_1,\dots,j_k\} =: J$, and $\varepsilon^* := \max\{\varepsilon_{c_{I\setminus[q]}(q)}, \varepsilon_{c_{J\setminus[q]}(q)}\}$, we have that
$$\left|E_{\mathscr{g},\mathscr{h},\phi_q}(\mathcal{A}^q)\right| \leq \max\left\{n^{k-|(I\cup J)\cap[q]|+\varepsilon^*}, n^{\varepsilon^*}\right\};$$

(d) for all $(\omega, \mathscr{c}) \in \mathcal{W}_{hit}$ with $\mathscr{c} \in V_{\sqcup I}$ and all non-empty $J \subseteq I$, we have that
$$\omega\left(\{\mathscr{x} \in \mathscr{X}_{\sqcup I} \colon \mathscr{x} \subseteq \mathscr{e} \in E(\mathcal{H}), \phi_q^+(\mathscr{x} \cap \mathscr{X}_{\cup(J\cap[q])}) \subseteq \mathscr{c}\}\right)$$
$$\leq \omega(\mathscr{X}_{\sqcup I})/n^{|J\cap[q]|-\varepsilon_{c_{J\setminus[q]}(q)}} + n^{\varepsilon_{c_{J\setminus[q]}(q)}};$$

(e) $\left|(V_i \setminus \phi_q(X_i^H)) \cap N_{G_B}(\mathcal{S})\right| \leq \varepsilon_T \left|V_i \cap N_{G_B}(\mathcal{S})\right|$ for all $H \in \mathcal{H}$, $i \in [q]$ and $\mathcal{S} \subseteq \bigcup_{\mathscr{r}\in E(R)\colon i\in\mathscr{r}} V_{\sqcup\mathscr{r}\setminus\{i\}}$ with $|\mathcal{S}| \leq t$;

(f) $|W \cap \bigcap_{j\in[\ell]} \phi_q(Y_j)| = |W||Y_1|\cdots|Y_\ell|/n^\ell \pm \alpha^2 n$ for all $(W, Y_1, \dots, Y_\ell) \in \mathcal{W}_{set}$ with $W \subseteq V_i$, $i \in [q]$;

(g) $\omega(\phi_q^{-1}(\mathscr{c})) = (1 \pm \varepsilon_T)\omega(\mathscr{X}_{\sqcup I})/n^{|I|} \pm n^{\varepsilon_T}$ for all $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ with centres $\mathscr{c} \in V_{\sqcup I}$ for $I \subseteq [q]$.

Let us first explain $\mathbf{S}(q)$(a)–(g). Properties $\mathbf{S}(q)$(a)–(c) are used to establish $\mathbf{S}(q+1)$ by applying Lemma 5.18. In particular, these properties will imply that the assumptions (P2), (P3) and (P5) are satisfied, respectively, in order to apply Lemma 5.18. Properties $\mathbf{S}(q)$(b) and (e) can be used to control the leftover for the completion in Step 5. Properties $\mathbf{S}(r)$(f) and (g) will imply the conclusions (ii) and (iii) of Lemma 5.19 as we merely modify the r-partial packing ϕ_r during the completion in Step 5.

We now inductively prove that $\mathbf{S}(q)$ holds for all $q \in [r]_0$. The statement $\mathbf{S}(0)$ holds for ϕ_0 and ϕ_0^+ being the empty function and $Z_{I_Z}^{H,0}$ being complete multipartite $2|I_Z|$-graphs: Clearly, for all $Z \in \{A, B\}$, and all index sets $I_A \subseteq [r]$ and $I_B \in [r]$, the candidacy graph $Z_{I_Z}^H(\phi_0^+)$ is complete $2|I_Z|$-partite. For $\mathbf{S}(0)$(b), consider an edge tester $(\omega, \omega_\iota, J, J_X, J_X, \mathscr{c}, \mathscr{p}) \in \mathcal{W}^0_{edge}$ with centres $\mathscr{c} \in V_{\sqcup I}$ and note that, by the definition of an edge tester (see Definition 5.15), we have $\omega(E(\mathcal{A}_{I\setminus J}) \sqcup \bigsqcup_{j\in J} E(\mathcal{B}_j)) = \omega_\iota(\mathscr{X}_{\sqcup I})$. For $\mathbf{S}(0)$(c), we observe that for all $\mathscr{g} = \{v_{i_1},\dots,v_{i_k}\}, \mathscr{h} = \{w_{j_1},\dots,w_{j_k}\} \in E(G_A)$ with $v_{i_k} = w_{j_k}$, we have that $|E_{\mathscr{g},\mathscr{h},\phi_0}(\mathcal{A}^0)| \leq \alpha^{-1} e_{\mathcal{H}}(\mathscr{X}_{i_1},\dots,\mathscr{X}_{i_k}) \leq \alpha^{-1}n^k \leq n^{k+\varepsilon_0}$ because $e_{\mathcal{H}}(\mathscr{X}_{\sqcup\mathscr{r}}) \leq (1-\alpha)dn^k$ and $\Delta(H) \leq \alpha^{-1}$ for each $H \in \mathcal{H}$ by assumption. (Recall Definition 5.17 of $E_{\mathscr{g},\mathscr{h},\phi_0}(\mathcal{A}^0)$.) $\mathbf{S}(0)$(d)–(g) are vacuously true.

Hence, we assume the truth of $\mathbf{S}(q)$ for some $q \in [r-1]_0$ and let $\phi_q \colon \mathscr{X}_q^{\phi_q} \to \mathcal{V}_q$, ϕ_q^+, and $\mathcal{A}^q$ and $\mathcal{B}^q$ be as in $\mathbf{S}(q)$. Any function $\sigma \colon \mathscr{X}^\sigma_{q+1} \to V_{q+1}$ with $\mathscr{X}^\sigma_{q+1} \subseteq \mathscr{X}_{q+1}$ naturally extends ϕ_q to a function $\phi_{q+1} := \phi_q \cup \sigma$ with $\phi_{q+1} \colon \mathscr{X}_q^{\phi_q} \cup \mathscr{X}^\sigma_{q+1} \to \mathcal{V}_{q+1}$.

We now make a key observation based on the Definition 5.11 of candidacy graphs: By definition of the candidacy graphs $\mathcal{A}^q_{q+1} = \bigcup_{H\in\mathcal{H}} A^{H,q}_{q+1}$ where $A^{H,q}_{q+1} \subseteq A^H_{q+1}(\phi_q^+)$, if σ is a conflict-free packing in $\mathcal{A}^q_{q+1}$ as defined in (5.5.3), then ϕ_{q+1} is a $(q+1)$-partial packing. See also Figure 5.4 in Section 5.5.1.

We aim to apply Lemma 5.18 in order to obtain a conflict-free packing σ in $\mathcal{A}^q_{q+1}$. To this end, we consider subgraphs $\mathcal{H}_{q+1}, G_{A,q+1}, G_{B,q+1}, R_{q+1}$ of $\mathcal{H}, G_A, G_B, R$, respectively, that consist only of the 'relevant' clusters when finding a conflict-free packing in $\mathcal{A}^q_{q+1}$. Note that all relevant clusters lie in $N_{R^3_*}[q+1]$. That is, by considering all clusters in $N_{R^3_*}[q+1]$, we also account for hyperedges $\not r \in E(R)$ and all R-edges that intersect $\not r$ where $q+1 \notin \not r$ but $\not r \cap \not r_{q+1} \neq \emptyset$ for some $\not r_{q+1} \in E(R)$ with $q+1 \in \not r_{q+1}$. Let $\mathcal{Q} := [q] \cap N_{R^3_*}(q+1)$ and for each $Z \in \{A, B\}$, let

$$\begin{aligned}\mathcal{H}_{q+1} &:= \bigcup_{H\in\mathcal{H}} H\left[\bigcup\nolimits_{i\in N_{R^3_*}[q+1]} X^H_i\right];\\ G_{Z,q+1} &:= G_Z\left[\bigcup\nolimits_{i\in N_{R^3_*}[q+1]} V_i\right];\\ R_{q+1} &:= R\left[N_{R^3_*}[q+1]\right].\end{aligned}$$

Correspondingly, for $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$, we also define a subset $\mathcal{Z}^q[R_{q+1}]$ of $\mathcal{Z}^q$. Let $\mathcal{Z}^q[R_{q+1}]$ be the collection of all candidacy graphs $\mathcal{Z}^q_{I_Z}$ for index sets $I_A \subseteq N_{R^3_*}[q+1] \setminus [q]$, $I_B \in N_{R^3_*}[q+1]$.

Following the definition of a packing instance in Section 5.5.1, we observe that

$$\mathscr{P} := (\mathcal{H}_{q+1}, G_{A,q+1}, G_{B,q+1}, R_{q+1}, \mathcal{A}^q[R_{q+1}], \mathcal{B}^q[R_{q+1}], \phi_q|_{\mathscr{X}_{\cup\mathcal{Q}}}, \phi^+_q|_{\mathscr{X}_{\cup\mathcal{Q}}})$$

is a packing instance of size

$$(n, k, |N_{R^3_*}(q+1) \setminus [q]|, |\mathcal{Q}|).$$

Further, we claim that $\mathscr{P}$ is an $(\varepsilon_{c(q+1)-1}, (c(q+1)-1)t^{1/2}, t, \mathbf{d})$-packing instance with suitable edge testers $\mathcal{W}^q_{edge}$, where

$$\mathbf{d} = (d_A, d_B, (d_A^{m_i(q)})_{i\in N_{R^3_*}[q+1]\setminus[q]}, (d_B^{m_i(q)})_{i\in N_{R^3_*}[q+1]}).$$

To establish this claim, we first make some important observations. By the definition of $c_i(q)$ in (5.6.1) and $c_I(q)$ in (5.6.2), we have:

(5.6.9)
If $i \in N_{R_*}(q+1)$, then $c(q+1) = c_i(q+1) > c_I(q)$ for every index set $I \subseteq [r]$ with $i \in I$.

Note that for the inequality in (5.6.9) we used that an index set I is contained in some hyperedge $\not r \in E(R)$, and no vertex of a hyperedge $\not r$ in R has two neighbours in R_* that are coloured alike as we have chosen the vertex colouring as a colouring in R^3_*. In particular, we infer from (5.6.9) that

(5.6.10)
for all $i \in N_{R_*}(q+1)$, we have $\varepsilon_{c(q+1)-1} = \varepsilon_{c_i(q+1)-1} \geq \varepsilon_{c_i(q)}$ and $\varepsilon_{c(q+1)} = \varepsilon_{c_i(q+1)}$. For every index set $I \subseteq [r] \setminus [q+1]$ with $I \cap \not r \neq \emptyset$ for some $\not r \in E(R)$ and $q+1 \in \not r$, we have $\varepsilon_{c(q+1)-1} = \varepsilon_{c_I(q+1)-1} \geq \varepsilon_{c_I(q)}$ and $\varepsilon_{c(q+1)} = \varepsilon_{c_I(q+1)}$.

Similar, by the definition of $m_i(q)$ in (5.6.3), we have:

(5.6.11)
If $i \in N_{R_*}(q+1)$, then

$$m_i(q+1) = m_i(q) + \left|\left\{\not r \in E(R)\colon \{q+1, i\} = \not r \cap \big(([r] \setminus [q]) \cup \{i\}\big)\right\}\right|.$$

If $i \in [r] \setminus N_{R_*}(q+1)$, then $m_i(q+1) = m_i(q)$.

Hence to see that $\mathscr{P}$ is an $(\varepsilon_{c(q+1)-1}, (c(q+1)-1)t^{1/2}, t, \mathbf{d})$-packing instance, note that (P1) follows from (5.6.4), property (P2) follows from $\mathbf{S}(q)$(a), property (P3) follows from $\mathbf{S}(q)$(b), property (P4) holds by the definition of the edge testers in (5.6.6), and (P5) follows from $\mathbf{S}(q)$(c).

Observe further that by assumption, we have $|\mathcal{H}| \leq n^{2k}$, and $e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) \leq (1-\alpha)dn^k \leq d_A n^k$ for all $\mathscr{r} \in E(R_{q+1})$. Hence, we can apply Lemma 5.18 to $\mathscr{P}$ with

parameter	n	$\varepsilon_{c(q+1)-1}$	$\varepsilon_{c(q+1)}$	$(c(q+1)-1)t^{1/2}$	α^{-1}	$\lvert N_{R^3_*}(q+1) \setminus [q]\rvert$	$\lvert\mathcal{Q}\rvert$
replaces	n	ε	ε'	q	s	r	$r_\circ$

and with

- local α^{-1}-testers in $\mathcal{W}_{local}$ that we will define explicitly in Steps 4.3–4.7 when establishing $\mathbf{S}(q+1)$(c)–(g);
- tuples (ω, c) in $\mathcal{W}_0$ that we will define explicitly in Step 4.4 when establishing $\mathbf{S}(q+1)$(d);
- edge testers in $\mathcal{W}^q_{edge}$.

Let $\sigma\colon \mathscr{X}^\sigma_{q+1} \to V_{q+1}$ be the conflict-free packing in $\mathcal{A}^q_{q+1}$ obtained from Lemma 5.18 with $|\mathscr{X}^\sigma_{q+1} \cap X^H_{q+1}| \geq (1-\varepsilon_{c(q+1)})n$ for all $H \in \mathcal{H}$, which extends ϕ_q to $\phi_{q+1} = \phi_q \cup \sigma$ with $\phi_{q+1}\colon \mathcal{X}^{\phi_q}_q \cup \mathscr{X}^\sigma_{q+1} \to \mathcal{V}_{q+1}$, and let $M = M(\sigma)$ be the corresponding edge set to σ defined as in (5.5.1). Further, let σ^+ be the cluster-injective extension of σ obtained from Lemma 5.18. Analogously, σ^+ extends ϕ^+_q to a cluster-injective extension $\phi^+_{q+1} := \phi^+_q \cup \sigma^+$ of ϕ_{q+1} with $\phi^+_{q+1}\colon \mathcal{X}_{q+1} \to \mathcal{V}_{q+1}$.

For all $H \in \mathcal{H}$, $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$, and all index sets $I_A \subseteq [r] \setminus [q+1]$, $I_B \in [r]$ with $I_Z \cap \mathscr{r} \neq \emptyset$ for some $\mathscr{r} \in E(R)$ with $q+1 \in \mathscr{r}$, let $Z^{H,q+1}_{I_Z} \subseteq Z^H_{I_Z}(\phi^+_q|_{\mathscr{X}_{\cup \mathcal{Q}}} \cup \sigma^+) = Z^H_{I_Z}(\phi^+_q \cup \sigma^+)$ be the candidacy graph $Z^{H,new}_{I_Z} =: Z^{H,q+1}_{I_Z}$ obtained from Lemma 5.18 satisfying $(\mathrm{I})_{\mathrm{L5.18}}$–$(\mathrm{III})_{\mathrm{L5.18}}$.

For all $H \in \mathcal{H}$, $(Z, \mathcal{Z}) \in \{(A, \mathcal{A}), (B, \mathcal{B})\}$, and all index sets $I_A \subseteq [r] \setminus [q+1]$, $I_B \in [r]$ with $I_Z \cap \mathscr{r} = \emptyset$ for all $\mathscr{r} \in E(R)$ with $q+1 \in \mathscr{r}$, note that $m_i(q) = m_i(q+1)$ for all $i \in I_Z$ and $Z^H_{I_Z}(\phi^+_q) = Z^H_{I_Z}(\phi^+_{q+1})$. Thus, in such a case we set $Z^{H,q+1}_{I_Z} := Z^{H,q}_{I_Z}$. Let $\mathcal{Z}^{q+1}_{I_Z} := \bigcup_{H \in \mathcal{H}} Z^{H,q+1}_{I_Z}$ and let $\mathcal{Z}^{q+1}$ be the collection of all $\mathcal{Z}^{q+1}_{I_Z}$ for all index sets $I_A \subseteq [r] \setminus [q+1]$, $I_B \in [r]$. We will employ Lemma 5.18$(\mathrm{I})_{\mathrm{L5.18}}$–$(\mathrm{III})_{\mathrm{L5.18}}$ to establish $\mathbf{S}(q+1)$(a)–(g).

<u>*Step 4.1.*</u> *Checking* $\mathbf{S}(q+1)$(a)

We fix some $H \in \mathcal{H}$, $Z \in \{A, B\}$ and establish $\mathbf{S}(q)$(a) for the candidacy graph $Z^{H,q+1}_{i_Z}$. For all $i_A \in N_{R_*}(q+1) \setminus [q]$ and $i_B \in N_{R_*}(q+1)$, we have by our observation (5.6.11) for $m_{i_Z}(q+1)$ that $d_A^{m_{i_Z}(q+1)} = d^{new}_{Z,i_Z}$ for d^{new}_{Z,i_Z} in $(\mathrm{I})_{\mathrm{L5.18}}$ as defined in (5.5.4). Hence with $(\mathrm{I})_{\mathrm{L5.18}}$, (5.6.9) and (5.6.11), we obtain that the candidacy graph $Z^{H,q+1}_{i_Z}$ is $(\varepsilon_{c_{i_Z}(q+1)}, d_Z^{m_{i_Z}(q+1)})$-super-regular and $(\varepsilon_{c_{i_Z}(q+1)}, c_{i_Z}(q+1)t^{1/2})$-well-intersecting for all $i_A \in N_{R_*}(q+1) \setminus [q]$ and $i_B \in N_{R_*}(q+1)$. For all $i_A \in [r] \setminus (N_{R_*}(q+1) \cup [q+1])$ and $i_B \in [r] \setminus N_{R_*}[q+1]$, we have $m_{i_Z}(q) = m_{i_Z}(q+1)$ and $Z^{H,q+1}_{i_Z} = Z^{H,q}_{i_Z}$. Hence with $\mathbf{S}(q)$(a), we also obtain in this case that $Z^{H,q+1}_{i_Z}$ is $(\varepsilon_{c_{i_Z}(q+1)}, d_Z^{m_{i_Z}(q+1)})$-super-regular and $(\varepsilon_{c_{i_Z}(q+1)}, c_{i_Z}(q+1)t^{1/2})$-well-intersecting. This establishes $\mathbf{S}(q+1)$(a).

<u>*Step 4.2.*</u> *Checking* $\mathbf{S}(q+1)$(b)

In order to establish $\mathbf{S}(q+1)$(b), we fix an edge tester $(\omega_{q+1}, \omega_\iota, J, J_X, J_V, \mathscr{c}, \mathscr{p}) \in \mathcal{W}^{q+1}_{edge}(\phi_{q+1}, \phi^+_{q+1}, \mathcal{A}^{q+1}, \mathcal{B}^{q+1})$ with centres $\mathscr{c} \in V_{\sqcup I}$ for $I \subseteq [r]$, non-empty $I_{q+1} := (I \setminus [q+1]) \cup J$, and patterns $\mathscr{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$.

If $I_{q+1} \cap r = \emptyset$ for all $r \in E(R)$ with $q+1 \in r$, then the conflict-free packing σ does not have an effect at all on the considered edge tester; that is, $(\omega_q, \omega_\iota, J, J_X, J_V, c, p) = (\omega_{q+1}, \omega_\iota, J, J_X, J_V, c, p)$ by Definition 5.15, and thus, $\mathbf{S}(q+1)$(b) holds by $\mathbf{S}(q)$(b). In this case, note in particular that if p is such that $\omega_\iota(\mathscr{X}_{\sqcup I}) > 0$, then $\|\mathbf{p}^Z_{[q]}\| = \|\mathbf{p}^Z_{[q+1]}\|$ and $\|\mathbf{p}^{Z,2nd}_{[q]}\| = \|\mathbf{p}^{Z,2nd}_{[q+1]}\|$ for all $Z \in \{A, B\}$ because by Definition 5.12 of a 1^{st}-pattern and 2^{nd}-pattern we have for the $(q+1)$th entries that $\mathbf{p}^Z_{q+1} = \mathbf{p}^{Z,2nd}_{q+1} = 0$ as $I_{q+1} \cap r = \emptyset$ for all $r \in E(R)$ with $q+1 \in r$.

Hence, we may assume that $I_{q+1} \cap r \neq \emptyset$ for some $r \in E(R)$ with $q+1 \in r$. It is important to note that the edge tester $(\omega_{q+1}, \omega_\iota, J, J_X, J_V, c, p) \in \mathcal{W}^{q+1}_{edge}(\phi_{q+1}, \phi^+_{q+1}, \mathcal{A}^{q+1}, \mathcal{B}^{q+1})$ is defined in the same way as the edge tester $(\omega^{new}, \omega_\iota, J, J_X, J_V, c, p)$ that we obtain from $(\mathrm{II})_{\mathrm{L5.18}}$, and thus, they are identical. As in Step 4.1, for $i \in I_{q+1}$, we have by our observation (5.6.11) for $m_i(q+1)$ that $d_Z^{m_i(q+1)} = d^{new}_{i,Z}$ for $Z \in \{A, B\}$ and $d^{new}_{i,Z}$ in $(\mathrm{I})_{\mathrm{L5.18}}$ as defined in (5.5.4). Hence with (5.6.10), (5.6.11) and $(\mathrm{II})_{\mathrm{L5.18}}$, we obtain that the edge tester $(\omega_{q+1}, \omega_\iota, J, J_X, J_V, c, p) \in \mathcal{W}^{q+1}_{edge}(\phi_{q+1}, \phi^+_{q+1}, \mathcal{A}^{q+1}, \mathcal{B}^{q+1})$ with respect to $(\omega_\iota, J, J_X, J_V, c, p)$, ϕ_{q+1}, $\mathcal{A}^{q+1}$ and $\mathcal{B}^{q+1}$ satisfies $\mathbf{S}(q+1)$(b).

Step 4.3. Checking $\mathbf{S}(q+1)$(c)

In order to establish $\mathbf{S}(q+1)$(c), we fix $g = \{v_{i_1}, \ldots, v_{i_k}\}, h = \{w_{j_1}, \ldots, w_{j_k}\} \in E(G)$ with $v_{i_k} = w_{j_k}$, $I := \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} =: J$, and $\varepsilon^*_q := \max\{\varepsilon_{c_{I\setminus[q]}(q)}, \varepsilon_{c_{J\setminus[q]}(q)}\}$.

If $q+1 \notin I \cup J$, we have $|(I \cup J) \cap [q]| = |(I \cup J) \cap [q+1]|$, and thus $\mathbf{S}(q+1)$(c) holds by $\mathbf{S}(q)$(c).

Hence, by symmetry, we may assume that $q+1 = i_\ell$ for some $\ell \in [k]$. Our strategy is to define a weight function that bounds from above the number of elements in $E_{g,h,\phi_q}(\mathcal{A}^q)$ that still can be present in $E_{g,h,\phi_{q+1}}(\mathcal{A}^{q+1})$ by employing $(\mathrm{III})_{\mathrm{L5.18}}$. For $(e, f) \in E_{g,h,\phi_q}(\mathcal{A}^q)$, let $\{x_{i_\ell}\} := e \cap \mathscr{X}_{i_\ell}$, and we define a weight function $\omega_{e,f}\colon E(\mathcal{A}^q_{q+1}) \to \{0,1\}$ by $\omega_{e,f}(xv) := \mathbb{1}_{\{xv = x_{i_\ell} v_{i_\ell}\}}$. Note that $(e, f) \in E_{g,h,\phi_{q+1}}(\mathcal{A}^{q+1})$ only if $\omega_{e,f}(M) = 1$ as it is necessary that σ embeds x_{i_ℓ} onto v_{i_ℓ}. Let $\omega_{g,h} := \sum_{(e,f)\in E_{g,h,\phi_q}(\mathcal{A}^q)} \omega_{e,f}$. A key observation is that

$$\left|E_{g,h,\phi_{q+1}}(\mathcal{A}^{q+1})\right| \leq \omega_{g,h}(M). \tag{5.6.12}$$

By the definition of $\omega_{g,h}$, we have that

$$\omega_{g,h}(E(\mathcal{A}^q_{q+1})) = |E_{g,h,\phi_q}(\mathcal{A}^q)|. \tag{5.6.13}$$

By adding $\omega_{g,h}$ to $\mathcal{W}_{local}$ and by employing $(\mathrm{III})_{\mathrm{L5.18}}$, we obtain with (5.6.13) that

$$\omega_{g,h}(M) = (1 \pm \varepsilon^2_{c(q+1)})(d_A^{m_{q+1}(q)} n)^{-1} |E_{g,h,\phi_q}(\mathcal{A}^q)| \pm n^{\varepsilon_{c(q+1)-1}}. \tag{5.6.14}$$

We further observe that

$$|E_{g,h,\phi_q}(\mathcal{A}^q)| \overset{\mathbf{S}(q)(\mathrm{c})}{\leq} \max\left\{n^{k-|(I\cup J)\cap[q]|+\varepsilon^*_q}, n^{\varepsilon^*_q}\right\},$$

and thus by (5.6.10), (5.6.14) and because $d \geq n^{-\varepsilon}$, we finally obtain

$$\omega_{g,h}(M) \leq \max\left\{n^{k-|(I\cup J)\cap[q]|-1+\varepsilon_{c(q+1)}}, n^{\varepsilon_{c(q+1)}}\right\}.$$

By our key observation (5.6.12), this establishes $\mathbf{S}(q+1)$(c).

Step 4.4. Checking $\mathbf{S}(q+1)$(d)

In order to establish $\mathbf{S}(q+1)$(d), we fix $(\omega, c) \in \mathcal{W}_{hit}$ with $c \in V_{\sqcup I}$ and $J \subseteq I$. In view of the statement, we may assume that $q+1 \in J$, otherwise $\mathbf{S}(q+1)$(d) holds by

$\mathbf{S}(q)$(d). Our general strategy is to define two weight functions and employ (III)$_{\text{L5.18}}$ and (IV)$_{\text{L5.18}}$ to derive the desired upper bound.

For $p \in \{q, q+1\}$, let $W_p := \{\boldsymbol{x} \in \mathscr{X}_{\sqcup I} \colon \boldsymbol{x} \subseteq \boldsymbol{e}$ for some $\boldsymbol{e} \in E(\mathcal{H})$, $\phi_p^+(\boldsymbol{x} \cap \mathscr{X}_{\cup(J\cap[p])}) \subseteq \boldsymbol{e}\}$, and let $c := \boldsymbol{e} \cap V_{q+1}$. We define a local tester $\omega_\sigma \colon E(\mathcal{A}^{q+1}) \to [0, \alpha^{-1}]$ by

$$\omega_\sigma(uv) := \sum_{x \in W_q \colon u \in x} \mathbb{1}\{v = c\}\omega(\boldsymbol{x}),$$

and we add ω_σ to $\mathcal{W}_{local}$. We also define a tuple (ω_{σ^+}, c) with $\omega_{\sigma^+} \colon \mathscr{X}_{q+1} \to [0, \alpha^{-1}]$ by

$$\omega_{\sigma^+}(u) := \sum_{x \in W_q \colon u \in x} \omega(\boldsymbol{x}),$$

and we add (ω_{σ^+}, c) to $\mathcal{W}_0$. We make the following observation

$$\omega(W_{q+1}) = \omega_\sigma(M) + \omega_{\sigma^+}(\{x \in \mathscr{X}_{q+1} \setminus \mathscr{X}^\sigma_{q+1} \colon \sigma^+(x) = c\}). \tag{5.6.15}$$

We first employ (III)$_{\text{L5.18}}$ to derive an upper bound on $\omega_\sigma(M)$

$$\begin{aligned}
\omega_\sigma(M) &\overset{\text{(III)}_{\text{L5.18}}}{\leq} 2(d_A^{m_{q+1}(q)} n)^{-1}\omega_\sigma(E(\mathcal{A}^{q+1})) + n^{\varepsilon_{c(q+1)-1}} \\
&\overset{\mathbf{S}(q)(\text{d})}{\leq} 2(d_A^{m_{q+1}(q)} n)^{-1}\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|J\cap[q]|-\varepsilon_{c(q+1)-1}}} + 2n^{\varepsilon_{c(q+1)-1}} \\
&\leq \frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|J\cap[q+1]|-\varepsilon^{1/2}_{c(q+1)-1}}} + 2n^{\varepsilon_{c(q+1)-1}}.
\end{aligned} \tag{5.6.16}$$

Next, we employ (IV)$_{\text{L5.18}}$ to derive an upper bound for the last term of (5.6.15)

$$\begin{aligned}
\omega_{\sigma^+}(\{x \in \mathscr{X}_{q+1} \setminus \mathscr{X}^\sigma_{q+1} \colon \sigma^+(x) = c\}) &\overset{\text{(IV)}_{\text{L5.18}}}{\leq} \omega_{\sigma^+}(\mathscr{X}_{q+1})/n^{1-\varepsilon_{c(q+1)-1}} + n^{\varepsilon_{c(q+1)-1}} \\
&\overset{\mathbf{S}(q)(\text{d})}{\leq} \frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|J\cap[q+1]|-2\varepsilon_{c(q+1)-1}}} + 2n^{\varepsilon_{c(q+1)-1}}.
\end{aligned} \tag{5.6.17}$$

Finally, plugging (5.6.16) and (5.6.17) into (5.6.15), yields that

$$\omega(W_{q+1}) \leq \frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|J\cap[q+1]|-\varepsilon_{c(q+1)}}} + n^{\varepsilon_{c(q+1)}}.$$

Together with (5.6.10), this establishes $\mathbf{S}(q+1)$(d).

<u>*Step 4.5.*</u> *Checking* $\mathbf{S}(q+1)$(e)

In order to establish $\mathbf{S}(q+1)$(e), we fix $H \in \mathcal{H}$ and $\mathcal{S} \subseteq \bigcup_{\boldsymbol{r} \in E(R) \colon q+1 \in \boldsymbol{r}} V_{\sqcup \boldsymbol{r} \setminus \{q+1\}}$ with $|\mathcal{S}| \leq t$. Let $W := V_{q+1} \cap N_{G_B}(\mathcal{S})$. Our general strategy is to define a weight function that estimates the number of vertices in $W \cap \sigma(X^H_{q+1})$ from which we can derive an upper bound for $|W \setminus \sigma(X^H_{q+1})|$.

Hence, let $\omega_W \colon E(\mathcal{A}^q_{q+1}) \to \{0, 1\}$ be defined by $\omega_W(e) := \mathbb{1}\{w \in e, w \in W\}$. A key observation is that

$$|W \setminus \sigma(X^H_{q+1})| \leq |W| - \omega_W(M). \tag{5.6.18}$$

By the definition of ω_W and because $A^{H,q}_{q+1}$ is $(\varepsilon_{c_{q+1}(q)}, d_A^{m_{q+1}(q)})$-super-regular for every $H \in \mathcal{H}$ by $\mathbf{S}(q)$(a), we have that

$$\omega_W(E(\mathcal{A}^q_{q+1})) = (1 \pm 3\varepsilon_{c_{q+1}(q)}) d_A^{m_{q+1}(q)} n|W|. \tag{5.6.19}$$

By adding ω_W to $\mathcal{W}_{local}$ and by employing $(\mathrm{III})_{\mathrm{L5.18}}$, we obtain with (5.6.19) that

$$\omega_W(M) \overset{(\mathrm{III})_{\mathrm{L5.18}}}{=} (1 \pm \varepsilon^2_{c(q+1)}) \frac{\omega_W(E(\mathcal{A}^q_{q+1}))}{d_A^{m_{q+1}(q)} n} \pm n^{\varepsilon_{c_{q+1}(q)}} \overset{(5.6.19)}{=} (1 \pm \varepsilon_T)|W|.$$

Now, this together with (5.6.18) implies that $|W \setminus \sigma(X^H_{q+1})| \leq \varepsilon_T |W|$. Together with $\mathbf{S}(q)$(e) this establishes $\mathbf{S}(q+1)$(e).

<u>*Step 4.6.*</u> *Checking* $\mathbf{S}(q+1)$(f)

Let $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{set}$ be a set tester with $W \subseteq V_{q+1}$ and $Y_j \subseteq X^{H_j}_{q+1}$ for all $j \in [\ell]$. We define

$$E_{(W,Y_1,\ldots,Y_\ell)} := \left\{ \{e_1, \ldots, e_\ell\} \in \bigsqcup_{j \in [\ell]} E\left(A^{H_j,q}_{q+1}[W, Y_j]\right) : \bigcap_{j\in[\ell]} e_j \neq \emptyset \right\}$$

and a weight function $\omega_{(W,Y_1,\ldots,Y_\ell)} \colon \binom{E(\mathcal{A}^q_{q+1})}{\ell} \to \{0,1\}$ by $\omega_{(W,Y_1,\ldots,Y_\ell)}(\mathbf{e}) := \mathbb{1}\{\mathbf{e} \in E_{(W,Y_1,\ldots,Y_\ell)}\}$. Note that

$$\omega_{(W,Y_1,\ldots,Y_\ell)}(M) = \left|W \cap \textstyle\bigcap_{j\in[\ell]} \sigma(Y_j)\right|. \tag{5.6.20}$$

In view of the statement, we may assume that $|W|, |Y_j| \geq \varepsilon_{c(q+1)} n$ for all $j \in [\ell]$. Since $A^{H,q}_{q+1}$ is $(\varepsilon_{c_{q+1}(q)}, d_A^{m_{q+1}(q)})$-super-regular for every $H \in \mathcal{H}$ by $\mathbf{S}(q)$(a), we obtain by Fact 1.11 and because $\ell \leq \alpha^{-1}$ that there are at most $\varepsilon^{1/2}_{c_{q+1}(q)} n$ vertices in W that do not have $(1 \pm \varepsilon_{c_{q+1}(q)}) d_A^{m_{q+1}(q)} |Y_j|$ neighbours in Y_j for every $j \in [\ell]$. Hence, we obtain that

$$\omega_{(W,Y_1,\ldots,Y_\ell)}(E(\mathcal{A}^q_{q+1})) = |E_{(W,Y_1,\ldots,Y_\ell)}| = (1 \pm \varepsilon_{c(q+1)})(d_A^{m_{q+1}(q)})^\ell |W||Y_1| \cdots |Y_\ell|. \tag{5.6.21}$$

We check that $\omega_{(W,Y_1,\ldots,Y_\ell)}$ is a local tester: For all $\{e_1, \ldots, e_{\ell'}\} \in \binom{E(\mathcal{A}^q_{q+1})}{\ell'}$, $\ell' \in [\ell]$, the number of edges $\{e_{\ell'+1}, \ldots, e_\ell\}$ such that $\mathbf{e} := \{e_j\}_{j\in[\ell]} \in \binom{E(\mathcal{A}^q_{q+1})}{\ell}$ with $\omega_{(W,Y_1,\ldots,Y_\ell)}(\mathbf{e})$ is at most $(2n)^{\ell-\ell'}$, implying that $\|\omega_{(W,Y_1,\ldots,Y_\ell)}\|_{\ell'} \leq n^{\ell-\ell'+\varepsilon^2_{c_{q+1}(q)}}$.

Hence, $\omega_{(W,Y_1,\ldots,Y_\ell)}$ is a local tester and we can add $\omega_{E(W,Y_1,\ldots,Y_\ell)}$ to $\mathcal{W}_{local}$. By $(\mathrm{III})_{\mathrm{L5.18}}$, we conclude that

$$\begin{aligned} \omega_{(W,Y_1,\ldots,Y_\ell)}(M) &= (1 \pm \varepsilon^2_{c(q+1)}) \frac{\omega_{(W,Y_1,\ldots,Y_\ell)}(E(\mathcal{A}^q_{q+1}))}{(d_A^{m_{q+1}(q)} n)^\ell} \pm n^{\varepsilon_{c(q+1)}} \\ &\overset{(5.6.21)}{=} \frac{|W||Y_1| \cdots |Y_\ell|}{n^\ell} \pm \alpha^2 n, \end{aligned}$$

which establishes $\mathbf{S}(q+1)$(f) by (5.6.20).

<u>*Step 4.7.*</u> *Checking* $\mathbf{S}(q+1)$(g)

In order to establish $\mathbf{S}(q+1)$(g), let $(\omega_{ver}, \mathscr{c}) \in \mathcal{W}_{ver}$ be an α^{-1}-vertex tester with centres $\mathscr{c} = \{c_i\}_{i\in I} \in V_{\sqcup I}$ where $I \subseteq [q+1]$ and $q+1 \in I$. By (5.6.5), we defined in particular for all $\mathbf{p}^A, \mathbf{p}^{A,2nd} \in [k\alpha^{-1}]^r_0$ and $\mathscr{p} := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{0}, \mathbf{0})$ a tuple $(\omega_\iota, J = \emptyset, J_X = \emptyset, J_V = \emptyset, \mathscr{c}, \mathscr{p})$ with initial weight function $\omega_\iota =: \omega_{\iota,\mathscr{p}}$ corresponding to $(\omega_{ver}, \mathscr{c})$. That is, by (5.6.5), we have

$$\omega_{ver}(\mathscr{X}_{\sqcup I}) = \sum_{\mathscr{p} \colon \mathbf{p}^A, \mathbf{p}^{A,2nd} \in [k\alpha^{-1}]^r_0} \omega_{\iota,\mathscr{p}}(\mathscr{X}_{\sqcup I}). \tag{5.6.22}$$

Note that $\omega_{\iota,\not p}(\mathscr{X}_{\sqcup I}) > 0$ only if $\|\mathbf{p}^A_{[q]}\| = \|\mathbf{p}^{A,2nd}_{[q]}\|$ by (5.4.5) since $|I| \leq k-1$. For $\mathbf{p}^A, \mathbf{p}^{A,2nd} \in [k\alpha^{-1}]^r_0$ and $\not p := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{0}, \mathbf{0})$ with $\|\mathbf{p}^A_{[q]}\| = \|\mathbf{p}^{A,2nd}_{[q]}\|$, let $\tau_{\not p} := (\omega_q, \omega_\iota, J = \emptyset, J_X = \emptyset, J_V = \emptyset, \not c, \not p)$ be the edge tester with respect to $(\omega_\iota, J = \emptyset, J_X = \emptyset, J_V = \emptyset, \not c, \not p)$, (ϕ_q, ϕ_q^+), $\mathcal{A}^q$ and $\mathcal{B}^q$, which is contained in $\mathcal{W}^q_{edge}$, and $\omega_q \colon E(\mathcal{A}^q_{q+1}) \to [0, \alpha^{-1}]$, and let $\omega_{q,\not p} := \omega_q$. By $\mathbf{S}(q)$(b), we obtain

$$\omega_{q,\not p}(E(\mathcal{A}^q_{q+1})) = (1 \pm \varepsilon_{c_{q+1}(q)}) d_A^{m_{q+1}(q)} \frac{\omega_{\iota,\not p}(\mathscr{X}_{\sqcup I})}{n^{|I|-1}} \pm n^{\varepsilon_{c_{q+1}(q)}}. \tag{5.6.23}$$

A key observation is that

$$\omega_{ver}(\phi^{-1}_{q+1}(\not c)) = \sum_{\not p \colon \mathbf{p}^A, \mathbf{p}^{A,2nd} \in [k\alpha^{-1}]^r_0} \omega_{q,\not p}(M), \tag{5.6.24}$$

which follows from the definition of the edge tester $\tau_{\not p}$ for $\not p = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{0}, \mathbf{0})$ as in Definition 5.15. Note that $\mathcal{A}^q_{q+1}$ is $(\varepsilon_{c_{q+1}(q)}, d_A^{m_{q+1}(q)})$-super-regular by $\mathbf{S}(q)$(a). For all $\mathbf{p}^A, \mathbf{p}^{A,2nd} \in [k\alpha^{-1}]^r_0$ with $\|\mathbf{p}^A_{[q]}\| = \|\mathbf{p}^{A,2nd}_{[q]}\|$ and $\not p := (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{0}, \mathbf{0})$, we add $\omega_{q,\not p} \colon E(\mathcal{A}^q_{q+1}) \to [0, \alpha^{-1}]$ to $\mathcal{W}_{local}$ and obtain by (III)$_{\text{L5.18}}$ that

$$\begin{aligned} \omega_{q,\not p}(M) &\overset{\text{(III)}_{\text{L5.18}}}{=} (1 \pm \varepsilon^2_{c(q+1)}) \frac{\omega_{q,\not p}(E(\mathcal{A}^q_{q+1}))}{d_A^{m_{q+1}(q)} n} \pm n^{\varepsilon_{c_{q+1}(q)}} \\ &\overset{(5.6.23)}{=} (1 \pm \varepsilon_{c(q+1)}) \frac{\omega_{\iota,\not p}(\mathscr{X}_{\sqcup I})}{n^{|I|}} \pm 2n^{\varepsilon_{c_{q+1}(q)}}. \end{aligned}$$

Together with (5.6.22) and (5.6.24), this implies that

$$\omega_{ver}(\phi^{-1}_{q+1}(\not c)) = (1 \pm \varepsilon_T)\omega_{ver}(\mathscr{X}_{\sqcup I})/n^{|I|} \pm n^{\varepsilon_T},$$

which establishes $\mathbf{S}(q+1)$(g).

Step 5. Completion

Let $\phi_r \colon \bigcup_{H \in \mathcal{H}, i \in [r]} \widehat{X}^H_i \to \mathcal{V}_r$ be an r-partial packing satisfying $\mathbf{S}(r)$ with (ε_T, d_i)-super-regular and $(\varepsilon_T, t^{2/3})$-well-intersecting candidacy graphs $B^H_i := B^{H,r}_i \subseteq B^H_i(\phi^+_r)$ where $d_i := d_B^{\deg_R(i)} = d_B^{m_i(r)}$ for all $i \in [r]$ and $\mathcal{X}^{\phi_r}_r = \bigcup_{H \in \mathcal{H}, i \in [r]} \widehat{X}^H_i$. We will apply a random packing procedure in order to complete the partial packing ϕ_r using the edges in G_B. Recall that $\varepsilon_T \ll \mu \ll \gamma \ll 1/t \ll \alpha, 1/k$ and we often call the vertices $\bigcup_{H \in \mathcal{H}, i \in [r]} (X^H_i \setminus \widehat{X}^H_i)$ *unembedded (by ϕ_r)* or the *leftover (of ϕ_r)*. Our general strategy is as follows. For every $H \in \mathcal{H}$ in turn, we choose a set $Y^H_i \subseteq \widehat{X}^H_i$ for all $i \in [r]$ of size roughly μn by selecting every vertex uniformly at random with probability μ and adding $X^H_i \setminus \widehat{X}^H_i$ deterministically. Afterwards we apply a random matching argument to pack $H[Y^H_{\cup[r]}]$ into G_B, which together with ϕ_r yields a complete packing of H into $G_A \cup G_B$. Before we proceed with the details of our random packing procedure in Claim 6 (Steps 5.6–5.11), we verify in Claim 4 (Steps 5.3–5.5) that we can indeed pack a subgraph of one single $H \in \mathcal{H}$ into G_B using another random embedding argument as long as our random packing procedure does not deviate too much from its expected behaviour. To that end, we collect some more notation in Step 5.1, and establish several important leftover conditions in Step 5.2.

Step 5.1. Notation for the completion

We introduce some more notation. We arbitrarily enumerate the graphs in $\mathcal{H}$ and write $\mathcal{H} = \{H_1, \ldots, H_{|\mathcal{H}|}\}$. For $G^\circ \subseteq G_B$ and B^H_i with $H \in \mathcal{H}$, $i \in [r]$,

(5.6.25)

let $(B^H_i)^{G^\circ}$ be the subgraph of B^H_i where $N_{(B^H_i)^{G^\circ}}(x) = V_i \cap N_{G_B - G^\circ}(\mathcal{S}_x)$ for every $x \in X^H_i$, and $\mathcal{S}_x \subseteq \binom{V(G)}{k-1}$ is the set such that $N_{B_i^H}(x) = N_{G_B}(\mathcal{S}_x)$. Note that $\mathcal{S}_x$ exists because B_i^H is $(\varepsilon_T, t^{2/3})$-well-intersecting. This implies in particular that we removed every edge xv from B_i^H for which there exists an edge $e \in E(H)$ such that $\phi_r(e \setminus \{x\}) \cup \{v\} \in E(G^\circ)$. We may think of $E(G^\circ)$ as the edge set in G_B that we have already used in our completion step for packing some other graphs of $\mathcal{H}$ into $G_A \cup G_B$. Consequently, $(B_i^H)^{G^\circ}$ is the subgraph of the candidacy graph B_i^H that only contains an edge xv if we do not use an edge in G° when we would map x onto v. To count the number of removed edges incident to a vertex $v \in V_i$ in B_i^H, we define

$$\rho^H_{G^\circ}(v) := \left|\left\{x \in N_{B_i^H}(v) \colon S \cup \{v\} \in E(G^\circ) \text{ for some } S \in \mathcal{S}_x\right\}\right|, \tag{5.6.26}$$

where $\mathcal{S}_x \subseteq \binom{V(G)}{k-1}$ is the set such that $N_{B_i^H}(x) = N_{G_B}(\mathcal{S}_x)$. Note that

$$\deg_{B_i^H}(v) - \deg_{(B_i^H)^{G^\circ}}(v) = \rho^H_{G^\circ}(v) \text{ for every } v \in V_i,\ i \in [r]. \tag{5.6.27}$$

During our random packing procedure we will guarantee that $\rho^H_{G^\circ}(v)$ is negligibly small (Step 5.10) and that the probability for a G_B-edge to be 'used' during the completion is appropriately small (at most $\mu^{3/4}$, see Claim 7). To that end, we control certain conditions for the leftover of ϕ_r in the next step.

Step 5.2. *Controlling the leftover for the completion*

In this step we make some important observations for the completion. In general, we will employ $\mathbf{S}(r)$(b) multiple times in order to suitably control the leftover, that is, the structure of the vertices that are left unembedded by ϕ_r.

We start with an observation how to control the number of neighbours of a vertex v in B_j^H that are left unembedded by ϕ_r.

Claim 1. $|N_{B_j^H}(v) \cap (X_j^H \setminus \widehat{X}_j^H)| \leq 2\varepsilon_T d_j n$ *for all* $H \in \mathcal{H}$, $j \in [r]$ *and* $v \in V_j$.

Proof of claim: Recall that we defined edge testers in (5.6.7) for all $H \in \mathcal{H}$, $j \in [r]$ and $v \in V_j$ to count the number of neighbours of v in B_j^H that are left unembedded by ϕ_r. Let $(\omega, \omega_{\iota,H}, J = \{j\}, J_X = \{j\}, J_V = \emptyset, e = \{v\}, (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}))$ be the edge tester in $\mathcal{W}^r_{edge}(\phi_r, \phi_r^+, \mathcal{A}^r, \mathcal{B}^r)$ that we obtain from $\mathbf{S}(r)$(b). Definition 5.15 of this edge tester implies that

$$|N_{B_j^H}(v) \cap (X_j^H \setminus \widehat{X}_j^H)| = \omega(E(\mathcal{B}_j^r)) = \omega(E(B_j^H)) \overset{\mathbf{S}(r)(\mathrm{b})}{\leq} 2\varepsilon_T d_j n.$$

This establishes Claim 1. $-$

Next, we control the number of neighbours of a vertex v in B_i^H that are embedded but lie in an H-edge that contains unembedded vertices.

Claim 2. $|N_{B_i^H}(v) \cap \mathcal{X}_r^{\phi_r} \cap N_{H_*}(\mathcal{X}_r \setminus \mathcal{X}_r^{\phi_r})| \leq \varepsilon_T^{1/2} d_i n$ *for all* $H \in \mathcal{H}$, $i \in [r]$ *and* $v \in V_i$.

Proof of claim: Our general strategy is as follows. We do not only consider a single vertex $v \in V_i$ but also a second vertex $w \in V_j$ for some $j \in N_{R_*}(i)$. We can use our defined edge testers and employ $\mathbf{S}(r)$(b) to count for a fixed vertex w the number of 2-sets $\{x_i, x_j\}$ where $x_i \in N_{B_i^H}(v)$ and $x_j \in X_j^H$ is left unembedded. Hence, by summing over all possible choices of w, we count all such 2-sets but multiple times. Hence, by a double counting argument, for one fixed j and all choices of $w \in V_j$, we can establish an upper bound for $|N_{B_i^H}(v) \cap \mathcal{X}_r^{\phi_r} \cap N_{H_*}(X_j^H \setminus \mathcal{X}_r^{\phi_r})|$. In the end, this implies Claim 2 as there are at most $k\alpha^{-1}$ choices for j. We proceed with the details.

We fix $H \in \mathcal{H}$, $i \in [r]$, $v \in V_i$, $j \in N_{R_*}(i)$ and consider $w \in V_j$. For all $\mathscr{p} = (\mathbf{0}, \mathbf{0}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$, we defined a tuple $(\omega_\iota, J = \{i,j\}, J_X = \{j\}, J_V = \emptyset, \{v,w\}, \mathscr{p})$ in (5.6.8). Let $(\omega, \omega_\iota, J, J_X, J_V, \{v,w\}, \mathscr{p})$ be the edge tester in $\mathcal{W}^r_{edge}(\phi_r, \phi_r^+, \mathcal{A}^r, \mathcal{B}^r)$ with respect to $(\omega_\iota, J, J_X, J_V, \{v,w\}, \mathscr{p})$, (ϕ_r, ϕ_r^+), $\mathcal{A}^r$, and $\mathcal{B}^r$ that we obtain from $\mathbf{S}(r)$(b). In order to be able to distinguish these edge testers according to the pattern vector $\mathscr{p}$, we write $\omega_{\iota,\mathscr{p}} := \omega_\iota$ and $\omega_{\mathscr{p}} := \omega$. Note that $\sum_{\mathscr{p}} \omega_{\iota,\mathscr{p}}(X_i^H \sqcup X_j^H) \leq 2\alpha^{-1} n$ by (5.6.8). Thus, by Definition 5.15 of an edge tester and by summing over all patterns $\mathscr{p} = (\mathbf{0}, \mathbf{0}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$ with $\mathbf{p}^B, \mathbf{p}^{B,2nd} \in [k\alpha^{-1}]_0^r$, we can employ conclusion $\mathbf{S}(r)$(b) to count the tuples $\{x_i, x_j\} \in X_i^H \sqcup X_j^H$ where x_j is left unembedded but $\{x_i, x_j\}$ could still be mapped onto $\{v, w\}$; that is, $\{\{x_i, v\}, \{x_j, w\}\} \in E(B_i^H) \sqcup E(B_j^H)$. Note that we count each such tuple $\{x_i, x_j\}$ multiple times, namely, for every $w \in V_j$ such that $\{\{x_i, v\}, \{x_j, w\}\} \in E(B_i^H) \sqcup E(B_j^H)$. By the Definition 5.11 of the candidacy graphs and because B_j^H is (ε_T, d_j)-super-regular, there are $|N_{B_j^H}(x_j)| = (1 \pm 2\varepsilon_T) d_j n$ choices for $w \in V_j$ such that $\{\{x_i, v\}, \{x_j, w\}\} \in E(B_i^H) \sqcup E(B_j^H)$. Altogether, this implies that

$$
\begin{aligned}
\left| N_{B_i}(v) \cap \mathcal{X}_r^{\phi_r} \cap N_{H_*}(X_j^H \setminus \mathcal{X}_r^{\phi_r}) \right| &\leq ((1 - 2\varepsilon_T) d_j n)^{-1} \sum_{w \in V_j} \sum_{\mathscr{p}} \omega_{\mathscr{p}} \left(E(\mathcal{B}_i^r) \sqcup E(\mathcal{B}_j^r) \right) \\
&\overset{\mathbf{S}(r)(\mathrm{b})}{\leq} ((1 - 2\varepsilon_T) d_j n)^{-1} \, 2n \sum_{\mathscr{p}} \left(\varepsilon_T d_i d_j \omega_{\iota,\mathscr{p}}(\mathscr{X}_{\sqcup J}) + n^{\varepsilon_T} \right) \\
&\leq 4 d_j^{-1} \cdot \left(2\alpha^{-1} \varepsilon_T d_i d_j n + n^{\varepsilon_T} \right) \\
&\leq \varepsilon_T^{2/3} d_i n.
\end{aligned}
$$

Summing over all $j \in N_{R_*}(i)$ establishes Claim 2. —

The last observation for controlling the leftover concerns the number of $\mathcal{H}$-edges that contain unembedded vertices and could still be mapped onto an edge $\mathscr{g} \in E(G_B)$. We define the following set in a slightly more general way as we will use this definition again in Step 5.11, where we also consider subsets of edges. For all $\mathscr{r} \in E(R)$, $I \subseteq \mathscr{r}$, $\mathscr{c} \in V_{\sqcup I}$, non-empty $J \subseteq I$, and all pairs of disjoint sets $J_X, J_V \subseteq J$, let

(5.6.28)

$$
\begin{aligned}
E_{\phi_r}(\mathscr{c}, J, J_X, J_V) := \bigcup_{H \in \mathcal{H}} \Big\{ & \mathscr{x} \in X_{\sqcup I}^H : \mathscr{x} \subseteq \mathscr{e} \text{ for some } \mathscr{e} \in E(\mathcal{H}), \mathscr{c} \cap V_{\cup(I \setminus J)} \subseteq \phi_r(\mathscr{x}), \\
& \phi_r(\mathscr{x}) \neq \mathscr{c}, \mathscr{c} \cap V_{\cup J_V} \subseteq \mathscr{c} \setminus \phi_r(\widehat{X}_{\sqcup I}^H), \mathscr{x} \setminus \mathcal{X}_r^{\phi_r} = \mathscr{x} \cap X_{\cup J_X}^H, \\
& \{\mathscr{x} \cap X_j^H, \mathscr{c} \cap V_j\} \in E(B_j^H) \text{ for all } j \in J \Big\}.
\end{aligned}
$$

Let us first explain this definition in words for the following more special case. For $I = \mathscr{r} \in E(R)$, $\mathscr{c} = \mathscr{g} \in E(G_B)$, J_X, J_V, J as above, $E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$ is the set of all H-edges $\mathscr{e}$ for all $H \in \mathcal{H}$ such that

(i)$_{(5.6.28)}$ $\{\mathscr{e} \cap X_j^H, \mathscr{g} \cap V_j\}$ is an edge in B_j^H for every $j \in J$,

(ii)$_{(5.6.28)}$ the $k - |J|$ vertices $\mathscr{e} \cap X_{\cup(\mathscr{r} \setminus J)}^H$ are mapped onto $\mathscr{g} \cap V_{\cup(\mathscr{r} \setminus J)}$,

(iii)$_{(5.6.28)}$ $\mathscr{g} \cap V_{\cup J_V}$ is a subset of the vertices of $\mathscr{g}$ onto which no H-vertex is embedded by ϕ_r, and

(iv)$_{(5.6.28)}$ all vertices in $\mathscr{e}$ but the $|J_X|$ vertices $\mathscr{e} \cap X_{\cup J_X}^H$ of $\mathscr{e}$ are embedded by ϕ_r.

This means, that if we modified ϕ_r and allowed the vertices $\mathscr{e} \cap X_{\cup(J \setminus J_X)}^H$ to be embedded somewhere else, we could potentially embed $\mathscr{e}$ onto $\mathscr{g}$. Note that for an edge $\mathscr{g} \in$

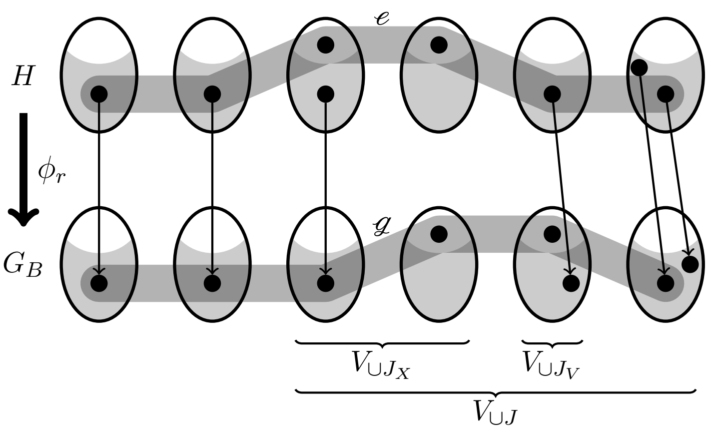


Figure 5.7: Illustration of one edge $\mathscr{e}$ in $E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$ for $k = 6$. Note that the vertices $\mathscr{e} \cap X^H_{\cup J_X}$ of $\mathscr{e}$ are left unembedded by ϕ_r, and no H-vertex is embedded onto $\mathscr{g} \cap X_{\cup J_V}$.

$E(G_B)$, we naturally have that $\phi_r(\mathscr{e}) \neq \mathscr{g}$ because ϕ_r maps $\mathcal{H}$-edges onto G_A-edges. For an illustration, see Figure 5.7.

Claim 3. $|E_{\phi_r}(\mathscr{g}, J, J_X, J_V)| \leq (\mathbb{1}\{J_{XV} = \emptyset\} + 2\varepsilon_T)\gamma^{-1} n^{|J|} \prod_{j\in J} d_j$*, for all* $\mathscr{r} \in E(R)$*,* $\mathscr{g} \in E(G_B[V_{\cup \mathscr{r}}])$*, non-empty* $J \subseteq \mathscr{r}$*, all pairs of disjoint sets* $J_X, J_V \subseteq J$*, and* $J_{XV} = J_X \cup J_V$*. Note that* $\varepsilon_T \gamma^{-1} \leq \varepsilon_T^{1/2}$*. Hence* $|E_{\phi_r}(\mathscr{g}, J, J_X, J_V)| \leq \varepsilon_T^{1/2} n^{|J|} \prod_{j\in J} d_j$ *if* $J_{XV} \neq \emptyset$*.*

Proof of claim: We fix $\mathscr{r} = I$, $\mathscr{g}$, J, J_X and J_V as in the statement of Claim 3, and recall that in (5.6.6) we defined a tuple $(\omega_\iota, J, J_X, J_V, \mathscr{g}, \mathscr{p})$ for each $\mathscr{p} \in ([k\alpha^{-1}]_0^r)^4$. Let $(\omega, \omega_\iota, J, J_X, J_V, \mathscr{g}, \mathscr{p})$ be the edge tester in $\mathcal{W}^r_{edge}(\phi_r, \phi_r^+, \mathcal{A}^r, \mathcal{B}^r)$ with respect to $(\omega_\iota, J, J_X, J_V, \mathscr{g}, \mathscr{p})$, (ϕ_r, ϕ_r^+), $\mathcal{A}^r$, and $\mathcal{B}^r$ that we obtain from $\mathbf{S}(r)$(b). In order to be able to distinguish these edge testers according to the patterns $\mathscr{p} \in ([k\alpha^{-1}]_0^r)^4$, we write $\omega_{\iota,\mathscr{p}} := \omega_\iota$ and $\omega_{\mathscr{p}} := \omega$. Note that for $\mathscr{p} = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd})$ such that $\omega_{\iota,\mathscr{p}}(\mathscr{X}_{\cup\mathscr{r}}) > 0$, we have that $\mathrm{supp}(\omega_{\iota,\mathscr{p}}) \subseteq E_{\mathcal{H}}(\mathscr{p}, \mathscr{r}, J)$ by Definition 5.15 and thus, by (5.4.5) and because $I = \mathscr{r} \in E(R)$ and $J \subseteq \mathscr{r}$ is non-empty, it holds that $\|\mathbf{p}^A\| - \|\mathbf{p}^{A,2nd}\| = 0$ and $\|\mathbf{p}^B\| - \|\mathbf{p}^{B,2nd}\| \in \{-1, 0\}$. By Definition 5.15 of an edge tester and by summing over all patterns $\mathscr{p} \in ([k\alpha^{-1}]_0^r)^4$, we can utilize our general edge testers to count the edges in $E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$. Note that the properties (i)$_{(5.6.28)}$–(iv)$_{(5.6.28)}$ of the definition of $E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$ in (5.6.28) correspond to the properties (i)$_{\text{D5.15}}$–(iv)$_{\text{D5.15}}$ of Definition 5.15 for a general edge tester, respectively. Due to (v)$_{\text{D5.15}}$ of Definition 5.15, the edge tester $(\omega_{\mathscr{p}}, \omega_{\iota,\mathscr{p}}, J, J_X, J_V, \mathscr{p})$ additionally requires for an element $\mathscr{e} \in E_{\mathcal{H}}(\mathscr{p}, I, J)$ that $\phi_r^+(\mathscr{e} \cap \mathscr{X}_{\cup J}) \cap \mathscr{g} = \emptyset$. That is, elements $\mathscr{e} \in E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$ where some vertices $\mathscr{e} \cap \mathscr{X}_{\cup J}$ are already embedded by ϕ_r^+ onto $\mathscr{g}$ during the partial packing procedure, are not counted by any edge tester. However, our intuition is that the number of such edges only yields a minor order contribution, and in fact, we can employ $\mathbf{S}(r)$(d) to also account for those edges. We make the following observation

$$
\begin{aligned}
&|E_{\phi_r}(\mathscr{g}, J, J_X, J_V)| \\
&\leq \sum_{\mathscr{p}} \omega_{\mathscr{p}} \left(\textstyle\bigsqcup_{j\in J} E(\mathcal{B}_j^r) \right) + \sum_{j\in J} \left| \left\{ \mathscr{e} \in E(\mathcal{H}[\mathscr{X}_{\cup\mathscr{r}}]) \colon \phi_r^+(\mathscr{e} \cap \mathscr{X}_{\cup(\{j\}\cup(\mathscr{r}\setminus J))}) \subseteq \mathscr{g} \right\} \right|. \qquad (5.6.29)
\end{aligned}
$$

We obtain an upper bound on the first term in (5.6.29) by employing $\mathbf{S}(r)$(b) as follows

$$\begin{aligned}\sum_{\mathscr{p}}\omega_{\mathscr{p}}\left(\bigsqcup_{j\in J}E(\mathcal{B}_j^r)\right) &\overset{\mathbf{S}(r)(\mathrm{b})}{\leq} \sum_{\mathscr{p}}\left((\mathbb{1}\{J_{XV}=\emptyset\}+\varepsilon_T)\,d_B^{-1}\prod_{j\in J}d_j\cdot\frac{\omega_{\iota,\mathscr{p}}(\mathscr{X}_{\sqcup\mathscr{r}})}{n^{k-|J|}}+n^{\varepsilon_T}\right)\\ &\leq (\mathbb{1}\{J_{XV}=\emptyset\}+\varepsilon_T)\,d_B^{-1}\prod_{j\in J}d_j\cdot\frac{\sum_{\mathscr{p}}|E_{\mathcal{H}}(\mathscr{p},\mathscr{r},J)|}{n^{k-|J|}}+n^{2\varepsilon_T}\\ &\leq (\mathbb{1}\{J_{XV}=\emptyset\}+\varepsilon_T)\,\gamma^{-1}n^{|J|}\prod_{j\in J}d_j, \qquad (5.6.30)\end{aligned}$$

where we used that $\sum_{\mathscr{p}}|E_{\mathcal{H}}(\mathscr{p},\mathscr{r},J)|\leq d_An^k$ and $d_B^{-1}d_A=(\gamma d)^{-1}(1-\gamma)d\leq\gamma^{-1}$.

An upper bound on the second term in (5.6.29) can be obtained by employing $\mathbf{S}(r)$(d)

(5.6.31)
$$\sum_{j\in J}\left|\left\{\mathscr{e}\in E(\mathcal{H}[\mathscr{X}_{\cup\mathscr{r}}])\colon \phi_r^+(\mathscr{e}\cap\mathscr{X}_{\cup(\{j\}\cup(\mathscr{r}\setminus J))})\subseteq\mathscr{g}\right\}\right|\leq\sum_{j\in J}n^{k-|\{j\}\cup(\mathscr{r}\setminus J)|+2\varepsilon_T}\leq n^{|J|-1+3\varepsilon_T}.$$

Substituting (5.6.30) and (5.6.31) in (5.6.29) establishes Claim 3. –

Step 5.3. Embedding one $H\in\mathcal{H}$ by a random argument – Claim 4

We proceed with our argument for embedding one subgraph of a fixed graph $H\in\mathcal{H}$. Suppose we are given $Y_i^H\subseteq X_i^H$ for all $i\in[r]$. For all $\mathscr{r}\in E(R)$ and $i\in[r]$, let

$$\begin{aligned}E_{\mathscr{r}}^{bad}&:=\left\{\mathscr{e}\in E(H[X_{\cup\mathscr{r}}^H])\colon|\mathscr{e}\cap Y_{\cup\mathscr{r}}^H|\geq 2\right\};\\ \mathcal{E}^{bad}&:=\textstyle\bigcup_{\mathscr{r}\in E(R)}E_{\mathscr{r}}^{bad};\\ Y_i^{bad}&:=\{\mathscr{e}\cap Y_i^H\colon\mathscr{e}\in\mathcal{E}^{bad}\};\\ \mathcal{E}^{good}&:=\{\mathscr{e}\in E(H)\colon|\mathscr{e}\cap Y_{\cup[r]}^H|=1\};\\ Y_i^{good}&:=Y_i^H\setminus Y_i^{bad}.\end{aligned}$$

Claim 4. Suppose $G^\circ\subseteq G_B$, $H\in\mathcal{H}$, and $Y_i^H\subseteq X_i^H$ for all $i\in[r]$ such that the following hold for all $i\in[r]$, where $W_i^H:=V_i\setminus\phi_r(\widehat{X}_i^H\setminus Y_i^H)$:

(A)$_{\mathrm{C4}}$ *$X_i^H\setminus\widehat{X}_i^H\subseteq Y_i^H\subseteq X_i^H$ and $|Y_i^H|=|W_i^H|=(1\pm\varepsilon_T^{1/2})\mu n$;*

(B)$_{\mathrm{C4}}$ *$|E_{\mathscr{r}}^{bad}|\leq\mu^{3/2}n$ for all $\mathscr{r}\in E(R)$;*

(C)$_{\mathrm{C4}}$ *$|N_{B_i^H}(v)\cap Y_i^{bad}|\leq\mu^{3/2}d_in$ for all $i\in[r]$, $v\in W_i^H$;*

(D)$_{\mathrm{C4}}$ *$(B_i^H)^{G^\circ}$ is $(\mu^{1/5},d_i)$-super-regular for all $i\in[r]$;*

(E)$_{\mathrm{C4}}$ *$(B_i^H)^{G^\circ}[Y_i^H,W_i^H]$ is $(\mu^{1/6},d_i)$-super-regular for all $i\in[r]$;*

(F)$_{\mathrm{C4}}$ *G_B-G° is $(\mu^{1/2},t,d_B)$-typical with respect to R;*

(G)$_{\mathrm{C4}}$ *for all $\mathcal{S}\subseteq\bigcup_{\mathscr{r}\in E(R)\colon i\in\mathscr{r}}V_{\sqcup\mathscr{r}\setminus\{i\}}$ with $|\mathcal{S}|\leq t$, we have*

$$\left|W_i^H\cap N_{G_B-G^\circ}(\mathcal{S})\right|=(1\pm\varepsilon_T^{1/2})\mu\left|V_i\cap N_{G_B-G^\circ}(\mathcal{S})\right|.$$

Then there exists a probability distribution of the embeddings $\phi^{\widetilde{H}}$ of $\widetilde{H}:=H[Y_{\cup[r]}^H]$ into $\widetilde{G}:=G_B[W_{\cup[r]}^H]-G^\circ$ with $\phi^H:=\phi^{\widetilde{H}}\cup\phi_r|_{V(H)\setminus V(\widetilde{H})}$ such that

(H)$_{\text{C4}}$ *ϕ^H is an embedding of H into G where $\phi^H(X_i^H) = V_i$ for all $i \in [r]$;*

(J)$_{\text{C4}}$ *ϕ^H embeds all H-edges that contain a vertex in $Y^H_{\cup[r]}$ onto an edge in $G_B - G^\circ$;*

(K)$_{\text{C4}}$ $\mathbb{P}\left[\phi^H(e \cap X^H_{\cup I}) = \{v_i\}_{i \in I}\right] \le \prod_{i \in I\colon e \cap Y_i^H \neq \emptyset} 2(\mu d_i n)^{-1}$

for all $m \in [k]$, index sets $I \in \binom{[r]}{m}$, $\{v_i\}_{i\in I} \in V_{\sqcup I}$, and H-edges e that contain a vertex in $Y^H_{\cup I}$.

Proof of claim: We split the proof of Claim 4 into two parts, Step 5.4 and 5.5. In Step 5.4, we greedily embed all the H-edges of $\mathcal{E}^{bad}$ into $G_B - G^\circ$ by considering the clusters in turn. Afterwards, in Step 5.5, we are left with the H-edges in $\mathcal{E}^{good}$; that is, where only a single vertex is not yet embedded. The assumptions (B)$_{\text{C4}}$ and (C)$_{\text{C4}}$ will guarantee that we only used few edges in $G_B - G^\circ$ in Step 5.4, such that we merely have to modify our candidacy graphs. Hence, we can easily find a perfect matching in each of these candidacy graphs to embed the H-edges of $\mathcal{E}^{good}$, which will complete the embedding of H. This matching procedure can be performed independently for each cluster as the H-edges in $\mathcal{E}^{good}$ only contain a single vertex that is not yet embedded. Clearly, this approach will establish (H)$_{\text{C4}}$ and (J)$_{\text{C4}}$.

In both steps, we will embed the vertices of H by a random procedure in order to establish (K)$_{\text{C4}}$. We will do so by making several random choices sequentially which naturally yields a probability distribution. Since some of these choices lead to instances that do not yield a valid or good embedding, we will discard some of these instances and say the random procedure terminates with failure in these cases. We will show that the proportion of choices with failure instances is exponentially small, that is, the probability that the random procedure terminates with failure is exponentially small in n. This allows us to discard some of these choices / instances and to restrict our probability space to the remaining 'nice' outcomes. Since the failure probability is exponentially small in n, this does not have a significant effect on the probability in (K)$_{\text{C4}}$.

Step 5.4. Proof of Claim 4 – Embedding the H-edges in $\mathcal{E}^{bad}$

Suppose $\phi_q^{bad}\colon Y^{bad}_{\cup[q]} \to W^H_{\cup[q]}$ is an injective function. Similarly as we defined candidacy graphs in Definition 5.11, we will also define updated candidacy graphs of B_i^H with respect to ϕ_q^{bad} for $i \in [r] \setminus [q]$. To that end, for all $i \in [r]\setminus[q]$ and $B \subseteq B_i^H$, let $B(\phi_q^{bad})$ be the spanning subgraph of B, where we keep the edge xv of B in $B(\phi_q^{bad})$ if all $e \in \mathcal{E}^{bad}$ with $e \cap Y^{bad}_{\cup([r]\setminus[q])} = \{x\}$ satisfy that

$$\phi_r|_{V(H)\setminus V(\widetilde{H})}(e \setminus V(\widetilde{H})) \cup \phi_q^{bad}(e \cap Y^{bad}_{\cup[q]}) \cup \{v\} \in E(G_B - G^\circ). \tag{5.6.32}$$

Observe that (5.6.32) is very similar to (5.4.1) in Definition 5.11. For all $q \in [r]$, $i \in [r]\setminus[q]$, $y \in Y_i^{bad}$, let

$$b_q(y) := \left|\left\{e \in \mathcal{E}^{bad}\colon e \cap Y^{bad}_{\cup([r]\setminus[q])} = \{y\}\right\}\right|.$$

That is, $b_q(y)$ is the number of edges e in $\mathcal{E}^{bad}$ containing y whose $(k-1)$-set $e \setminus \{y\}$ has already been embedded by $\phi_r|_{V(H)\setminus V(\widetilde{H})} \cup \phi_q^{bad}$.

We make one more observation and fix $y \in Y^{bad}_{q+1}$. Since B^H_{q+1} is $(\varepsilon_T, d_B^{\deg_R(q+1)})$-super-regular and $(\varepsilon_T, t^{2/3})$-well-intersecting by $\mathbf{S}(r)$(a), we have that for all $r \in E(R)$ and $e \in E_r^{bad}$ with $e \cap Y^{bad}_{\cup([r]\setminus[q])} = \{y\}$, there exists a $(k-1)$-set $S_r = \phi_r^+(e \setminus \{y\}) \in$

$V_{\sqcup(\mathscr{r}\setminus\{q+1\})}$, as well as there exists a set $\mathcal{S}_y$ of $(k-1)$-sets with $S_{\mathscr{r}} \in \mathcal{S}_y$ and $|\mathcal{S}_y| = \deg_R(q+1)$ such that $N_{B^H_{q+1}}(y) = V_{q+1} \cap N_{G_B}(\mathcal{S}_y)$ and $|N_{B^H_{q+1}}(y)| = (1 \pm \varepsilon_T) d_B^{|\mathcal{S}_y|} n$. (See (5.4.2) for the definition of well-intersecting.) Note that ϕ^+_r embeds $\mathscr{e}\setminus\{y\}$ onto $S_{\mathscr{r}}$ but $\phi_r|_{V(H)\setminus V(\widetilde{H})}$ does not. Hence, $S_{\mathscr{r}}$ only serves as a 'dummy' $(k-1)$-set for updating the candidacy graph B^H_{q+1} conveniently and to artificially restrict the candidates for y. That is, $V_{q+1} \cap N_{G_B}(\mathcal{S}_y \setminus \{S_{\mathscr{r}}\})$ are also suitable candidates where we could embed y, assuming that $\mathscr{e} \setminus \{y\}$ has not been embedded yet. Let $\mathcal{S}^{dummy}_y$ be the set of all these $(k-1)$-sets $S_{\mathscr{r}}$ for y, and note that $|\mathcal{S}^{dummy}_y| = b_q(y)$.

During our process of embedding the H-edges in $\mathcal{E}^{bad}$, we will have to drop this artificial restriction of the candidate sets. To that end, suppose we are given a spanning subgraph $B \subseteq B^H_{q+1}$, $y \in Y^{bad}_{q+1}$, and there exists a set $\mathcal{S}'_y$ of $(k-1)$-sets such that we can write

$$N_B(y) = V_{q+1} \cap N_{G_B - G^\circ}(\mathcal{S}'_y).$$

Then let $B^{\ominus dummy}$ be the spanning supergraph of B where the neighbourhood of each vertex $y \in Y^{bad}_{q+1}$ is given by

$$N_{B^{\ominus dummy}}(y) = V_{q+1} \cap N_{G_B - G^\circ}(\mathcal{S}'_y \setminus \mathcal{S}^{dummy}_y). \tag{5.6.33}$$

We inductively prove that the following statement $\mathbf{C}(q)$ holds for all $q \in [r]_0$, which will extend $\phi_r|_{V(H)\setminus V(\widetilde{H})}$ by embedding the edges in $\mathcal{E}^{bad}$ into $G_B - G^\circ$. To that end, we define a set of good pairs of vertices.

(5.6.34)

For every $i \in [r]$, let $Y^{good\ pairs}_i \subseteq \binom{Y^{good}_i}{2}$ be the set containing all pairs $\{y, y'\}$ with $\mathcal{S}_y \cap \mathcal{S}_{y'} = \emptyset$ where $\mathcal{S}_x \subseteq \bigcup_{\mathscr{r}\in E(R)\colon i \in \mathscr{r}} V_{\sqcup \mathscr{r}\setminus\{i\}}$ is such that $N_{B^H_i}(x) = V_i \cap N_{G_B}(\mathcal{S}_x)$ for each $x \in \{y, y'\}$.

We note for future reference that

$$\left| \binom{Y^{good}_i}{2} \setminus Y^{good\ pairs}_i \right| \le 2n \cdot n^{1/4+\varepsilon_T} \le n^{4/3}, \tag{5.6.35}$$

since B^H_i is $(\varepsilon_T, t^{2/3})$-well-intersecting as defined in (5.4.2).

C(q). There exists a probability distribution of the injective functions $\phi^{bad}_q \colon Y^{bad}_{\cup[q]} \to W^H_{\cup[q]}$ with $\phi'_q := \phi^{bad}_q \cup \phi_r|_{V(H)\setminus V(\widetilde{H})}$ such that

- (I)$_{C4}$ ϕ'_q is an embedding of $H'_q := H[Y^{bad}_{\cup[q]} \cup (V(H) \setminus V(\widetilde{H}))]$ into G;
- (II)$_{C4}$ all edges in $H[Y^{bad}_{\cup[q]} \cup (V(H) \setminus V(\widetilde{H}))]$ that contain a vertex in $Y^{bad}_{\cup[q]}$ are embedded on an edge in $G_B - G^\circ$;
- (III)$_{C4}$ for every vertex $x \in Y^{good}_i$ and $i \in [q]$, there are at most $\mu^{4/3} d_i n$ vertices $y \in Y^{bad}_i$ with $\phi^{bad}_q(y) \in N_{B^H_i}(x)$;
- (IV)$_{C4}$ $\mathbb{P}\left[\phi'_q(\mathscr{e} \cap X^H_{\cup I}) = \{v_i\}_{i\in I}\right] \le \prod_{i \in I\colon \mathscr{e}\cap Y^H_i \neq \emptyset} 2(\mu d_i n)^{-1}$

  for all $\mathscr{r} \in E(R)$, $m \in [k]$, $I \in \binom{\mathscr{r}\cap[q]}{m}$, $\{v_i\}_{i\in I} \in V_{\sqcup I}$ and $\mathscr{e} \in E^{bad}_{\mathscr{r}}$.

The statement $\mathbf{C}(0)$ clearly holds for ϕ_0^{bad} being the empty function. Hence, we assume the truth of $\mathbf{C}(q)$ for some $q \in [r-1]_0$. Our strategy to establish $\mathbf{C}(q+1)$ is as follows. We extend the probability space given in $\mathbf{C}(q)$ by making further random choices. For ϕ_q^{bad} as in $\mathbf{C}(q)$, we aim to find a matching $\sigma_{q+1}^{bad}\colon Y_{q+1}^{bad} \to W_{q+1}^H$ in a suitable candidacy graph between Y_{q+1}^{bad} and W_{q+1}^H that extends ϕ_q^{bad} to $\phi_{q+1}^{bad} := \phi_q^{bad} \cup \sigma_{q+1}^{bad}$. If we can find such a matching σ_{q+1}^{bad}, then $\mathbf{C}(q+1)(\mathrm{I})_{\mathrm{C4}}$ and $(\mathrm{II})_{\mathrm{C4}}$ will hold by the definition of this suitable candidacy graph. In particular, we will find σ_{q+1}^{bad} by a random procedure to also ensure $(\mathrm{IV})_{\mathrm{C4}}$. We will discard an exponentially small proportion of random choices during this procedure in order to satisfy $(\mathrm{III})_{\mathrm{C4}}$ and to obtain a suitable embedding σ_{q+1}^{bad}.

Let us describe this suitable candidacy graph. We will choose σ_{q+1}^{bad} randomly in

$$\widetilde{B} := (B_{q+1}^H)^{G^\circ}(\phi_q^{bad})^{\ominus dummy}[Y_{q+1}^{bad}, W_{q+1}^H].$$

That is, $\widetilde{B}$ arises from B_{q+1}^H as follows.

- First, we restrict the candidate sets in B_{q+1}^H to those edges whose corresponding edges in G_B have not been used in G° for packing graphs $H_1, \ldots, H_h$ in previous rounds.
- Second, we restrict the candidate sets with respect to the packing ϕ_q^{bad} of the vertices in $Y_{\cup[q]}^{bad}$ according to (5.6.32).
- Third, we drop the restriction of the candidate sets to the dummy $(k-1)$-sets as in (5.6.33).
- In the end, we consider the induced subgraph of this candidacy graph on the bad vertices Y_{q+1}^{bad} and the vertices W_{q+1}^H that can be used for the completion.

For the sake of a better readability, let $B := (B_{q+1}^H)^{G^\circ}$.

To guarantee the existence of σ_{q+1}^{bad}, we will show that the degree of every vertex $y \in Y_{q+1}^{bad}$ is sufficiently large in B and also in $\widetilde{B}$. Let $y \in Y_{q+1}^{bad}$ be fixed. Since B_{q+1}^H is $(\varepsilon_T, t^{2/3})$-well-intersecting by $\mathbf{S}(r)(\mathrm{a})$, there exists a set $\mathcal{S}_y \subseteq \bigcup_{\mathscr{r} \in E(R)\colon q+1 \in \mathscr{r}} V_{\sqcup \mathscr{r} \setminus \{q+1\}}$ of $(k-1)$-sets with $|\mathcal{S}_y| \leq t^{2/3}$ such that $N_{B_{q+1}^H}(y) = V_{q+1} \cap N_{G_B}(\mathcal{S}_y)$. Hence, by the definition of $B = (B_{q+1}^H)^{G^\circ}$ in (5.6.25), we have

$$N_B(y) = V_{q+1} \cap N_{G_B - G^\circ}(\mathcal{S}_y), \quad \text{and} \quad \deg_B(y) = (1 \pm 2\mu^{1/5}) d_{q+1} n, \tag{5.6.36}$$

because B is $(\mu^{1/5}, d_{q+1})$-super-regular by $(\mathrm{D})_{\mathrm{C4}}$. Of course we have to restrict the potential images of y according to the vertices we already embedded by ϕ_q^{bad}. To this end, let

$$\mathcal{S}_y^{bad} := \left\{ \phi_q^{bad}(\mathscr{e} \cap Y_{\cup[q]}^{bad}) \cup \phi_r|_{V(H)\setminus V(\widetilde{H})}(\mathscr{e} \setminus V(\widetilde{H}))\colon \mathscr{e} \in \mathcal{E}^{bad}, \mathscr{e} \cap Y_{\cup([r]\setminus[q])}^{bad} = \{y\} \right\}$$

and note that $|\mathcal{S}_y^{bad}| = b_q(y)$. By the definition of the candidacy graph $B(\phi_q^{bad})$ in (5.6.32) and by (5.6.36), we obtain

$$N_{B(\phi_q^{bad})}(y) = V_{q+1} \cap N_{G_B - G^\circ}(\mathcal{S}_y \cup \mathcal{S}_y^{bad}),$$

and thus, together with $(\mathrm{F})_{\mathrm{C4}}$ and (5.6.36), we have that

$$\deg_{B(\phi_q^{bad})}(y) = (1 \pm \mu^{1/6}) d_{q+1} d_B^{b_q(y)} n.$$

Since $|\mathcal{S}_y^{dummy}| = b_q(y)$, this implies

(5.6.37)
$$\deg_{B(\phi_q^{bad})^{\ominus dummy}}(y) = \left|V_{q+1} \cap N_{G_B - G^\circ}\big((\mathcal{S}_y \setminus \mathcal{S}_y^{dummy}) \cup \mathcal{S}_y^{bad}\big)\right| = (1 \pm \mu^{1/6})d_{q+1}n.$$

Now, we obtain

$$N_{\widetilde{B}}(y) = W_{q+1}^H \cap N_{G_B - G^\circ}\big((\mathcal{S}_y \setminus \mathcal{S}_y^{dummy}) \cup \mathcal{S}_y^{bad}\big) \tag{5.6.38}$$

and thus, by (G)$_{\text{C4}}$ and (5.6.37), we have that

$$\deg_{\widetilde{B}}(y) = (1 \pm \mu^{1/7})d_{q+1}\mu n. \tag{5.6.39}$$

In order to guarantee (IV)$_{\text{C4}}$, we find σ_{q+1}^{bad} via the following random procedure:

- for every vertex $y \in Y_{q+1}^{bad}$ in turn, we choose a neighbour in $N_{\widetilde{B}}(y)$ uniformly at random among all neighbours that have not been chosen in previous turns;
- we terminate the random procedure with failure at some step of the procedure, say at the turn of some vertex $y \in Y_{q+1}^{bad}$,
  - if we have less than $2(\mu d_{q+1}n)/3$ choices to select an image for y in $N_{\widetilde{B}}(y)$, or
  - if there is a vertex $x \in Y_{q+1}^{good}$ such that (III)$_{\text{C4}}$ is violated, that is, we mapped in previous turns already more than $\mu^{4/3}d_{q+1}n$ vertices of Y_{q+1}^{bad} into $N_{B_{q+1}^H}(x)$.

We show in the following claim that this random procedure terminates with failure only with exponentially small probability. If the procedure does not terminate with failure, we obtain a random Y_{q+1}^{bad}-saturating matching $\sigma_{q+1}^{bad}\colon Y_{q+1}^{bad} \to W_{q+1}^H$ in $\widetilde{B}$, which by definition of the candidacy graph B_{q+1}^H and $\widetilde{B}$ implies $\mathbf{C}(q+1)$(I)$_{\text{C4}}$–(III)$_{\text{C4}}$. Further, σ_{q+1}^{bad} satisfies the following.

(5.6.40)
For all $y \in Y_{q+1}^{bad}$, $w \in W_{q+1}^H$, we have that $\sigma_{q+1}^{bad}(y) = w$ with probability at most $2(\mu d_{q+1}n)^{-1}$.

Hence, the following claim together with $\mathbf{C}(q)$(IV)$_{\text{C4}}$ establishes $\mathbf{C}(q+1)$(IV)$_{\text{C4}}$.

Claim 5. The random procedure for computing σ_{q+1}^{bad} terminates with failure with probability at most $\mathrm{e}^{-n^{1/2}}$.

Proof of claim: Let $Y_{q+1}^{bad} = \{y_1, \dots, y_m\}$ and we consider every vertex in turn, that is, $y_{\ell+1}$ will be treated after y_ℓ. For all $x \in Y_{q+1}^H$ and $\ell \in [m]$, let $\xi_\ell(x)$ be the random variable that counts the number of covered neighbours so far, that is, the number of vertices $v \in N_{B_{q+1}^H}(x)$ such that $\sigma_{q+1}^{bad}(y) = v$ for some $y \in \{y_i\}_{i\in[\ell]}$. We say the random procedure fails at step $\ell \in [m]$, if ℓ is the smallest integer such that

- $\xi_\ell(z) > \mu^{2/5}d_{q+1}\mu n$ for some vertex $z \in Y_{q+1}^{good} \cup \{y_i\}_{i\in[m]\setminus[\ell]}$.

We show that the random procedure fails at some step $\ell \in [m]$ with probability at most $\mathrm{e}^{-n^{2/3}}$. A union bound then establishes Claim 5.

We fix $\ell \in [m]$ and a vertex $z \in Y_{q+1}^{good} \cup \{y_i\}_{i\in[m]\setminus[\ell]}$. By employing (5.6.38), (5.6.39) and that B_{q+1}^H is $(\varepsilon_T, t^{2/3})$-well-intersecting, we obtain

(5.6.41)
$$|N_{\widetilde{B}}(y \wedge z)| = (1 \pm \mu^{1/7})d_{q+1}^2\mu n \text{ for all but at most } n^{1/3} \text{ vertices } y \in \{y_i\}_{i\in[\ell]}.$$

Further, at the turn of each y_i, $i \in [\ell]$, we have at least

$$|N_{\widetilde{B}}(y_i)| - \mu^{2/5}d_{q+1}\mu n \overset{(5.6.39)}{\geq} (1 - 2\mu^{1/7})d_{q+1}\mu n \tag{5.6.42}$$

choices for the embedding of y_i. Hence, by (5.6.41) and (5.6.42), the probability that a vertex y_i, $i \in [\ell]$ which satisfies (5.6.41) is mapped into $N_{\widetilde{B}}(z)$ is at most

$$\frac{(1 + \mu^{1/7})d_{q+1}^2\mu n}{(1 - 2\mu^{1/7})d_{q+1}\mu n} \leq 2d_{q+1}.$$

This implies that

$$\mathbb{E}\left[\xi_\ell(z)\right] \leq 2d_{q+1}\ell + n^{1/3} \leq 2d_{q+1}|Y_{q+1}^{bad}| + n^{1/3} \leq 3d_{q+1}\alpha^{-1}\mu^{3/2}n,$$

where we used that $|Y_{q+1}^{bad}| \leq \alpha^{-1}\mu^{3/2}n$ by $(\mathrm{B})_{\mathrm{C4}}$. Hence, Theorem 1.8 implies that $\xi_\ell(z) > \mu^{2/5}d_{q+1}\mu n$ with probability, say, at most $\mathrm{e}^{-n^{2/3}}$ for some $z \in Y_{q+1}^{good} \cup \{y_i\}_{i\in[m]\setminus[\ell]}$. Thus, the random procedure fails at step ℓ with probability at most $\mathrm{e}^{-n^{2/3}}$. A simple union bound completes the proof of Claim 5. —

Step 5.5. Proof of Claim 4 – Embedding the H-edges in $\mathcal{E}^{good}$

Let $\phi_r^{bad}\colon Y_{\cup[r]}^{bad} \to W_{\cup[r]}^H$ and $\phi_r' = \phi_r^{bad} \cup \phi_r|_{V(H)\setminus V(\widetilde{H})}$ be as in $\mathbf{C}(r)$ obtained in Step 5.4. Recall that for all $i \in [r]$, we have $Y_i^{good} = Y_i^H \setminus Y_i^{bad}$ and let $W_i^{good} := W_i^H \setminus \phi_r^{bad}(Y_i^{bad})$, and thus clearly, $|Y_i^{good}| = |W_i^{good}|$. For every $i \in [r]$, we aim to embed the vertices Y_i^{good} onto W_i^{good} by finding a perfect matching in $(B_i^H)^{G^\circ}$.

Let $i \in [r]$ be fixed. By $(\mathrm{E})_{\mathrm{C4}}$, we have that $\widehat{B}_i := (B_i^H)^{G^\circ}[Y_i^H, W_i^H]$ is $(\mu^{1/6}, d_i)$-super-regular. We show that for every vertex in $Y_i^{good} \cup W_i^{good}$, we only removed few incident edges when take the subgraph $B_i^{good} := \widehat{B}_i[Y_i^{good}, W_i^{good}]$ of $\widehat{B}_i$. For a vertex $v \in W_i^{good}$, we removed at most $\mu^{3/2}d_i n$ incident edges by $(\mathrm{C})_{\mathrm{C4}}$. For a vertex $x \in Y_i^{good}$, we removed at most $\mu^{4/3}d_i n$ incident edges by $(\mathrm{III})_{\mathrm{C4}}$. Hence, by employing Fact 1.12, we obtain that B_i^{good} is $(\mu^{1/19}, d_i)$-super-regular for every $i \in [r]$.

Our strategy is to apply Lemma 5.8 that allows us to find a regular spanning subgraph of B_i^{good} from which we can easily take a random perfect matching. In order to satisfy the assumptions of Lemma 5.8, we show that the common neighbourhood of most of the pairs in Y_i^{good} is also not too large in B_i^{good}. To that end, we fix a pair of vertices $\{y, y'\} \in Y_i^{good\ pairs}$ as defined in (5.6.34) and let $\mathcal{S}_y, \mathcal{S}_{y'}$ be the sets of $(k-1)$-sets such that $N_{B_i^H}(x) = V_i \cap N_{G_B}(\mathcal{S}_x)$ for each $x \in \{y, y'\}$. We have that

$$N_{\widehat{B}_i}(y \wedge y') = W_i^H \cap N_{(B_i^H)^{G^\circ}}(y \wedge y') = W_i^H \cap N_{G_B - G^\circ}(\mathcal{S}_y \cup \mathcal{S}_{y'}). \tag{5.6.43}$$

Hence, we obtain

$$\begin{aligned} |N_{B_i^{good}}(y \wedge y')| &\leq |N_{\widehat{B}_i}(y \wedge y')| \\ &\overset{(5.6.43),(\mathrm{G})_{\mathrm{C4}}}{\leq} (1 + \varepsilon_T^{1/2})\mu\left|V_i \cap N_{G_B - G^\circ}(\mathcal{S}_y \cup \mathcal{S}_{y'})\right| \\ &\overset{(\mathrm{F})_{\mathrm{C4}}}{\leq} (1 + 2\mu^{1/2})\mu d_i^2 n, \end{aligned} \tag{5.6.44}$$

where we used for the last equality that $\mathcal{S}_y \cap \mathcal{S}_{y'} = \emptyset$ since $\{y, y'\} \in Y_i^{good\ pairs}$, and thus, $d_B^{|\mathcal{S}_y \cup \mathcal{S}_{y'}|} = d_i^2$. Hence, (5.6.35) implies that all but at most $n^{4/3}$ pairs of vertices in Y_i^{good} satisfy (5.6.44).

Finally, we can apply Lemma 5.8 and obtain a spanning $\mu d_i n/2$-regular subgraph of B_i^{good}. In particular, B_i^{good} contains $\mu d_i n/2$ edge-disjoint perfect matchings, from which we choose one perfect matching $\sigma_i^{good}\colon Y_i^{good} \to W_i^{good}$ for each $i \in [r]$ uniformly and independently at random. We crucially observe:

(5.6.45)
For all $i \in [r]$, $y_i \in Y_i^{good}$, $w_i \in W_i^{good}$, we have that $\sigma_i^{good}(y_i) = w_i$ with probability at most $2(\mu d_i n)^{-1}$.

Further, since all the vertices in $H_*[Y_{\cup[r]}^{good}]$ are isolated, we have that $\phi^{good} := \bigcup_{i\in[r]} \sigma_i^{good}$ is an injective function $\phi^{good}\colon Y_{\cup[r]}^{good} \to W_{\cup[r]}^{good}$ such that $\phi_r|_{V(H)\setminus V(\widetilde{H})} \cup \phi^{good}$ is a random packing of $H[Y_{\cup[q]}^{good} \cup (V(H) \setminus V(\widetilde{H}))]$ into G that embeds all edges in $H[Y_{\cup[r]}^{good} \cup (V(H) \setminus V(\widetilde{H}))]$ that contain a vertex in $Y_{\cup[r]}^{good}$ on an edge in $G_B - G^\circ$. In particular, since we find the perfect matchings in the candidacy graphs $(B_i^H)^{G^\circ}$ induced on $Y_i^{good} \cup W_i^{good}$ for each $i \in [r]$, we have for $\phi^H := \phi_r|_{V(H)\setminus V(\widetilde{H})} \cup \phi_r^{bad} \cup \phi^{good}$ that

- ϕ^H is an embedding of H into G where $\phi^H(X_i^H) = V_i$ for all $i \in [r]$, which establishes $(\mathrm{H})_{\mathrm{C4}}$;
- all H-edges that contain a vertex in $Y_{\cup[r]}^H$ are embedded on an edge in $G_B - G^\circ$, which establishes $(\mathrm{J})_{\mathrm{C4}}$;
- for all $m \in [k]$, index sets $I \in \binom{[r]}{m}$, $\{v_i\}_{i\in I} \in V_{\sqcup I}$, and H-edges e that contain a vertex in $Y_{\cup I}^H$, we have that $\phi^H(e \cap X_{\cup I}^H) = \{v_i\}_{i\in I}$ with probability at most $\prod_{i\in I\colon e\cap Y_i^H \neq \emptyset} 2(\mu d_i n)^{-1}$ by $\mathbf{C}(r)(\mathrm{IV})_{\mathrm{C4}}$ and (5.6.45), which establishes $(\mathrm{K})_{\mathrm{C4}}$.

This completes the proof of Claim 4. —

Step 5.6. *The random packing procedure – Claim 6*

We now proceed to our Random Packing Procedure (RPP). Let G_0° be the edgeless graph on $V(G)$ and let ϕ^0 be the empty function. We perform the following random procedure.

Random Packing Procedure (RPP)

For $h = 1, \ldots, |\mathcal{H}|$ do:

- *Set $H := H_h$. For all $i \in [r]$, independently activate every vertex in $\widehat{X}_i^H$ with probability μ and let Y_i^H be the union of $X_i^H \setminus \widehat{X}_i^H$ and all activated vertices in $\widehat{X}_i^H$.*
- *If the assumptions of Claim 4 are satisfied, apply Claim 4 and obtain a random packing ϕ^H that satisfies $(\mathrm{H})_{\mathrm{C4}}$–$(\mathrm{K})_{\mathrm{C4}}$; otherwise terminate with failure.*
- *Set $\phi^h := \phi^{h-1} \cup \phi^H$ and $G_h^\circ := G_{h-1}^\circ \cup (\phi^H(H) \cap G_B)$.*

Claim 6. *With probability at least* $1-1/n$*, the RPP terminates without failure and satisfies conclusion* (iii) *of Lemma 5.19.*

Proof of claim: We prove Claim 6 in Steps 5.7–5.11. Our general strategy is to guarantee that we can apply Claim 4 in each turn of the procedure. To that end, we will introduce a collection of random variables that we call *identifiers*. Such an identifier indicates an unlikely event and if this event happens, we say the identifier *detects alarm* and we simply terminate the RPP with failure. This means, if an identifier detects alarm at some turn $h \in [|\mathcal{H}|]$, we terminate the RPP and deactivate all further identifiers; that is, the probability that they detect alarm is set to 0. We show that the probability that an individual identifier detects alarm is exponentially small in n. In the end, a union bound over all identifiers will imply that with probability at least $1-1/n$, none of the identifiers will detect alarm and thus, the RPP terminates without failure.

For most of the identifiers, it follows by a standard application of Theorem 1.8 that the probability to detect alarm is exponentially small (in fact, often Chernoff's inequality suffices). To that end, in the subsequent Steps 5.7–5.11, we often describe only the random variables to which we apply Theorem 1.8.

First, let us observe that $\mathbf{S}(r)$ implies that

$$\big|X_i^H \setminus \widehat{X}_i^H\big| = \big|V_i \setminus \phi_r(\widehat{X}_i^H)\big| \leq 2\varepsilon_T n \text{ for all } H \in \mathcal{H},\ i \in [r]. \tag{5.6.46}$$

Further, for Y_i^H as in the RPP, let $W_i^H := V_i \setminus \phi_r(\widehat{X}_i^H \setminus Y_i^H)$. For convenience, we also call a vertex $w \in W_i^H \cap \phi_r(\widehat{X}_i^H)$ *activated* by $H \in \mathcal{H}$. (Recall that we only activate vertices in $\widehat{X}_i^H$.)

<u>*Step 5.7.*</u> *Proof of Claim 6 – Establishing* (A)$_{\mathrm{C4}}$–(G)$_{\mathrm{C4}}$

In this step, in order to establish (A)$_{\mathrm{C4}}$–(G)$_{\mathrm{C4}}$ at each turn of the RPP, we consider several random variables for which we individually introduce an identifier that detects alarm if the considered random variable is not within a factor of $(1\pm\varepsilon)$ of its expectation. As mentioned above, for each identifier a standard application of Theorem 1.8 implies that the probability to detect alarm is exponentially small, say, $\mathrm{e}^{-n^{1/2}}$. Let us only describe the random variables that we consider for establishing (A)$_{\mathrm{C4}}$–(G)$_{\mathrm{C4}}$.

To establish (A)$_{\mathrm{C4}}$, for each $H \in \mathcal{H}$, $i \in [r]$, we consider the sum of indicator variables which indicate whether a vertex is activated in X_i^H. Together with (5.6.46), this implies (A)$_{\mathrm{C4}}$.

To establish (B)$_{\mathrm{C4}}$, for each $H \in \mathcal{H}$ and $\mathscr{r} \in E(R)$, we consider the random variable that counts how many H-edges $\mathscr{e}$ lie in $X_{\cup\mathscr{r}}^H$ where at least two vertices of $\mathscr{e}$ are activated; in view of the statement, we may assume that $e_H(X_{\cup\mathscr{r}}^H) \geq \mu^{3/2}n$. Note that the probability for an edge $\mathscr{e}$ that at least two vertices are activated is at most $k^2\mu^2$. Together with (5.6.46), this implies (B)$_{\mathrm{C4}}$.

To establish (C)$_{\mathrm{C4}}$, for all $H \in \mathcal{H}$, $i \in [r]$, and $v \in V_i$, we consider the random variable ξ that counts how many B_i^H-neighbours of v in X_i^H are activated and lie in an H-edge $\mathscr{e}$ where a second vertex in $\mathscr{e}$ is either activated or left unembedded. With Claim 2 we have that $\mathbb{E}[\xi] \leq 2k^2\mu^2\alpha^{-1}d_in + 2\mu\varepsilon_T^{1/2}d_in \leq \mu^{5/3}d_in$. Together with Claim 1, this implies (C)$_{\mathrm{C4}}$.

To establish (G)$_{\mathrm{C4}}$, for all $i \in [r]$, $\mathcal{S} \subseteq \bigcup_{\mathscr{r}\in E(R)\colon i\in\mathscr{r}} V_{\sqcup\mathscr{r}\setminus\{i\}}$, $|\mathcal{S}| \leq t$, we consider the sum of indicator variables which each indicates whether a vertex in $\phi_r(\widehat{X}_i^H) \cap N_{G_B-G_h^\circ}(\mathcal{S})$ is activated. By $\mathbf{S}(r)$(e), we further have that

$$\big|(V_i \setminus \phi_r(\widehat{X}_i^H)) \cap N_{G_B}(\mathcal{S})\big| \leq \varepsilon_T\big|V_i \cap N_{G_B}(\mathcal{S})\big|. \tag{5.6.47}$$

Altogether, this implies $(\mathrm{G})_{\mathrm{C4}}$.

In order to establish $(\mathrm{D})_{\mathrm{C4}}$–$(\mathrm{F})_{\mathrm{C4}}$, we claim that if no identifier detected alarm until turn $h \in [|\mathcal{H}|]$, then for all $i \in [r]$

(5.6.48)
$$\left|\bigcup_{S\in\mathcal{S}}\left(V_i \cap N_{G_h^\circ}(S) \cap N_{G_B}(\mathcal{S})\right)\right| \le \mu^{2/3} d_B^{|\mathcal{S}|} n \text{ for all } \mathcal{S} \subseteq \bigcup_{\mathscr{r}\in E(R)\colon i\in\mathscr{r}} V_{\cup\mathscr{r}\setminus\{i\}},\ |\mathcal{S}| \le t;$$

$$\rho^H_{G_h^\circ}(v) \le \mu^{2/3} d_i n \text{ for all } H \in \mathcal{H},\ v \in V_i, \tag{5.6.49}$$

where $\rho^H_{G_h^\circ}(v)$ is defined as in (5.6.26). We verify (5.6.48) and (5.6.49) in the subsequent Steps 5.9 and 5.10, respectively, and first establish $(\mathrm{D})_{\mathrm{C4}}$–$(\mathrm{F})_{\mathrm{C4}}$ assuming (5.6.48) and (5.6.49).

To establish $(\mathrm{F})_{\mathrm{C4}}$, recall that G_B is $(2\varepsilon_0, t, d_B)$-typical with respect to R by (5.6.4). Hence, we obtain from (5.6.48) and the definition of typicality that $G_B - G_h^\circ$ is $(\mu^{1/2}, t, d_B)$-typical with respect to R, which implies $(\mathrm{F})_{\mathrm{C4}}$.

To establish $(\mathrm{D})_{\mathrm{C4}}$ for $H = H_{h+1}$, recall that B_i^H is (ε_T, d_i)-super-regular and $(\varepsilon_T, t^{2/3})$-well-intersecting with respect to G_B for all $i \in [r]$. Observe that there exists a set $\mathcal{S}_x$ for every $x \in X_i^H$ with $|\mathcal{S}_x| \le t^{2/3}$ and $N_{B_i^H}(x) = V_i \cap N_{G_B}(\mathcal{S}_x)$. By the definition of $(B_i^H)^{G_h^\circ}$ in (5.6.25), we have that $N_{(B_i^H)^{G_h^\circ}}(x) = V_i \cap N_{G_B - G_h^\circ}(\mathcal{S}_x)$. This together with (5.6.48), and (5.6.49) together with (5.6.27), implies that we removed at most $\mu^{2/3} d_i n$ edges incident to every vertex to obtain $(B_i^H)^{G_h^\circ}$ from B_i^H. Now Fact 1.12 yields that $(B_i^H)^{G_h^\circ}$ is $(\mu^{1/5}, d_i)$-super-regular. This implies $(\mathrm{D})_{\mathrm{C4}}$.

To establish $(\mathrm{E})_{\mathrm{C4}}$ for $H = H_{h+1}$, we exploit that $(B_i^H)^{G_h^\circ}$ is $(\mu^{1/5}, d_i)$-super-regular and only have to show that every vertex in $(B_i^H)^{G_h^\circ}[Y_i^H, W_i^H]$ has the appropriate degree. To that end, we consider the following random variables for all $x \in X_i^H$, $v \in V_i$, $i \in [r]$. For x, we consider the random variable that counts the number of activated vertices by H in $N_{(B_i^H)^{G_h^\circ}}(x)$, and by employing (5.6.47) and that B_i^H is (ε_T, t)-well-intersecting with respect to G_B, we expect that $N_{(B_i^H)^{G_h^\circ}}(x) \cap W_i^H$ has size $\mu|N_{(B_i^H)^{G_h^\circ}}(x)| \pm 2\varepsilon_T d_i n$. This yields the appropriate degree for x. For v, we consider the random variable that counts the number of activated vertices in $N_{(B_i^H)^{G_h^\circ}}(v)$, and by employing Claim 1, we expect that $N_{(B_i^H)^{G_h^\circ}}(v) \cap Y_i^H$ has size $\mu|N_{(B_i^H)^{G_h^\circ}}(v)| \pm 2\varepsilon_T d_i n$. This yields the appropriate degree for v. Altogether this implies $(\mathrm{E})_{\mathrm{C4}}$.

Step 5.8. *Probability to use a G_B-edge during the completion*

We say an edge $\mathscr{g} \in E(G_B)$ is *used* during the RPP if there exists a graph $H_h \in \mathcal{H}$ and an edge $\mathscr{e} \in H_h$ such that $\phi^h(\mathscr{e}) = \mathscr{g}$. In this step we show the following claim that we will apply to establish (5.6.48) and (5.6.49) in Steps 5.9 and 5.10.

Claim 7. For every edge $\mathscr{g} \in E(G_B)$, the probability that $\mathscr{g}$ is used during the RPP is at most $\mu^{3/4}$.

Proof of claim: Let $\mathscr{g} \in E(G_B)$ and $\mathscr{r} \in E(R)$ with $\mathscr{g} = \{v_i\}_{i\in\mathscr{r}} \in E(G_B[V_{\cup\mathscr{r}}])$ be fixed. We consider different cases and sets of $\mathcal{H}$-edges that could potentially be embedded onto $\mathscr{g}$, say in each case with probability at must $\mu^{4/5}$. In the end, a union bound will establish Claim 7. Therefore, for all $m \in [k]$, $J \in \binom{\mathscr{r}}{m}$, we consider different sets of edges in $\mathcal{H}[\mathscr{X}_{\cup\mathscr{r}}]$ where the m vertices corresponding to the clusters in J are either unembedded or activated and can potentially be mapped onto $\{v_i\}_{i\in J}$, and where the remaining $k - m$ vertices corresponding to the clusters in $\mathscr{r} \setminus J$ have already been embedded onto $\{v_i\}_{i\in\mathscr{r}\setminus J}$. Let $m \in [k]$ and $J \in \binom{\mathscr{r}}{m}$ be fixed.

We first consider the set of H-edges $\mathscr{e} = \{x_i\}_{i\in\mathscr{r}}$ for all $H \in \mathcal{H}$ where no vertex of $\mathscr{e}$ is left unembedded by ϕ_r and for every vertex of $\mathscr{g}$, there is an H-vertex that is mapped onto $\mathscr{g}$. That is, the m vertices $\{x_i\}_{i\in J}$ as well as the m vertices $\{v_i\}_{i\in J}$ are activated, the vertices $\{x_i\}_{i\in J}$ can potentially be mapped onto $\{v_i\}_{i\in J}$, and $\phi_r(\{x_i\}_{i\in\mathscr{r}\setminus J}) = \{v_i\}_{i\in\mathscr{r}\setminus J}$. To that end, we first consider the set $\widetilde{E} := E_{\phi_r}(\mathscr{g}, J, J_X = \emptyset, J_V = \emptyset) \setminus E_{\phi_r}(\mathscr{g}, J, J_X = \emptyset, J_V \neq \emptyset)$ as defined in (5.6.28). (Note that we defined this set in (5.6.28) only in the more convenient way that $\mathscr{g} \cap V_{\cup J_V} \subseteq \mathscr{g} \setminus \phi_r(X^H_{\cup\mathscr{r}})$, which is the reason why we remove the set $E_{\phi_r}(\mathscr{g}, J, J_X = \emptyset, J_V \neq \emptyset)$ from the current consideration.) For an edge $\mathscr{e} = \{x_i\}_{i\in\mathscr{r}} \in \widetilde{E}$, in order that $\{x_i\}_{i\in J}$ can be mapped onto $\{v_i\}_{i\in J}$ during the completion, it must hold that the vertices $\{x_i\}_{i\in J}$ and the vertices $\{v_i\}_{i\in J}$ must become activated. Since $\phi_r(\mathscr{x}) \neq \mathscr{g}$ and $\phi_r(\{x_i\}_{i\in\mathscr{r}\setminus J}) = \{v_i\}_{i\in\mathscr{r}\setminus J}$, it holds that $\phi_r(\{x_i\}_{i\in J}) \neq \{v_i\}_{i\in J}$. Hence, the probability that $\{x_i\}_{i\in J}$ and $\{v_i\}_{i\in J}$ become activated is at most μ^{m+1} because every vertex in $\widehat{X}^H_i$ for all $i \in [r]$, $H \in \mathcal{H}$ is activated independently with probability μ. Further, by $(\mathrm{K})_{\mathrm{C4}}$, activated vertices $\{x_i\}_{i\in J}$ are mapped onto $\{v_i\}_{i\in J}$ with probability at most $2^m \prod_{i\in J}(\mu d_i n)^{-1}$. Altogether, this implies that that the probability that some edge in $\widetilde{E}$ is mapped onto $\mathscr{g}$ is at most

$$\mu^{m+1}2^m\prod_{i\in J}(\mu d_i n)^{-1}\big|E_{\phi_r}(\mathscr{g}, J, J_X = \emptyset, J_V = \emptyset)\big|$$

$$\overset{\text{Claim 3}}{\leq} \mu^{m+1}2^{m+1}\prod_{i\in J}(\mu d_i n)^{-1}\cdot\gamma^{-1}n^m\prod_{i\in J}d_i \leq \mu^{4/5}, \tag{5.6.50}$$

where we used for the last inequality that $\mu \ll \gamma$.

Next, we consider the sets $E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$ where $J_{XV} = J_X \cup J_V$ is non-empty for disjoint $J_X, J_V \subseteq J$. For an edge $\mathscr{e} = \{x_i\}_{i\in\mathscr{r}} \in E_{\phi_r}(\mathscr{g}, J, J_X, J_V)$, the vertices $\{x_i\}_{i\in J}$ are mapped onto $\{v_i\}_{i\in J}$ with probability at most $2^m\prod_{i\in J}(\mu d_i n)^{-1}$ by $(\mathrm{K})_{\mathrm{C4}}$. This implies that the probability that some edge in $E_{\phi_r}(\mathscr{g}, J, J_X = \emptyset, J_V)$ is mapped onto $\mathscr{g}$ is at most

$$2^m\prod_{i\in J}(\mu d_i n)^{-1}\big|E_{\phi_r}(\mathscr{g}, J, J_X, J_V)\big|$$

$$\overset{\text{Claim 3}}{\leq} 2^m\prod_{i\in J}(\mu d_i n)^{-1}\cdot\varepsilon_T^{1/2}n^m\prod_{i\in J}d_i \leq \varepsilon_T^{1/3}. \tag{5.6.51}$$

Now, Claim 7 is established by a union bound over all $m \in [k]$, $J \in \binom{\mathscr{r}}{m}$, and all possible sets J_X, J_V that we considered to be fixed in (5.6.50) and (5.6.51). $-$

Step 5.9. Proof of Claim 6 – Bound in (5.6.48)

We use Claim 7 to verify the claimed bound in (5.6.48). We fix $i \in [r]$, $\mathcal{S} \subseteq \bigcup_{\mathscr{r}\in E(R):\, i\in\mathscr{r}} V_{\sqcup\mathscr{r}\setminus\{i\}}$ with $|\mathcal{S}| \leq t$ and $S \in \mathcal{S}$. By Claim 7, each G_B-edge is used with probability at most $\mu^{3/4}$ during the RPP, and thus, by an application of Lemma 1.10, we have that $|V_i \cap N_{G^\circ_h}(S) \cap N_{G_B}(\mathcal{S})| \leq \mu^{7/10}d_B^{|\mathcal{S}|}n$ with probability at least, say, $1 - \mathrm{e}^{-n^{3/4}}$. Otherwise we detect alarm. Together with a union bound this implies the claimed bound in (5.6.48) because $t \cdot \mu^{7/10}d_B^{|\mathcal{S}|}n \leq \mu^{2/3}d_B^{|\mathcal{S}|}n$.

Step 5.10. Proof of Claim 6 – Bound for $\rho^H_{G^\circ_h}(v)$

We also use Claim 7 to verify the claimed bound for $\rho^H_{G^\circ_h}(v)$ in (5.6.49). (Recall the definition of $\rho^H_{G^\circ_h}(v)$ in (5.6.26).) For all $H \in \mathcal{H}$, $i \in [r]$ and $v \in V_i$, we have that v has at most $(1 + 2\varepsilon_T)d_i n$ neighbours in B^H_i. For each such neighbour $x \in N_{B^H_i}(v)$, there exists a set $\mathcal{S}_x \subseteq \binom{V(G)}{k-1}$ with $|\mathcal{S}_x| \leq t^{2/3}$ such that $N_{B_i^H}(x) = N_{G_B}(\mathcal{S}_x)$. Hence, there exist at most $2t^{2/3}d_in$ edges $g = S \cup \{v\}$ in G_B for some $S \in \bigcup_{x \in N_{B_i^H}(v)} \mathcal{S}_x$. By Claim 7, each such G_B-edge g is used with probability at most $\mu^{3/4}$ during the entire RPP. We therefore expect that for each $h \in [|\mathcal{H}|]$ at most $\mu^{3/4}2t^{2/3}d_in$ edges incident to v in B_i^H have to be removed when we obtain $(B_i^H)^{G_h^\circ}$. Hence, by an application of Lemma 1.10 and a union bound, we obtain that $\rho^H_{G_h^\circ}(v) \leq \mu^{2/3}d_in$ for all $h \in [|\mathcal{H}|]$, $H \in \mathcal{H}$, $i \in [r]$, $v \in V_i$ with probability at least, say, $1 - \mathrm{e}^{-n^{3/4}}$. Otherwise we detect alarm. This implies the claimed bound in (5.6.49).

<u>*Step 5.11.*</u> *Proof of Claim 6 – Establishing* (iii) *of Lemma 5.19*

In order to establish conclusion (iii) of Lemma 5.19, let $(\omega, c) \in \mathcal{W}_{ver}$ with centres $c = \{c_i\}_{i \in I}$ in I be fixed. We split the proof into two parts, depending on whether we activate a vertex of a tuple $x \in \mathscr{X}_{\sqcup I}$ that ϕ_r already mapped onto c, or whether a tuple $x \in \mathscr{X}_{\sqcup I}$ contains vertices in $\mathcal{X}_r \setminus \mathcal{X}_r^{\phi_r}$ and x is mapped onto c during the completion.

Let us first consider tuples $x \in \mathscr{X}_{\sqcup I}$ that we have already embedded onto c, that is, $\phi_r(x) = c$, and where some vertices of x become activated. We claim that during the entire RPP not too many such tuples become activated. That is, we claim that

$$\omega\left(\bigcup_{H \in \mathcal{H}} \{x \in X^H_{\sqcup I} \colon \phi_r(x) = c, x \cap Y^H_{\sqcup I} \neq \emptyset\}\right) \leq \mu^{1/2}\omega(\phi_r^{-1}(c)) + n^{\varepsilon_T}, \tag{5.6.52}$$

with high probability. To see (5.6.52), note that for every $x \in \mathscr{X}_{\sqcup I}$, the probability that x contains an activated vertex is at most $|I|\mu$. Hence, an application of Theorem 1.8 and a union bound yield (5.6.52) with probability, say, at least $1 - \mathrm{e}^{-n^{\varepsilon}}$.

Next, we consider tuples $x \in \mathscr{X}_{\sqcup I}$ that have not been embedded onto c by ϕ_r. To that end, we fix $m \in [|I|]$ and $J \in \binom{I}{m}$, and a pair of disjoint sets $J_X, J_V \subseteq J$. We aim to control the ω-weight on the tuples in $E_{\phi_r}(c, J, J_X, J_V)$ as defined in (5.6.28). Analogously as in Claim 3, we can employ $\mathbf{S}(r)$(b) and (d) for the edge testers that we defined for (ω, c) in (5.6.5). Proceeding as in Claim 3 yields that

$$\omega\left(E_{\phi_r}(c, J, J_X, J_V)\right) \leq (\mathbb{1}\{J_{XV} = \emptyset\} + 2\varepsilon_T) \prod_{i \in J} d_i \frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|-|J|}} + n^{2\varepsilon_T}. \tag{5.6.53}$$

We can now proceed similarly as in the proof of Claim 7 in Step 5.8; that is, we consider different cases and sets of tuples $x \in \mathscr{X}_{\sqcup I}$ that could potentially be embedded onto c, and derive an upper bound on the expected ω-weight used in such a case. In the end, a union bound over all these cases gives us an upper bound on the total expected ω-weight of tuples that are embedded onto c during the completion. Therefore, we consider different choices for $J_X, J_V \subseteq J$.

For $x = \{x_i\}_{i \in I} \in E_{\phi_r}(c, J, J_X = \emptyset, J_V = \emptyset) \setminus E_{\phi_r}(c, J, J_X = \emptyset, J_V \neq \emptyset)$, in order that $\{x_i\}_{i \in J}$ can be mapped onto $\{c_i\}_{i \in J}$ during the RPP, it must hold that the vertices $\{x_i\}_{i \in J}$ and $\{c_i\}_{i \in J}$ become activated because $J_V = \emptyset$. Since $\phi_r(\{x_i\}_{i \in J}) \neq \{c_i\}_{i \in J}$, this happens with probability at most μ^{m+1}. Further, by $(\mathrm{K})_{\mathrm{C4}}$, the activated vertices $\{x_i\}_{i \in J}$ are mapped onto $\{c_i\}_{i \in J}$ with probability at most $2^m \prod_{i \in J}(\mu d_i n)^{-1}$. Altogether, this implies that the expected weight of edges in $E_{\phi_r}(c, J, J_X = \emptyset, J_V =$

$\emptyset) \setminus E_{\phi_r}(\mathscr{c}, J, J_X = \emptyset, J_V \neq \emptyset)$ that are mapped onto $\mathscr{c}$ during the RPP is at most

$$\mu^{m+1}2^m \prod_{i\in J}(\mu d_i n)^{-1}\omega\left(E_{\phi_r}\left(\mathscr{c}, J, J_X = \emptyset, J_V = \emptyset\right)\right)$$

$$\stackrel{(5.6.53)}{\leq} \mu^{m+1}2^m \prod_{i\in J}(\mu d_i n)^{-1}\left(2\prod_{i\in J} d_i \frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|-|J|}} + n^{2\varepsilon_T}\right) \leq \mu^{4/5}\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|}} + n^{3\varepsilon_T}. \tag{5.6.54}$$

Next, we consider the set $E_{\phi_r}(\mathscr{c}, J, J_X, J_V)$ for disjoint but fixed $J_X, J_V \subseteq J$ such that $J_{XV} = J_X \cup J_V \neq \emptyset$. The vertices $\{x_i\}_{i\in J}$ are mapped onto $\{c_i\}_{i\in J}$ with probability at most $2^m \prod_{i\in J}(\mu d_i n)^{-1}$ by $(\mathrm{K})_{\mathrm{C4}}$. Hence, the expected weight of edges in $E_{\phi_r}(\mathscr{c}, J, J_X, J_V)$ that are mapped onto $\mathscr{c}$ during the RPP is at most

$$2^m \prod_{i\in J}(\mu d_i n)^{-1}\omega\left(E_{\phi_r}\left(\mathscr{c}, J, J_X, J_V\right)\right) \stackrel{(5.6.53)}{\leq} \varepsilon_T^{1/2}\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|}} + n^{3\varepsilon_T}. \tag{5.6.55}$$

Altogether, a union bound over all $m \in [|I|]$, $J \in \binom{I}{m}$, and all sets $J_X, J_V \subseteq J$ that we considered to be fixed in (5.6.54) and (5.6.55) together with an application of Lemma 1.10 yields that

$$\omega\left(\left\{\mathscr{x} \in \mathscr{X}_{\sqcup I} \colon \phi_r(\mathscr{x}) \neq \mathscr{c}, \phi^{|\mathcal{H}|}(\mathscr{x}) = \mathscr{c}\right\}\right) \leq \mu^{1/2}\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|}} + n^{4\varepsilon_T}$$

with probability at least, say, $1 - \mathrm{e}^{-n^{2\varepsilon_T}}$.

Combining this with (5.6.52) yields

$$\begin{aligned}\omega((\phi^{|\mathcal{H}|})^{-1}(\mathscr{c})) &= (1 \pm \mu^{1/2})\omega(\phi_r^{-1}(\mathscr{c})) \pm \mu^{1/2}\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|}} \pm 2n^{4\varepsilon_T} \\ &\stackrel{\mathbf{S}(r)(\mathrm{g})}{=} (1 \pm \alpha)\frac{\omega(\mathscr{X}_{\sqcup I})}{n^{|I|}} \pm n^{\alpha}.\end{aligned}$$

This establishes conclusion (iii) of Lemma 5.19 and completes the proof of Claim 6.

–

Step 5.12. Finishing the completion

As the RPP outputs $\phi^{|\mathcal{H}|}$ with positive probability by Claim 6, we obtain a packing $\phi := \phi^{|\mathcal{H}|}$ of $\mathcal{H}$ into G which clearly satisfies conclusions (i) and (iii) of Lemma 5.19. Note that by the construction of $\phi^{|\mathcal{H}|}$, we have that $\phi|_{X_i^H} = \phi_r|_{X_i^H \setminus Y_i^H} \cup \phi^H|_{Y_i^H}$ for all $H \in \mathcal{H}$, $i \in [r]$. Since $|Y_i^H| = (1 \pm \varepsilon_T^{1/2})\mu n$, we therefore merely modified ϕ_r to obtain ϕ and thus, $\mathbf{S}(r)$(f) easily implies conclusion (ii) of Lemma 5.19. This completes the proof of Lemma 5.19. □

5.7 Proof of the main results

In this section we prove Theorem 5.2 and Theorem 5.3.

Proof of Theorem 5.2. Our general approach is as follows. Given a blow-up instance $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$, we refine the vertex partition $\mathcal{X}$ of the graphs in $\mathcal{H}$ using Lemma 5.9 and we randomly refine the vertex partition $\mathcal{V}$ of G accordingly. Afterwards we can apply Lemma 5.19 to obtain the required packing of $\mathcal{H}$ into G.

Suppose $1/n \ll \varepsilon \ll 1/t \ll \beta \ll \alpha, 1/k$ for a new parameter β. For each $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$ with $W \subseteq V_i$, $i \in [r]$, and each $\ell \in [m]$, let $\omega_{Y_\ell} \colon \bigcup_{H\in\mathcal{H}} X_i^H \to$

$\{0,1\}$ be such that $\omega_{Y_\ell}(x) = \mathbb{1}\{x \in Y_\ell\}$, and let $\mathcal{W}_Y$ be the set containing all those weight functions. Further, for all $\mathscr{r} \in E(R)$, let $\omega_{\mathscr{r}} \colon \mathscr{X}_{\sqcup \mathscr{r}} \to \{0,1\}$ be defined by $\omega_{\mathscr{r}}(\mathscr{x}) := \mathbb{1}\{\mathscr{x} \in E(\mathcal{H})\}$, and let $\mathcal{W}_{\mathcal{H}}$ be the set containing all those weight functions. We apply Lemma 5.9 to $\mathcal{H}$ with weight functions $\{\omega \colon (\omega, \mathscr{c}) \in \mathcal{W}_{ver}\} \cup \mathcal{W}_Y \cup \mathcal{W}_{\mathcal{H}}$. This yields a refined partition $\mathcal{X}' = (X^H_{i,j})_{H \in \mathcal{H}, i \in [r], j \in [\beta^{-1}]}$ of $\mathcal{H}$ such that for all $H \in \mathcal{H}$ and $i \in [r]$, the partitions $(X^H_{i,j})_{j \in [\beta^{-1}]}$ of X^H_i satisfy the conclusions (i)–(iv) of Lemma 5.9.

Let R' be the k-graph with vertex set $[r] \times [\beta^{-1}]$ and edge set

$$\left\{ \{(i_\ell, j_\ell)\}_{\ell \in [k]} \colon \{i_\ell\}_{\ell \in [k]} \in E(R), j_\ell \in [\beta^{-1}] \right\}.$$

Note that $\Delta(R') \leq \alpha^{-1}\beta^{-(k-1)}$ because $\Delta(R) \leq \alpha^{-1}$. Let $n' := \beta n$.

Employing conclusion (iv) of Lemma 5.9 for the weight functions in $\mathcal{W}_{\mathcal{H}}$ implies for all $\{(i_\ell, j_\ell)\}_{\ell \in [k]} \in E(R')$ with $\mathscr{r} := \{i_\ell\}_{\ell \in [k]} \in E(R)$ that

$$\begin{aligned}\sum_{H \in \mathcal{H}} e_H(X^H_{i_1,j_1}, \ldots, X^H_{i_k,j_k}) &\leq (1+\varepsilon)\beta^k \omega_{\mathscr{r}}(\mathscr{X}_{\sqcup \mathscr{r}}) + n^{1+\varepsilon} \\ &= (1+\varepsilon)\beta^k e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) + n^{1+\varepsilon} \leq (1-\alpha/2)dn'^k,\end{aligned}$$

because by assumption, $e_{\mathcal{H}}(\mathscr{X}_{\sqcup \mathscr{r}}) \leq (1-\alpha)dn^k$ and $d \geq n^{-\varepsilon}$.

Further, note that Lemma 5.9(i) implies for each $H \in \mathcal{H}$ that $H[X^H_{\cup \mathscr{r}}]$ is a matching if $\mathscr{r} \in E(R')$ and empty if $\mathscr{r} \in \binom{[r] \times [\beta^{-1}]}{k} \setminus E(R')$.

According to the refinement $\mathcal{X}'$ of $\mathcal{X}$, we claim that there exists a refined partition $\mathcal{V}' = (V_{i,j})_{i \in [r], j \in [\beta^{-1}]}$ of $\mathcal{V}$, where $(V_{i,j})_{j \in [\beta^{-1}]}$ is a partition of V_i for every $i \in [r]$ such that

(a) $|W \cap V_{i,j}| = \beta|W| \pm \beta^{3/2}n$ for all $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$ and $j \in [\beta^{-1}]$ with $W \subseteq V_i$;

(b) $\mathscr{B}' := (\mathcal{H}, G, R', \mathcal{X}', \mathcal{V}')$ is an $(\varepsilon^{1/2}, t, d)$-typical, β^{-k}-bounded blow-up instance of size $(n', k, \beta^{-1}r)$ with $n' = \beta n$.

The existence of such a partition $\mathcal{V}'$ can be seen by a probabilistic argument. For all $i \in [r]$ and $j \in [\beta^{-1}]$, we take disjoint subset $V_{i,j}$ of V_i of size exactly $|X^H_{i,j}|$ uniformly and independently at random. We analyse the probability that (a) or (b) are not satisfied. To that end, we consider the slightly different random experiment where we assign for all $i \in [r]$, every vertex in V_i uniformly and independently at random to some $V_{i,j}$ for $j \in [\beta^{-1}]$. For this experiment and a fixed $i \in [r]$, we consider the bad events that (a) or (b) are not satisfied; in such a case, we say the experiment *fails* (in step i). Standard properties of the multinomial distribution yield that $|V_{i,j}| = |X^H_{i,j}|$ for all $j \in [\beta^{-1}]$, $H \in \mathcal{H}$ with probability at least $\Omega(n^{-\beta^{-1}})$. Hence, together with Theorem 1.8 and a union bound, this yields that the original experiment fails in step i with probability, say, at most $\mathrm{e}^{-n^{1/2}}$. Since in fact we take $V_{i,j}$ of size exactly $|X^H_{i,j}|$, this altogether implies the existence of a refined partition $\mathcal{V}'$ of $\mathcal{V}$ satisfying (a) and (b) with positive probability.

We show how to adapt the vertex and set testers from the original blow-up instance to the blow-up instance $\mathscr{B}'$. For each $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$ and distinct $H_1, \ldots, H_m \in \mathcal{H}$ such that $W \subseteq V_i$ for some $i \in [r]$ and $Y_\ell \subseteq X^{H_\ell}_i$ for all $\ell \in [m]$, we define $(W_j, Y_{1,j}, \ldots, Y_{m,j})$ by setting $W_j := W \cap V_{i,j}$ and $Y_{\ell,j} := Y_\ell \cap X^{H_\ell}_{i,j}$ for all $j \in [\beta^{-1}], \ell \in [m]$. By (a), we conclude that $|W_j| = \beta|W| \pm \beta^{3/2}n$. Employing conclusion (iii) of Lemma 5.9 for the weight function $\omega_{Y_\ell} \in \mathcal{W}_Y$, we have that

$$|Y_{\ell,j}| = \omega_{Y_\ell}(X^{H_\ell}_{i,j}) = \beta\omega_{Y_\ell}(X^{H_\ell}_i) \pm \beta^{3/2}n = \beta|Y_\ell| \pm \beta^{3/2}n. \tag{5.7.1}$$

Let $\mathcal{W}'_{set} := \{(W_j, Y_{1,j}, \dots, Y_{m,j}) \colon j \in [\beta^{-1}], (W, Y_1, \dots, Y_m) \in \mathcal{W}_{set}\}$. For each $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ with centres $\mathscr{c} = \{c_i\}_{i\in I}$ in $I \subseteq [r]$ and multiset $\{j_i\}_{i\in I} \subseteq [\beta^{-1}]$ such that $c_i \in V_{i,j_i}$ for each $i \in I$, let $\omega' := \omega|_{\bigcup_{H\in\mathcal{H}, i\in I} X^H_{i,j_i}}$ and $\mathcal{W}'_{ver} := \{(\omega', \mathscr{c}) \colon (\omega, \mathscr{c}) \in \mathcal{W}_{ver}\}$.

Hence, we can apply Lemma 5.19 to $\mathscr{B}'$ with set testers $\mathcal{W}'_{set}$ and vertex testers $\mathcal{W}'_{ver}$ as follows:

parameter	n'	$\varepsilon^{1/2}$	t	β^k	d	$r\beta^{-1}$
plays the role of	n	ε	t	α	d	r

This yields a packing of $\mathcal{H}$ into G such that

(I) $\phi(X^H_{i,j}) = V_{i,j}$ for all $i \in [r]$, $j \in [\beta^{-1}]$, $H \in \mathcal{H}$;

(II) $|W_j \cap \bigcap_{\ell\in[m]} \phi(Y_{\ell,j})| = |W_j||Y_{1,j}| \cdots |Y_{m,j}|/n'^m \pm \beta^k n'$ for all $(W_j, Y_{1,j}, \dots, Y_{\ell,j}) \in \mathcal{W}'_{set}$;

(III) $\omega'(\phi^{-1}(\mathscr{c}))) = (1 \pm \beta^k)\omega'(\bigcup_{H\in\mathcal{H}}(\bigsqcup_{i\in I} X^H_{i,j_i}))/n'^{|I|} \pm n'^{\beta^k}$ for all $(\omega', \mathscr{c}) \in \mathcal{W}'_{ver}$ with centres $\mathscr{c} = \{c_i\}_{i\in I}$ in $I \subseteq [r]$ and multiset $\{j_i\}_{i\in I} \subseteq [\beta^{-1}]$ such that $c_i \in V_{i,j_i}$ for each $i \in I$.

For $(W, Y_1, \dots, Y_m) \in \mathcal{W}_{set}$, we conclude that

$$\begin{aligned}
\left|W \cap \textstyle\bigcap_{\ell\in[m]} \phi(Y_\ell)\right| &= \sum_{j\in[\beta^{-1}]} \left|W_j \cap \textstyle\bigcap_{\ell\in[m]} \phi(Y_{\ell,j})\right| \\
&\overset{\text{(II),(a),(5.7.1)}}{=} \sum_{j\in[\beta^{-1}]} \left(\frac{\beta^{m+1}\left(|W||Y_1|\cdots|Y_m| \pm \beta^{1/3} n^{m+1}\right)}{(\beta n)^m} \pm \beta^k n' \right) \\
&= |W||Y_1|\cdots|Y_m|/n^m \pm \alpha n.
\end{aligned}$$

This establishes Theorem 5.2(i).

In order to establish Theorem 5.2(ii), we fix $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ with centres $\mathscr{c} = \{c_i\}_{i\in I}$ for $I \subseteq [r]$ and multiset $\{j_i\}_{i\in I} \subseteq [\beta^{-1}]$ such that $c_i \in V_{i,j_i}$ for each $i \in I$, and we fix the corresponding tuple $(\omega', \mathscr{c}) \in \mathcal{W}'_{ver}$. We conclude that

$$\begin{aligned}
\omega(\phi^{-1}(\mathscr{c})) &\overset{\text{(I)}}{=} \omega'(\phi^{-1}(\mathscr{c})) \overset{\text{(III)}}{=} (1 \pm \beta^k) \frac{\omega'\left(\bigcup_{H\in\mathcal{H}}\left(\bigsqcup_{i\in I} X^H_{i,j_i}\right)\right)}{n'^{|I|}} \pm n'^{\beta^k} \\
&= (1 \pm \beta^k) \frac{\omega\left(\bigcup_{H\in\mathcal{H}}\left(\bigsqcup_{i\in I} X^H_{i,j_i}\right)\right)}{n'^{|I|}} \pm n'^{\beta^k} \\
&\overset{\text{(iv)}}{=} (1 \pm \beta^k) \frac{(1\pm\varepsilon)\beta^{|I|}\omega(\mathscr{X}_{\sqcup I}) \pm n^{1+\varepsilon}}{(\beta n)^{|I|}} \pm n'^{\beta^k} = (1 \pm \alpha)\omega(\mathscr{X}_{\sqcup I})/n^{|I|} \pm n^\alpha,
\end{aligned}$$

where we employed conclusion (iv) of Lemma 5.9 in the penultimate equation. This establishes Theorem 5.2(ii) and completes the proof. □

We now proceed to the proof of Theorem 5.3. The highlevel strategy is similar as in the proof of Theorem 5.2. Additionally, we group the hypergraphs in $\mathcal{H}$ into $P = \operatorname{polylog} n$ many collections of hypergraphs $\mathcal{H}_1, \dots, \mathcal{H}_P$ and accordingly we partition the edge set of the host graph G into $G_1, \dots, G_P$ subgraphs. Afterwards, we partition the vertex sets of the graphs in $\mathcal{H}$ via Lemma 5.10 and randomly partition the vertex set of each G_p for $p \in [P]$ accordingly. Then, we can iteratively apply Lemma 5.19 for each $p \in [P]$ to map $\mathcal{H}_p$ into G_p. Note that this yields a packing of $\mathcal{H}$ into G, and considering $P = \operatorname{polylog} n$ partitions enables us to establish conclusions (i) and (ii) of Theorem 5.3.

Proof of Theorem 5.3. We set $P := \log^t n$ and suppose $1/n \ll \varepsilon \ll 1/t \ll \beta \ll \alpha, 1/k$ for a new parameter β.

First, we group the graphs in $\mathcal{H}$ into P collections $\mathcal{H}_1, \ldots, \mathcal{H}_P$ with roughly equally many edges. That is, we claim that there exists a partition of $\mathcal{H}$ into P collections of graphs $\mathcal{H}_1, \ldots, \mathcal{H}_P$ such that each $H \in \mathcal{H}$ belongs to exactly one $\mathcal{H}_p$ for $p \in [P]$, and for each $p \in [P]$, we have

$$e(\mathcal{H}_p) \leq (1+\varepsilon)P^{-1}e(\mathcal{H}) + n^{1+\varepsilon}, \tag{5.7.2}$$

and for every $\omega \in \mathcal{W}_{ver}$ with $\omega(V(\mathcal{H})) \geq n^{1+\varepsilon}$ and each $p \in [P]$, we have

$$\omega(V(\mathcal{H}_p)) = (1 \pm \varepsilon)P^{-1}\omega(V(\mathcal{H})). \tag{5.7.3}$$

The existence of $\mathcal{H}_1, \ldots, \mathcal{H}_P$ can be easily seen by assigning every graph $H \in \mathcal{H}$ to one collection $\mathcal{H}_p$ for $p \in [P]$ uniformly and independently at random.

We now aim to apply Lemma 5.10 to each $\mathcal{H}_p$. Let $p \in [P]$ be fixed. For each $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$ and each $\ell \in [m]$ such that $Y_\ell \subseteq V(H_\ell)$ for $H_\ell \in \mathcal{H}_p$, let $\omega_{Y_\ell} \colon V(H_\ell) \to \{0,1\}$ be such that $\omega_{Y_\ell}(x) = \mathbb{1}\{x \in Y_\ell\}$, and let $\mathcal{W}_Y$ be the set containing all those weight functions. Further, let $\omega_{edges} \colon \bigcup_{H \in \mathcal{H}} \binom{V(H)}{k}_{\prec} \to \{0,1\}$ be defined by $\omega_{edges}((x_1, \ldots, x_k)) := \mathbb{1}\{\{x_1, \ldots, x_k\} \in E(\mathcal{H}_p)\}$, and let $\mathcal{W}_{\mathcal{H}_p}$ be the set containing all those weight functions. We apply Lemma 5.10 to $\mathcal{H}_p$ with weight functions $\{\omega|_{\cup V(\mathcal{H}_p)} \colon (\omega, \mathscr{c}) \in \mathcal{W}_{ver}\} \cup \mathcal{W}_Y \cup \mathcal{W}_{\mathcal{H}_p}$. This yields a partition $\mathscr{X}_p = (X_j^H)_{H \in \mathcal{H}_p, j \in [\beta^{-1}]}$ of $\mathcal{H}_p$ such that for all $H \in \mathcal{H}_p$, the partitions $(X_j^H)_{j \in [\beta^{-1}]}$ of $V(H)$ satisfy the conclusions (i)–(iv) of Lemma 5.10.

For each $p \in [P]$, let R_p be the k-graph with vertex set $[\beta^{-1}]$ and $\mathscr{r} \in \binom{[\beta^{-1}]}{k}$ is an edge in R_p if $H[X_{\sqcup \mathscr{r}}^H]$ is non-empty for some $H \in \mathcal{H}_p$. Clearly, $\Delta(R_p) \leq \beta^{-k}$. Let $n' := \beta n$.

Employing conclusion (iv) of Lemma 5.10 for the weight functions in $\mathcal{W}_{\mathcal{H}_p}$ yields for all $\mathscr{r} = \{i_1, \ldots, i_k\} \in E(R_p)$ that

$$\begin{aligned}\sum_{H \in \mathcal{H}_p} e_H(X_{\sqcup \mathscr{r}}^H) &= \omega_{edges}(\textstyle\bigcup_{H \in \mathcal{H}_p}(X_{i_1}^H \times \ldots \times X_{i_k}^H)) \leq (1+\beta^{1/2})\beta^k \omega_{edges}(V(\mathcal{H}_p)) + n^{1+\varepsilon} \\ &= (1+\beta^{1/2})\beta^k k! e(\mathcal{H}_p) + n^{1+\varepsilon} \overset{(5.7.2)}{\leq} (1-\alpha/2)P^{-1}dn'^k \end{aligned} \tag{5.7.4}$$

because by assumption, $e(\mathcal{H}) \leq (1-\alpha)e(G)$.

Now, we want to prepare G accordingly to $\mathcal{H}_1, \ldots, \mathcal{H}_P$ and their partitions. To that end, we first partition G into P edge-disjoint spanning subgraphs $G_1, \ldots, G_P$ such that G_p is $(\varepsilon^{1/2}, t, P^{-1}d)$-typical for every $p \in [P]$. The existence of $G_1, \ldots, G_P$ can be seen by assigning every edge in G to one subgraph G_p for $p \in [P]$ uniformly and independently at random.

Further, we claim that there exist partitions $\mathcal{V}_p = (V_j)_{j \in [\beta^{-1}]}$ of $V(G_p)$ according to the partition $\mathscr{X}_p$ of $\mathcal{H}_p$, such that for every $p \in [P]$ and $\mathcal{V}_p = (V_j)_{j \in [\beta^{-1}]}$, we have

(a) $|W \cap V_j| = \beta|W| \pm \beta^{3/2}n$ for all $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{set}$ and $j \in [\beta^{-1}]$,

(b) $\mathscr{B}_p := (\mathcal{H}_p, G_p, R_p, \mathscr{X}_p, \mathcal{V}_p)$ is an $(\varepsilon^{1/2}, t, P^{-1}d)$-typical, β^{-k}-bounded blow-up instance of size (n', k, β^{-1}) with $n' = \beta n$,

as well as

(c) $\sum_{p\in[P]\colon \text{centres collide}} \omega(V(\mathcal{H}_p)) \le \beta^{1/2}\omega(V(\mathcal{H}))$ for all $(\omega, \mathscr{c}) \in \mathcal{W}_{ver}$ with $\omega(V(\mathcal{H})) \ge n^{1+\varepsilon}$, where we say that the *centres $\mathscr{c}$ collide (with respect to $\mathcal{V}_p$)* if $|\mathscr{c} \cap V_j| \ge 2$ for some $V_j \in \mathcal{V}_p$.

The existence of such partitions $\mathcal{V}_p$ can be seen by assigning for every $p \in [P]$, every vertex to some V_j for $j \in [\beta^{-1}]$ uniformly and independently at random. Theorem 1.8 and a union bound establish (a) with probability, say, at least $1 - e^{-n^{1/2}}$. For (b), note that standard properties of the multinomial distribution yield for $p \in [P]$ that $|V_j| = |X_j^H|$ for all $j \in [\beta^{-1}]$ and $H \in \mathcal{H}$ with probability at least $\Omega(n^{-\beta^{-1}})$. For (c), note that for $p \in [P]$, the probability that the centres $\mathscr{c}$ collide with respect to $\mathcal{V}_p$ is at most $k^2\beta$. By (5.7.3), we therefore expect that at most $\sum_{p\in[P]} k^2\beta\omega(V(\mathcal{H}_p)) \le 2k^2\beta\omega(V(\mathcal{H}))$ weight of ω collides in (c). Since P grows sufficiently fast in terms of n, we can establish concentration. That is, Theorem 1.8 and a union bound that yield (c) with probability, say, at least $1-n^{-\log n}$. Hence, a final union bound yields the existence of these partitions $\mathcal{V}_p$ for $p \in [P]$ satisfying (a)–(c) with positive probability.

Next, we iteratively apply Lemma 5.19 to $\mathscr{B}_p$ for $p \in [P]$ which yields a packing ϕ_p of $\mathcal{H}_p$ into G_p. Let us first explain how we adapt the vertex and set testers from the original blow-up instance to the blow-up instance $\mathscr{B}_p$.

For $p \in [P]$, we define

$$\mathcal{W}_{ver}(p) := \Big\{(\omega_p, \mathscr{c})\colon (\omega, \mathscr{c}) \in \mathcal{W}_{ver},\ \text{centres } \mathscr{c} = \{c_i\}_{i\in I} \text{ do not collide with respect to } \mathcal{V}_p,$$
$$\omega_p := \omega|_{\bigcup_{H\in\mathcal{H}_p, i\in I} X_{j_i}^H}\Big\},$$

where we define $\{j_i\}_{i\in I} \subseteq [\beta^{-1}]$ as the indices such that $c_i \in V_{j_i}$ for centres $\mathscr{c} = \{c_i\}_{i\in I}$ that do not collide w.r.t. $\mathcal{V}_p = (V_j)_{j\in[\beta^{-1}]}$.

For all $j \in [\beta^{-1}]$, $(W, Y_1, \dots, Y_m) \in \mathcal{W}_{set}$ and distinct $H_1, \dots, H_m \in \mathcal{H}$ such that $Y_\ell \subseteq V(H_\ell)$, we define $Y_{\ell,j} := Y_\ell \cap X_j^{H_\ell}$ for each $\ell \in [m]$. For $j \in [\beta^{-1}], p \in [P]$, we define

$$\mathcal{W}_{set}(j,p) := \Big\{\big(W_j(p), \{Y_{\ell,j}\}_{\ell\in[m]\colon Y_{\ell,j}\subseteq V(\mathcal{H}_p)}\big)\colon (W, Y_1, \dots, Y_m) \in \mathcal{W}_{set},\ Y_{\cup[m]} \cap V(\mathcal{H}_p) \ne \emptyset\Big\},$$

where we define $W_j(p)$ for $(W, Y_1, \dots, Y_m) \in \mathcal{W}_{set}$ recursively by

$$W_j(p) := W_j(p-1) \cap \bigcap_{\ell\in[m]\colon Y_{\ell,j}\subseteq V(\mathcal{H}_{p-1})} \phi_{p-1}(Y_{\ell,j})$$

with ϕ_0 being the empty function and thus, $W_j(1) = W_j(0) := W \cap V_j$. By (a), we have that $|W \cap V_j| = \beta|W| \pm \beta^{3/2}n$. By employing conclusion (iii) of Lemma 5.10 for the weight function $\omega_{Y_\ell} \in \mathcal{W}_Y$, we have that

$$|Y_{\ell,j}| = \omega_{Y_\ell}(X_j^{H_\ell}) \overset{\text{(iii)}}{=} \beta\omega_{Y_\ell}(V(H_\ell)) \pm \beta^{3/2}n = \beta|Y_\ell| \pm \beta^{3/2}n. \tag{5.7.5}$$

Hence, we iteratively apply Lemma 5.19 for every $p \in [P]$ to $\mathscr{B}_p$ with set testers $\bigcup_{j\in[\beta^{-1}]} \mathcal{W}_{set}(j,p)$ and vertex testers $\mathcal{W}_{ver}(p)$ as follows:

parameter	n'	$\varepsilon^{1/2}$	t	β^k	$P^{-1}d$	β^{-1}
plays the role of	n	ε	t	α	d	r

For every $p \in [P]$, this yields a packing ϕ_p of $\mathcal{H}_p$ into G_p such that

(I) $\phi_p(X_j^H) = V_j$ for all $j \in [\beta^{-1}]$, $H \in \mathcal{H}_p$;

(II) $|W_j(p)\cap\bigcap_{\ell\in[m]\colon Y_{\ell,j}\subseteq V(\mathcal{H}_p)}\phi_p(Y_{\ell,j})| = |W_j(p)|\prod_{\ell\in[m]\colon Y_{\ell,j}\subseteq V(\mathcal{H}_p)} n'^{-1}|Y_{\ell,j}| \pm \beta^k n'$ for all $(W_j(p),\{Y_{\ell,j}\}_{\ell\in[m]\colon Y_\ell\subseteq V(\mathcal{H}_p)})\in\mathcal{W}_{set}(j,p)$, $j\in[\beta^{-1}]$;

(III) $\omega_p(\phi_p^{-1}(\mathscr{c}))) = (1\pm\beta^k)\omega_p(\bigcup_{H\in\mathcal{H}_p}(\bigsqcup_{i\in I}X^H_{j_i}))/n'^{|I|}\pm n'^{\beta^k}$ for all $(\omega_p,\mathscr{c})\in\mathcal{W}_{ver}(p)$ with centres $\mathscr{c}=\{c_i\}_{i\in I}$ that do not collide with respect to $\mathcal{V}_p$ and $\{j_i\}_{i\in I}\subseteq[\beta^{-1}]$ such that $c_i\in V_{j_i}$ for each $i\in I$.

Let $\phi := \bigcup_{p\in[P]}\phi_p$ and note that ϕ is a packing of $\mathcal{H}$ into G.

For $(W,Y_1,\ldots,Y_m)\in\mathcal{W}_{set}$, we conclude that

$$\begin{aligned}\left|W\cap\bigcap\nolimits_{\ell\in[m]}\phi(Y_\ell)\right| &= \sum_{j\in[\beta^{-1}]}\left|W_j(P)\cap\bigcap\nolimits_{\ell\in[m]\colon Y_{\ell,j}\subseteq V(\mathcal{H}_P)}\phi_P(Y_{\ell,j})\right|\\ &\overset{\text{(II)}}{=} \sum_{j\in[\beta^{-1}]}\Big(|W_j(0)||Y_{1,j}|\cdots|Y_{m,j}|/n'^m \pm m\beta^k n'\Big)\\ &\overset{\text{(a),(5.7.5)}}{=} |W||Y_1|\cdots|Y_m|/n^m \pm \alpha n.\end{aligned}$$

This establishes Theorem 5.3(i).

In order to establish Theorem 5.3(ii), we fix $(\omega,\mathscr{c})\in\mathcal{W}_{ver}$ with centres $\mathscr{c}=\{c_i\}_{i\in I}$. Recall that $\mathrm{supp}(\omega)\subseteq E(\mathcal{H})$ for $|I|\geq 2$, and thus, if the centres $\mathscr{c}$ collide with respect to one of the partitions $\mathcal{V}_p$, then $\omega(\phi_p^{-1}(\mathscr{c}))=0$. Therefore, we can consider the corresponding tuples $(\omega_p,\mathscr{c})\in\mathcal{W}_{ver}(p)$ for $p\in[P]$ and conclude that

$$\begin{aligned}\omega(\phi^{-1}(\mathscr{c})) &= \sum_{p\in[P]}\omega(\phi_p^{-1}(\mathscr{c})) = \sum_{p\in[P]\colon\text{ no collision}}\omega_p(\phi_p^{-1}(\mathscr{c}))\\ &\overset{\text{(III)}}{=} \sum_{p\in[P]\colon\text{ no collision}}\left((1\pm\beta^k)\frac{\omega_p(\bigcup_{H\in\mathcal{H}_p}(\bigsqcup_{i\in I}X^H_{j_i}))}{n'^{|I|}}\pm n'^{\beta^k}\right)\\ &\overset{\text{(iv)}}{=} \sum_{p\in[P]\colon\text{ no collision}}\left((1\pm\beta^k)\frac{(1\pm\beta^{1/2})\beta^{|I|}\omega(V(\mathcal{H}_p))}{n'^{|I|}}\pm n'^{\beta^k}\right)\\ &\overset{\text{(c)}}{=} (1\pm\alpha)\frac{\omega(V(\mathcal{H}))}{n^{|I|}}\pm n^{\alpha}.\end{aligned}$$

This establishes Theorem 5.3(ii) and completes the proof. □

5.8 Applications

We believe that our main results will be useful for fruitful outcomes in forthcoming applications. We illustrate and discuss some of these applications in the following sections. Indeed, Theorem 1.6 can be applied to several natural questions on hypergraph decompositions, which we consider in Section 5.8.1. In particular, a direct consequence of our main result is an asymptotic solution to a hypergraph Oberwolfach problem asked by Glock, Kühn and Osthus [53].

Closely linked to hypergraphs are simplicial complexes which are equivalent to downward closed hypergraphs H; that is, whenever $\mathscr{e}$ is an edge of H, then H contains also all subsets of $\mathscr{e}$ as edges. Gowers was one of the first who suggested the investigation of topological analogues of 1-dimensional graph structures in higher dimensions as we may consider a Hamilton cycle in a graph as a spanning 1-dimensional simplicial complex that is homeomorphic to $\mathbb{S}^1$ and hence a Hamilton cycle in higher dimensions may be viewed as spanning k-dimensional simplicial complex that is homeomorphic to $\mathbb{S}^k$. In particular, Linial has considered further questions of this type under the term 'high dimensional combinatorics' and has achieved several new insights. We discuss implications of Theorem 1.6 to this type of questions in Section 5.8.2.

5.8.1 Applications to hypergraph decompositions

As we pointed out in the introduction, Conjectures 1.1–1.3 intrigued mathematicians for decades. In the following we propose several conjectures of similar spirit for k-graphs. Our main results will imply approximate versions thereof.

Recall that the Oberwolfach problem asks for a decomposition of K_n into $(n-1)/2$ copies of a graph on n vertices that is the disjoint union of cycles. There are many definitions for cycles in k-graphs and tight cycles are among the most well studied cycles. A k-graph is a tight cycle if its vertex set can be cyclically ordered and the edge set consists of all k-sets that appear consecutively in this ordering. We refer to the number of vertices in a tight cycle as its length. One potential version of a hypergraph Oberwolfach problem has recently been asked by Glock, Kühn and Osthus.

Conjecture 5.20 (Hypergraph Oberwolfach problem; Glock, Kühn and Osthus [53]). *Let $k \geq 3$ and suppose n is sufficiently large in terms of k and k divides $\binom{n-1}{k-1}$. Suppose F is a k-graph on n vertices that is the disjoint union of tight cycles each of length at least $2k-1$. Then there is a decomposition of $K_n^{(k)}$ into copies of F.*

Clearly, Theorem 1.6 yields an approximate solution of Conjecture 5.20.

We think that an even stronger result is true.

Conjecture 5.21 (Hypergraph Oberwolfach problem [31]). *Let $k \geq 3$ and suppose n is sufficiently large in terms of k and k divides $\binom{n-1}{k-1}$. Suppose F is a k-graph on n vertices that is the disjoint union of tight cycles each of length at least $k+2$. Then there is a decomposition of $K_n^{(k)}$ into copies of F.*

Observe that Conjecture 5.20 includes the natural generalisation of Walecki's theorem to hypergraphs, namely decompositions into Hamilton cycles. This has already been conjectured by Bailey and Stevens [11] (and when n and k are coprime by Baranyai [13] and independently by Katona) and there are a few results that provide approximate decompositions of quasirandom graphs into Hamilton cycles (of various types); see for example [12, 45, 46].

Whenever we allow cycles of length $k+1$ and the cycle factor consists (essentially) of cycles of length $k+1$, we suspect that there are more divisibility obstructions present. Hence we pose the following problem.

Problem 5.22 (Hypergraph Oberwolfach problem [31]). *Let $k \geq 3$ and suppose n is sufficiently large in terms of k and k divides $\binom{n-1}{k-1}$. Which disjoint unions of tight cycles whose length add up to n admit a decomposition of $K_n^{(k)}$?*

It immediately follows from Theorem 1.6 that the hypergraph Oberwolfach problems are approximately true in a sense that $K_n^{(k)}$ contains $(1-o(1))\binom{n-1}{k-1}/k$ disjoint copies of F (for any choice of F as above); in fact, we can we take any collection of $(1-o(1))\binom{n-1}{k-1}/k$ cycle factors.

Similarly as for cycles, there is more then one notion for trees in k-graphs. Let us stick to the following recursive definition of tree to which we refer as a k-tree. A single edge is a k-tree. A k-tree with ℓ edges can be constructed from a k-tree with $\ell-1$ edges T by adding a vertex v and an edge that contains v and a $(k-1)$-set that is contained in an edge of T. For this definition, we propose the following generalisation of Ringel's conjecture.

Conjecture 5.23 ([31])**.** *Let $k, n \in \mathbb{N} \setminus \{1\}$. Suppose T is a k-tree with n edges. Then $K^{(k)}_{kn+k-1}$ admits a decomposition into copies of T.*

Observe that similarly as for Ringel's conjecture, the order of the complete graph needs to be at least $kn + k - 1$ if we allow T to be any tree with n edges as the natural generalisation of a star shows. It is an easy exercise to show this conjecture for stars.

There is a conjecture related to Ringel's conjecture for bipartite graphs due to Graham and Häggkvist stating that $K_{n,n}$ can be decomposed into n copies of any tree with n edges. We propose here the following strengthening.

Conjecture 5.24 ([31])**.** *Suppose $k, n \in \mathbb{N} \setminus \{1\}$ and T is a k-tree with n edges. Then there is a decomposition of the complete balanced k-partite graph on kn vertices.*

The tree packing conjecture has arguably the least obvious strengthening to k-graphs and there may be more than one. We propose the following one.

Conjecture 5.25 ([31])**.** *Suppose $k, n \in \mathbb{N} \setminus \{1\}$. Let $\mathcal{T}$ be a family of k-trees such that $\mathcal{T}$ contains $\binom{n-i-1}{k-2}$ trees with i edges for $i \in [n - k + 1]$. Then $K^{(k)}_n$ admits a decomposition into $\mathcal{T}$.*

It follows directly from Theorem 1.6 that Conjectures 5.23 and 5.25 are approximately true when restricted to bounded degree trees (and similarly an approximate version of Conjecture 5.24 follows from our main theorem, see Theorem 5.1 in Section 5.1.1).

5.8.2 Applications to simplicial complexes

Generalizing long-studied and nowadays classical combinatorial questions to higher dimensions appears to be a challenging but insightful theme. There are several results considering k-dimensional permutations and it was Linial and Meshulam [96] who introduced a random model for simplicial complexes whose probability measure is the same as those of a binomial random $(d+1)$-graph; simply add all d'-faces for $d' \leq d-1$ with points in $[n]$ and add every potential d-face independently with probability p. Recently, Linial and Peled investigated and determined the threshold in $Y_d(n, p)$ for the emergence of a what may considered as an analogoue of a giant component [97].

With this topological viewpoint of treating a k-graph as a simplicial complex, a cycle in a graph is simply an object homomorphic to $\mathbb{S}^1$ and hence a 2-dimensional Hamilton cycle a collection of 2-faces containing every vertex and which is homomorphic to $\mathbb{S}^2$. In [98], Luria and Tessler determined the threshold in $Y_2(n, p)$ for the appearance of such a Hamilton cycle (another suitable term may be spanning triangulation of the sphere).

An analogue of Dirac's theorem was proved by Georgakopoulos, Haslegrave, Narayanan and Montgomery; to be precise, when every pair of vertices is contained in at least $n/3 + o(n)$ edges/2-faces of a 3-graph G, then there is a spanning triangulation of the sphere in G. This bound remains the same when we replace $\mathbb{S}^2$ by any other compact surface without boundary [49].

Instead of only asking for a single triangulation of some surface, we can of course also investigate decompositions into (spanning) triangulations of surfaces. Our results imply that every quasirandom simplicial complex (this in particular includes almost all graphs in $Y_d(n, p)$) can even be almost decomposed into any list of triangulations of any kind of manifolds provided the density does not vanish too quickly with n

and as long as every vertex is contained in at most a bounded number of d-faces. The triangulations may even be chosen in advance. Hence a precise statement for 2-complexes is as follows.

Corollary 5.26 ([31])**.** *For all $\alpha > 0$, there exist $n_0, h \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose G is an (ε, h, d)-typical 3-graph on n vertices with $d \geq n^{-\varepsilon}$ and $H_1, \ldots, H_\ell$ are spanning triangulations of $\mathbb{S}^2$ where every vertex is contained in at most α^{-1} 2-faces and $\ell \leq (1-\alpha)dn^2/12$. Then G contains edge-disjoint copies of $H_1, \ldots, H_\ell$ such that every 2-face of G is contained in at most one H_i.*

We omit statements for higher dimensions as they follow in the obvious way from Theorem 1.6. We wonder whether there is always an actual decomposition (subject to certain divisibility conditions). This might be easier than a decomposition into tight Hamilton cycles as the structure of tight Hamilton cycles seems to be more restrictive.

Conclusion

In this thesis I provided new versatile tools for graph and hypergraph decomposition problems. In general, one expects that each of these results is useful for further applications.

The presented result on pseudorandom hypergraph matchings in Chapter 2 is a convenient and flexible generalization of a classical result due to Pippenger and allows to find an almost perfect hypergraph matching with pseudorandom properties in a hypergraph with small codegrees. Since many combinatorial problems can be stated as a hypergraph matching problem (in particular, not only in the area of decompositions), we believe that this will be useful for future research in different areas of extremal combinatorics. We note that our result has already been used in the resolutions of Conjecture 1.1 in [76] and of Conjecture 1.3 in [75].

Problems on rainbow embeddings and rainbow colourings are a vibrant research area nowadays, not only from the perspective of graph colouring but also because many problems can be phrased as a rainbow problem (as we alluded in Chapter 3). The presented rainbow blow-up lemma in Chapter 3 is a general tool that allows to find a rainbow spanning subgraph in a quasirandom host graph whose edge-colouring is almost optimally bounded. We provided applications of this result in Section 3.7 to graph decompositions, graph labelings and orthogonal double covers. It is very likely that our result can be applied to further problems. Recently, our rainbow blow-up lemma has already been used in [40].

In Chapter 4 we strengthened the blow-up lemma for approximate decompositions and provided a short proof. The blow-up lemma for approximate decompositions has already shown its versatility and has been applied successfully in [21, 65, 77]. It is a key ingredient for the resolution of Conjectures 1.1 and 1.2 for bounded degree trees in [65], and it has been used to establish a bandwith theorem for approximate decompositions [21] which is in turn a key ingredient in the resolution [51] of Conjecture 1.3. Our result simplifies the statement of the main result and provides stronger quasirandom properties, which leads to an easier applicability.

By providing a new and significantly shorter proof of the blow-up lemma for approximate decompositions, we were able to overcome the obstacle for further generalizations of this decomposition result. Building upon our developed proof methods, we lifted the blow-up lemma for approximate decompositions to the setting of hypergraphs in Chapter 5. This is the first result on hypergraph decompositions into arbitrary spanning hypergraphs H, provided that H has bounded maximum degree. Similar as in the graph case, we believe that this approximate decomposition result will be suitable to obtain perfect hypergraph decompositions. To that end, it might be useful to apply it in combination with Keevash's results on designs and to use absorbing techniques for hypergraphs as pioneered by Glock, Kühn, Lo and Osthus [52]. In particular, it would be interesting to obtain deeper insights into the decomposition of simplicial complexes, and to make further progress on Conjectures 5.21–5.25, which are hypergraph analogues of Conjectures 1.1–1.3.

Parts of this thesis have already been published:

- The content of Chapter 2 is based on

  [29] S. Ehard, S. Glock, and F. Joos, *Pseudorandom hypergraph matchings*, Combin. Probab. Comput. **29** (2020), 868–885.

  DOI: `https://doi.org/10.1017/S0963548320000280`

- The content of Chapter 3 is based on

  [30] S. Ehard, S. Glock, and F. Joos, *A rainbow blow-up lemma for almost optimally bounded edge-colourings*, Forum Math. Sigma **8** (2020), e37.

  DOI: `https://doi.org/10.1017/fms.2020.38`

- The content of Chapter 4 is based on

  [32] S. Ehard and F. Joos, *A short proof of the blow-up lemma for approximate decompositions*, arXiv:2001.03506 (2020).

- The content of Chapter 5 is based on

  [31] S. Ehard and F. Joos, *Decompositions of quasirandom hypergraphs into hypergraphs of bounded degree*, arXiv:2011.05359 (2020).

Bibliography

[1] A. Adamaszek, P. Allen, C. Grosu, and J. Hladký, *Almost all trees are almost graceful*, Random Structures Algorithms **56** (2020), 948–987.

[2] R. Aharoni, A. Georgakopoulos, and P. Sprüssel, *Perfect matchings in r-partite r-graphs*, European J. Combin. **30** (2009), 39–42.

[3] M. Ajtai, J. Komlós, and E. Szemerédi, *A dense infinite Sidon sequence*, European J. Combin. **2** (1981), 1–11.

[4] M. Albert, A. Frieze, and B. Reed, *Multicoloured Hamilton cycles*, Electron. J. Combin. **2** (1995), Art. 10, 13 pages.

[5] P. Allen, J. Böttcher, D. Clemens, and A. Taraz, *Perfectly packing graphs with bounded degeneracy and many leaves*, arXiv:1906.11558 (2019).

[6] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Blow-up lemmas for sparse graphs*, arXiv:1612.00622 (2016).

[7] P. Allen, J. Böttcher, J. Hladký, and D. Piguet, *Packing degenerate graphs*, Adv. Math. **354** (2019).

[8] P. Allen, J. Böttcher, J. Skokan, and M. Stein, *Regularity inheritance in pseudo-random graphs*, Random Structures Algorithms **56** (2020), 306–338.

[9] N. Alon, V. Rödl, and A. Ruciński, *Perfect matchings in ε-regular graphs*, Electron. J. Combin. **5** (1998), 1–4.

[10] N. Alon and R. Yuster, *On a hypergraph matching problem*, Graphs Combin. **21** (2005), 377–384.

[11] R. F. Bailey and B. Stevens, *Hamiltonian decompositions of complete k-uniform hypergraphs*, Discrete Math. **310** (2010), 3088–3095.

[12] D. Bal and A. Frieze, *Packing tight Hamilton cycles in uniform hypergraphs*, SIAM J. Discrete Math. **26** (2012), 435–451.

[13] Z. Baranyai, *The edge-coloring of complete hypergraphs. I*, J. Combin. Theory Ser. B **26** (1979), 276–294.

[14] J. Baste, S. Ehard, and D. Rautenbach, *Non-monotone target sets for threshold values restricted to* 0, 1, *and the vertex degree*, arXiv:2007.03959 (2020).

[15] S. Bessy, S. Ehard, L.D. Penso, and D. Rautenbach, *Dynamic monopolies for interval graphs with bounded thresholds*, Discrete Appl. Math. **260** (2019), 256–261.

[16] J. Böttcher, J. Hladký, D. Piguet, and A. Taraz, *An approximate version of the tree packing conjecture*, Israel J. Math. **211** (2016), 391–446.

[17] J. Böttcher, Y. Kohayakawa, and A. Procacci, *Properly coloured copies and rainbow copies of large graphs with small maximum degree*, Random Structures Algorithms **40** (2012), 425–436.

[18] J. Böttcher, Y. Kohayakawa, A. Taraz, and A. Würfl, *An extension of the blow-up lemma to arrangeable graphs*, SIAM J. Discrete Math. **29** (2015), 962–1001.

[19] J. Böttcher, M. Schacht, and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós*, Math. Ann. **343** (2009), 175–205.

[20] C. Brause, S. Ehard, and D. Rautenbach, *Bounds and extremal graphs for degenerate subsets, dynamic monopolies, and partial incentives*, Discrete Math. **281** (2020), 216–223.

[21] P. Condon, J. Kim, D. Kühn, and D. Osthus, *A bandwidth theorem for approximate decompositions*, Proc. Lond. Math. Soc. **118** (2019), 1393–1449.

[22] C. Cooper, A. Frieze, M. Molloy, and B. Reed, *Perfect matchings in random r-regular, s-uniform hypergraphs*, Combin. Probab. Comput. **5** (1996), 1–14.

[23] M. Coulson and G. Perarnau, *Rainbow matchings in Dirac bipartite graphs*, Random Structures Algorithms **55** (2019), 271–289.

[24] B. Csaba, *On the Bollobás-Eldridge conjecture for bipartite graphs*, Combin. Probab. Comput. **16** (2007), 661–691.

[25] B. Csaba, D. Kühn, A. Lo, D. Osthus, and A. Treglown, *Proof of the 1-factorization and Hamilton decomposition conjectures*, Mem. Amer. Math. Soc. **244** (2016), monograph 1154, 164 pages.

[26] M. C. Dourado, S. Ehard, L.D. Penso, and D. Rautenbach, *Partial immunization of trees*, Discrete Optimization **35** (2020), 100568.

[27] M. Drmota and A. Lladó, *Almost every tree with m edges decomposes $K_{2m,2m}$*, Combin. Probab. Comput. **23** (2014), 50–65.

[28] R.A. Duke, H. Lefmann, and V. Rödl, *A fast approximation algorithm for computing the frequencies of subgraphs in a given graph*, SIAM J. Comput. **24** (1995), 598–620.

[29] S. Ehard, S. Glock, and F. Joos, *Pseudorandom hypergraph matchings*, Combin. Probab. Comput. **29** (2020), 868–885.

[30] ———, *A rainbow blow-up lemma for almost optimally bounded edge-colourings*, Forum Math. Sigma **8** (2020), e37.

[31] S. Ehard and F. Joos, *Decompositions of quasirandom hypergraphs into hypergraphs of bounded degree*, arXiv:2011.05359 (2020).

[32] ———, *A short proof of the blow-up lemma for approximate decompositions*, arXiv:2001.03506 (2020).

[33] S. Ehard and E. Mohr, *Rainbow triangles and cliques in edge-colored graphs*, European J. Combin. **84** (2020), 103037.

[34] S. Ehard, E. Mohr, and D. Rautenbach, *Biholes in balanced bipartite graphs*, arXiv:2004.03245 (2020).

[35] ———, *Low weight perfect matchings*, arXiv:2010.15418 (2020).

[36] S. Ehard and D. Rautenbach, *On some tractable and hard instances for partial incentives and target set selection*, Discrete Optimization **34** (2019), 100547.

[37] ———, *Vaccinate your trees!*, Theoretical Computer Science **772** (2019), 46–57.

[38] ———, *Approximating connected safe sets in weighted trees*, Discrete Appl. Math. **281** (2020), 216–223.

[39] P. Erdős, J. Nešetřil, and V. Rödl, *On some problems related to partitions of edges of a graph*, Graphs and other combinatorial topics, Teubner-Texte Math. 59, Teubner, 1983, pp. 54–63.

[40] T. Feng, D. Horsley, and X. Wang, *Novák's conjecture on cyclic steiner triple systems and its generalization*, arXiv:2001.06995 (2020).

[41] A. Ferber, C. Lee, and F. Mousset, *Packing spanning graphs from separable families*, Israel J. Math. **219** (2017), 959–982.

[42] A. Ferber and W. Samotij, *Packing trees of unbounded degrees in random graphs*, J. Lond. Math. Soc. **99** (2019), 653–677.

[43] P. Frankl and V. Rödl, *Near perfect coverings in graphs and hypergraphs*, European J. Combin. **6** (1985), 317–326.

[44] D.A. Freedman, *On tail probabilities for martingales*, Ann. Probab. **3** (1975), 100–118.

[45] A. Frieze and M. Krivelevich, *Packing Hamilton cycles in random and pseudo-random hypergraphs*, Random Structures Algorithms **41** (2012), 1–22.

[46] A. Frieze, M. Krivelevich, and P.-S. Loh, *Packing tight Hamilton cycles in 3-uniform hypergraphs*, Random Structures Algorithms **40** (2012), 269–300.

[47] Z. Füredi, *Matchings and covers in hypergraphs*, Graphs Combin. **4** (1988), 115–206.

[48] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **5** (1998), Dynamic Survey 6, 43.

[49] A. Georgakopoulos, J. Haslegrave, R. Montgomery, and B. Narayanan, *Spanning surfaces in* 3*-graphs*, arXiv:1808.06864 (2018).

[50] S. Glock and F. Joos, *A rainbow blow-up lemma*, Random Structures Algorithms **56** (2020), 1031–1069.

[51] S. Glock, F. Joos, J. Kim, D. Kühn, and D. Osthus, *Resolution of the Oberwolfach problem*, arXiv:1806.04644 (2018).

[52] S. Glock, D. Kühn, A. Lo, and D. Osthus, *The existence of designs via iterative absorption: hypergraph F-designs for arbitrary F*, Mem. Amer. Math. Soc. (to appear).

[53] S. Glock, D. Kühn, and D. Osthus, *Extremal aspects of graph and hypergraph decomposition problems*, Surveys in Combinatorics, London Mathematical Society Lecture Note Series, to appear.

[54] R.J. Gould, *Advances on the Hamiltonian problem—a survey*, Graphs Combin. **19** (2003), 7–52.

[55] ———, *Recent advances on the Hamiltonian problem: Survey III*, Graphs Combin. **30** (2014), 1–46.

[56] R.L. Graham and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Algebraic Discrete Methods **1** (1980), 382–404.

[57] H.-D. O. F. Gronau, R. C. Mullin, and A. Rosa, *Orthogonal double covers of complete graphs by trees*, Graphs Combin. **13** (1997), 251–262.

[58] A. Gyárfás and J. Lehel, *Packing trees of different order into K_n*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), 1, North-Holland New York, 1978, pp. 463–469.

[59] A. Hajnal and E. Szemerédi, *Proof of a conjecture of Erdős*, In: Combinatorial Theory and its Applications II (P. Erdős, A. Rényi, and V.T. Sós, eds.), North Holland, 1970, pp. 601–623.

[60] H. Hàn, J. Han, and P. Morris, *Factors and loose Hamilton cycles in sparse pseudo-random hypergraphs*, Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, 2020, pp. 702–717.

[61] H. Hàn, Y. Person, and M. Schacht, *On perfect matchings in uniform hypergraphs with large minimum vertex degree*, SIAM J. Discrete Math. **23** (2009), 732–748.

[62] P. Hatami and P.W. Shor, *A lower bound for the length of a partial transversal in a latin square*, J. Combin. Theory Ser. A **115** (2008), 1103–1113.

[63] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Intersci. Ser. Discrete Math. Optim., Wiley-Interscience, 2000.

[64] A. Johansson, J. Kahn, and V. Vu, *Factors in random graphs*, Random Structures Algorithms **33** (2008), 1–28.

[65] F. Joos, J. Kim, D. Kühn, and D. Osthus, *Optimal packings of bounded degree trees*, J. Eur. Math. Soc. **21** (2019), 3573–3647.

[66] J. Kahn, *Asymptotically good list-colorings*, J. Combin. Theory Ser. A **73** (1996), 1–59.

[67] ———, *A linear programming perspective on the Frankl-Rödl-Pippenger theorem*, Random Structures Algorithms **8** (1996), 149–157.

[68] ———, *Asymptotics for Shamir's problem*, arXiv:1909.06384 (2019).

[69] R. Karp, *Reducibility among combinatorial problems*, Complexity of computer computations, Springer, 1972, pp. 85–103.

[70] P. Keevash, *A hypergraph blow-up lemma*, Random Structures Algorithms **39** (2011), 275–376.

[71] ———, *The existence of designs*, arXiv:1401.3665 (2014).

[72] ———, *The existence of designs II*, arXiv:1802.05900 (2018).

[73] ———, *Hypergraph matchings and designs*, Proc. Int. Cong. of Math. **3** (2018), 3099–3122.

[74] P. Keevash and R. Mycroft, *A geometric theory for hypergraph matching*, Mem. Amer. Math. Soc. **233** (2015), vi+95.

[75] P. Keevash and K. Staden, *The generalised Oberwolfach problem*, arXiv:2004.09937 (2020).

[76] ———, *Ringel's tree packing conjecture in quasirandom graphs*, arXiv:2004.09947 (2020).

[77] J. Kim, Y. Kim, and H. Lui, *Tree decompositions of graphs without large bipartite holes*, Random Structures Algorithms **57** (2020), 150–168.

[78] J. Kim, D. Kühn, A. Kupavskii, and D. Osthus, *Rainbow structures in locally bounded colourings of graphs*, Random Structures Algorithms **56** (2020), 1171–1204.

[79] J. Kim, D. Kühn, D. Osthus, and M. Tyomkyn, *A blow-up lemma for approximate decompositions*, Trans. Amer. Math. Soc. **371** (2019), 4655–4742.

[80] A. Klenke, *Probability theory*, second ed., Springer, London, 2014.

[81] J. Komlós, J. Pintz, and E. Szemerédi, *A lower bound for Heilbronn's problem*, J. London Math. Soc. (2) **25** (1982), 13–24.

[82] J. Komlós, G.N. Sárközy, and E. Szemerédi, *Blow-up lemma*, Combinatorica **17** (1997), 109–123.

[83] ———, *On the Pósa-Seymour conjecture*, J. Graph Theory **29** (1998), 167–176.

[84] ———, *Proof of the Seymour conjecture for large graphs*, Ann. Comb. **2** (1998), 43–60.

[85] ———, *Proof of the Alon-Yuster conjecture*, Discrete Math. **235** (2001), 255–269.

[86] ———, *Spanning trees in dense graphs*, Combin. Probab. Comput. **10** (2001), 397–416.

[87] A. Kotzig, *On certain vertex-valuations of finite graphs*, Utilitas Math. **4** (1973), 261–290.

[88] D. Král', B. Lidický, T. Martins, and Y. Pehova, *Decomposing graphs into edges and triangles*, Combin. Probab. Comput. **28** (2019), 465–472.

[89] D. Kühn and D. Osthus, *Embedding large subgraphs into dense graphs*, Surveys in Combinatorics, Cambridge University Press, 2009, pp. 137–167.

[90] ———, *The minimum degree threshold for perfect graph packings*, Combinatorica **29** (2009), 65–107.

[91] ———, *A survey on Hamilton cycles in directed graphs*, European J. Combin. **33** (2012), 750–766.

[92] ______, *Hamilton decompositions of regular expanders: A proof of Kelly's conjecture for large tournaments*, Adv. Math. **237** (2013), 62–146.

[93] ______, *Hamilton decompositions of regular expanders: applications*, J. Combin. Theory Ser. B **104** (2014), 1–27.

[94] H. Lenz and G. Ringel, *A brief review on Egmont Köhler's mathematical work*, Discrete mathematics **97** (1991), 3–16.

[95] J. Lenz and D. Mubayi, *Perfect packings in quasirandom hypergraphs i*, J. of Combin. Theory Ser. B **119** (2016), 155–177.

[96] N. Linial and R. Meshulam, *Homological connectivity of random 2-complexes*, Combinatorica **26** (2006), 475–487.

[97] N. Linial and Y. Peled, *On the phase transition in random simplicial complexes*, Ann. of Math. **184** (2016), 745–773.

[98] Z. Luria and R.J. Tessler, *A sharp threshold for spanning 2-spheres in random 2-complexes*, Proc. Lond. Math. Soc. **119** (2019), 733–780.

[99] C. McDiarmid, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), London Math. Soc. Lecture Note Ser. 141, Cambridge Univ. Press, 1989, pp. 148–188.

[100] S. Messuti, V. Rödl, and M. Schacht, *Packing minor-closed families of graphs into complete graphs*, J. Combin. Theory Ser. B **119** (2016), 245–265.

[101] M. Molloy and B. Reed, *Near-optimal list colorings*, Random Structures Algorithms **17** (2000), 376–402.

[102] R. Montgomery, A. Pokrovskiy, and B. Sudakov, *Decompositions into spanning rainbow structures*, Proc. Lond. Math. Soc. **119** (2019), 899–959.

[103] ______, *A proof of Ringel's Conjecture*, arXiv:2001.02665 (2020).

[104] ______, *Embedding rainbow trees with applications to graph labelling and decomposition*, J. Eur. Math. Soc. (to appear).

[105] C.St.J.A. Nash-Williams, *Hamiltonian lines in graphs whose vertices have sufficiently large valencies*, Combinatorial Theory and its Applications III (Proc. Colloq., Blatonfüred, 1969), North Holland, 1970, pp. 813–819.

[106] ______, *Hamiltonian arcs and circuits*, Recent Trends in Graph Theory (Proc. Conf., New York, 1970), Springer, 1971, pp. 157–183.

[107] N. Pippenger and J. Spencer, *Asymptotic behaviour of the chromatic index for hypergraphs*, J. Combin. Theory Ser. A **51** (1989), 24–42.

[108] D.K. Ray-Chaudhuri and R.M. Wilson, *Solution of Kirkman's schoolgirl problem*, Combinatorics (Proc. Sympos. Pure Math., 1971, pp. 187–203.

[109] ______, *The existence of resolvable block designs*, In: A Survey of Combinatorial Theory (J.N. Srivastava, ed.), North-Holland, 1973, pp. 361–375.

[110] G. Ringel, *Theory of graphs and its applications*, Proceedings of the Symposium Smolenice, 1963.

[111] V. Rödl, *On a packing and covering problem*, European J. Combin. **6** (1985), 69–78.

[112] V. Rödl and A. Ruciński, *Perfect matchings in ϵ-regular graphs and the blow-up lemma*, Combinatorica **19** (1999), 437–452.

[113] ______, *Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve)*, In: An irregular mind (I. Bárány and J. Solymosi, eds.), Bolyai Soc. Math. Stud. 21, János Bolyai Math. Soc., 2010, pp. 561–590.

[114] V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in large uniform hypergraphs with large minimum collective degree*, J. Combin. Theory Ser. A **116** (2009), 613–636.

[115] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Sympos., Rome, 1966), Gordon and Breach, New York; Dunod, Paris, 1967, pp. 349–355.

[116] E. Szemerédi, *Regular partitions of graphs*, Problèmes Combinatoires et Théorie des Graphes (Colloq. Internat.CNRS, Univ. Orsay, Orsay, 1976), pp. 339–401.

[117] R. Wilson, *The early history of block designs*, Rend. Sem. Mat. Messina Ser. II **9** (2003), 267–276.

[118] R.M. Wilson, *An existence theory for pairwise balanced designs I. Composition theorems and morphisms*, J. Combin. Theory Ser. A **13** (1972), 220–245.

[119] ______, *An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures*, J. Combin. Theory Ser. A **13** (1972), 246–273.

[120] ______, *An existence theory for pairwise balanced designs III. Proof of the existence conjectures*, J. Combin. Theory Ser. A **18** (1975), 71–79.

[121] M. Woźniak, *Packing of graphs and permutations—a survey*, Discrete Math. **276** (2004), 379–391.

[122] H. Yap, *Packing of graphs—a survey*, Ann. of Discrete Math. **38** (1988), 395–404.

[123] A. Żak, *Harmonious order of graphs*, Discrete Math. **309** (2009), 6055–6064.

[124] Y. Zhao, *Recent advances on Dirac-type problems for hypergraphs*, In: Recent trends in combinatorics (A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker, and P. Tetali, eds.), IMA Vol. Math. Appl. 159, Springer, 2016, pp. 145–165.