Risk Management of Variable Annuities

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Research Papers

Overview of Research Papers

Research papers included in this dissertation

1. Kling, A., Ruez, F. and Ruß, J., 2014. The impact of policyholder behavior on pricing, hedging, and hedge efficiency of withdrawal benefit guarantees in variable annuities. *European Actuarial Journal*, 4(2), pp. 281–314.
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2. Kling, A., Ruez, F. and Ruß, J., 2016. Guaranteed minimum surrender benefits in variable annuities: the impact of regulator-imposed guarantees.
   The final version of this paper will appear in the *European Actuarial Journal*.
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3. Ruez, F., 2016. Variable annuities with guaranteed lifetime withdrawal benefits: an analysis of risk-based capital requirements.
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Research Context and Summary of Research Papers

1 Field of Research

Variable annuities

Variable annuities are unit-linked life insurance contracts where typically an initial investment amount is invested in one or several mutual funds. On top of this basic structure, certain guarantee riders are offered by the insurer, adding different types of financial protection to the contract. Therefore, variable annuities allow policyholders to benefit from the upside potential of the underlying fund investment and, at the same time, offer some kind of protection when the fund loses value. Variable annuities have experienced a growth in sales in US and Japan since the 1990s and are also becoming increasingly widespread over Europe (cf. EIOPA, 2011).

Variable annuity providers offer a variety of guarantee riders: Besides guaranteed minimum death benefit riders (GMDB), three main types of guaranteed living benefit riders (GLB) exist: guaranteed minimum accumulation benefit riders (GMAB), guaranteed minimum income benefit riders (GMIB) and guaranteed minimum withdrawal benefit riders (GMWB). GMAB and GMIB offer the policyholder some guaranteed maturity value or some guaranteed annuity benefit, respectively, while GMWB allow policyholders to (temporarily or lifelong) withdraw money from their account, even after the account's cash value has dropped to zero. GMWB with a lifelong guarantee are called "GMWB for life" or guaranteed lifetime withdrawal benefit riders (GLWB). They offer policyholders a lifelong income and, thus, protection from outliving their savings, with the invested amount still benefitting from potential fund growth and remaining under the control of the policyholder. In contrast to, for instance, traditional life insurance in Germany, GLWB do not require the policyholder to annuitize their

savings in order to hedge against longevity risk: The initial investment amount remains in control of the policyholder and may be cashed out (via surrendering of the contract) at any time during the contract's lifetime. Also, beneficiaries will receive a potential remaining account value in case of death of the policyholder. In other words, this type of variable annuity embeds a variant of ruin-contingent life annuity (cf. Huang et al., 2014), where the guarantee provider starts to pay a lifelong annuity as soon as the account value (reduced by pre-defined withdrawals) hits zero. Modern GLWB riders typically also include a form of ratchet mechanism, through which the guaranteed withdrawal amount may increase during the lifetime of the contract if the underlying fund performs well.

In contrast to more traditional offers, variable annuity providers usually receive an explicit compensation for the guarantees: Typically, they receive a guarantee charge that is periodically deducted from the policyholder's account, for instance a certain percentage of the invested amount, annually.

Risk profile of variable annuities

From a risk manager's perspective, the complexity of the guarantees offered within variable annuities also means that there are several important risks that need to be managed at the same time, including financial risk, behavioral risk, biometric risk, as well as regulatory risk. These risks are accompanied by a variety of additional risks that come with most insurance contracts (e.g. operational and reputational risk) and are amplified by the usually very long term of typical (variable) annuity contracts.

Financial risk inherent in variable annuities with guarantees comes from the direct exposure to market movements via the fund investment as well as from the impact interest rates have on the present value of future benefits. A decrease of interest rates, for instance, causes the present value of future guaranteed benefits to increase, which could negatively affect the variable annuity provider's (market value) balance sheet if the provider is not hedged against such changes. Movements in the spot prices of the underlying fund's assets, like, for instance, equity shares, directly affect the likelihood (and the extent) of the guarantee coming into effect: An increase in spot prices increases the account value and, thus, usually reduces the likelihood of the variable annuity provider needing to make guarantee payments. If the guarantee comes into effect, the extent usually is reduced by the increase in the account value. As a consequence, the value of the guarantee (and therefore

the value of liabilities on the variable annuity provider's market-value balance sheet) decreases with increasing spot prices. Vice versa, decreasing spot prices typically result in a higher value of liabilities on the market-value balance sheet of a variable annuity provider.

In order to lessen the impact market movements have on the balance sheet of the provider, risk management typically also includes the administration and rebalancing of a so-called hedging portfolio on the asset side. The purpose of such a hedging portfolio is to replicate changes in the value of liabilities on the liabilities side of the balance sheet with according increases or decreases in value on the asset side of the balance sheet. Ideally, with such a hedging program in place, the provider's (market-value) equity is not affected by changes in the value of liabilities. Such hedging programs can be quite effective in mitigating the financial risks inherent in variable annuity riders, but they usually do not allow for a perfect replication of the changes in the value of liabilities, due to discrete rebalancing and other imperfections (cf. Ledlie et al., 2008). On top of imperfections in the hedging program, there are risks that influence the provider's profit and loss attribution (P&L) that are not easily hedgeable, like for instance behavioral risks. Hence, the provider's P&L with regard to its variable annuity business remains subject to fluctuations, even if a hedging program is implemented.

Policyholder behavior risk stems from the fact that variable annuities usually offer the policyholder many choices, e.g. surrender, partial surrender, the decision whether or not and when to annuitize (in GMIB products) or the decision whether or not and how much to withdraw each year (in GMWB products). Several authors (cf. e.g. Milvesky & Salisbury, 2006, or Bauer et al., 2008) come to the conclusion that insurers assume what they call "suboptimal" policyholder behavior when pricing the guarantees. This means that (at least some) policyholders are assumed to not behave in a way that would maximize the value of the insurer's liabilities arising from the financial guarantees embedded in the products. From an insurer's risk management perspective, "optimal" policyholder behavior in this sense would constitute a worst-case scenario with respect to policyholder behavior. Therefore, this kind of behavior can also be described as "loss-maximizing" behavior from the viewpoint of the provider of the guarantee (cf. Azimzadeh et al., 2014). Such behavior is to be expected from institutional investors who buy policies in a secondary market and, subsequently, optimize the options embedded in the policies.

Bauer et al., 2008, state in particular that the value of certain guarantees under optimal policyholder behavior significantly exceeds typical prices charged in many

insurance markets, whereas the value of the same guarantees assuming suboptimal behavior (using e.g. typical surrender probabilities and independence between surrender behavior and financial markets) are in line with observed prices. This appears to bear significant risks for the insurers. There are several examples where insurance companies had to update their policyholder behavior assumptions leading to significant increases in liabilities, for instance ING, Manulife Financial and Sun Life Financial (cf. Knoller et al., 2016). Other insurers even completely stopped their variable annuity business in certain markets, cf. for instance The Hartford, 2009.

On top of market and behavioral risks, variable annuity providers also face regulatory challenges: First, providers have to comply to capital requirements, which can trigger the need for capital injections, lessen the return on equity of the company and, as a result, can make it harder to run a profitable variable annuity business. Capital requirements, in particular risk-based capital requirements, can also change over time and may be market-dependent, such that difficult market conditions may be accompanied by additional stress to the provider from a simultaneous increase in capital requirements. Second, for reasons of consumer protection, regulators may impose changes to the way some benefits of variable annuities are calculated, for instance surrender benefits. Such mandatory changes that are stipulated by a regulator can also apply to contracts that are already in force, i.e., effectively, the product design of the variable annuity is changed after inception of the contract and, therefore, was likely neither considered in the pricing nor the (initial) hedging of the contract.

In conclusion, the risk management of variable annuities with guarantees covers a variety of aspects and risks, which are hard to control and which are not easily hedged against. Therefore, risk management of a variable annuity product already starts in the product development process, where the design of the product's features has to be carefully analyzed and weight against different objectives, both, from the provider's perspective as well as from the perspective of a potential future customer. To accommodate for the complexity of the product and its inherent risks, usually a quantitative analysis is necessary to assess the consequences of certain design choices. In this process, future risk management and the risk-mitigating effect of future hedging should be considered. While for the economic risk only the true effectiveness of the hedging program seems relevant, for the calculation of (future) capital requirements (and therefore, future capital costs) it is also relevant to which extent this risk-mitigating effect is allowed to be considered in the calculation of future capital requirements. Also, possible future mandatory changes made

to the product design of already existing contracts imposed by the regulator can have a substantial impact on the profitability and should be assessed when designing a new product. Behavioral risk is a main risk that risk managers of variable annuities have to consider when assessing the risk profile and when setting up a hedging portfolio. This risk can be intensified by the presence and actions of institutional investors that try to profit from (from their perspective) underpriced contracts, which they subsequently optimize – from the provider's perspective this represents loss-maximizing behavior, which should be assessed and considered in the risk management of variable annuities.

2 Motivation and Objectives

The pricing and valuation of variable annuity contracts with guarantees have been studied in great detail, with Milvesky & Salisbury, 2006, being the first to analyze the valuation of guaranteed minimum withdrawal benefits, and with Bauer et al., 2008, as well as Bacinello et al., 2011, providing general frameworks for the valuation of variable annuities with all types of guarantees. Regarding the risk management of variable annuities, Cathcart et al., 2015 provide schemes to efficiently calculate the "Greeks" of a variable annuity liability via Monte Carlo simulation, while Coleman et al., 2006, and Coleman et al., 2007, provide schemes to hedge variable annuities under different assumptions regarding the capital market. Forsyth & Vetzal, 2014, present an optimal stochastic control framework, in which they analyze the sensitivity of the cost of hedging a variable annuity with GLWB to various economic and contractual assumptions. The hedging costs for variable annuities with combined guaranteed lifelong withdrawal and death benefits (GLWDB) is analyzed in Azimzadeh et al., 2014, in which the authors also argue that, when analyzing dynamic policyholder behavior from an insurer's perspective, it is better to use the term "loss-maximizing strategy" instead of "optimal strategy".

In Kling et al., 2011, the authors analyze the pricing and risk profile (from a provider's perspective) of GLWB riders with different product designs. In particular, they analyze how results change if equity volatility is modeled stochastic instead of deterministic. In their analysis, capital market models are used for several purposes: First, they are used in the pricing of the contract, i.e. the calculation of a "fair" guarantee charge as compensation for the guarantee of a GLWB rider. Second, after inception of a variable annuity contract, market models are used as a means to

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calculate the constitution of the hedging portfolio, i.e. to calculate the weights of the (financial) instruments used for hedging. Depending on the considered hedging strategy, the instruments they modeled in their analysis include a money market account, a position in the underlying equity fund and a position in a put option (with the fund as underlying). Third, they use market models to calculate risk measures of a stylized pool of policies of variable annuity contracts with GLWB riders. In their simulation study, the considered hedging programs are projected over the lifetime of the variable annuity contracts and risk measures of the provider's resulting P&L are computed.

For all three applications, Kling et al., 2011, analyze how the modeling of equity volatility influences the results and how these results are affected by the differences in the considered product designs. They find that both, the probability that guaranteed payments have to be paid and their amount vary significantly for the different considered product designs. The development over time of delta, rho and vega – i.e. the sensitivity of the value of liabilities with respect to changes in the underlying's price, the interest rate level and the level of equity volatility, respectively – was found to be also significantly different between the product designs, resulting in both, the constitution of a hedging portfolio (following a certain hedging strategy) and the provider's risk after hedging to differ significantly for the different product designs. Thus, risk management already starts during the development process of a new variable annuity product. The authors also find that the fair prices of the considered guarantees hardly change when stochastic volatility is introduced, while the provider's risk changes dramatically. They analyze different hedging strategies (no hedging, delta hedging, and delta and vega hedging) to deal with this risk and analyze the distribution of the provider's P&L and certain risk measures thereof. They find that the provider's risk can be reduced significantly by implementing suitable hedging strategies. Risks caused by policyholder behavior, however, is not part of their analysis.

The impact of policyholder behavior on the pricing of guarantees embedded in insurance contracts has been analyzed by several authors, e.g. by Grosen & Jorgensen, 2000, Steffensen, 2002, Bacinello, 2003, Bacinello, 2005, Bacinello et al., 2010, Bacinello et al., 2011, and Gao & Ulm, 2012, and with focus on the optimal stopping time within the context of GMWB guarantees for example by Chen et al., 2008, and Yang & Dai, 2013. Bernard et al., 2014, analyze so-called "optimal" policyholder behavior for variable annuities with GMAB riders. De Giovanni, 2010, uses a "Rational Expectation" model describing the policyholder's behavior in surrendering the con-

tract, which also allows for irrational policyholder behavior (as opposed to "optimal" behavior). Knoller et al., 2016, analyze individual policy data from a Japanese variable annuity product and find evidence that confirms their "moneyness hypothesis": In their statistical analysis, the fund performance and hence the value of the financial options and guarantees has the largest explanatory power for the surrender rate.

However, to our knowledge, there exists no simultaneous analysis of the impact of policyholder behavior on the pricing, hedging and hedge efficiency of GLWB riders with particular emphasis on different product designs. Therefore, in our first research paper, we extend the analysis conducted in Kling et al., 2011, with regard to the modeling of policyholder behavior and with a special focus on the risks that arise from behavior that differs from anticipated behavior. In our second research paper, we perform a similar analysis for variable annuities with GMAB riders and analyze the risks that arise if a regulator imposes certain guaranteed surrender benefits for variable annuity contracts that are already in force. We also analyze the impact such a mandatory change would have on the pricing and the risk profile of new variable annuity contracts. The impact the presence of institutional investors has on the results is also part of the analysis in our second research paper. Finally, in our third research paper, we have a closer look on the risk profile of a pool of variable annuities with GLWB riders and analyze the corresponding capital requirements under a risk-based regulatory regime like Solvency II in the European Union. In particular, we analyze the risk of simultaneous changes in the value of liabilities (which are likely to be hedged) and changes in capital requirements (which are likely not hedged against) und different assumptions regarding the extent to which the risk-mitigating effect of the hedging program can be considered in the calculation of the capital requirements.

In summary, the following research questions are considered in this thesis:

(1) From a variable annuity provider's perspective, how does policyholder behavior impact the risk profile (before and after hedging) of a pool of variable annuities with GLWB riders under different assumptions regarding the product design of the GLWB riders?

(2) What is the impact of regulator-imposed guaranteed minimum surrender benefits on the risk profile of existing contracts with GMAB riders and how are new contracts affected?

(3) How does the risk profile of variable annuities with GMAB and the impact of mandatory guaranteed minimum surrender benefits change if institutional

investors buy contracts from policyholders who are willing to surrender their contract?

(4) How do risk-based capital requirements for a pool of variable annuities with GLWB riders change depending on the market environment and the level of recognition of the actual hedging program?

3 Summary of Research Papers

Research Paper 1:
The Impact of Policyholder Behavior on Pricing, Hedging, and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities

The pricing of guarantees in variable annuities is usually performed under certain assumptions for future surrender rates. Such assumptions can be, for instance, deterministic surrender or (typically) path-dependent surrender (where assumed surrender rates depend on market parameters and/or the value of the guarantee). However, the pricing is usually not performed under the assumption of "optimal" surrender. This reduces the price of such guarantees since – in simplified terms – future profits the insurer expects from sub-optimal policyholder behavior are given to the client by means of a reduced price for the guarantee. The possibility to allow for sub-optimal policyholder behavior in pricing and hedging of such products is a reason why these (often primarily financial) guarantees can be offered by insurers at competitive prices when compared to similar products offered by banks. This opens opportunities for institutional investors to purchase such policies in a secondary market at a price that exceeds the surrender benefit from policyholders who are willing to surrender their contract. In this situation, selling the contract to the institutional investor – instead of surrendering it – is beneficial for the policyholder. After acquiring the contract, the institutional investor then maximizes (optimizes) the value of the contract, which typically results in loss-maximizing behavior from the insurer's perspective.

In Kling et al., 2014, we extend the model used in Kling et al., 2011, and incorporate different models of dynamic, so-called path-dependent policyholder behavior. The analyzed models include purely deterministic surrender rates, path-dependent

surrender rates that are influenced by a certain observable quantity (for instance the "moneyness" of the guarantee), as well as a model of "optimal" policyholder behavior, i.e. behavior that maximizes the value of liabilities of the guarantee.

We find that the popular method of using the "moneyness", while being a huge improvement (from a risk manager's perspective) over purely deterministic behavior, is not enough to capture the full risk optimal behavior poses to the insurer. We also found that the product design is a powerful tool to minimize behavior risk: with an appropriate design of the ratchet mechanism of the GLWB rider, the guarantee is never fully "out of the money" (i.e. with little value) and therefore, the value of liabilities, from the provider's perspective, is less sensitive to surrender rates. In these cases, for pricing purposes, optimal behavior was very close to the assumption of the policyholders not surrendering at all, as the surrender benefit and the value of continuing the contract (the so-called continuation value) are more closely together. However, in the analysis of the risk profile, a proper modeling of loss-maximizing behavior – while certainly a worst-case scenario – seems indispensable for a full risk analysis.

This paper is joint work with Alexander Kling and Jochen Ruß and has been published in the European Actuarial Journal. It answers the first research question listed above.

Research Paper 2:
Guaranteed Minimum Surrender Benefits in Variable Annuities: The Impact of Regulator-Imposed Guarantees

Surrender risk is not only influenced by dynamic policyholder behavior, but also by the way surrender benefits themselves are calculated. The product design of variable annuities usually stipulates that the surrender value of such products coincides with the policyholder's account value (minus surrender charges, if applicable). The "fair value" of the guaranteed benefits or the market value of certain hedge assets is typically not part of the individual policyholder's account value and thus, with the usual product design, not part of the surrender value. This in consequence means that the surrender value in general is different from a "fair" market value of the contract. In particular, the surrender benefit will not be reduced if interest rates rise, although both, the assets backing the contract and the "fair value" of the contract,

would drop. The resulting risk has been discussed e.g. in Feodoria & Förstemann, 2015.

Additionally, regulator-imposed minimum surrender benefits, like they are discussed and imposed in Germany, pose a relevant risk to the providers of variable annuities. This is especially the case if the guaranteed minimum surrender benefits (GMSB) are imposed after inception of the contracts and, thus, were not incorporated in the pricing and (initial) hedging process of the product. We analyze the accompanying risk in Kling et al., 2016, where we analyze different discussed and proposed models for determining a "time value" of the guarantee as a minimum for the surrender benefit a policyholder would receive in case of surrendering their contract. For this purpose, we analyze variable annuity contracts with a GMAB rider under different assumptions regarding the GMSB as well as different policyholder behavior models. A model for determining a GMSB is especially harmful to the provider if it is systematically utilized by an institutional investor like hedge funds in a potential secondary market (cf. e.g. Central Bank of Ireland, 2010). We introduce a model where policyholders who are willing to surrender their contract sell them instead to an institutional investor, if they will receive a selling price that is higher than the current surrender benefit. Of course, the institutional investor is only able to offer a higher price in an economically sound way, if the continuation value of the contract exceeds the surrender benefit.

We find that, while the impact of GMSBs on market risk is relatively low, the impact on the fair guarantee charge, the value of liabilities and the risk resulting from changes in policyholder behavior is substantial. If the GMSB is already considered when pricing the contract, the resulting advantage for policyholders who surrender the contract comes at the price of increased guarantee charges for all policyholders, adversely affecting especially those who keep the contract until maturity. As a consequence, the same protection level with regard to old-age provision becomes more expensive when GMSBs are in place.

If a GMSB is introduced after inception of the contract, e.g. because of a regulatory change, the insurer will suffer an immediate loss on its market-value balance sheet. While the value of the contract increases with the value added by the GMSB, the sensitivity with regard to surrender rates decreases, as, from a valuation perspective, it becomes less important whether policyholders decide to surrender or not. As a consequence, the potential for mispricing of the contracts with respect to incorrect surrender assumptions is reduced.

Our analyses with regard to the impact of a secondary market show that, in a market without GMSBs, the presence of an institutional investor creates a loss for the insurer and also increases market risk. At the same time, the impact of introducing GMSBs is reduced and the specific design of the GMSB is less relevant. On the other hand, if GMSBs are already in place, the potential for a successful secondary market is reduced, since the difference between the surrender benefit of a contract and its continuation value is typically lower and, thus, institutional investors less likely are able to offer prices that exceed the surrender benefit.

This paper is joint work with Alexander Kling and Jochen Ruß and answers the second and third research question listed above.

Research Paper 3:

Variable Annuities with Guaranteed Lifetime Withdrawal Benefits: An Analysis of Risk-Based Capital Requirements

Under risk-based regulatory regimes like Solvency II in the EU and the Swiss Solvency Test in Switzerland, the risk profile of a variable annuity directly affects the amount of capital that providers are required to hold. Therefore, providers of variable annuities not only face the challenge to hedge against changes in the value of embedded guarantees (i.e. the value of liabilities), but are also exposed to potential additional capital needs due to changes in their capital requirements. Both, the (market) value of liabilities as well as corresponding risk-based capital requirements, are dependent on market parameters and, thus, subject to changes.

Therefore, under risk-based regulatory regimes, not only is the valuation and hedging relevant when designing and profit-testing new variable annuity products, but also (future) capital requirements. Not only does the product design directly influence the risk profile and, thereby, the capital requirements under risk-based regulatory regimes, but it also influences the risk of changing capital requirements in the future. A change in the value of the guarantee rider from the provider's perspective, for instance, is likely to be accompanied by a change in the risk-based capital requirements. Furthermore, while the hedging program might prove reliable in reality, it is not clear to which extent it can be considered when calculating the capital requirements. If certain regulatory requirements are not fulfilled it is likely that the risk-mitigating effect of future hedging is only allowed to be partially considered in

the calculation. While the change in the value of liabilities is likely (at least partially) hedged, the change in capital requirements typically is not.

In Ruez, 2016, we analyze the risk profile and corresponding risk-based requirements of a pool of variable annuity policies with Guaranteed Lifetime Withdrawal Benefit (GLWB) riders with regard to the pool's key financial risk drivers: equity returns, implied equity volatility and interest rates. In a simulation study, we analyze the effectiveness of different stylized hedging programs over a one-year time horizon and compute indicators for risk-based capital requirements. The approach we use is comparable to an internal model type approach under Solvency II. We also analyze the impact changing market environments have on risk profile, hedge effectiveness and capital requirements, similar to a forward-looking analysis in the context of the mandatory Own Risk and Solvency Assessment (ORSA) under Solvency II.

We find that, in addition to the stress from potentially unhedged increases in the value of liabilities, changes in the market environment can have a substantial impact on capital requirements. As a result, GLWB providers face the risk of increases in their risk-based capital requirements and, thus, the need for capital injections – even without pricing errors or malfunctioning of the hedging program. We also find that, while the impact of the level of interest rates on the effectiveness of the modeled hedging program is rather low, a higher volatility level has a distinct adverse effect on the hedge effectiveness, leading to a further increase of risk-based capital requirements. However, there are also cases where an increase in the value of liabilities was accompanied by a decrease of capital requirements, reducing the overall impact on the insurer. This is the case for some risk measures if no allowance of the hedging program is made in the calculation of capital requirements and equity volatility is increased.

As the sensitivity of capital requirements to market parameters is not easily assessable, thorough numerical analyses appear necessary for a proper assessment of this risk. In such analyses, also the effect of a potentially reduced hedge performance in adverse market environments and a reduced level of recognition of the hedging program's risk-mitigating effect should be considered, as this may lead to substantial additional increases of capital requirements during the lifetime of the variable annuity contract.

In summary, this research paper answers the fourth research question listed above.

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Research Papers

1 The Impact of Policyholder Behavior on Pricing, Hedging, and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities

The Impact of Policyholder Behavior on Pricing, Hedging, and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities

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Abstract

We analyze the impact of policyholder behavior on pricing, hedging and hedge efficiency of variable annuities with guaranteed lifetime withdrawal benefits. We consider different product designs, market models and approaches for modeling policyholder behavior in our analyses, covering deterministic behavior, behavior depending on the 'moneyness' of the guarantee, and optimal (value maximizing) behavior. First, we assess the risk of mispricing the guarantee due to inaccurate assumptions regarding future policyholder behavior. Comparing products with different ratchet mechanisms, we find that this potential for mispricing is the smallest for the product design with the most valuable ratchet mechanism. We further quantify the impact of different behavior models on the efficiency of the insurer's hedging strategy and the risk that results if the insurer's assumption for policyholder behavior deviates from actual behavior. Our analyses indicate significant differences between the considered products in terms of hedgeability and the sensitivity of the guarantee's value towards policyholder behavior and towards changes in the underlying asset's volatility. Also, we show that a simple path-dependent behavior model may not be suitable to fully assess the risk arising from policyholder behavior.

1 Introduction

Variable annuities are fund-linked annuities where the policyholder typically pays a single premium into the policy and the money is then invested in one or several mutual funds. Variable annuities usually offer a wide range of investment options for the policyholder to choose from. On top of this basic structure, certain guarantee riders are offered by the insurer, adding different types of financial protection to the contract. There are several types of guarantee riders that come with variable annuities, including guaranteed minimum death benefits (GMDB) as well as guaranteed minimum living benefits, which can be categorized into three main subcategories: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and guaranteed minimum withdrawal benefits (GMWB). A GMAB guarantee provides the policyholder with some guaranteed value at one or several future points in time, while the GMIB guarantee provides a guaranteed annuity benefit, starting after a certain deferment period. With the GMWB rider, if certain conditions are met, the policyholders may continue to withdraw money from their account, even after the value of the account has dropped to zero. Such withdrawals are guaranteed as long as both, the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy, stay within certain limits.

Insurers also started to include additional features in GMWB products. The most prominent is called "GMWB for Life" (also known as guaranteed lifetime withdrawal benefits, GLWB). With this type of guarantee, the total amount of withdrawals is unlimited. However, the annual amount that may be withdrawn while the insured is still alive may not exceed some maximum value; otherwise the guarantee will be affected. The withdrawals made by the policyholder are deducted from their account value as long as this value is positive. Afterwards, the insurer has to provide the guaranteed withdrawals for the rest of the insured's life. In return for this guarantee, the insurer receives guarantee charges, which are deducted from the policyholder's account value (as long as this value is positive). These charges are typically calculated as a fixed annual percentage of the so-called withdrawal benefit base (explained below) or of the account value. In a few products, annual guarantee charges are calculated as a fixed percentage of the single premium. In contrast to a conventional annuity, where the assets covering the liabilities are owned by the pool of insured, in a GLWB policy, the fund units of the contract are owned by the individual policyholder and remain accessible to the policyholder even in the payout phase. The policyholder may access the remaining fund assets at any time by (partially) surrendering the contract. In case of death of the insured, any remaining fund value (or a guaranteed minimum death benefit if such a rider is included and the corresponding value exceeds the fund value) is paid out to the beneficiary.

From an insurer's point of view, such products contain an interesting and challenging combination of several risks, resulting from policyholder behavior (with regard to surrender and withdrawal), financial markets, and longevity, alongside a variety of additional risks that come with most insurance contracts (e.g. operational and reputational risk). This combination of risks makes these guarantees challenging to hedge and has been in the focus of both, academics and practitioners.

Policyholder behavior risk stems from the fact that variable annuities usually offer the policyholder many choices, e.g. surrender, partial surrender, the decision whether or not and when to annuitize (in GMIB products) or the decision whether or not and how much to withdraw each year (in GMWB products). Several authors (cf. e.g. Milevsky and Salisbury, 2006, or Bauer et al., 2008) come to the conclusion that insurers assume what they call "suboptimal"

policyholder behavior when pricing the guarantees. This means that (at least some) policyholders are assumed to not behave in a way that would maximize the value of the insurer's liabilities arising from the financial guarantees embedded in the products. From an insurer's risk management perspective, "optimal" policyholder behavior in this sense would constitute a worst-case scenario with respect to policyholder behavior. Bauer et al. (2008) state in particular that the value of certain guarantees under optimal policyholder behavior significantly exceeds typical prices charged in many insurance markets, whereas the value of the same guarantees assuming suboptimal behavior (using e.g. typical surrender probabilities and independence between surrender behavior and financial markets) are in line with observed prices. This appears to bear significant risks for the insurers. There are several examples where insurance companies had to update their policyholder behavior assumptions leading to significant increases in liabilities, see e.g. ING (2011), Manulife Financial (2011), and Sun Life Financial (2011). Other insurers even completely stopped their variable annuity business in certain markets, cf. for instance The Hartford (2009).

The effect of policyholder behavior not only on pricing but also – and much more importantly – on hedging and hedge efficiency of variable annuity guarantees should therefore be of interest to academics, product providers and regulators. The impact of policyholder behavior on the pricing of guarantees embedded in insurance contracts has been analyzed by several authors, e.g. by Grosen and Jørgensen (2000), Steffensen (2002), Bacinello et al. (2003, 2005, 2011) and Gao and Ulm (2012) and with focus on the optimal stopping time within the context of GMWB guarantees for example by Chen et al. (2008) and Yang and Dai (2013). Bernard et al. (2014) analyze optimal policyholder behavior for variable annuities with a GMAB. De Giovanni (2010) uses a 'Rational Expectation' model describing the policyholder's behavior in surrendering the contract, which also allows for irrational policyholder behavior. Knoller et al. (2013) analyze individual policy data from a Japanese variable annuity product and find evidence that confirms their "moneyness hypothesis": In their statistical analysis the fund performance and hence the value of the financial options and guarantees has the largest explanatory power for the surrender rate. They find that surrender rates increase with decreasing value of the guarantee and that policyholders' apparent rationality increases with increasing contract volume.

To our knowledge, there exists no simultaneous analysis of the impact of policyholder behavior on the pricing, hedging and hedge efficiency of GLWB riders with particular emphasis on different product designs. The present paper fills this gap: We extend the model presented in Kling et al. (2011) to incorporate non-deterministic policyholder behavior and – for different product designs – analyze the impact policyholder behavior has on pricing, hedging and hedge efficiency, and how results change if the capital market model incorporates stochastic instead of deterministic equity volatility.

The remainder of this paper is organized as follows. In Section 2, we describe our model framework that consists of three parts: the financial model, where for the sake of comparison we use both, the classic Black-Scholes model (with deterministic equity volatility) and the Heston model for the evolution of an underlying with stochastic equity volatility; the liability model that describes the different considered variable-annuity contracts with different GLWB options; and the valuation framework including the policyholder-behavior model, which allows for different policyholder strategies with regard to surrendering the contract. We particularly consider "optimal" policyholder behavior, as well as several "suboptimal" strategies, where, in both cases, "optimal" as explained above denotes the behavior that maximizes the value of the insurer's liabilities. In Section 3, we present the results of our analyses regarding the pricing of

the guarantee. In particular, we analyze the differences in the option value for different product designs and how the option value depends on assumed policyholder behavior. This is a first indication for an insurer's potential loss arising from an inaccurate assessment of policyholder behavior. Section 4 deals with hedging strategies and hedge efficiency. Here, we particularly analyze how the insurer's expected profit and risk change if actual policyholder behavior deviates from the behavior assumed within the hedging strategy. Finally, Section 5 concludes.

2 Model Framework

In Bauer et al. (2008), a general framework for modeling and valuation of variable annuity contracts was introduced. Within this framework, any contract with one or several living benefit guarantees and/or a guaranteed minimum death benefit can be represented. In their numerical analysis however, only contracts with a rather short finite time horizon were considered. Holz et al. (2012) describe how GLWB products can be included in this model. In what follows, we apply the general framework of Bauer et al. (2008). However, in our concrete specification, additionally to the simple Black-Scholes model used in Bauer et al. (2008), we also consider a model which allows for stochastic equity volatility (Section 2.1). In Section 2.2, we introduce and define the specific product designs considered within our numerical analyses. Different models for policyholder behavior are introduced in Section 2.3, where also our valuation approach is summarized.

2.1 Financial Market

The valuation framework in this section follows in some parts the one used in Bacinello et al. (2010) and in others Bauer et al. (2008). We take as given a filtered probability space (Ω, Σ, F, P) in which P is the real-world (or physical) probability measure and $F \doteq (F_t)_{t \geq 0}$ is a filtration with $F_0 = \{\varnothing, \Omega\}$ and $F_t \subset \Sigma \ \forall \ t \geq 0$. We assume that trading takes place continuously over time and without any transaction costs or spreads. Furthermore, we assume that the price processes of the traded assets in the market are adapted and of bounded variation. For our analyses we assume two primary tradable assets: the underlying fund (or basket of funds), whose spot price at time t will be denoted by S_t, and the money-market account, whose value at time t will be denoted by B_t. We assume the money-market account to evolve at a constant risk-free rate of interest r:

$$dB_t = rB_t dt$$
$$\Rightarrow B_t = B_0 \exp(rt) \tag{1}$$

For the dynamics of S_t, we use two different models. First, we assume the equity volatility to be deterministic and constant over time, and hence use the Black-Scholes model for our simulations. To allow for a more realistic equity volatility model, we also use the Heston model, in which both, the underlying and its (instantaneous) variance, are stochastic processes. These two models will be explained in the following two subsections.

2.1.1 Black-Scholes Model

In the Black-Scholes (1973) model, the underlying's spot price S_t follows a geometric Brownian motion whose dynamics under the real-world measure P are given by

$$dS_t = \mu S_t dt + \sigma_{BS} S_t dW_t, \quad S_0 \geq 0 \ , \tag{2}$$

where μ is the (constant) drift of the underlying, σ_{BS} its constant volatility and W_t denotes a P-Brownian motion. By Itō's lemma, S_t has the solution

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma_{BS}^2}{2}\right)t + \sigma_{BS}W_t\right), \quad S_0 \geq 0 \ . \tag{3}$$

2.1.2 Heston Model

There are various extensions to the Black-Scholes model that allow for a more realistic modeling of the underlying's volatility. We use the Heston (1993) model in our analyses where the instantaneous (or local) volatility of the asset is stochastic. Under the Heston model, the market is assumed to be driven by two stochastic processes: the underlying's price S_t, and its instantaneous variance V_t, which is assumed to follow a one-factor square-root process identical to the one used in the Cox-Ingersoll-Ross interest rate model (Cox et al., 1985). The dynamics of the two processes under the real-world measure P are given by the following system of stochastic differential equations:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1, \quad S_0 \geq 0 \tag{4}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^2, \quad V_0 \geq 0, \tag{5}$$

where μ again is the drift of the underlying, V_t is the local variance at time t, κ is the speed of mean reversion, θ is the long-term variance, σ_v is the so-called "volatility of volatility", and $W_t^{1/2}$ are correlated P-Brownian motion processes (with correlation parameter ρ). The condition $2\kappa\theta \geq \sigma_v^2$ ensures that the variance process will remain strictly positive almost surely (see Cox et al., 1985).

2.1.3 Equivalent Martingale Measure

Assuming the absence of arbitrage opportunities in the financial market, there exists a probability measure Q that is equivalent to P and under which the gain from holding a traded asset is a Q-martingale after discounting with the chosen numéraire process, in our case the money-market account. Q is called equivalent martingale measure. While – under the usual assumptions – the transformation to such a measure is unique under the Black-Scholes model (cf. e.g. Bingham and Kiesel, 2004), it is not under the Heston model. Within the Heston model, since there are two sources of risk, there are also two market-price-of-risk processes, denoted by γ_t^1 and γ_t^2 (corresponding to W_t^1 and W_t^2). Heston (1993) proposed the following restriction on the market price of volatility risk process, assuming it to be linear in volatility,

$$\gamma_t^1 = \lambda \sqrt{V_t} \ . \tag{6}$$

Provided both measures, P and Q, exist, the Q-dynamics of S_t and V_t, again under the assumption that no dividends are paid, are then given by

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^{Q,1}, \quad S_0 \geq 0 \tag{7}$$

$$dV_t = \kappa^* \left(\theta^* - V_t\right) dt + \sigma_v \sqrt{V_t} dW_t^{Q,2}, \quad V_0 \geq 0 \tag{8}$$

where $W_t^{Q,1}$ and $W_t^{Q,2}$ are two correlated Q-Brownian motion processes (with correlation parameter ρ) and where

$$\kappa^* = \left(\kappa + \lambda \sigma_v\right), \quad \theta^* = \frac{\kappa \theta}{\left(\kappa + \lambda \sigma_v\right)} \tag{9}$$

are the risk-neutral counterparts to κ and θ (cf., for instance, Wong and Heyde, 2006).

2.2 Model of the Liabilities

With variable annuities, the single premium P is invested in one or several mutual funds. We call the value of the policyholder's individual portfolio the *account value* and denote its value at time t by AV_t. All charges are taken from the account value by cancellation of fund units. Furthermore, the policyholder has the possibility to surrender the contract or to withdraw a portion of the account value.

Products with a GMWB option give the policyholder the possibility to perform guaranteed withdrawals. In this paper, we focus on the case where such withdrawals are guaranteed lifelong ("GMWB for Life" or guaranteed lifetime withdrawal benefits, GLWB). The initially guaranteed withdrawal amount is usually a certain percentage x_{WL} of the single premium P. In most products, x_{WL} depends on the age when withdrawals start. Any remaining account value at the time of death is paid to the beneficiary as death benefit. If, however, the account value of the policy drops to zero while the insured is still alive, the policyholder can still continue to withdraw the guaranteed amount until death of the insured. The insurer charges a fee for this guarantee, which is usually a pre-specified annual percentage of the withdrawal benefit base, the account value or the single premium. In what follows, we will assume that the guarantee charge is a percentage of the account value and that withdrawals may only occur on the policy's anniversary dates.

Often, GLWB products contain certain features that lead to an increase of the guaranteed withdrawal amount if the underlying funds perform well. Typically, on every policy anniversary, the current account value is compared to a certain reference value, which we refer to as 'withdrawal benefit base'. Whenever the account value exceeds the withdrawal benefit base, the guaranteed annual withdrawal amount is increased (step-up or ratchet). In our numerical analyses in Sections 3 and 4, we consider three different product designs that can be observed in the market:

- **No Ratchet (Product I)**: The first and simplest alternative is one where no ratchets exist at all. In this case, the guaranteed annual withdrawal amount is constant and does not depend on market movements.

- **Lookback Ratchet (Product II)**: The second alternative is a ratchet mechanism where the withdrawal benefit base at outset is given by the single premium paid. During the contract term, on each policy anniversary date, the withdrawal benefit base is increased

to the account value, if the account value exceeds the previous withdrawal benefit base. The guaranteed annual withdrawal amount is increased accordingly to x_{WL} multiplied by the new withdrawal benefit base. This effectively means that the fund performance needs to compensate for charges and annual withdrawals in order to increase future guaranteed withdrawals. Increases in the guaranteed withdrawal amount are permanent, i.e. over time, the guaranteed withdrawal amount may only increase, never decrease.

- **Remaining WBB Ratchet (Product III)**: The basic idea of the third product is to provide a ratchet mechanism where, in order to increase guaranteed annual withdrawals, the fund performance needs to compensate only for charges, but not for annual withdrawals. In this product, the withdrawal benefit base at outset is also given by the single premium paid. However, at each withdrawal date, the withdrawal benefit base is reduced by the withdrawn amount (if this amount does not exceed the guaranteed withdrawal amount). If on a policy anniversary the current account value exceeds this reduced withdrawal benefit base by a certain amount Δ, the guaranteed annual withdrawal is increased by $x_{WL} \cdot \Delta$. After such an increase, the withdrawal benefit base is reset to the account value. This ratchet mechanism is therefore c.p. somewhat "richer" than the Lookback Ratchet. As a consequence, the initially guaranteed withdrawal amount should c.p. be lower than with a product offering a Lookback Ratchet. As with the Lookback Ratchet design, increases in the guaranteed amount are permanent.

Throughout the paper, we assume that administration charges and guarantee charges are deducted at the end of each policy year as a percentage φ^{adm} and φ^{guar} of the account value. Additionally, we allow for upfront acquisition charges φ^{acq} as a percentage of the single premium P. This leads to $AV_0 = P \cdot \left(1 - \varphi^{acq}\right)$.

We denote the guaranteed withdrawal amount at time t by W_t^{guar} and the corresponding withdrawal benefit base by WBB_t. At inception, for each of the considered products, the initial withdrawal benefit base is set to P and hence the guaranteed withdrawal amount for the initial withdrawal is given by $W_0^{guar+} = x_{WL} \cdot WBB_0 = x_{WL} \cdot P$. The amount actually withdrawn by the client is denoted by W_t.[1]

Since we restrict our analyses to single premium contracts, policyholder actions during the life of the contract are limited to withdrawals and (partial) surrender.

During the year, all processes are subject to capital market movements. As mentioned above, we allow for withdrawals at policy anniversaries only. Also, we assume that death benefits are paid out at policy anniversaries if the insured person has died during the previous year. Thus, at each policy anniversary $t = 1, 2, ..., T$, we have to distinguish between the value of a variable $(\cdot)_t^-$ immediately before and the value $(\cdot)_t^+$ after withdrawals, (partial) surrender, and death

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[1] Note that the client can choose to withdraw less than the guaranteed amount, thereby increasing the probability of future ratchets. If the client wants to withdraw more than the guaranteed amount, any exceeding withdrawal would be considered a partial surrender.

benefit payments. For the latter, we assume that no additional guaranteed minimum death benefit rider is included in the policy, i.e. in case of death the remaining fund value is paid out.

In what follows, in the spirit of Bauer et al. (2008), we describe the development between two policy anniversaries and the transition at policy anniversaries for the considered contract designs. From these, we are finally able to determine all benefits for any given policyholder strategy and any capital market path. This allows for an analysis of such contracts in a Monte-Carlo framework.

2.2.1 Development between two Policy Anniversaries

We assume that the annual fees φ^{adm} and φ^{guar} are deducted from the policyholder's account value at the end of each policy year. Thus, the development of the account value between two policy anniversaries is given by

$$AV_{t+1}^- = AV_t^+ \cdot \frac{S_{t+1}}{S_t} \cdot \exp\left(-\varphi^{adm} - \varphi^{guar}\right). \tag{10}$$

At the end of each year, the different ratchet mechanisms are applied after deduction of charges and before any other actions are taken. Thus W_t^{guar} develops as follows:

- **No Ratchet:** $WBB_{t+1}^- = WBB_t^+ = P$ and $W_{t+1}^{guar-} = W_t^{guar+} = x_{WL} \cdot P$.

- **Lookback Ratchet:** $WBB_{t+1}^- = \max\left\{WBB_t^+, AV_{t+1}^-\right\}$
  and $W_{t+1}^{guar-} = x_{WL} \cdot WBB_{t+1}^- = \max\left\{W_t^{guar+}, x_{WL} \cdot AV_{t+1}^-\right\}$.

- **Remaining WBB Ratchet**: Since withdrawals are only possible on policy anniversaries, the withdrawal benefit base during the year develops like in the Lookback Ratchet case. Thus, we have $WBB_{t+1}^- = \max\left\{WBB_t^+, AV_{t+1}^-\right\}$ and
  $W_{t+1}^{guar-} = W_t^{guar+} + x_{WL} \cdot \max\left\{AV_{t+1}^- - WBB_t^+, 0\right\}$.

2.2.2 Transition at a Policy Anniversary t

At the policy anniversaries, we have to distinguish the following four cases:

a) The insured has died within the previous year (t-1,t]

If the insured has died within the previous policy year, the account value is paid out as death benefit. With the payment of the death benefit, the insurance contract matures. Thus, $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t = 0$, and $W_t^{guar+} = 0$.

b) The insured has survived the previous policy year and does not withdraw any money from the account at time t

If no death benefit is paid out to the policyholder and no withdrawals are made from the contract, i.e. $W_t = 0$, we get $AV_t^+ = AV_t^-$, $WBB_t^+ = WBB_t^-$, and $W_t^{guar+} = W_t^{guar-}$.

c) The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the withdrawal guarantee

If the insured has survived the past year, no death benefits are paid. Any withdrawal W_t up to the guaranteed annual withdrawal amount W_t^{guar-} reduces the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $AV_t^+ = \max\left\{0; AV_t^- - W_t\right\}$.

For the alternatives "No Ratchet" and "Lookback Ratchet", the withdrawal benefit base and the guaranteed annual withdrawal amount remain unchanged, i.e. $WBB_t^+ = WBB_t^-$, and $W_t^{guar+} = W_t^{guar-}$. For the alternative "Remaining WBB Ratchet", the withdrawal benefit base is reduced by the withdrawal taken, i.e. $WBB_t^+ = \max\left\{0; WBB_t^- - W_t\right\}$ and the guaranteed annual withdrawal amount remains unchanged, i.e. $W_t^{guar+} = W_t^{guar-}$.

d) The insured has survived the previous policy year and at the policy anniversary withdraws an amount exceeding the limits of the withdrawal guarantee

In this case again, no death benefits are paid. For the sake of brevity, we only give the formulas for the case of full surrender, since partial surrender is not analyzed in what follows.[2] In case of full surrender, the complete account value is withdrawn. We then set $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t = AV_t^-$, and $W_t^{guar+} = 0$ and the contract terminates. However, the policyholder does not receive the full asset value as surrender benefit, since surrender fees φ_t^{surr} are deducted from the cash amount exceeding the guaranteed withdrawal amount.

2.3 Valuation

Let Q be an equivalent martingale measure of the financial market (cf. section 2.1.3). Assuming independence between financial markets and mortality as well as risk-neutrality of the insurer with respect to mortality and behavioral risk, we are able to use the product measure of Q and the mortality measure. In what follows, we denote this measure by $\hat{Q}$.

As mentioned earlier, for the contracts considered within our analyses, policyholder actions are limited to withdrawals and (partial) surrender. In our numerical analyses in Sections 3 and 4, we only consider two possible policyholder actions: withdrawal of the guaranteed withdrawal amount, i.e. $W_t = W_t^{guar-}$, or full surrender, i.e. $W_t = AV_t^-$. This also means that we assume that withdrawals begin at the earliest anniversary possible and, hence, that there is no initial waiting period before the first withdrawal. To keep notation simple, we only give formulas for the considered cases (cf. Bauer et al. (2008) for formulas for the other cases).

We denote by x_0 the insured's age at the start of the contract, $_t p_{x_0}$ the probability under $\hat{Q}$ for a x_0-year old to survive the next t years, q_{x_0+t} the probability under $\hat{Q}$ for a $(x_0 + t)$-year old

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[2] For details on partial surrender, we refer the reader to Bauer et al. (2008).

to die within the next year, and let ω be the limiting age of the mortality table, i.e. the age beyond which survival is deemed impossible. The probability under $\hat{Q}$ that an insured aged x_0 at inception passes away in the year $(t,t+1]$ is thus given by $_t p_{x_0} \cdot q_{x_0+t}$. The limiting age ω allows for a finite time horizon $T = \omega - x_0 + 1$.

For pricing purposes, we consider a pool of policyholders who hold identical contracts and in which each insured has the same age, same gender and same mortality probability. We assume the number of policyholders to be large enough such that the assumption that deaths occur exactly according to the probabilities q_{x_0+t} is justified. The policyholders in the pool may however differ in their (surrender) behavior.

We model the (surrender) behavior of the policyholders in the pool as a $\hat{F}$-adapted family of random variables $\xi = (\xi_t)_{t=1,...,T}$, where $0 \leq \xi_t \leq 1, \forall t = 1,...,T$, represents the fraction of the remaining policyholders at time t who surrender their contract at time t. After the guarantee has been triggered, i.e. $W_t^{guar-} > AV_t^-$ for some $t=1,...,T$, there is no rational reason for a policyholder to surrender their contract, hence we set $\xi_t = 0, \forall t \geq \tau_G$, where τ_G represents a $\hat{F}$-stopping time indicating the policy anniversary at which the guarantee of the GLWB rider triggers, i.e. the smallest $t=1,...,T$ for which $W_t^{guar-} > AV_t^-$ holds. If the guarantee does not trigger during the contract's lifetime, we set $\tau_G = T$.

For a given behavior assumption $\xi = (\xi_t)_{t=1,...,T}$, all contractual cash flows of the pool of policies are specified for any given capital market scenario. Thus both, the guarantee payments (i.e. payments made by the insurer after the account value has dropped to zero) at times $i \in \{1,2,...,T\}$, denoted by $G_i^P(\xi)$, and the guarantee fee payments $G_i^F(\xi)$ made by the policyholder (including surrender fee payments), again at times $i \in \{1,2,...,T\}$, are known. For any given ξ, the time-t value $V_t^G(\xi)$ of the GLWB rider is then given by the expected present value of all future guarantee payments $G_i^P(\xi)$, $i \in \{1,2,...,T\}$, minus future guarantee fees $G_i^F(\xi)$, $i \in \{1,2,...,T\}$,

$$V_t^G(\xi) = E_{\hat{Q}}\left[\sum_{i=t+1}^{T} e^{-r(i-t)} \left(G_i^P(\xi) - G_i^F(\xi) \right) \middle| \hat{F}_t \right]. \tag{11}$$

In the following numerical section, this value is calculated using (nested) Monte-Carlo simulations.

Within our numerical analyses, we consider five different assumptions regarding policyholder behavior:

a) No surrender

Under this assumption, policyholders never surrender their contract, i.e.

$$\xi^0 = \left(\xi_t^0\right)_{t=1,\dots,T},$$

$$\xi_t^0 = 0, \quad \forall\, t = 1,\dots,T.$$

b) Deterministic surrender

Surrender under this assumption occurs according to pre-specified deterministic (time-dependent) percentages $(s_t)_{t=1,\dots,T}$, $0 \le s_t \le 1$, $\forall\, t = 1,\dots,T$, as long as the guarantee has not been triggered. In formulas,

$$\xi^d = \left(\xi_t^d\right)_{t=1,\dots,T},$$

$$\xi_t^d := \begin{cases} s_t, & 1 \le t < \tau_G \\ 0, & else \end{cases}$$

c) Longstaff-Schwartz approximation to optimal surrender

In order to compute the fair value of an American option using Monte-Carlo techniques, Longstaff and Schwartz (2001) introduced a method in which optimal behavior is approximated via least-squares regression of the conditional expectation of the option's payoff, given some path- and time-dependent variables. Essentially, when applied to pricing of the GLWB rider, their algorithm works as follows:

1. Define a set of base functions that take some state variables of the contract and the scenario as argument and return a real number.
2. Create a set of N scenarios under $\hat{Q}$.
3. Starting at T-1, at each policy anniversary t, compute the present value of the cash flow between t and T for each scenario in which the guarantee has not been triggered at time t. Fit the linear least-squares regression with these present values as dependent variables and the base functions with the corresponding state variables of the contract and the scenario as input variables.
4. Evaluate the resulting approximation of the GLWB rider's continuation value for each scenario and decide whether the policyholder should surrender or not. If they surrender, the cash flow following t is set to zero and the cash flow at t to minus the surrender fee paid.
5. Repeat steps 3-5 for t-1 until t=0 is reached.

As base functions we use weighted Hermite polynomials up to a degree of three for each state variable and cross products hereof, again up to a degree of three, as well as a constant. Before simulation and/or pricing, we first execute the Longstaff-Schwartz algorithm with a separate set of scenarios in order to avoid an upward bias.

The surrender behavior of the policyholder can be considered optimal – in the sense that it maximizes the option value V_t^G of the GLWB rider – if the policyholder decides to surrender the contract whenever the benefit from discontinuing the contract (i.e. the negative of the continuation value) exceeds the surrender fees.

With $\hat{V}_t^G$ denoting the approximated continuation value of the GLWB rider at time t, the policyholder behavior is modeled as follows:

$$\xi^{LS} = \left(\xi_t^{LS}\right)_{t=1,\dots,T},$$

$$\xi_t^{LS} := \begin{cases} 1, & \hat{V}_t^G < -\varphi_t^{surr} AV_t^-, \ 1 \le t < \tau_G \\ 0, & else \end{cases}$$

In what follows, we refer to surrender behavior according to this algorithm as being "optimal", although we have to keep in mind that it is only an approximation for the value maximizing strategy defined as $\xi^* = \arg\max_{\xi \in \Xi} V_0^G(\xi)$, where Ξ denotes the set of all admissible strategies.

d) Function of moneyness

Within this approach, we model the fraction of the policyholders who surrender their contract as a function of time and the "in-the-moneyness" of the guarantee (as, for example, described in American Academy of Actuaries, 2005).

We define the moneyness θ_t of the guarantee at time t as the ratio of the surrender value (account value less surrender fees) and the 'strike price' of the guarantee, for which we use the net present value of an immediate annuity paying the current guaranteed withdrawal amount annually until the insured's death. Because this annuity's net present value is a lower limit for the sum of asset value and option value of the GLWB rider, θ_t will be upward biased and not reside around 1 ("at-the-money") as desired. To correct for this, we use θ_0, the moneyness at inception of the contract, as benchmark and use the relative deviation of θ_t hereof as measure.

The basis for the surrender function is a set of given pre-specified deterministic percentages $(s_t)_{t=1,\dots,T}$, $0 \le s_t \le 1$, $\forall\, t = 1,\dots,T$ (as in the deterministic surrender scenario). However, we now model the fraction of the surrendering policyholders at time t as s_t multiplied by a factor that depends on the moneyness-variable θ_t. In detail, we model the behavior according to the following formulas:

$$\xi^{ITM} = \left(\xi_t^{ITM}\right)_{t=1,\dots,T},$$

$$\xi_t^{ITM} := \begin{cases} s_t \cdot \eta_1(\theta_t / \theta_0), & 1 \le t < \tau_G \\ 0, & else \end{cases},$$

$$\eta_1(x) := \begin{cases} 1/3, & x < 0.95 \\ 1, & 0.95 \le x < 1.05 \\ 3, & 1.05 \le x < 1.15 \\ 5, & x \ge 1.15 \end{cases}$$

e) Function of option value

Here, we use a similar approach as for the 'function of moneyness', except that we now use the sum of the rider's (approximated) continuation value and the surrender charge as decision

variable. Within the Longstaff-Schwartz algorithm, it is optimal for the policyholder to discontinue the contract whenever this value becomes negative. Using again pre-specified probabilities $(s_t)_{t=1,...,T}$, $0 \le s_t \le 1$, $\forall\, t = 1,...,T$, this modeling approach is defined as follows:

$$\xi^{OV} = \left(\xi_t^{OV}\right)_{t=1,...,T}$$

$$\xi_t^{OV} := \begin{cases} s_t \cdot \eta_2(\hat{V}_t^G(\xi^0) + \varphi_t^{surr} AV_t^-), & 1 \le t < \tau_G \\ 0, & else \end{cases}$$

$$\eta_2(x) := \begin{cases} 1/3, & x > 0.01 \\ 1, & 0.01 \ge x > -0.01 \\ 3, & -0.01 \ge x > -0.03 \\ 5, & x \le -0.03 \end{cases}$$

Note that the last two models for policyholder behavior, d) and e), allow for the following interpretation which appears to be the motivation for the use of such models in practice: If a certain percentage of the policyholders follow a more or less optimal strategy (in the sense that they intuitively or with the help of professional advisors aim at maximizing the value V_t^G of the embedded guarantee) and the rest of the policyholders are assumed to follow a suboptimal strategy with deterministic surrender rates, then a pool would show patterns similar to models d) and e).

3 Contract Analysis

3.1 Assumptions

For all of the analyses we use the fee structure given in Table 1.

Acquisition charges	4.00 % of single premium
Management charges	1.50 % p.a. of AV
Guarantee charges	1.50 % p.a. of AV

Table 1: Fee structure for the considered contracts.

We further assume the policyholder to be a 65 year old male. For pricing purposes, we use best-estimate annuitant mortality probabilities given in the DAV 2004R table published by the German Actuarial Society (DAV).

As described in Section 2.3, we use different assumptions for the policyholder behavior. In the case where surrender is assumed to be deterministic, we use the surrender pattern given in Table 2. As observed for many products in many markets, we assume higher surrender rates in earlier years and some base surrender in later years.

Year	Surrender rate $p_t{}^S$
1	6 %
2	5 %
3	4 %
4	3 %
5	2 %
≥ 6	1 %

Table 2: Assumed deterministic surrender rates.

Besides deterministic surrender (in what follows denoted by *DS*), we also analyze the other types of policyholder behavior introduced in Section 2.3, i.e. Longstaff-Schwartz-optimal surrender behavior (*optimal*), surrender behavior depending on the option value (*OV*), and surrender behavior depending on the "in-the-moneyness" of the option (*ITM*). We also consider the case without any surrender (*NS*).

3.2   Determination of the Fair Guaranteed Withdrawal Rate

For the pricing of the contract, i.e. for the determination of the guaranteed withdrawal rate x_{WL} that makes the contract fair at inception in the sense that $V_0^G = 0$ holds, we perform a root search with x_{WL} as argument and the value of the option V_0^G as function value, cf. e.g. Bauer et al. (2008) or Kling et al. (2011). In this process, V_0^G is computed via Monte-Carlo simulation, where 100,000 paths are used per valuation.

3.2.1   Results for the Black-Scholes model

In Table 3, we show the fair guaranteed withdrawal rates x_{WL} for different ratchet mechanisms, volatilities, rates of interest, surrender fees and policyholder-behavior assumptions. Note that we here analyze the impact of the policyholder behavior assumptions used for pricing the contract. Effects resulting from a potential deviation between actual policyholder behavior and behavior assumed in pricing and hedging will be analyzed in Sections 3.3 and 4.

| | | Ratchet mechanism | | | | | |
| | | Product I (No Ratchet) | | Product II (Lookback) | | Product III (Remaining WBB) | |
σ_{BS}, r	Behavior	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$
$\sigma_{BS} = 15\%$, $r = 4\%$	Optimal	4.75	4.95	4.69	4.77	4.45	4.45
	OV	5.13	5.23	4.86	4.90	4.51	4.53
	ITM	5.14	5.24	4.87	4.91	4.52	4.54
	NS	5.27	5.27	4.82	4.82	4.45	4.45
	DS	5.48	5.54	5.04	5.09	4.65	4.70
$\sigma_{BS} = 20\%$, $r = 4\%$	Optimal	4.23	4.49	4.17	4.26	4.02	4.02
	OV	4.75	4.87	4.36	4.40	4.09	4.10
	ITM	4.75	4.87	4.37	4.42	4.09	4.12
	NS	5.00	5.00	4.34	4.34	4.03	4.03
	DS	5.20	5.25	4.54	4.59	4.22	4.26
$\sigma_{BS} = 22\%$, $r = 4\%$	Optimal	4.03	4.31	3.96	4.06	3.84	3.86
	OV	4.59	4.72	4.17	4.21	3.92	3.93
	ITM	4.60	4.72	4.17	4.22	3.92	3.95
	NS	4.89	4.89	4.14	4.14	3.86	3.86
	DS	5.08	5.13	4.34	4.39	4.05	4.09
$\sigma_{BS} = 25\%$, $r = 4\%$	Optimal	3.74	4.04	3.67	3.78	3.59	3.61
	OV	4.37	4.51	3.88	3.92	3.67	3.69
	ITM	4.37	4.50	3.88	3.94	3.68	3.70
	NS	4.72	4.72	3.87	3.87	3.62	3.62
	DS	4.90	4.95	4.05	4.10	3.80	3.84
$\sigma_{BS} = 22\%$, $r = 2\%$	Optimal	3.11	3.33	3.08	3.18	3.06	3.10
	OV	3.56	3.67	3.28	3.32	3.16	3.18
	ITM	3.57	3.67	3.28	3.33	3.16	3.19
	NS	3.78	3.78	3.27	3.27	3.12	3.12
	DS	3.96	4.00	3.44	3.48	3.28	3.32
$\sigma_{BS} = 22\%$, $r = 3\%$	Optimal	3.55	3.80	3.51	3.61	3.45	3.48
	OV	4.06	4.17	3.71	3.75	3.53	3.55
	ITM	4.07	4.18	3.71	3.76	3.54	3.56
	NS	4.32	4.32	3.70	3.70	3.48	3.48
	DS	4.50	4.55	3.88	3.93	3.66	3.70
$\sigma_{BS} = 22\%$, $r = 5\%$	Optimal	4.53	4.85	4.44	4.55	4.24	4.25
	OV	5.16	5.30	4.65	4.69	4.31	4.32
	ITM	5.17	5.31	4.65	4.70	4.32	4.34
	NS	5.49	5.49	4.62	4.62	4.25	4.25
	DS	5.69	5.75	4.83	4.88	4.45	4.49

Table 3: Fair guaranteed withdrawal rates x_{WL} in percent under the Black-Scholes model for different ratchet mechanisms, policyholder behavior assumptions, volatilities, rates of interest and surrender fees.

Obviously, the product design without any ratchet allows for the highest withdrawal rates throughout, while the remaining WBB ratchet (which constitutes the 'richest' type of ratchet) allows for the lowest. It is also obvious that fair withdrawal rates are decreasing with increasing volatility and/or decreasing interest rates, since the corresponding guarantees increase in value

with increasing volatility or decreasing interest rates. Our main focus, however, is on the analysis of different assumptions about policyholder behavior:

As defined in Section 2.3, optimal surrender behavior maximizes the value V_t^G of the contract. Thus, the fair withdrawal rates are the lowest in this case. For all considered parameter combinations, the assumption of deterministic surrender rates leads to the highest fair withdrawal rate. The difference between the fair withdrawal rates in these two cases can exceed a full percentage point. For a volatility of 25%, for example, and in the product without ratchet, the fair withdrawal rate assuming deterministic surrender amounts to 4.90% in the case of a surrender fee of 1% (and 4.95% for a surrender fee of 3%) while for optimal policyholder behavior, the fair withdrawal rate is only 3.74% (4.04%). Hence, an insurer assuming deterministic behavior would be willing to provide policyholders lifelong guaranteed withdrawal amounts that are (c.p.) more than 30% higher than the rates offered by a more conservative insurer assuming optimal policyholder behavior.

For the product design with no ratchet, this difference in withdrawal rates is increasing with increasing volatility and with increasing interest rates. Thus, the potential for mispricing resulting from too aggressive assumptions for policyholder behavior is also increasing. For the product designs with ratchet, the difference of the fair withdrawal rate assuming deterministic policyholder behavior and optimal policyholder behavior, respectively, is significantly smaller and much less sensitive to changes in volatility or interest rates. For the lookback ratchet, the difference is about 30 to 40 basis points, in the case of the remaining WBB ratchet, the difference amounts to 20 to 25 basis points. Thus, the potential for mispricing by assuming incorrect policyholder behavior is the smallest for the product design with the most valuable ratchet mechanism.

One reason for this can be seen by comparing the fair withdrawal rate assuming no surrender and optimal surrender. If the ratchet mechanism is quite valuable (i.e. remaining WBB ratchet), there is very little or even no difference in the corresponding fair withdrawal rates. Thus, no surrender seems to be very close to an optimal policyholder behavior. Hence, by assuming some deterministic (and fairly low) surrender rate, the assumption basically is that almost all policyholders behave optimally (by not surrendering) and only very few behave suboptimally. For a product design without any ratchet on the other hand, surrender can become optimal if funds perform well. In all these scenarios, deterministic surrender rates imply the assumption of a high portion of customers behaving suboptimally by not surrendering and only a low portion of customers displaying optimal behavior.

The two path-dependent assumptions about policyholder behavior (*OV* and *ITM*) show a rather similar pattern. For the product design without ratchet, both show a significant potential for mispricing. Even if volatility is only 15%, the difference in the fair withdrawal rates between path-dependent assumptions and optimal behavior is around 40 basis points (roughly 30 basis points for a surrender fee of 3%). Again, with increasing volatility or interest rates, this difference also increases. However, for this product design, the considered path-dependent assumptions lead to lower guaranteed withdrawal rates than assuming no surrender. Thus, in this case the potential for mispricing is lower if path-dependent assumptions are made. This changes if ratchets are included into the product. Then again, the differences in withdrawal rates decrease. At the same time, the fair withdrawal rates assuming no surrender are lower than the fair withdrawal rates assuming path-dependent surrender. Thus, within this modeling framework, even though an insurer assumes surrender behavior that is somehow linked to the

option value or the in-the-moneyness of the option, the potential for mispricing is higher than when assuming no surrender.

Fair withdrawal rates obviously increase with increasing surrender fees. It is, however, worth noting that for product I the difference in fair withdrawal rates between a surrender fee of 3% and 1% increases with increasing "optimality" of the policyholders' behavior. If a strong ratchet mechanism is included (e.g. product III), however, there is almost no difference if policyholders behave optimally. This again is due to the fact that for this product design, not surrendering is close to optimal, even for the lower surrender fee.

3.2.2 Results for the Heston model

For the Heston model, we use the model parameters given in Table 4, that were derived by Eraker (2004), and stated in annualized form for instance by Ewald et al. (2007).

Parameter	Numerical value
r	0.04
θ	$(0.22)^2$
κ	4.75
σ_v	0.55
ρ	-0.569
$V(0)$	θ

Table 4: Parameters for the Heston model.

One of the key parameters in the Heston model is the market price of volatility risk λ. Since absolute λ-values are hard to interpret, in the following table we show the values of the long-term variance and the speed of mean reversion for different values of λ.

Market price of volatility risk	Speed of mean reversion κ^*	Long-term variance θ^*
$\lambda = 3$	6.40	$(0.190)^2$
$\lambda = 2$	5.85	$(0.198)^2$
$\lambda = 1$	5.30	$(0.208)^2$
$\lambda = 0$	4.75	$(0.220)^2$
$\lambda = -1$	4.20	$(0.234)^2$
$\lambda = -2$	3.65	$(0.251)^2$
$\lambda = -3$	3.10	$(0.272)^2$

Table 5: Q-parameters for different values of the market price of volatility risk.

Higher values of λ correspond to a lower volatility and a higher mean-reversion speed, while lower (and negative) values of λ correspond to high volatilities and a lower speed of mean reversion. E.g., $\lambda = 2$ implies a long-term volatility of 19.8% and $\lambda = -2$ implies a long-term volatility of 25.1%.

In the following table, we show the fair annual guaranteed withdrawal rates under the Heston model for all different product designs using the same assumptions regarding policyholder behavior and interest rates as for the Black-Scholes model, and values of λ between -2 and 2.

| | | Ratchet mechanism | | | | | |
| | | Product I (No Ratchet) | | Product II (Lookback) | | Product III (Remaining WBB) | |
λ, r	Behavior	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$	$\varphi_t^{surr} = 1\%$	$\varphi_t^{surr} = 3\%$
	Optimal	4.26	4.54	4.20	4.32	4.04	4.04
	OV	4.77	4.90	4.41	4.45	4.11	4.12
$\lambda = 2,$ $r = 4\%$	*ITM*	4.78	4.91	4.42	4.47	4.11	4.14
	NS	5.02	5.02	4.39	4.39	4.05	4.05
	DS	5.21	5.27	4.59	4.64	4.24	4.28
	Optimal	4.17	4.46	4.10	4.22	3.96	3.96
	OV	4.70	4.83	4.32	4.36	4.02	4.04
$\lambda = 1,$ $r = 4\%$	*ITM*	4.71	4.83	4.32	4.37	4.03	4.06
	NS	4.96	4.96	4.30	4.30	3.97	3.97
	DS	5.16	5.21	4.50	4.55	4.15	4.20
	Optimal	4.06	4.36	3.99	4.11	3.86	3.87
	OV	4.61	4.75	4.21	4.25	3.93	3.95
$\lambda = 0,$ $r = 4\%$	*ITM*	4.62	4.75	4.21	4.27	3.94	3.97
	NS	4.90	4.90	4.19	4.19	3.87	3.87
	DS	5.09	5.14	4.39	4.44	4.06	4.10
	Optimal	3.95	4.24	3.86	3.98	3.75	3.76
	OV	4.52	4.66	4.08	4.13	3.82	3.84
$\lambda = -1,$ $r = 4\%$	*ITM*	4.52	4.66	4.09	4.14	3.83	3.86
	NS	4.82	4.82	4.07	4.07	3.77	3.77
	DS	5.01	5.07	4.26	4.31	3.95	3.99
	Optimal	3.78	4.10	3.71	3.83	3.61	3.64
	OV	4.40	4.55	3.93	3.98	3.69	3.71
$\lambda = -2,$ $r = 4\%$	*ITM*	4.40	4.54	3.94	3.99	3.70	3.73
	NS	4.73	4.73	3.92	3.92	3.64	3.64
	DS	4.91	4.97	4.11	4.16	3.82	3.86
	Optimal	3.14	3.38	3.12	3.23	3.09	3.13
	OV	3.58	3.69	3.32	3.37	3.19	3.20
$\lambda = 0,$ $r = 2\%$	*ITM*	3.59	3.69	3.32	3.37	3.19	3.22
	NS	3.79	3.79	3.32	3.32	3.14	3.14
	DS	3.97	4.01	3.49	3.53	3.31	3.34
	Optimal	3.59	3.85	3.54	3.66	3.48	3.50
	OV	4.08	4.20	3.75	3.80	3.56	3.57
$\lambda = 0,$ $r = 3\%$	*ITM*	4.09	4.21	3.76	3.81	3.56	3.59
	NS	4.33	4.33	3.74	3.74	3.50	3.50
	DS	4.51	4.56	3.93	3.97	3.68	3.72
	Optimal	4.56	4.90	4.47	4.59	4.24	4.25
	OV	5.18	5.33	4.69	4.73	4.31	4.33
$\lambda = 0,$ $r = 5\%$	*ITM*	5.19	5.34	4.69	4.75	4.32	4.35
	NS	5.51	5.51	4.66	4.66	4.25	4.25
	DS	5.71	5.77	4.87	4.93	4.45	4.50

Table 6: Fair guaranteed withdrawal rates x_{WL} in percent under the Heston model for different ratchet mechanisms, policyholder behavior assumptions, market price of volatility risk parameters λ, rates of interest and surrender fees.

The results under the Heston model are very similar to those observed within the Black-Scholes model: Fair withdrawal rates are higher for the product design without ratchet and for lower long-term volatility assumptions. The potential for mispricing is also higher for the product design without any ratchet.

Comparing the results of the Heston model with the corresponding results using the Black-Scholes model shows that the assumption of stochastic equity volatility seems to have only little influence on pricing results for GLWB riders (which is consistent to findings in Kling et al., 2011).

3.3 Quantifying the Risk resulting from Behavioral Assumptions

In a next step, we analyze the loss potential an insurer faces if pricing assumptions for policyholder behavior deviate from actual policyholder behavior.

3.3.1 Results for the Black-Scholes model

Table 7 shows the GLWB rider's value at inception from the insurance company's perspective as a percentage of the single premium if the actual future policyholder behavior as well as equity volatility or interest rates differ from the pricing assumptions. Negative values therefore represent the equivalent of an immediate loss for the insurance company if the insurer charges a certain price for the guarantee that was calculated using assumptions that differ from actual behavior and/or market parameters in a negative way. The results in this table are given for a surrender fee of 3%. We assume that the products are priced assuming an equity volatility in the Black-Scholes model of 22% alongside an interest rate of 4%, and that the actual parameters are either 22% or 25% for the volatility and either 4% or 3% for the rate of interest.

Behavior		Pricing: $\sigma_{BS} = 22\%$, $r = 4\%$ Actual: $\sigma_{BS} = 22\%$, $r = 4\%$			Pricing: $\sigma_{BS} = 22\%$, $r = 4\%$ Actual: $\sigma_{BS} = 25\%$, $r = 4\%$			Pricing: $\sigma_{BS} = 22\%$, $r = 4\%$ Actual: $\sigma_{BS} = 22\%$, $r = 3\%$		
Pricing	Actual	I	II	III	I	II	III	I	II	III
Optimal	Optimal	0.0	0.0	0.0	-1.3	-2.2	-2.5	-3.1	-4.0	-4.2
	OV	2.2	1.1	0.7	1.1	-1.2	-1.6	-0.9	-2.8	-3.2
	ITM	2.3	1.1	0.8	1.1	-1.0	-1.4	-0.8	-2.6	-2.9
	NS	4.3	0.7	0.0	3.1	-1.8	-2.5	0.1	-3.7	-4.2
	DS	4.9	2.2	1.8	3.9	0.3	-0.1	1.7	-1.1	-1.4
OV	Optimal	-2.2	-1.1	-0.7	-3.7	-3.5	-3.3	-6.1	-5.4	-5.1
	OV	0.0	0.0	0.0	-1.2	-2.3	-2.3	-3.9	-4.2	-4.0
	ITM	0.0	0.1	0.1	-1.2	-2.1	-2.1	-3.6	-3.8	-3.7
	NS	1.3	-0.6	-0.7	0.0	-3.2	-3.3	-3.6	-5.2	-5.1
	DS	2.5	1.3	1.2	1.5	-0.8	-0.8	-1.2	-2.3	-2.1
ITM	Optimal	-2.2	-1.2	-0.9	-3.7	-3.6	-3.5	-6.1	-5.6	-5.3
	OV	0.0	-0.1	-0.2	-1.3	-2.5	-2.5	-3.9	-4.3	-4.2
	ITM	0.0	0.0	0.0	-1.3	-2.2	-2.2	-3.6	-3.9	-3.8
	NS	1.3	-0.7	-0.9	0.0	-3.3	-3.5	-3.6	-5.4	-5.3
	DS	2.5	1.2	1.1	1.4	-0.9	-0.9	-1.2	-2.4	-2.2
NS	Optimal	-3.2	-0.6	0.0	-4.7	-2.9	-2.5	-7.4	-4.8	-4.3
	OV	-1.0	0.5	0.7	-2.3	-1.8	-1.6	-5.2	-3.6	-3.2
	ITM	-0.9	0.6	0.8	-2.2	-1.6	-1.4	-4.8	-3.2	-2.9
	NS	0.0	0.0	0.0	-1.3	-2.6	-2.5	-5.2	-4.5	-4.3
	DS	1.5	1.7	1.8	0.4	-0.3	-0.2	-2.4	-1.7	-1.4
DS	Optimal	-4.7	-2.5	-2.4	-6.3	-5.1	-5.0	-9.5	-7.3	-7.0
	OV	-2.6	-1.5	-1.5	-3.9	-3.9	-3.9	-7.2	-6.0	-5.8
	ITM	-2.4	-1.3	-1.3	-3.7	-3.5	-3.6	-6.6	-5.4	-5.3
	NS	-2.0	-2.2	-2.3	-3.3	-4.9	-5.0	-7.6	-7.2	-7.0
	DS	0.0	0.0	0.0	-1.1	-2.1	-2.1	-4.3	-3.8	-3.5

Table 7: GLWB rider value at inception as percentage of the single premium if actual policyholder behavior and/or parameters in the Black-Scholes model differ from pricing assumptions.

Pricing assumption *DS*

We first look at the case where deterministic surrender probabilities are assumed in the pricing of the contract. Clearly, the potential loss is the highest if policyholders behave optimally. In particular, if assumptions about equity volatility and interest rates are correct, the insurance company's loss is 4.7% of the single premium paid if no ratchet is included, 2.5% of the single premium paid in the case of the lookback ratchet and 2.4% of the single premium paid for the remaining WBB ratchet. In line with the results from Section 3.2, the loss potential if only the assumption about policyholder behavior is incorrect is significantly lower if ratchets are included into the product and is the lowest for the product design with the most valuable ratchet mechanism. If policyholders in reality do not behave optimally but either surrender according to one of the path-dependent rules (*OV* or *ITM*) or do not surrender at all (*NS*), then the potential loss roughly lies between 1% and 3% of the single premium paid if the assumptions regarding the market parameters are correct. Again, the riskiest product design is the one without ratchet. However, the product designs with ratchets are more sensitive to changes in volatility:

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If (additionally to policyholder behavior assumptions) volatility assumptions are also wrong (i.e. $\sigma_{BS} = 25\%$), the insurance company's loss increases by at least 2% for the products with ratchet, while for the product without ratchet, the increase is between 1.1% and 1.6% of the single premium paid.

If (additionally to policyholder behavior assumptions) interest rate assumptions are also wrong (i.e. $r = 3\%$), the insurance company's loss increases by at least 3.5% of the single premium paid. This increase is rather similar across the three product designs. Losses can now go up to almost 10% of the premium paid if deterministic policyholder behavior is assumed and actual behavior is optimal.

If policyholders in reality do not surrender at all (*NS*), independent of volatility and interest rates, the loss potential is higher than with any path-dependent behavior (*OV* and *ITM*) for the product designs with ratchets and lower for the product design without ratchet. At the same time, not surrendering seems to be very close to the optimal strategy for product III, which has a rich ratchet mechanism. Also, the interest rate sensitivity is the highest if policyholders do not surrender.

Pricing assumption *NS*

We now look at the case where no surrender is assumed in pricing. In this case (if assumed and actual market parameters coincide), for the products with ratchet, the insurer would realize a gain if any of the other non-optimal policyholder behavior patterns occurs, i.e. *OV*, *ITM* or *DS*. Thus, the insurance company can reduce the risk resulting from policyholder behavior by including a strong ratchet mechanism into the product and at the same time assuming no surrender in pricing the contract. A rich ratchet mechanism can prevent high values of the option to surrender under almost all circumstances. This can be a very effective means to manage policyholder behavior risk.

The effect of wrong volatility assumptions on the insurance company's loss is similar to the one observed when deterministic surrender is assumed: The loss increases by at least 2% for the products with ratchet, while for the product without ratchet the increase is between 1.1% and 1.5% of the single premium paid. Similar increases can also be observed for all other pricing assumptions.

The absolute increase caused by wrong interest rate assumptions, however, is less pronounced than if deterministic surrender is assumed in pricing. The increase is still similar for the different product designs.

Pricing assumptions *OV* and *ITM*

A common path-dependent assumption about policyholder behavior suggests that surrender rates are influenced by the in-the-moneyness (as e.g. described in American Academy of Actuaries, 2005) or (more directly) the value of the guarantee. Although more conservative than assuming purely deterministic behavior, these assumptions can still be quite dangerous: If, for instance, a remaining WBB ratchet is in place and policyholders do not surrender at all, the potential loss amounts to almost 1% of the single premium paid even if assumed market parameters are correct. A slightly smaller loss occurs in case of a lookback ratchet, whereas if no ratchet is in place, there even is a profit. However, if policyholders behave optimally, the potential loss, again, is the highest for the product design without ratchet and amounts to 2.2% of the single premium paid.

If additionally actual market parameters deviate from assumed market parameters, potential losses increase up to over 5% of the single premium paid. The structure of the increase is similar to the effects observed previously, only the differences between the different product designs with respect to their interest rate sensitivity are now slightly higher. This also holds for the following optimal behavior assumption.

Pricing assumption *Optimal*

The most conservative assumption about the policyholders' behavior is of course to assume that they follow an optimal surrender strategy. As a result, losses due to mispricing only occur if additionally to policyholder behavior also market parameters are different than assumed. In this case, the profit from the potentially over-conservative behavior assumption is reduced by these losses. Of course, the losses are the highest if actual behavior is either optimal or, in the case of the remaining WBB ratchet, where not surrendering yields very similar results to optimal behavior, if policyholders do not surrender at all.

Summarizing, we find that the product design without ratchet shows the highest sensitivity to deviations from assumed policyholder behavior. On the other hand, it is the design with the least sensitivity to deviations from assumed volatility, while the negative effect of an overestimated level of interest rates is roughly the same for all three product designs.

3.3.2 Results for the Heston model

Table 8 shows similar results to those presented in the previous section but now using the Heston model. For all results in this table, the surrender fee was set to 3% and, for pricing, the market-price-of-risk factor was set to $\lambda=0$ and a rate of interest of 4% was assumed.

Behavior		Pricing: $\lambda=0, r = 4\%$ Actual: $\lambda=0, r = 4\%$			Pricing: $\lambda=0, r = 4\%$ Actual: $\lambda=-2, r = 4\%$			Pricing: $\lambda=0, r = 4\%$ Actual: $\lambda=0, r = 3\%$		
Pricing	Actual	I	II	III	I	II	III	I	II	III
Optimal	Optimal	0.0	0.0	0.0	-1.2	-2.1	-2.4	-3.1	-3.9	-4.1
	OV	2.0	1.0	0.7	1.0	-1.1	-1.5	-1.0	-2.8	-3.1
	ITM	2.1	1.1	0.8	1.0	-0.9	-1.3	-1.0	-2.5	-2.8
	NS	4.0	0.6	0.0	2.8	-1.7	-2.4	-0.3	-3.7	-4.1
	DS	4.6	2.2	1.8	3.6	0.3	-0.1	1.4	-1.1	-1.3
OV	Optimal	-2.0	-1.0	-0.7	-3.4	-3.3	-3.1	-5.9	-5.3	-5.0
	OV	0.0	0.0	0.0	-1.2	-2.2	-2.2	-3.8	-4.1	-3.9
	ITM	0.0	0.1	0.2	-1.1	-1.9	-1.9	-3.5	-3.7	-3.5
	NS	1.1	-0.5	-0.7	-0.1	-3.0	-3.1	-3.8	-5.1	-5.0
	DS	2.4	1.2	1.2	1.3	-0.7	-0.7	-1.3	-2.2	-2.0
ITM	Optimal	-2.0	-1.1	-0.9	-3.4	-3.4	-3.3	-5.9	-5.4	-5.2
	OV	0.0	-0.1	-0.2	-1.2	-2.3	-2.4	-3.9	-4.2	-4.1
	ITM	0.0	0.0	0.0	-1.2	-2.0	-2.1	-3.6	-3.8	-3.7
	NS	1.1	-0.7	-0.9	-0.2	-3.1	-3.3	-3.8	-5.2	-5.2
	DS	2.3	1.1	1.1	1.3	-0.8	-0.8	-1.4	-2.3	-2.2
NS	Optimal	-2.8	-0.5	0.0	-4.3	-2.7	-2.4	-7.1	-4.7	-4.1
	OV	-0.9	0.5	0.7	-2.1	-1.7	-1.5	-5.0	-3.5	-3.1
	ITM	-0.8	0.5	0.8	-2.0	-1.4	-1.3	-4.6	-3.1	-2.8
	NS	0.0	0.0	0.0	-1.3	-2.4	-2.4	-5.1	-4.4	-4.1
	DS	1.5	1.7	1.8	0.4	-0.2	-0.1	-2.4	-1.7	-1.3
DS	Optimal	-4.4	-2.5	-2.3	-5.9	-4.8	-4.8	-9.1	-7.1	-6.8
	OV	-2.4	-1.4	-1.5	-3.6	-3.7	-3.8	-7.0	-5.8	-5.6
	ITM	-2.2	-1.2	-1.2	-3.4	-3.3	-3.4	-6.3	-5.2	-5.1
	NS	-1.9	-2.1	-2.3	-3.2	-4.6	-4.8	-7.5	-7.0	-6.8
	DS	0.0	0.0	0.0	-1.1	-2.0	-2.0	-4.2	-3.7	-3.4

Table 8: GLWB rider value at inception as percentage of the single premium if actual policyholder behavior and/or parameters in the Heston model differ from pricing assumptions.

Again, the results observed under the Heston model are very similar to those observed under the Black-Scholes model. Thus, the potential for mispricing arising from wrong assumptions about policyholder behavior or a wrong level of volatility or interest rates seems to be much higher than the potential loss arising from ignoring the stochasticity of equity volatility. However, by solely calculating the rider value of the guarantee we implicitly assume perfect hedge effectiveness which is not given in reality.

Kling et al. (2011) have shown that the impact of stochastic volatility on hedge efficiency of such products is typically much higher than on pricing. Therefore, we now analyze how this result relates to different assumptions about policyholder behavior.

4 Analysis of Hedge Efficiency

In this section, we analyze the performance of a hedging program an insurer might apply in order to reduce the financial risk – and thus also the required economic capital – resulting from selling GLWB guarantees. We analyze this performance under different assumptions regarding policyholder behavior and particularly analyze the case where the policyholder behavior assumed by the insurer for pricing and hedging differs from the actual behavior of the policyholders.

In what follows, we first describe the analyzed delta-hedging strategy; we then define the risk measures that we use to compare the (simulated) hedge performance, and finally, we present the simulation results in the last part of this section. The methodology we use is similar to the one used by Kling et al. (2011).

4.1 Hedge Portfolio

We assume that an insurer has sold a pool of policies with GLWB guarantees. We denote by Ψ_t the cumulative option value for that pool of guarantees, i.e. the sum of the option values V_t^G of each policy as defined in Section 2.3. We assume that the insurer cannot influence the value of Ψ_t by changing the underlying fund (e.g. changing the fund's exposure to risky assets or forcing the policyholder to switch to a different, e.g. less volatile, fund). We further assume that the insurer invests the guarantee fees as well as surrender fees in a hedge portfolio Π_t^H and applies some hedging strategy within this portfolio. In case the guarantee of a policy is triggered, the guaranteed payments due are deducted from this portfolio. Thus,

$$\Pi_t := -\Psi_t + \Pi_t^H \tag{12}$$

is the insurer's cumulative profit/loss (in what follows sometimes just denoted as the insurer's profit) at time t stemming from the guarantee and the corresponding hedging strategy. We assume the value of the guarantee to be marked-to-model, where the same model the insurer uses for hedging is used for the valuation of Ψ_t.

For the simulations in the following section, we assume that the insurer uses the Black-Scholes model for hedging purposes and applies a simple delta-hedging strategy within the hedge portfolio Π_t^H: In order to immunize the portfolio against small changes in the underlying's spot price S_t (i.e. to attain delta-neutrality), the quantity of exposure to the underlying within the insurer's hedge portfolio is determined as the delta of Ψ_t, i.e. the partial derivative of Ψ_t with respect to S_t.

We assume that the hedge portfolio is rebalanced on a monthly basis, using central finite differences calculated via Monte-Carlo simulation as approximation for the partial derivative of Ψ_t with respect to S_t.

4.2 Risk Measures

We use the following three measures to compare the different hedging strategies. All measures will be normalized as a percentage of the premium volume at t=0:

- $E_P\left[e^{-rT}\Pi_T\right]$, the expectation of the discounted final value of the insurer's profit under the real-world measure P. This is a measure for the insurer's expected profit and constitutes the "performance" measure in our context. A value of 1 means that, in expectation, for a single premium of 100 paid by the client, the insurance company's discounted profit from selling and hedging the guarantee is 1.

- $CTE_{1-\alpha}(\chi) = E_P\left[-\chi\big|-\chi \geq VaR_\alpha(\chi)\right]$, the conditional tail expectation of the random variable χ, where $\chi = \min\left\{e^{-rt}\Pi_t\big|t = 0,1,...,T\right\}$ is defined as the minimum of the discounted values of the insurer's profit/loss at all policy calculation dates and $VaR_\alpha(\chi)$ denotes the Value at Risk of the variable χ at the level α. This is a measure for the insurer's risk resulting from a certain hedging strategy: it can be interpreted as the additional amount of money that would be necessary at outset such that the insurer's portfolio would never become negative over the life of the contract, even if the market develops according to the average of the α (e.g. 10%) worst scenarios in the stochastic model. Thus a value of 1 means that, in expectation over the α worst scenarios, for a single premium of 100 paid by the client, the insurance company would need to hold 1 additional unit of capital upfront.

- $CTE_{1-\alpha}(e^{-rT}\Pi_T) = E_P\left[-e^{-rT}\Pi_T\big|-e^{-rT}\Pi_T \geq VaR_\alpha(e^{-rT}\Pi_T)\right]$, the conditional tail expectation of the discounted profit/loss' final value. This is also a risk measure which, however, focuses on the value of the profit/loss at time T, i.e. after all liabilities have been met, and does not account for negative portfolio values over time. Thus, a value of 1 means that, in expectation over the α worst scenarios, for a premium of 100 paid by the client, the insurance company's expected loss is 1. By definition, of course, $CTE_{1-\alpha}(\chi) \geq CTE_{1-\alpha}(e^{-rT}\Pi_T)$.

4.3 Simulation Results

In the numerical analyses below, we set α=10% for both risk measures and assume a pool of identical policies with parameters as given in Section 3. We assume that mortality within the population of insured occurs according to the best-estimate probabilities given in the DAV 2004R table. As our analysis focuses on model risk rather than parameter risk, we use the parameters for the capital market models presented in Section 3 for both, the hedging and the data-generating model. The results are calculated using 10,000 Monte-Carlo paths for the simulation, 1,000 paths for each of the valuations used in the calculation of the central finite differences and 10,000 paths for each of the valuations of Ψ_t used to calculate χ.

4.3.1 Results for the Black-Scholes model

Table 9 gives the results for different combinations of behavioral assumptions made by the insurer and actual behavior within the pool of policies. The Black-Scholes model with σ_{BS}=22%

and $r=4\%$ is thereby used as the hedging model of the insurer and as a model of the real-world progression of the capital market (with $\mu=7\%$).

Behavior		Product I			Product II			Product III		
Pricing / Hedging	Actual	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$
	Optimal	-0.1	2.4	2.2	0.1	3.1	2.8	0.0	3.2	2.8
	OV	3.4	1.8	0.8	1.7	2.5	1.5	0.7	2.8	2.0
Optimal	*ITM*	3.5	1.9	0.9	1.7	2.3	1.4	0.8	2.3	1.6
	NS	8.7	2.5	1.8	1.9	3.1	2.3	0.1	3.4	2.8
	DS	7.8	1.8	0.9	2.9	2.1	0.9	1.8	2.0	0.9
	Optimal	-2.7	7.5	7.5	-1.3	4.5	4.5	-0.7	3.4	3.3
	OV	-0.1	1.6	1.4	0.0	2.7	2.3	0.0	2.6	2.2
OV	*ITM*	0.0	1.5	1.4	0.1	2.4	2.0	0.2	2.2	1.9
	NS	2.9	2.2	2.1	-0.3	3.2	2.9	-0.7	3.3	3.1
	DS	3.2	0.9	0.5	1.2	2.0	1.1	1.2	1.6	0.9
	Optimal	-2.8	7.5	7.5	-1.4	4.8	4.8	-0.9	3.7	3.6
	OV	-0.1	1.7	1.5	-0.1	2.9	2.6	-0.2	2.9	2.5
ITM	*ITM*	0.0	1.6	1.4	0.0	2.5	2.3	0.0	2.4	2.1
	NS	2.9	2.3	2.2	-0.4	3.3	3.1	-0.9	3.6	3.4
	DS	3.2	1.0	0.5	1.1	2.1	1.3	1.0	1.9	1.1
	Optimal	-4.5	15.4	15.4	-1.0	5.1	5.0	-0.1	3.0	2.8
	OV	-2.1	11.0	11.0	0.3	3.0	2.7	0.6	2.2	1.7
NS	*ITM*	-2.0	10.8	10.8	0.3	2.6	2.3	0.8	1.9	1.4
	NS	0.0	1.3	1.2	0.0	3.0	2.6	0.0	2.7	2.4
	DS	0.8	3.9	3.8	1.4	2.1	1.3	1.7	1.4	0.7
	Optimal	-5.5	13.3	13.3	-2.6	5.9	5.9	-2.3	5.1	5.1
	OV	-2.9	8.0	8.0	-1.4	4.0	4.0	-1.4	3.9	3.8
DS	*ITM*	-2.7	7.6	7.6	-1.1	3.6	3.6	-1.1	3.4	3.4
	NS	-1.2	5.3	5.3	-1.9	5.0	5.0	-2.2	5.0	5.0
	DS	0.0	1.2	1.0	0.0	2.5	2.2	0.0	2.3	2.0

Table 9: Hedge efficiency results using the Black-Scholes model as data-generating model.

Pricing assumption *DS*

We first look at the case where the insurer assumes deterministic surrender in pricing and hedging the contract. If, in reality, the pool of policyholders behaves exactly according to the same pattern, the insurer's expected profit is close to zero for all different product designs. (Note that we assume that the insurer priced the contracts without incorporating any profit margin.) On average over the 10% worst scenarios, the present value of the insurer's final loss averages to 1.0% of the single premium for product design I. The corresponding values are 2.2% and 2.0% for product designs II and III, respectively. The CTE of the present value of the maximum loss over all policy calculation dates is slightly higher. Similar results are observed for other assumptions about the policyholder behavior as long as assumed policyholder behavior and realized policyholder behavior coincide.

If the insurer assumes deterministic surrender but policyholders actually behave according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option

value (*OV*), the insurer's expected loss significantly increases. The expected loss for the product design without ratchet is roughly 3% of the single premium paid and hence more than twice as high than for the product designs with ratchet. In the case of optimal surrender, the expected loss further increases to 5.5% of the single premium paid in the case without ratchet and about half that value for the products with ratchet. With the expected loss, also the risk increases. Assuming deterministic surrender for product I results in a CTE of final losses between 7.6% and 13.3% of the single premium paid if actual policyholder behavior is path-dependent or even optimal. This risk is reduced by roughly 50% by including ratchets into the product design. If policyholders do not surrender at all (*NS*), the risk is almost the same for all product designs. Consistent with the pricing results above, for product design III, the results for no surrender and optimal policyholder behavior are almost identical since optimal surrender for this product is close to no surrender. For product design I, the risk if policyholders do not surrender is lower than for any path-dependent behavior.

Pricing assumption *NS*

If no surrender is assumed in pricing and hedging, the results are rather diverse. Actual deterministic policyholder behavior leads to a positive expected profit for all product designs and rather limited risk for product designs II and III. However, risk measures for product design I are almost 4%. If policyholders actually behave according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option value (*OV*), the insurer's expected profit is slightly positive for product designs II and III and around -2% for product design I. Interestingly, while for the product designs with ratchet, the risk also is rather limited, the risk measures for product design I exceed 10%. Thus, if no ratchet is included in the product design, the assumption of no surrender is rather risky. If policyholder behavior is optimal, the risk for this product even increases to 15%.

Pricing assumptions *OV* and *ITM*

If policyholder behavior is assumed to occur according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option value (*OV*) and ratchets are included into the product design (products II and III), the expected loss for the insurer (even in the case of optimal policyholder behavior) is below 1.5% of the single premium paid. Furthermore, the considered risk measures remain below 5%. Again, for product design III, actual policyholder behavior without surrender turns out to be almost as risky as optimal policyholder behavior. For product design I, however, no surrender leads to expected profits of 2.9% of the single premium paid and risk measures below 2.3% while optimal behavior leads to a risk of 7.5% and a negative expected profit. Deterministic behavior under both assumptions and for all product designs leads to expected profits and rather low risk.

Pricing assumption *Optimal*

Not very surprisingly, the most conservative assumption of optimal policyholder behavior always leads to the highest expected profit. If actual behavior is deterministic or no surrender, for product design I the expected profit reaches 7.8% and 8.7%, respectively. Also, risk is rather limited and for all product designs below 3.4%. However, it is worth noting that the risk if policyholders do not surrender for product design III (3.4%) is slightly higher than in the case of optimal surrender. We attribute this to the fact that in the case of optimal surrender, all policyholders surrender at the same time. Thus, hedging is needed for a potentially shorter period of time, resulting in a reduced hedging error. Also, for product III, assuming optimal surrender results in a slightly higher risk than if no surrender is assumed. We attribute this to a

more stable hedging in the case of no surrender (as there are no decisions whether all of the policyholders either stay or leave) and some imperfections in the Longstaff-Schwartz algorithm we used.

4.3.2 Results for the Heston model

Table 10 shows the same results as in Table 9, but now using the Heston model as data-generating model instead of the Black-Scholes model, with parameters as stated in Table 4 and with $\mu=7\%$.

Behavior		Product I			Product II			Product III		
Pricing / Hedging	Actual	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$
	Optimal	0.1	3.3	2.9	0.3	4.1	3.7	0.0	4.5	4.0
	OV	3.4	2.5	1.5	1.9	3.5	2.5	0.7	3.9	3.1
Optimal	*ITM*	3.5	2.7	1.6	1.9	3.4	2.4	0.8	3.5	2.7
	NS	8.7	3.5	2.8	2.2	4.3	3.6	0.1	4.6	4.0
	DS	7.8	2.4	1.1	3.1	3.0	1.4	1.7	2.9	1.7
	Optimal	-2.4	6.4	6.4	-1.0	5.0	4.9	-0.7	4.9	4.7
	OV	0.1	2.5	2.2	0.3	3.9	3.4	0.0	4.1	3.6
OV	*ITM*	0.1	2.4	2.2	0.3	3.6	3.2	0.2	3.7	3.3
	NS	2.9	3.4	3.3	0.0	4.7	4.4	-0.7	4.9	4.7
	DS	3.2	1.7	0.8	1.4	3.1	2.0	1.1	3.0	2.0
	Optimal	-2.5	6.4	6.4	-1.1	5.1	5.1	-0.9	5.1	4.9
	OV	0.1	2.5	2.2	0.1	4.0	3.6	-0.2	4.3	3.8
ITM	*ITM*	0.1	2.5	2.2	0.2	3.7	3.3	0.0	3.8	3.4
	NS	2.9	3.5	3.3	-0.1	4.8	4.5	-0.9	5.1	4.9
	DS	3.2	1.7	0.8	1.3	3.2	2.1	1.0	3.1	2.2
	Optimal	-4.2	13.4	13.4	-0.7	4.8	4.6	-0.1	4.3	3.9
	OV	-1.9	9.3	9.3	0.5	3.7	3.1	0.6	3.7	2.9
NS	*ITM*	-1.8	9.2	9.2	0.6	3.5	2.9	0.7	3.3	2.6
	NS	0.0	2.3	2.0	0.3	4.2	3.7	0.0	4.3	3.8
	DS	0.8	3.3	3.2	1.6	3.0	1.8	1.6	2.7	1.5
	Optimal	-5.2	11.7	11.7	-2.3	6.8	6.8	-2.3	6.7	6.7
	OV	-2.8	6.8	6.8	-1.1	5.4	5.3	-1.4	5.5	5.4
DS	*ITM*	-2.6	6.6	6.6	-0.9	5.0	5.0	-1.1	5.0	4.9
	NS	-1.1	6.4	6.4	-1.5	6.7	6.6	-2.3	6.7	6.7
	DS	0.0	2.0	1.8	0.3	3.6	3.2	0.0	3.7	3.3

Table 10: Hedge efficiency results using the Heston model as data-generating model.

Changing the data-generating model from Black-Scholes to Heston does not have any substantial impact on the expected profit, independent of the product design and the assumed policyholder behavior. Also, the structure of the results, i.e. the relation between the results for the different products and the different client behavior patterns is very similar. However, the absolute values of the risk measures change. While product design I appears to be less risky in case of path-dependent behavior under the Heston model if deterministic or no surrender is assumed, the risk for product designs II and III increases. The results show that product designs II and III display a higher sensitivity to volatility than the design without ratchet (I). This is in line with the results of Section 3.3.

We can conclude that assumptions about policyholder behavior can bear significant risk for the insurer, especially if such assumptions are too aggressive, i.e. if policyholders' behavior is closer to optimal behavior than assumed. However, this risk can be significantly reduced by means of product design and making appropriate behavioral assumptions. The latter, however, also increases the price of the product and may result in a lower competitiveness of the product. While the product designs with ratchet features (II and III) appear to be less sensitive to policyholder behavior, our results indicate that they may be harder to hedge and are more sensitive to changes in volatility and/or model risk, respectively.

5 Conclusions

In the present paper, we have analyzed the impact of policyholder behavior on pricing, hedging and hedge efficiency of different GLWB guarantees in variable annuities. We have considered several types of policyholder behavior ranging from deterministic surrender over path-dependent surrender to optimal strategies. We have found that the price of the guarantee strongly depends on the assumed policyholder behavior and there is a significant potential for mispricing if actual policyholder behavior deviates from assumed behavior. Comparing products with different ratchet mechanisms, we find that this potential for mispricing is the smallest for the product design with the most valuable ratchet mechanism.

Analyses of an insurer's hedging strategy showed that both, the insurer's expected profit and the insurer's risk (quantified by CTE measures), depend heavily on the deviation between assumed and actual policyholder behavior as well as the chosen product design. We find that the product design without ratchet shows the highest sensitivity to changes in policyholder behavior. On the other hand, it is the design with the least sensitivity to changes in volatility and the potentially easiest one to hedge. We also find that the impact of stochastic volatility on hedging (and the insurer's risk) is much higher than on pricing (and the insurer's expected profit).

In future research, it would be interesting to combine the analyses of model risk performed in Kling et al. (2011) with the analyses of policyholder behavior risk and quantify how the insurer's risk depends on a simultaneous deviation from reality of assumptions regarding policyholder behavior and the capital market model. It might also be worthwhile to analyze different types of variable annuity guarantees and see whether different types of guarantees (e.g. GMAB or GMIB) display higher or lower behavioral risk than the GLWB designs considered in this paper.

Our analyses so far have been performed on the level of an individual policy. Since hedging errors are not necessarily additive over a pool of policies, it would be worthwhile to analyze how the results with respect to risk management and hedge efficiency change for a heterogeneous pool of policies.

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2 Guaranteed Minimum Surrender Benefits in Variable Annuities: The Impact of Regulator-Imposed Guarantees

Guaranteed Minimum Surrender Benefits in Variable Annuities: The Impact of Regulator-Imposed Guarantees

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Abstract

We analyze the impact of regulator-imposed minimum surrender benefits on variable annuities with a Guaranteed Minimum Accumulation Benefit (GMAB) rider. Based on recent discussions in the German market, we consider different models how these Guaranteed Minimum Surrender Benefits (GMSB) are determined: A minimum surrender benefit given by the present value of the GMAB calculated using market interest rates, the present value of the GMAB calculated using some technical rate of interest, and the market-consistent value of the GMAB. We look at the case where the GMSB is introduced before the contract is sold and considered in the pricing of the GMAB rider. We also consider the case if the GMSB is imposed after the contract has been sold and analyze the impact on the technical provisions and capital requirements of already existing contracts. Finally, we analyze how our results change in the presence of a secondary market. Our results show that (if considered in the pricing of the contract) a GMSB can significantly affect the fair guarantee charge of variable annuities. We also find a significant impact on the technical provisions and capital requirements of already existing contracts. Finally, our results indicate that a secondary market adversely affects the insurer's profitability but reduces the impact of the considered GMSBs on the insurers.

1   Introduction

Variable annuities are unit-linked life insurance contracts that often come with investment guarantees. Therefore, they allow policyholders to benefit from the upside potential of the underlying fund and, at the same time, offer protection when the fund loses value (cf. EIOPA, 2011). Such products offer a variety of guarantees. Besides guaranteed minimum death benefits (GMDB), three main types of guaranteed living benefits (GLB) exist: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and guaranteed minimum withdrawal benefits (GMWB). GMAB and GMIB offer the policyholder some guaranteed maturity value or some guaranteed annuity benefit, respectively, while GMWB allow policyholders to (temporarily or lifelong) withdraw money from their account, even after its cash value has dropped to zero. Variable annuities have experienced a growth in sales in US and Japan since the 1990s and are also becoming increasingly widespread over Europe (cf. EIOPA, 2011).

The product design of variable annuities usually stipulates that the surrender value of such products coincides with the policyholder's account value (minus surrender charges, if applicable). The "fair value" of the guaranteed benefits or the market value of certain hedge assets is typically not part of the individual policyholder's account value and thus, with the usual product design, not part of the surrender value.

The pricing of the guarantees in variable annuities is usually performed under certain assumptions for future surrender rates. Such assumptions can be, for instance, deterministic surrender or (typically) path-dependent surrender (where assumed surrender rates depend on market parameters and/or the value of the guarantee). However, the pricing is usually not performed under the assumption of "optimal" surrender (in the sense of loss-maximizing behavior from the insurer's perspective, cf. Azimzadeh et al., 2014). This reduces the price of such guarantees since – in simplified terms – future profits the insurer expects from sub-optimal policyholder behavior are given to the client by means of a reduced price for the guarantee. The possibility to allow for sub-optimal policyholder behavior in pricing and hedging of such products is a reason why these (often primarily financial) guarantees can be offered by insurers at competitive prices when compared to similar products offered by banks. This opens opportunities for institutional investors to purchase such policies in a secondary market at a price that exceeds the surrender benefit from policyholders who are willing to surrender their contract. In this situation, selling the contract to the institutional investor instead of surrendering it is beneficial for the policyholder. After acquiring the contract, the institutional investor then maximizes (optimizes) the value of the contract, which typically results in loss-maximizing behavior from the insurer's perspective.

Of course, this creates risks for the insurer, most notably the risk that policyholders behave differently than assumed. In Kling et al., 2014, the authors have analyzed the resulting risk in detail.

In this paper, we use their model to additionally analyze a new, regulator-imposed risk that might arise in certain insurance markets: The risk arising from Guaranteed Minimum Surrender Benefits (GMSB). To analyze this risk, we use different exemplary approaches a regulator might choose for including GMSBs in variable annuities. The models result from discussions in Germany where different approaches have been discussed among consumer protection organizations, insurance companies and the local Actuarial Association. Even though the specific approaches are based on this discussion in Germany, the basic intuition behind each model could be introduced in any market.

Since its revision in 2008, § 169 of the German Insurance Contract Law (Versicherungsvertragsgesetz) requires guaranteed minimum surrender values for all insurance contracts where both, in case of death or survival, an insurance benefit is paid. For traditional life insurance contracts with an interest rate guarantee, the law even prescribes how this guaranteed minimum surrender value has to be calculated:

it is given by the prospective policy reserve which is the present value of future benefits. As a discount rate, the technical rate used for the calculation of premiums and benefits of the contract has to be applied. Therefore, the surrender value does not allow for any kind of adjustments to changing market conditions. This in consequence means that the surrender value in general is different from a "fair" market value of the contract. In particular, the surrender value will not be reduced if interest rates rise, although both, the assets backing the contract and the "fair value" of the contract, would drop. The resulting risk has been discussed e.g. in Feodoria & Förstemann, 2015. For unit-linked contracts without guarantees, the law requires the surrender value to coincide with the net asset value of the fund. For unit-linked contracts with guarantees, however, the law is not very clear, since it uses the term "time value" of the contract, which is not properly defined.

In particular, the question if and how this law has to be applied to guaranteed minimum benefits in variable annuities is controversially discussed. We consider four different potential interpretations of the law: First, there is a group of legal experts stating that the corresponding section of the insurance contract law defining guaranteed minimum surrender values is not applicable at all for typical "US-style" variable annuities. Under this interpretation, the surrender value is given by the policyholder's fund value and neither future guaranteed benefits nor guarantee charges are taken into account. Second, since the law uses the word "time value", some market participants demand that a market-consistent value of the contract has to be paid out as surrender benefit. Third, the German Actuarial Association (Deutsche Aktuarvereinigung e.V.) has issued a paper introducing an easy-to-implement method that could serve as an approximation for this market-consistent value if the value of the guarantee has to be considered in the surrender benefit (see Deutsche Aktuarvereinigung e.V., 2011). Note that this paper does not give an opinion on the question whether the value of the guarantee has to be considered or not. Finally, based on the interpretation of minimum reserves required in the German Insurance Supervisory Law (Versicherungsaufsichtsgesetz) given in Herde, 1996, for certain other unit-linked insurance products with a maturity guarantee, a minimum reserve for the guaranteed maturity value (which is given by the guaranteed maturity value discounted with some technical interest rate) might also have to be paid out as a minimum surrender benefit. This technical interest rate is set when the contract is concluded and will not change with changing market interest rates – a similar approach as described above for traditional life insurance contracts. We therefore consider this minimum surrender benefit as a further GMSB-model in our analysis.

We analyze the effect of the considered GMSB-models on pricing, profitability, and market as well as behavioral risk. We particularly consider the effect on an insurer's profitability if a GMSB is imposed after a product has been sold. Furthermore, when assessing policyholder behavior and lapse risk, insurers are required to consider activity by institutional investors like hedge funds in a potential secondary market (cf. e.g. Central Bank of Ireland, 2010). Therefore, we also investigate the impact of the different types of GMSB on a secondary market for variable annuities. To our knowledge, such analyses, in particular with respect to variable annuities with regular premium payment, have not yet been performed.

The paper is organized as follows. In Section 2, we present the model framework that we use to conduct our analyses, including the modeling of the pool of policies, the assumed hedging strategy of the insurer, and, of course, the considered models of the GMSB. In Section 3, we present our numerical results regarding the impact of the considered GMSBs on different key figures from the insurer's perspective, such as market risk, sensitivity to changes in surrender rates and the guarantee value of the contract. In Section 4 we introduce institutional investors into our model framework. We first present an extension of the model given in Section 2 and, subsequently, present numerical results for the extended model. Finally, Section 5 concludes.

2 Model

2.1 Product design of the considered variable annuity

We consider a variable annuity contract that offers the policyholder a Guaranteed Minimum Accumulation Benefit (GMAB, see, e.g., Bauer et al., 2008), where the policyholder is entitled to a minimum account value $B^{A,g}$ at maturity T of the contract. We assume all transactions and events within the contract to happen at one of the contract anniversary dates, represented by the set $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$, where $t_0 = 0$ and $t_N = T$. At these dates, potential premium payments are made by the policyholder or benefits are paid out by the insurer in case the policyholder decided to surrender the contract, the insured person has died or the contract has matured.

At a contract anniversary t prior to T, the premium P_t is paid by the policyholder, provided the contract is still active (i.e. the insured person is still alive and the contract has not been surrendered) and the policyholder has not decided to surrender the contract at time t. For a single premium contract, we let $P_0 > 0, P_t = 0 \; \forall \, t > 0$.

The minimum accumulation benefit $B^{A,g}$ guaranteed at maturity T is defined as a percentage γ of the sum of premiums paid by the policyholder, i.e.

$$B^{A,g} := \gamma \cdot \sum_{i=0}^{N-1} P_{t_i}$$

and the accumulation benefit B^A is the larger of the account value F_T and the guaranteed minimum accumulation benefit:

$$B^A := \max(F_T, B^{A,g}).$$

In return for this guarantee, the insurer receives an ongoing guarantee charge as a percentage η^g of the policyholder's account value. The ongoing administration charges, also deducted from the account value, are denoted by the percentage η^a. Additionally, acquisition and administration charges are deducted from each premium payment, denoted by the percentage $\eta^{a,u}$.

The account value directly after inception is therefore given by

$$F_{t_0} := P_{t_0} \cdot (1 - \eta^{a,u}).$$

At any contract anniversary $t_i \in \mathcal{T} \setminus \{t_0\}$, the account value is calculated as

$$F_{t_i} := \left(F_{t_{i-1}} + P_{t_{i-1}} \cdot (1 - \eta^{a,u}) \right) \cdot \frac{S_{t_i}}{S_{t_{i-1}}} \cdot e^{-(\eta^a + \eta^g) \cdot (t_i - t_{i-1})},$$

where S_t denotes the price of one share of the variable annuity's underlying fund at time t.

In case of death of the insured, the policyholder receives the stipulated death benefit at the subsequent contract anniversary and the contract expires. In what follows, B_t^D denotes the death benefit paid at $t \in \mathcal{T}$ and τ^D denotes the first contract anniversary after the insured's death. If $\tau^D > T$ then the insured is still alive at the contract's maturity. With the considered product design, $B_t^D := F_t, \; \forall \, t \in \mathcal{T}$.

We assume that the policyholder has the right to (fully) surrender the contract at any time during the contract's lifetime. If the policyholder decides to surrender the contract, the stipulated surrender benefit B_t^S is paid out by the insurer at the subsequent date $t \in \mathcal{T}$ and the contract expires.

We assume that the policyholder always waits until a contract anniversary before deciding whether to surrender or continue the contract. This implies that the policyholder knows the exact amount of the potential surrender benefit before deciding on whether to surrender or not. The date at which the policyholder surrenders the contract is denoted by τ^S with $t_1 \leq \tau^S \leq T$. If the policyholder does not surrender the contract, we set $\tau^S = T$. We also let $B_T^S := B^A$.

Note that the contract expires at the time $\tilde{\tau} := \min(\tau^D, \tau^S)$ with $t_1 \leq \tau \leq T$ and the cash flow to the policyholder (or the beneficiaries) is nonzero only at time $\tilde{\tau}$ and equals either $B_{\tilde{\tau}}^S$ or $B_{\tilde{\tau}}^D$.

2.1.1 Guaranteed minimum surrender benefits

We consider four different types of guaranteed minimum surrender benefit (GMSB). As explained in Section 1, these four approaches are based on ideas discussed in Germany. Since these ideas cover a wide range of potential ideas, the results may be of relevance for any market where the introduction of GMSBs is discussed. Even if in some market a different GMSB-model is being considered, qualitatively, the effects will likely be similar to our results.

The GMSB at time t (before deduction of surrender charges) is denoted by $B_t^{S,g,j}$, where the superscript $j \in \{1,2,3,4\}$ indicates the type of the considered GMSB. In all four cases, the surrender benefit is calculated as follows:

$$B_t^S := (1 - \eta^S) \cdot \max(F_t, B_t^{S,g,j}), \quad \forall\, t \in \mathcal{T} \setminus \{t_N\}, j \in \{1,2,3,4\},$$

where η^S represents a time-constant surrender charge.

In the first considered case, denoted as "**no GMSB**", there is no guaranteed surrender benefit, i.e.

$$B_t^{S,g,1} \equiv 0, \quad \forall\, t \in \mathcal{T} \setminus \{t_N\}.$$

The second case is denoted as "**market-rate GMSB**". This model is similar to the approximation for the fair value given by the German Actuarial Association. Here, the policyholder receives at least the discounted guaranteed accumulation benefit $\hat{B}_t^{A,g}$ that would result if all following premium payments P_t were zero, i.e. only premium payments made prior to t are considered:

$$\hat{B}_t^{A,g} := \gamma \cdot \sum_{i=0}^{N-1} P_{t_i} \cdot \mathbf{1}_{t_i < t} \, ,$$

where $\mathbf{1}_{t_i < t}$ denotes the indicator function.

In order to calculate the guaranteed minimum surrender benefit, this hypothetical guaranteed minimum accumulation benefit $\hat{B}_t^{A,g}$ is discounted with the then-current market rate and multiplied by the probability of the insured to survive until maturity of the contract. Let $Z_t(T - t)$ denote the price of a riskless zero-coupon bond at time t with maturity at time T and let $q(s, t)$ represent the expected percentage of the insured who are alive at time s and die within the time interval $]s, t]$. The guaranteed minimum surrender benefit is then defined as

$$B_t^{S,g,2} := \hat{B}_t^{A,g} \cdot Z_t(T - t) \cdot \left(1 - q(t, T)\right), \quad \forall\, t \in \mathcal{T} \setminus \{t_N\}.$$

The third version of the GMSB is denoted as "**technical-rate GMSB**". This is the model that is based on the prospective minimum reserve using a technical interest rate. Here, again the present value of the hypothetical guaranteed minimum accumulation benefit $\hat{B}_t^{A,g}$ is used; however, it is now discounted with a technical, time-constant rate ξ and again weighted with the survival probability. The guaranteed surrender benefit is then defined as

$$B_t^{S,g,3} := \hat{B}_t^{A,g} \cdot e^{-(T-t)\cdot\xi} \cdot \left(1 - q(t,T)\right), \ \ \forall\, t \in \mathcal{T} \setminus \{t_N\}.$$

In the fourth design, denoted as "**MCV GMSB**", the GMSB is the market-consistent value of the GMAB from the insurer's perspective, i.e. the market-consistent value of the guaranteed minimum accumulation benefit less the market-consistent value of the future guarantee charges to be received by the insurer (not considering the option to surrender at a future date). As with the previous two types of GMSB, the valuation implies that there are no future premium payments.

In our analyses, this market-consistent value of the GMAB is approximated by the value of a European put option on the account value with strike price $\hat{B}_t^{A,g}$ minus the present value of future guarantee charges. Both, the guarantee at maturity as well as the future guarantee charges, are weighted with the corresponding survival probabilities of the insured. Let $\hat{F}_{t,s}$ denote the account value at time s assuming that since $t < s$ there were no more premium payments, i.e.

$$\hat{F}_{t,s} := F_t \cdot \frac{S_s}{S_t} \cdot e^{-(\eta^a+\eta^g)(s-t)}.$$

At a contract anniversary $t \in \mathcal{T}$, for the purpose of this GMSB, the value of the guarantee is calculated as

$$
\begin{aligned}
\hat{V}_t^g &:= \mathbb{E}_Q\left[\frac{C_t}{C_T}\max(\hat{B}_t^{A,g} - \hat{F}_{t,T}\,\Big|\, t\right] \cdot \left(1 - q(t,T)\right) \\
&= \frac{F_t}{S_t} \cdot O_t^P\left(T-t, \frac{\hat{B}_t^{A,g}}{F_t}\cdot S_t, \eta^g + \eta^a\right) \cdot \left(1 - q(t,T)\right),
\end{aligned}
$$

where C_t denotes the value of the cash account at time t and $O_t^P(s,K,\phi)$ denotes the price of a European put option on the underlying S_t with time to maturity s, strike price K and a drain ϕ due to charges (that are assumed to have the same effect on the option price as a dividend yield).

At any time $t \in \mathcal{T}$, for the purpose of this GMSB, the value of the future guarantee charges deducted from the policyholder's account value, denoted by $\hat{V}_t^c$, is defined as[1]

$$
\begin{aligned}
\hat{V}_t^c &:= \sum_{i=1}^{N-1}\left(\frac{\eta^g}{\eta^g+\eta^a}\cdot\left(\mathbb{E}_Q\left[\frac{C_t}{C_{t_i}}\hat{F}_{t,t_i}\,\Big|\, t\right] - \mathbb{E}_Q\left[\frac{C_t}{C_{t_{i+1}}}\hat{F}_{t,t_{i+1}}\,\Big|\, t\right]\right)\cdot\left(1 - q(t,t_i)\right)\cdot\mathbf{1}_{t_i>t}\right) \\
&= \sum_{i=1}^{N-1}\left(\frac{\eta^g}{\eta^g+\eta^a}\cdot\left(F_t\cdot e^{-(\eta^a+\eta^g)(t_i-t)} - F_t\cdot e^{-(\eta^a+\eta^g)(t_{i+1}-t)}\right)\cdot\left(1 - q(t,t_i)\right)\cdot\mathbf{1}_{t_i>t}\right)
\end{aligned}
$$

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[1] The reasoning behind the term $\frac{\eta^g}{\eta^g+\eta^a}$ is the following (cf. DAV 2011): $\mathbb{E}_Q\left[\int_{t_{i-1}}^{t_i} F_s \cdot \eta^g \cdot \frac{C_t}{C_s}dt\,\Big|\, t\right] = F_t \cdot \frac{\eta^g}{\eta^g+\eta^a}\cdot\left(1 - e^{-(\eta^a+\eta^g)(t_{i+1}-t_i)}\right).$

$$= F_t \cdot \sum_{i=1}^{N-1} \left(\frac{\eta^g}{\eta^g + \eta^a} \cdot \left(1 - e^{-(\eta^a + \eta^g)(t_{i+1} - t_i)}\right) \cdot e^{-(\eta^a + \eta^g)(t_i - t)} \cdot \left(1 - q(t, t_i)\right) \cdot \mathbf{1}_{t_i > t} \right).$$

The market-consistent value of the GMAB, from the policyholder's viewpoint, is then $\hat{V}_t^g - \hat{V}_t^c$, which is added to the account value F_t in order to determine the fourth type of considered GMSB:

$$B_t^{S,g,4} := F_t + \hat{V}_t^g - \hat{V}_t^c, \ \ \forall \, t \in \mathcal{T} \setminus \{t_N\}.$$

This GMSB is a representation of an actual (market-consistent) "time value" of the GMAB. However, as the surrender benefit is the maximum of the fund value and GMSB less surrender charges, the GMSB only increases the surrender benefit if the time value is positive, but does not lead to a reduced surrender benefit if the time value is negative. Note also, that the actual liability of the insurer includes future premium payments, while this GMSB only considers the sum of premiums paid so far.

2.2 Pool of policies

For our analyses on a portfolio level, we assume a pool of policies with identical contract parameters with regard to inception and maturity date, guarantee level, charges, etc. We also assume the pool of insured to be homogeneous and large enough to justify the application of the law of large numbers such that mortality henceforth is only expressed as a percentage of the pool of insured. We denote the number of contracts in the considered pool of policies at time t by π_t.

The total number of contracts that expire at time t_i due to death of the insured person is given by

$$\pi_{t_i}^D := q(t_{i-1}, t_i) \cdot \pi_{t_i}.$$

The policyholders are assumed to surrender according to deterministic base probabilities, which are increased by a factor of 2 if the contract is "out-of-the-money", i.e. if the contract's discounted guarantee (assuming no future premium payments) is lower than the current surrender benefit. This represents policyholders who are not able or willing to "fully optimize" their contract, but will have an increasing tendency to surrender their contract if the guarantee appears less valuable.[2]

The fraction of policyholders who surrender their contract at the end of the time interval $]t_{i-1}, t_i]$ is given by

$$s_i := \begin{cases} 2 \cdot \tilde{s}_i, & \text{if } B_{t_i}^S > \hat{B}_t^{A,g} \cdot Z_{t_i}(T - t_i), \\ \tilde{s}_i, & \text{else} \end{cases},$$

where $\tilde{s}_i$ represent the deterministic base surrender probabilities. The total number of policyholders who surrender their contract at time $t_i < T$, is then given by

$$\pi_{t_i}^S = s_i \cdot \left(\pi_{t_{i-1}} - \pi_{t_i}^D \right).$$

In line with the approach in Section 2.1, we model the maturity of the contract as all remaining policyholders leaving the pool via "surrender", i.e. we set $\pi_T^S = \pi_{t_{N-1}} - \pi_{t_N}^D$.

The number of contracts immediately after t_i is given by

$$\pi_{t_i} = \pi_{t_{i-1}} - \pi_{t_i}^D - \pi_{t_i}^S.$$

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[2] See Section 4 for an extension of this modeling approach, where "optimal" (loss-maximizing) behavior by institutional investors is explicitly modeled.

2.3 Hedging

We assume the insurer to have a hedging program in place that aims at mitigating the effects the key financial risk drivers have on the insurer's P&L. Guarantee charges from the pool of policies are used to finance the hedge portfolio. In return, guarantee payments are taken from the hedge portfolio's funds. At inception, there is no cash flow to or from the hedge portfolio, i.e.

$$\Phi_{t_0}^\pi = 0$$

The cash flow $\Phi_{t_i}^\pi$ at subsequent dates $t_i \in \mathcal{T}, t_i > 0$ is given by

$$\Phi_{t_i}^\pi = -\pi_{t_i} \cdot F_{t_i} \cdot \frac{\eta^g}{\eta^g + \eta^a} \cdot \left(1 - e^{-(\eta^a + \eta^g)(t_i - t_{i-1})}\right) + \pi_{t_i}^S \cdot \left(B_{t_i}^S - F_{t_i}\right)$$

If the surrender benefit is less than the account value (due to surrender charges), this is also used for financing the hedge portfolio.

Let $\widetilde{\Phi}_t^\pi$ denote the cash flow at an arbitrary time t between inception and maturity, i.e. $0 \le t \le T$,

$$\widetilde{\Phi}_t^\pi = \begin{cases} \Phi_t^\pi, \text{ if } t \in \mathcal{T} \\ 0, \text{ else} \end{cases}.$$

We denote by V_t^π the market-consistent value at time t of the cash flow $\{\Phi_u^\pi, \ u \in \mathcal{T}, \ u > t\}$, i.e. the value for which the pool's guarantee-related cash flows at all future dates can be traded.

In order to replicate the changes in the value of V_t^π due to movements in the underlying fund and changing interest rate environment, we assume a hedging strategy using three hedging instruments and the hedge portfolio to be rebalanced on a regular basis. The considered hedging instruments are: cash (overnight lending/borrowing), the underlying fund (long/short exposures) and a zero-coupon paying bond with the same maturity as the variable annuity contract.

The value of the hedge portfolio at time t is denoted by Ψ_t. We assume the hedge portfolio to start with a value of zero, i.e.

$$\Psi_{t_0} = \Phi_{t_0}^\pi = 0.$$

We assume a simple dynamic hedging strategy that aims at offsetting changes in the value of the pool's liability resulting from changes in the underlying's price ("Delta") and shifts in the interest rate structure ("Rho"). For an arbitrary time t, let λ_t^S denote the number of shares of the underlying fund in the hedge portfolio, λ_t^Z denote the corresponding number of zero-coupon bonds with maturity in T and λ_t^C denote the sum invested in a cash account. For a rebalancing date t, let s denote the time when the last rebalancing occurred. The value Ψ_t then calculates as

$$\Psi_t = \Psi_s + \widetilde{\Phi}_t^\pi + \lambda_s^C \cdot \frac{C_t}{C_s} + \lambda_s^S \cdot \frac{S_t}{S_s} + \lambda_s^Z \cdot \frac{Z_t(T-t)}{Z_s(T-s)}.$$

At a rebalancing date t, the weights of the hedge positions in the underlying and the bond, λ_t^S and λ_t^Z, are calculated as follows:

$$\lambda_t^S = \frac{\partial V_t^\pi}{\partial S_t} \quad, \lambda_t^Z = \frac{\dfrac{\partial V_t^\pi}{\partial r_t}}{\dfrac{\partial Z_t(T-t)}{\partial r_t}}.$$

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In the simulation, λ_t^S and λ_t^Z are calculated numerically by computing the finite difference with respect to changes in the market-consistent value V_t^π.

The position in cash, λ_t^C, is a function of the other two positions:

$$\lambda_t^C = \Psi_t - \left(\lambda_t^S \cdot S_t + \lambda_t^Z \cdot Z_t(T-t) \right).$$

We assume that the insurer neither injects nor extracts any money from the hedge portfolio, such that the final value of the hedge portfolio, Ψ_T gives an indication of the insurer's profit or loss with regard to pricing and hedging the contracts' guarantees. We assume that acquisition and administration charges and expenses are not part of this consideration.

In the following analyses, we will analyze the probability distribution of Ψ_T and use corresponding (risk) measures as indicators for the market risk of the insurer.

2.4 Financial market model

For the valuation as well as the simulation, we need to project the price dynamics of the following assets: the price of one share of the variable annuity's underlying fund, S_t; the price of a risk-free (with regard to default) zero-coupon bond with time to maturity of τ, $Z_t(\tau)$; the price of the cash account, C_t; and the prices of simple put options on the underlying fund, $O_t^P(\tau, K, \phi)$, where K denotes the strike level of the respective option and ϕ the drain due to charges (or the dividend yield, respectively).

We use a similar approach and similar model as in Ruez, 2016. However, we use an extension of the Black-Scholes model (Black & Scholes, 1973) with stochastic interest rates via the Cox-Ingersoll-Ross model ("CIR", Cox et al., 1985). Therefore, the dynamics of the market's state variables under the risk-neutral measure Q are given by

$$dr_t = \kappa_r^Q(\theta_r^Q - r_t)dt + \sigma_r^Q \sqrt{r_t} dW_t^{Q,r},$$
$$dS_t = r_t S_t dt + \sigma_S^Q S_t dW_t^{Q,S}$$
$$dC_t = r_t C_t dt$$

where $W_t^{Q,r}$ and $W_t^{Q,S}$ are two independent Wiener processes under the risk-neutral measure Q. The fair value of a zero-coupon bond can be computed by closed-form formulas given in Cox et al., 1985. The value of the put option is approximated via the Black-Scholes formulas with the assumption of deterministic interest rates.

For the real-world simulation used to project the hedging program of the insurer, we use the same system of stochastic differential equations for the dynamics under the real-world measure P,

$$dr_t = \kappa_r^P(\theta_r^P - r_t)dt + \sigma_r^P \sqrt{r_t} dW_t^{P,r},$$
$$dS_t = (r_t + \mu)S_t dt + \sigma_S^P S_t dW_t^{P,S}$$
$$dC_t = r_t C_t dt$$

where $W_t^{P,r}$ and $W_t^{P,S}$ are two independent Wiener processes under P.

3 Numerical results

In this chapter, we analyze the impact of the considered GMSB on the product's pricing, expected profit and its risk profile with respect to market and lapse risk. The parameters for the interest-rate model used for valuation are taken from Bacinello et al., 2011. We use the same set of parameters for the real-world

projection – this implies a market that is risk-neutral with regard to interest-rate risk. For the equity process, we set the volatility to 10% for both, valuation and real-world projection, and we use 3% for the parameter μ in the real-world projection (as in Kling et al., 2014). Table 1 summarizes the parameters used for the market model in the base case (sensitivity analyses follow).

Parameter	Value
$r_0, \theta_r^P, \theta_r^Q$	0.03
κ_r^P, κ_r^Q	0.60
σ_r^P, σ_r^Q	0.03
σ_S^P, σ_S^Q	0.10
μ	0.03

Table 1: Market parameters used in the base case.

The parameters for the variable annuity contract are summarized in Table 2.

Parameter	Value
T	10
γ	100.0%
$\eta^{a,u}$	5.0%
η^a	1.0%
η^S	1.0%

Table 2: Contract parameters used in the base case.

The rebalancing of the hedge portfolio is assumed to happen on a monthly basis.

We assume the insured person to be 60 years old and male. We use the best-estimate mortality probabilities for annuitants given in the DAV 2004R mortality table published by the German Actuarial Association (DAV). We assume a base surrender rate of 10% in the first year that is subsequently reduced by 1% per year until it reaches 2%, i.e.

$$\tilde{s}_i = \max(2\%, 10\% - (i - 1) \cdot 1\%), \qquad i = 1, 2, \dots$$

For the analyses with regard to surrender risk, we also use scenarios with increased and decreased lapse, where the base surrender rates are multiplied by 1.5 and 0.5, respectively, as well as a scenario where no surrender occurs at all.

For the technical-rate GMSB, we use $\xi = 1.25\%$.

We use 25,000 Monte Carlo paths for the valuations and 10,000 paths for the real-world simulation. Within the real-world simulation, we use 1,000 paths to compute the finite differences used in the modeling of the hedging program.

3.1 Impact of the GMSB on contract pricing

We start our analyses with a comparison of the fair guarantee charges for the four considered GMSBs (cf. Section 2.1.1). For the purpose of this analysis, the "fair guarantee charge" is the guarantee charge for which the market-consistent value V_0^π of the variable annuity's guarantee-related cash flow is zero, i.e. at inception of the contract, the value of the guarantee charges coincides with the value of the potential guarantee payments. In order to illustrate the sensitivity with regard to lapse, we use the different lapse assumptions defined above.

Note that in this section, the fair guarantee charge η^g is calculated under the assumption that the type of GMSB is already known at the time the insurer prices the variable annuity, i.e. the product is being offered after the GMSB has been required by the regulator.

Figure 1 shows the fair guarantee charges for a single premium product (left) and a regular premium product (right).
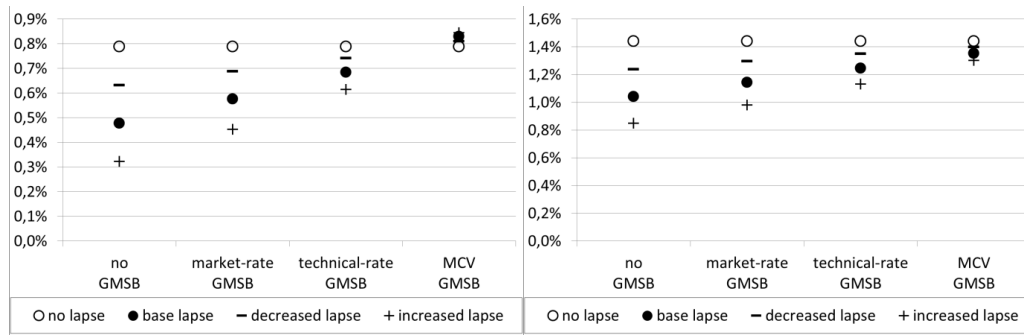

Figure 1 Fair guarantee charges for the single-premium product (left) and the regular-premium product (right).

Even though the fair guarantee charge significantly differs between the single premium and the regular premium contract, the pattern with respect to changes in surrender rates and the different types of GMSB is similar.

As expected, if no GMSB is in place, the fair guarantee charge is the lowest for all considered surrender assumptions. Under the base-lapse assumption it amounts to 0.48% for a single premium contract and 1.04% for regular premium payments. Surrender is on average profitable for the insurer and, thus, the fair guarantee charge decreases if the likelihood of surrender increases.

The addition of a GMSB causes surrender to be less profitable for the insurer. Consequently, the fair charge increases by roughly 10 bp if a market-rate GMSB is enforced and by 20 bp if a technical-rate GMSB is enforced. In case of the MCV GMSB, the fair guarantee charge increases by more than 30 bp. Thus, if the regulator imposes a GMSB in order to treat surrendering customers better, the products will become more expensive for all customers.

While a change in surrender assumptions has a considerable impact on the fair guarantee charge if no GMSB is in place, this sensitivity is reduced if a GMSB is in place. As a consequence, the "potential for mispricing" (by using incorrect surrender assumptions) is the highest if no GMSB is in place. Without GMSB, the fair guarantee charge changes by 15 bp for the single premium case and 20 bp for regular premiums if surrender rates are increased or decreased.

In the case of the MCV GMSB, the fair guarantee charge is the highest out of all considered GMSBs and almost independent of surrender assumptions. In turn, the potential for mispricing with regard to the assumed future surrender rates is fairly low. For the single premium product, the fair guarantee charge with surrender is even higher than without surrender. This means that, on average, surrender with this GMSB means a loss for the insurer, despite the earned surrender charges.

3.2 Impact of the GMSB on the guarantee value of existing contracts

In a next step, we analyze how the value of the variable annuity contract changes if a certain GMSB is enforced by the regulator (immediately) after the contract has been sold. For this and all following analyses, we assume a fixed guarantee charge of $\eta^g = 1.0\%$ p.a. for the single premium contract and

$\eta^g = 1.5\%$ p.a. for the regular premium contract has been used by the insurer when the contract was sold.

We then analyze the change in the value V_0^π caused by the introduction of a GMSB. This can be interpreted as an immediate loss (in case of an increase of V_0^π) or an immediate profit (in case of a decrease of V_0^π) for the insurer caused by the regulatory change.

Figure 2 shows the value of the guarantee for the single-premium product (left) and for the regular-premium product (right) assuming a fixed guarantee charge. Here and in all following figures, values are given as a percentage of the single premium or the sum of premiums, respectively.
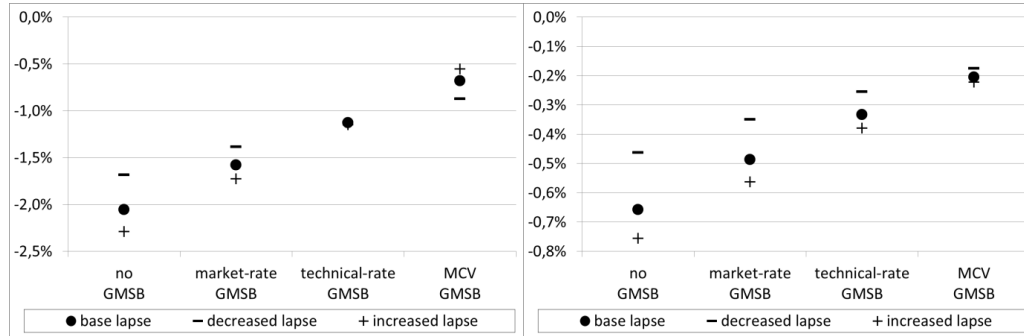
Figure 2 Value of the guarantee for the single-premium product (left) and the regular-premium product (right).

We first have a look at the results for the base lapse rates. The initial value of the guarantee if no GMSB is in place is roughly -2% of the premium for the single-premium contract and roughly -0.65% of the sum of premiums for the regular-premium contract. The introduction of a market-rate GMSB immediately after the start of the contract causes an increase of the value of the guarantee and hence a loss for the insurance company of roughly 0.5% for the single-premium contract and almost 0.2% for the regular-premium contract. The loss caused by the introduction of a technical-rate GMSB is twice as high while that caused by the introduction of a MCV GMSB is roughly three times as high.

Thus, requiring GMSBs for business in force can have a significant impact on the insurer's profitability and can in particular immediately wipe out any safety loadings or profit margins.

The loss caused by the introduction of a GMSB obviously depends on the surrender assumptions used when the contract was priced. The higher the assumed surrender rates, the higher the impact of a GMSB on the value of the guarantee, which is consistent with the results in the previous section.

We finally analyze the sensitivity of the value of the guarantee with respect to changes in surrender rates, i.e. we consider the immediate loss / profit caused by higher / lower surrender rates. As with the fair guarantee charges (cf. Figure 1), the sensitivity to surrender differs significantly between the four GMSB models and also between single and regular premium payment. Without GMSB, surrender on average is profitable for the insurer. For regular premiums surrender is always profitable although any GSMB reduces the profitability. For the single premium product, this changes: if the most valuable GMSB, the MCV GMSB, is considered, additional surrender causes a loss for the insurer, i.e. the additional value of this GMSB outweighs the surrender charges.

The difference between single premium and regular premium payments mostly results from the calculation of the GMSB in the regular premiums case. For the purpose of calculating the GMSBs, the contract is assumed to receive no more premiums after the surrender date. Simply put, if a flat guarantee

charge is used, then the guarantee is overpriced for early premiums while it is underpriced for later contributions. If future premiums are not considered and the guarantee charge remains the same, the contract tends to be overpriced and, thus, often becomes less valuable to the policyholder. This is reflected in the value of the MCV GMSB in the regular-premiums case and makes surrender less valuable for the policyholder than in the single-premium case.

3.3   Impact of the GMSB on capital requirements

In market-based solvency regimes, solvency capital requirements are often calculated as an immediate loss resulting from some stress scenario. The above sensitivities can therefore also be interpreted as an indication for solvency capital requirement (SCR) for lapse risk, where only the "direction of change" (increase or decrease of lapse rates) that causes the highest loss for the insurer has to be considered.

Additionally, we will now give an indication for the SCR for market risk via the following approach:

In contrast to e.g. the Solvency II standard formula, where only the risk resulting from an immediate (or one year) shock is being considered, we consider a full lifetime projection of the pool of policies in order to assess the market risk resulting from the total remaining lifetime of the contracts, cf. Section 164 in EIOPA, 2011 in connection with Article 122 of the Directive 2009/138/EC. This includes the risk from accumulated hedging errors, particularly hedging errors that occur close to maturity of the contracts.

We account for the longer time horizon by setting the VaR level at 95% and consider the respective percentile of the real-world distribution of the (discounted) profit/loss $\frac{\Psi_T}{C_T}$ for the insurer after hedging is applied (cf. Section 2.3). We interpret the difference of this 95th percentile and the present value of the guarantee V_0^π as (an indicator for) the SCR for market risk, cf. also Central Bank of Ireland, 2010.

The resulting number shows how much solvency capital needs to be added on the value of the guarantee in order to meet all liabilities until the maturity of the contract with a probability of 95%.

In Figure 3, we show the surrender sensitivities for lapse risk as well as the considered capital requirement for market risk for the single-premium product (left) and for the regular-premium product (right).
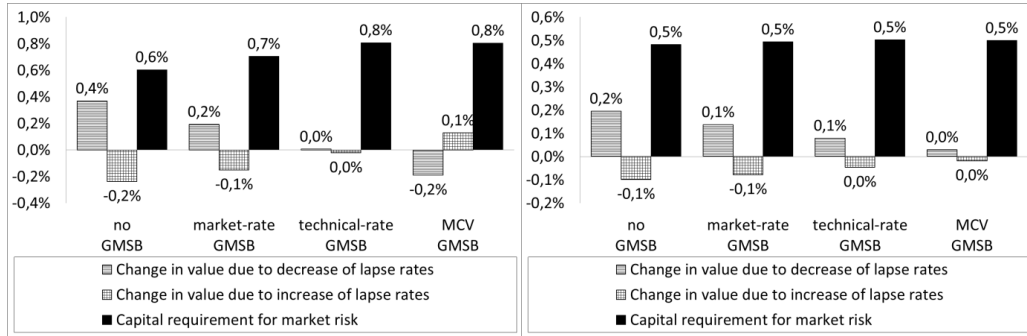
Figure 3 Surrender sensitivities for lapse risk and capital requirement for market risk for the single-premium product (left) and the regular-premium product (right).

For all considered GSMB models, the SCR for market risk remains below 1% of the sum of premiums. While for the regular-premium product market risk is more or less the same for all types of GMSB, it increases with the introduction of GMSBs for the single-premium product. This is in line with the findings regarding the impact on the guarantee value. For regular-premium products the SCR for market

risk is roughly 0.5% of the sum of premiums for all GMSBs. For the single premium product, it fluctuates between 0.6% and 0.8% of the single premium.

The considered SCR for surrender risk overall is smaller, but depends more heavily on the GMSB. Again, we can observe that, depending on the GMSB type, the risk of the insurer can lie in either increased or decreased surrender. It is worth noting that the risk arising from the introduction of a certain GMSB after the product has been sold (see Figure 2) is significantly higher than the considered SCR for the risk arising from changes in surrender behavior once a GMSB is in force.

3.4 Sensitivity to interest rates

In a next step, we will perform capital market sensitivities. We start with an analysis of the impact of lower interest rates on the value of the guarantee and capital requirements. For this, we set r_0, θ_r^P and θ_r^Q to 1.5% (as opposed to 3.0% in the base case). As with the introduction of the GMSBs, we assume that the change happens after the variable annuity has been sold, i.e. the pricing is not adjusted to the new interest rate level. This represents a scenario in which, after the variable annuity has been sold with the guarantee charge used in the previous sections, the embedded guarantee (and also the modeled hedging portfolio) becomes rather valuable. In such a scenario, without a GMSB, it is highly profitable for the insurer if the policyholder decides to surrender, since in this case the value of the guarantee remains with the insurer. The addition of a GMSB reduces this effect and is thus potentially especially harmful for the insurer in such a scenario.

Figure 4 shows the value of the guarantee for the single-premium product (left) and for the regular-premium product (right) for low interest rates. In Figure 5, we show surrender sensitivities for lapse risk as well as the considered capital requirement for market risk for the single-premium product (left) and for the regular premium product (right) for low interest rates.
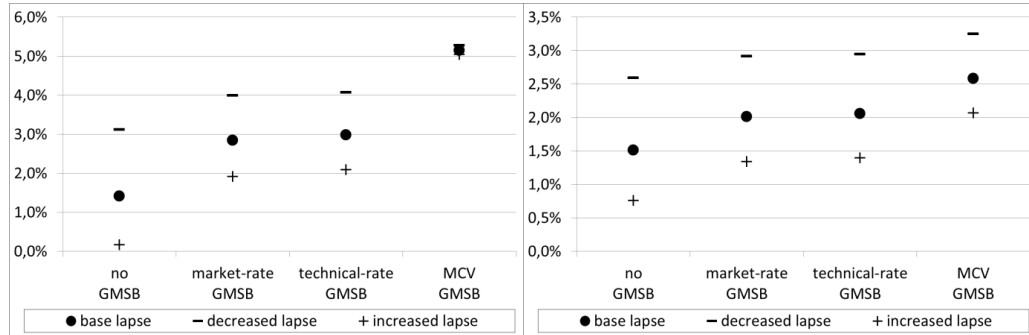
Figure 4 Value of the guarantee for low interest rates, single-premium product (left) and regular-premium product (right).
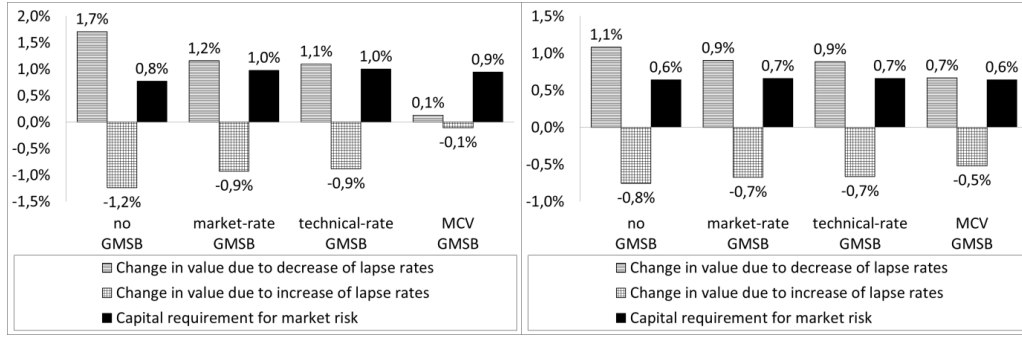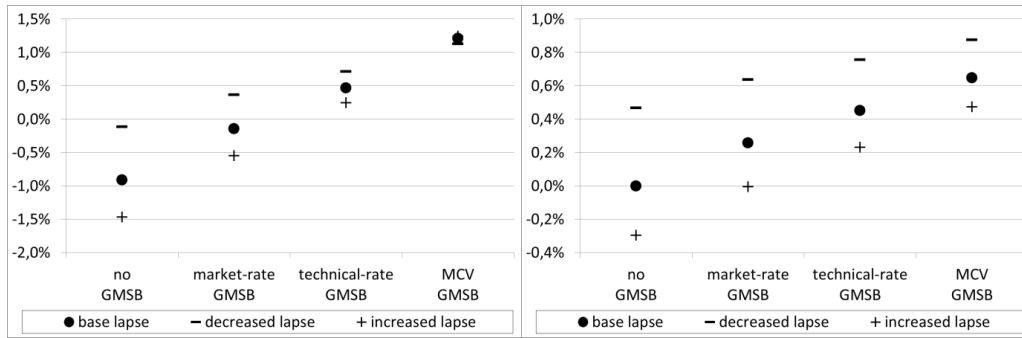
Figure 5 Surrender sensitivities for lapse risk and capital requirement for market risk for low interest rates, single-premium product (left) and regular-premium product (right).

Obviously, a change of interest rates has a tremendous effect on the results. We can see from Figure 4 that the value of the guarantee has increased and is now positive, independent of the assumed surrender behavior and the GMSB model. This means the present value of future expected guarantee payments exceeds the present value of future expected guarantee charges. As such, surrender is highly profitable for the insurer. Overall, without a GMSB, the sensitivity with respect to surrender has strongly increased in comparison to the base case. While market risk has increased only slightly in comparison with the base case and still does not exceed 1% of the sum of premiums, surrender risk can be as high as 1.7% if no GMSB is in place.

As expected, with the higher value of the guarantee, the immediate loss resulting from introducing a GMSB also increases. In the case of the single-premium product, the immediate loss from introducing the MCV GMSB is roughly 4% of the premium. Also the sensitivity for lapse risk is significantly reduced, i.e. from the insurer's perspective it is now more or less irrelevant if the policyholders do surrender or not. This does not hold for regular premiums, again due to the assumption of a contract with no more future premium payments. With the other, less valuable, GMSB models, surrender is still profitable for the insurer. That means the corresponding guaranteed surrender benefits are below the market value of the contract.

3.5 Sensitivity to equity volatility

Finally, we perform a similar sensitivity analysis with respect to the equity volatility. We assume the equity volatility (i.e. σ_S^P and σ_S^Q) to be 12.5% (as opposed to 10.0% in the base case). As with the scenario of lower interest rates, we assume the change to happen immediately after the variable annuity has been sold with the guarantee charge used in the previous sections.

Figure 6 shows the value of the guarantee for the single-premium product (left) and for the regular-premium product (right) for the increased equity volatility. In Figure 7, we show surrender sensitivities for lapse risk as well as the considered capital requirement for market risk for the single-premium product (left) and for the regular-premium product (right).

Figure 6 Value of the guarantee for increased equity volatility, single-premium product (left) and regular-premium product (right).
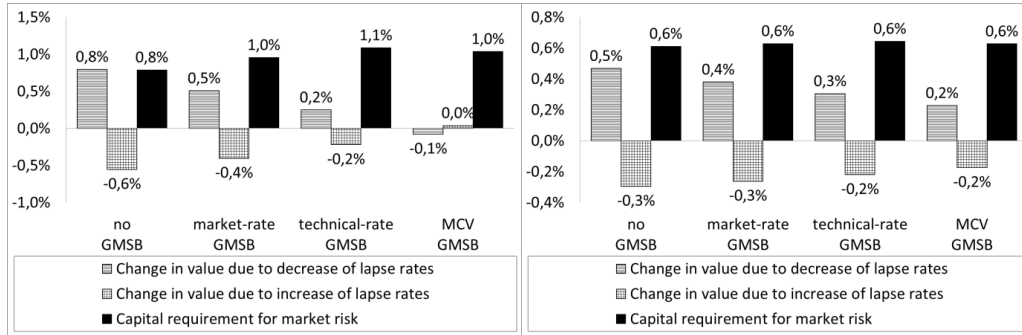

Figure 7 Surrender sensitivities for lapse risk and capital requirement for market risk for increased equity volatility, single-premium product (left) and regular-premium product (right).

The value of the guarantee as well as both considered risk indicators increase with higher equity volatility. The considered SCR for lapse risk increases more strongly than the SCR for market risk and, for some GMSBs, both reach similar levels.

The immediate loss caused by the introduction of a GMSB increases only slightly but, at the same time, sensitivity with respect to surrender increases. Changing volatility, therefore, not only has an impact on the insurer's market risk, but can have an even higher impact on the insurer's lapse risk.

4   Impact of a secondary market

In this section, we analyze the impact of a secondary market with "rational", i.e. value maximizing, investors. We assume that those policyholders who are willing to surrender their contract have the possibility to alternatively sell their policy to some institutional investor in the secondary market. After purchasing the contract, the investor then acts "rationally".

We assume that the presence of a secondary market has not been considered in the pricing of the contract and therefore use the same contract and guarantee charges as in Chapter 3.

4.1   Model description

We use the model from Section 2 unless stated otherwise in this section.

We assume that the pool of policyholders consists of two groups: Policyholders of "type A" behave according to the model used in Section 2.2. We denote the number of contracts in this group at time t by π_t^A.

The policyholders of type "B" are the investors purchasing policies in the secondary market. They will surrender their contract as soon as the contract's continuation value (assumed to be the market-consistent value of the contract) drops below the surrender benefit (after deduction of surrender charges). The market-consistent value includes the value of all guarantees, i.e. the GMAB as well as the GMSB. From the insurer's perspective these investors represent a worst-case "loss maximizing" behavior (cf. Azimzadeh et al., 2014). We denote the number of contracts in this group at time t by π_t^B.

The total number of active contracts at time t is therefore given by $\pi_t := \pi_t^A + \pi_t^B$.

The total number of contracts that expire at time t_i due to death of the insured person is given by

$$\pi_{t_i}^D := \pi_{t_i}^{A,D} + \pi_{t_i}^{B,D}, \text{ where}$$
$$\pi_{t_i}^{*,D} = q(t_{i-1}, t_i) \cdot \pi_{t_i}^*, \text{ with } * = A, B.$$

Similarly, let $s_i^* = s^*(t_{i-1}, t_i)$ represent the fraction of policyholders in the sub-pools $\pi_{t_{i-1}}^A$ and $\pi_{t_{i-1}}^B$, respectively, who want to surrender their contract at the end of the time interval $]t_{i-1}, t_i]$.

The policyholders in sub-pool A are assumed to be "willing to surrender" as explained in Section 2.2. However, we now assume that there exists a secondary market and a certain percentage λ of the policyholders of type A who are willing to surrender the contract, will instead sell the contract to an institutional investor (i.e. a policyholder of type B) if such an investor offers to pay more than the surrender benefit. Since an institutional investor would only offer a price exceeding the surrender value if the continuation value exceeds the surrender benefit, the number of transitioning contracts, denoted by $\pi_{t_i}^{A \to B}$, then is given by

$$\pi_{t_i}^{A \to B} := \begin{cases} \lambda \cdot s_i^A \cdot \left(\pi_{t_{i-1}}^A - \pi_{t_i}^{A,D} \right), & \text{if } B_{t_i}^S < CV_{t_i}, \\ 0, & \text{else} \end{cases}$$

where $CV(t_i)$ represents the market-consistent continuation value of the contract at time t_i (assuming loss-maximizing behavior of the policyholder). Note that, with regard to the pool of policies, surrender is assumed to happen right before the next contract anniversary and, thus, only contracts where the insured is still alive at the end of the interval can be surrendered.

The total number of policyholders in sub-pool A that surrender their contract at time $t_i < T$, denoted by $\pi_{t_i}^{A,S}$, then is given by

$$\pi_{t_i}^{A,S} = s_i^A \cdot \left(\pi_{t_{i-1}}^A - \pi_{t_i}^{A,D} \right) - \pi_{t_i}^{A \to B}.$$

Policyholders in the sub-pool B surrender their contract if and only if the surrender benefit exceeds the continuation value:

$$s_i^B := \begin{cases} 1, & \text{if } B_{t_i}^S > CV_{t_i}, \\ 0, & \text{else} \end{cases}.$$

The total number of policyholders in sub-pool B that surrender their contract at time $t_i < T$, denoted by $\pi_{t_i}^{B,S}$, then is given by

$$\pi_{t_i}^{B,S} = \left(\pi_{t_{i-1}}^{B} - \pi_{t_i}^{B,D} + \pi_{t_i}^{A \to B}\right) \cdot s_i^{B}.$$

The total number of contracts that expire at time $t_i < T$ due to surrender, $\pi_{t_i}^{S}$, then is given by

$$\pi_{t_i}^{S} := \pi_{t_i}^{A,S} + \pi_{t_i}^{B,S}$$

The number of contracts in the two sub-pools immediately after t_i, i.e. after contracts that matured due to surrender or death of the insured have left the respective pool, are given by

$$\pi_{t_i}^{A} = \pi_{t_{i-1}}^{A} - \pi_{t_i}^{A,D} - \pi_{t_i}^{A,S} - \pi_{t_i}^{A \to B} \text{ and}$$
$$\pi_{t_i}^{B} = \pi_{t_{i-1}}^{B} - \pi_{t_i}^{B,D} - \pi_{t_i}^{B,S} + \pi_{t_i}^{A \to B}.$$

4.2 Numerical results

We use the parameters from Section 3 and set $\lambda = 0.5$. This means we assume that half of the policyholders of pool A who are willing to surrender their contract would rather sell it to an institutional investor (if the investor offers a price that exceeds the surrender benefit). For the purpose of calculating the loss-maximizing behavior and the corresponding continuation value, we use the Longstaff-Schwarz method (cf. Longstaff & Schwartz, 2001 and Bacinello et al., 2011).

We first compare the value of the guarantee with and without a secondary market. Figure 8 shows the value of the guarantee for the single-premium contract without secondary market (left) and with a secondary market (right). Figure 9 shows similar charts for the regular-premium contract.
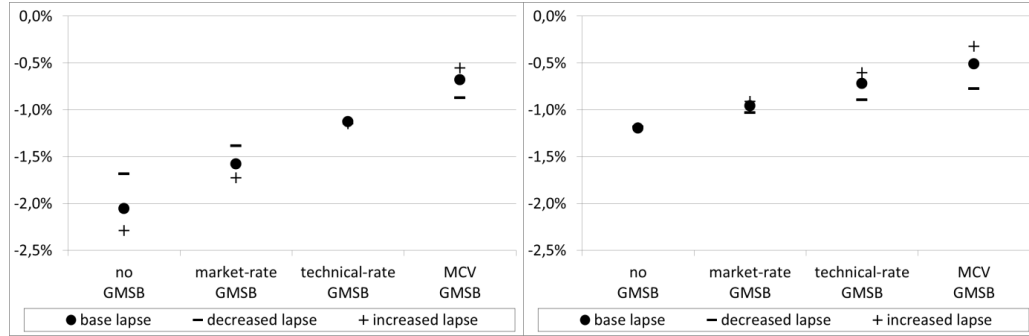
Figure 8 Value of the guarantee for the single-premium contract without secondary market (left) and with secondary market (right)
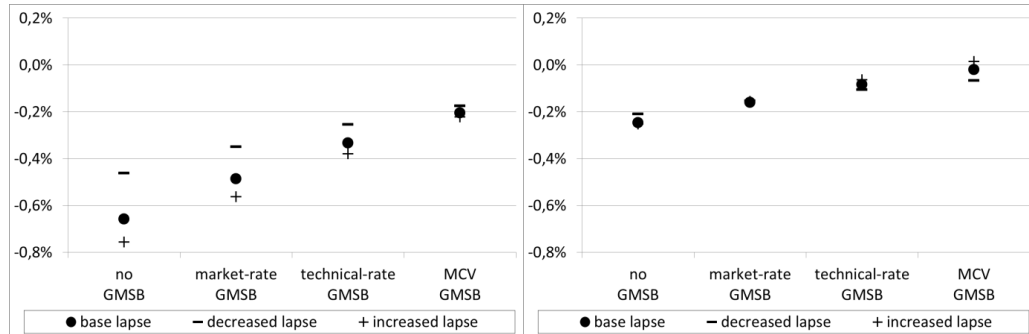

Figure 9 Value of the guarantee for the regular-premium contract without secondary market (left) and with secondary market (right)

For both, the single premium and the regular premium contract, the value of the guarantee is significantly increased by the introduction of a secondary market. This means that – as expected – a secondary market can significantly reduce the insurer's profitability.

In the single premium case, the introduction of a secondary market leads to an immediate loss of almost 1% of the sum of premiums for the insurance company if no GMSB is in place. For the market-rate GMSB and the technical-rate GMSB, the immediate loss caused by a secondary market is roughly 0.5% of the single premium. For all these GMSB models, there are situations where the insurer can profit from surrender if no secondary market exists. The introduction of the secondary market reduces these profits due to the "rational" behavior of the investor.

A secondary market can only exist if surrender benefits are lower than the continuation value. Otherwise it is not possible to offer the policyholders a price that exceeds the surrender benefit. Since in case of the MCV GMSB the surrender value is rather close to the continuation value, the impact of a secondary market on the value of the guarantee is much lower.

For the regular-premium product, the effects are similar but in general less pronounced. An exception is the MCV GMSB where (in contrast to the single premium case) the introduction of a secondary market has a relatively pronounced effect. This shows that in case of regular-premium payment the MCV GMSB does not fully represent the market-consistent value of the whole contract, which is consistent to the findings in Section 3.2.

In Figure 10 we show surrender sensitivities for lapse risk as well as the considered capital requirement for market risk for the single-premium contract without secondary market (left) and with secondary market (right). Figure 11 shows similar charts for the regular-premium contract.
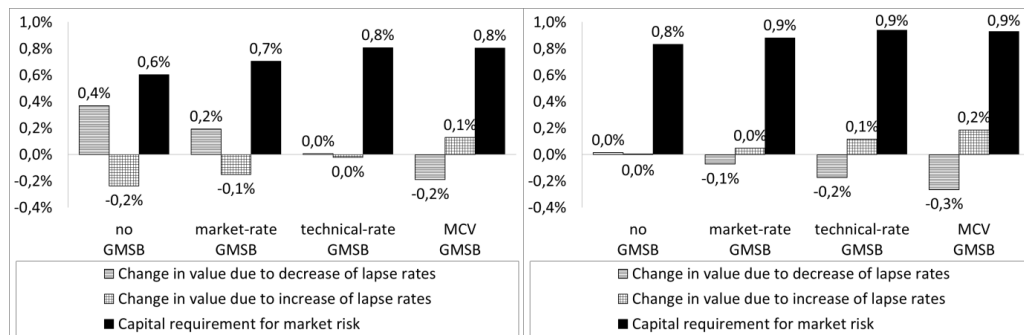

Figure 10 Surrender sensitivities for lapse risk and capital requirement for market risk for the single-premium contract without secondary market (left) and with secondary market (right)
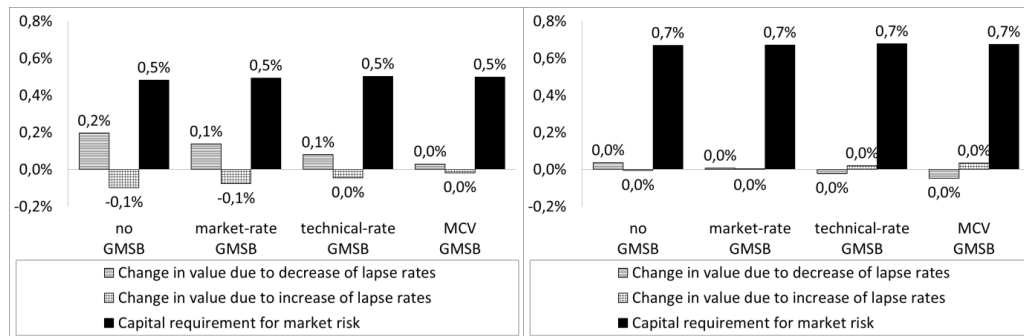
Figure 11 Surrender sensitivities for lapse risk and capital requirement for market risk for the regular-premium contract without secondary market (left) and with secondary market (right)

Figure 10 and Figure 11 confirm the effects described above: lapse risk in general decreases if a secondary market exists with the exception of the MCV GMSB.

Also, the investor typically does not surrender in scenarios where the guarantee is valuable and (absolute) hedging errors tend to be higher and vice versa. This emphasizes hedging errors and therefore, additionally increases market risk in the presence of a secondary market for both considered contracts and all considered GMSBs.

5   Conclusion

In the present paper, we have analyzed the impact of the introduction of GMSBs to a variable annuity contract with a GMAB. We have considered both scenarios: An introduction of GMSBs after a contract has been sold as well as offering new contracts in a market with mandatory GMSBs. We have analyzed the impact on key figures such as the fair guarantee charge of the contract, the guarantee value of the contract, capital requirements with respect to market risk and the sensitivity with regard to surrender rates. We found that, while the impact of GMSBs on market risk is relatively low, the impact on the fair guarantee charge, the guarantee value and the risk resulting from changes in policyholder behavior is substantial.

If the GMSB model is already known and considered when pricing the contract, the resulting advantage for policyholders who surrender the contract comes at the price of increased guarantee charges for all policyholders, adversely affecting especially those who keep the contract until maturity. The fair charge increases only slightly if a market-rate GMSB is enforced, twice as much if a technical-rate GMSB is enforced and roughly three times as much in case of the MCV GMSB. As a consequence, the same protection level with regard to old-age provision becomes more expensive when GMSBs are in place.

If a GMSB is introduced after inception of the contract, e.g. because of a regulatory change, the insurer will suffer an immediate loss on its market-value balance sheet. This loss is the highest if a MCV GMSB is introduced and the lowest if a market-rate GMSB is introduced. While the value of the contract increases with the value added by the GMSB, the sensitivity with regard to surrender rates decreases, as, from a valuation perspective, it becomes less important whether policyholders decide to surrender or not. As a consequence, the potential for mispricing of the contracts with respect to incorrect surrender assumptions is reduced. The market-rate GMSB shows the lowest potential for mispricing with respect to surrender assumptions.

Our analyses with regard to the impact of a secondary market show that, in a market without GMSBs, the presence of an institutional investor creates a loss for the insurer and also increases market risk. At

the same time, the impact of introducing GMSBs is reduced and the specific design of the GMSB is less relevant. On the other hand, if GMSBs are already in place, the potential for a successful secondary market is reduced, since the difference between the surrender benefit of a contract and its continuation value is typically lower.

With a GMSB in place, institutional investors less likely are able to offer prices that exceed the surrender benefit. On the other hand, after the investor has bought a contract, GMSBs offer them additional value that can be exploited by optimized surrender behavior.

Our results strongly indicate that regulators considering the introduction of mandatory GMSBs should carefully analyze the potential impact – also on contracts with different forms of guarantees than the simple GMAB guarantee used in this paper. In particular, the following effects should be considered: For new business, GMSBs will cause increased guarantee charges for the policyholder. This creates redistribution effects from policyholders not surrendering their contract towards surrendering policyholders. Imposing mandatory GMSBs also on already existing contracts increases insurers' risk and can have severe adverse effects on insurers' profitability.

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3 Variable Annuities with Guaranteed Lifetime Withdrawal Benefits: An Analysis of Risk-Based Capital Requirements

Variable Annuities with Guaranteed Lifetime Withdrawal Benefits:
An Analysis of Risk-Based Capital Requirements

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ABSTRACT

Under risk-based regulatory regimes like Solvency II in the EU, the risk profile of a variable annuity directly affects the amount of capital that providers are required to hold. Therefore, providers of variable annuities not only face the challenge to hedge against changes in the value of embedded guarantees, but are also exposed to potential additional capital needs due to changes in their capital requirements. Both, the value of embedded guarantees as well as corresponding capital requirements, are dependent on market parameters and, thus, subject to changes.

We analyze the risk profile of a pool of variable annuity policies with Guaranteed Lifetime Withdrawal Benefit (GLWB) riders with regard to the pool's key financial risk drivers: equity returns, implied equity volatility and interest rates. In a simulation study, we analyze the effectiveness of different stylized hedging programs over a one-year time horizon and compute indicators for risk-based capital requirements. The approach we use is comparable to an internal model type approach under Solvency II. We also analyze the impact changing market environments have on risk profile, hedge effectiveness and capital requirements, similar to a forward-looking analysis in the context of the mandatory Own Risk and Solvency Assessment (ORSA) under Solvency II.

We find that, in addition to the stress from potentially unhedged increases in the value of liabilities, changes in the market environment can have a substantial impact on capital requirements. As a result, GLWB providers face the risk of increases in their risk-based capital requirements and, thus, the need for capital injections – even without pricing errors or malfunctioning of the hedging program. However, there are also cases where an increase in the value of liabilities is accompanied by a decrease of capital requirements, reducing the overall impact on the provider.

1 INTRODUCTION

Variable annuities with Guaranteed Lifetime Withdrawal Benefit (GLWB) riders provide policyholders an opportunity to combine fund investments with a protection against outliving their retirement savings. While the savings remain invested in funds, the GLWB gives them the right to lifelong regular withdrawals from their account, even after the invested amount is depleted and the account balance dropped to zero. Usually, variable annuity providers receive ongoing guarantee charges deducted from the policyholder's account in compensation for this guarantee. In other words, this type of variable annuity embeds a variant of ruin-contingent life annuity (cf. Huang et al., 2014), where the guarantee provider starts to pay a lifelong annuity as soon as the account value (reduced by pre-defined withdrawals) hits zero. Modern GLWB riders typically also include a form of ratchet mechanism, through which the guaranteed withdrawal amount may increase during the lifetime of the contract if the underlying fund performs well. To counter the financial risks that come with this guarantee, insurers and other providers typically implement hedging programs, which aim to mitigate the influence market movements and changing market conditions have on the provider's profit and loss (P&L).

Hedging programs can be quite effective in mitigating the financial risks inherent in GLWB riders, but they usually do not allow for a perfect replication of the changes in the value of liabilities, due to discrete rebalancing and other imperfections (cf. Ledlie et al., 2008). Hence, the provider's P&L with regard to its GLWB business remains subject to fluctuations, even if a hedging program is implemented. Since it is also not feasible to control for every single influencing market parameter, hedging programs normally only aim to control the influence of key risk drivers like the underlying fund's return, interest rate levels and implied equity volatility up to a certain degree. Additionally, policyholder behavior can have a substantial impact on the variable annuity provider's P&L (cf. Kling et al., 2014). As a consequence, variable annuity providers face considerable risks and, under risk-based regulatory regimes like Solvency II, need to equip their variable annuity business with adequate financial resources in order to meet capital requirements.

In the context of GLWB, while the analysis of the impact of key financial risk drivers on the value of liabilities is rather straightforward, it is not entirely clear to which extent this impact transfers to the magnitude of the inherent risk and, thus, to the risk-based capital requirements stipulated by a regulator. To our knowledge, this has not yet been analyzed in the scientific literature, which is why this paper aims to fill this gap by analyzing and illustrating the risk profile and resulting risk-based capital requirements in the context of pools of variable annuity contracts with GLWB riders. The approach we use is comparable to an internal model type approach under Solvency II and the analyses of the impact changing market environments have on the risk profile, hedge effectiveness and capital requirements are comparable to forward-looking analyses in the context of the mandatory Own Risk and Solvency Assessment (ORSA) under Solvency II.

The pricing and valuation of variable annuity contracts with guarantees have been studied in great detail, with Milvesky & Salisbury, 2006, being the first to analyze the valuation of guaranteed minimum withdrawal benefits, and with Bauer et al., 2008, as well as Bacinello et al., 2011, providing general frameworks for the valuation of variable annuities with all types of guarantees. Regarding the risk management of variable annuities, Cathcart et al., 2015 provide schemes to efficiently calculate the "Greeks" of a variable annuity liability via Monte Carlo simulation, while Kling et al., 2011, show that it is important to include stochastic volatility in the modeling when assessing the risk inherent in variable annuity contracts. The same authors analyze the impact of policyholder behavior on the valuation and risk assessment of GLWB (cf. Kling et al., 2014). Forsyth & Vetzal, 2014, present an optimal stochastic control framework, in which they analyze the sensitivity of the cost of hedging a variable annuity with

GLWB to various economic and contractual assumptions. The hedging costs for variable annuities with combined guaranteed lifelong withdrawal and death benefits (GLWDB) is analyzed in Azimzadeh et al., 2014, in which the authors also argue that, when analyzing dynamic policyholder behavior from an insurer's perspective, it is better to use the term "loss-maximizing strategy" instead of "optimal strategy".

The paper is organized as follows. In Section 2, we describe the model framework we use for our analysis, including the product design of the considered GLWB (Section 2.1), the modeling of the pool of policies (Section 2.2), the used market models (Section 2.3) and the modeling of the stylized hedging programs implemented by the insurer (Section 2.4). Our numerical results are presented in Section 3, where we first present the parameters and assumptions used in the numerical modeling of the pool of policies in Section 3.1 and present the results of our analysis regarding the risk profile of the modeled pool of policies and its sensitivity to different financial risk factors in Section 3.2. The results of our simulation study are presented in Sections 3.3 and 3.4, respectively, where we analyze the effectiveness of the stylized hedging programs and, based on the distribution of the provider's resulting P&L, compute different indicators for risk-based capital requirements. Finally, Section 4 concludes.

2 MODEL FRAMEWORK

In this section, we describe the model used in our analysis of the risk profile of a pool of variable annuities with GLWB riders. Mainly, we follow the modeling approach used in Bacinello et al., 2011, and Bauer et al., 2008. We consider only two product designs of the GLWB rider: a GLWB rider with a ratchet mechanism, i.e. a product design where increases in the guaranteed withdrawal amount are possible and are regularly checked for, as well as a GLWB rider without ratchet mechanism. We use the product design without ratchet in order to illustrate and extract the effect the considered ratchet mechanism of the first product has on the results. We refer the interest reader to Kling et al., 2011, and Kling et al., 2014, for a broader analysis of different designs of the GLWB rider.

Please note that, although we set surrender rates to zero in our computations in Section 3, we still give formulas for surrender in this section – for the sake of model completeness and in order to set the basis for potential subsequent analyses, where surrender might be considered. The reason we do not consider surrender in our computations is that we want to purely focus on the effect financial risk drivers have on risk-based capital requirements. Depending on the approach used to model dynamic policyholder behavior, results may show effects that are not easily interpretable and that are highly dependent on the specific parametrization of the used behavior model (for instance a model where surrender rates are modeled as a function of the "moneyness" of the contract). To consider this in our analyses would be out of the scope of this paper. However, we refer the interested reader to Forsyth & Vetzal, 2014, and Kling et al., 2014, for an analysis of surrender risk, especially due to dynamic and "loss-maximizing" policyholder behavior.

2.1 VARIABLE ANNUITY CONTRACT

We consider a sequence of withdrawal (or "calculation") dates, represented by the points in time $(t_i)_{i=0}^N$, where $t_i \in \mathbb{R}_{\geq 0}$ and $t_0 = 0$ represents the inception of the contract. While the number of such calculation dates is not limited by means of the contract, it is limited by (the assumption of) a limiting age, i.e. an age after which survival is deemed impossible.

Let $\mathcal{T} := \{t_i | \, i = 0, \dots, N\}$ denote the set of all calculation dates. On a calculation date after the inception of the contract, a potential ratchet mechanism is applied and the policyholder is allowed to withdraw money up to a specified amount (the guaranteed withdrawal benefit) from the account without affecting the guarantee of the contract. If the withdrawn amount exceeds the guaranteed withdrawal benefit, this is interpreted as partial surrender, which usually reduces the guarantee. Also, on a calculation date of the contract, a potential surrender of the contract is settled (by payout of the surrender benefit) and, in case the insured person has died, a death benefit is paid out to the beneficiaries.

The value of the policyholder's account at time t is denoted by F_t, where F_0 represents the initial investment amount after deduction of acquisition and other upfront charges, i.e. F_0 is the amount that is actually invested in the fund at inception of the contract. The amount of upfront charges (such as acquisition charges) is not relevant in our analysis and therefore not explicitly denoted in the formulas.

For any calculation date t_{i-1}, the account value F_{t_i} at the following calculation date t_i (while the contract is still in force) is calculated as follows

$$F_{t_i} := \max\left(0, \left(F_{t_{i-1}} - B_{t_{i-1}}^W\right) \cdot \frac{S_{t_i}}{S_{t_{i-1}}} \cdot e^{-(\eta^{mc} + \eta^g) \cdot (t_i - t_{i-1})}\right),$$

where

- $B_{t_{i-1}}^W$ denotes the withdrawal amount paid out to the policyholder at time t_{i-1},
- $S_{t_{i-1}}$ and S_{t_i} denote the share prices of the variable annuity's underlying fund at time t_{i-1} or t_i, respectively,
- η^{mc} denotes the continuously deducted management charge as a percentage of the account value, and
- η^g denotes the continuously deducted guarantee charge as a percentage of the account value.

We denote the guaranteed withdrawal benefit at time t by $B_t^{W,g}$ and, in the case of the product design with ratchet, define it as

$$B_t^{W,g} := \begin{cases} \omega \cdot wbb_t, & \text{if } t \in \mathcal{T}, t > 0 \\ 0, & \text{else} \end{cases},$$

where

- ω is the guaranteed withdrawal rate stipulated at inception, and
- wbb_t denotes the "withdrawal benefit base", which, at a calculation date $t \in \mathcal{T}, t > 0$, is defined as

$$wbb_t := \max_{s \in T, s \le t, F_s > 0}\left\{F_s + \sum_{u \in \mathcal{T}, u < s} B_u^W\right\}.$$

The withdrawal benefit base wbb_t implements the ratchet mechanism of the GLWB rider, i.e. the potential for increases of the guaranteed withdrawal amount if the fund performs well.

In words, the considered ratchet mechanism works as follows: At each calculation date where the account value is still positive, the then-current account value and the sum of withdrawals are added; The withdrawal benefit base then is defined as the highest of these values in the time span between inception and the current calculation date. Thus, the withdrawal benefit base is non-decreasing over time and can

only increase as long as the account value is positive – after the account value has fallen to zero no more increases of wbb_t are possible, i.e. the guaranteed withdrawal amount remains the same until the insured person's death. As a consequence, the guaranteed withdrawal amount $B_t^{W,g}$ can only increase and never decrease (not considering the effects of partial surrenders on the guarantee).

Note: an alternative product design to using ongoing guarantee charges that are proportional to the account value (i.e. η^g) are ongoing charges that are proportional to the withdrawal benefit base (e.g. 1% of the wbb_t, annually), but are still deducted from the account value. This way, the guarantee charges received by the insurer are non-decreasing (until the guarantee triggers) and only increase when a ratchet occurs.

With regard to the fund performance needed to trigger an increase of wbb_t, in this product design, the fund performance only has to compensate for the charges of the variable annuity contract, but not for the withdrawals itself (as long as they remain within the limit of the guaranteed amount $B_t^{W,g}$, which we assume to be the case in our analysis).

In our analysis of the characteristics of a pool of GLWB policies, we also consider an alternative product design without a ratchet, in order to illustrate the effect of the considered ratchet mechanism. In this case, the withdrawal benefit base remains constant over the lifespan of the contract and takes the value of the initial investment:

$$\widehat{wbb}_t := F_0, \qquad t \in \mathcal{T}, t > 0.$$

All other formulas remain the same for this alternative product design without ratchet.

With both product designs, we assume the policyholder to always withdraw exactly the guaranteed withdrawal benefit, i.e. the highest amount that can be withdrawn without negatively affecting the guarantee (in typical product designs the guarantee is reduced proportionally if the withdrawal amount exceeds the guaranteed withdrawal benefit). That is, we set $B_t^W = B_t^{W,g}$ for all $t \in \mathcal{T}, t > 0$ in our analysis.

The considered variable annuity contract (irrespective of the ratchet mechanism) does not include a guaranteed minimum death benefit, but rather pays to the beneficiaries the remaining account value at the calculation date following the insured's death. Therefore, the death benefit B_t^D at time $t \in \mathcal{T}, t > 0$ is defined as

$$B_t^D := F_t.$$

If the policyholder decides to surrender (i.e. "cash out") the contract, surrender charges apply. The portion of the account value corresponding to the guaranteed withdrawal amount at that calculation date is not subject to surrender charges. The amount that exceeds the guaranteed withdrawal benefit, however, is reduced by a proportional surrender charge η^S. Therefore, the surrender benefit at time $t \in \mathcal{T}, t > 0$, denoted by B_t^S, is given by

$$B_t^S := F_t - \max\big(0, \big(F_t - B_t^{W,g}\big) \cdot \eta^S\big).$$

2.2 POOL OF POLICIES

For our analyses on a portfolio level, we assume a pool of policies with identical contract parameters with regard to inception date, underlying fund, age as well as gender of insured, guarantee, charges, etc.

We assume the pool of insured to be homogeneous and large enough to apply the law of large numbers such that mortality henceforth is only expressed as percentage of the pool of insured.

We denote the total number of active contracts within the pool at time t by π_t^A.

Let $q_i := q(t_{i-1}, t_i)$ represent the percentage of the insured who are alive at time t_{i-1} and die within the time interval $]t_{i-1}, t_i]$. The total number of contracts where the insured person dies within the time interval $]t_{i-1}, t_i]$, denoted by $\pi_{t_i}^D$, then is given by

$$\pi_{t_i}^D = \pi_{t_{i-1}}^A \cdot q_i.$$

Similarly, let $s_i := s(t_{i-1}, t_i)$ represent the fraction of policyholders who want to surrender their contracts at the end of the time interval $]t_{i-1}, t_i]$. The total number of policyholders that surrender their contracts at time t_i, denoted by $\pi_{t_i}^S$, then is given by

$$\pi_{t_i}^S = \left(\pi_{t_{i-1}}^A - \pi_{t_i}^D\right) \cdot s_i.$$

Immediately after the calculation date t_i, i.e. after contracts that were surrendered or that ended due to the death of the insured have left the pool of policies, the number of active contracts is given by

$$\pi_{t_i}^A = \pi_{t_{i-1}}^A - \pi_{t_i}^D - \pi_{t_i}^S.$$

Cash flow

From the viewpoint of the insurer, the cash flows from and to the pool of policies with regard to the guarantee (the GLWB rider) are as follows.

In order to define the guarantee charges received by the insurer at a calculation date t_i, we introduce an auxiliary variable that defines the hypothetical account value after fund performance, but without deduction of charges, denoted by $\hat{F}_{t_i}$ and given by

$$\hat{F}_{t_i} := \max\left(0, \left(F_{t_{i-1}} - B_{t_{i-1}}^W\right) \cdot \frac{S_{t_i}}{S_{t_{i-1}}}\right).$$

Surrender charges are treated as a "contribution" to financing the guarantee, and therefore are counted towards the guarantee charges received by the insurer. At a calculation date t_i, the guarantee charges $G_{t_i}^C$ received by the insurer from the pool of policies are then given by

$$G_{t_i}^C = \pi_{t_i}^S \cdot \left(F_{t_i} - B_{t_i}^S\right) + \pi_{t_{i-1}}^A \cdot \left(\frac{\eta^g}{\eta^{mc} + \eta^g} \cdot \left(\hat{F}_{t_i} - F_{t_i}\right)\right).$$

In return, the insurer has to continue the guaranteed withdrawal payments for the lifetime of the insured person after the account value has fallen to zero (i.e. the guarantee has triggered). At a calculation date t_i, the guarantee payment $G_{t_i}^P$ to be made by the insurer with regard to the pool of policies is given by

$$G_{t_i}^P = \pi_{t_i}^A \cdot \max\left(0, B_{t_i}^{W,g} - F_{t_i}\right).$$

2.3 MARKET MODEL, VALUATION AND REAL-WORLD PROJECTION

Market model

For our simulation, we need to project the price dynamics of the following assets: the price of one share of the variable annuity's underlying fund, denoted by S_t, the price of a risk-free (with regard to default) zero-coupon bond with time to maturity of τ, denoted by $Z_t(\tau)$, the price of the cash account, denoted by C_t, and the prices of simple put options on the underlying fund, denoted by $O_t^P(\tau, K)$, where τ is the time to maturity and K the strike level of the respective option.

For the valuation of these assets, we use the same system of stochastic differential equations as in Cathcart et al., 2015, in which Heston's stochastic volatility model (Heston, 1993) is used for the equity process and combined with stochastic interest rates via the Cox-Ingersoll-Ross model ("CIR", Cox et al., 1985). Therefore, the dynamics of the market's state variables under the risk-neutral measure Q are given by

$$
\begin{aligned}
dV_t &= \kappa_V^Q\big(\theta_V^Q - V_t\big)dt + \sigma_V^Q\sqrt{V_t}\,dW_t^{Q,V}, \\
dr_t &= \kappa_r^Q\big(\theta_r^Q - r_t\big)dt + \sigma_r^Q\sqrt{r_t}\,dW_t^{Q,r}, \\
dS_t &= r_t S_t\,dt + \sqrt{V_t}\,S_t\,dW_t^{Q,S} \\
dC_t &= r_t C_t\,dt
\end{aligned}
$$

where $W_t^{Q,V}, W_t^{Q,r}$ and $W_t^{Q,S}$ are Wiener processes under the risk-neutral measure Q.

We assume the Wiener process $W_t^{Q,r}$ of the interest rate process to be independent of the two equity processes. The correlation between $W_t^{Q,V}$ and $W_t^{Q,S}$ is denoted by the correlation factor $\rho_{S,V}$.

The prices of the bond and the equity option are then given by the following expectations

$$
\begin{aligned}
Z_t(\tau) &= \mathbb{E}_Q\left[\frac{C_t}{C_{t+\tau}}\bigg| t\right], \\
O_t^P(\tau, K) &= \mathbb{E}_Q\left[\frac{C_t}{C_{t+\tau}}\max(0, K - S_{t+\tau})\bigg| t\right].
\end{aligned}
$$

Within our simulation, the bond price $Z_t(\tau)$ is calculated via the formulas given in Cox et al., 1985. For the Heston stochastic volatility model, Heston, 1993, found a semi-analytical solution for pricing European call and put options using Fourier inversion techniques. In our analyses, we use the numerical scheme proposed in Kahl & Jäckel, 2005, with the approximation that the interest rates are assumed to be deterministic when pricing the put option used for hedging.

Valuation

In order to determine the value of liabilities, we assume a market-consistent valuation of the guarantee within the considered pool of policies. We define this value of liabilities at time t, denoted by V_t^π, as the difference between the expected present value of future guarantee payments made by the insurer and the expected present value of future guarantee charges to be received by the insurer, i.e.

$$
V_t^\pi := \mathbb{E}_Q\left[\sum_{s \in \mathcal{T}, s > t} \frac{C_t}{C_s}(G_s^P - G_s^C)\bigg| t\right].
$$

In our simulation, this expectation is approximated via Monte Carlo estimation, using the QE scheme developed by Andersen, 2008. See Chan & Joshi, 2013, for a further development of this scheme, allowing for long-stepping intervals. However, as we consider weekly rebalancing of the hedge portfolio, this would be not beneficial for our simulation.

Real-world projection of the market

In our simulation, we project the state of the market over time under the real-world measure P. The state of the market at time t is given by the set of state variables (V_t, r_t, S_t, C_t). For this set of state variables, we assume the dynamics under the real-world measure P to be

$$dV_t = \kappa_V^P(\theta_V^P - V_t)dt + \sigma_V^P\sqrt{V_t}dW_t^{P,V},$$
$$dr_t = \kappa_r^P(\theta_r^P - r_t)dt + \sigma_r^P\sqrt{r_t}dW_t^{P,r},$$
$$dS_t = (r_t + \mu)S_tdt + \sqrt{V_t}S_tdW_t^{P,S}$$
$$dC_t = r_tC_tdt$$

where $W_t^{P,V}, W_t^{P,r}$ and $W_t^{P,S}$ are Wiener processes under P.

The correlation structure between the Wiener processes is assumed to be identical under both measures.

2.4 HEDGING

We assume the insurer to have implemented a dynamic hedging program in order to mitigate the financial risk resulting from the GLWB rider within the pool of policies. The hedge portfolio is assumed to consist of four instruments that are regularly reallocated: units of the underlying fund, put options on the underlying fund, zero-coupon paying bonds, and a cash position. Each position may be long (positive weight) or short (negative weight). The respective weights are determined according to the Greeks of the pool of policies, i.e. derivatives of V_t^π with regard to different variables.

Note that this means we ignore basis risk, i.e. the variable annuity's underlying fund and the equity indices used within the hedging program are assumed to be the same or to be at least perfectly correlated.

At each hedging (or rebalancing) date, existing bonds in the portfolio are sold and replaced with zero-coupon paying bonds that have a time to maturity of d_Z. Similarly, put options in the portfolio are sold and replaced with "new" put options that have a strike price of 100% of the then-current spot price (i.e. $K_t = S_t$) and a time to maturity of d_O.

The hedge portfolio's value at time t is denoted by Ψ_t. The inflows into the hedge portfolio are given by the guarantee charges paid by the pool of policies, G_t^C, and its outflows by the guarantee payments G_t^P. We assume the hedge portfolio's value to be zero at inception, i.e. we set $\Psi_0 = 0$.

The weights of the hedge portfolio are calculated as follows, where λ_t^S denotes the number of fund shares, λ_t^Z the number of zero-coupon bonds, λ_t^O the number of options and λ_t^C the amount that is invested in the cash account:

$$\lambda_t^O = \frac{\frac{\partial V_t^\pi}{\partial V_t}}{\frac{\partial O_t^P(d_O, K_t)}{\partial V_t}},$$

$$\lambda_t^S = \frac{\partial V_t^\pi}{\partial S_t} - \lambda_t^O \cdot \frac{\partial O_t^P(d_O, K_t)}{\partial S_t},$$

$$\lambda_t^Z = \frac{\frac{\partial V_t^\pi}{\partial r_t}}{\frac{\partial Z_t(d_Z)}{\partial r_t}} - \lambda_t^O \cdot \frac{\frac{\partial O_t^P(d_O, K_t)}{\partial r_t}}{\frac{\partial Z_t(d_Z)}{\partial r_t}},$$

$$\lambda_t^C = \Psi_t - \left(\lambda_t^S \cdot S_t + \lambda_t^Z \cdot Z_t(d_Z) + \lambda_t^O \cdot O_t^P(d_Z, K_t) \right).$$

While it is used for controlling (implied) volatility risk, the option also gives exposure to the other two considered risk factors, the fund price S_t and the short rate r_t. Therefore, with the amount of options in the hedge portfolio, λ_t^O, being calculated such that the current sensitivity of V_t^π to volatility is matched, λ_t^S and λ_t^Z have to be calculated considering the exposure already attained via the options in order for the hedge portfolio to match the current sensitivity of V_t^π to S_t and to r_t, respectively.

The derivatives of V_t^π are approximated via the "bump and revalue" approach (cf. for instance Cathcart et al., 2015), where we use the central finite difference with regard to the fund price S_t, and the forward and backward finite difference for V_t and r_t, respectively. The same approach is used for the derivatives of the option value.

3 NUMERICAL RESULTS

In order to understand the risk profile of a pool of GLWB policies, it helps to analyze the inherent guarantees with regard to their sensitivity to several influencing factors, such as valuation assumptions, market movements, policyholder behavior and demographic risks like longevity. In our analyses, we focus on the purely financial risk drivers equity returns, equity (implied) volatility and interest rates. First, we present the parameters and assumptions used in the subsequent analyses in Section 3.1. In Section 3.2, we present the results of an analysis of the "Greeks" of V_t^π, i.e. the sensitivity of the value of liabilities to the considered financial risk drivers. In Section 3.3, we analyze the risk profile of the pool of policies by means of a simulation study and compute indicators for corresponding capital requirements with regard to the inherent market risk. How this risk profile and, thereby, capital requirements change with variations of interest and volatility levels is analyzed in Section 3.4.

3.1 PARAMETERS AND ASSUMPTIONS

For the remainder of this section, we use the following parameters and assumptions, unless stated otherwise.

Market parameters

Table 1 gives the values of the parameters used for the market model – for valuation purposes under the risk-neutral measure Q as well as the parameters used for the real-world projection (under the measure P) of our risk analysis. The parameters under Q are those used by Bacinello et al., 2011, which we also

use for the parameters of the real-world projection – this implies the assumption of a market that is risk-neutral with regard to interest-rate risk as well as equity volatility risk. The parameter μ for the equity process under P is chosen similar to Kling et al., 2014.

Parameter	Value
$r_0, \theta_r^P, \theta_r^Q$	0.03
κ_r^P, κ_r^Q	0.60
σ_r^P, σ_r^Q	0.03
$V_0, \theta_V^P, \theta_V^Q$	$(0.2)^2$
κ_V^P, κ_V^Q	1.50
σ_V^P, σ_V^Q	0.40
$\rho_{S,V}$	-0.70
μ	0.03

Table 1: Values of the market and simulation parameters used in the base-case simulation.

Contract parameters

The assumptions regarding the parameters of the variable annuity contract are stated in Table 2. We use annual calculation dates in our analysis, i.e. the set $\mathcal{T}$ represents anniversaries of the contract's inception date.

Parameter	Value
η^{mc}	1.50%
η^g	1.00%
ω	3.25%

Table 2: Values of the product parameters used in the simulation.

Compare the results shown in Figure 3 in the following section for the relationship between the value of future guarantee charges (influenced by η^g) and the value of future guarantee payments (influenced by ω). As we do not consider surrender in our analysis, the value of η^s is not relevant and therefore not given here. Also, neither the initial investment amount of a single contract, F_0, nor the number of active contracts at $t = 0$, π_0^A, is relevant, as we state all results as percentages of the pool's total initial investment amount ($\pi_0^A \cdot F_0$).

Policyholder behavior and mortality parameters

As stated in the beginning of Section 2, we assume that no lapses occur in the pool of policies – neither for the purpose of valuation of the liabilities nor for the projection of the pool of policies. That is, we set $s_i = s\,(t_{i-1}, t_i) = 0$ for all $i = 1, \ldots, N$.

We also assume the policyholder to withdraw exactly the guaranteed withdrawal amount at each withdrawal date, i.e. we assume there is neither partial surrender nor any kind of deferral or similar delays in withdrawals (withdrawals begin immediately at the first withdrawal date t_1, i.e. one year after inception).

Regarding mortality, we use the best-estimate probabilities for both valuation and the projection of the pool of policies. The persons insured within the pool of policies are assumed to be male and aged 65 at the contract's inception date. We use the best-estimate probabilities for annuities given in the DAV 2004R table published by the German Actuarial Association (DAV). The mortality table has a limiting age (i.e. the age after which survival is assumed to be impossible) of 121, which leads to a maximum

projection horizon of 57 years $(122 - 65)$, which also gives the value of N, the last index of the calculation dates.

Hedging parameters

As to the modeling of the hedge instruments, we use $d_O = \frac{3}{12}$, i.e. 3 months for the option's term to maturity, and $d_Z = 10$, i.e. 10 years for the bond's duration. As stated in Section 2.4, the options used in the hedging program have a strike set at the spot price of the underlying fund at the time of buying them, i.e. they are bought "at the money".

In our simulation study, we consider rebalancing of the hedge portfolio (i.e. a hedge frequency) on a monthly and a weekly basis, as well as no hedging at all. The latter gives an indication for the capital requirements if the hedging program is not considered at all.

3.2 ANALYSIS OF THE "GREEKS"

In this first analysis of the risk profile of a pool of variable annuities with GLWB riders, we consider an exemplary pool of homogeneous policies and analyze the Greeks of the pool's value of liabilities at $t = 0$, i.e. V_0^π. We also analyze how these Greeks at inception change if an instantaneous shock is applied to the variable annuity's account value, i.e. if the account value changes a theoretical second after inception. This helps to get a better understanding of the pool's risk profile, as it illustrates how the sensitivity of the pool's value of liabilities changes with market movements. This sensitivity to (market) risk factors directly affects risk measures and, therefore, risk-based capital requirements.

In order to illustrate the impact of the ratchet mechanism that is included in the modeled GLWB rider, we also show results for a stylized GLWB rider without a ratchet mechanism, i.e. a GLWB rider whose guaranteed withdrawal amount is constant and cannot increase after inception. For this GLWB rider without ratchet mechanism, we use the same parameters as given in Table 2, i.e., with the exception of the ratchet mechanism, both products are identical. Note also that this means that, in comparison to the product with ratchet, the product without ratchet is "overpriced".

For all computations in this section, we used 100,000 Monte-Carlo paths under the risk-neutral measure Q. In Figure 1, we show the expected trigger time of the guarantee (i.e. the contract year in which the account value hits zero) as a function of the account value immediately after inception. In other words, we analyze the impact an instantaneous shock to the variable annuity's underlying fund has on the point in time when the insurer has to start guaranteed payments. Note that the analyzed expected trigger time is an expected value under the risk-neutral measure Q, which is why the values are not to be taken as a best estimate, but rather as an explanatory aid to understand the characteristics of the value of liabilities and its corresponding risk profile.
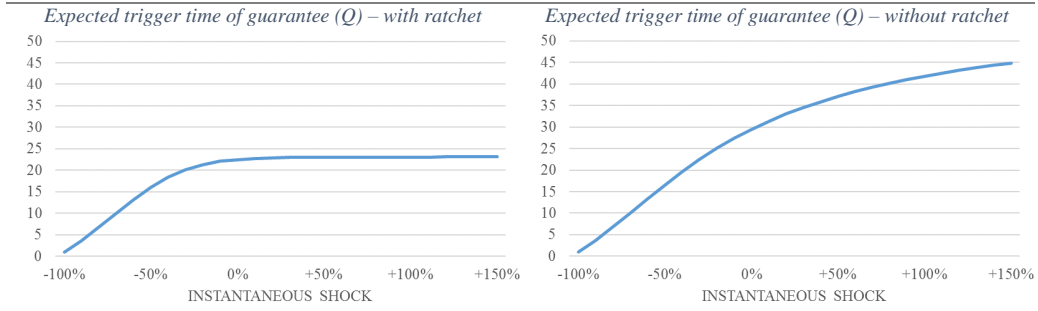
Figure 1: Expected trigger time of the guarantee (time of ruin) in years after inception, analyzed in the context of the valuation under Q for different levels of the instantaneous shock applied to the fund value. With a ratchet mechanism as in the normal design of the GLWB (left-hand side) and, as variation, without ratchet (right-hand side).

In the calculation of the expected trigger time, the trigger time of the guarantee is taken into account as $57 \, (122 - 65)$ if the guarantee does not trigger before the insured person reaches the limiting age of the mortality table. For the variable annuity without ratchet (right-hand side in Figure 1), the higher the account value the later the guarantee triggers, i.e. the later the account value hits zero. For the design with ratchet (left-hand side in Figure 1), a higher account value also leads to higher guaranteed withdrawal benefits (due to the ratchet mechanism), which counteracts the prolonging effect and leads to a "top out" of the expected trigger time at around 23 years.

In Figure 2, we show similar plots of the value of liabilities and its two components, the value of future guarantee payments (to be made by the insurer) and the value of future guarantee charges (to be received by the insurer). We also show plots of the derivatives of these values: the derivative with regard to the underlying fund's share price (the Delta) multiplied by the share price (i.e. the "Dollar-Delta" or "EUR-Delta" of V_0^{π}), the derivative with regard to the short rate (similar to Rho in the Black-Scholes-Merton model) and the derivative with regard to the instantaneous volatility (similar to Vega in the Black-Scholes-Merton model). All values are stated as a fraction of the initial investment amount, i.e. as a percentage of $\pi_0^A \cdot F_0$.
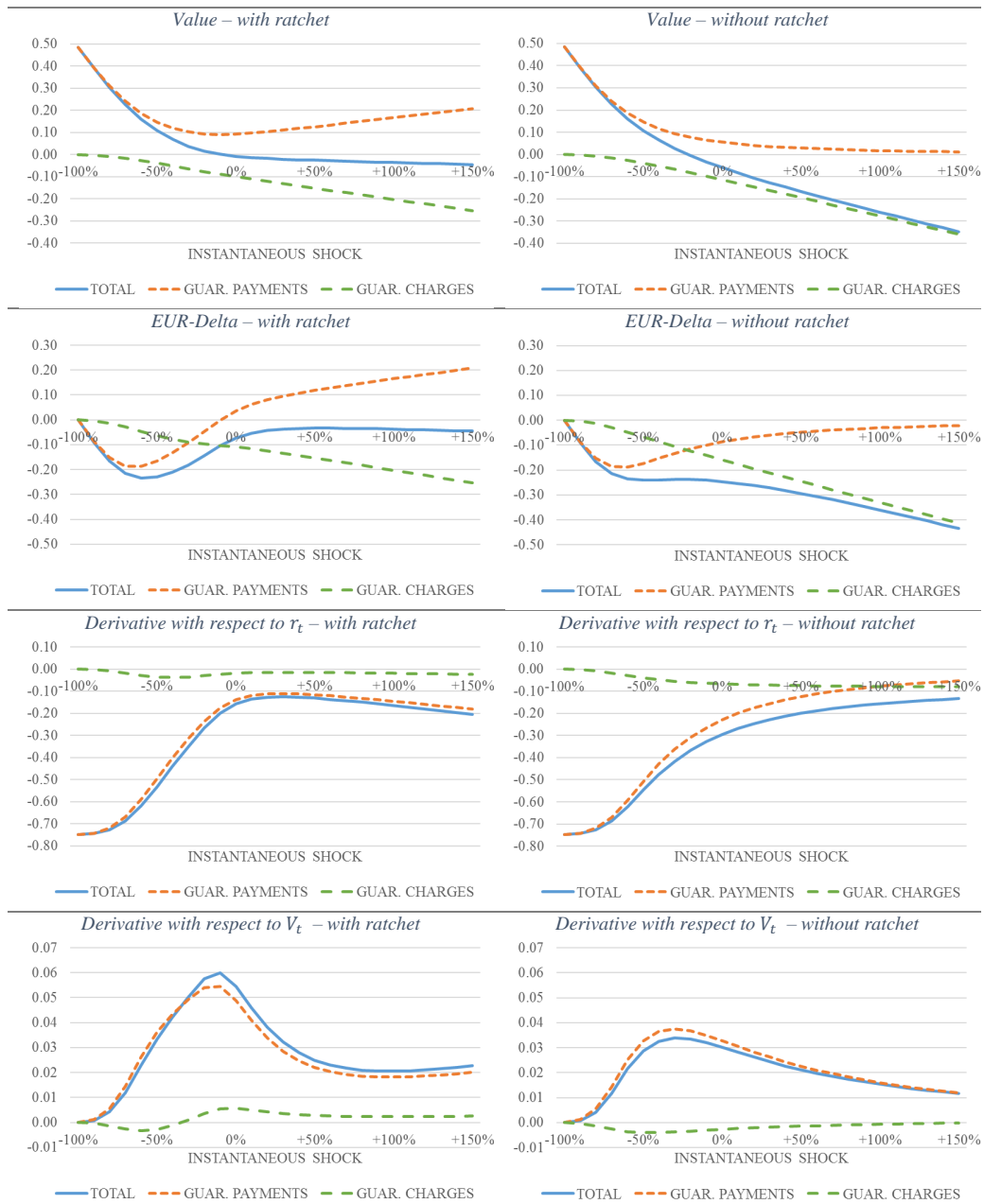
Figure 2: Value and "Greeks" at $t = 0$ as a function of the instantaneous shock applied to the account value. Charts on the left-hand side show the results for a GLWB design with a ratchet mechanism and charts on the right-hand side results for a GLWB design without ratchet (for comparison purposes). Values are stated as a fraction of the initial investment amount.

Note that Figure 2 shows values as liabilities, i.e. future guarantee charges have a negative value, as they represent future earnings of the insurer. Intuitively, if the account loses value, the value of liabilities increases. When the account value increases, however, it is not intuitively clear what happens, as now the GLWB's ratchet comes into play. Without a ratchet mechanism, the value of liabilities would strictly decrease, as future guarantee payments become less likely and future guarantee charges become more valuable. With a ratchet in place, however, positive fund returns can potentially lead to increases in the

value of future guarantee payments, due to an increase in the guaranteed withdrawal amount. Which of these two effects dominates the value of liabilities depends on several factors, including the pricing and the design of the GLWB rider, as well as other factors relevant for the valuation, e.g. interest rates and volatility. In the considered case, the value of future guarantee charges increases faster than the value of future guarantee payments. As a result, the total value of liabilities slightly decreases if the fund value increases.

For the GLWB rider without ratchet, the EUR-Delta of the guarantee payments is similar to the EUR-Delta of a simple put option. For the GLWB rider with a ratchet, however, the EUR-Delta of the guarantee payments becomes positive if the account value increases, reflecting the increase in value caused by the ratchet mechanism. In comparison, the ratchet also causes the value of future guarantee charges to increase (from the insurer's perspective) less with higher account values, because over time, higher account values are also reduced by higher future withdrawal amounts. Therefore, the EUR-Delta of future guarantee charges is less pronounced for the rider with ratchet. Also, for the product with ratchet, the EUR-Delta of future guarantee charges more or less counter-balances the EUR-Delta of future guarantee payments for higher account values, resulting in a total EUR-Delta of near zero in good scenarios (as opposed to the product without ratchet, where the total EUR-Delta in good scenarios is dominated by the sensitivity of the value of future guarantee charges). The increase in value caused by the ratchet mechanism is also visible in the sensitivity to the short rate and the instantaneous volatility: in both cases, there is a slight increase in sensitivity at the positive end of the considered fund shocks.

For both riders, the sensitivity to the short rate sharply increases with increasing "moneyness" of the guarantee, i.e. with declining account values. This is also similar to the pattern of the Rho of a simple put option. An adverse market movement therefore causes a simultaneous increase in the exposure to interest rate risk – and therefore likely an increase in corresponding risk-based capital requirements. However, in the base position with an unchanged account value, the sensitivity to the short rate seems rather low. This may also be due to the choice of parameters for the interest-rate model, where lower values for the speed of mean-reversion or higher interest-rate volatility could lead to significantly higher sensitivities to the short rate.

The sensitivity of the value of liabilities to the instantaneous volatility shows the typical pattern of a bump around the "strike" of the guarantee. The sensitivity to the instantaneous volatility is similar for both considered riders if the account loses more than 50% (i.e. for shocks between -100% and -50%) but changes as soon as the likelihood for the ratchet having an effect increases, with the design with ratchet showing a much more pronounced sensitivity to volatility. This is in line with the findings in Kling et al., 2011, in which the impact of stochastic volatility on different product designs is analyzed. Also, for the rider with ratchet, the value of future guarantee charges shows a change of sign, causing an overall even higher sensitivity to volatility of the value of liabilities for moderate to positive shocks of the account value.

Figure 3 shows a similar analysis, but now, instead of the account value, the GLWB's guaranteed withdrawal rate ω is changed.
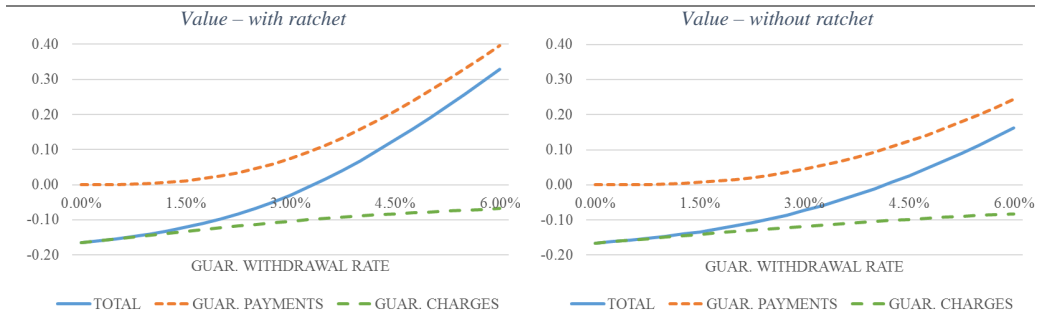
Figure 3: Value (as fraction of the initial investment amount) at $t = 0$ for different levels of the guaranteed withdrawal rate ω.With a ratchet mechanism as in the normal design of the GLWB (left-hand side) and as a variation without ratchet (right-hand side).

Finding the level of ω for which the total value is (close to) zero is typically part of the pricing process. For instance, at a guaranteed withdrawal rate of 3.25%, the total value of liabilities (from the insurer's perspective) amounts to -0.8% of the initial investment for the GLWB with ratchet, where the value of future guarantee payments of 9.2% is (more than) compensated by the value of future guarantee charges of -10.0%. This guaranteed withdrawal rate, $\omega = 3.25\%$, is the pricing assumption used in the following analyses of the risk profile (cf. Table 2).

Of course, the value of the guaranteed payments is zero if ω is zero. For positive values of ω, the impact of the ratchet is clearly visible: the value of future guarantee payments increases much faster if the GLWB has a ratchet mechanism. This results from the withdrawal rate affecting two components at once: the minimum guaranteed withdrawal amount at inception and the amplitude of potential future increases of the guaranteed withdrawal amount due to the ratchet. Note that with the considered ratchet design (cf. Section 2.1), the fund performance only has to compensate for charges, not for withdrawals (within the guaranteed limit), and, thus, the impact of higher withdrawal rates on the potential for future ratchets is reduced.

3.3 ANALYSIS OF RISK PROFILE AND CAPITAL REQUIREMENTS

In this section, we analyze the distribution of the insurer's one-year P&L with regard to the modeled pool of GLWB policies. Using this distribution, we calculate risk metrics and, thereby, indicators for risk-based capital requirements, as, for instance, stipulated under Solvency II (99.5%-Value-at-Risk on a one-year basis) or the Swiss Solvency Test (99%-Tail-Value-at-Risk, also on a one-year basis). Using the results for the 99.5%-Value-at-Risk, our approach is comparable to a Solvency II internal model type approach (cf. e.g. Central Bank of Ireland, 2010).

We assume different levels of consideration of the insurer's hedging program, ranging from no credit at all (i.e. no allowance of the risk-mitigating effect of future hedging), over hedging with monthly rebalancing to weekly reallocations of the hedge portfolio. In reality, the "Future Trading Offset", i.e. the difference in capital requirements calculated without and with (full) consideration of the risk-mitigating effect of future hedging, is likely not allowed to be fully applied when calculating the capital requirements. Instead, it will likely only be partially considered, taking into account (amongst others) the effectiveness of the hedging program (considering e.g. basis risk) and how close to reality the modeling of the hedging program is (cf. e.g. Central Bank of Ireland, 2010).

We used 10,000 runs (paths) under the real-world measure P for each simulation in this section. Within each of these 10,000 real-world paths, we used 10,000 risk-neutral paths under the measure Q in order

to re-evaluate the liabilities at the end of the one-year projection as well as 1,000 risk-neutral paths for the calculation of the Greeks at each rebalancing date of the modeled hedging program. We only consider the product design with a ratchet mechanism in this analysis.

Distribution of the present value of liabilities

First, we have a look at the distribution of the (discounted) one-year change of the value of liabilities, stated as a fraction of the initial investment amount, i.e. the pool's assets under management at the beginning of the year. In formal terms, we look at the distribution of

$$\frac{\frac{C_0}{C_1} V_1^\pi - V_0^\pi}{\pi_0^A \cdot F_0}.$$

Figure 4 shows scatter plots in which this one-year change is plotted against the fund's one-year return, $\frac{S_1}{S_0} - 1$, the short rate at the end of the year (r_1), and the instantaneous volatility at the end of the year $\left(\sqrt{V_1}\right)$, respectively.
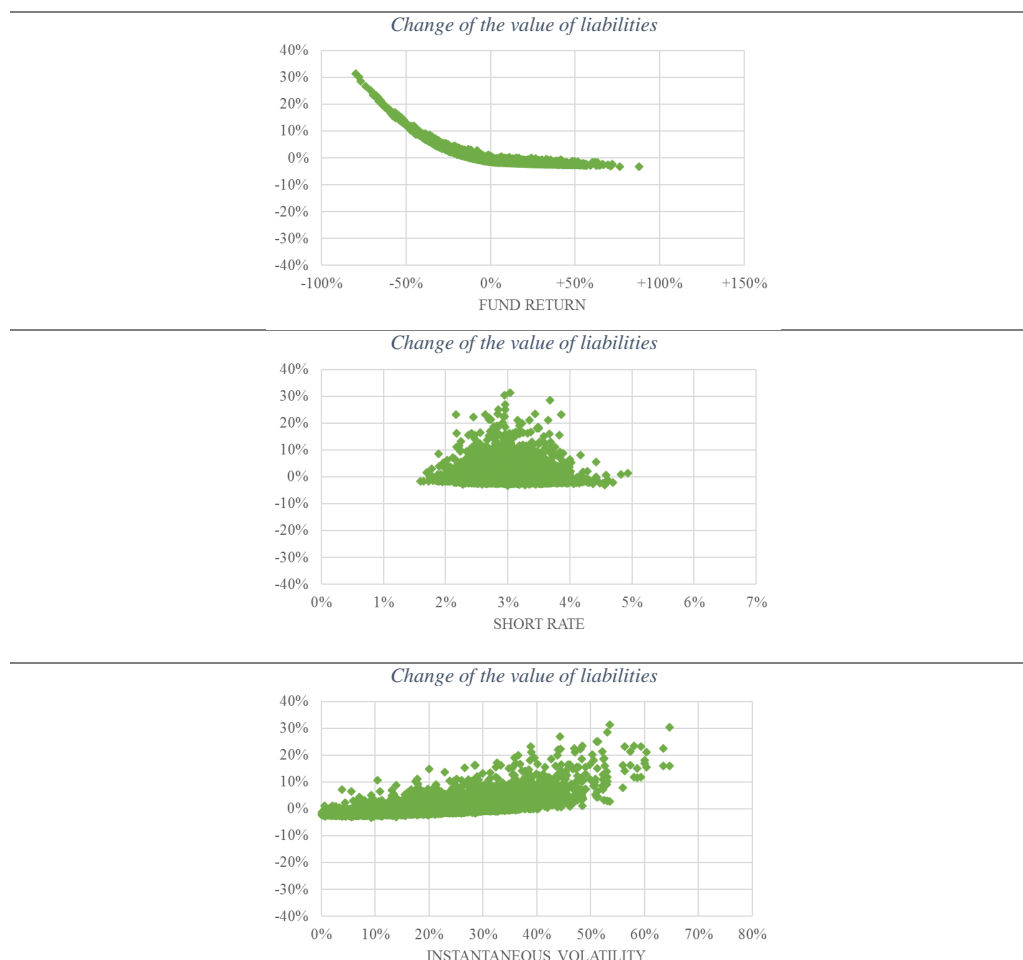
Figure 4: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the present value of liabilities (value of future guarantee payments minus future guarantee charges) as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation.

Of course, the value of liabilities is strongly affected by the underlying fund's one-year return, with a maximum increase of around 30% of the initial AuM in the worst path. The value is strictly decreasing with respect to positive fund returns and shows the same pattern as in Figure 2.

In comparison, the short rate at the end of the year, r_1, seems to have a negligible effect, as there is no clear influence of r_1 on the level of changes – the higher levels of changes around the mean-reversion level of 3% can be explained by the majority of the runs lying in that area. This is in line with the results in Section 3.2, where, at inception, the sensitivity of the value of liabilities to the short rate is rather low – as long as the fund value hasn't decreased significantly. Also, note that, while the negative correlation between equity returns and instantaneous volatility has a systematic cumulative effect on risk, the Wiener process of the short rate is independent of the other two and, thus, there is no similar cumulative effect with equity returns and the short rate.

Ceteris paribus, the higher the volatility, the higher the value of liabilities – the derivative with respect to the instantaneous volatility is non-negative, irrespective of the current account value (see Figure 2).

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This dependence is also visible in the scatter plot against the instantaneous volatility at the end of the year, i.e. against $\sqrt{V_1}$. Another contributing factor is the modeled negative correlation between the instantaneous volatility and equity returns, which increases both, the dependence between the value of liabilities and the instantaneous volatility at the end of the year as well as the dependence between the value of liabilities and the fund's one-year return.

Next, we analyze how well the modeled hedging programs are capable of replicating the observed changes in the value of liabilities.

Hedge performance

In order to assess the hedge performance, we look at the (discounted) P&L at the end of the projected year, stated as a percentage of the pool's assets under management at the beginning of the year. We denote this value by $\Pi_{0,1}$ and define it as follows:

$$\Pi_{0,1} := \frac{\frac{C_0}{C_1}(\Psi_1 - V_1^{\pi}) - (\Psi_0 - V_0^{\pi})}{\pi_0^A \cdot F_0}.$$

This means that $\Pi_{0,1}$ is the (discounted) net P&L at the end of the projection year, i.e. the (discounted) change of the difference in value of the hedge portfolio and the value of liabilities (as a percentage of the initial AuM).

We also look at the absolute discounted change of the hedge portfolio separately, i.e. we consider the value

$$\frac{\frac{C_0}{C_1}\Psi_1 - \Psi_0}{\pi_0^A \cdot F_0}.$$

We assume two different rebalancing frequencies of the hedging program: monthly and weekly. We also consider the resulting P&L if no hedging program is in place or the existing hedging program is not recognized for the purpose of calculating the risk-based capital requirements.

Weekly rebalancing

Figure 5 shows scatter plots of the one-year change in the value of the hedge portfolio with assumed weekly rebalancing of the hedge portfolio (left-hand side), as well as the scatter plots for the resulting P&L $\Pi_{0,1}$ (right-hand side).
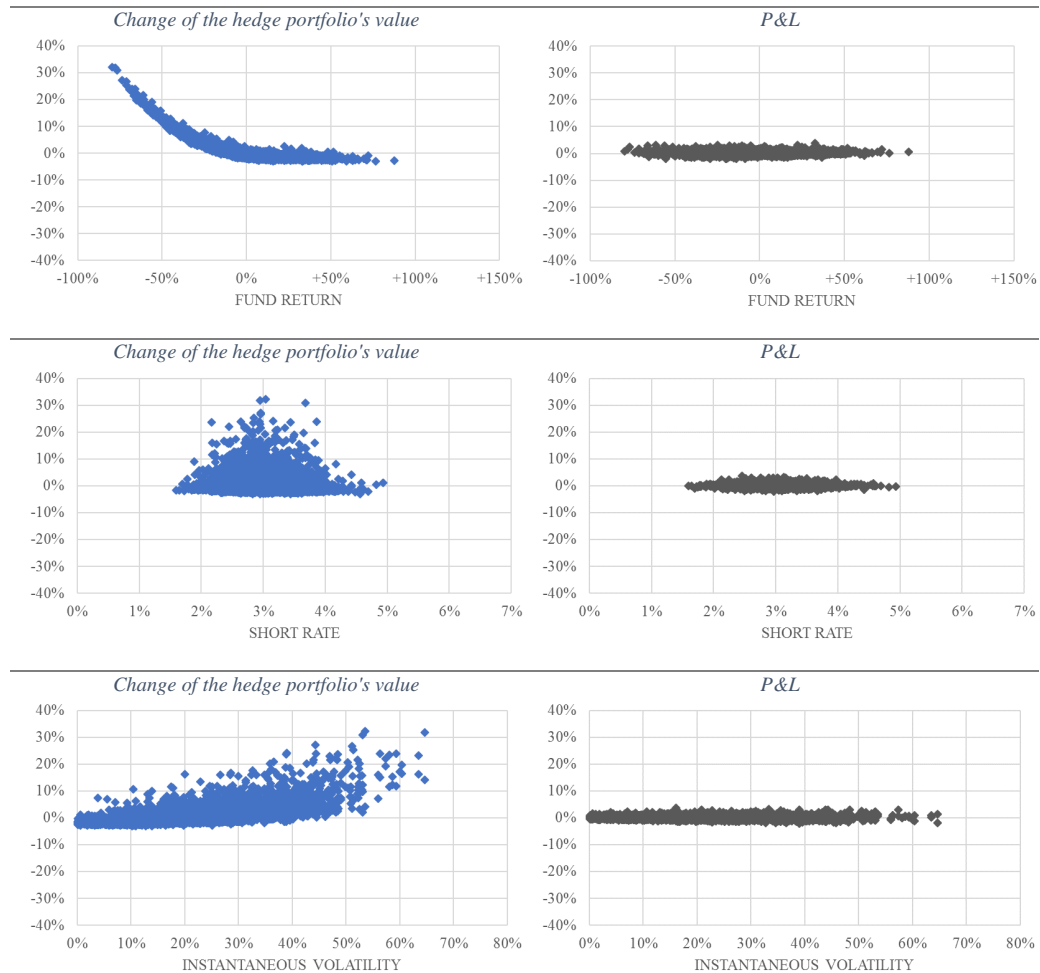

Figure 5: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the hedge portfolio's value over the one-year projection (left-hand side) and resulting discounted net P&L (right-hand side), both as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation with weekly rebalancing of the hedge portfolio.

The modeled hedging program seems to effectively offset the change of the value of liabilities, resulting in no visible dependency between the P&L and the fund's return. The resulting P&L also does not seem to be dependent on the short rate or the instantaneous volatility at the end of the year.

Monthly rebalancing

Figure 6 shows scatter plots of the one-year change in the value of the hedge portfolio with assumed monthly rebalancing (left-hand side), as well as the resulting P&L $\Pi_{0,1}$ (right-hand side).

Figure 6: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the hedge portfolio's value over the one-year projection (left-hand side) and resulting discounted net P&L (right-hand side), both as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation with monthly rebalancing of the hedge portfolio.

In comparison to the results for weekly rebalancing, the replication is less accurate, resulting in a broader distribution of the P&L. The overall impact of the considered risk factors still seems to be rather limited, but the higher variance of the P&L is noticeable when comparing the resulting risk measures (see next section).

No Hedge

We also analyze the resulting P&L if no hedging program is in place or an existing hedging program is not considered in the calculation of the capital requirements. In this case, the change of the hedge

portfolio's (discounted) value is zero in all paths and the net P&L equals the (discounted) negative of the change in the value of liabilities (cf. Figure 4).

Comparison of the P&L distribution and risk measures

In Figure 7, we have a look at the empirical (cumulative) density functions of $\Pi_{0,1}$ for the three different modeled hedging programs.
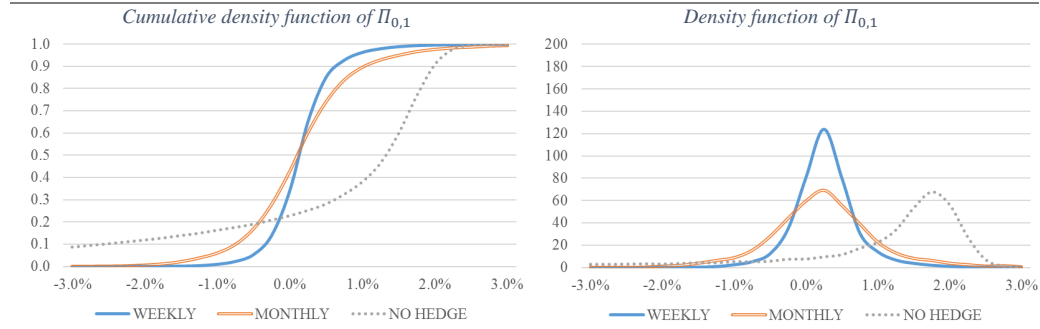

Figure 7: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the base-case simulation.

For the hedging programs with weekly and monthly rebalancing, most of the distribution's mass is located between -2% and +2%, with the monthly rebalancing showing a higher deviation than the weekly rebalancing. If no hedging program is considered, around 10% of the paths result in a P&L of less than -3%, whereas most of the paths seem to result in a P&L that is located around +2%.

These patterns can also be found in the results shown in Table 3, in which (risk) measures of $\Pi_{0,1}$, including the value-at-risk (VaR) and the conditional-value-at-risk (CVaR), are shown.

Characteristics of $\Pi_{0,1}$	Weekly Hedge	Monthly Hedge	No Hedge
mean	0.1%	0.1%	0.3%
median	0.1%	0.1%	1.3%
standard deviation	0.4%	0.8%	2.8%
$VaR_{99.5\%}$	1.2%	2.1%	16.0%
$CVaR_{99.5\%}$	1.4%	2.5%	20.2%
$VaR_{95\%}$	0.5%	1.1%	5.0%
$CVaR_{95\%}$	0.8%	1.6%	9.3%
$VaR_{90\%}$	0.3%	0.7%	2.6%
$CVaR_{90\%}$	0.6%	1.2%	6.5%
$VaR_{99.5\%} + V_0^\pi$	0.4%	1.3%	15.2%

Table 3: Risk measures of the discounted net P&L, $\Pi_{0,1}$, for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the base-case simulation.

Depending on the chosen risk measure, the difference between monthly and weekly rebalancing makes up to a factor equal or greater than 2 (for a confidence level of 90% or 95%, respectively) and around 1.75 for a confidence level of 99.5%. If no hedging program is considered, the risk measures are, of course, considerably increased. If we use the results for no hedging as a benchmark to compute a measure of hedge effectiveness, we obtain – using the value-at-risk at a confidence level of 99.5% – a hedge effectiveness of 93% for the weekly hedge ($\frac{16.0\% - 1.2\%}{16.0\%}$, cf. Morrison & Tadrowski, 2015) and 87% ($\frac{16.0\% - 2.1\%}{16.0\%}$) for the monthly hedge.

With regard to risk-based capital requirements, the results mean that – depending on the allowance of the risk-mitigating effect of future hedging – the insurer would be required to hold additional 1.2% to 16.0% of the initial investment amount as capital for the considered market risks (using the value-at-risk at a confidence level of 99.5%). If the rebalancing of the hedging program is only considered on a monthly basis (for instance due to delays in trading or an otherwise slow reaction time), the capital requirements are almost doubled in comparison to weekly rebalancing. Note that these measures only consider the three purely financial risk drivers equity returns, interest rates and equity implied volatility – the results, thus, only represent a part of the total capital requirements for all relevant risks.

We also give the sum of the value-at-risk at a level of 99.5% and the value of liabilities (V_0^π) in the last row of Table 3, in order to provide a better understanding of the total impact on the balance sheet of the insurer – this will be relevant in the next section, where – in order to gain a better understanding how the results are influenced by the parameter assumptions made in Section 3.1 – we conduct similar analyses for different assumptions regarding the interest-rate level as well as the level of equity volatility.

3.4 IMPACT OF INTEREST RATE AND VOLATILITY LEVELS ON THE RISK PROFILE

After inception of the variable annuity, the insurer usually aims to offset the changes of the value of liabilities via a hedging program. However, even if a well-functioning hedging program – that is able to effectively offset the changes of the value of liabilities – is in place, the insurer still faces the risk of having to increase its capital resources due to changes in risk-based capital requirements, driven by changes in market parameters. Therefore, we analyze how the indicators for risk-based capital requirements examined in the previous section change with different levels of interest rates and volatility.

We consider four variations of the market environment: higher and lower interest, as well as higher and lower equity volatility. In all four considered variations, we simultaneously change r_0, θ_r^P and θ_r^Q (cf. Table 1) for different interest-rate levels and do the same with V_0, θ_V^P and θ_V^Q in the variations with different levels of equity volatility. See Table 4 for the used parameter values.

Variation	Parameter	New value
Lower interest	$r_0, \theta_r^P, \theta_r^Q$	0.015
Higher interest	$r_0, \theta_r^P, \theta_r^Q$	0.045
Lower volatility	$V_0, \theta_V^P, \theta_V^Q$	$(0.1)^2$
Higher volatility	$V_0, \theta_V^P, \theta_V^Q$	$(0.3)^2$

Table 4: Values of the market and simulation parameters used in the variations. All other parameters are as in the base case.

With the changed parameter values for interest rates and equity volatility, respectively, we conduct similar analyses as in the previous section. We also use the same pricing as previously, i.e. the same guarantee charge and the same guaranteed withdrawal rate as stated in Table 2, meaning that the value of the GLWB rider at inception changes. This represents either a situation where the market environment changes a theoretical second after inception of the contract (with the change in the value of liabilities potentially being hedged) or a situation where the pricing for the offered GLWB rider is only updated after a certain time period and hence does not (immediately) reflect changes in the market environment. In the latter situation, from the insurer's perspective, the value of liabilities of newly sold contracts with otherwise identical parameters fluctuates. Table 5 gives an overview over the value of liabilities at inception, as well as its components, the value of future guarantee payments and the value of future guarantee charges.

Variation	Value guar. payments	Value guar. charges	Total value (V_0^π)
Base case	9.2%	-10.0%	-0.8%
Lower interest	14.6%	-9.7%	4.9%
Higher interest	5.7%	-10.2%	-4.5%
Lower volatility	3.2%	-10.4%	-7.2%
Higher volatility	17.2%	-9.6%	7.6%

Table 5: Resulting values of liabilities at inception and its components for the considered parameter variations. Values are given as fraction of the invested amount at inception.

For instance, in the variation with a lower interest-rate level, the total value of liabilities is 4.9% with the value of future guarantee payments of 14.6% not being compensated by the value of future guarantee charges of -9.7%. While the value of future guarantee charges is rather stable, the value of future guarantee payments shows noticeable differences between the considered variations, ranging between 3.2% and 17.2% of the pool's AuM at the beginning of the year.

We do not show all scatter plots shown in Section 3.3, but instead only show the scatter plots for the change of the value of liabilities (Figure 8, Figure 9 and Figure 10), the comparison of the (cumulative) density functions (Figure 11 and Figure 12), as well as the resulting risk measures (Table 6 and Table 7) for each variation.

Distribution of the present value of liabilities – variations

Figure 8 shows scatter plots of the one-year change of the value of liabilities plotted against the underlying fund's one-year return in all four considered parameter variations (cf. Figure 4 for corresponding plots of the base-case parametrization).

Figure 8: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of future guarantee payments minus future guarantee charges) as a function of the one-year fund return. Based on 10,000 simulation paths for each parameter variation.

The effect of the volatility's parameterization on the distribution of the fund's return is clearly observable, with the width of the fund returns significantly widening with higher volatility. With the

lower volatility level, the fund returns only reach around +50%, while they reach up to +150% in the simulation with a higher volatility level.

In a lower interest-rate environment, the increase in future guarantee payments (due to the ratchet) is more valuable than the increase in future guarantee charges, resulting in an increase in the value of liabilities with positive fund returns. The opposite can be noticed in the variation with higher interest rates: here the ratchet has a lower value and, thus, the value of liabilities decreases with increasing fund returns. Similar effects can be observed with the volatility variations: While the change in value for negative fund returns is similar for both volatility levels, the value increases with positive fund returns for the higher volatility level and decreases for the lower volatility level.
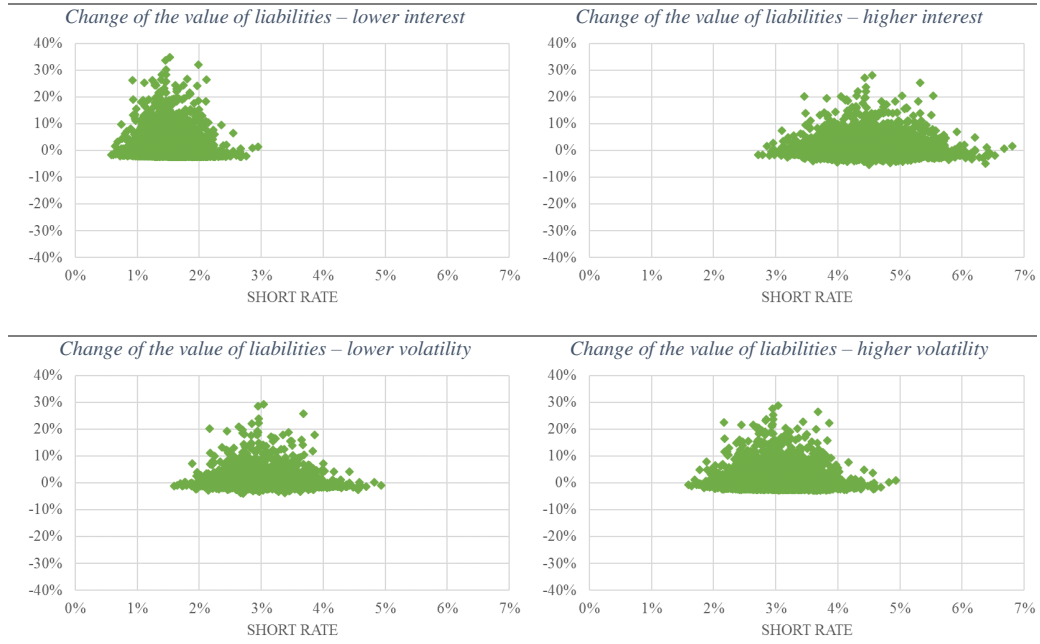

Figure 9: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of the future guarantee payments minus future guarantee charges) as a function of the short rate at the end of the year. Based on 10,000 simulation paths for each parameter variation.

The scatter plots against the short rate r_1 (shown in Figure 9) show a much broader lateral scatter in the variation with higher interest rates, which illustrates the CIR model's volatility structure, where (ceteris paribus) higher levels of interest are accompanied by higher levels of interest-rate volatility. Still, the influence of r_1 on the overall distribution of the value of liabilities seems to be of secondary nature.

Similarly, Figure 10 shows scatter plots of the one-year change of the value of liabilities with respect to the instantaneous volatility at the end of the projection year, $\sqrt{V_1}$.
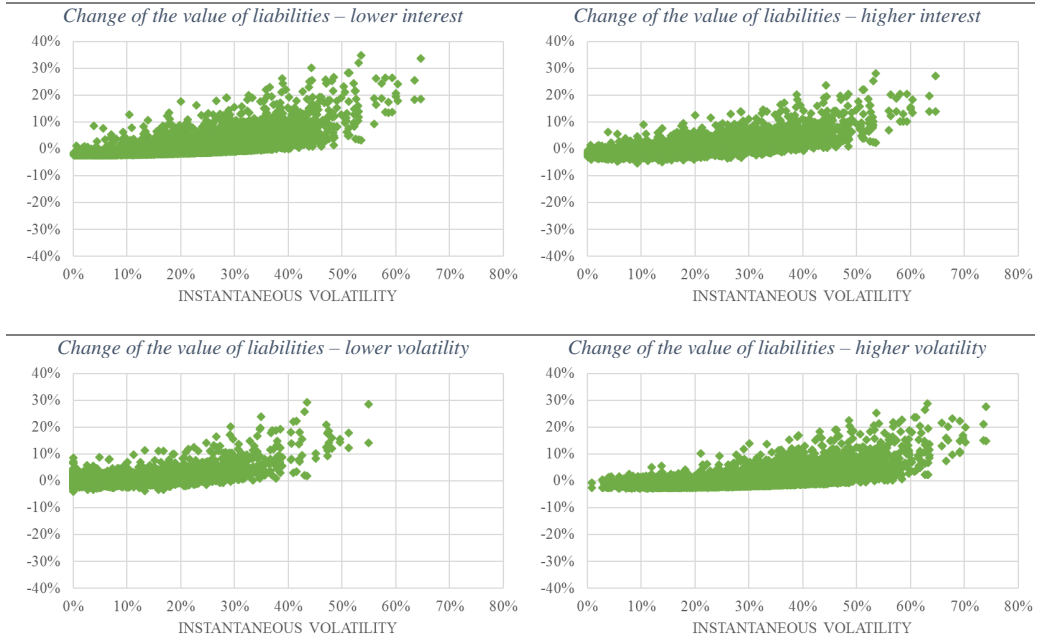
Figure 10: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of the future guarantee payments minus future guarantee charges) as a function of the instantaneous volatility at the end of the year. Based on 10,000 simulation paths for each parameter variation.

For the two interest-rate variations, the scatter plots against the instantaneous volatility seem to be similar to the base case, with the importance of volatility in the "good scenarios" (reduction in the value of liabilities, i.e. the lower edge in the scatter plots), being higher if interest rates are low.

Similar to the higher volatility of interest-rates in the variation with a higher level of interest, an increase of the level of volatility also causes an increase of the observed volatility of volatility (i.e. a wider lateral scatter of $\sqrt{V_1}$), potentially making hedging much harder. However, this effect does not seem to be very pronounced in the variation with a higher volatility level, with the highest observed values of $\sqrt{V_1}$ only shifting by around 10 percentage points in comparison to the base case (which is the same as the increase in the volatility level).

It is noticeable that, in the variation with a lower volatility level, there seem to be quite a few paths where the instantaneous volatility reaches (or gets very close) to zero. This is to be expected, as the parameters in this variation clearly do not fulfill the condition under which V_t cannot reach zero ($\sigma_V^{P\,2} > 2\kappa_V^P \theta_V^P$), therefore zero is accessible (cf. Cox et al., 1985).

Comparison of the P&L distribution and risk measures – variations

The (cumulative) density functions given in Figure 11 show a considerably greater deviation if the interest-rate level is lower, accounting for the increase in the value of liabilities and the increase in the guarantee's "moneyness", as the fund's performance now is less likely to be able to compensate for withdrawals and, thus, the guarantee is more likely to trigger. The opposite can be said for the variation with higher interest, where a narrower distribution of the P&L can be observed. Note that $\Pi_{0,1}$ only measures the P&L after inception of the contract (i.e. the deviation) and does not reflect the initial impact on the balance sheet when the value of liabilities V_0^π is added. Hence, a potential considerable loss or

profit from V_0^π is not shown in the density functions and, as a consequence, there is no considerable shift to the left or right.
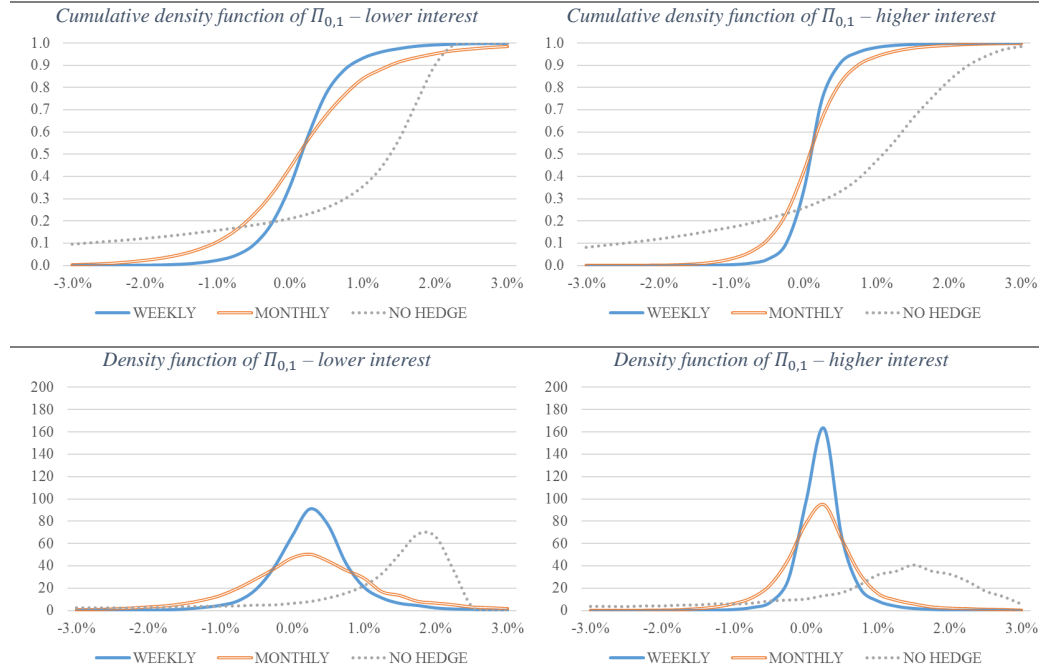

Figure 11: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the simulation with a lower (left-hand side) and a higher interest-rate level (right-hand side).

Similar effects can be observed with the equity-volatility variations, with the higher volatility level leading to a significant broadening of the respective density functions (cf. Figure 12). Judging from the density functions in the lower volatility variation, the difference between monthly and weekly hedging seems much less pronounced than in the other considered market environments. However, as can be observed in Table 7, this is only true for the body of the distribution, and not for its tail.
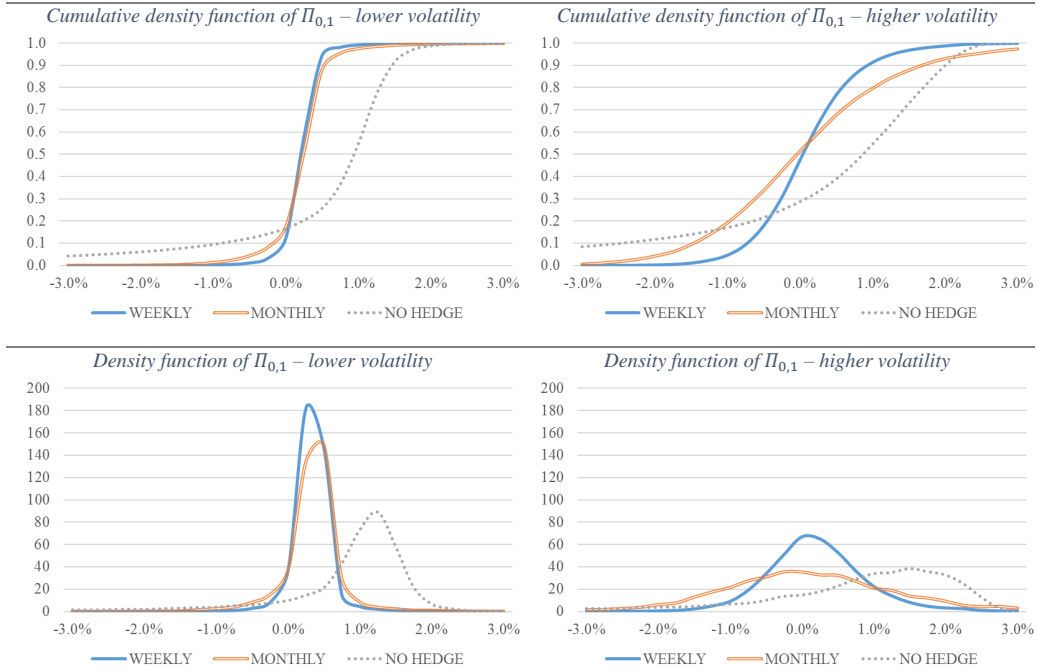
Figure 12: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the simulation with a lower (left-hand side) and a higher equity-volatility level (right-hand side).

The risk measures given in Table 6 show that higher interest rates not only reduce the value of liabilities, but also reduce risk measures and, thereby, risk-based capital requirements. In the lower interest-rate environment, however, the greater deviation of the P&L is also noticeable in the resulting risk measures – meaning that, additionally to the value of liabilities increasing from -0.8% to 4.9% (see Table 5), the risk measures increase and, thus, lead to higher capital requirements. From the insurer's perspective, this means a "double hit", increasing both the value of liabilities and the corresponding capital requirements. For instance, the sum of the value-at-risk at a confidence level of 99.5% and the value of liabilities V_0^π increases from 0.4% to 6.4% if weekly hedging is considered and from 15.2% to 23.3% without hedging.

Characteristics of $\Pi_{0,1}$	Weekly Hedge			Monthly Hedge			No Hedge		
Interest level	lower	base	higher	lower	base	higher	lower	base	higher
mean	0.2%	0.1%	0.1%	0.2%	0.1%	0.1%	0.3%	0.3%	0.3%
median	0.2%	0.1%	0.1%	0.1%	0.1%	0.1%	1.4%	1.3%	1.1%
std. deviation	0.6%	0.4%	0.3%	1.1%	0.8%	0.6%	3.2%	2.8%	2.6%
VaR$_{99.5\%}$	1.5%	1.2%	0.9%	2.7%	2.1%	1.6%	18.4%	16.0%	13.7%
CVaR$_{99.5\%}$	1.8%	1.4%	1.1%	3.2%	2.5%	1.9%	23.2%	20.2%	17.6%
VaR$_{95\%}$	0.7%	0.5%	0.4%	1.5%	1.1%	0.8%	5.8%	5.0%	4.5%
CVaR$_{95\%}$	1.1%	0.8%	0.6%	2.1%	1.6%	1.2%	10.9%	9.3%	8.1%
VaR$_{90\%}$	0.5%	0.3%	0.2%	1.0%	0.7%	0.5%	2.8%	2.6%	2.5%
CVaR$_{90\%}$	0.8%	0.6%	0.4%	1.7%	1.2%	0.9%	7.5%	6.5%	5.7%
VaR$_{99.5\%}$ + V_0^π	6.4%	0.4%	-3.6%	7.6%	1.3%	-2.9%	23.3%	15.2%	9.2%

Table 6: Risk measures of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all) and all three variations of the interest-rate level. Based on 10,000 paths in each corresponding simulation.

The measured hedge effectiveness, again using the results for no hedging as a benchmark and the value-at-risk at a confidence level of 99.5%, is only slightly affected by the change in interest rates, ranging around 92% (lower interest) to 93% (higher interest) for the weekly hedge and 85% (lower interest) to 88% (higher interest) for the monthly hedge.

Similar to the decrease in interest rates, an increase of the volatility level leads to a significant increase in the deviation of the P&L and, thereby, to an increase of the potential capital requirements derived from the considered risk measures. The value-at-risk at a level of 99.5%, for instance, increases from 1.2% to 1.7% if a hedging program with weekly rebalancing is considered (cf. Table 7). This comes additionally to the increase of the value of liabilities from -0.8% to 7.6% (cf. Table 5).

Characteristics of $\Pi_{0,1}$	Weekly Hedge			Monthly Hedge			No Hedge		
Volatility level	lower	base	higher	lower	base	higher	lower	base	higher
mean	0.2%	0.1%	0.1%	0.2%	0.1%	0.1%	0.4%	0.3%	0.1%
median	0.2%	0.1%	0.0%	0.2%	0.1%	0.0%	0.9%	1.3%	0.9%
std. deviation	0.3%	0.4%	0.7%	0.4%	0.8%	1.3%	1.9%	2.8%	2.8%
$VaR_{99.5\%}$	0.7%	1.2%	1.7%	1.3%	2.1%	3.1%	11.8%	16.0%	15.3%
$CVaR_{99.5\%}$	1.1%	1.4%	2.0%	1.7%	2.5%	3.5%	16.7%	20.2%	19.6%
$VaR_{95\%}$	0.2%	0.5%	1.0%	0.4%	1.1%	1.9%	2.6%	5.0%	5.0%
$CVaR_{95\%}$	0.4%	0.8%	1.3%	0.8%	1.6%	2.4%	6.2%	9.3%	9.4%
$VaR_{90\%}$	0.0%	0.3%	0.7%	0.2%	0.7%	1.5%	0.9%	2.6%	2.5%
$CVaR_{90\%}$	0.2%	0.6%	1.1%	0.6%	1.2%	2.0%	3.9%	6.5%	6.4%
$VaR_{99.5\%} + V_0^\pi$	-6.5%	0.4%	9.3%	-5.9%	1.3%	10.7%	4.6%	15.2%	22.9%

Table 7: Risk measures of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all) and all three variations of the equity-volatility level. Based on 10,000 paths in each corresponding simulation.

Also, in contrast to the change in interest rate levels, the measured hedge effectiveness is noticeably affected by the higher volatility. Using again the results for no hedging as a benchmark and the value-at-risk at a confidence level of 99.5%, the hedge effectiveness decreases from 93% to 89% for the weekly hedge and from 87% to 80% for the monthly hedge if the higher volatility environment is considered. On the other hand, the effectiveness only slightly increases in the reduced volatility environment (to 94% for the weekly hedge and to 89% for the monthly hedge).

If no hedging is considered, the risk measures at a confidence level of 99.5% (as well as those at the 90% level) actually decrease with higher volatility, meaning that, after the increase of the value of liabilities from -0.8% to 7.6% (see Table 5) and – if unhedged – a potentially significant loss for the insurer, the P&L in the subsequent year has a less pronounced downside risk as in the base case. Therefore, in comparison to a stand-alone comparison of the change in the value of liabilities, the overall impact on the insurer is reduced, with the sum of $VaR_{99.5\%}$ and V_0^π increasing only by 7.7 percentage points from 15.2% to 22.9%.

4 CONCLUSION

After inception of a variable annuity, the provider usually aims to compensate changes in the value of liabilities by means of a hedging program. However, even if a well-functioning hedging program is in place that is able to effectively replicate (most) changes in the value of liabilities, risk-based capital requirements can still fluctuate due to their dependence on market parameters. As a result, variable

annuity providers face the risk of increases in their risk-based capital requirements and, thus, the need for capital injections – even without pricing errors or malfunctioning of the hedging program.

By means of a simulation study, we assessed the effectiveness of a stylized hedging program with different rebalancing frequencies over a one-year time horizon and analyzed the distribution of the insurer's resulting P&L. Using this P&L, we computed different risk measures as indicators for risk-based capital requirements. The resulting capital requirements highly depend on the degree to which the risk-mitigating effect of the hedging program is allowed for in the calculation, with our results showing an increase by a factor of more than ten if no allowance of the risk-mitigating effect is made. The approach we used is comparable to an internal model type approach under Solvency II and, thus, our results can be used as indicator for risk-based capital requirements (with respect to market risk) a variable annuity provider needs to meet in the EU (cf. Section 3.3).

We analyzed how hedge effectiveness and the considered indicators for capital requirements change with different assumptions regarding the interest-rate and equity-volatility environment (cf. Section 3.4). We found that, additionally to the potentially unhedged changes of the value of liabilities that such a change in the market environment causes, the changed parameters also have a considerable impact on risk measures, meaning that in these cases, the insurer faces two stresses at once: the change in the value of liabilities (this might be hedged) and the change in capital requirements (likely unhedged). We also found that, while the impact of the level of interest rates on the effectiveness of the modeled hedging program is rather low, a higher volatility level has a distinct adverse effect on the hedge effectiveness, leading to a further increase of risk-based capital requirements. However, there are also cases where an increase in the value of liabilities was accompanied by a decrease of capital requirements, reducing the overall impact on the insurer. This is the case for some risk measures if no allowance of the hedging program is made and equity volatility is increased.

In conclusion, if the insurer assesses its risk situation with regard to its (new) variable annuity business, it should also incorporate an analysis of future capital requirements, as those may pose an economic risk and can potentially reduce the profitability of the insurer's variable annuity business. Under Solvency II, for instance, such analyses are mandatory in the context of the Own Risk and Solvency Assessment (ORSA). Furthermore, as the sensitivity of capital requirements to market parameters is not easily assessable, thorough numerical analyses appear necessary for a proper assessment of this risk. In such analyses, also the effect of a potentially reduced hedge performance in adverse market environments and a reduced level of recognition of the hedging program's risk-mitigating effect should be considered, as this may lead to additional increases of the capital requirements.

Regarding future research, it seems worthwhile to investigate ways to incorporate changes of risk-based capital requirements into the value that is being replicated by the hedging program, as well as ways to adequately account for future capital requirements in the profit testing of variable annuity products. Also, an analysis that extends the stand-alone analysis of a homogeneous pool of policies (as presented in this paper) to a model with different lines of businesses and heterogeneous pools of policies appears promising. The same applies to a potential analysis that incorporates surrender and/or biometric risk factors (for instance longevity risk), as well as additional market risk factors as, for instance, the level of interest-rate volatility.

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Curriculum Vitae

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