



ON PROBLEMS RELATED TO GRAPH COLOURING

Dissertation

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Abstract

In this thesis we study various problems from the prominent area of graph colourings.

We provide a characterisation of t -perfect triangulations and quadrangulations of the projective plane. For the latter class, a novel method to transform quadrangulations of the sphere is developed. The involved operations are simple and minor-preserving, in contrast to other known methods.

We conjecture that any graph with treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has chromatic index Δ . In support of the conjecture we prove its fractional version by developing a new upper bound on the edge number of such graphs.

We further prove the list colouring conjecture for generalised Petersen graphs of the form $GP(3k, k)$ and $GP(4k, k)$. In doing so, we discover an interesting connection between the number of 1-factorisations of $GP(3k, k)$ and the Jacobsthal numbers.

Finally, we develop new techniques to construct cycle decompositions. They work on the common neighbourhood of two degree-6 vertices. With these techniques we find structures that cannot occur in a minimal counterexample to Hajós' conjecture and verify the conjecture for Eulerian graphs of pathwidth at most 6. This is the first time the conjecture has been verified for graphs that are not 4-degenerate.

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Introduction

Graph colourings are an important area of graph theory. Problems from various domains such as logistics, finance, traffic, and communication systems can be expressed in terms of graph colourings. Basically, this can be done whenever the problem at hand is equivalent to partitioning the elements (eg vertices, edges or faces) of a graph according to some given criteria. Consequently, considerable effort was put into research of this area in the past. Interesting connections between colourings and various other characteristics of graphs were discovered. Prominent examples include the Four Colour Theorem and the Strong Perfect Graph Theorem.

This thesis examines the relation of specific graph properties with t -perfection which is closely related to vertex colourings (Part I), with edge colourings, list edge-colourings, and fractional edge colourings (Part II) as well as with colourings of the edges such that every colour class is a cycle (Part III).

The original definition of a perfect graph is in terms of the graph's vertex colouring: A graph is called *perfect* if and only if every subgraph H has a vertex colouring with clique number of H many colours. As early as 1960, Berge conjectured that this is equivalent to the graph not containing an odd hole or an odd anti-hole. This was finally proven in 2002 by Chudnovsky et al. and is nowadays known as *Strong Perfect Graph Theorem*.

While looking for a proof of the Strong Perfect Graph Theorem another characterisation of perfect graphs was found: A graph is perfect, if and only if its stable set polytope is completely determined by non-negativity inequalities and clique inequalities.

The definition of t -perfect graphs is inspired by the latter characterisation. A graph is called *t -perfect* if and only if its stable set polytope is determined by non-negativity, edge and odd-cycle inequalities.

A characterisation of t -perfect graphs similar to the Strong Perfect Graph Theorem still seems far away. However, in Part I we provide such characterisations at least for two classes of graphs: triangulations and quadrangulations of the projective plane.

In Chapter 3 we prove that a triangulation is t -perfect, if and only if it is perfect and contains no clique of size 4.

In Chapter 2 we show that a quadrangulation of the projective plane is t -perfect, if and only if it is bipartite. To prove this we devised a novel method. It allows to transform every quadrangulation of the sphere into a quadrangle by deletions of degree-2 vertices and so called *t -contractions*. The involved operations are simple and minor-preserving, in contrast to other known techniques. Coming from a completely different area of research, namely the theory of t -perfection, our method implies the results of several works dealing with transformations of quadrangulations of the sphere.

Part II revolves around different variants of edge colourings.

In Chapter 6 we analyse graphs with bounded treewidth. We determine a novel upper bound for the number of edges in graphs of maximum degree Δ and treewidth at most k . Using this bound, we prove that graphs of maximum degree $\Delta \geq k + \sqrt{k}$ can not be overfull. This means that these graphs have fractional chromatic index Δ . We conjecture that all graphs of treewidth k and maximum degree $\Delta(G) \geq k + \sqrt{k}$ also have chromatic index Δ .

Chapter 5 deals with list edge-colourings. The well-known *list colouring conjecture* asserts that the chromatic index of any graph equals its choice index. This conjecture remains wide open for most graph classes, among them cubic graphs. We prove that the conjecture holds for two classes of cubic graphs: for generalised Petersen graphs of the form $GP(3k, k)$ and $GP(4k, k)$.

Thereto we use the algebraic colouring criterion by Alon and Tarsi. It says that a d -regular graph has choice index d if the numbers of its positive and negative 1-factorisations differ.

For graphs of the form $GP(4k, k)$, we only need to consider the graphs with odd k . For them, we show that all 1-factorisations have the same sign.

For graphs of the form $GP(3k, k)$ we show that the number of 1-factorisations corresponds to the *Jacobsthal number* $J(k)$ if k is odd. The Jacobsthal numbers are closely related to the Fibonacci numbers. They match the count of various combinatorial objects, ie tilings of $3 \times (k - 1)$ -rectangles with 1×1 and 2×2 -squares or certain meets in lattices.

Finally, in Part III we consider decompositions of Eulerian graphs into cycles.

Hajós' conjecture asserts that a simple Eulerian graph on n vertices can be decomposed into at most $\lfloor (n - 1)/2 \rfloor$ cycles. The conjecture is only proved for graph classes in which every element contains vertices of degree 2 or 4, eg planar and projective-planar graphs.

By regarding a cycle decomposition as a colouring of the edges in which every colour class is a cycle we develop new techniques to construct cycle decompositions. The techniques work on the common neighbourhood of two degree-6 vertices. Applying them, we deduce various substructures for degree-6 vertices that cannot occur in any minimal counter-example to Hajós' conjecture.

This enables us to verify the conjecture for a class of graphs that is not 4-degenerate: the class of Eulerian graphs with pathwidth at most 6. This implies that these graphs satisfy another conjecture, the *small cycle double cover conjecture*.

Part I.

On t -perfect Embeddings

1. Introduction to Perfection and t -Perfection

In this chapter, we provide basic definitions and knowledge about perfect and t -perfect graphs that are required for Chapter 2 and 3.

All graphs considered in this thesis are finite and simple. We use standard graph theory notation as found in the book of Diestel [Die00].

Perfect graphs received considerable attention in graph theory. A graph is *perfect* if for every induced subgraph the chromatic number and the size of a largest clique coincide. (The *chromatic number* of a graph is the smallest number of colours needed to colour its vertices.) This purely combinatorial definition is equivalent to two other characterisations.

First, it was conjectured by Berge in 1960 that a graph is perfect if and only if it contains no odd hole or odd anti-hole. An *odd hole* C_k is an induced odd cycle on $k \geq 5$ vertices. An *odd anti-hole* $\overline{C_k}$ is the complement of an odd hole. This conjecture remained open until 2002 when it was shown by Chudnovsky, Robertson, Seymour, and Thomas [CRST06].

Theorem 1.1 (Strong Perfect Graph Theorem). *A simple graph G is perfect if and only if G contains no induced odd hole or odd anti-hole.*

On the long way to prove this theorem, the second, polyhedral characterisation of perfect graphs was found: Clearly, every colour class of a vertex colouring is a *stable set*, ie a set of vertices that are pairwise non-adjacent. The *characteristic vector* of a subset $S \subseteq V(G)$ is the vector $\chi_S \in \{0, 1\}^{V(G)}$ defined by $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. For a stable set S in G , the characteristic vector $\chi_S \in \mathbb{R}^{V(G)}$ satisfies the linear inequalities

$$x \geq 0, \tag{1.1}$$

$$\sum_{v \in V(C)} x_v \leq 1 \text{ for every clique } C \text{ in } G. \tag{1.2}$$

They are known as *non-negativity inequalities* and *clique inequalities* respectively.

We call the convex hull of the characteristic vectors of stable sets of a graph G the *stable set polytope* $\text{SSP}(G)$:

$$\text{SSP}(G) := \text{conv}(\{\chi_S : S \text{ stable set of } G\}) \subseteq \mathbb{R}^{V(G)}$$

Note that all vectors in the stable set polytope $\text{SSP}(G)$ satisfy the above described non-negativity inequalities (1.1) and clique inequalities (1.2).

Results of Lovász [Lov72], Fulkerson [Ful72], and Chvátal [Chv75] showed that G is perfect if and only if non-negativity inequalities and clique inequalities suffice to describe the stable set polytope $\text{SSP}(G)$. This is the case if and only if the system (1.2) is totally dual integral (see [Sch03, Vol. B, Ch. 65.4]). Thus,

$$G \text{ is perfect} \Leftrightarrow \text{the system of (1.1) and (1.2) is totally dual integral.} \quad (1.3)$$

Besides non-negativity inequalities and clique inequalities, the characteristic vector $\chi_S \in \mathbb{R}^{V(G)}$ of a stable set S in a graph G satisfies

$$\sum_{v \in V(C)} x_v \leq \lfloor |V(C)|/2 \rfloor \text{ for every induced odd cycle } C \text{ in } G, \quad (1.4)$$

the so called *odd-cycle inequalities*. The most simple clique inequalities consider cliques of size 2, ie edges, and are thus called *edge inequalities*:

$$x_u + x_v \leq 1 \text{ for every edge } uv \in E \quad (1.5)$$

A graph G is called *t-perfect* if its stable set polytope $\text{SSP}(G)$ is described by non-negativity inequalities (1.1), edge inequalities (1.5) and odd-cycle inequalities (1.4).

We define $\text{TSTAB}(G) \subseteq \mathbb{R}^V$ as the polyhedron determined by these three types of inequalities.

$$\text{TSTAB}(G) := \left\{ x \in \mathbb{R}^{V(G)} : x \text{ satisfies (1.1), (1.5), and (1.4)} \right\}$$

Note that G is *t-perfect* if and only if $\text{SSP}(G)$ and $\text{TSTAB}(G)$ coincide. Equivalently, G is *t-perfect* if and only if $\text{TSTAB}(G)$ is an integral polytope, ie if all its vertices are integral vectors.

Evidently, the polytope $\text{TSTAB}(G)$ is integral if the system of the linear inequalities (1.1), (1.5), and (1.4) is totally dual integral. A graph for which this system is totally dual integral is called *strongly t-perfect*. We want to emphasise that

$$\text{strong } t\text{-perfection implies } t\text{-perfection.} \quad (1.6)$$

It is an open question whether a *t-perfect* graph is always *strongly t-perfect*. A brief discussion of this question can be found in Schrijver [Sch03, Vol. B, Ch. 68].

It is easy to verify that vertex deletion preserves *t-perfection*. For every $v \in V(G)$, the polytopes $\text{TSTAB}(G - v)$ and $\text{SSP}(G - v)$ are lattice isomorphic to the facets $\text{TSTAB}(G) \cap \{x \in \mathbb{R}^{V(G)} : x_v = 0\}$ and $\text{SSP}(G) \cap \{x \in \mathbb{R}^{V(G)} : x_v = 0\}$ of $\text{TSTAB}(G)$ and $\text{SSP}(G)$ respectively. In contrast, edge deletion does not always keep *t-perfection* (see [Sch03, Vol. B, Ch. 68] for an example).

Another operation that maintains *t-perfection* was found by Gerards and Shepherd [GS98]: whenever there is a vertex v , so that its neighbourhood is stable, we may contract all edges incident with v simultaneously. Again, the two polytopes $\text{SSP}(G')$ and $\text{TSTAB}(G')$ of the obtained graph G' are faces of $\text{SSP}(G)$ and $\text{TSTAB}(G)$ respectively. We will call this operation a *t-contraction at v*.

Any graph obtained from G by a sequence of vertex deletions and t -contractions is called a t -minor of G . Let us point out that

$$\text{any } t\text{-minor of a } t\text{-perfect graph is again } t\text{-perfect.} \quad (1.7)$$

The same holds for strong t -perfection (see eg [Sch03, Vol. B, Ch. 68.4]).

The smallest t -imperfect graph is the complete graph K_4 . This graph is isomorphic to the wheel W_3 . A *wheel* W_p is a graph consisting of a p -cycle w_1, \dots, w_p, w_1 (with $p \geq 3$) and a center vertex v adjacent to w_i for $i = 1, \dots, p$; see Figure 1.1. The wheel W_p is an *odd wheel* if p is odd. It is well-known (see eg [Sch03, Vol. B, Ch. 68.4]) that

$$\text{odd wheels are } t\text{-imperfect.} \quad (1.8)$$

Indeed, the vector $(1/3, \dots, 1/3)$ is contained in $\text{TSTAB}(W_{2k+1})$ but not in $\text{SSP}(W_{2k+1})$ for $k \geq 1$. Another t -imperfect graph is the anti-hole $\overline{C_7}$ (see [BS12]):

$$\overline{C_7} \text{ is not (strongly) } t\text{-perfect.} \quad (1.9)$$

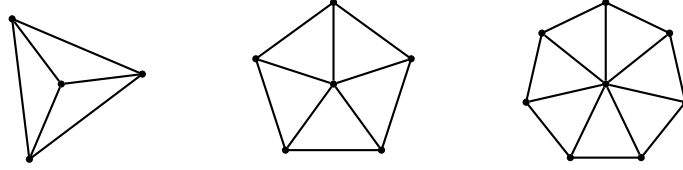


Figure 1.1.: The odd wheels W_3, W_5 and W_7

The knowledge about perfect graphs is often helpful for the study of t -perfect graphs. This can be seen in the next lemma.

Lemma 1.2. [FG17] *Every perfect graph without K_4 is (strongly) t -perfect.*

Proof. A perfect graph contains no odd hole by Theorem 1.1. Thus, for a perfect graph G with no clique of size 4, the system of (1.1) and (1.2) is equivalent to the system of (1.1) and (1.5) and (1.4). The first system is totally dual integral as G is perfect (see (1.3)). This shows that G is (strongly) t -perfect. \square

A general treatment on t -perfect graphs may be found in Schrijver [Sch03, Ch. 68] and in Grötschel, Lovász and Schrijver [GLS88, Ch. 9.1]. The class of t -perfect graphs is of interest as the fractional stable set polytope (that is the polytope defined by non-negativity inequalities and edge inequalities) of these graphs has Chvátal rank of at most 1 (see [Sch03]). Finding a maximum-weight stable set of a t -perfect graph can be done in polynomial time with a combinatorial algorithm described by Eisenbrand et al. [EFGK03]. The problem whether a given graph is t -perfect belongs to Co-NP. A polynomial-time algorithm for non- t -perfection is not known.

Chvátal [Chv75] conjectured the t -perfection of series-parallel graphs. This was proved by Clancy in 1977 and by Mahjoub [Mah88] in 1988. Boulala and Uhry [BU79] established the strong t -perfection of these graphs.

Gerards and Schrijver [GS86] showed that any graph not containing an *odd K_4 -subdivision* is t -perfect. It was shown by Gerards [Ger89] that these graphs are also strongly t -perfect. (An odd K_4 -subdivision is a subdivision in which each triangle has become an odd cycle.)

Gerards and Shepherd [GS98] characterised the graphs with all subgraphs t -perfect, while Barahona and Mahjoub [BM94] described the subdivisions of K_4 that are not t -perfect. Gerards and Shepherd [GS98] showed that any graph without such a K_4 -subdivision is t -perfect. Schrijver [Sch02] showed that all graphs without such a subdivision are strongly t -perfect.

Fonlupt and Uhry [FU82] proved that all almost bipartite graphs, ie graphs G having a vertex v such that $G - v$ is bipartite, are t -perfect. This was implicit in Sbihi and Uhry [SU84]. Cao and Nemhauser [CN98] studied the t -perfection of line graphs. Bruhn and Stein [BS12] characterised t -perfect claw-free graphs.

Bruhn and Benchetrit analysed t -perfection of triangulations of the sphere [BB15]. Against this background, in Chapter 2 of this thesis t -perfect quadrangulations of the projective plane are characterised. Moreover, in Chapter 3 t -perfection of triangulations of the projective plane is considered.

2. Quadrangulations of the Sphere and the Projective Plane

The content of this chapter is based on the paper [FG16] by Elke Fuchs and the author of this thesis.

We show that every quadrangulation of the sphere can be transformed into a quadrangle by deletions of degree-2 vertices and by t -contractions at degree-3 vertices. A t -contraction simultaneously contracts all incident edges at a vertex with stable neighbourhood. The operation is used mainly in the field of t -perfect graphs.

We further show that a non-bipartite quadrangulation of the projective plane can be transformed into an odd wheel by t -contractions and deletions of degree-2 vertices. This implies that a quadrangulation of the projective plane is (strongly) t -perfect if and only if the graph is bipartite.

2.1. Introduction

A quadrangulation of a surface is a finite simple graph embedded on the surface such that every face is bounded by a walk of four edges.

In this chapter, we consider quadrangulations of the sphere and the projective plane. For characterising quadrangulations, it is very useful to transform a quadrangulation into a slightly smaller one. Such reductions for quadrangulations of the sphere are mainly based on the following idea: Given a class of quadrangulations, a sequence of particular face-contractions transforms every member of the class into a 4-cycle; see eg Brinkmann et al [BGG⁺05], Nakamoto [Nak99], Negami and Nakamoto [NN93], and Broersma et al. [BDG93]. A *face-contraction* identifies two non-adjacent vertices v_1, v_3 of a 4-face v_1, v_2, v_3, v_4 in which the common neighbours of v_1 and v_3 are only v_2 and v_4 and deletes multiple edges.

A somewhat different approach was made by Bau et al. [BMNZ14]. They showed that any quadrangulation of the sphere can be transformed into a 4-cycle by a sequence of deletions of degree-2 vertices and so called hexagonal contractions. A *hexagonal contraction* at a vertex x with neighbourhood $\{a_1, a_2, a_3\}$ deletes the edge xa_1 and contracts the edges xa_2 and xa_3 . The obtained graph is a minor of the previous graph.

Both operations, hexagonal contractions and deletions of degree-2 vertices can be obtained from face-contractions.

We provide a new way to reduce arbitrary quadrangulations of the sphere to a quadrangle. Our operations are minor-operations — in contrast to face-contractions. We use deletions of degree-2 vertices and t -contractions. A t -contraction simultaneously

contracts all incident edges of a vertex with stable neighbourhood and deletes all multiple edges. The operation is mainly used in the field of t -perfection (see Chapter 1 for more details). Face-contractions cannot be obtained from t -contractions. We restrict ourselves to t -contractions at vertices that are only contained in 4-cycles whose interior does not contain a vertex;

$$\begin{aligned} &\text{these } t\text{-contractions and deletions of degree-2 vertices} \\ &\text{can be obtained from a sequence of face-contractions.} \end{aligned} \tag{2.1}$$

Figure 2.1 illustrates this. Note that the restriction on the applicable t -contractions makes sure that all face-contractions can be applied, ie that all identified vertices are non-adjacent and have no common neighbours besides the two other vertices of their 4-face.

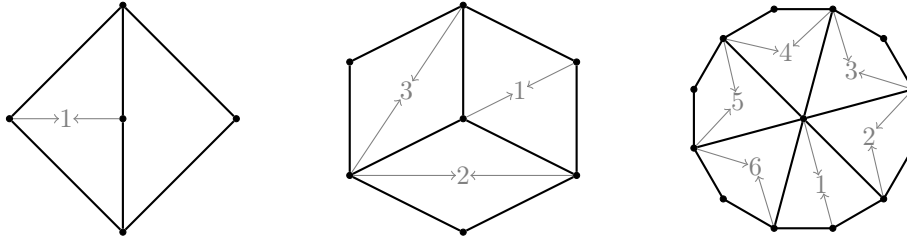


Figure 2.1.: Face-contractions that give a deletion of a degree-2 vertex, a t -contraction at a degree-3 and a degree-6 vertex

We prove:

Theorem 2.1. [FG16] *Let G be a quadrangulation of the sphere. Then, there is a sequence of t -contractions at degree-3 vertices and deletions of degree-2 vertices that transforms G into a 4-cycle. During the whole process, the graph remains a quadrangulation.*

The proof of Theorem 2.1 follows directly from Lemma 2.7 and can be found in Section 2.3. It is easy to see that both operations used in Theorem 2.1 are necessary. By (2.1), Theorem 2.1 implies:

Any quadrangulation of the sphere can be transformed
into a 4-cycle by a sequence of face-contractions.

Via the dual graph, quadrangulations of the sphere are in one-to-one correspondence with planar 4-regular (not necessarily simple) graphs. Theorem 2.1 thus implies a method to reduce all 4-regular planar graphs to the multigraph on two vertices and four parallel edges.

Besides quadrangulations of the sphere, we consider quadrangulations of the projective plane. We use Theorem 2.1 to reduce all non-bipartite quadrangulations of the projective plane to an odd wheel. (See Chapter 1 for a definition of an odd wheel.)

Theorem 2.2. [FG16] *Let G be a non-bipartite quadrangulation of the projective plane. Then, there is a sequence of t -contractions and deletions of degree-2 vertices that transforms G into an odd wheel. During the whole process, the graph remains a non-bipartite quadrangulation.*

The proof of this theorem can be found in Section 2.3. It is easy to see that both operations used in this theorem are necessary.

Section 2.4 provides an application of Theorem 2.2 to the theory of t -perfect graphs. It is shown that a quadrangulation of the projective plane is (strongly) t -perfect if and only if the graph is bipartite.

Negami and Nakamoto [NN93] showed that any non-bipartite quadrangulation of the projective plane can be transformed into a K_4 by a sequence of face-contractions. This result can be deduced from Theorem 2.2: By (2.1), Theorem 2.2 implies that any non-bipartite quadrangulation of the projective plane can be transformed into an odd wheel by a sequence of face-contractions. The odd wheel W_{2k+1} can now be transformed into W_{2k-1} — and finally into $W_3 = K_4$ — by face-contractions (see Figure 2.2).

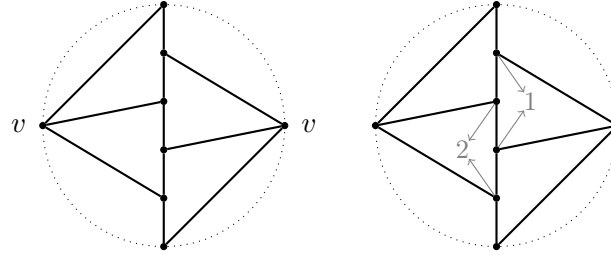


Figure 2.2.: An even embedding of W_5 in the projective plane and face-contractions that produce a smaller odd wheel. Opposite points on the dotted cycle are identified.

Nakamoto [Nak99] gave another reduction method based on face-contractions and so called 4-cycle deletions for non-bipartite quadrangulations of the projective plane with minimum degree 3. Matsumoto et al. [MNY16] analysed quadrangulations of the projective plane with respect to hexagonal contractions while Nakamoto considered face-contractions for quadrangulations of the Klein bottle [Nak95] and the torus [Nak96]. Youngs [You96] showed that all non-bipartite quadrangulations of the projective plane have chromatic number equal to 4 and Esperet and Stehlík [ES15] gave bounds for edge- and face-width of non-bipartite quadrangulations.

2.2. Basic Definitions and Facts

We begin by recalling several useful definitions related to surface-embedded graphs. For further background on topological graph theory, we refer the reader to Gross and Tucker [GT87] or Mohar and Thomassen [MT01].

2. Quadrangulations of the Sphere and the Projective Plane

An *embedding* of a simple graph G on a surface is a continuous one-to-one function from a topological representation of G into the surface. For our purpose, it is convenient to abuse the terminology by referring to the image of G as *the graph* G . The *faces* of an embedding are the connected components of the complement of G . An embedding G is *even* if all faces are bounded by an even circuit. A *quadrangulation* is an embedding in which each face is bounded by a circuit of length 4.

A cycle C is *contractible* if C separates the surface into two sets S_C and $\overline{S_C}$ where S_C is homeomorphic to an open disk in \mathbb{R}^2 . Note that for the sphere, S_C and $\overline{S_C}$ are homeomorphic to an open disk. In contrast, for the plane and the projective plane, $\overline{S_C}$ is not homeomorphic to an open disk. For the plane and the projective plane, we call S_C the *interior* of C and $\overline{S_C}$ the *exterior* of C .

Using the stereographic projection, it is easy to switch between embeddings in the sphere and the plane. In order to have an interior and an exterior of a contractible cycle, we will concentrate on quadrangulations of the plane (and the projective plane). Note that by the Jordan curve theorem,

$$\text{all cycles in the plane are contractible.} \quad (2.2)$$

A cycle in a non-bipartite quadrangulation of the projective plane is contractible if and only if it has even length (see e.g. [KS15, Lemma 3.1]). As every non-bipartite even embedding is a subgraph of a non-bipartite quadrangulation, one can easily generalise this result.

Observation 2.3. [FG16] *A cycle in a non-bipartite even embedding in the projective plane is contractible if and only if it has even length.*

Proof. Let G be an even embedding. We construct a quadrangulation G' from G as follows: In every face bounded by a circuit $(v_1, v_2, \dots, v_k, v_1)$ with $k \geq 6$ we insert a smaller concentric cycle $C = (w_1, w_2, \dots, w_k, w_1)$ and the edges $v_1w_1, v_2w_2, \dots, v_kw_k$. Further, we add a vertex x into the interior of C and connect x with all vertices of C that have an odd index. As the vertices w_1, w_2, \dots, w_k are pairwise different, the obtained quadrangulation is a simple graph.

Because all cycles of G are contained in the constructed quadrangulation, we are done by [KS15, Lemma 3.1]. \square

In quadrangulations of the plane, we do not have to consider odd cycles. It is easy to see that

$$\text{all quadrangulations of the plane are bipartite.} \quad (2.3)$$

An embedding is a *2-cell embedding* if each face is homeomorphic to an open disk. It is well-known that embeddings of 2-connected graphs in the plane are 2-cell embeddings. A non-bipartite quadrangulation of the projective plane contains a non-contractible cycle; see Observation 2.3. The complement of this cycle in the projective plane is homeomorphic to an open disk. Thus, we observe:

Observation 2.4. [FG16] *Every quadrangulation of the plane and every non-bipartite quadrangulation of the projective plane is a 2-cell embedding.*

This observation makes sure that we can apply Euler's formula to all the considered quadrangulations. A simple graph cannot contain a 4-circuit that is not a 4-cycle. Thus, note that every face of a quadrangulation is bounded by a cycle.

We will now take a closer look at deletions of degree-2 vertices.

Observation 2.5. *[FG16] Let $G \neq C_4$ be a quadrangulation of the plane or the projective plane that contains a vertex v of degree 2. Then, $G - v$ is again a quadrangulation.*

Proof. Let u and u' be the two neighbours of v . Then, there are distinct vertices s, t such that the cycles (u, v, u', s, u) and (u, v, u', t, u) are bounding a face. Thus, (u, s, u', t, u) is a contractible 4-cycle whose interior contains only v and $G - v$ is again a quadrangulation. \square

2.3. Quadrangulations and t -Contractions

In this section, we take a closer look at t -contractions and prove the main theorems of this chapter.

Lemma 2.6. *[FG16] Let G be a quadrangulation of the plane or a non-bipartite quadrangulation of the projective plane. Let G' be obtained from G by a t -contraction at v . If v is not a vertex of a contractible 4-cycle with some vertices in its interior, then G' is again a quadrangulation.*

Proof. Let G'' be obtained from G by the operation that identifies v with all its neighbours but does not delete multiple edges. This operation leaves every cycle not containing v untouched, transforms every other cycle C into a cycle of length $|C| - 2$, and creates no new cycles. Therefore, all cycles bounding faces of G'' are of size 4 or 2. The graphs G' and G'' differ only in the property that G'' has some double edges. These double edges form 2-cycles that arise from 4-cycles containing v . As all these 4-cycles are contractible (see (2.2) and Observation 2.3) with no vertex in their interior, the 2-cycles are also contractible and contain no vertex in its interior. Deletion of all double edges now gives G' — an embedded graph where all faces are of size 4. \square

Lemma 2.6 enables us to prove the following lemma that directly implies Theorem 2.1.

Lemma 2.7. *[FG16] For every quadrangulation G of the plane, there is a sequence of*

- *t -contractions at degree-3 vertices that are only contained in 4-cycles whose interior does not contain a vertex and*
- *deletions of degree-2 vertices*

that transforms G into a 4-cycle. During the whole process, the graph remains a quadrangulation.

Proof. Let \mathcal{C} be the set of all contractible 4-cycles whose interior contains some vertices of G . Note that \mathcal{C} contains the 4-cycle bounding the outer face unless $G = C_4$.

Let $C \in \mathcal{C}$ be a contractible 4-cycle whose interior does not contain another element of \mathcal{C} . We will first see that the interior of C contains a vertex of degree 2 or 3: Deletion of all vertices in the exterior of C gives a quadrangulation G' of the plane. As G is connected, one of the vertices in C must have a neighbour in the interior of C and thus must have degree at least 3. Euler's formula now implies that $\sum_{v \in V(G')} \deg(v) = 2|E(G')| \leq 4|V(G')| - 8$. As no vertex in G' has degree 0 or 1, there must be a vertex of degree 2 or 3 in $V(G') - V(C)$. This vertex has the same degree in G and is contained in the interior of C .

We now use deletions of degree-2 vertices and t -contractions at degree-3 vertices in the interior of the smallest cycle of \mathcal{C} to successively get rid of all vertices in the interior of 4-cycles. By Observation 2.5 and Lemma 2.6, the obtained graphs are quadrangulations.

Now, suppose that no more t -contraction at a degree-3 vertex and no more deletion of a degree-2 vertex is possible. Assume that the obtained graph is not a 4-cycle. Then, there is a cycle $C' \in \mathcal{C}$ whose interior does not contain another cycle of \mathcal{C} . As we have seen above, $C' \in \mathcal{C}$ contains a vertex v of degree 3. Since no t -contraction can be applied to v , the vertex v has two adjacent neighbours. This contradicts (2.3). \square

In the rest of the chapter, we will consider the projective plane.

A quadrangulation of the projective plane is *nice* if no vertex is contained in the interior of a contractible 4-cycle.

Lemma 2.8. [FG16] *Let G be a non-bipartite quadrangulation of the projective plane. Then, there is a sequence of t -contractions and deletions of vertices of degree 2 that transforms G into a nice quadrangulation. During the whole process, the graph remains a quadrangulation.*

Proof. Let C be a contractible 4-cycle whose interior contains at least one vertex. Delete all vertices that are contained in the exterior of C . The obtained graph is a quadrangulation of the plane. By Lemma 2.7, there is a sequence of t -contractions (as described in Lemma 2.6) and deletions of degree-2 vertices that eliminates all vertices in the interior of C . With this method, it is possible to transform G into a nice quadrangulation. \square

Similar as in the proof of Theorem 2.1, Euler's formula implies that a non-bipartite quadrangulation of the projective plane contains a vertex of degree 2 or 3. As no nice quadrangulation has a degree-2 vertex (see Observation 2.5), we deduce:

Observation 2.9. [FG16] *Every nice non-bipartite quadrangulation of the projective plane has minimum degree 3.*

In an even embedding of an odd wheel W , every odd cycle must be non-contractible, see Observation 2.3. Thus, it is easy to see that there is only one way (up to topological isomorphism) to embed an odd wheel in the projective plane. (This can easily be deduced

from [MRV96] — a paper dealing with embeddings of planar graphs in the projective plane.) The embedding is illustrated in Figure 2.2. Noting that this embedding is a quadrangulation, we observe:

Observation 2.10. [FG16] *Let G be a quadrangulation of the projective plane that contains an odd wheel W . If G is nice, then G equals W .*

Note that every graph containing an odd wheel also contains an induced odd wheel. Next, we consider even wheels.

Lemma 2.11. [FG16] *Even wheels W_{2k} for $k \geq 2$ do not have an even embedding in the projective plane.*

The statement follows directly from [MRV96]. We nevertheless give an elementary proof of the lemma.

Proof. First assume that the 4-wheel W_4 has an even embedding. As all triangles of $W_4 - w_3w_4$ must be non-contractible by Observation 2.3, it is easy to see that the graph must be embedded as in Figure 2.3. Since the insertion of w_3w_4 will create an odd face, W_4 is not evenly embeddable.

Now assume that W_{2k} for $k \geq 3$ is evenly embedded. Delete the edges vw_i for $i = 5, \dots, 2k$ and note that w_5, \dots, w_{2k} are now of degree 2, ie the path $P = w_4w_5 \dots w_{2k}w_1$ bounds either two faces or one face from two sides.

Deletion of the edges vw_i preserve the even embedding: Deletion of an edge bounding two faces F_1, F_2 merges the faces into a new face of size $|F_1| + |F_2| - 2$. Deletion of an edge bounding a face F from two sides leads to a new face of size $|F| - 2$. In both cases, all other faces are left untouched.

Next, replace the odd path P by the edge w_4w_1 . The two faces F_3, F_4 adjacent to P are transferred into two new faces of size $|F_3| - (2k - 3) + 1$ and $|F_4| - (2k - 3) + 1$. This yields an even embedding of W_4 which is a contradiction. \square

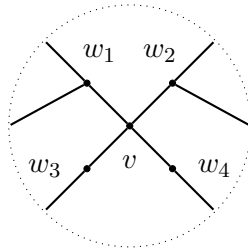


Figure 2.3.: The only even embedding of $W_4 - w_3w_4$ in the projective plane. Opposite points on the dotted cycle are identified.

Recall that a t -contraction at a vertex v is only allowed if its neighbourhood is stable, that is, if v is not contained in a triangle. The next lemma characterises the quadrangulations to which no t -contraction can be applied.

Lemma 2.12. [FG16] *Let G be a non-bipartite nice quadrangulation of the projective plane where each vertex is contained in a triangle. Then G is an odd wheel.*

Proof. By Observation 2.9, there is a vertex v of degree 3 in G . Let $\{x_1, x_2, x_3\}$ be its neighbourhood and let x_1, x_2 and v form a triangle.

Recall that each two triangles are non-contractible (see Observation 2.3). Consequently each two triangles intersect. As x_3 is contained in a triangle intersecting the triangle vx_1x_2v and as v has no further neighbour, we can suppose without loss of generality that x_3 is adjacent to x_1 . The graph induced by the two triangles vx_1x_2v and vx_1x_3v is not a quadrangulation. If $x_2x_3 \in E(G)$, G contains a K_4 . By Observation 2.10, G then equals the odd wheel $W_3 = K_4$.

If $x_2x_3 \notin E(G)$, the graph contains a further vertex and this vertex is contained in a further triangle T . Since v has degree 3, the vertex v cannot be contained in T . If further $x_1 \notin V(T)$, then the vertices x_2 and x_3 must be contained in T . But then $x_2x_3 \in E(G)$ and, as above, v, x_1, x_2 and x_3 form a K_4 . Therefore, x_1 is contained in T and consequently in every triangle of G . Since every vertex is contained in a triangle, x_1 must be adjacent to all vertices of $G - x_1$. As $|E(G)| = 2|V(G)| - 2$ by Euler's formula, the graph $G - x_1$ has $2|V(G)| - 2 - (|V(G)| - 1) = |V(G)| - 1 = |V(G - x_1)|$ many edges. By Observation 2.9, no vertex in G has degree smaller than 3. Consequently, no vertex in $G - x_1$ has degree smaller than 2. Thus, $G - x_1$ is a cycle and G is a wheel. By Lemma 2.11, G is an odd wheel. \square

Finally, we can prove our second main result:

Proof of Theorem 2.2. Transform G into a nice quadrangulation (Lemma 2.8). Now, consecutively apply t -contractions (as described in Lemma 2.6) as long as possible. In each step, the obtained graph is a quadrangulation. By Lemma 2.8 we can assume that the quadrangulation is nice. If no more t -contraction can be applied, then every vertex is contained in a triangle. By Lemma 2.12, the obtained quadrangulation is an odd wheel. \square

2.4. On t -Perfect Quadrangulations of the Projective Plane

Theorem 2.2 allows a direct application to the theory of t -perfection.

Theorem 2.13. [FG16] *For every quadrangulation G of the projective plane the following assertions are equivalent:*

- (a) G is t -perfect
- (b) G is strongly t -perfect
- (c) G is bipartite

Proof. If G is bipartite, then G is perfect and contains no K_4 . Lemma 1.2 implies that G is (strongly) t -perfect.

If G is not bipartite, then Theorem 2.2 implies that G has an odd wheel as a t -minor. As odd wheels are not (strongly) t -perfect (see (1.8)), G is not (strongly) t -perfect by (1.7). \square

3. On t -Perfect Triangulations of the Projective Plane

The content of this chapter is based on the paper [FG17] by Elke Fuchs and the author of this thesis.

We prove that a triangulation of the projective plane is (strongly) t -perfect if and only if the graph is perfect and contains no K_4 .

3.1. Introduction

As we have seen in Chapter 1, there is no structural characterisation for (strongly) t -perfect graphs known. In this chapter, we give such a characterisation for triangulations of the projective plane. We show:

Theorem 3.1. [FG17] *For every triangulation G of the projective plane the following assertions are equivalent:*

- (a) G is t -perfect
- (b) G is strongly t -perfect
- (c) G is perfect and contains no K_4
- (d) G contains no loose odd wheel and no $\overline{C_7}$ as an induced subgraph.

One of the main open questions about t -perfection is, whether a t -perfect graph can always be coloured with few colours. Standard polyhedral methods assert that the fractional chromatic number of a t -perfect graph is at most 3. Laurent and Seymour as well as Benchetrit found examples of a t -perfect graph that is not 3-colourable (see [Sch02, p. 1207] and [Ben15]).

Conjecture 3.2 (Shepherd, Sebő). *Every t -perfect graph is 4-colourable.*

The conjecture is known to hold in a number of graph classes, for instance in P_6 -free graphs (Benchetrit [Ben16]), claw-free graphs (Bruhn and Stein [BS12]), and in P_5 -free graphs (Bruhn and Fuchs [BF15]). In the last two classes, the graphs are even 3-colourable.

The verification of the conjecture for triangulations of the projective plane follows directly from Theorem 3.1: A t -perfect triangulation is perfect without K_4 and thus 3-colourable.

Corollary 3.3. [FG17] *Every t -perfect triangulation of the projective plane is 3-colourable.*

In order to prove Theorem 3.1, we mainly study *Eulerian triangulations*, ie triangulations where all vertices are of even degree¹. Colourings of Eulerian triangulations were studied by Hutchinson et al. [HRS02] and by Mohar [Moh02]. Suzuki and Watanabe [SW07] determined a family of Eulerian triangulations of the projective plane such that every Eulerian triangulation of the projective plane can be transformed into one of its members by application of two operations (see Section 3.4 for more details). Barnette [Bar82] showed that one can obtain each triangulation of the projective plane from one of two minimal triangulations using two kinds of splitting operation.

3.2. Triangulations

A *triangulation* G of the projective plane is a finite simple graph embedded on the surface such that every face of G is bounded by a 3-cycle. A cycle C in the projective plane is *contractible* if C separates the projective plane into two sets S_C and $\overline{S_C}$ where S_C is homeomorphic to an open disk in \mathbb{R}^2 . We call S_C the *interior* of C . A triangulation is *nice* if no vertex is contained in the interior of a contractible 3-cycle. As a contractible cycle separates G , the following theorem of Chvátal implies that

$$\text{it suffices to analyse } t\text{-(im)perfection of nice triangulations.} \quad (3.1)$$

Theorem 3.4 (Chvátal [Chv75]). *Let G be a graph with a clique separator X , and let C_1, \dots, C_k be the components of $G - X$. Then, G is t -perfect respectively perfect if and only if $G[C_1 \cup X], \dots, G[C_k \cup X]$ are t -perfect respectively perfect.*

Ringel [Rin78] showed that the neighbourhood of any vertex in a triangulation contains a Hamilton cycle:

Theorem 3.5 (Ringel [Rin78]). *Let G be a triangulation of a closed surface and let v be a vertex with neighbourhood $\{v_0 = v, v_1, \dots, v_{d-1}\}$ where $vv_iv_{i+1}v$ is a contractible triangle for $i = 0, \dots, d-1$. Then, the cycle $v_0v_1 \dots v_{d-1}v_0$ is a contractible Hamilton cycle in $N_G(v)$.*

We denote the contractible Hamilton cycle of v described in Theorem 3.5 by $HC(v)$ and observe:

Observation 3.6. [FG17] *Let G be a nice triangulation with a non-contractible cycle C . Let u, v, w be three consecutive vertices on C . Then the two paths between u and w along the Hamilton cycle $HC(v)$ are induced.*

Proof. Suppose that the path $P = v_1v_2 \dots v_k$ with $u = v_1$ and $w = v_k$ on the Hamilton cycle of v is not induced. Then, there are vertices v_i and $v_{i+\ell}$ of P with $\ell \geq 2$ that are adjacent in G . As C is non-contractible, the triangle $vv_iv_{i+\ell}v$ must be contractible and its interior contains the vertices $v_{i+1}, \dots, v_{i+\ell-1}$. This shows that G is not a nice triangulation. \square

¹Sometimes, an Eulerian triangulation is referred to as an *even triangulation*.

A triangulation is called *Eulerian* if each vertex has even degree. The next observation implies that a t -perfect triangulation must be Eulerian.

Observation 3.7. [FG17] *Let G be a triangulation of any surface and let G contain a vertex whose neighbourhood is not bipartite. Then, G does not satisfy any of the assertions given in Theorem 3.1.*

Proof. Let the neighbourhood of $v \in V(G)$ contain an odd induced cycle C . Then, C and v form an odd wheel W and G is not (strongly) t -perfect by (3.3). If C is a triangle, W is a K_4 ; otherwise, C is an odd hole and G is imperfect (see Theorem 1.1). \square

In a triangulation, a vertex of odd degree has a non-bipartite neighbourhood (Theorem 3.5). It thus follows from Observation 3.7 that in order to prove Theorem 3.1,

$$\text{it suffices to consider Eulerian triangulations.} \quad (3.2)$$

The next theorem follows directly from a characterisation of t -perfect triangulations of the plane given by Bruhn and Benchetrit [BB15, Theorem 2].

Theorem 3.8. *Every triangulation G of the projective plane that contains a contractible odd hole also contains a loose odd wheel.*

A *loose wheel* W is a graph consisting of a cycle $C = w_1 \dots w_p w_1$ and a *center vertex* $v \notin V(C)$ where v has at least three neighbours in C . For a cycle C and three vertices $v_1, v_2, v_3 \in V(C)$ we denote by P_{C, v_1-v_2, v_3} the path connecting v_1 and v_2 along C that does not contain v_3 . Figure 3.1 illustrates the definition; wiggly lines represent paths. A path P in C between two neighbours x, y of v is a *segment* of the loose wheel W if P equals $P_{C, x-y, z}$ for every neighbour $z \notin \{x, y\}$ of v (see also Figure 3.1). A cycle, a path or a segment is *odd* if it has an odd number of edges.

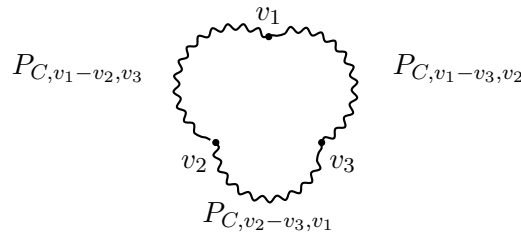


Figure 3.1.: The paths P_{C, v_1-v_2, v_3} , P_{C, v_1-v_3, v_2} and P_{C, v_2-v_3, v_1} on a cycle C

Let C be an odd cycle and let $v \notin V(C)$ be a vertex. Three vertices v_1, v_2, v_3 are called three *odd neighbours* of v on C , if they are neighbours of v and the paths P_{C, v_1-v_2, v_3} , P_{C, v_1-v_3, v_2} and P_{C, v_2-v_3, v_1} on C are odd. A *loose odd wheel* consists of an odd cycle C and a vertex $v \notin V(C)$ that has three odd neighbours on C . Evidently, every odd wheel is a loose odd wheel and every graph that contains an odd wheel as a subgraph also contains an odd wheel as an induced subgraph. Further, one can

3. On t -Perfect Triangulations of the Projective Plane

see that every loose odd wheel has an odd wheel as a t -minor. From (1.7) and (1.8) follows directly that

$$\text{a } t\text{-perfect graph contains no loose odd wheel.} \quad (3.3)$$

To prove Theorem 3.1, we thus can assume (see Theorem 3.8 and (3.3)) that in the considered triangulations

$$\text{all odd cycles are non-contractible.} \quad (3.4)$$

There is another useful property of a graph that forces a loose odd wheel.

Observation 3.9. [FG17] *Let G be a triangulation of the projective plane. If G contains an odd hole with a vertex of degree 4, then G contains a loose odd wheel.*

Proof. Let u, v, w be three consecutive vertices on an odd hole and let v have neighbourhood $N_G(v) = \{u, w, x, y\}$. As u and w cannot be adjacent, the Hamilton cycle around v equals $uxwyu$ and C forms a loose odd wheel together with x . \square

By (3.2), it suffices to analyse Eulerian triangulations of the projective plane in order to prove Theorem 3.1. Suzuki and Watanabe [SW07] introduced the following operations that modify Eulerian triangulations.

Definition 3.10. *Let G be an Eulerian triangulation of the projective plane. Let $x \in V(G)$ be a vertex with Hamilton cycle $aba'b'a$ where the set of common neighbours of b and b' equals $\{a, a', x\}$. An even-contraction at x together with b and b' identifies the vertices x, b, b' to a new vertex y and removes loops as well as multiple edges. The inverse operation is called an even-splitting at y .*

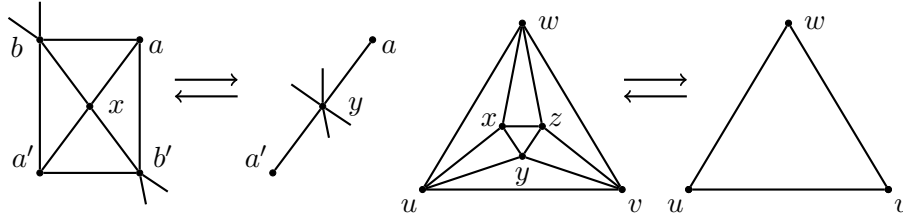


Figure 3.2.: Even-contraction and deletion of an octahedron

Figure 3.2 shows an even-contraction. Note that the graph obtained from an Eulerian triangulation by an even-contraction or an even-splitting is again an Eulerian triangulation of the projective plane. An even-splitting is always unique: The embedding directly yields the partition of the neighbours of y into neighbours of b and of b' .

We put some useful observations down: If we can apply an even-contraction, then

$$N_G(b) \cap N_G(b') = \{a, a', x\}. \quad (3.5)$$

Further, not only the common neighbours of b and b' are restricted. If $bb' \in E(G)$ then $\{b, b', a, x\}$ induces a K_4 , thus

$$\text{we may assume that } bb' \notin E. \quad (3.6)$$

Suzuki and Watanabe [SW07] pointed out that even-contraction and even-splitting preserve 3-colourability. This leads to the following observation:

Observation 3.11. [FG17] *Let G' be obtained from G by an even-contraction. If G' is perfect without K_4 , then G is 3-colourable.*

Suzuki and Watanabe [SW07] defined one more operation:

Definition 3.12. *Let G be an Eulerian triangulation of the projective plane. Let uvw be a contractible triangle in G whose interior contains only the vertices x, y, z and the edges $xy, xz, yz, ux, uy, vy, vz, wx, wz$. The deletion of the vertices x, y, z is called a deletion of an octahedron.*

Figure 3.2 shows a deletion of an octahedron.

We call an Eulerian triangulation *irreducible* if no deletion of an octahedron and no even-contraction can be applied. Suzuki and Watanabe [SW07] listed all irreducible Eulerian triangulations. These graphs are treated in Section 3.4.

3.3. Proof of Theorem 3.1

In this section, we prove Theorem 3.1. The proof is inductive. Lemma 3.13 provides the induction start. For the induction step, we consider a nice triangulation with an odd hole to which we apply an even-contraction. Lemma 3.14, Lemma 3.15 and Lemma 3.16 treat the different structures of the obtained graph.

Lemma 3.13. [FG17] *Every irreducible triangulation G of the projective plane that contains no loose odd wheel and no \overline{C}_7 as an induced subgraph is perfect and contains no K_4 .*

This lemma will be shown in Section 3.4.

Lemma 3.14. [FG17] *Let G be a nice Eulerian triangulation and let G' be obtained from G by an even-contraction. If G contains an odd hole and G' is perfect, then G contains a loose odd wheel.*

The proof of this lemma can be found in Section 3.6. The proofs of the following two lemmas appear in Section 3.5.

Lemma 3.15. [FG17] *Let G be a nice Eulerian triangulation and let G' be obtained from G by an even-contraction. If G' contains a loose odd wheel, then G contains a loose odd wheel.*

Lemma 3.16. [FG17] *Let G be a nice Eulerian triangulation and let G' be obtained from G by an even-contraction. If G' contains an induced \overline{C}_7 , then G contains a \overline{C}_7 or a loose odd wheel as an induced subgraph.*

Now, we are able to prove the main theorem.

Proof of Theorem 3.1. If a graph G is perfect without K_4 , then G is strongly t -perfect and thus t -perfect by Lemma 1.2. Consequently, (c) implies (b) and (b) implies (a).

If a graph G is t -perfect, then G contains no loose odd wheel or $\overline{C_7}$; see (3.3) and (1.9). Thus, (a) implies (d).

In order to show that (d) implies (c), we now consider an imperfect triangulation G of the projective plane and prove that G contains $\overline{C_7}$ or a loose odd wheel as an induced subgraph. Note that K_4 is a loose odd wheel.

We can assume that G is a nice triangulation (see (3.1)). Further, as G is imperfect, G contains an odd hole or an odd anti-hole (Theorem 1.1). The Euler characteristic of the projective plane implies that $2|E(H)| \leq 6|V(H)| - 6$ for every embeddable graph H . Thus, no odd anti-hole on nine or more vertices is embeddable. As the anti-hole $\overline{C_5}$ is isomorphic to the hole C_5 , an imperfect graph in the projective plane contains an odd hole or the anti-hole $\overline{C_7}$. If G contains an anti-hole $\overline{C_7}$, G is t -imperfect by (1.9). Thus, suppose that G contains an induced odd hole C . If the odd cycle C is contractible, G contains a loose odd wheel by Theorem 3.8. As G is a nice triangulation, no deletion of an octahedron can be applied.

If no even-contraction can be applied to G , then the graph is irreducible and the claim follows from Lemma 3.13. Assume that an even-contraction can be applied. If G' is perfect, G contains a loose odd wheel by Lemma 3.14. If the obtained graph G' contains an induced $\overline{C_7}$, then by Lemma 3.16, G contains a loose odd wheel or an induced $\overline{C_7}$. If G' contains an odd hole then by induction, G' contains a loose odd wheel. Thus, G also contains a loose odd wheel ; see Lemma 3.15. \square

3.4. Irreducible Triangulations

In this section, we analyse irreducible Eulerian triangulations of the projective plane, ie Eulerian triangulations to which no deletion of an octahedron and no even-contraction can be applied.

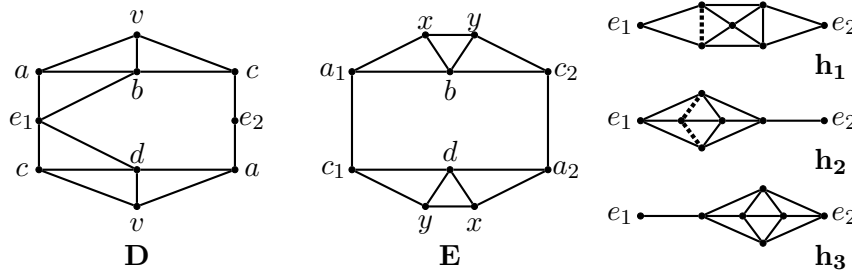


Figure 3.3.: The pieces D , E , h_1 , h_2 and h_3

Following Suzuki and Watanabe [SW07] we define three families of graphs.

The graph $I_{16}[s_1, \dots, s_n]$ with $s_i \in \{1, 2, 3\}$ for $i \in [n]$ is obtained from the graph D by inserting the pieces h_1 , h_2 and h_3 (see Figure 3.3) into the hexagonal region

of D as follows: Insert h_{s_1}, \dots, h_{s_n} one below the other into the hexagon $e_1 b c e_2 a d$ (with $e_1 = e_2$). Then identify the paths between e_1 and e_2 in each pair of consecutive pieces. Further, identify the path $e_1 b c e_2$ in D with the path connecting e_1 and e_2 in h_{s_1} that has not been connected to another piece. Analogously, identify e_1, d, a, e_2 in D with the path in h_{s_n} that has not been connected to another piece. Figure 3.4 shows $I_{16}[1, 2, 3]$ as an example where the identified paths are dotted.

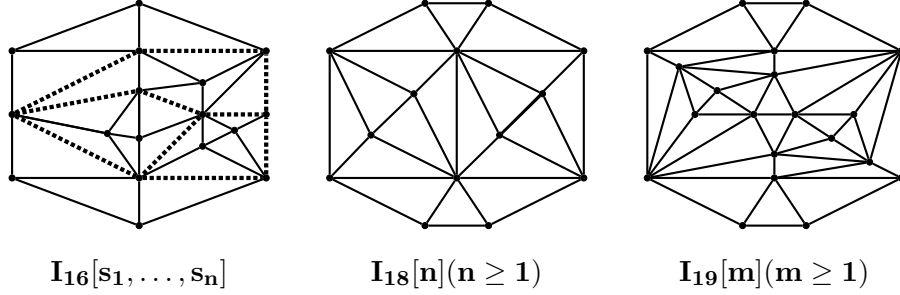


Figure 3.4.: The three infinite families of irreducible graphs. Opposite points on the outer cycles are identified.

The graph $I_{18}[n]$, with $n \in \mathbb{N}$ is obtained from the graph E in Figure 3.3 by inserting the pieces h_2 and h_3 of Figure 3.3 one below the other a total of n times alternatingly into the hexagonal region $a_1 b c_1 a_2 d c_2$ (with $a_1 = a_2$ and $c_1 = c_2$). The paths between $e_1 = c_1$ and $e_2 = c_2$ in each pair of consecutive pieces are identified. Figure 3.4 shows $I_{18}[1]$ as an example.

The graph $I_{19}[m]$ for $m \in \mathbb{N}$ is obtained from E (see Figure 3.3) by inserting m copies of h_2 and m copies of h_3 alternatingly — starting with h_2 — and identifying c_1 and e_1 as well as c_2 and e_2 . Each pair of consecutive pieces bounds a hexagonal region. This region may be triangulated. Note that this happens in a unique way as all vertices are required to be of even degree. Figure 3.4 shows $I_{19}[1]$ as an example.

Theorem 3.17 (Suzuki and Watanabe [SW07]). *An Eulerian triangulation G of the projective plane is irreducible if and only if G is one of the graphs in Figure 3.5 or belongs to one of the families $I_{16}[s_1, \dots, s_n]$, $I_{18}[n]$, and $I_{19}[m]$.*

This characterisation enables us to prove Lemma 3.13.

Proof of Lemma 3.13. The graphs $I_5, I_8, I_{11}, I_{13}, I_{15}$ and I_{20} contain no loose odd wheel (and thus no K_4) and no $\overline{C_7}$. One can check that these graphs are perfect.

Figure 3.5 shows that $I_1, I_2, I_3, I_4, I_6, I_7, I_9, I_{10}, I_{12}, I_{14}$ and I_{17} have a loose odd wheel as subgraph.

Now consider $I_{16}[s_1, \dots, s_n]$. First, suppose that the graph contains an even number of copies of h_1 . The graphs D, h_1, h_2 and h_3 are easily seen to be 3-colourable. The colourings can be combined to obtain a 3-colouring of the vertices of $I_{16}[s_1, \dots, s_n]$. Thus, the graphs contains no K_4 . One can check easily that every triangle-free subgraph of G can be coloured with two colours. Therefore, the clique number and the

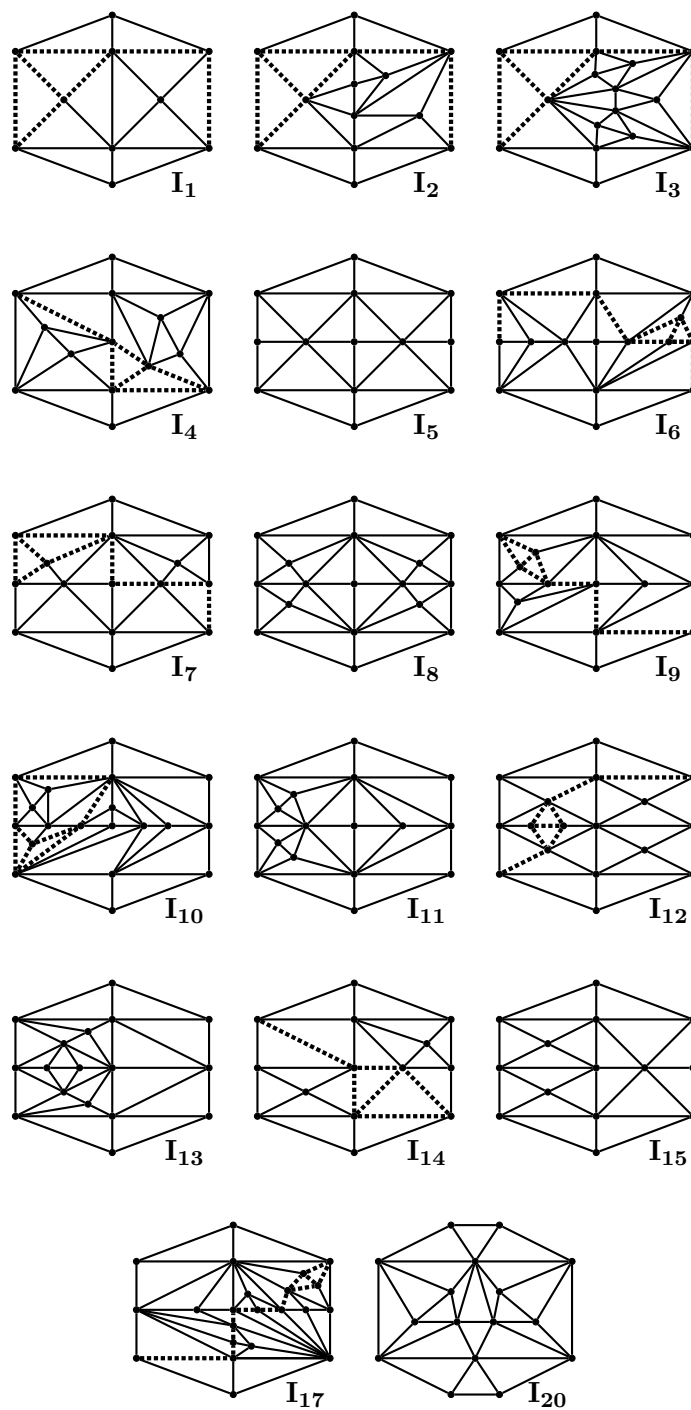


Figure 3.5.: Some irreducible Eulerian triangulations of the projective plane. Opposite points on the outer cycles are identified.

chromatic number coincide for each subgraph of $I_{16}[s_1, \dots, s_n]$. This shows that the graph is perfect.

Second, let $I_{16}[s_1, \dots, s_n]$ contain an odd number of copies of h_1 . Consider the path P from b to d consisting of the dotted edges in each copy of h_1 and h_2 . This path is induced and odd and forms a loose odd wheel together with v and a .

Next, consider $I_{18}[n]$ for $n \in \mathbb{N}$. Each consecutive pair of pieces has exactly one edge in common that is not adjacent to $c_1 = c_2$. The union of these edges forms an induced odd path between b and d . (For an illustration see Figure 3.6.) This path together with x and y gives a loose odd wheel.

Last, consider $I_{19}[m]$ for $m \in \mathbb{N}$. An induced induced odd path from b to d together with the vertices x and y forms a loose odd wheel in $I_{19}[m]$. Such a path is eg dotted in Figure 3.6 for $I_{19}[2]$. For the general case take for the first pair of h_1 and h_2 the dotted path from v_1 to v_2 which is odd. For all other pairs take the dotted path from v_2 to v_3 which is even. Sticking these paths together gives an odd induced path from b to d . \square

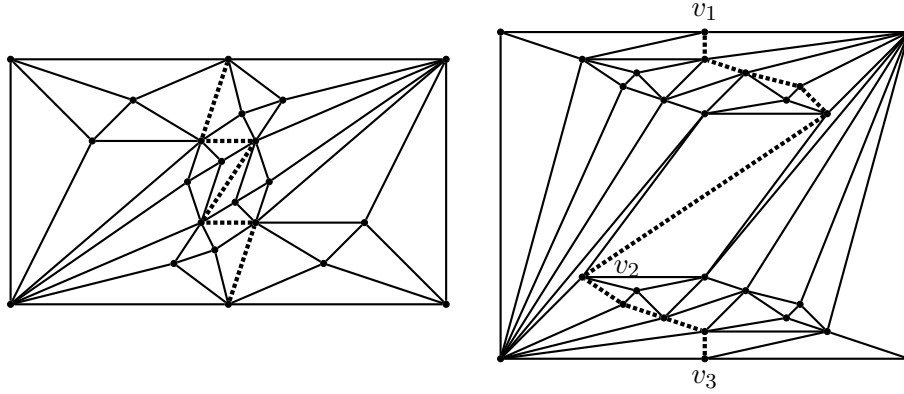


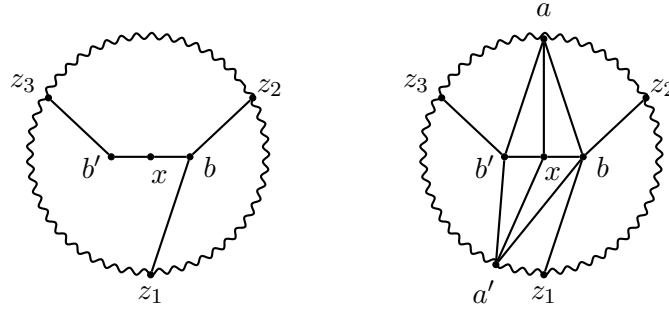
Figure 3.6.: Two odd paths described in the proof of Lemma 3.13. Opposite points on the outer cycles are identified.

3.5. Even-Contraction Creating a Loose Odd Wheel or a 7-Anti-Hole

In this section we prove Lemma 3.15 and Lemma 3.16.

Proof of Lemma 3.15. Let W be a loose odd wheel in G' with central vertex v and induced cycle C . Let the even-contraction in G be applied at x together with b and b' , let $N_G(x) = \{a, a, b, b'\}$ and let $V(G') = V(G) \setminus \{x, a, a, b, b'\} \cup \{y\}$. We distinguish between the different positions of y in the loose odd wheel W and show that G contains a loose odd wheel in all cases.

Evidently, G still contains W if $y \notin V(W)$.

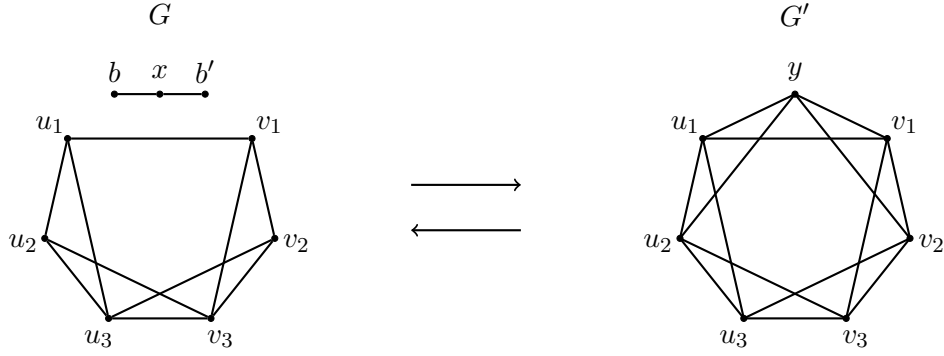

 Figure 3.7.: The situation after the splitting if y equals the central vertex v

Suppose that $y \in V(C)$. Let u, u' be the two cycle vertices adjacent to y . Let C' be the cycle obtained from C by deletion of y and addition of the edges ub, bu' (if $ub, bu' \in E(G')$) or $ub, bx, xb', b'u'$ (if $ub, u'b' \in E(G)$). C' is an odd induced cycle in G' . If C' forms a loose odd wheel together with v , we are done. Otherwise, b is adjacent to u and u' , but not to v . The vertex b' is adjacent to v and neither adjacent to u nor to u' (or vice versa). But then, $\{u, u'\}$ is different from $\{a, a'\}$, say $u \notin \{a, a'\}$. Since C' together with v does not form a loose odd wheel, there is a vertex z such that the path $P_{C', b-z, u'}$ together with x and v forms an odd hole. By Observation 3.9 we obtain a loose odd wheel.

Next, suppose that y equals the center vertex v of W . Let z_1, z_2 and z_3 be three odd neighbours of y in W . If b respectively b' is adjacent to all the three vertices in G , then b respectively b' forms a loose odd wheel together with C . Assume that $bz_1, bz_2, b'z_3 \in E(G)$ and $b'z_1, bz_3 \notin E(G)$ (see Figure 3.7). The paths P_{C, z_2-z_3, z_1} and P_{C, z_1-z_3, z_2} form odd cycles C_1 and C_2 together with b, x, b' . By Observation 3.9, G contains a loose odd wheel if C_1 or C_2 is induced. Thus, assume that C_1 and C_2 have chords. Then, C_i yields an induced odd subcycle C'_i . If C'_i contains b, x, b' , it leads to a loose odd wheel by Observation 3.9. Thus, x is contained in a chord of C_i for $i = 1, 2$. As x has only four neighbours, the chords are of the form ax and $a'x$. Consequently, b and b' are also contained in chords of C_1 and C_2 . (Note that it is possible that a and a' and coincide with vertices in $\{z_1, z_2, z_3\}$.) See Figure 3.7 for an illustration. If $P_{C, a'-z_1, z_2}$ or $P_{C, a-z_2, z_1}$ is odd, then b has three odd neighbours on the cycle C and yields a loose odd wheel. Otherwise, the paths P_{C, z_3-a', z_1} and P_{C, z_3-a, z_2} are odd and consequently, b' has three odd neighbours on C .

This shows that G always contains a loose odd wheel. \square

Proof of Lemma 3.16. Let the even-contraction in G be applied at x together with b and b' , let $N_G(x) = \{a, a, b, b'\}$ and let $V(G') = V(G) \setminus \{x, a, a, b, b'\} \cup \{y\}$. Assume that G' contains an induced \overline{C}_7 . Then, either G also contains an induced \overline{C}_7 (and we are done) or y is one of the vertices of \overline{C}_7 in G' . Let $u_1 u_2 u_3 v_3 v_2 v_1 y u_1$ be a cycle of \overline{C}_7 in G' (see Figure 3.8). Then, $N_{G'}(y) \cap V(\overline{C}_7) = \{u_1, u_2, v_1, v_2\}$ (see also Figure 3.8). In G , the vertices u_1, u_2, v_1, v_2 are neighbours of b or b' . If a vertex is adjacent to both,


 Figure 3.8.: Even-splitting at \bar{C}_7

b and b' , then this vertex equals a or a' . We now assume that G contains no loose odd wheel and no induced \bar{C}_7 and deduce some useful observations:

At most three of the vertices in $\{u_1, u_2, v_1, v_2\}$ are adjacent to b in G . (3.7)

Otherwise, b induces a \bar{C}_7 in G together with $u_1, u_2, u_3, v_3, v_2, v_1$.

If u_1b and u_2b' are edges in G , then $\{u_1, u_2\} \cap \{a, a'\} \neq \emptyset$. (3.8)

Otherwise, the edges u_1b' and u_2b are not contained in G . Then, b, x, b', u_1, u_2, b is an odd hole that forms a loose odd wheel in G together with a .

If u_1b and v_1b' are edges in G , then $\{u_1, v_1\} \cap \{a, a'\} \neq \emptyset$. (3.9)

Otherwise, the edges u_1b' and v_1b are not contained in G . Then, b, x, b', v_1, u_1, b forms a loose odd wheel in G together with a . Next, we note that

$$\{u_1, v_2\} \neq \{a, a'\}. \quad (3.10)$$

Otherwise, $u_1, x, v_2, v_3, u_2, u_1$ is an odd hole that forms a loose odd wheel in G together with b .

If u_1 and v_2 are adjacent to b' in G , then u_2 is also a neighbour of b' . (3.11)

Otherwise, the vertices $b', u_1, u_2, u_3, v_3, v_2$ form a loose odd wheel in G with center vertex u_3 .

Note that in all observations, we can exchange u and v as well as b and b' . We now use our observations in order to get a contradiction:

First, we assume that u_1 and v_2 are neighbours of b' . Then, by (3.11), u_2 is adjacent to b' and by (3.7), v_1 is not adjacent to b' . Further, because of (3.8) and (3.9), $\{u_1, v_2\} = \{a, a'\}$. This is a contradiction to (3.10). Consequently,

$$u_1 \text{ and } v_2 \text{ do not have a common neighbour in } \{b, b'\}. \quad (3.12)$$

Let u_1b and v_2b' be edges of G , and let $u_1b', v_2b \notin E(G)$.

First, assume that $v_1 \in N_G(b')$. Then by (3.9), $v_1 = a$. If u_2 is a neighbour of b' , then u_2 equals a' by (3.8). This is a contradiction to (3.10). Thus, $u_2 \in N_G(b)$ and $v_2b \in E(G)$ by (3.11). This contradicts (3.12).

Now let v_1 be adjacent to b but not to b' . This contradicts (3.8) because of (3.12). \square

3.6. Even-Contraction Destroying an Odd Hole

This section is dedicated to the proof of Lemma 3.14. To prove Lemma 3.14, let $C = v_1v_2 \dots v_kv_1$ be an odd hole in G and let G' be obtained by an even-contraction identifying b, b' and x . To the even-contraction applied at x together with b and b' we associate the map

$$\gamma : V(G) \mapsto V(G) \setminus \{x, b, b'\} \cup \{y\}$$

with $\gamma(x) = \gamma(b) = \gamma(b') = y$ and $\gamma(v) = v$ for $v \in V(G) \setminus \{x, b, b'\}$. We will abuse notation and will apply γ also to subgraphs of G and G' .

Our aim is to show that G' contains an odd hole or G contains a loose odd wheel. We split up this proof into several cases concerning the form of $\{a, b, a', b'\} \cap V(C)$.

If $x \in V(C)$, then G contains an odd wheel by Observation 3.9.

If $\{a, a', b, b'\} \cap V(C)$ equals $\{b, b'\}$, then $\gamma(C)$ consists of an odd and an even induced cycle intersecting each other in the vertex y . G' still contains an odd hole, if the odd cycle has length at least 5. Otherwise, the cycle is a triangle y, v, w, y . Then, b, v, w, b', x form a 5-cycle. If this cycle is induced, G contains a loose odd wheel by Observation 3.9. Otherwise, x has a neighbour among the cycle vertices v, w . Such a neighbour must equal a or a' . This contradicts the assumption that $V(C) \cap \{a, a', b, b'\} = \{b, b'\}$.

If $\{a, b, a', b'\} \cap V(C)$ is of size 3, then x has three odd neighbours on C .

If a, b, a', b' are all contained in $V(C)$, then C is not an induced odd cycle. This is a contradiction.

$$\text{If } \{a, b, a', b'\} \cap V(C) \subseteq \{a, a'\} \text{ then } C = \gamma(C) \text{ is an odd hole in } G'. \quad (3.13)$$

The cases where $\{x, a, a', b, b'\} \cap V(C)$ equals $\{a', b'\}$ or $\{b'\}$ are treated in Section 3.6.1 and Section 3.6.2 respectively.

3.6.1. Even-Contraction if a', b' Are Contained in the Odd Hole

In this subsection, we prove the following lemma.

Lemma 3.18. [FG17] *Let G be a nice Eulerian triangulation. Let G' be obtained from G by an even-contraction at x and let $\{x, a, a', b, b'\} \cap V(C)$ equal $\{a', b'\}$. If G contains an odd hole C and if G' is perfect, then G contains a loose odd wheel.*

Proof of Lemma 3.18. Let G and G' be as described. Application of the even-contraction translates the odd hole C of G into a new odd cycle K in G' with vertex set $\gamma(V(C)) = V(C) \setminus \{b'\} \cup \{y\}$.

Suppose that G' does not contain an odd hole. Then, K is not an odd hole, ie K has chords. Further, all induced subcycles of K are triangles or even cycles.

Note that each chord splits an odd cycle into an odd and an even subcycle. Therefore,

$$\text{every odd cycle has an induced odd subcycle.} \quad (3.14)$$

If G' does not contain an odd hole, the induced odd subcycle is a triangle.

As a' and b' are adjacent and both contained in the induced cycle C , the vertices a' and b' consecutively appear on C . We define z, w, b', a' and v to be five consecutive vertices of C .

As C is induced, each chord of K may have y and a vertex that is adjacent to b in G as end vertices. Thus, the cycle K may only have three different triangles as a subcycle: The vertex y can be contained in a triangle together with w, z (Case A) or with a', v (Case B) or with two other vertices that are adjacent on K (Case C). Figure 3.9 shows these three cases and the associated configurations in G and G' . Note that the cases are not exclusive.

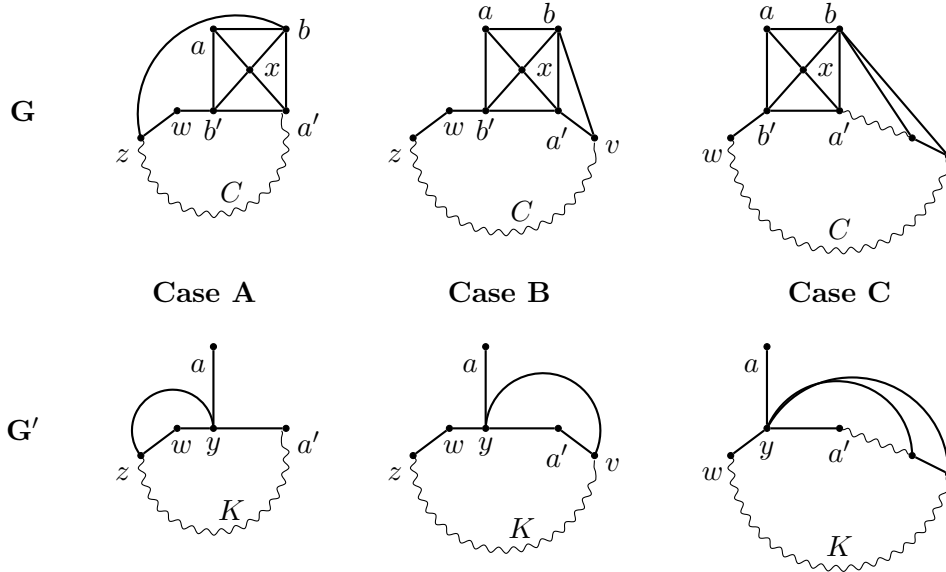


Figure 3.9.: The three possible constellations in G which lead to triangles in K

In Case A, ie if $ywzy$ is a triangle in G' , the vertices b, z, w, b', a' form a 5-cycle in G . This cycle is induced as C is induced and as b can by assumption not be adjacent to b' or w . Thus, the 5-cycle forms a loose odd wheel together with x .

For the two other cases, we denote by v_0 the vertex contained in a triangle with y and a further vertex of K for which $P_{K, a' - v_0, y}$ is of shortest odd length. This also implies that

$$P_{C, a' - v_0, b'} \text{ is of shortest odd length} \quad (3.15)$$

and means that v_0 equals v in Case B. In Case C, denote by v_1 the neighbour of v_0 on C respectively K that forms a triangle with v_0 and y .

Note that b has three odd neighbours on C in G if the neighbour of v_0 that is not contained in $P_{C,a'-v_0,b'}$ is adjacent to y in G' . Thus, we may assume that v_0 forms a triangle together with y and its neighbour v_1 that is contained in $P_{C,a'-v_0,b'}$.

There are two paths connecting w and a' on the Hamilton cycle of b' . Both of these paths are induced (Observation 3.6) and have the same parity (Observation 3.7).

Depending on parity and length of these paths, we prove the statement of this lemma independently. Claim 3.19 applies if the paths along $HC(b')$ connecting a' and w are even. If the paths are odd, see Claim 3.20 and Claim 3.21.

The proof strategy of all the three claims is as follows: Starting from the subgraph of G described in Case B and Case C, we analyse the surrounding vertices and edges until we find an odd cycle. We are done if such an odd cycle is an odd hole that is not destroyed by the even-contraction or if this cycle is part of a loose odd wheel. If the cycle we found is not induced, the chords lead to a new odd cycle in G . After some steps, we always find a loose odd wheel or an odd hole that is not affected by the even-contraction in G . This means, we show that in every possible triangulation satisfying Case B or Case C, the graph G contains a loose odd wheel if G' contains no odd hole.

Claim 3.19. *Let the two paths along $HC(b')$ connecting a' and w be even. If G' does not contain an odd hole, then G contains a loose odd wheel.*

Proof. We first analyse G . The path $P_{HC(b'),w-a',a}$ forms an odd cycle together with the path $C - b'$ in G . If this cycle is induced, G contains a loose odd wheel with center b' . If the cycle contains chords, these chords have one end vertex in $P_{HC(b'),w-a',a}$ and one endvertex in $C - b'$. This comes from the fact that $C - b'$ and $P_{HC(b'),w-a',a}$ (see Observation 3.6) are induced. The chords lead to an induced odd subcycle (see (3.14)). If a longest such cycle \tilde{C} is not a triangle, G contains an odd hole that is not affected by the even-contraction. Thus, suppose that \tilde{C} is a triangle with either two vertices of $C - b'$ and one vertex of $P_{HC(b'),w-a',a}$ or vice versa.

If \tilde{C} is a triangle with one vertex p of $P_{HC(b'),w-a',a}$ and two vertices c, c' of $C - b'$, the vertex p either has three odd neighbours on C (namely the c, c' and b') or one of the two neighbours of p , say c , is of the form $c \in V(C) \setminus \{b'\}$ where the path $P_{C,v_0-c,a'}$ is even and does not contain c' .

$$\begin{aligned} &\text{If } p \in P_{HC(b'),w-a',a} \text{ and } c \in C - b' \text{ and } pc \in E(G) \text{ and} \\ &P_{C,v_0-c,a'} \text{ is even, then } G \text{ contains a loose odd wheel} \end{aligned} \quad (3.16)$$

by the following observations:

If $P_{C,v_0-c,a'}$ is even, then the path $\{c, p, b', x, b, v_0\}$ together with $P_{C,v_0-c,a'}$ gives an odd cycle C' . If C' is induced, then G contains a loose odd wheel with center a . We will now see that every possible chord of C' also yields a loose odd wheel: Since C is induced, there is no chord between two vertices of C . Further, no chord may have x as an endvertex. By (3.5) and (3.6), the vertices b and p are not adjacent and G contains a loose odd wheel if b is adjacent to b' . If b or p is adjacent to a vertex of $P_{C,v_0-c,a'}$,

then either b respectively p has three odd neighbours on C , or the corresponding odd subcycle contains x, b, b' . In both cases, we obtain a loose odd wheel with a .

Let \tilde{C} now be a triangle with two vertices p_1, p_2 of $P_{HC(b'), w-a', a}$ and one vertex c of $C - b'$. Without loss of generality, select c such that $P_{C, v_0-c, b'}$ is of minimal length. By choosing $p_1 = p$, it follows from (3.16) that G contains a loose odd wheel if $P_{C, v_0-c, a'}$ is even. Otherwise, p_1 and p_2 form odd cycles C_1, C_2 together with the path $C - P_{C, c-b', a'}$; see (3.16). If one of these cycles is induced, $C - P_{C, c-b', a'}$, p_1 and p_2 form a loose odd wheel. Assume that p_i ($i \in \{1, 2\}$) is adjacent to an inner vertex u of $C - P_{C, c-b', a'}$. Choose u in such a way that $P_{C, u-w, b'}$ is of minimal length. Again apply (3.16) where $p_i = p$ and $u = c$ to see that G either contains a loose odd wheel or the smallest induced odd subcycle of C_i contains the vertices w, b', p_i, u . This odd cycle is induced and contains at least five vertices. As the cycle is not affected by the even-contraction, the graph G' contains an odd hole. \square

Note that we can assume that C is non-contractible (see (3.4)), but do not know in which way the chords are embedded. Figure 3.10 shows all possible embeddings up to topological isomorphism. In Case B, there are two ways to embed the edge bv : the odd cycle $ba'vb$ may be contractible (Case I) or non-contractible (Case II). Similarly, there are four embeddings in Case C, differing in the (non)-contractability of the cycles $ba'P_{C, a'-v_1, b'}v_1b$ and $ba'P_{C, a'-v_0, b'}v_0b$.

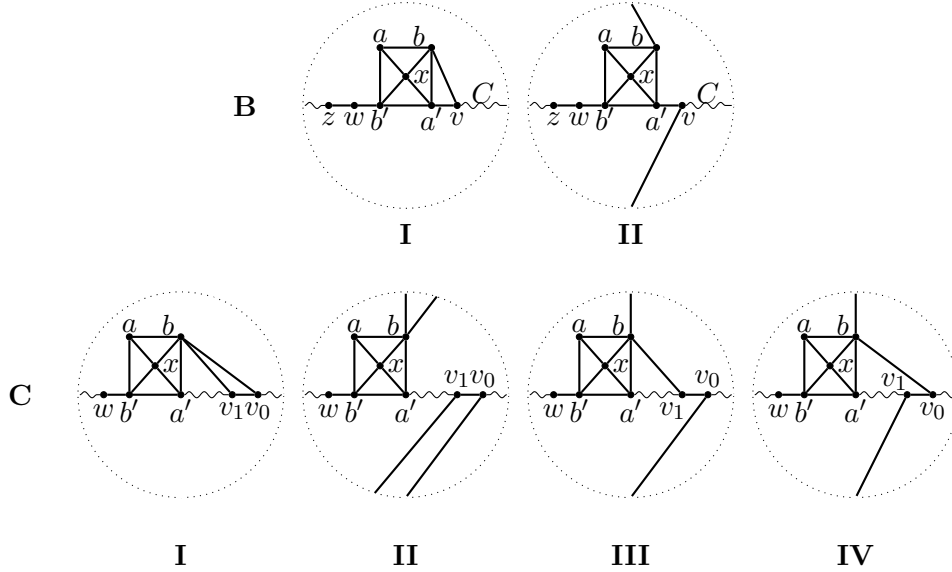


Figure 3.10.: Different ways of embedding the chords of K . (Opposite points on the dotted cycle are identified.)

Claim 3.20. *Let the a' - w -path along $HC(b')$ that contains a be odd and of length 3. If G' does not contain an odd hole, then G contains a loose odd wheel.*

Proof. Note that in this case, the vertices a and w are adjacent.

3. On t -Perfect Triangulations of the Projective Plane

We first treat Case B, ie we suppose that $v_0 = v$ is adjacent to b . The path $P_{C,w-a',b'}$ forms a z' - b' -path together with the edges $a'b, bx, xb'$. This path and the path $V(C) - w$ form cycles $C_{Q,x}$ and C_Q of different parity together with $Q = P_{HC(w),b'-z,a}$. If the arising odd cycle is induced, G contains a loose odd wheel with center w .

We analyse the different types of chords:

First, note that x cannot be an endvertex of a chord and that $b \notin V(Q)$ by (3.5). If b is the endvertex of a chord, the chord must be of the form bq with $q \in V(Q)$. Then, G contains the 5-cycle q, w, b', x, b, q . As $bw, b'q \notin E(G)$ by (3.5), either b is adjacent to b' or q, w, b', x, b, q is induced. In both cases, G contains a loose odd wheel; see (3.6) and Observation 3.9. If a' is the endvertex of a chord, the chord must be of the form $a'q$ with $q \in V(Q)$. This gives the 5-cycle w, a, x, a', q, w . If this cycle is induced, it forms a loose odd wheel with b . We are done unless aq is a chord.

In embedding II (Figure 3.10), the cycle obtained from C by replacing the edge $a'v$ with $a'b, bv$ separates a from q . Thus, a and q cannot be adjacent. In embedding I, the edge aq yields the contractible cycle w, a, x, a', q, w . The interior of this cycle contains the path $Q' = P_{HC(b'),w-a',a}$. Note that Q' is odd and that we are done if $q \in V(Q')$; then $\{w, b', a'\}$ is a set of odd neighbours of q on C . Let Q'' be the path joining w and the other neighbour of z on C along the Hamilton cycle of z such that q is not contained in $V(Q'')$. Then, $V(C) \setminus \{z\}$ and $Q' \cup V(C) \setminus \{z, b'\}$ form cycles C' and C'' of different parity together with Q'' .

We will now consider possible chords of the associated odd cycle. The vertices of Q' are not adjacent to further vertices of C or to a vertex of Q'' since they are contained in the interior of the contractible cycle w, q, a', b', w . If a vertex of Q'' is adjacent to a vertex of C , take the smallest induced cycle in C' and C'' that contains w . Then, again one of the cycles is odd. If the odd and induced cycle contains at least three vertices of Q'' , then it forms a loose odd wheel with z . Otherwise, the neighbour $q_1'' \in V(Q'')$ of w is adjacent to a vertex of C and forms an induced odd cycle. Then, either q_1'' has three odd neighbours on C or the cycle using Q' is odd. In both cases, we obtain a loose odd wheel with center vertex b' .

We now treat Case C, ie we assume that v_0 is not adjacent to b . Recall that v_1 is the vertex adjacent to v_0 on $P_{C,a'-v_0,b'}$. Let $\tilde{v} \neq a'$ be adjacent to v on C . There are two paths between a' and \tilde{v} along $HC(v)$. In all of the four possible embeddings of bv_0 and bv_1 (see Figure 3.10), one of the paths is contained in a region whose boundary does not contain a . Depending on the embedding, denote this path by $P = w_0, w_1, \dots, w_k$ with $a' = w_0$ and $w_k = \tilde{v}$. Let $j \in \{1, \dots, k\}$ be the first index such that w_j is adjacent to a vertex of $C - \{a', \tilde{v}\}$ (if such a chord exists, otherwise set $j = k$). Let w' be the vertex closest to v_1 along $P_{C,a'-v_1,v_0}$ that is adjacent to w_j . Note that the cycles formed by $a' = w_0, w_1, \dots, w_j$ together with $P_{C,w'-w,\tilde{v}} \cup \{a, x\}$ respectively with $P_{C,w'-a',\tilde{v}}$ are of different parity. The only possible chords of the two cycles are edges between a and $V(C)$. If ac is a chord with $c \in V(C)$, then a has three odd neighbours on C or the arising subcycle that contains $x, a' = w_0, w_1, \dots, w_j$ is odd. This cycle yields an odd wheel with center vertex \tilde{v} if $j \geq 2$. If $j = 1$, the cycle formed by a' and w_1 together with $P_{C,w'-w,\tilde{v}} \cup \{a, x\}$ is odd or w_1 has three odd neighbours on C . In both cases, G contains a loose odd wheel. \square

Claim 3.21. *Let the a' - w -path along $HC(b')$ that contains a be odd and of length at least 5. If G' is perfect, then G contains a loose odd wheel.*

Proof. The path $P_{HC(b'),a-w,x}$ connecting a and w along $HC(b')$ is odd and has length at least 3. First note that we can assume that b is neither contained in $P_{HC(b'),a-w,x}$ nor adjacent to a vertex of $HC(b')$ (see (3.6) and (3.5)). The path $P_{HC(b'),a-w,x}$ together with b and $P_{C,v_0-w,b'}$ forms an odd cycle C' . If C' is induced in G , it forms a loose odd wheel together with b' .

Thus, suppose that the cycle has chords. With the same arguments as for (3.16) we can show the following. Let $p \in V(P_{HC(b'),a-w,x}) \setminus \{a\}$ and $c \in V(C) \setminus \{b'\}$ be adjacent in G and let the path $P_{C,v_0-c,a'}$ be even. Then, G contains a loose odd wheel. As before, G does not contain a loose odd wheel and we can conclude that

$$G \text{ has no triangle with vertices of } V(P_{HC(b'),a-w,x}) \setminus \{a\} \text{ and } V(C). \quad (3.17)$$

Suppose there is a chord from a to a vertex of C . Let c be the vertex adjacent to a that is closest to b' on $P_{C,b'-v_0,a'}$. If $P_{C,w-c,a'}$ is odd, we get an odd cycle with $a, V(P_{HC(b'),a-w,x})$ and $V(P_{C,w-c,a'})$. If the cycle is induced we obtain a loose odd wheel with center vertex b' . Otherwise, the smallest subgraph is an odd hole or a triangle, and we are done by (3.17). Thus, we can assume that $P_{C,w-c,a'}$ is even.

Let R be the path connecting b' and z along $HC(w)$ such that an edge between a vertex of R and a always gives a contractible cycle. Let $C_{R,a}$ be the cycle formed by $V(P_{C,b'-c,a'}) \setminus \{w\}$, $V(R)$ and a , and let $C_{R,x}$ be the cycle formed by $V(R)$, x, b and $P_{C,v_0-z,a'}$. One of the cycles $C_{R,a}$ and $C_{R,x}$ is odd. If this cycle is induced, G contains a loose odd wheel with w as center.

We now consider possible chords in the two cycles. The vertex b cannot be adjacent to a vertex of R as $ac \in E(G)$. Further, a cannot be adjacent to a vertex of $C \cap C_{R,a}$, by the definition of ac . The only possible chords are edges from R to C , edges from R to a and edges from C to b . If there is a chord between C and b , then b has three odd neighbours on C (and G has a loose odd wheel) or there is still an induced odd cycle containing b, x and $V(R)$. A chord from a vertex of R to C leads to an odd hole either in $C_{R,a}$ or in $C_{R,x}$. If the odd hole contains three vertices of R , we are done. Otherwise, the neighbour r_1 of b' on R is contained in a chord from R to C . But then, either r_1 has three odd neighbours in C or G has an odd hole that contains x . In both cases we obtain a loose odd wheel (see Observation 3.9).

Assume that there is a chord from a to a vertex r of R . Recall that G has a 3-colouring by Observation 3.11. Thus, in G , the colours of the vertices on a Hamilton cycle $HC(u)$ alternate for every $u \in V(G)$. If $P_{HC(w),r-b',z}$ is odd, this means that r and b' have different colours. As a and w are adjacent to r and b' , the vertices a and w then have the same colour. This contradicts our assumption that the a' - w -path along $HC(b')$ that contains a is odd and shows that $P_{HC(w),r-b',z}$ is even. Consequently, $C_{R,a}$ has an odd induced subcycle that contains a . This cycle is not affected by the even-contraction and we are done if this cycle is of length at least five.

Assume that this odd cycle is a triangle. Then, there is a vertex $r \in R$ with $ra \in E(G)$ and $rc \in E(G)$ where c is the vertex adjacent to a that is closest to

b' on $P_{C,b'-v_0,a'}$ (as described in the beginning of the subcase). Choose r such that its distance to b' on the path R is minimal. Since $P_{C,b'-c,w}$ is even, the vertices of $P_{C,b'-c,w}$ together with the vertices of $P_{R,b'-r,z}$ form an odd cycle $C_{r,R}$. If this cycle is induced, it yields an odd wheel with center vertex w . The vertices of $C_{r,R} \cap C$ cannot form chords, since C is induced. The vertices of $R - r$ cannot be adjacent to further vertices of $C_{r,R} \cap C$ since they lie in a contractible cycle which is closed by ra . If there is a chord from r to a vertex of $C_{r,R} \cap C$, then either this edge or the edge ac form a contractible cycle that includes a part of the Hamilton cycle of z — the path R' . Thus, in that case no vertex of R' is adjacent to a or b . We obtain two cycles of different parity: the cycle with vertices $V(R')$, b' , x , b and vertices of C , and the cycle with vertices $V(R')$, b' , a and vertices of C . Both cycles can contain chords from R' to C . But the, one of the induced cycles that includes b' is odd and of length at least 5. Consequently, we obtain a loose odd wheel.

Finally, if there is no chord from a to a vertex of C , the only possible chords that can occur in C' are edges from a vertex $r \in V(P_{HC(b'),a-w,x})$ or from b to a vertex of C . If there is a chord bc with $c \in V(C)$, then b has either three odd neighbours on C or there is still an odd cycle of length at least 5 containing $V(P_{HC(b'),a-w,x})$, a and b .

Suppose there is a chord from $r \in V(P_{HC(b'),a-w,x})$ to a vertex of C . If the induced cycle in C' that contains a and b is even, there is an odd cycle with vertices of $P_{HC(b'),a-w,x}$ and C . Then, as we have seen in (3.17), G contains a loose odd wheel. If the induced cycle in C' that contains a and b is odd and contains at least three vertices of $P_{HC(b'),a-w,x}$, then it forms a loose odd wheel with center vertex b' . Otherwise, there is a vertex $r_1 \in V(P_{HC(b'),a-w,x})$ with $ar_1 \in E(G)$ that is adjacent to a vertex c_r of C such that $P_{C,c_r-w,b'}$ is odd. Choose c_r such that $P_{C,c_r-w,b'}$ is of maximal length. The path $P_{C,c_r-v_0,b'}$ is also odd and forms an odd cycle together with b, a and r_1 . The vertex r_1 cannot be contained in a chord of this cycle by choice of c_r and by (3.5). Further, there is no chord from a to a vertex of C . The vertex b can be adjacent to vertices of C . But then, b either has three odd neighbours or there is an odd hole that contains b, a and r_1 . If the hole is contractible, Theorem 3.8 assures that G contains a loose odd wheel. Otherwise, the edge r_1c_r closes a contractible cycle containing a part of the Hamilton cycle of w — the path R'' . Using the fact that the vertices b and a' do not lie in the interior of this cycle, we get a loose odd wheel with R'' , the vertices of C , and a' (respectively $\{x, b\}$) similar to the cases we have seen before. \square

This finishes the proof of Lemma 3.18. \square

3.6.2. Even-Contraction if b' Is Contained in the Odd Hole

Lemma 3.22. [FG17] *Let G be a nice Eulerian triangulation. Let G' be obtained from G by an even-contraction at x and let $\{x, a, a', b, b'\} \cap V(C)$ equal $\{b'\}$. If G contains an odd hole C and G' is perfect, then G contains a loose odd wheel.*

Proof. Suppose that G' is perfect. Then, every odd induced subcycle of $\gamma(C) = K$ in G' is a triangle. As all chords of $\gamma(C)$ contain b , there are two possibilities: the

triangle may contain one or two chords.

If the triangle contains one chord, then b is adjacent to a vertex v_1 of distance 2 from b' on C . The vertices y, v_1 and the common neighbour v_2 of v_1 and y on K now give a triangle. In this case, the vertices b', x, b, v_1 and v_2 form an odd cycle in G . Since C is induced and b is not adjacent to v_2 (see (3.5)), this cycle is induced. It forms a loose odd wheel together with a .

If the triangle is of the form y, c_1, c_2, y where yc_1 and yc_2 are chords of K , then, bc_1 and bc_2 are edges in G . Without loss of generality, we choose c_1 and c_2 in such a way that $P_{C, b'-c_1, c_2}$ is of minimal length. Note that this choice implies that

$$P_{C, b'-c_1, c_2} \text{ is odd.} \quad (3.18)$$

Otherwise, $P_{C, b'-c_1, c_2}$ forms an odd cycle together with b and x which has an odd subcycle of length at least 5.

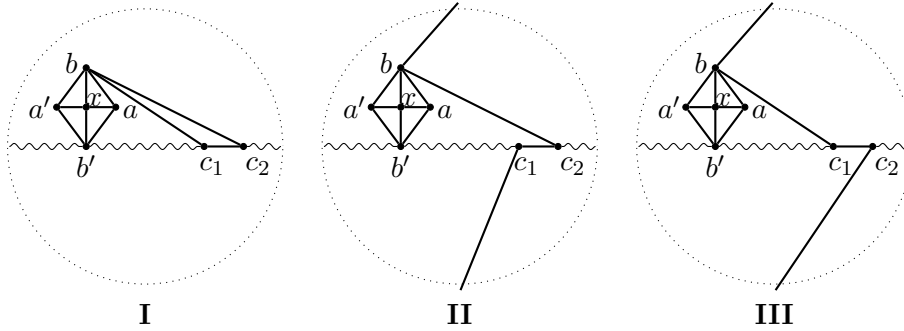


Figure 3.11.: The three possible embeddings of the edges bc_1 and bc_2 in G

There are three ways of embedding the edges bc_1 and bc_2 (up to switching the vertices c_1 and c_2 and the vertices a and a' and up to topological isomorphism). They are shown in Figure 3.11 and treat the (non-)contractibility of the cycles $P_{C, b'-c_1, c_2} \cup \{b'a, ab, bc_1\}$ and $P_{C, b'-c_2, c_1} \cup \{b'a, ab, bc_2\}$. Note that C is non-contractible by Theorem 3.8. If a and a' have no common neighbour besides $\{b, b', x\}$, then we can switch the roles of b, b' with the roles of a, a' . As we have seen in (3.13), this means C is not affected by the even-contraction. Thus, we can assume that a and a' have a common neighbour besides b, x, b' .

First suppose that the graph is embedded as in III (see Figure 3.11). Then the cycle $P_{C, b'-c_1, c_2} b'x bc_1$ is contractible and contains a in its interior. Further the cycle $P_{C, b'-c_2, c_1} b'x bc_2$ is contractible and contains a' in its interior. For this reason, a common neighbour of a and a' must be contained in the intersection of both cycles, ie in $\{b, b'\}$ and a and a' have no further common neighbour if the graph is embedded as in III.

Next, suppose that the graph is embedded as in I (see Figure 3.11). Similar to the arguments before, one can see that all common neighbours of a and a' besides b, x, b' lie on the path $P_{C, b'-c_1, c_2}$.

3. On t -Perfect Triangulations of the Projective Plane

First assume that there is a vertex v' in $P_{C,b'-c_1,c_2}$ that is adjacent to a and that there is a vertex w on $P_{C,v'-b',c_1}$ that is not adjacent to a . There are two paths along $HC(w)$ that connect v' and z . Let P' be the path that lies in the contractible cycle formed by $P_{C,b'-v',c_1}$ and a . Consider the cycle C_1 formed by P' together with $P_{C,b'-z,w}$, the vertices x, b and the path $P_{C,c_1-v',w}$. Further, consider the cycle C_2 formed by P' together with $P_{C,v'-z,w}$. Notice that either C_1 or C_2 is odd. If the odd cycle is induced, we get a loose odd wheel with center vertex x respectively w . Assume that the odd cycle contains chords. If there is a chord from b to a vertex of $P_{C,b'-c_1,c_2}$ (in the case if C_1 is odd), the vertex b has three odd neighbours on C or there still remains an induced odd cycle containing b, x . If there is a chord from a vertex p of P' to a vertex q of C , consider the induced cycle C'_i in C_i which contains v' for $i = 1, 2$. Either C'_1 or C'_2 is odd. If C'_1 is an odd induced cycle, we get a loose odd wheel by Observation 3.9. Otherwise, C'_2 is an odd cycle. If w has at least three neighbours on C'_2 we get a loose odd wheel. If w has only two neighbours on C'_2 , it directly follows that p has three odd neighbours on C , namely v', w and q .

Now assume, that no vertex v' as above exists. Then, either a has three odd neighbours (and we get a loose odd wheel) or the only vertex in the interior of $P_{C,b'-c_1,c_2}$ that is adjacent to a equals the neighbour v of b' . But then b, a and $P_{C,v-c_1,c_2}$ form an odd cycle C' . Note that all chords of C' have b as an endvertex. Further, either b has three odd neighbours on C (and G contains a loose odd wheel) or all chords bw satisfy that P_{C,c_1-w,c_2} is of even length. In the second case, every odd induced subcycle of C' contains a and b . If the largest induced odd subcycle of C' is of length at least 5, then G has a further odd hole which contains a and b . Then, G contains a loose odd wheel by Lemma 3.18. Otherwise, b is adjacent to v which contradicts (3.5).

Last, suppose that the graph is embedded as in II (see Figure 3.11). In this case, one can see that c_1 or c_2 are the only possible common neighbours of a and a' besides b, x, b' . If c_1 is adjacent to a and a' , we obtain an odd cycle formed by a' and $P_{C,b'-c_1,c_2}$. This means that G has an odd hole which contains a' and b' . If c_2 is the common neighbour of a and a' , we obtain the cycle consisting of $P_{C,b'-c_2,c_1}$ and a that contains a and b' . Both cycles are odd (see (3.18)), of length at least 5 (see (3.5)), and induced (as C is induced and the embedding does not allow any chords). With Lemma 3.18, the proof is finished. \square

Part II.

On Edge Colourings

4. Introduction to Edge Colourings and Tree Decompositions

In this chapter, we provide basic definitions and knowledge about the types of edge colourings addressed in Chapter 5 and 6. Additionally, we introduce tree decompositions in Section 4.4.

4.1. Edge Colouring Basics

An *edge colouring* of a graph G is an assignment of colours to the edges of G such that adjacent edges receive different colours. Of particular interest are *minimum edge colourings* — edge colourings containing the smallest possible number of colours. The *chromatic index* or *edge chromatic number* $\chi'(G)$ is the number of colours of a minimum edge colouring.

It is clear that an edge colouring of G contains at least maximum degree $\Delta(G)$ many colours. Perhaps the most celebrated and important result on edge colourings is the following theorem due to Vizing.

Theorem 4.1 (Vizing [Viz65a]). *For any graph G holds $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

We call G a *class I* graph if $\chi'(G) = \Delta(G)$; if $\chi'(G) = \Delta(G) + 1$, then G is of *class II*. It is a difficult algorithmic problem to decide whether G is of class I or class II.

For an edge colouring $c : E(G) \rightarrow \{1, \dots, k\}$ the colour class $c^{-1}(i)$ (with $i \in \{1, \dots, k\}$) is the set of all edges of G that are coloured with i . All colour classes of an edge colouring are matchings. Thus, every colour class contains at most $\lfloor |V(G)|/2 \rfloor$ edges. Consequently, a graph G with $\Delta(G) \cdot \lfloor |V(G)|/2 \rfloor < |E(G)|$ cannot have an edge colouring with $\Delta(G)$ colours. Such a graph is called *overfull*. Note that the property implies that an overfull graph has an odd number of vertices.

Chetwynd and Hilton [CH86] conjectured¹:

Conjecture 4.2 (Overfull Conjecture). *Let $G = (V, E)$ be a simple graph with $\Delta(G) > 1/3 \cdot |V|$. Then G is of class II if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

The conjecture is verified for a few graph classes (see [Hil87]) such as graphs with $\Delta(G) \geq |V| - 3$.

To analyse classes of graphs with regard to their chromatic index, it is useful to consider Δ -critical graphs. A graph G of maximum degree Δ is Δ -critical, if all proper subgraphs have an edge colouring using no more than Δ colours.

Vizing's popular adjacency lemma asserts:

¹Originally, they conjectured a bound of $3/10$.

Theorem 4.3 (Vizing’s adjacency lemma). *Let uv be an edge in a Δ -critical graph. Then v has at least $\Delta - \deg(u) + 1$ neighbours of degree Δ .*

Zhang [Zha00] and Sanders and Zhao [SZ01] supplemented this with slightly more complicated adjacency lemmas:

Theorem 4.4 (Zhang [Zha00]). *Let G be a Δ -critical graph, and let uvw be a path in G . If $\deg(u) + \deg(w) = \Delta + 2$ then all neighbours of v but u and w have degree Δ .*

Theorem 4.5 (Sanders and Zhao [SZ01]). *Let G be a Δ -critical graph, and let v be a common neighbour of u and w such that $\deg(u) + \deg(v) + \deg(w) \leq 2\Delta + 1$. Then there are at most $\deg(u) + \deg(v) - \Delta - 3$ common neighbours $x \neq u$ of v and w .*

A 1-factorisation of a graph $G = (V, E)$ is a partition of the edge set into perfect matchings. (A *perfect matching* is a set of $|V|/2$ edges, no two of which share an endvertex.) Such factorisations are closely linked to edge colourings: indeed, a d -regular graph G has a 1-factorisation if and only if its edge set can be coloured with d colours. That is, the chromatic index is equal to d .

4.2. Fractional Edge Colourings

Fractional edge colourings can be seen as a relaxation of edge colourings. A *fractional edge colouring* is an assignment of a non-negative weight λ_M to each matching M of G such that for every edge $e \in E(G)$ holds $\sum_{M \ni e} \lambda_M \geq 1$. The *fractional chromatic index* of a graph G is defined as

$$\chi'_f(G) = \min \left\{ \sum_{M \in \mathcal{M}} \lambda_M : \lambda_M \in \mathbb{R}_+, \sum_{M \in \mathcal{M}} \lambda_M \mathbb{1}_M(e) = 1 \quad \forall e \in E(G) \right\},$$

where \mathcal{M} denotes the collection of all matchings in G and $\mathbb{1}_M$ the characteristic vector of M . The fractional chromatic index satisfies $\chi'_f(G) \leq \chi'(G)$. If $\chi'_f(G)$ is greater than $\Delta(G)$ then clearly G is class II. It follows from Edmonds’ matching polytope theorem that the fractional chromatic index of a graph on at least three vertices satisfies

$$\chi'_f(G) = \max\{\Delta(G), \Lambda(G)\} \quad \text{where } \Lambda(G) = \max_H \frac{2|E(H)|}{|V(H)|-1}$$

and the maximisation is over all induced subgraphs H of G on an odd number $|V(H)| \geq 3$ of vertices that have maximum degree $\Delta(G)$ (see eg [Thm. 28.5] [Sch02]). With this, it is not hard to see that a graph G that does not contain an overfull subgraph H with $\Delta(H) = \Delta(G)$ has fractional chromatic index $\chi'_f(G) = \Delta(G)$. For more details on the fractional chromatic index, see for instance Scheinerman and Ullman [SU13].

4.3. List Edge-Colourings

List edge-colourings generalise edge colourings. Given lists L_e of allowed colours at every edge $e \in E(G)$, the task is to colour the edges of G so that every edge e receives

a colour from its list L_e . The *choice index* (or *list chromatic index*) of G is the smallest number ℓ so that any collection of lists L_e of size ℓ permits a list colouring.

Clearly, the choice index is at least as large as the chromatic index. The famous *list colouring conjecture* asserts that the two indices are in fact the same:

Conjecture 4.6 (List colouring conjecture). *The chromatic index of every simple graph equals its list chromatic index.*

Similar to list edge-colourings, one can define a list vertex colouring and a list chromatic number. The analogue conjecture for list vertex colourings is not true. The graph $K_{3,3}$ is bipartite, ie has a 2-colouring. Nevertheless, there is a choice of lists of size 2 for every vertex such that the graph cannot be coloured from these lists.

4.4. Tree Decompositions

For a graph G a *tree decomposition* (T, \mathcal{B}) consists of a tree T and a collection $\mathcal{B} = \{B_t : t \in V(T)\}$ of *bags* $B_t \subset V(G)$ such that

- (i) $V(G) = \bigcup_{t \in V(T)} B_t$,
- (ii) for each edge $vw \in E(G)$ there exists a vertex $t \in V(T)$ such that $v, w \in B_t$, and
- (iii) if $v \in B_s \cap B_t$, then $v \in B_r$ for each vertex r on the path between s and t in T .

A tree decomposition (T, \mathcal{B}) has *width* k if each bag has a size of at most $k + 1$. The *treewidth* of G is the smallest integer k for which there is a width k tree decomposition of G .

A tree decomposition (T, \mathcal{B}) of width k is *smooth* if

- (iv) $|B_t| = k + 1$ for all $t \in V(T)$ and
- (v) $|B_s \cap B_t| = k$ for all $st \in E(T)$.

A graph with treewidth of at most k always has a smooth tree decomposition of width k ; see Bodlaender [Bod98].

5. List Edge-Colouring in Generalised Petersen Graphs

The content of Section 5.1, 5.2, and 5.3 is based on the paper [BGG17] by Henning Bruhn, Jacob Günther, and the author of this thesis. Section 5.4 is inspired by the previous sections. The results are unpublished and work by the author of this thesis.

The first three sections serve to prove that the number of 1-factorisations of a generalised Petersen graph of the type $GP(3k, k)$ is equal to the k th Jacobsthal number $J(k)$ if k is odd and equal to $4J(k)$ if k is even. Moreover, we verify the list colouring conjecture for $GP(3k, k)$.

In the last section, we verify the list colouring conjecture for $GP(4k, k)$.

5.1. Introduction

Often, combinatorial objects that on the surface seem quite different nevertheless exhibit a deeper, somewhat hidden, connection. This is, for instance, the case for tilings of $3 \times (k - 1)$ -rectangles with 1×1 and 2×2 -squares [Heu99], certain meets in lattices [DKW79], and the number of walks of length k between adjacent vertices in a triangle [Bar07]: in all three cases the cardinality is equal to the k th *Jacobsthal number*. Their sequence

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341 \dots$$

is defined by the recurrence relation $J(k) = J(k - 1) + 2J(k - 2)$ and initial values $J(0) = 0$ and $J(1) = 1$. Jacobsthal numbers also appear in the context of alternating sign matrices [FS00], the Collatz problem and in the study of necktie knots [FM00]; see [Slo, A001045] for much more.

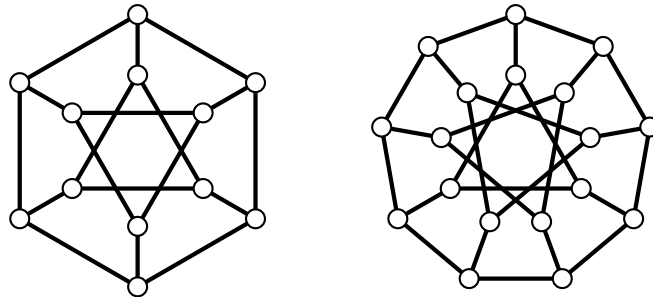


Figure 5.1.: The Dürer graph $GP(6, 2)$ and the generalised Petersen graph $GP(9, 3)$

In this article, we add to this list by describing a relationship to certain *generalised Petersen graphs* $GP(3k, k)$. These graphs arise from matching k disjoint triangles to triples of equidistant vertices on a cycle of $3k$ vertices; see below for a precise definition and Figure 5.1 for two examples.

Theorem 5.1. [BGG17] *For odd k , the number of 1-factorisations of the generalised Petersen graph $GP(3k, k)$ equals the Jacobsthal number $J(k)$; for even k , the number is equal to $4J(k)$.*

The famous List colouring conjecture (Conjecture 4.6) asserts that the the choice index of a graph equals its chromatic index. While the conjecture (Conjecture 4.6) has been verified for some graph classes, regular planar graphs [EG96] and bipartite graphs [Gal95] for instance, the conjecture remains wide open for most graph classes, among them cubic graphs. We prove:

Theorem 5.2. [BGG17] *The list-colouring conjecture is true for generalised Petersen graphs $GP(3k, k)$.*

Our proof is based on the algebraic colouring criterion of Alon and Tarsi [AT92]. In our setting, it suffices to check that, for a suitable definition of a sign, the number of positive 1-factorisations differs from the number of negative 1-factorisations. In this respect our second topic ties in quite nicely with our first, and we will be able to re-use some of the observations leading to Theorem 5.1.

Generalised Petersen graphs were first studied by Coxeter [Cox50]. For $k, n \in \mathbb{N}$ with $k < \frac{n}{2}$, the graph $GP(n, k)$ is defined as the graph on vertex set $\{u_i, v_i : i \in \mathbb{Z}_n\}$ with edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i \in \mathbb{Z}_n\}$. Generalised Petersen graphs are cubic graphs. All of them, except the Petersen graph itself, have chromatic index 3; see Watkins [Wat69], and Castagna and Prins [CP72]. In particular, this means that the list colouring conjecture for them does not follow from the list version of Brooks' theorem. We focus in the first three sections (Section 5.1, 5.2, and 5.3) on the graphs $GP(3k, k)$, the smallest of which, $GP(6, 2)$, is called the *Dürer graph*. Section 5.4 will deal with the graphs $GP(4k, k)$ and will use the ideas explained and used in the first three sections.

5.2. Counting 1-Factorisations

In the rest of the article we consider a fixed generalised Petersen graph $GP(3k, k)$. The *outer cycle* C_O of $GP(3k, k)$, the cycle $u_0 u_1 \dots u_{3k-1} u_0$, and the *spokes*, the edges $u_i v_i$ for $i = 0, \dots, 3k - 1$, will play a key role. The indices of u_i and v_i are taken modulo $3k$. See Figure 5.2 for an illustration.

Our objective is to count the number of 1-factorisations of $GP(3k, k)$. Rather than counting them directly, we will consider edge colourings, and here we will see that it suffices to focus on certain edge colourings of the outer cycle.

Let ϕ be an edge colouring with colours $\{1, 2, 3\}$ of either the whole graph $GP(3k, k)$ or only of the outer cycle C_O . We split ϕ into k triples

$$\phi_i = (\phi(u_i u_{i+1}), \phi(u_{k+i} u_{k+i+1}), \phi(u_{2k+i} u_{2k+i+1})) \text{ for } i = 1, \dots, k.$$

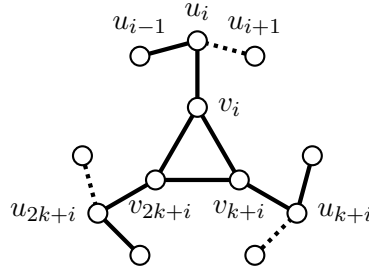


Figure 5.2.: A small part of $GP(3k, k)$; the colours of the dotted edges make up ϕ_i

To keep notation simple, we will omit the parentheses and commas, and only write $\phi_i = 123$ to mean $\phi_i = (1, 2, 3)$. We define $\phi_{k+1} = (\phi(u_{k+1}u_{k+2}), \phi(u_{2k+1}u_{2k+2}), \phi(u_1u_2))$, and note that ϕ_{k+1} is obtained from ϕ_1 by a cyclic shift.

It turns out that the colours on the outer cycle already uniquely determine the edge colouring on the whole graph. Moreover, it is easy to describe which colourings of the outer cycle extend to the rest of the graph:

Lemma 5.3. [BGG17] *Let $\phi : E(C_O) \rightarrow \{1, 2, 3\}$ be an edge colouring of C_O . Then the following two statements are equivalent:*

- (a) *there is an edge colouring γ of $GP(3k, k)$ with $\gamma|_{C_O} = \phi$; and*
- (b) *there is a permutation (a, b, c) of $(1, 2, 3)$ so that ϕ_i and ϕ_{i+1} are for all $i = 1, \dots, k$ adjacent vertices in one of the graphs T and H in Figure 5.3.*

Furthermore, if there is an edge colouring γ of $GP(3k, k)$ as in (a) then it is unique.

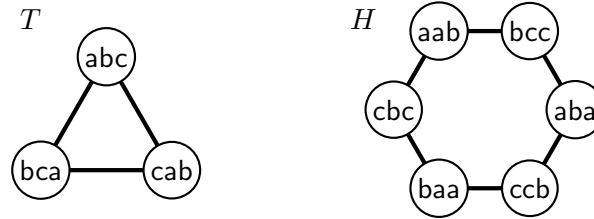


Figure 5.3.: The graphs T and H capture the possible combinations of consecutive colour triples

Proof. First assume (a), that is, that there is an edge colouring γ of $GP(3k, k)$ with $\gamma|_{C_O} = \phi$. Note that for all $i \in \mathbb{Z}_{3k}$

$$\text{the spokes } u_i v_i, u_{i+k} v_{i+k} \text{ and } u_{i+2k} v_{i+2k} \text{ receive distinct colours.} \quad (5.1)$$

Indeed, the three edges of the triangle $v_i v_{i+k} v_{i+2k} v_i$ need to be assigned three different colours, which then also must be the case for the corresponding spokes.

From (5.1) follows that no ϕ_i is monochromatic, i.e., that $\phi_i \notin \{111, 222, 333\}$. If there was such an ϕ_i , say $\phi_i = 111$, then none of the spokes $u_i v_i$, $u_{i+k} v_{i+k}$ and $u_{i+2k} v_{i+2k}$ could be coloured with 1 under γ .

In particular, ϕ_1 will contain at least two distinct colours and we can choose distinct a , b and c so that $\phi_1 \in \{abc, aab, aba, baa\}$. Then ϕ_1 is a vertex of either T or H . Consider inductively ϕ_i to be such a vertex as well. By rotational symmetry of $GP(3k, k)$ and permutation of colours, we may assume that $\phi_i \in \{abc, aab\}$.

If $\phi_i = \{abc\}$ then, by (5.1), the spokes $u_i v_i$, $u_{i+k} v_{i+k}$ and $u_{i+2k} v_{i+2k}$ can only be coloured b, c, a (in that order) or c, a, b . In the first case, the colour of the edge $u_{i+1} u_{i+2}$ needs to be c , and so on, resulting in $\phi_{i+1} = cab$. In the other case, we get $\phi_{i+1} = bca$. Both of these colour triples are adjacent to ϕ_i in T . The proof for $\phi_i = aab$ is similar.

For the converse direction suppose now that (b) holds. Consider a pair of colour triples ϕ_i and ϕ_{i+1} . By rotation symmetry of $GP(3k, k)$ and by symmetry of the three colours, we only need to check the cases that

$$(\phi_i, \phi_{i+1}) = (abc, bca) \text{ and } (\phi_i, \phi_{i+1}) = (aab, cbc).$$

In the first case, the spokes $u_i v_i$, $u_{i+k} v_{i+k}$ and $u_{i+2k} v_{i+2k}$ can be coloured with c , a and b (in that order), which then permits to colour the triangle $v_i v_{i+k} v_{i+2k}$ with abc . Observe that neither for the spokes nor for the triangle there was an alternative colouring. For the other case, colour the spokes with bca and then the triangle accordingly. Again, all the colours are forced. Extending the colouring ϕ in this way for all i yields an edge colouring γ of all of $GP(3k, k)$, and as all colours are forced, γ is uniquely determined by ϕ . \square

The lemma implies : Any edge colouring γ of $GP(3k, k)$ corresponds to a walk $\gamma_1 \gamma_2 \dots \gamma_{k+1}$ of length k in either T or in H . Where does such a walk start and end? By symmetry, we may assume that the walk starts at $\gamma_1 = abc$ or $\gamma_1 = aab$. It then ends in γ_{k+1} , which is either bca or aba . Conversely, all such walks define edge colourings of $GP(3k, k)$.

To count the number of these walks, consider two vertices x, y of T , respectively of H , that are at distance ℓ from each other in T (respectively in H). We define

$$\begin{aligned} t_k(\ell) &:= \# \{ \text{walks of length } k \text{ between } x \text{ and } y \text{ in } T \} \\ h_k(\ell) &:= \# \{ \text{walks of length } k \text{ between } x \text{ and } y \text{ in } H \} \end{aligned}$$

Then every edge colouring of $GP(3k, k)$ corresponds to a walk that is either counted in $t_k(1)$ (as abc and bca have distance 1 in T) or counted in $h_k(2)$.

Lemma 5.4. [BGG17] *The number of 1-factorisations of $GP = GP(3k, k)$ equals $t_k(1) + 3h_k(2)$.*

Proof. First, we note that 1-factorisations are basically the same as edge colourings $\gamma : E(GP) \rightarrow \{1, 2, 3\}$ where colours of some edges are restricted. In more detail, there is a bijection between the 1-factorisations and the edge colourings $\gamma : E(GP) \rightarrow \{1, 2, 3\}$

where u_1u_2 is coloured with 1, $u_{1+k}u_{2+k}$ coloured with 1 or 2, and $u_{1+2k}u_{2+2k}$ is only coloured with 3 if $u_{1+k}u_{2+k}$ was coloured with 2. Note that 111 is no possible choice. By Lemma 5.3, the number of such γ is equal to the number of edge colourings $\phi : E(C_O) \rightarrow \{1, 2, 3\}$ satisfying (b) of Lemma 5.3 and for which $\phi_1 \in \{123, 112, 121, 211\}$.

How many such edge colourings ϕ are there with $\phi_1 = 123$? Since $\phi_{k+1} = 231$, Lemma 5.3 implies that this number is $t_k(1)$. Each of the numbers of edge colourings ϕ with $\phi_1 \in \{112, 121, 211\}$ is equal to $h_2(2)$, which means that, in total, we get $t_k(1) + 3h_k(2)$ edge colourings. \square

We need the closed expression of the Jacobsthal numbers:

$$J(k) = \frac{1}{3} \left(2^k + (-1)^{k+1} \right) \quad \text{for every } k \geq 0. \quad (5.2)$$

Lemma 5.5. *For any k*

$$t_k(0) = \frac{1}{3}(2^k + 2(-1)^k) \text{ and } t_k(1) = J(k)$$

The second equation can be found in [Bar07]. Since it follows directly from the first one, we will still include a proof.

Proof. A classic question in algebraic graph theory is to count the number of pairs (W, v) for a graph G , where W is a closed walk of length k in a graph G and v the first vertex of W . It turns out, see for instance [BW04, Section 1.4], that this number is equal to $\lambda_1^k + \dots + \lambda_n^k$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix of G . For the triangle these eigenvalues are 2 and twice -1 (see eg [BW04, Section 1.2]). Our aim, however, is to count the number of closed walks of length k that start at a specific vertex, which means that we have to divide by 3. This gives $t_k(0) = \frac{1}{3}(2^k + 2(-1)^k)$ for every k , and thus the first assertion of the lemma.

Since every walk on $k + 1$ edges has to visit a vertex adjacent to the end vertex after the k -th step, for which there are two possibilities, we obtain $t_{k+1}(0) = 2t_k(1)$ and thus

$$2t_k(1) = t_{k+1}(0) = \frac{1}{3} \left(2^{k+1} + 2(-1)^{k+1} \right)$$

By (5.2), we get $2t_k(1) = 2J(k)$ for every k . \square

For the next lemma, we label the vertices of T as x_0, x_1, x_2 in clockwise order. The vertices of H are $y_0, y_1, y_2, z_0, z_1, z_2$ in clockwise order.

Lemma 5.6. [BGG17] *If k is even, there is a bijection between the set of walks of length k from x_0 to x_2 in T and the set of walks from y_0 to y_2 in H . Moreover, the ℓ th edge of the walk in T is traversed in clockwise direction if and only if this is also the case for ℓ th edge of the corresponding walk in H .*

The projection of the set $\{y_i, z_i\}$ in H to the set $\{x_i\}$ in T for $i \in \mathbb{Z}_3$ yields a covering map. The bijection between the considered walks of length k in T and H follows immediately by the path lifting property for covering spaces (see eg [Hat02, Section 1.3]). We give, nevertheless, an elementary proof of the lemma.

Proof of Lemma 5.6. Every vertex with index $i \in \mathbb{Z}_3$ in H is adjacent to exactly one vertex with index $i + 1$ in clockwise direction and one with index $i - 1$ in counter-clockwise direction. Therefore, the following rule translates a walk $W = w_0 \dots w_k$ in T starting in x_0 to a walk $W' = w'_0 \dots w'_k$ in H starting in y_0 while maintaining the directions of edge traversals: let $w'_0 = y_0$; for $\ell = 1, \dots, k$, if $w_\ell = x_i$ then pick w'_ℓ to be the neighbour of $w'_{\ell-1}$ among y_i, z_i .

As $w_k = x_2$, the last vertex w'_k of W' has to be one of y_2, z_2 . Since k is even and the distance between y_0 and z_2 in H is odd, W' must terminate in y_2 . Clearly, the described rule yields a bijection. \square

Proof of Theorem 5.1. By Lemma 5.4 the number of 1-factorisations of $GP(3k, k)$ is $t_k(1) + 3h_k(2)$. Lemma 5.6 yields that $h_k(2)$ equals $t_k(1)$ for even k . Since there is no walk of odd length in H that connects two vertices of even distance, $h_k(2)$ is zero for odd k . By Lemma 5.5 the number of 1-factorisations now equals $J(k) + 3J(k)$ if k is even and $J(k) + 3 \cdot 0$ otherwise. \square

5.3. List Edge-Colouring

In order to show the list edge-colouring conjecture for $GP(3k, k)$, we will use the method of Alon and Tarsi [AT92], or rather its specialisation to regular graphs [EG96].

To define a local orientation, we consider $GP(3k, k)$ always to be drawn as in Figure 5.1: the vertices u_i for $i = 1, \dots, 3k$ are placed on an outer circle in clockwise order, the vertices v_i for $i = 1, \dots, 3k$ on a smaller concentric circle in such a way that u_i and v_i match up, and all edges are straight. We define the sign of γ at a vertex w as $+$ if the colours 1, 2, 3 appear in clockwise order on the incident edges; otherwise the sign is $-$. More formally,

$$\begin{aligned} \text{sgn}_\gamma(u_i) &= \begin{cases} + & \text{if } (\gamma(u_{i-1}u_i), \gamma(u_iu_{i+1}), \gamma(u_iv_i)) \in \{123, 231, 312\} \\ - & \text{otherwise.} \end{cases} \\ \text{sgn}_\gamma(v_i) &= \begin{cases} + & \text{if } (\gamma(v_{k+i}v_i), \gamma(v_iv_{2k+i}), \gamma(v_iu_i)) \in \{123, 231, 312\} \\ - & \text{otherwise} \end{cases} \end{aligned}$$

The sign of the colouring γ is then

$$\text{sgn}(\gamma) := \prod_{v \in V(GP(3k, k))} \text{sgn}_\gamma(v).$$

Permuting colours in our context does not change the sign of an edge colouring. This is true in all regular graphs, see for instance [EG96]:

Lemma 5.7. [BGG17] *Let G be a d -regular graph, and let γ be an edge colouring of G with d colours. If γ' is obtained from γ by exchanging two colours, then $\text{sgn}(\gamma) = \text{sgn}(\gamma')$.*

For d -regular graphs with odd d , such as cubic graphs, Lemma 5.7 is easy to see: the signs of γ and γ' differ at every vertex of G , and there is an even number of vertices in total.

Lemma 5.7 allows to define a sign $\text{sgn}(f)$ for any 1-factorisation f by fixing it to the sign of any edge colouring that induces f . The Alon-Tarsi colouring criterion now takes a particularly simple form in d -regular graphs; see Ellingham and Goddyn [EG96] or Alon [Alo93].

Theorem 5.8. *Let G be a d -regular graph with*

$$\sum_{f \text{ 1-factor of } G} \text{sgn}(f) \neq 0.$$

Then, G is d -list-edge-colourable.

Applying Theorem 5.8 to $GP(3k, k)$ with odd k , we can now see that the list edge-colouring conjecture holds:

Corollary 5.9. [BGG17] *For odd k , the graph $GP(3k, k)$ has choice index 3.*

Proof. By Theorem 5.1, $GP(3k, k)$ has $J(k) = \frac{2^k+1}{3}$ distinct 1-factorisations, if k is odd. Since this number is odd, the sum of the signs of all 1-factorisations cannot be zero. Theorem 5.8 finishes the proof. \square

Unfortunately, for even k the number of 1-factorisations is even. That means, we have to put a bit more effort into showing that the sum of the signs of all 1-factorisations is not zero. In particular, we will need to count the positive and negative 1-factorisations separately.

As a first step, we refine the colour triple graphs T and H , and endow them with signs on the edges. Figure 5.4 shows the graphs T_{\pm} and H_{\pm} , which we obtain from T and H by replacing each edge by two inverse directed edges, each having a sign. Note that in T_{\pm} all edges in clockwise direction are positive, while clockwise edges in H_{\pm} are negative.

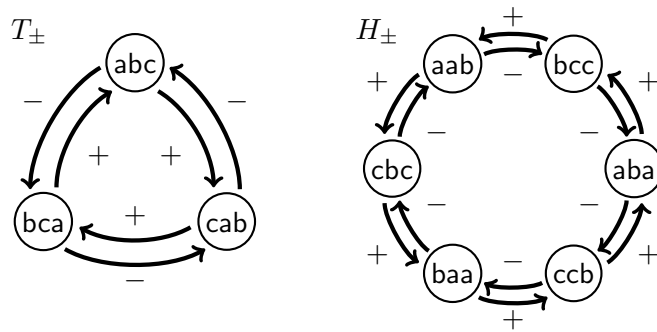


Figure 5.4.: Signs of the possible combinations of consecutive colour triples

Let x, y be two adjacent vertices in T_{\pm} or in H_{\pm} . We denote the *sign of the edge* pointing from x to y by $\text{sgn}(x, y)$. The next lemma shows that the signs on the edges capture the signs of edge colourings.

Lemma 5.10. [BGG17] Let $\gamma : E(GP) \rightarrow \{1, 2, 3\}$ be an edge colouring of $GP = GP(3k, k)$, and let (a, b, c) be a permutation of $(1, 2, 3)$ so that γ_1 is a vertex in T_\pm or in H_\pm . Then

$$\text{sgn}(\gamma) = \prod_{i=1}^k \text{sgn}(\gamma_i, \gamma_{i+1}).$$

Proof. We partition the vertices of GP into k parts, namely into the sets

$$V_i := \{u_i, v_i, u_{k+i}, v_{k+i}, u_{2k+i}, v_{2k+i}\} \text{ for } i = 1, \dots, k.$$

See Figure 5.2 for the vertices in V_i . Factorising

$$\text{sgn}(\gamma) = \prod_{w \in V} \text{sgn}_\gamma(w) = \prod_{i=1}^k \prod_{w \in V_i} \text{sgn}_\gamma(w),$$

we see that the lemma is proved if

$$\prod_{w \in V_i} \text{sgn}_\gamma(w) = \text{sgn}(\gamma_i, \gamma_{i+1}) \quad (5.3)$$

holds true for all $i = 1, \dots, k$.

That the total sign on V_i depends only on γ_i and γ_{i+1} is clear from Lemma 5.3: γ_i and γ_{i+1} determine the colours of the edges incident with vertices in V_i . Therefore, there is a function f on the edges of $T_\pm \cup H_\pm$ to $\{+, -\}$ so that

$$\prod_{w \in V_i} \text{sgn}_\gamma(w) = f(\gamma_i, \gamma_{i+1})$$

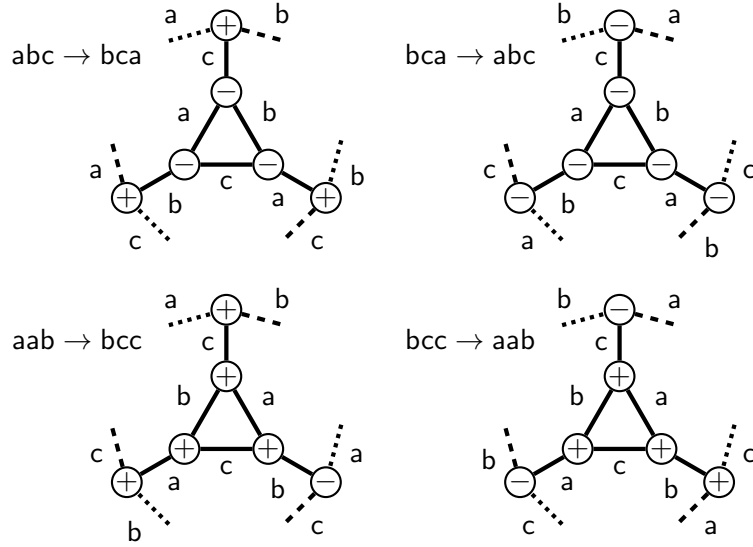
Our task reduces to verifying that $f(\gamma_i, \gamma_{i+1}) = \text{sgn}(\gamma_i, \gamma_{i+1})$. In principle, we could now check all edges in T_\pm and H_\pm , one by one, to see whether the signs are correct. Instead, we exploit the fact that all vertices in T_\pm (or in H_\pm) are in some sense the same.

A clockwise orientation of GP by k vertices induces a shift in a colour triple γ_i from $(\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$ to $(\gamma_{i2}, \gamma_{i3}, \gamma_{i1})$. Note that a rotation of GP obviously does not change the sign of γ . Moreover, permutation of colours preserves the total sign of V_i since swapping two colours changes the sign at all six vertices. Therefore we may assume that $\{\gamma_i, \gamma_{i+1}\} = \{\mathbf{abc}, \mathbf{bca}\}$ (if $\gamma_1 \in T_\pm$) or that $\{\gamma_i, \gamma_{i+1}\} = \{\mathbf{aab}, \mathbf{bcc}\}$ (if $\gamma_1 \in H_\pm$).

This gives four configurations to check, as the sign can (and does) depend on the direction of the edge from γ_i to γ_{i+1} . The four constellations are shown in Figure 5.5, where we can see, for instance, that the edge from \mathbf{abc} to \mathbf{bca} has a net negative sign under f , while the inverse edge is positive.

Since, on these four constellations, f coincides with the edge signs of T_\pm and H_\pm , it coincides everywhere, which proves (5.3). \square

Lemmas 5.3 and 5.10 imply that every positive 1-factorisation corresponds to a walk in either T_\pm or H_\pm whose edge signs multiply to $+$. We call such a walk *positive*; whereas a walk whose signs multiply to $-$ is *negative*.


 Figure 5.5.: Signs of the vertices in V_i for some consecutive colour triples

To count such walks, we observe that a walk and its reverse walk might have different signs. Not only the distance between two vertices has an influence, but also the rotational direction of the shortest path.

For two vertices x and y for which the clockwise path from x to y is of length ℓ , we define

$$t_k^+(\ell) := \# \{ \text{positive walks of length } k \text{ from } x \text{ to } y \text{ in } T_{\pm} \}$$

$$h_k^+(\ell) := \# \{ \text{positive walks of length } k \text{ from } x \text{ to } y \text{ in } H_{\pm} \}$$

and $t_k^-(\ell)$ and $h_k^-(\ell)$ analogously. Note that $t_0^+(0) = h_0^+(0) = 1$ whereas $t_0^-(0) = h_0^-(0) = 0$.

Similarly as in Section 5.2 for unsigned colourings, every positive edge colouring of $GP(3k, k)$ now corresponds to a positive walk in T_{\pm} or in H_{\pm} . Since all edge colourings with the same associated 1-factorisation have the same sign, we thus have a way to count positive and negative 1-factorisations via walks in signed graphs:

Lemma 5.11. [BGG17] *The number of positive respectively negative 1-factorisations of $GP(3k, k)$ is equal to $t_k^{\pm}(2) + 3h_k^{\pm}(2)$.*

Proof. As before, in order to count 1-factorisations it suffices to count edge colourings γ with $\gamma_1 \in \{123, 112, 121, 211\}$. Lemma 5.10 in conjunction with Lemma 5.3 shows that there is a one-to-one correspondence between positive (resp. negative) edge colourings and certain positive (resp. negative) walks of length k in T_{\pm} and in H_{\pm} . Namely, these are the $t_k^{\pm}(2)$ walks in T_{\pm} from 123 to 231 plus the $3h_k^{\pm}(2)$ walks in H_{\pm} with starting point 112, 121, 211, and respective end point 121, 211, 112. \square

As in Lemma 5.6 we can state a connection between walks in T_{\pm} and H_{\pm} .

Lemma 5.12. [BGG17] $h_k^\pm(2) = t_k^\pm(2)$ for even k .

Proof. We can canonically extend the map of Lemma 5.6 to a bijection between walks in T_\pm and H_\pm . Then the bijection maps walks counted by $t_k(1) = t_k^+(2) + t_k^-(2)$ to walks counted by $h_k(2) = h_k^+(2) + h_k^-(2)$. Since the signs of the clockwise (resp. anti-clockwise) arcs are different in T_\pm and H_\pm , any arc in a walk in T_\pm has a different sign from its image in H_\pm . However, as we consider walks of even length k , the total sign of the walks is preserved by the bijection, and the assertion follows. \square

In order to show that the numbers of positive and negative 1-factorisations differ, it remains to compute t_k^+ and t_k^- :

Lemma 5.13. For any integer $k \geq 1$

$$\begin{aligned} t_k^+(2) &= \frac{1}{6} \left(2^k - (-1)^k \left(1 + (-3)^{\lceil \frac{k}{2} \rceil} \right) \right) \\ t_k^-(2) &= \frac{1}{6} \left(2^k - (-1)^k \left(1 - (-3)^{\lceil \frac{k}{2} \rceil} \right) \right) \end{aligned}$$

Proof. Every walk in T_\pm from a vertex v to a vertex v' induces a reflected walk from v' to v . In that walk, every arc is replaced by its reversed arc, which has opposite sign. Furthermore, the shortest path from v to v' in clockwise direction is of length 1 if and only if the shortest path from v' to v in clockwise direction has length 2. Therefore

$$t_k^\pm(1) = \begin{cases} t_k^\pm(2) & \text{if } k \text{ is even} \\ t_k^\mp(2) & \text{if } k \text{ is odd} \end{cases} \quad (5.4)$$

Note that the signs swap for odd k .

In the same way follows for odd k that $t_k^+(0) = t_k^-(0)$. Since $t_k^+(0) + t_k^-(0) = t_k(0)$ we get $t_k^\pm(0) = \frac{1}{2}t_k(0)$ and thus with Lemma 5.5 that

$$t_k^\pm(0) = \frac{1}{3}(2^{k-1} - 1) \quad \text{for odd } k. \quad (5.5)$$

We use Lemma 5.5 together with (5.2) and note for later that

$$t_k^+(\ell) + t_k^-(\ell) = t_k(1) = J(k) = \frac{1}{3} \left(2^k - (-1)^k \right) \quad \text{for } \ell \in \{1, 2\} \quad (5.6)$$

Trivially, a walk of length k must visit a neighbour of its last vertex in the $(k-1)$ th step, and a vertex adjacent to its penultimate vertex in the $(k-2)$ th step. Therefore

$$\begin{aligned} t_k^\pm(2) &= t_{k-1}^\mp(0) + t_{k-1}^\pm(1) && \text{for all } k \geq 1 \\ t_k^\pm(2) &= 2t_{k-2}^\mp(2) + t_{k-2}^\pm(0) + t_{k-2}^\pm(1) && \text{for all } k \geq 2. \end{aligned}$$

Applying (5.4) and (5.5), we obtain

$$\begin{aligned} t_k^\pm(2) &= t_{k-1}^\mp(0) + t_{k-1}^\pm(1) \\ &= \frac{1}{3} \left(2^{k-2} - 1 \right) + t_{k-1}^\mp(2) && \text{if } k \text{ is even.} \end{aligned} \quad (5.7)$$

For odd k we get a recurrence relation by using again (5.4), (5.5) and additionally (5.6)

$$\begin{aligned}
 t_k^\pm(2) &= 2t_{k-2}^\mp(2) + t_{k-2}^\pm(0) + t_{k-2}^\pm(1) \\
 &= 3t_{k-2}^\mp(2) + \frac{1}{6} \left(2^{k-2} - 2 \right) \\
 &= -3t_{k-2}^\pm(2) + \frac{1}{6} \left(6 \left(2^{k-2} + 1 \right) + 2^{k-2} - 2 \right) \\
 &= -3t_{k-2}^\pm(2) + \frac{1}{6} \left(7 \cdot 2^{k-2} + 4 \right) \quad \text{if } k \text{ is odd.} \quad (5.8)
 \end{aligned}$$

It is straightforward to check that

$$t_k^+(2) = \frac{1}{6} \left((-3)^{\frac{k+1}{2}} + 2^k + 1 \right) \quad \text{for odd } k$$

satisfies the recurrence relation (5.8) and the initial condition $t_1^+(2) = 0$.

Therefore, we deduce with (5.6) that

$$\begin{aligned}
 t_k^-(2) &= \frac{1}{6} \left(2^{k+1} + 2 \right) - \frac{1}{6} \left((-3)^{\frac{k+1}{2}} + 2^k + 1 \right) \\
 &= \frac{1}{6} \left(-(-3)^{\frac{k+1}{2}} + 2^k + 1 \right) \quad \text{if } k \text{ is odd.}
 \end{aligned}$$

The transition to even k is now made by using (5.7). We obtain

$$\begin{aligned}
 t_k^+(2) &= \frac{1}{3} \left(2^{k-2} - 1 \right) + t_{k-1}^-(2) \\
 &= \frac{1}{6} \left(2^{k-1} - 2 \right) + \frac{1}{6} \left(-(-3)^{\frac{k}{2}} + 2^{k-1} + 1 \right) \\
 &= \frac{1}{6} \left(-(-3)^{\frac{k}{2}} + 2^k - 1 \right) \quad \text{if } k \geq 2 \text{ is even} \\
 t_k^-(2) &= \frac{1}{3} \left(2^{k-2} - 1 \right) + t_{k-1}^+(2) \\
 &= \frac{1}{6} \left(2^{k-1} - 2 \right) + \frac{1}{6} \left((-3)^{\frac{k}{2}} + 2^{k-1} + 1 \right) \\
 &= \frac{1}{6} \left((-3)^{\frac{k}{2}} + 2^k - 1 \right) \quad \text{if } k \geq 2 \text{ is even.}
 \end{aligned}$$

The different values of $(-1)^k$ and $\lceil \frac{k}{2} \rceil$ for odd and even k yield the formulas for t_k^+ and t_k^- . \square

We have finally collected all necessary facts to finish the proof of Theorem 5.2.

Proof of Theorem 5.2. By Corollary 5.9, it remains to consider the case of even k .

By Lemma 5.11, the number of positive/negative 1-factorisations is equal to $t_k^\pm(2) + 3c_k^\pm(2)$, which is the same as $4t_k^\pm(2)$, by Lemma 5.12. Applying Lemma 5.13, we see that $t_k^+(2) \neq t_k^-(2)$, which shows that the sum of the signs of all 1-factorisations is not zero. Thus, the Alon-Tarsi criterion, Theorem 5.8, concludes the proof. \square

5.4. Generalised Petersen Graphs $GP(4k, k)$

In this section, we focus on generalised Petersen graphs of the form $GP(4k, k)$. These graphs arise from matching k disjoint quadrangles to 4-tuples of equidistant vertices on a cycle of $4k$ vertices; Figure 5.6 shows two examples.

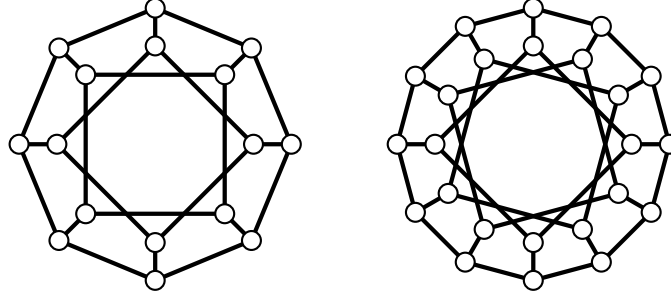


Figure 5.6.: The generalised Petersen graphs $GP(8, 2)$ and $GP(12, 3)$

We show that every graph $GP(4k, k)$ has choice index 4, ie:

Theorem 5.14. *The list-colouring conjecture is true for generalised Petersen graphs $GP(4k, k)$.*

It is easily seen that $GP(4k, k)$ is bipartite for odd k . Thus, it follows from [Gal95] that for odd k the graph $GP(4k, k)$ has choice index 4 and it suffices to analyse the graphs $GP(4k, k)$ with even k .

The main ideas to prove the list-colouring conjecture for $GP(4k, k)$ are the same as those for Dürer-type graphs in the previous sections. While there are significantly more ways to colour the spokes, we do not need to count the 1-factorisations. Instead, we show that all 1-factorisations have the same sign.

First, we show that the colours of the outer cycle of $GP(4k, k)$ determine the edge colouring of the whole graph in a way. Next, we use the algebraic colouring criterion of Alon and Tarsi [AT92] (see Theorem 5.8) to assert that $GP(4k, k)$ is 3-list-edge-colourable if the sum of the signs of its 1-factors is not zero.

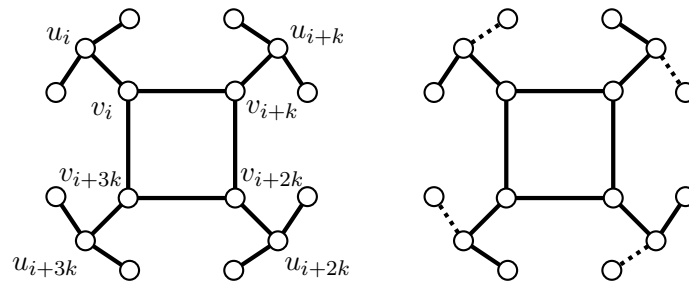


Figure 5.7.: The graph G_i ; the colours of the dotted edges define ϕ_i

To start with, we provide necessary definitions. Note that these definitions are analogues of those for Dürer-type graphs.

For every $i \in \{1, \dots, k\}$, our particular interest lies on the subgraph G_i of $GP(4k, k)$ shown in Figure 5.7. More precisely, we define G_i as the induced subgraph of $GP(4k, k)$ whose vertex set consists of $\{v_i, v_{i+k}, v_{2i+k}, v_{3i+k}\}$ together with its first neighbourhood $\{u_i, u_{i+k}, u_{2i+k}, u_{3i+k}\}$ and with its second neighbourhood that consists of the vertices $u_{i-1}, u_{i-1+k}, u_{2i-1+k}, u_{3i-1+k}$ and $u_{i+1}, u_{i+1+k}, u_{2i+1+k}, u_{3i+1+k}$.

The *outer cycle* C_O of $GP(4k, k)$, the cycle $u_0 u_1 \dots u_{4k-1} u_0$, and the *spokes*, ie the edges $u_i v_i$ for $i = 0, \dots, 4k-1$, will play a key role in the following. All vertex indices are taken modulo $4k$.

Let ϕ be an edge colouring with colours $1, 2, 3$ of either the whole graph $GP(4k, k)$ or only of the outer cycle C_O . We split ϕ into k tuples

$$\phi_i = \begin{pmatrix} \phi(u_i u_{i+1}) \\ \phi(u_{i+k} u_{i+k+1}) \\ \phi(u_{i+2k} u_{i+2k+1}) \\ \phi(u_{i+3k} u_{i+3k+1}) \end{pmatrix} \text{ for } i = 1, \dots, k.$$

Figure 5.7 shows the edges whose colours are given in ϕ_i . Additionally, we define

$$\phi_{k+1} = \begin{pmatrix} \phi(u_{k+1} u_{k+2}) \\ \phi(u_{2k+1} u_{2k+2}) \\ \phi(u_{3k+1} u_{3k+2}) \\ \phi(u_1 u_2) \end{pmatrix}$$

and note that ϕ_{k+1} is obtained from ϕ_1 by a cyclic shift.

In order to analyse the 1-factorisations of $GP(4k, k)$ we must analyse its edge colourings. Similar to Dürer-type graphs, the colours on the outer cycle determine the edge colouring on the whole graph in a way. Moreover, it is easy to characterise the colourings of the outer cycle that extend to the rest of the graph:

Lemma 5.15. *Let $\phi : E(C_O) \rightarrow \{1, 2, 3\}$ be an edge colouring of C_O . Then the following two statements are equivalent:*

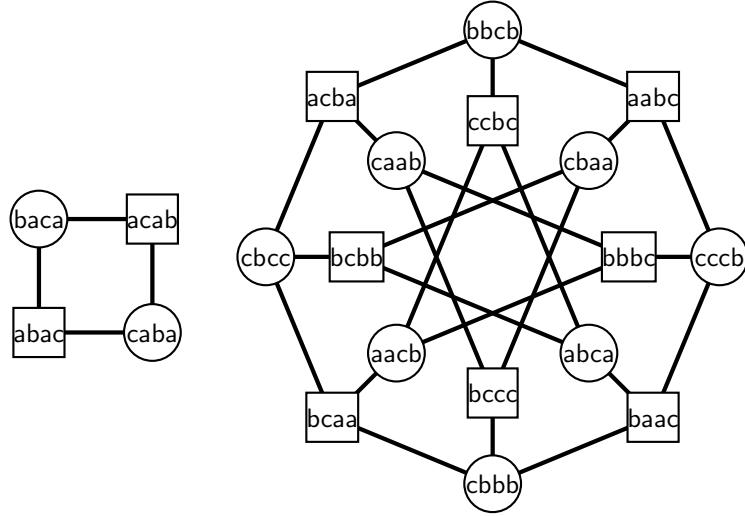
- (a) *there is an edge colouring γ of $GP(4k, k)$ with $\gamma|_{C_O} = \phi$; and*
- (b) *there is a permutation (a, b, c) of $(1, 2, 3)$ so that ϕ_i and ϕ_{i+1} are adjacent vertices in either G_a or in G_0 (see Figure 5.8 and 5.9) for all $i = 1, \dots, k$.*

To better understand the proof of Lemma 5.15, we introduce the *Klein four-group*

$$K = (\{a, b, c, 0\}, +).$$

K is the group where two non-zero elements add up to the third element (eg $a+b=c$) and the sum of an element with itself gives zero (eg $a+a=0$). Note that the difference of two elements always equals their sum.

As all considered graphs are cubic, it is convenient for us to use this group for modelling 3-edge colourings. Let (a, b, c) be a permutation of $(1, 2, 3)$ and let the edges of a cubic graph be coloured with $1, 2, 3$. Then, at each vertex, all three colours


 Figure 5.8.: The graph G_a

appear. Further, if the colours (say a and b) of two incident edges are known, then the colour of the third edge is the sum of the other two colours, ie $c = a + b$.

The vertices of G_a and G_0 are labelled with four elements, eg with **baca**. This vertex corresponds to the tuple $(\mathbf{baca})^T$. Note that all vertices in G_0 are tuples whose elements sum up to 0. Similarly, all vertices in G_a have sum a .

Note furthermore that the graphs G_a and G_0 are both isomorphic to themselves under cyclic shifts of the entries of the tuples associated with the vertices.

Proof of Lemma 5.15. First assume (a), ie there is an edge colouring γ of $GP(4k, k)$ with $\gamma|_{C_0} = \phi$ where (a, b, c) is a permutation of $(1, 2, 3)$.

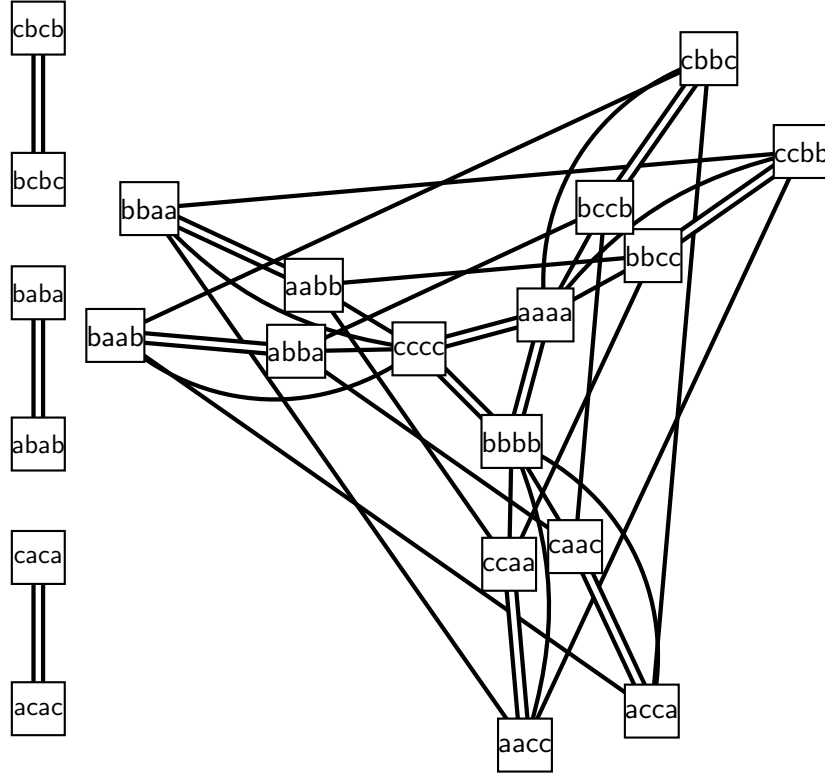
For a 4-cycle $Q_{i+1} = v_{i+1}v_{i+1+k}v_{i+1+2k}v_{i+1+3k}v_i$ (with $i \in \mathbb{Z}_{4k}$), there are only two ways to colour its edges. Either the quadrangle is coloured with only two colours that alternate, eg with 1, 2, 1, 2, or the quadrangle is coloured with three colours, eg with 1, 2, 3, 2. Figure 5.10 illustrates the two possible colourings for Q_{i+1} .

We can deduce that the adjacent spokes need to be coloured either all with the same colour or with exactly two colours such that spokes of the same colour are consecutive. Up to cyclic shifts and permutation of colours, we can thus assume that

$$\begin{pmatrix} \gamma(u_{i+1}v_{i+1}) \\ \gamma(u_{i+1+k}v_{i+1+k}) \\ \gamma(u_{i+1+2k}v_{i+1+2k}) \\ \gamma(u_{i+1+3k}v_{i+1+3k}) \end{pmatrix} \in \left\{ \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Suppose that all four spokes have the same colour 3 in γ . Then the adjacent outer edges must be coloured with 1 and 2.

We thus have four possibilities (up to cyclic shifts and permutation of 1 and 2) for


 Figure 5.9.: The graph G_0

the colours of the adjacent outer edges and the matrix $(\phi_i | \phi_{i+1})$ is an element of

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}.$$

The first three elements $(\phi_i | \phi_{i+1})$ correspond to adjacent vertices in G_0 . To see this, set $(a, b, c) = (1, 2, 3)$. The last element corresponds to two adjacent vertices in G_a ; set $(b, c, a) = (1, 2, 3)$. One can easily verify that all pairs (ϕ'_i, ϕ'_{i+1}) emerging from (ϕ_i, ϕ_{i+1}) by cyclic shifts in ϕ_i and ϕ_{i+1} and by permutation of colours also correspond to adjacent vertices of G_a or G_0 .

Can a tuple ϕ_i be contained in both, G_a and G_0 ? No. The elements of the tuples in G_0 sum up to 0. Any permutation of a, b, c will again lead to a sum of zero. In contrast, the elements in G_a have tuple sum a . Permutation of a, b, c can never yield sum zero.

Why do the elements of adjacent ϕ_i and ϕ_{i+1} always sum up to the same number? Let $x_1, x_2, x_3, x_4 \in \{a, b, c\}$ be the entries of ϕ_i . Then, the entries of ϕ_{i+1} are $c + x_1, c + x_2, c + x_3, c + x_4$ and add up to

$$(c+x_1)+(c+x_2)+(c+x_3)+(c+x_4) = (c+c+c+c)+(x_1+x_2+x_3+x_4) = x_1+x_2+x_3+x_4.$$

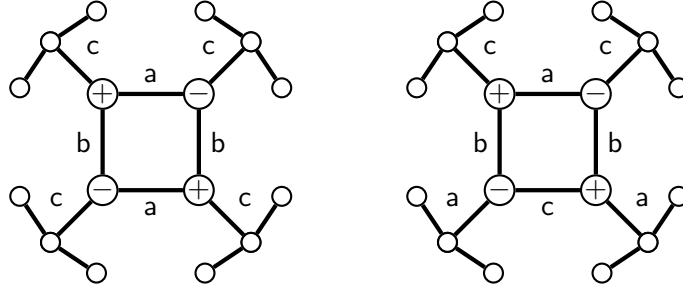


Figure 5.10.: The two ways to colour a quadrangle G_i and the colours of the incident spokes

Now, assume that the four spokes are coloured $(3, 3, 1, 1)$. In this case, there are slightly a few more possibilities for the matrix $M_i = (\phi_i | \phi_{i+1})$. In the first and in the second row of M_i , the entries must be 1 and 2 while in the third and the fourth row of M_i , the entries must be 2 and 3. This means that M_i is contained in the set

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 3 & 2 \\ 3 & 2 \end{pmatrix} \right\}.$$

Similar to the above, the entries of ϕ_i and ϕ_{i+1} must add up to the same number. Again, one can check that all pairs (ϕ_i, ϕ_{i+1}) correspond (after suitable permutation of colours) to adjacent vertices in G_0 or G_a . The same holds for all pairs (ϕ'_i, ϕ'_{i+1}) emerging from (ϕ_i, ϕ_{i+1}) by cyclic shifts and permutation of a, b, c .

For the converse direction suppose now that (b) holds. First, consider a pair of adjacent colour quadruples ϕ_i and ϕ_{i+1} in G_a . To extend the colouring of C_O to the spokes, note that $\phi_i + \phi_{i+1}$ determines the colours of the spokes of G_{i+1} :

$$\phi_i + \phi_{i+1} = \begin{pmatrix} \phi(u_{i+1}v_{i+1}) \\ \phi(u_{i+1+k}v_{i+1+k}) \\ \phi(u_{i+1+2k}v_{i+1+2k}) \\ \phi(u_{i+1+3k}v_{i+1+3k}) \end{pmatrix}$$

Note that $\phi_i + \phi_{i+1}$ equals a cyclic shift of **bbcc** if the edge is contained in the 4-cycle on the left hand side of Figure 5.8. Further, $\phi_i + \phi_{i+1}$ equals **aaaa** or a cyclic shift of **bbcc** if the edge connects a vertex of the inner and the outer cycle of the graph component on the right hand side of Figure 5.8. If the edge connects two inner or two outer vertices, then $\phi_i + \phi_{i+1}$ equals a cyclic shift of **aabb** or of **aacc**. In all cases, $\phi_i + \phi_{i+1}$ is a colouring of the spokes that can be extended to the quadrangle $v_i v_{i+k} v_{i+2k} v_{i+3k} v_i$.

(Choose a suitable permutation and a suitable rotation of the quadrangle; then see Figure 5.10.)

It remains to consider G_0 . Note that $\phi_i + \phi_{i+1}$ equals **aaaa**, **bbbb** or **cccc** if the edge is contained in one of the three graphs on two nodes shown on the left hand side of Figure 5.9. The graph on the right hand side of Figure 5.9 contains three different kinds of vertices. Its vertices are arranged on three concentric cycles of different radius. We call the vertices on the smallest cycle, ie the vertices **aaaa**, **bbbb** and **cccc**, the *inner vertices*. The six vertices on the largest cycle are the *outer vertices* while the other six vertices are called the *intermediate vertices*. Note that $\phi_i + \phi_{i+1}$ equals **aaaa**, **bbbb** or **cccc** if the edge connects two inner vertices or if the edge connects an outer vertex with an intermediate vertex. If the edge connects an inner vertex with an intermediate vertex or an outer vertex, then $\phi_i + \phi_{i+1}$ equals a cyclic shift of **baab**, **cbbc** or **acca**. If the edge connects two outer vertices or two intermediate vertices, then $\phi_i + \phi_{i+1}$ also equals a cyclic shift of **baab**, **cbbc** or **acca**. In all cases, $\phi_i + \phi_{i+1}$ is a colouring of the spokes that can be extended to the quadrangle Q_{i+1} . (Again, choose a suitable permutation and a suitable rotation of the quadrangle; then see Figure 5.10.)

We have seen that any adjacent vertices ϕ_i and ϕ_{i+1} in G_a and G_0 permit to colour the 4-cycle Q_i . Extending the colouring ϕ in this way for all i yields an edge colouring γ for $GP(4k, k)$. \square

The graph G_0 contains some double edges. For the end vertices ϕ_i and ϕ_{i+1} of these edges, all corresponding spokes have the same colour. Thus, there are two ways to colour the associated quadrangle Q_{i+1} . Therefore, Lemma 5.15 implies: Any edge colouring γ of $GP(4k, k)$ corresponds to a walk $\gamma_1\gamma_2 \dots \gamma_{k+1}$ of length k in either G_0 or G_a .

Recall that we can concentrate on the graphs $GP(4k, k)$ with even k . In this case, the walk of length k cannot be in G_a (and therefore must be in G_0): Assume that there is an edge colouring γ of $GP(4k, k)$ with even k with $\gamma|_{C_O} = \phi$ so that ϕ_i and ϕ_{i+1} are for all $i = 1, \dots, k$ adjacent vertices in G_a . Then, there is a walk of even length k between ϕ_1 and its cyclic shift ϕ_{k+1} in G_a . Taking a closer look at G_a , one can see that the graph is bipartite. The two bipartition classes are represented by the vertices of circular and rectangular shape. Further, for all vertices ϕ_1 , the vertex ϕ_1 and its cyclic shift ϕ_{k+1} are contained in different partition classes. This means, all paths $\phi_1, \phi_2 \dots \phi_{k+1}$ contain an even number $k + 1$ of vertices. This contradicts the assumption that k is even. We can conclude:

Lemma 5.16. *Let k be even and let $\phi : E(C_O) \rightarrow \{1, 2, 3\}$ be an edge colouring of C_O . Then the following two statements are equivalent:*

- (a) *there is an edge colouring γ of $GP(4k, k)$ with $\gamma|_{C_O} = \phi$; and*
- (b) *there is a permutation (a, b, c) of $(1, 2, 3)$ so that ϕ_i and ϕ_{i+1} are adjacent vertices in G_0 (see Figure 5.9) for all $i = 1, \dots, k$.*

We have just seen that all edge colourings respectively 1-factorisations of $GP(4k, k)$ can be obtained from paths in G_0 .

Since we want to use the method of Alon and Tarsi (Theorem 5.8) we must first define a local orientation: We consider $GP(4k, k)$ always to be drawn as in Figure 5.6. The vertices u_i for $i = 1, \dots, 4k$ are placed on a circle in clockwise order, the vertices v_i for $i = 1, \dots, 4k$ are placed on a smaller concentric circle in such a way that u_i and v_i match up, and all edges are straight. We define the sign of an edge colouring γ at a vertex w as $+$ if the colours $1, 2, 3$ appear in clockwise order on the incident edges; otherwise the sign is $-$. More formally,

$$\begin{aligned} \text{sgn}_\gamma(u_i) &= \begin{cases} + & \text{if } (\gamma(u_{i-1}u_i), \gamma(u_iu_{i+1}), \gamma(u_iv_i)) \in \{123, 231, 312\} \\ - & \text{otherwise.} \end{cases} \\ \text{sgn}_\gamma(v_i) &= \begin{cases} + & \text{if } (\gamma(v_{k+i}v_i), \gamma(v_iv_{2k+i}), \gamma(v_iu_i)) \in \{123, 231, 312\} \\ - & \text{otherwise} \end{cases} \end{aligned}$$

The sign of the colouring γ is then

$$\text{sgn}(\gamma) := \prod_{v \in V(GP(4k, k))} \text{sgn}_\gamma(v).$$

As we have seen in Lemma 5.7, permuting colours in our context does not change the sign of an edge colouring and we are able to define a sign $\text{sgn}(f)$ for any 1-factorisation f by fixing it to the sign of any edge colouring that induces f . As for Dürer-type graphs we use Theorem 5.8 to show that $GP(4k, k)$ is 3-list-edge-colourable.

Lemma 5.17. *Let $\gamma : E(GP(4k, k)) \rightarrow \{1, 2, 3\}$ be an edge colouring of $GP(4k, k)$ with even k , and let (a, b, c) be a permutation of $(1, 2, 3)$ so that γ_1 is a vertex in G_0 . Then*

$$\text{sgn}(\gamma) = +.$$

Proof. We partition the vertices of $GP(4k, k)$ into k parts, namely into the sets

$$V_i := \{u_i, v_i, u_{k+i}, v_{k+i}, u_{2k+i}, v_{2k+i}, u_{3k+i}, v_{3k+i}\}$$

for $i = 1, \dots, k$. Factorising

$$\text{sgn}(\gamma) = \prod_{w \in V} \text{sgn}_\gamma(w) = \prod_{i=1}^k \prod_{w \in V_i} \text{sgn}_\gamma(w),$$

we see that the lemma is proved if

$$\prod_{w \in V_i} \text{sgn}_\gamma(w) = \text{sgn}(\gamma_i, \gamma_{i+1}) = + \tag{5.9}$$

holds true for all $i = 1, \dots, k$.

It is clear from Lemma 5.15 that the total sign on V_i depends only on γ_i and γ_{i+1} : γ_i and γ_{i+1} determine the colours of the edges incident with vertices in V_i . Our task narrows down to verifying that $\text{sgn}(\gamma_i, \gamma_{i+1})$ is positive for every pair (γ_i, γ_{i+1}) of

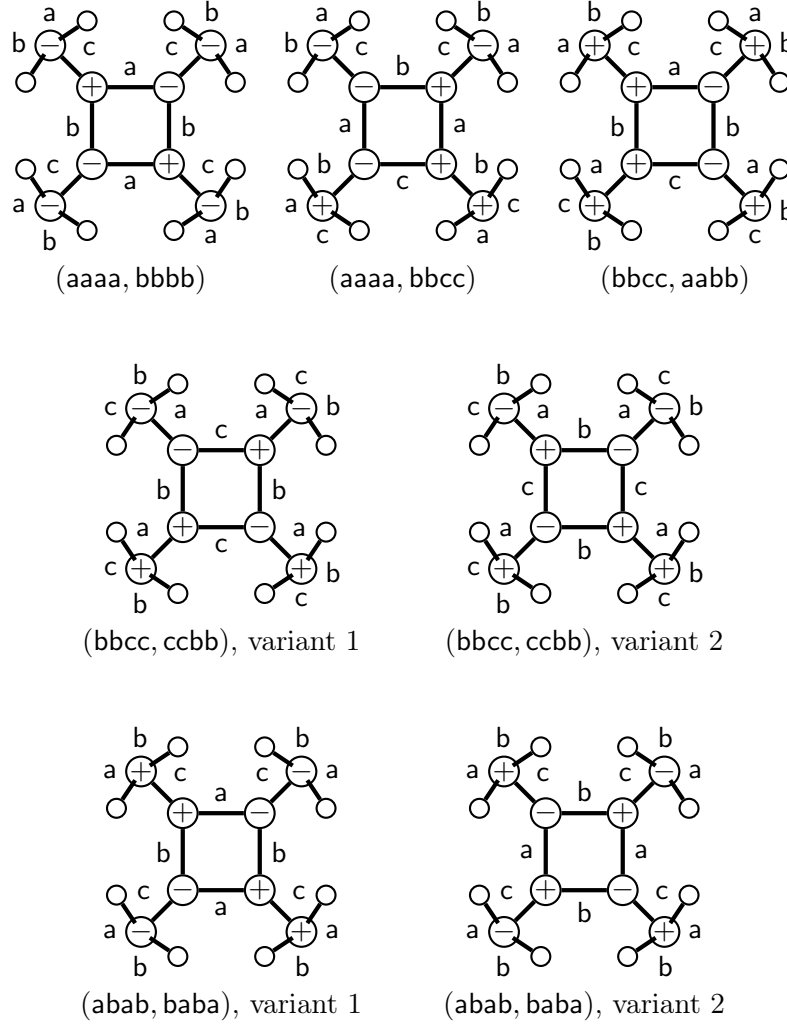


Figure 5.11.: Signs of the vertices in each quadrangle

adjacent vertices in G_0 . In principle, we could check all edges in G_0 , one by one, to see whether the sign is positive. Instead, we exploit the fact that there are only five kinds of edges.

A clockwise rotation of $GP(4k, k)$ by k vertices induces a shift in the colour quadruple γ_i from $(\gamma_{i1} \gamma_{i2} \gamma_{i3} \gamma_{i4})^T$ to $(\gamma_{i2} \gamma_{i3} \gamma_{i4} \gamma_{i1})^T$. Such a rotation of $GP(4k, k)$ obviously does not change the sign of γ . Moreover, permutation of colours preserves the total sign of V_i since swapping two colours changes the sign at all eight vertices. Therefore we may assume that $(\gamma_i | \gamma_{i+1})$ is contained in

$$\left\{ \begin{pmatrix} a & b \\ a & b \\ a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \\ a & c \\ a & c \end{pmatrix}, \begin{pmatrix} b & a \\ b & a \\ c & b \\ c & b \end{pmatrix}, \begin{pmatrix} b & c \\ b & c \\ c & b \\ c & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \\ a & b \\ b & a \end{pmatrix} \right\}.$$

These five constellations of $(\gamma_i|\gamma_{i+1})$ are shown in Figure 5.11. All of them have a positive sign. This proves (5.9) and thus the lemma. \square

We have now collected all results necessary to prove Theorem 5.14.

Proof of Theorem 5.14. If k is even, the graph $GP(4k, k)$ is bipartite. By [Gal95], the graph satisfies the list-colouring conjecture.

If k is odd, every 1-factorisation of $GP(4k, k)$ has the same sign (see Lemma 5.17). Further, there exists at least one 1-factorisation; see Watkins [Wat69]. Thus, the Alon-Tarsi criterion(Theorem 5.8) concludes the proof. \square

6. Chromatic Index, Treewidth and Maximum Degree

The content of this chapter is based on the paper [BGL16] by Henning Bruhn, Richard Lang and the author of this thesis.

We conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geq k + \sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version.

6.1. Introduction

The least number $\chi'(G)$ of colours necessary to properly colour the edges of a (simple) graph G is either the maximum degree $\Delta(G)$ or $\Delta(G) + 1$. But to decide whether $\Delta(G)$ or $\Delta(G) + 1$ colours suffice is a difficult algorithmic problem [Hol81].

Often, graphs with a relatively simple structure can be edge-coloured with only $\Delta(G)$ colours. This is the case for bipartite graphs (König's theorem) and for cubic Hamiltonian graphs. Arguably, one measure of simplicity is *treewidth*, how closely a graph resembles a tree. (See next section for a definition.)

Vizing [Viz65b] (see also Zhou et al. [ZN96]) observed a consequence of his adjacency lemma (Theorem 4.3): any graph with treewidth k and maximum degree at least $2k$ has chromatic index $\chi'(G) = \Delta(G)$. Is this tight? No, it turns out. Using two recent adjacency lemmas, the requirement on the maximum degree can be dropped to $\Delta(G) \geq 2k - 1$ whenever $k \geq 4$; see Section 6.3. This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

Conjecture 6.1. [BGL16] *Any graph of treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has chromatic index Δ .*

The bound is close to best possible: in Section 6.4 we construct, for infinitely many k , graphs with treewidth k , maximum degree $\Delta = k + \lfloor \sqrt{k} \rfloor < k + \sqrt{k}$, and chromatic index $\Delta + 1$. For other values k the conjecture (if true) might be off by 1 from the best bound on Δ . This is, for instance, the case for $k = 2$, where the conjecture is known to hold. Indeed, Juvan et al. [JMR99] show that series-parallel graphs with maximum degree $\Delta \geq 3$ are even Δ -edge-choosable.

In support of the conjecture we prove its fractional version:

Theorem 6.2. [BGL16] *Any simple graph of treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has fractional chromatic index Δ .*

The theorem follows from a new upper bound on the number of edges:

$$2|E(G)| \leq \Delta|V(G)| - (\Delta - k)(\Delta - k + 1)$$

The bound is proved in Proposition 6.3. It implies quite directly that no graph with treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ can be overfull.

Thus, for certain parameters our conjecture coincides with the overfull conjecture of Chetwynd and Hilton [CH86] (Conjecture 4.2) that asserts that every graph G on less than $3\Delta(G)$ vertices can be edge-coloured with $\Delta(G)$ colours unless it contains an overfull subgraph.

Because we can exclude that graphs with treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ are overfull, the overfull conjecture (as well as our conjecture) implies that such graphs on less than 3Δ vertices can always be edge-coloured with Δ colours.

Graphs of treewidth k are in particular k -degenerate (see Section 4.4 for the definition and some discussion). Indeed, Vizing [Viz65b] originally showed that k -degenerate graphs, rather than treewidth k graphs, of maximum degree $\Delta \geq 2k$ have an edge colouring with Δ colours. We briefly list some related work on edge colourings and their variants in k -degenerate graphs. Isobe et al. [IZN07] showed that any k -degenerate graph of maximum degree $\Delta \geq 4k + 3$ has a total colouring with only $\Delta + 1$ colours. For graphs that are not only k -degenerate but also of treewidth k , a maximum degree of $\Delta \geq 3k - 3$ already suffices [BLS16]. Noting that they are 5-degenerate, we include some results on planar graphs as well. Borodin, Kostochka and Woodall [BKW97a, BKW97b] showed that planar graphs have list-chromatic index $\Delta(G)$ and total chromatic number $\chi''(G) = \Delta(G) + 1$ if $\Delta(G) \geq 11$ or if the maximum degree and the girth are at least 5. Vizing [Viz65b] proved that a planar graph G has a $\Delta(G)$ -edge-colouring if $\Delta(G) \geq 8$. Sanders and Zhao [SZ01] and independently Zhang [Zha00] extended this to $\Delta(G) \geq 7$.

6.2. A Bound on the Number of Edges

Theorem 6.2 follows quickly from a bound on the number of edges:

Proposition 6.3. [BGL16] *A graph G of treewidth k satisfies*

$$2|E(G)| \leq \Delta(G)|V(G)| - (\Delta(G) - k)(\Delta(G) - k + 1). \quad (6.1)$$

Before proving Proposition 6.3 we present one of its consequences:

Lemma 6.4. [BGL16] *Let G be a graph of treewidth at most k and maximum degree $\Delta \geq k + \sqrt{k}$. Then G is not overfull.*

Proof. By Proposition 6.3 holds

$$\frac{2|E(G)|}{|V(G)| - 1} \leq \frac{\Delta|V(G)| - (\Delta - k)(\Delta - k + 1)}{|V(G)| - 1} = \frac{\Delta|V(G)| - (\Delta - k)^2 - \Delta + k}{|V(G)| - 1}$$

and as $\Delta \geq k + \sqrt{k}$ we obtain

$$\frac{2|E(G)|}{|V(G)| - 1} \leq \frac{\Delta|V(G)| - k - \Delta + k}{|V(G)| - 1} = \Delta.$$

This finishes the proof. \square

It follows from Edmonds' matching polytope theorem that $\chi'_f(G) = \Delta(G)$, if the graph G does not contain any overfull subgraph of maximum degree Δ ; see [Sch03, Ch. 28.5]. (See Section 4.2 for more details.) As the treewidth of a subgraph is never larger than the treewidth of the original graph, Theorem 6.2 is a consequence of Lemma 6.4.

The proof of Proposition 6.3 rests on two lemmas. We defer their proofs to the end of the section. For a tree T we write $|T|$ to denote the number of its vertices. If $st \in E(T)$ is an edge of T then we let $T_{(s,t)}$ be the component of $T - st$ containing s . For any number k we set $[k]^+ = \max(k, 0)$.

Lemma 6.5. [BGL16] *For a tree T and a positive integer $d \leq |T|$ it holds that*

$$\sum_{(s,t): st \in E(T)} [d - |T_{(s,t)}|]^+ \geq d(d-1).$$

If T^* is a subtree of T then let $\delta^+(T^*)$ be the set of (s, t) so that st is an edge of T with $s \in V(T^*)$ but $t \notin V(T^*)$. (That is, $\delta^+(T^*)$ may be seen as the set of oriented edges leaving T^* .)

Lemma 6.6. *Let T be a tree and let $d \leq |T|$ be a positive integer. Then for any subtree $T^* \subset T$ it holds that*

$$\sum_{(s,t) \in \delta^+(T^*)} [d - |T_{(s,t)}|]^+ \leq [d - |T^*|]^+. \quad (6.2)$$

We introduce one more piece of notation. If (T, \mathcal{B}) is a tree decomposition of the graph G , then for any vertex v of G we denote by $T(v)$ the subtree of T that consists of those bags that contain v .

Proof of Proposition 6.3. Let (T, \mathcal{B}) be a smooth tree decomposition of G of width k . First note that for any vertex v of G , the number of vertices in the union of all bags containing v is at most $|T(v)| + k$ since the tree decomposition is smooth. Thus $\deg(v) \leq |T(v)| + k - 1$.

Set $d := \Delta - k + 1 \geq 1$, and observe that $d \leq |V(G)| - k = |T|$ as the tree decomposition is smooth. We calculate

$$\begin{aligned} \Delta - \deg(v) &\geq [\Delta - k + 1 - |T(v)|]^+ \\ &= [d - |T(v)|]^+ \geq \sum_{(s,t) \in \delta^+(T(v))} [d - |T_{(s,t)}|]^+, \end{aligned}$$

where the last inequality follows from Lemma 6.6.

Consider an edge $st \in E(T)$. Since the tree decomposition is smooth there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. Setting $\phi((s, t)) = v$ then defines a function from the set of all (s, t) with $st \in E(T)$ into $V(G)$. Note that $\phi((s, t)) = v$ if and only if $(s, t) \in \delta^+(T(v))$. Summing the previous inequality over all vertices, we get

$$\begin{aligned} \sum_{v \in V(G)} (\Delta - \deg(v)) &\geq \sum_{v \in V(G)} \sum_{(s, t) \in \phi^{-1}(v)} [d - |T_{(s, t)}|]^+ \\ &= \sum_{(s, t) : st \in E(T)} [d - |T_{(s, t)}|]^+ \geq d(d-1), \end{aligned}$$

where the last inequality is due to Lemma 6.5. This directly implies (6.1). \square

It remains to prove Lemma 6.5 and 6.6.

Proof of Lemma 6.5. We proceed by induction on $|T| - d$. The induction starts when $d = |T|$. Then $[d - |T_{(s, t)}|]^+ = d - |T_{(s, t)}|$ and thus

$$\begin{aligned} \sum_{(s, t) : st \in E(T)} [d - |T_{(s, t)}|]^+ &= \sum_{st \in E(T)} (|T| - |T_{(s, t)}| + |T| - |T_{(t, s)}|) \\ &= \sum_{st \in E(T)} |T_{(t, s)}| + |T_{(s, t)}| = (|T| - 1)|T|. \end{aligned}$$

Now, let $d \leq |T| - 1$, which implies in particular $|T| \geq 2$. Then T has a leaf ℓ . We set $T' := T - \ell$ and note that $d \leq |T| - 1 = |T'|$.

Observe that for any edge $st \in E(T')$ we get

$$|T_{(s, t)}| = \begin{cases} |T'_{(s, t)}| + 1 & \text{if } \ell \in V(T_{(s, t)}), \\ |T'_{(s, t)}| & \text{if } \ell \notin V(T_{(s, t)}). \end{cases}$$

We denote by F the set of all (s, t) for which st is an edge in T' with $\ell \in V(T_{(s, t)})$ and with $|T'_{(s, t)}| \leq d - 1$. Then

$$[d - |T'_{(s, t)}|]^+ = \begin{cases} [d - |T_{(s, t)}|]^+ - 1 & \text{if } (s, t) \in F, \\ [d - |T_{(s, t)}|]^+ & \text{if } (s, t) \notin F. \end{cases} \quad (6.3)$$

Among the $(s, t) \in F$ choose (x, y) such that y maximises the distance to ℓ . This means, that $st \in E(T'_{(x, y)})$ for any $(s, t) \in F \setminus \{(x, y)\}$. Consequently, $|T'_{(x, y)}| = |E(T'_{(x, y)})| + 1 \geq |F| - 1 + 1$.

Let r be the unique neighbour of the leaf ℓ . Then $|T_{(\ell, r)}| = 1$, and we obtain

$$[d - |T_{(\ell, r)}|]^+ = d - 1 \geq |T'_{(x, y)}| \geq |F|. \quad (6.4)$$

We conclude

$$\begin{aligned}
 \sum_{(s,t):st \in E(T)} [d - |T_{(s,t)}|]^+ &= [d - |T_{(\ell,r)}|]^+ + [d - |T_{(r,\ell)}|]^+ \\
 &\quad + \sum_{(s,t):st \in E(T')} [d - |T_{(s,t)}|]^+ \\
 &\stackrel{(6.4)}{\geq} |F| + 0 + \sum_{(s,t):st \in E(T')} [d - |T_{(s,t)}|]^+ \\
 &\stackrel{(6.3)}{=} \sum_{(s,t):st \in E(T')} [d - |T'_{(s,t)}|]^+ \\
 &\geq d(d-1),
 \end{aligned}$$

where the last inequality follows by induction. \square

Proof of Lemma 6.6. We proceed by induction on $|T| - d$. For the induction start, consider the case when $d = |T|$. Then

$$[d - |T_{(s,t)}|]^+ = [|T| - |T_{(s,t)}|]^+ = |T_{(t,s)}|,$$

which yields

$$\sum_{(s,t) \in \delta^+(T^*)} [d - |T_{(s,t)}|]^+ = \sum_{(s,t) \in \delta^+(T^*)} |T_{(t,s)}| = |T| - |T^*| = [d - |T^*|]^+.$$

Now assume $|T| - d \geq 1$. If every vertex in $T - V(T^*)$ is a leaf of T then t is a leaf for every $(s, t) \in \delta^+(T^*)$. This implies $|T_{(s,t)}| = |T| - 1 \geq d$ and the left hand side of (6.2) vanishes.

Therefore we may assume that there is a leaf $\ell \notin T^*$ of T whose neighbour is not in $V(T^*)$. Set $T' = T - \ell$, and observe that, by choice of ℓ , the set $\delta^+(T^*)$ of edges leaving T^* is the same in T and in T' . Moreover, $|T_{(s,t)}| \geq |T'_{(s,t)}|$ holds for every $(s, t) \in \delta^+(T^*)$. The desired inequality

$$\sum_{(s,t) \in \delta^+(T^*)} [d - |T_{(s,t)}|]^+ \leq \sum_{(s,t) \in \delta^+(T^*)} [d - |T'_{(s,t)}|]^+ \leq [d - |T^*|]^+$$

now follows by induction. \square

6.3. A Lower Bound on the Maximum Degree

Vizing [Viz65b] (see also Zhou et al. [ZN96]) proved that every graph of treewidth k and maximum degree $\Delta \geq 2k$ has an edge colouring with Δ colours. We show that this is not tight.

Proposition 6.7. [BGL16] *For any graph G of treewidth $k \geq 4$ and maximum degree $\Delta(G) \geq 2k - 1$ it holds that $\chi'(G) = \Delta(G)$.*

For the proof of Proposition 6.7 we use Vizing's adjacency lemma (Theorem 4.3), as well as two other adjacency lemmas (Theorem 4.4 and Theorem 4.5) that involve the second neighbourhood.

Proof. Assume Proposition 6.7 to be wrong. Then there is a Δ -critical graph G of treewidth at most k for $\Delta = 2k - 1$. Let (T, \mathcal{B}) be a tree-decomposition of G of width $\leq k$. By picking an arbitrary root, we may consider T as a rooted tree. For any $s \in V(T)$, we denote by $[s]$ the sub tree of T rooted at s , that is, the sub tree of T consisting of the vertices $t \in V(T)$ for which s is contained in the path between t and the root of T .

Set $L := \{v \in V(G) : \deg(v) \geq k + 2\}$, and choose a vertex $v^* \in L$ that maximises the distance of $T(v^*)$ to the root (among the vertices in L). Let q be the vertex of $T(v^*)$ that achieves this distance. For $S := N(q) \cap T(v^*)$ and any $s \in S$, define $X_s := \bigcup_{t \in V([s])} B_t$, and let $X := B_q \cup \bigcup_{s \in S} X_s$. Note that by the definition of v^* and q

$$N(v^*) \subset X \text{ and } X \cap L \subset B_q. \quad (6.5)$$

Claim 6.8. *All vertices of $X \setminus B_q$ have degree at most k .*

Proof of Claim 6.8. Suppose the statement to be false. Then there is an $s \in S$ for which $X_s \setminus B_q$ contains a vertex of degree at least $k + 1$. Fix a vertex $w^* \in \{w \in X_s \setminus B_q : \deg(w) \geq k + 1\} =: L'$ that maximises the distance of $T(w^*)$ to s . Let p be the vertex of $T(w^*)$ that achieves this distance. Set $Y = \bigcup_{t \in V([p])} B_t$. As in (6.5) we have $N(w^*) \subset Y$ and $Y \cap L' \subset B_p$. Since, moreover, w^* has degree $k + 1$, it has a neighbour u^* outside B_p , which then has degree at most k (by choice of w^*).

Vizing's adjacency lemma implies that w^* has at least $\Delta - \deg(u^*) + 1 \geq 2k - 1 - k + 1 = k$ neighbours of degree Δ . By (6.5), all vertices of degree Δ of Y have to be in $B_q \cap B_s$. Since $B_q \cap B_s$ is a cutset of size at most k , the vertex w^* is adjacent to all vertices in $B_q \cap B_s$. As w^* is therefore adjacent to at most k vertices of degree Δ it holds $\deg(u^*) = k$. By definition of S , the set B_s contains v^* , which implies that v^* is adjacent to w^* and of degree Δ . As $k \geq 4$, it follows that v^* has degree $\Delta = 2k - 1 \geq k + 3$, which means by (6.5) that v^* has at least three neighbours of degree $\leq k + 1$. Thus, v^* has a neighbour of degree $\leq k + 1$, which is neither u^* nor w^* . This, however, contradicts Theorem 4.4 (applied to v^*, w^*, u^*). \square

By (6.5) and since v^* has degree at least $k + 2$, the vertex v^* has a neighbour $u \notin B_q$. (In fact, v^* has at least two such neighbours.) By Vizing's adjacency lemma, applied to uv^* , it follows that v^* has at least $\Delta - \deg(u) + 1 \geq k$ neighbours of degree Δ . In particular, by (6.5)

$$v^* \text{ is adjacent to every vertex in } B_q, \text{ each of which has degree } \Delta. \quad (6.6)$$

Claim 6.9. *Every $u \in N(v^*) \setminus B_q$ has exactly k neighbours, all of which are contained in B_q .*

Proof of Claim 6.9. By Vizing's adjacency lemma (applied to uv^*), u is of degree at least k . By (6.6), every vertex in B_q has degree Δ and thus $u \notin B_q$. The set B_q is a

cutset. This implies that u has all its neighbours in X . However, u cannot be adjacent to any vertex w of degree $\leq k$; otherwise we could extend any Δ -edge-colouring of $G - uw$ to G . It follows from Claim 6.8 that all of the k neighbours of u are in B_q . \square

Since the vertex v^* has degree at least $k + 2$, it has two neighbours u, w of degree at most $k + 1$ (again by (6.5)). By Claim 6.9, the degree of u and w is k . Thus, $\deg(u) + \deg(v^*) + \deg(w) \leq k + \Delta + k = 2\Delta + 1$. Moreover, by Claim 6.9 and (6.6), the vertices v^* and w have $k - 1$ common neighbours in B_q . As $k - 1 > \deg(u) + \deg(v^*) - \Delta - 3$, we get a contradiction to Theorem 4.5. This finishes the proof of Proposition 6.7. \square

6.4. Discussion

Proposition 6.3 bounds the number of edges in a graph G of fixed treewidth and maximum degree. A simpler bound – only considering the treewidth – is easily shown by induction (see Rose [Ros74]):

$$2|E(G)| \leq 2k|V(G)| - k(k + 1) \quad (6.7)$$

For $\Delta < 2k$ and $|V(G)| > \Delta + 1$ a straightforward computation shows that the bound of Proposition 6.3 is strictly better than (6.7). The bounds are the same if $\Delta = 2k$ or if $|V(G)| = \Delta + 1$. For $\Delta = 2k$ this is illustrated by the k th power P^k of a long path P .

The bound in Proposition 6.3 is tight. There are simple examples that show this: Take the complete graph K_k on k vertices and add $r \geq 1$ further vertices each adjacent to each vertex of K_k . These graphs also demonstrate that Conjecture 6.1 (if true) would be tight or almost tight. Indeed, if $k + \lfloor k \rfloor$ is even, and k not a square, then we obtain for $r = \lfloor k \rfloor + 1$ an overfull graph with maximum degree $\Delta = k + \lfloor k \rfloor$. If $k + \lfloor \sqrt{k} \rfloor$ is odd, then, by setting $r = \lfloor \sqrt{k} \rfloor$, we obtain an overfull graph with $\Delta = k + \lfloor \sqrt{k} \rfloor - 1$.

These tight graphs, however, have a very special structure. In particular, they all satisfy $|V(G)| = \Delta(G) + 1$. Both, Conjecture 6.1 and Proposition 6.3, stay tight for an arbitrarily large number of vertices:

Proposition 6.10. [BGL16] *For every $k_0 \geq 4$ there is a $k \in \{k_0, k_0 + 1, \dots, k_0 + 8\}$ such that for every $n \geq 4k$ there exists a graph G on n vertices with treewidth at most k and maximum degree $\Delta = k + \lfloor \sqrt{k} \rfloor < k + \sqrt{k}$ such that*

$$2|E(G)| = \Delta n - (\Delta - k)(\Delta - k + 1).$$

In particular, the graph G is overfull whenever n is odd.

We need the following lemma to prove the proposition.

Lemma 6.11. [BGL16] *Let $c, r \in \mathbb{N}$. Then there is a graph with degree sequence*

$$\mathbf{d} = (\underbrace{c, \dots, c}_{r+1}, c-1, c-2, \dots, 1) \in \mathbb{Z}^{c+r}$$

if and only if 4 divides $c(2r + c + 1)$ and if $r^2 \geq c$.

We defer the proof of Lemma 6.11 until the end of the section and only show sufficiency. A closer look at the arguments in the proof yields necessity.

Proof of Proposition 6.10. We start by showing with a case distinction that there is a $k \in \{k_0, k_0 + 1, \dots, k_0 + 8\}$ such that

$$k \equiv \lfloor \sqrt{k} \rfloor \pmod{8} \text{ and } \lfloor \sqrt{k} \rfloor < \sqrt{k}. \quad (6.8)$$

To this end, let i such that $\lfloor \sqrt{k_0} \rfloor \equiv k_0 + i \pmod{8}$ and $0 \leq i \leq 7$.

Firstly, let us assume that $i = 0$. If k_0 is not a square, then $k = k_0$ satisfies (6.8). Otherwise $k = k_0 + 8$ satisfies (6.8) as $k_0 \geq 4 > 1$, and consequently $\lfloor \sqrt{k_0 + 8} \rfloor = \sqrt{k_0}$.

Secondly, we consider the case that $i \neq 0$. If $\lfloor \sqrt{k_0 + i} \rfloor = \lfloor \sqrt{k_0} \rfloor$, then $k = k_0 + i$ satisfies $\lfloor \sqrt{k} \rfloor \equiv k \pmod{8}$ and $\sqrt{k} > \sqrt{k_0} \geq \lfloor \sqrt{k_0} \rfloor = \lfloor \sqrt{k} \rfloor$, which shows (6.8). If, on the other hand, $\lfloor \sqrt{k_0 + i} \rfloor > \lfloor \sqrt{k_0} \rfloor$, then $\lfloor \sqrt{k_0 + i} \rfloor = \lfloor \sqrt{k_0} \rfloor + 1 = \lfloor \sqrt{k_0 + i + 1} \rfloor$ as $k_0 \geq 4$. Set $k = k_0 + i + 1$. By choice of i , we have $\lfloor \sqrt{k_0 + 1} \rfloor \equiv k \pmod{8}$. Thus, we obtain $\lfloor \sqrt{k} \rfloor \equiv k \pmod{8}$ as desired. Moreover, $\sqrt{k} > \sqrt{k_0 + i} \geq \lfloor \sqrt{k_0 + i} \rfloor = \lfloor \sqrt{k_0} \rfloor + 1 = \lfloor \sqrt{k} \rfloor$.

In all cases an element of $\{k_0, k_0 + 1, \dots, k_0 + 8\}$ satisfies (6.8).

Next we show that for any $n \geq 4k$, there is a graph G of treewidth k whose degree sequence $(\deg_G(v_1), \deg_G(v_2), \dots, \deg_G(v_n))$ equals

$$(k, k + 1, \dots, \Delta - 1, \Delta, \dots, \Delta, \Delta - 1, \dots, k + 1, k) \quad (6.9)$$

with $\Delta = k + \lfloor \sqrt{k} \rfloor$. A computation similar to Lemma 6.4 shows that G is overfull if $|V(G)|$ is odd.

We construct G in three steps. First we take a power of a path, where all but the outer vertices have the right degree. We increase the degree of the outer vertices by connecting them to vertices towards the middle of the path. This will create some degree excess for the used vertices. We balance this by deleting a subgraph H provided by Lemma 6.11. The construction is illustrated in Figure 6.1. Note that for ease of exposition the parameters k and Δ are not as in this proof.

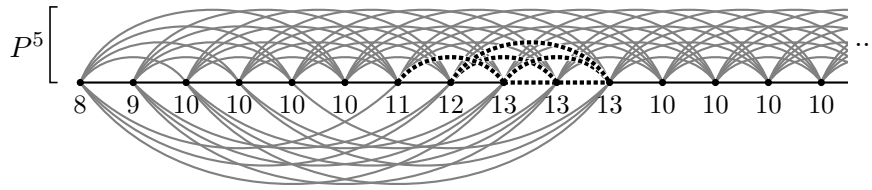


Figure 6.1.: Extreme example for $k = 8$ and $\Delta = 10$. The graph H is dotted.

Let P be a $\Delta/2$ -th power of a path on vertices v_1, \dots, v_n . This means, v_i and v_j are adjacent if and only if $0 < |i - j| \leq \Delta/2$. As P is symmetric, and as G will be symmetric as well, we concentrate on the part of P on the vertices $v_1, \dots, v_{\lceil n/2 \rceil}$. We tacitly agree that any additions and deletions of edges are also applied to the other half of P .

Comparing the degrees of P to (6.9) we see that all vertices have the target degree except for the initial vertices $v_1, \dots, v_{\Delta/2}$, whose degree is too small. For $i = 1, \dots, \Delta - k$ the vertex v_i has degree $\Delta/2 - 1 + i$ but should have degree $k - 1 + i$. We fix this by connecting v_i to $v_{i+\Delta/2+1}, \dots, v_{i+k+1}$. For $i = \Delta - k + 1, \dots, \Delta/2$, the vertex v_i should have degree Δ but has degree $\Delta/2 - 1 + i$. We make v_i adjacent to each of $v_{i+\Delta/2+1}, \dots, v_{\Delta+1}$.

Denote the obtained graph by P' and observe that its vertices in the range of $1, \dots, \lceil n/2 \rceil$ have the following degrees

$$\underbrace{k, k+1, \dots, \Delta}_{1, \dots, \Delta-k+1}, \underbrace{\Delta, \dots, \Delta}_{\Delta-k+2, \dots, \Delta/2+1}, \underbrace{\Delta+1, \dots, k+\Delta/2}_{\Delta/2+2, \dots, k+1}, \underbrace{k+\Delta/2, \dots, k+\Delta/2}_{k+2, \dots, \Delta+1}, \underbrace{\Delta, \dots, \Delta}_{\Delta+2, \dots, \lceil n/2 \rceil}$$

Hence all but the vertices v_i with index i between $\Delta/2+2$ and $\Delta+1$ have the correct degree. The difference between their degree in P' and the desired degree is

$$\mathbf{d} = (1, 2, \dots, k - \Delta/2 - 1, \underbrace{k - \Delta/2, \dots, k - \Delta/2}_{\Delta-k+1}). \quad (6.10)$$

Set $c = k - \Delta/2 = 1/2(k - \lfloor \sqrt{k} \rfloor)$ and $r = \Delta - k$. Note that k is chosen in such a way (see (6.8)) that c is divisible by 4. As furthermore $r^2 = (\Delta - k)^2 = \lfloor \sqrt{k} \rfloor^2 \geq \frac{1}{2}(k - \lfloor \sqrt{k} \rfloor) = c$, Lemma 6.11 yields that there is a graph H with degree sequence \mathbf{d} . Since the vertices $v_{\Delta/2+2}, \dots, v_{\Delta+1}$ induce a complete graph in P' there is a copy of H in P' , such that deleting its edges results in a graph G of the desired degree sequence. Note that for any two adjacent vertices v_i, v_j in P' it holds that $|i - j| \leq k$. This implies that P' is a subgraph of a k -th power of a path. Thus the subgraph G of P' has treewidth at most k . This finishes the proof. \square

To prove Lemma 6.11 we use the Erdős-Gallai-criterion:

Theorem 6.12 (Erdős and Gallai [EG60]). *There is a graph with degree sequence $d_1 \geq \dots \geq d_n$ if and only if $\sum_{i=1}^n d_i$ is even and if for all $\ell = 1, \dots, n$*

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell - 1) + \sum_{i=\ell+1}^n \min(d_i, \ell). \quad (6.11)$$

Proof of Lemma 6.11. We check the conditions of Theorem 6.12 for the degree sequence \mathbf{d} . The parity condition holds as 4 divides $c(2r + c + 1)$ and

$$\sum_{i=1}^{c+r} d_i = cr + \frac{c(c+1)}{2} = \frac{c}{2}(2r + c + 1).$$

Let us now verify (6.11). If $\ell > c$, then

$$\sum_{i=1}^{\ell} d_i \leq c\ell \leq \ell(\ell - 1) \leq \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell).$$

Thus we can assume that $\ell \leq c$. Two remarks: Firstly, $\min(d_i, \ell) = \ell$ for $i = 1, \dots, \leq c + r - \ell + 1$. Consequently, if $2\ell \leq c + r$ then

$$\begin{aligned} \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell) &= \ell(\ell - 1) + (c + r - 2\ell + 1)\ell + \frac{\ell(\ell - 1)}{2} \\ &= \frac{\ell}{2}(2r - 1 - \ell) + c\ell. \end{aligned} \quad (6.12)$$

Secondly, if $\ell > r$, then

$$\sum_{i=1}^{\ell} d_i = c\ell - \frac{(\ell - r - 1)(\ell - r)}{2} = c\ell + \frac{\ell}{2}(2r + 1 - \ell) - \frac{1}{2}(r^2 + r). \quad (6.13)$$

Now suppose that $2\ell \leq c + r$. For $\ell \leq r$, we have $\sum_{i=1}^{\ell} d_i = c\ell$ and hence (6.11) is easily seen to be satisfied in light of (6.12). On the other hand, for $\ell > r$ the assumption of $r^2 \geq c$ together with a comparison of (6.12) and (6.13) gives (6.11).

So let $2\ell > c + r$. This implies that $\ell > r$. Consequently, the right hand side of (6.11) is

$$\begin{aligned} \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell) &= \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} d_i \\ &= \ell(\ell - 1) + \frac{1}{2}(c + r - \ell)(c + r - \ell + 1). \end{aligned}$$

It follows from equation (6.13) that (6.11) is satisfied if the following expression is non-negative.

$$\begin{aligned} &2\ell(\ell - 1) + (c + r - \ell)(c + r - \ell + 1) - (2c\ell + \ell(2r + 1 - \ell) - (r^2 + r)) \\ &= 4\ell^2 - 4\ell(c + r) + (c + r)^2 + (c + 2r + r^2) - 4\ell \\ &= (2\ell - (c + r))^2 + (c + r) + (r + r^2) - 4\ell \\ &= (2\ell - (c + r))^2 - 2\left(2\ell - \frac{(c + r) + (r + r^2)}{2}\right) \end{aligned} \quad (6.14)$$

First, let $r^2 = c$. Then (6.14) equals

$$(2\ell - (c + r))^2 - 2(2\ell - (c + r)) \quad (6.15)$$

The term (6.15) is negative only if $2\ell - (c + r) = 1$. As $c + r = r^2 + r$ is even (for any integer r), (6.15) and thus (6.14) is non-negative.

Now let $r^2 > c$. Then (6.14) is strictly greater than (6.15) and hence non-negative. This shows that (6.11) is satisfied.

As (6.11) holds for all ℓ , there is a graph with degree sequence \mathbf{d} .

□

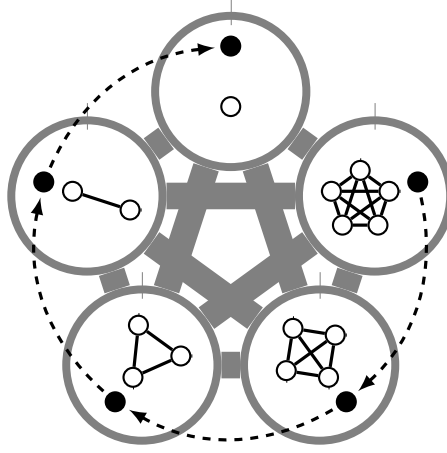


Figure 6.2.: The graph G_5 with the vertices v_i drawn in black; thick gray edges indicate that two vertex sets are complete to each other; elimination order of the v_i is shown in dashed lines

6.5. Degenerate Graphs

Recall that a graph G is k -degenerate if there is an enumeration v_n, \dots, v_1 of the vertices such that v_{i+1} has degree at most k in $G - \{v_n, \dots, v_i\}$ for every i . By simple induction following the elimination order we can obtain a bound with half the degree loss of (6.1):

$$2|E(G)| \leq \Delta|V(G)| - \frac{1}{2}(\Delta - k)(\Delta - k + 1). \quad (6.16)$$

The bound in (6.16) turns out to be tight for some Δ, k as the construction below shows. Consequently, Proposition 6.3 can easily be transferred: Any simple k -degenerate graph of maximum degree $\Delta \geq k + 1/2 + \sqrt{2k+1}/4$ is not overfull and therefore has fractional chromatic index $\chi'_f(G) = \Delta$.

Consider a positive integer p and let G_p be the complement of the disjoint union of p stars $K_{1,1}, K_{1,2}, \dots, K_{1,p}$; see Figure 6.2. Denote the centre of the i th star by v_i , and let W be the union of all leaves. The graph G_p has $n = p(p+1)/2 + p$ vertices and satisfies $\deg(v_i) = n - 1 - i$ for $i = 1, \dots, p$ and $\deg(w) = n - 2$ for $w \in W$. In particular, the maximum degree of G_p is $\Delta = n - 2$. Setting $k = n - 1 - p$, we note that G_p is k -degenerate as v_p, v_{p-1}, \dots, v_1 followed by an arbitrary enumeration of W is an elimination order. Finally, we observe that G_p satisfies (6.16) with equality.

Part III.

On Cycle Decompositions

7. Cycle Decompositions of Pathwidth-6 Graphs

The content of this chapter is based on the paper [FGH17] by Elke Fuchs, Irene Heinrich, and the author of this thesis.

Hajós’ conjecture asserts that a simple Eulerian graph on n vertices can be decomposed into at most $\lfloor (n-1)/2 \rfloor$ cycles. The conjecture is only proved for graph classes in which every element contains vertices of degree 2 or 4. We develop new techniques to construct cycle decompositions. They work on the common neighbourhood of two degree-6 vertices. With these techniques we find structures that cannot occur in a minimal counterexample to Hajós’ conjecture and verify the conjecture for Eulerian graphs of pathwidth at most 6. This implies that these graphs satisfy the *small cycle double cover conjecture*.

7.1. Introduction

It is well-known that the edge set of an Eulerian graph can be decomposed into cycles. In this context, a natural question arises: How many cycles are needed to decompose the edge set of an Eulerian graph? Clearly, a graph G with a vertex of degree $|V(G)|-1$ cannot be decomposed into less than $\lfloor (|V(G)|-1)/2 \rfloor$ many cycles. Thus, for a general graph G , we cannot expect to find a cycle decomposition with less than $\lfloor (|V(G)|-1)/2 \rfloor$ many cycles. Hajós’ conjectured that this number of cycles will always suffice.¹

Conjecture 7.1 (Hajós’ conjecture (see [Lov68])). *Every simple Eulerian graph G has a cycle decomposition with at most $\lfloor (|V(G)|-1)/2 \rfloor$ many cycles.*

Granville and Moisiadis [GM87] showed that for every $n \geq 3$ and for every $i \in \{1, \dots, \lfloor (n-1)/2 \rfloor\}$ there exists a connected graph with n vertices and maximum degree at most 4 whose minimal cycle decomposition consists of exactly i cycles. This shows that — even if the maximal degree is restricted to 4 — the bound $\lfloor (|V(G)|-1)/2 \rfloor$ is best possible.

A simple lower bound on the minimal number of necessary cycles is the maximum degree divided by 2. This bound is achieved by the complete bipartite graph $K_{2k,2k}$ that can be decomposed into k Hamiltonian cycles (see [LA76]). In general, all graphs with a Hamilton decomposition (for example complete graphs K_{2k+1} [Als08]) trivially satisfy Hajós’ conjecture.

¹Originally, Hajós’ conjectured a bound of $\lfloor |V(G)|/2 \rfloor$. Dean [Dea86] showed that Hajós’ conjecture is equivalent to the conjecture with bound $\lfloor (|V(G)|-1)/2 \rfloor$.

Hajós’ conjecture remains wide open for most classes. Heinrich, Natale and Streicher [HNS17] verified Hajós’ conjecture for small graphs by exploiting Lemma 7.6, 7.8, 7.10, and 7.11 of this chapter as well as random heuristics and integer programming techniques:

Theorem 7.2 (Heinrich, Natale and Streicher [HNS17]). *Every simple Eulerian graph with at most 12 vertices satisfies Hajós’ Conjecture.*

Apart from Hamilton decomposable (and small) graphs, the conjecture has (to our knowledge) only been shown for graph classes in which every element contains vertices of degree at most 4. Granville and Moisiadis [GM87] showed that Hajós’ conjecture is satisfied for all Eulerian graphs with maximum degree at most 4. Fan and Xu [FX02] showed that all Eulerian graphs that are embeddable in the projective plane or do not contain the minor K_6^- satisfy Hajós’ conjecture. To show this, they provided four operations involving vertices of degree less than 6 that transform an Eulerian graph not satisfying Hajós’ conjecture into another Eulerian graph not satisfying the conjecture that contains at most one vertex of degree less than 6. This statement generalises the work of Granville and Moisiadis [GM87]. As all four operations preserve planarity, the statement further implies that planar graphs satisfy Hajós’ conjecture. This was shown by Seyffarth [Sey92] before. The conjecture is still open for toroidal graphs. Xu and Wang [XW05] showed that the edge set of each Eulerian graph that can be embedded on the torus can be decomposed into at most $\lfloor (|V(G)| + 3)/2 \rfloor$ cycles. Heinrich and Krumke [HK17] introduced a linear time procedure that computes minimum cycle decompositions in treewidth-2 graphs of maximum degree 4.

We contribute to the sparse list of graph classes satisfying Hajós’ conjecture. Our class contains graphs without any vertex of degree 2 or 4 — in contrast to the above mentioned graph classes.

Theorem 7.3. [FGH17] *Every Eulerian graph G of pathwidth at most 6 satisfies Hajós’ conjecture.*

As graphs of pathwidth at most 5 contain two vertices of degree less than 6, it suffices to concentrate on graphs of pathwidth exactly 6. All such graphs with at most one vertex of degree 2 or 4 contain two degree-6 vertices that are either non-adjacent with the same neighbourhood or adjacent with four or five common neighbours. We use these structures to construct cycle decompositions.

With similar ideas, it is possible attack graphs of treewidth 6. As more substructures may occur, we restrict ourselves to graphs of pathwidth 6.

A *cycle double cover* of a graph G is a collection \mathcal{C} of cycles of G such that each edge of G is contained in exactly two elements of \mathcal{C} . The popular *cycle double cover conjecture* asserts that every 2-edge connected graph admits a cycle double cover. This conjecture is trivially satisfied for Eulerian graphs. Hajós’ conjecture implies a conjecture of Bondy regarding the Cycle double cover conjecture.

Conjecture 7.4 (Small Cycle Double Cover Conjecture (Bondy [Bon90])). *Every simple 2-edge connected graph G admits a cycle double cover of at most $|V(G)| - 1$ many cycles.*

As a cycle double cover may contain a cycle twice, we can conclude the following directly from Theorem 7.3.

Corollary 7.5. *[FGH17] Every Eulerian graph G of pathwidth at most 6 satisfies the small cycle double cover conjecture.*

7.2. Reducible Structures

All graphs considered in this chapter are Eulerian.

In order to prove our main theorem, we consider a cycle decomposition of a graph G as a colouring of the edges of G where each colour class is a cycle. We define a *legal colouring* c of a graph G as a map

$$c : E(G) \mapsto \{1, \dots, \lfloor (|V(G)| - 1)/2 \rfloor\}$$

where each colour class $c^{-1}(i)$ for $i \in \{1, \dots, \lfloor (|V(G)| - 1)/2 \rfloor\}$ is the edge set of a cycle of G . A legal colouring is thus associated to a cycle decomposition of G that satisfies Hajós' conjecture.

Using recolouring techniques, we show the following lemmas for two degree-6 vertices with common neighbourhood N of size 4, 5 or 6. All proofs can be found in Section 7.4.

Lemma 7.6. *[FGH17] Let G be an Eulerian graph with two degree-6 vertices u, v with*

$$N(u) = N \cup \{v\} \quad N(v) = N \cup \{u\}.$$

Let all Eulerian graphs obtained from $G - \{u, v\}$ by addition or deletion of edges with both end vertices in N have a legal colouring.

If $G[N]$ contains at least one edge, or if $G - \{u, v\}$ contains a vertex that is adjacent to at least three vertices of N then G also has a legal colouring.

Lemma 7.7. *[FGH17] Let G be an Eulerian graph with two degree-6 vertices u, v with*

$$N(u) = N \cup \{u, x_v\} \quad N(v) = N \cup \{v, x_u\}.$$

Let P be an x_u - x_v -path in $G - \{u, v\} - N$. Further let all Eulerian graphs obtained from $G - \{u, v\}$ by addition and deletion of edges with both end vertices in $N \cup \{x_u, x_v\}$ and by optional deletion of $E(P)$ have a legal colouring.

If $G[N \cup \{x_u, x_v\}]$ contains at least one edge not equal to $x_u x_v$, or if $G - \{u, v\}$ contains a vertex that is adjacent to at least three vertices of N then G also has a legal colouring.

Lemma 7.8. *[FGH17] Let G be an Eulerian graph with two degree-6 vertices u, v with*

$$N(u) = N(v) = N.$$

Let all Eulerian graphs obtained from $G - \{u, v\}$ by addition or deletion of edges with both end vertices in N have a legal colouring.

If $G[N]$ contains at least one edge, or if $G - \{u, v\}$ contains a vertex that is adjacent to at least three vertices of N then G also has a legal colouring.

The next two results are not required for the proof of Theorem 7.3. We nevertheless state them here.

The first lemma is useful for graphs with an odd number of vertices.

Lemma 7.9. *[FGH17] Let G be an Eulerian graph on an odd number n of vertices that contains a vertex u of degree 2 or 4 with neighbourhood N . Let G' be obtained from $G - \{u\}$ by addition or deletion of arbitrary edges in $G[N]$. If G' has a legal colouring, then G has a legal colouring.*

If a graph G contains a degree-2 vertex v with independent neighbours x_1, x_2 , then it is clear that a legal colouring of $G - v + x_1x_2$ can be transformed into a legal colouring of G . Granville and Moisiadis [GM87] observed a similar relation for a degree-4 vertex.

Lemma 7.10 (Granville and Moisiadis [GM87]). *Let G be an Eulerian graph containing a vertex v with neighbourhood $N = \{x_1, \dots, x_4\}$ such that $G[N]$ contains the edge x_1x_2 but not the edge x_3x_4 . If $G - \{vx_3, vx_4\} + \{x_3x_4\}$ has a legal colouring, then G also has a legal colouring.*

Generalising this idea, we analyse the neighbourhood of a degree-6 vertex.

Lemma 7.11. *[FGH17] Let G be an Eulerian graph that contains a degree-6 vertex u with neighbourhood $N_G(u) = \{x_1, \dots, x_6\}$ such that $\{x_1, x_2, x_3, x_4\}$ is a clique and $x_5x_6 \notin E(G)$. If $G' = G - \{x_5u, ux_6\} + \{x_5x_6\}$ has a legal colouring, then G has a legal colouring.*

7.3. Recolouring Techniques

In this section, we provide recolouring techniques necessary to prove Lemma 7.6, 7.7 and 7.8. For a path P or a cycle C we write $c(P) = i$ or $c(C) = i$ to express that all edges of P respectively C are coloured with colour i . We start with a statement about monochromatic triangles.

Lemma 7.12. *[FGH17] Let H be a graph with a clique $\{x_1, x_2, x_3, y\}$ and with a legal colouring c . Then there is a legal colouring c' of H in which the cycle $x_1x_2x_3x_1$ is not monochromatic.*

Proof. Figure 7.1 illustrates the recolourings described in this proof. Assume that $x_1x_2x_3x_1$ is monochromatic of colour i in c . First assume that

$$\text{an edge of colour } j := c(y_1y) \text{ is adjacent to } y_2 \quad (7.1)$$

for two distinct vertices y_1, y_2 in $\{x_1, x_2, x_3\}$. Without loss of generality, the path P' of colour j between y and y_2 along the path $c^{-1}(j) - \{yy_1\}$ does not contain the vertex y_3 (where $\{y_3\} = \{x_1, x_2, x_3\} - \{y_1, y_2\}$). Flip the colours of the monochromatic paths y_1y_2 and $y_1yP'y_2$, ie set $c'(y_1y_2) = j$, $c'(y_1yP'y_2) = c(y_1y_2)$ and $c'(e) = c(e)$ for all other edges $e \in E(H)$. The obtained colouring is legal: By construction, all colour classes are cycles and at most $\lfloor (|V(H)| - 1)/2 \rfloor$ many colours are used. Further, the cycle $x_1x_2x_3x_1$ is not monochromatic.

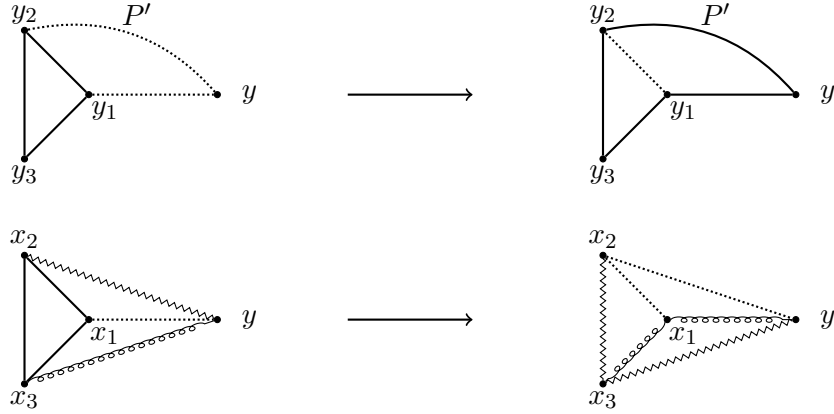


Figure 7.1.: The two possible cases in Lemma 7.12 to obtain a colouring in which a fixed triangle is not monochromatic; the different styles of the edges represent the colours

If (7.1) does not hold, we can get rid of one colour. Set $c'(x_1x_2y) = c(x_1y)$, $c'(x_2x_3y) = c(x_2y)$, $c'(x_3x_1y) = c(x_3y)$, and $c'(e) = c(e)$ for all other edges $e \in E(H)$. By construction, all colour classes are cycles and $x_1x_2x_3x_1$ is not monochromatic. \square

Figure 7.2 illustrates the following simple observation.

Observation 7.13. *Let P_1 be an x_1 - y_1 -path that is vertex-disjoint from an x_2 - y_2 -path P_2 . Then there are three possibilities to connect $\{x_1, y_1\}$ and $\{x_2, y_2\}$ by two vertex-disjoint paths that do not intersect $V(P_i) - \{x_i, y_i\}$ for $i = 1, 2$. Two of the possibilities yield a cycle — the third way leads to two cycles.*

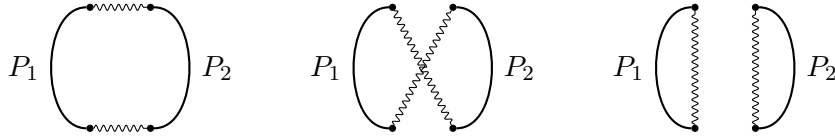


Figure 7.2.: The three possible ways to connect the end vertices of two paths P_1 and P_2 ; the connection between the end vertices is drawn with jagged lines

Lemma 7.14, 7.15 and 7.16 are all based on the same elementary fact: Let G and G' be graphs with $|V(G)| = |V(G')| + 2$. If G' allows for a cycle decomposition with at most $\lfloor (|V(G')| - 1)/2 \rfloor$ cycles, then any cycle decomposition of G that uses at most one cycle more than the cycle decomposition of G' shows that G is not a counterexample to Hajós' conjecture.

This fact leads us to the following inductive approach: Given a graph G with two vertices u and v of degree 6, we remove u and v from G and might remove or add edges to obtain a graph G' . If G' has a cycle decomposition with at most $\lfloor (|V(G')| - 1)/2 \rfloor$ cycles we construct a cycle decomposition of G from it. We reroute some of the cycles

in an appropriate way such that u and v are each touched by two cycles. Now, there remain some edges in G that are not covered. If those edges form a cycle, we have found a cycle decomposition of G . If a cycle is not rerouted to u or v twice, the cycle decomposition of G satisfies Hajós' conjecture.

To describe this inductive approach in a coherent way, we regard the cycle decomposition of G' as a legal colouring. Then we regard the above reroutings as recolourings where we have to make sure that no colour appears twice at u or v . If the edges that have not yet received a colour form a cycle, we associate the new colour $\lfloor (|V(G')| - 1)/2 \rfloor$ to this cycle. The obtained colouring of the edges then uses at most $\lfloor (|V(G')| - 1)/2 \rfloor$ many colours and each colour class is a cycle. Thus, we have constructed a legal colouring.

Lemma 7.14. *[FGH17] Let G be an Eulerian graph without legal colouring that contains two adjacent vertices u and v of degree 6 with common neighbourhood $N = \{x_1, \dots, x_5\}$. Define $G' = G - \{u, v\}$ and let c' be a legal colouring of G' .*

- (i) *If $G[N]$ contains a path $P' = y_1y_2y_3y_4$ of length 3 then P' is monochromatic in c' .*
- (ii) *Let $G[N]$ contain an independent set $S = \{y_1, y_2, y_3\}$ of size 3. If N is not an independent set or if there is a vertex in G' that is adjacent to y_1, y_2 and y_3 , then $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ does not have a legal colouring.*
- (iii) *If $G[N]$ contains an induced path $y_1y_2y_3y_4$ of length 3 then $G'' = G' - \{y_2y_3\} + \{y_2y_4, y_4y_1, y_1y_3\}$ does not have a legal colouring.*
- (iv) *If $G[N]$ contains a triangle $y_1y_2y_3y_1$, a vertex y_4 that is not adjacent to y_1 and y_3 and a vertex $y_5 \in N - \{y_1, y_2, y_3, y_4\}$ adjacent to y_4 then $G'' = G' - \{y_1y_3\} + \{y_1y_4, y_3y_4\}$ does not have a legal colouring.*

Proof of (i). If y_3y_4 has a colour different from y_1y_2 and y_2y_3 , then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vy_3) = c'(y_2y_3) \quad c(y_3uvy_4) = c'(y_3y_4).$$

If y_2y_3 has a colour different from y_1y_2 and y_3y_4 , then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vuy_3) = c'(y_2y_3) \quad c(y_3vy_4) = c'(y_3y_4).$$

The case distinction makes sure that the modified colour classes remain cycles. By further setting $c(y_1y_2y_3y_4uy_5vy_1) = \lfloor (|V(G)| - 1)/2 \rfloor$ and $c(e) = c'(e)$ for all other edges e we have constructed a legal colouring c of G . \square

Proof of (ii). Set $\{y_4, y_5\} = N - \{y_1, y_2, y_3\}$ and let c'' be a legal colouring of G'' . First assume that $c''(y_1y_2) \notin \{c''(y_2y_3), c''(y_3y_1)\}$. Then one can easily check that the following is a legal colouring of G .

$$\begin{aligned} c(y_2uvy_1) &= c''(y_2y_1) & c(y_2vy_3) &= c''(y_2y_3) & c(y_3uy_1) &= c''(y_3y_1) \\ c(y_4uy_5vy_4) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c''(e) \text{ for all other edges } e \end{aligned} \tag{7.2}$$

By symmetry, we are done unless the triangle $y_1y_2y_3y_1$ is monochromatic in c'' . By Lemma 7.12, we can suppose that there is no vertex y in G'' that is adjacent to y_1 , y_2 and y_3 . Suppose that N is not independent. Without loss of generality, we can assume that $G[N]$ contains an edge, say y_4y_1 incident to one of the vertices of the independent 3-set. (Otherwise, we can choose another suitable independent 3-set in $G[N]$). Then by construction the following is a legal colouring of G .

$$\begin{aligned} c(y_1uy_4) &= c''(y_1y_4) & c(y_2uy_3) &= c''(y_2y_3) & c(y_2vy_3) &= c''(y_2y_1y_3) \\ c(y_1y_4uy_5vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c''(e) \text{ for all other edges } e \end{aligned} \quad \square$$

Proof of (iii). Let G'' have a legal colouring c'' and let y_5 be the unique vertex in $N - \{y_1, y_2, y_3, y_4\}$.

If $c''(y_2y_4) = c''(y_4y_1) = c''(y_1y_3)$, set

$$\begin{aligned} c(y_1uy_2) &= c''(y_1y_2) & c(y_2vuy_3) &= c''(y_2y_4y_1y_3) & c(y_3vy_4) &= c''(y_3y_4) \\ c(uy_5vy_1y_2y_3y_4u) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

If $c''(y_1y_3)$ is different from $c''(y_2y_4)$ and $c''(y_4y_1)$, set

$$\begin{aligned} c(y_1uvy_3) &= c''(y_1y_3) & c(y_4vy_1) &= c''(y_4y_1) & c(y_2uy_4) &= c''(y_2y_4) \\ c(uy_5vy_2y_3u) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

If $c''(y_2y_4)$ is different from $c''(y_1y_3)$ and $c''(y_1y_4)$, the colouring is defined similarly by relabelling the vertices y_1, \dots, y_5 .

If $c''(y_4y_1)$ is different from $c''(y_1y_3)$ and $c''(y_2y_4)$, set

$$\begin{aligned} c(y_1vy_3) &= c''(y_1y_3) & c(y_4vuy_1) &= c''(y_4y_1) & c(y_2uy_4) &= c''(y_2y_4) \\ c(uy_5vy_2y_3u) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

Further set $c(e) = c''(e)$ for all other edges e in all cases. Again, the case distinction makes sure that all colour classes are cycles and we have constructed a legal colouring. \square

Proof of (iv). Let c'' be a legal colouring of G'' . First assume that $c''(y_2y_3)v$ is not contained in $\{c''(y_3y_4), c''(y_1y_4)\}$. Then set

$$\begin{aligned} c(y_2vuy_3) &= c''(y_2y_3) & c(y_1uy_4) &= c''(y_1y_4) & c(y_3vy_4) &= c''(y_3y_4) \\ c(uy_5vy_1y_3y_2u) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

If $c''(y_1y_2) \notin \{c''(y_3y_4), c''(y_1y_4)\}$, the colouring is defined as above by interchanging the roles of y_1 and y_3 .

Now assume that $c''(y_2y_3), c''(y_1y_2) \in \{c''(y_3y_4), c''(y_1y_4)\}$. If $c''(y_3y_4) = c''(y_1y_4)$, then the cycle $y_1y_2y_3y_4y_1$ is monochromatic. Set

$$\begin{aligned} c(y_4uvy_5) &= c''(y_4y_5) \\ c(y_1vy_3y_2uy_1) &= c''(y_1y_2y_3y_4y_1) & c(y_1y_3uy_5y_4vy_2y_1) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

If $c''(y_3y_4) \neq c''(y_1y_4)$, then either $c''(y_2y_3) = c''(y_3y_4)$ or $c''(y_2y_3) = c''(y_1y_4)$. If $c''(y_2y_3) = c''(y_3y_4)$, set

$$\begin{aligned} c(y_2uy_3vy_4) &= c''(y_2y_3y_4) & c(y_1vuy_4) &= c''(y_1y_4) \\ c(y_1uy_5vy_2y_3y_1) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

If $c''(y_2y_3) = c''(y_1y_4)$, set

$$\begin{aligned} c(y_2uy_3) &= c''(y_2y_3) & c(y_1vy_4) &= c''(y_1y_4) & c(y_3vuy_4) &= c''(y_3y_4) \\ c(y_1uy_5vy_2y_3y_1) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

By setting $c(e) = c''(e)$ for all other edges e we have constructed a legal colouring for G in all cases. \square

If u and v are adjacent degree-6 vertices that have a common neighbourhood N of size 4, we call the two vertices that are adjacent with exactly one of u, v the *private neighbours* of u and v . Here, we denote them by x_u and x_v . If there is a x_u - x_v -path P in $G - \{u, v\} - N$, it is possible to translate all techniques of Lemma 7.16. It suffices to delete u, v and $E(P)$ to obtain another Eulerian graph: In all recolourings of Lemma 7.16, the edges uy, vy for one vertex $y \in N$ were contained in the new colour class $\lfloor (|V(G)| - 1)/2 \rfloor$. If we have two private neighbours x_u and x_v it suffices to replace the path uyv by the path ux_uPx_vv in this colour class. This means, we can regard x_uPx_v as a single vertex y .

Lemma 7.15. [FGH17] *Let G be an Eulerian graph without legal colouring that contains two adjacent vertices u and v of degree 6 with common neighbourhood $N = \{x_1, \dots, x_4\}$ and $N_G(u) = N \cup \{x_u, v\}$ as well as $N_G(v) = N \cup \{x_v, u\}$. Let P be an x_u - x_v -path in $G - \{u, v\} - N$. Define $G' = G - \{u, v\} - E(P)$ and let c' be a legal colouring of G' .*

- (i) *If $G[N \cup \{x_u, x_v\}]$ contains a path $P' = y_1y_2y_3y_4$ with $y_2, y_3, y_4 \in N$ of length 3 then P' is monochromatic in c' .*
- (ii) *Let $G[N]$ contain an independent set $S = \{y_1, y_2, y_3\}$ of size 3. If $G[N \cup \{x_u, x_v\}]$ contains an edge $x_i x_j \neq x_u x_v$ or if there is a vertex in G' that is adjacent to y_1, y_2 and y_3 then $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ does not have a legal colouring.*
- (iii) *If $G[N \cup \{x_u, x_v\}]$ does not contain the edges $x_u y_1, y_1 y_2, y_2 x_v$ for two vertices $y_1, y_2 \in N$ but contains an edge with end vertex y_1 or y_2 then $G'' = G - \{u, v\} + \{x_u y_1, y_1 y_2, y_2 x_v\}$ does not have a legal colouring.*
- (iv) *If G contains the edges $y_1 y_2, y_3 y_4, y_1 y_5$ with $y_1, y_2, y_3, y_4 \in N$ and $y_5 \in \{x_u, x_v\}$ but not the edges $y_1 y_3, y_2 y_3$ then $G'' = G' - \{y_1 y_2\} + \{y_1 y_3, y_3 y_2\}$ does not have a legal colouring.*
- (v) *If $G[N \cup \{x_u, x_v\}]$ contains a triangle $y_1 y_2 y_3 y_1$ with $y_1, y_2, y_3 \in N$, a vertex $y_4 \in N - \{y_1, y_2, y_3\}$ that is not adjacent to y_1 and y_3 and a vertex $y_5 \in \{x_u, x_v\}$ adjacent to y_4 then $G'' = G' - \{y_1 y_3\} + \{y_1 y_4, y_3 y_4\}$ does not have a legal colouring.*

Proof of (i). The proof is very similar to the proof of Lemma 7.14.(i) if we regard x_uPx_v as one single vertex. We will nevertheless give a detailed proof. By symmetry of u and v (and thus of x_u and x_v), we can assume that y_1 is either contained in N or is equal to x_u . Suppose that P is not monochromatic.

If $c'(y_1y_2) \notin \{c'(y_2y_3), c'(y_3y_4)\}$, then set

$$c(y_1uvy_2) = c'(y_1y_2) \quad c(y_2uy_3) = c'(y_2y_3) \quad c(y_3vy_4) = c'(y_3y_4).$$

If $c'(y_2y_3) \notin \{c'(y_1y_2), c'(y_3y_4)\}$, then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vuy_3) = c'(y_2y_3) \quad c(y_3vy_4) = c'(y_3y_4).$$

If $c'(y_3y_4) \notin \{c'(y_1y_2), c'(y_2y_3)\}$, then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vy_3) = c'(y_2y_3) \quad c(y_3uvy_4) = c'(y_3y_4).$$

If $y_1 \in N$ the following completes by construction a legal colouring c of G :

$$\begin{aligned} c(y_1y_2y_3y_4x_uPx_vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Now suppose that $y_1 = x_u$ and that x_4, x_v are not contained in the path $y_1y_2y_3y_4$. Then, the following completes by construction a legal colouring c of G :

$$\begin{aligned} c(y_1y_2y_3y_4x_4vx_vPy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned} \quad \square$$

Proof of (ii). The proof is very similar to the proof of Lemma 7.14.(ii) if we regard x_uPx_v as one single vertex. \square

Proof of (iii). Assume that c'' is a legal colouring of G'' and let $\{y_3, y_4\} = N - \{y_1, y_2\}$. By symmetry of u and v (and thus of y_1 and y_2) we can suppose that $y_1y_4 \in E(G)$. If y_1x_u has a colour different from the colour of y_1y_2 and y_2x_v , set

$$\begin{aligned} c(x_uuvy_1) &= c''(x_uy_1) \quad c(y_1uy_2) = c''(y_1y_2) \quad c(y_2vx_v) = c''(y_2x_v) \\ c(y_3uy_4vy_3) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

An analogous colouring can be defined if x_vy_2 has a colour different from the colour of y_1y_2 and x_uy_1 .

If y_1y_2 has a colour different from the colour of y_1x_u and y_2x_v , then set

$$\begin{aligned} c(x_uuy_1) &= c''(x_uy_1) \quad c(y_1vuy_2) = c''(y_1y_2) \quad c(y_2vx_v) = c''(y_2x_v) \\ c(y_3uy_4vy_3) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

Now suppose that all three edges x_uy_1, y_1y_2, y_2x_v have the same colour. Then, y_1y_4 has a different colour. Set

$$\begin{aligned} c(x_uuy_2) &= c''(x_uy_1y_2) \quad c(y_1uvy_4) = c''(y_1y_4) \quad c(y_2vx_v) = c''(y_2x_v) \\ c(y_3vy_1y_4u) &= \lfloor (|V(G)| - 1)/2 \rfloor. \end{aligned}$$

In all cases, set $c(e) = c''(e)$ for all other edges e . The case distinction now makes sure that we constructed a legal colouring for G . \square

Proof of (iv). Assume that c'' is a legal colouring of G'' . Without loss of generality let $y_5 = x_u$.

First suppose that all three edges $x_u y_1, y_1 y_3, y_3 y_2$ have the same colour. Then, $y_3 y_4$ has a different colour and the following gives by construction a legal colouring for G :

$$\begin{aligned} c(x_u y_1) &= c''(x_u y_1) \quad c(y_1 v y_2) = c''(y_1 y_3 y_2) \quad c(y_3 v y_4) = c''(y_3 y_4) \\ c(u y_2 y_1 x_u P x_v v y_3 y_4 u) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Now suppose that $x_u y_1, y_1 y_3, y_3 y_2$ is not monochromatic.

If $x_u y_1$ has a colour different from the colours of $y_1 y_3$ and $y_3 y_2$, set

$$c(x_u v y_1) = c''(x_u y_1) \quad c(y_1 u y_3) = c''(y_1 y_3) \quad c(y_3 v y_2) = c''(y_3 y_2).$$

If $y_1 y_3$ has a colour different from the colours of $x_u y_1$ and $y_3 y_2$, set

$$c(x_u y_1) = c''(x_u y_1) \quad c(y_1 v u y_3) = c''(y_1 y_3) \quad c(y_3 v y_2) = c''(y_3 y_2).$$

If $y_3 y_4$ has a colour different from the colours of $x_u y_1$ and $y_1 y_3$, set

$$c(x_u y_1) = c''(x_u y_1) \quad c(y_1 v y_3) = c''(y_1 y_3) \quad c(y_3 v y_2) = c''(y_3 y_2).$$

By setting $c(u y_2 y_1 x_u P x_v v y_3 y_4 u) = \lfloor (|V(G)| - 1)/2 \rfloor$ and $c(e) = c''(e)$ for all other edges e , we obtain by construction in all cases a legal colouring for G . \square

Proof of (v). The proof is very similar to the proof of Lemma 7.14.(iv) if we regard $x_u P x_v$ as one single vertex. \square

In our last recolouring lemma, we consider two degree-6 vertices that are not adjacent but have six common neighbours x_1, \dots, x_6 . Some of the recolouring techniques of this lemma need a somewhat deeper look into the cycle decomposition. They rely on a generalisation of the recolourings used in Lemma 7.14 and 7.12. We introduce two pieces of notation. For two distinct vertices $x_i, x_j \in N = \{x_1, \dots, x_6\}$, a path $P_{x_i x_j}$ always denotes an x_i - x_j -path that is not intersecting with $N - \{x_i, x_j\}$.

For a cycle C and two distinct vertices $x_i, x_j \in N = \{x_1, \dots, x_6\} \cap V(C)$ there are two $x_i - x_j$ -paths along C . If there is a unique path that is not intersecting with $N - \{x_i, x_j\}$, we denote this path by $C_{x_i x_j}$.

Lemma 7.16. [FGH17] *Let G be an Eulerian graph without legal colouring and let G contain two degree-6 vertices u and v with common neighbourhood $N = \{x_1, \dots, x_6\}$. Define $G' = G - \{u, v\}$ and let c' be a legal colouring of G' .*

- (i) *If G' contains two vertex-disjoint paths $P_{y_1 y_2} P_{y_2 y_3}$ and $P_{y'_1 y'_2} P_{y'_2 y'_3}$ with $N = \{y_1, y_2, y_3, y'_1, y'_2, y'_3\}$ where the four paths $P_{y_1 y_2}, P_{y_2 y_3}, P_{y'_1 y'_2}, P_{y'_2 y'_3}$ are monochromatic in c' , then at least three of the four paths have the same colour in c' .*
- (ii) *Let G' contain a path $P' = P_{y_1 y_2} P_{y_2 y_3} P_{y_3 y_4} P_{y_4 y_5}$ with $\{y_1, \dots, y_5\} \subset N$ where $P_{y_i y_{i+1}}$ is monochromatic in c' for each $i \in \{1, 2, 3, 4\}$. Then $c'(P_{y_1 y_2}) = c'(P_{y_3 y_4})$ or $c'(P_{y_2 y_3}) = c'(P_{y_4 y_5})$.*

- (iii) If $G[N]$ contains an independent set $S = \{y_1, y_2, y_3\}$ of size 3 and if $G[N]$ contains at least one edge or there is a vertex in G' that is adjacent to y_1, y_2 and y_3 then $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ does not have a legal colouring.
- (iv) If $G[N]$ contains a path $P' = y_1y_2y_3y_4$ of length 3, then P' is monochromatic in c' .

Proof of (i). Suppose that less than three of the paths have the same colour. Then, without loss of generality $c'(P_{y'_1y'_2}) \neq c'(P_{y_1y_2})$ and $c'(P_{y'_2y'_3}) \neq c'(P_{y_2y_3})$ and the following is by construction a legal colouring of G :

$$\begin{aligned} c(y_1uy_2) &= c'(P_{y_1y_2}) & c(y_2vy_3) &= c'(P_{y_2y_3}) \\ c(y'_1uy'_2) &= c'(P_{y'_1y'_2}) & c(y'_2vy'_3) &= c'(P_{y'_2y'_3}) \\ c(y_1P_{y_1y_2}P_{y_2y_3}y_3uy'_3P_{y'_3y'_2}P_{y'_2y'_1}y'_1vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned} \quad \square$$

Proof of (ii). Suppose that $c'(P_{y_1y_2}) \neq c'(P_{y_3y_4})$ and $c'(P_{y_2y_3}) \neq c'(P_{y_4y_5})$ and let y_6 be the vertex of N not contained in P' . Then, the following is by construction a legal colouring of G :

$$\begin{aligned} c(y_1uy_2) &= c'(P_{y_1y_2}) & c(y_2vy_3) &= c'(P_{y_2y_3}) \\ c(y_3uy_4) &= c'(P_{y_3y_4}) & c(y_4vy_5) &= c'(P_{y_4y_5}) \\ c(y_1P_{y_1y_2}P_{y_2y_3}P_{y_3y_4}P_{y_4y_5}y_5uy_6vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned} \quad \square$$

Proof of (iii). The proof uses ideas of the proof of Lemma 7.14.(ii).

Let c'' be a legal colouring of G'' . First suppose that $i := c''(y_1y_2)$ is not contained in $\{c''(y_2y_3), c''(y_3y_1)\}$ and let $C = c^{-1}(i)$ be the monochromatic cycle in G'' with colour i .

If there is a vertex $y_4 \in N - \{y_1, y_2, y_3\}$ that is not contained in C set $\{y_5, y_6\} = N - \{y_1, y_2, y_3, y_4\}$ and use the recolouring (7.2) where the edge uv is replaced by the path uy_4v .

Otherwise, $\{y_4, y_5, y_6\} := N - \{y_1, y_2, y_3\}$ is a subset of $V(C)$. Then without loss of generality $C_{y_3y_4}$ and $C_{y_4y_5}$ exist. By construction, the following is a legal colouring of G :

$$\begin{aligned} c(y_2uy_3) &= c''(y_2y_3) & c(y_1vy_2) &= c''(y_1y_2) & c(y_1uy_4vy_5) &= c''(y_3y_1) \\ c(y_3C_{y_3y_4}C_{y_4y_5}y_5uy_6vy_3) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c''(e) \text{ for all other edges } e. \end{aligned}$$

Assume that the triangle $y_1y_2y_3y_1$ is monochromatic in c'' . By Lemma 7.12, there is no vertex y in G' that is adjacent to y_1, y_2 and y_3 . Suppose that N is not independent. Without loss of generality $G[N]$ contains the edge y_4y_1 . Set $\{y_5, y_6\} = N - \{y_1, y_2, y_3, y_4\}$.

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If there is a vertex in $\{y_5, y_6\}$, say y_6 , that is not contained in the cycle $C = c^{-1}(j)$ of colour $j := c'(y_1y_4)$, use the recolouring (7.2) where the edge uv is replaced by the path uy_6v .

If y_5 and y_6 are both contained in C , let S be the segment of $C - \{y_1y_4\}$ that connects y_4 with y_5 . By symmetry of y_5 and y_6 , we can suppose that $y_6 \notin S$. By construction, the following is a legal colouring of G :

$$\begin{aligned} c(y_1vy_4) &= c''(y_1y_4) & c(y_5uy_4) &= c''(S) \\ c(y_2uy_3vy_2) &= c(y_1y_2y_3y_1) & c(y_1y_4Sy_5vy_6uy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c''(e) \text{ for all other edges } e \end{aligned} \quad \square$$

Proof of (iv). Suppose that P is not monochromatic in c' and set $\{y_5, y_6\} = N - \{y_1, y_2, y_3, y_4\}$.

First assume that $c'(y_3y_4) \notin \{c'(y_1y_2), c'(y_2y_3)\}$. Let C be the cycle of colour $c'(y_3y_4)$ in G' .

If there is a vertex in $\{y_5, y_6\}$, say y_5 , that is not in C , then by construction the following is a legal colouring of G :

$$\begin{aligned} c(y_1uy_2) &= c'(y_1y_2) & c(y_2vy_3) &= c'(y_2y_3) & c(y_3uy_5vy_4) &= c'(y_3y_4) \\ c(y_1y_2y_3y_4uy_6vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Now assume that y_5 and y_6 are contained in C . If $C_{y_5y_1}$, $C_{y_6y_1}$, $C_{y_5y_4}$ or $C_{y_6y_4}$ exists then we can apply (ii). Thus, $C_{y_5y_6}$ must exist and by symmetry $C_{y_5y_2}$ and $C_{y_6y_3}$ exist. We can apply (ii) to $y_1y_2, y_2y_3, C_{y_3y_6}$ and $C_{y_5y_6}$.

Thus, for the rest of the proof we can assume that

$$c'(y_2y_3) =: i \notin \{c'(y_1y_2), c'(y_3y_4)\}.$$

Let $C' = c^{-1}(i)$ be the cycle of colour i in G' . If there is a vertex in $\{y_5, y_6\}$, say y_5 , that is not in C' , then by construction the following is a legal colouring of G :

$$\begin{aligned} c(y_1uy_2) &= c'(y_1y_2) & c(y_3vy_4) &= c'(y_3y_4) & c(y_2vy_5uy_3) &= c'(y_2y_3) \\ c(y_1y_2y_3y_4uy_6vy_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Thus, we can assume that

$$y_5 \text{ and } y_6 \text{ are contained in } C'.$$

Now, there are three cases up to symmetry: y_1, y_2 both are not contained in C' , y_1 is contained in C' but y_4 is not, and y_1, y_4 are both contained in C' .

First assume that y_1 and y_4 are not contained in C' . Then, by symmetry, C' is the cycle consisting of $y_2y_3, C'_{y_3y_6}, C'_{y_6y_5}, C'_{y_5y_2}$. We are done by applying (i) to the vertex-disjoint paths $y_1y_2, C'_{y_2y_5}$ and $y_4y_3, C'_{y_3y_6}$.

Next assume that y_1 is contained in C' and y_4 is not contained in C' . First suppose that $C'_{y_6y_3}$ exists. As $C'_{y_5y_1}$ or $C'_{y_5y_2}$ must exist, we are done with (i).

By symmetry, we can now suppose that neither $C'_{y_6y_3}$ nor $C'_{y_5y_3}$ exists. Then $C'_{y_3y_1}$ exists. We can suppose without loss of generality that C' is the cycle consisting of the paths $C'_{y_1y_3}, y_3y_2, C'_{y_2y_6}, C'_{y_6y_5}, C'_{y_5y_1}$ and by construction the following is a legal colouring of G :

$$\begin{aligned} c(y_1uy_2) &= c'(y_1y_2) & c(y_3vy_4) &= c'(y_3y_4) \\ c(y_3y_2vy_1C'_{y_1y_5}C'_{y_5y_6}y_6uy_4y_3) &= i \\ c(y_3uy_5vy_6C'_{y_6y_2}y_2y_1C'_{y_1y_3}y_3) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Last, assume that y_1 and y_4 are both contained in C' . First suppose that $C_{y_5y_6}$ does not exist. Without loss of generality, we can suppose that $C_{y_5y_1}$ exists. Now neither $C'_{y_6y_3}$ nor $C'_{y_6y_4}$ exists; otherwise we are done with (i). Thus, $C_{y_6y_1}$ and $C_{y_6y_2}$ must exist. Thus, $C'_{y_5y_4}$ exists and we are done with (i).

Now suppose that $C_{y_5y_6}$ exists. First suppose that $C_{y_5y_2}$ exists. Then, we are done with (i) if $C_{y_6y_3}$ or $C_{y_6y_4}$ exists. As C' is a cycle, $C_{y_6y_1}$ and thus also $C_{y_4y_1}$ and $C_{y_4y_3}$ exist. The following is by construction a legal colouring of G :

$$\begin{aligned} c(y_3uy_4) &= c'(y_3y_4) & c(y_1vy_2) &= c'(y_1y_2) \\ c(y_5vy_3C'_{y_3y_4}C'_{y_4y_1}y_2uC'_{y_6y_5}y_5) &= i \\ c(y_2y_3y_4vy_6C'_{y_6y_1}y_1uy_5C'_{y_5y_2}y_2) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Thus, we can suppose that none of $C_{y_5y_2}, C_{y_5y_3}, C_{y_6y_2}, C_{y_6y_3}$ exists. Without loss of generality, $C_{y_5y_1}$ exists. As $C_{y_6y_4}$ must exist we are done with (i). \square

7.4. Proofs of Reducibility

In this section we prove Lemma 7.6, 7.7, 7.8, 7.9 and 7.11. In the first three proofs, we use the following observation:

Observation 7.17. [FGH17] *Let x be a vertex of degree at least 3 in a graph H with a legal colouring. Then the neighbourhood N_x of x contains an independent set of size 3 or $G[\{x\} \cup N_x]$ contains a path of length 3 that is not monochromatic.*

Proof of Lemma 7.6. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 7.17 as well as Lemma 7.14.(i) and (ii).

Now, suppose that $G[N]$ contains a vertex, say x_1 of degree 0. As we have seen, $G[N]$ contains no vertex of degree 3 or 4. Thus, $G[N] - \{x_1\}$ contains two non-adjacent vertices, say x_2 and x_3 . Then, $\{x_1, x_2, x_3\}$ is an independent set and we are done by Lemma 7.14.(ii).

We can conclude that all vertices in $G[N]$ have degree 1 or 2. Consequently, the graph is isomorphic to $C_5, K_3 \dot{\cup} P_2, P_3 \dot{\cup} P_2$ or P_5 . The 5-cycle C_5 contains an induced

P_4 , the graph $K_3 \dot{\cup} P_2$ contains a triangle and a vertex that is not adjacent to two of the triangle vertices, the latter two graphs contain an independent set of size 3. Thus, we are done by (iii), (iv), and (ii) of Lemma 7.14. \square

Proof of Lemma 7.7. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 7.17 as well as Lemma 7.15.(i) and (ii). Thus, $G[N]$ must be isomorphic to one of the graphs that we will treat now.

First suppose that the edge set of $G[N]$ equals the empty set, $\{x_1x_2\}$ or $\{x_1x_2, x_1x_3\}$. Then, G contains an independent 3-set and has a legal colouring by Lemma 7.15.(ii).

Next suppose that the edge set of $G[N]$ equals $\{x_1x_2, x_3x_4\}$ or $\{x_1x_2, x_1x_3, x_3x_4\}$. If x_u is adjacent to x_2 , apply Lemma 7.15.(iv) to get a legal colouring: the edges $x_u x_2, x_2 x_1, x_3 x_4$ exist while x_4 is neither adjacent to x_1 nor to x_2 . Similarly, we can apply Lemma 7.15.(iv) if $x_3 x_v \in E(G)$. Thus, we can suppose that neither $x_2 x_u$ nor $x_3 x_v$ exists in G and we are done with Lemma 7.15.(iii): the edges $x_u x_2, x_2 x_3, x_3 x_v$ do not exist while $x_2 x_1 \in E(G)$.

Now suppose that the edge set of $G[N]$ consists of x_1x_2, x_2x_3, x_3x_1 . If x_u is adjacent to x_1 , not all paths of length 3 can be monochromatic and we can apply Lemma 7.15.(i). Thus we can suppose that $x_u x_1 \notin E(G)$. If $x_4 x_v \notin E(G)$ then we can apply Lemma 7.15.(iii) to $x_u x_1, x_1 x_4, x_4 x_v \notin E(G)$ and $x_1 x_3 \in E(G)$ to obtain a legal colouring of G . If $x_4 x_v \in E(G)$ we are done by Lemma 7.15.(v).

Last suppose that the edge set of $G[N]$ consists of $x_1x_2, x_2x_3, x_3x_4, x_4x_1$. If the 4-cycle is not monochromatic, the cycle contains a P_4 that is not monochromatic and we are done by Lemma 7.15.(i). Suppose that $x_1 x_u$ is an edge of G . Then, $x_u x_1 x_2 x_3$ is a P_4 that is not monochromatic. By symmetry, we get that neither x_u nor x_v is adjacent to a vertex of N . But then apply Lemma 7.15.(iii) to $x_u x_1, x_1 x_3, x_3 x_v \notin E(G)$ and $x_1 x_2 \in E(G)$ to obtain a legal colouring of G . \square

Proof of Lemma 7.8. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 7.17 as well as Lemma 7.16.(iii) and 7.16.(iv).

Now, suppose that $G[N]$ contains a vertex, say x_1 of degree 0. As we have seen, $G[N]$ contains no vertex of degree at least 3. Thus, $G[N] - \{x_1\}$ contains two non-adjacent vertices, say x_2 and x_3 . Then, $\{x_1, x_2, x_3\}$ is an independent set and we are done by Lemma 7.16.(iii).

We can conclude that all vertices in $G[N]$ have degree 1 or 2. Thus, $G[N]$ is isomorphic to one of the following graphs: $C_3 \dot{\cup} C_3$, C_6 , $C_4 \dot{\cup} P_2$, $C_3 \dot{\cup} P_3$, $P_3 \dot{\cup} P_3$, $P_4 \dot{\cup} P_2$, $P_2 \dot{\cup} P_2 \dot{\cup} P_2$.

If $G[N]$ is isomorphic to $C_3 \dot{\cup} C_3$, we can apply Lemma 7.16.(i). It is not possible that all pairs of 3-paths have three edges of the same colour. In all other cases, we can apply Lemma 7.16.(iii). \square

Proof of Lemma 7.9. The proof is based on the following observation: a legal colouring c' of G' consists of at most $\lfloor (|V(G)| - 2)/2 \rfloor = \lfloor (|V(G)| - 3)/2 \rfloor$ colours while a legal colouring of G can consist of $\lfloor (|V(G)| - 3)/2 \rfloor + 1 = \lfloor (|V(G)| - 1)/2 \rfloor$ many colours. We will now consider the neighbourhood of u in G .

If u has exactly two neighbours x_1 and x_2 that are non-adjacent, set $G' = G - \{u\} + \{x_1 x_2\}$ and set $c(x_1 u x_2) = c'(x_1 x_2)$.

If u has exactly two neighbours x_1 and x_2 that are adjacent, set $G' = G - \{u\} - \{x_1x_2\}$ and set $c(x_1ux_2x_1) = \lfloor (|V(G)| - 1)/2 \rfloor$. Further, set $c(e) = c'(e)$ for all other edges in both cases to obtain a legal colouring.

If u has exactly four neighbours x_1, \dots, x_4 such that $x_1x_2, x_3x_4 \notin E(G)$ set $G' = G - \{u\} + \{x_1x_2, x_3x_4\}$ and set $c(x_1ux_2) = c'(x_1x_2)$ and $c(x_3ux_4) = c'(x_3x_4)$. If $c'(x_1x_2) \neq c'(x_3x_4)$, setting $c(e) = c'(e)$ for all other edges gives a legal colouring. If $c'(x_1x_2) = c'(x_3x_4)$, we again set $c(e) = c'(e)$ for all other edges. Now, c is a colouring of G where one colour class consists of two cycles intersecting only at u . We can split up this colour class into two cycles to obtain a legal colouring of G .

By Lemma 7.10, we are done unless u has four neighbours x_1, x_2, x_3, x_4 that form a clique. In that case, set $G' = G - \{u\} - \{x_1x_3, x_2x_4\}$.

If x_1x_2 and x_3x_4 are of different colour, set

$$c(x_1ux_2) = c'(x_1x_2) \quad c(x_3ux_4) = c'(x_3x_4) \quad c(x_1x_2x_4x_3x_1) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

If the cycle $x_1x_2x_3x_4x_1$ is monochromatic, set

$$c(x_1ux_2x_3x_4x_1) = c'(x_1x_2x_3x_4x_1) \quad c(x_4ux_3x_1x_2x_4) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

If the cycle is not monochromatic but x_1x_2 and x_3x_4 are of the same colour i and x_1x_4 and x_2x_3 are of the same colour j , we need to distinguish two cases. By Observation 7.13, there are two ways for the shape of the cycle $C_i = c'^{-1}(i)$ with colour i . If $C_i - \{x_1x_2, x_3x_4\}$ consists of a x_1 - x_3 -path and a x_2 - x_4 -path, set

$$c(x_1ux_4) = i \quad c(x_2x_3) = i \quad c(x_2ux_3) = j \quad c(x_1x_2x_4x_3x_1) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

If $C_i - \{x_1x_2, x_3x_4\}$ consists of a x_4 - x_1 -path P_{41} and a x_2 - x_3 -path, set

$$c(x_1ux_4) = i \quad c(x_2ux_3) = j \quad c(x_1x_3x_2x_4P_{41}x_1) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

By setting $c(e) = c'(e)$ for all other edges e , c is a legal colouring of G . \square

Proof of Lemma 7.11. We transform the legal colouring c' of G' into a legal colouring c of G . For this, we first note that u has degree 4 in G' , ie $\{x_1, x_2, x_3, x_4\}$ splits up into two pairs $\{a, a'\}$ and $\{b, b'\}$ with $c'(ua) = c'(ua')$ and $c'(ub) = c'(ub')$ and $c'(ua) \neq c'(ub)$.

If the colour $c'(x_5x_6)$ is not incident with u in G' , set $c(x_5ux_6) = c'(x_5x_6)$ and leave all other edge colours untouched to get a legal colouring.

Now suppose that $c'(x_5x_6)$ is incident with u (say $c'(aua') = c'(x_5x_6)$), but the set $\{c'(ua), c'(aa'), c'(ub), c'(bb'), c'(x_5x_6)\}$ consists of at least three different colours. Then, there are two possible constellations. First, let $c'(aa') \neq c'(x_5x_6)$ and $c'(aa') \neq c'(bub')$. Then, set $c(x_5ux_6) = c'(x_5x_6)$, flip the colours of the edges aua' and aa' and leave all other edge colours untouched to get a legal colouring.

If $c'(aa') = c'(bub')$ and $c'(bb') \neq c'(x_5x_6)$, set $c(x_5ux_6) = c'(x_5x_6)$, flip the colours of the edges aua' and aa' and the colours of the edges bub' and bb' , and leave all other edge colours untouched to get a legal colouring.

Thus, without loss of generality $c'(x_5x_6) = c'(aua') = c'(bb')$ and $c'(aa') = c'(bub')$. That is, among the considered edges there are only two colours. We may assume

that $c'(a'b) \neq c'(x_5x_6)$ and $c'(a'b') \neq c'(x_5x_6)$. Because, if eg $c'(a'b) = c'(x_5x_6)$, then $c'(ab) \neq c'(x_5x_6)$ and $c'(ab') \neq c'(x_5x_6)$. This is symmetric to the assumption.

If $c'(a'b) = c'(bub')$ and $c'(a'b') \neq c'(bub')$ the following is a legal colouring for G :

$$\begin{aligned} c(aa') &= c'(a'ua') & c(a'ub') &= c'(a'b') \\ c(auba'b') &= c'(aa'bub') & c(x_5ux_6) &= c'(x_5x_6) \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

If $c'(a'b') = c'(bub')$, $c'(a'b) \neq c'(bub')$, the following colouring for G is legal:

$$\begin{aligned} c(aa') &= c'(a'ua') & c(a'ub) &= c'(a'b) \\ c(aub'a'b) &= c'(aa'b'ub) & c(x_5ux_6) &= c'(x_5x_6) \\ c(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

Otherwise by Observation 7.13, one of the following is a legal colouring for G :

$$\begin{aligned} c_1(aub') &= c'(aa') & c_1(a'b) &= c'(aa') \\ c_1(a'ub) &= c'(a'b) & c_1(aa') &= c'(a'ua') \\ c_1(x_5ux_6) &= c'(x_5x_6) \\ c_1(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

or

$$\begin{aligned} c_2(aub) &= c'(aa') & c_2(a'b') &= c'(aa') \\ c_2(a'ub') &= c'(a'b') & c_2(aa') &= c'(a'ua') \\ c_2(x_5ux_6) &= c'(x_5x_6) \\ c_2(e) &= c'(e) \text{ for all other edges } e \end{aligned}$$

□

7.5. Path Decompositions

In Section 4.4 the notion of tree decomposition and treewidth was introduced. In this section, we are interested in path decompositions and pathwidth. A *path decomposition* of width k is a tree decomposition $(\mathcal{P}, \mathcal{B})$ of width k where \mathcal{P} is a path. A path decomposition is *smooth* if the corresponding tree decomposition is smooth. A graph G of pathwidth at most k always has a smooth path decomposition of width k ; see Bodlaender [Bod98]. Note that this path decomposition has exactly $n' = |V(G)| - k$ many bags.

If $(\mathcal{P}, \mathcal{B})$ is a path decomposition of the graph G , then for any vertex set W of G we denote by $\mathcal{P}(W)$ the subpath of \mathcal{P} that consists of those bags that contain a vertex of W . Further, if $\mathcal{P}(W)$ is the path on vertex set $\{s, s+1, \dots, t-1, t\}$ with $s \leq t$ we denote s by $s(W)$ and t by $t(W)$. For $W = \{v\}$, we abuse notation and denote $\mathcal{P}(W)$, $s(W)$ and $t(W)$ by $\mathcal{P}(v)$, $s(v)$ and $t(v)$.

We note: in a smooth path decomposition, for an edge $st \in E(\mathcal{P})$, there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. We call this vertex $v(s, t)$. Thus for

any vertex v of G , the number of vertices in the union of all bags containing v is at most $|\mathcal{P}(v)| + k$ and

$$\deg(v) \leq |\mathcal{P}(v)| + k - 1. \quad (7.3)$$

Consequently,

$$\begin{aligned} &\text{for every } i \in \{1, \dots, \lfloor |V(G)|/2 \rfloor\} \text{ there are unique vertices} \\ &v(i, i+1) \text{ and } v(n'+1-i, n'-i) \text{ with degree at most } k+i-1 \end{aligned} \quad (7.4)$$

For the proof of Theorem 7.3 we last note a direct consequence of a theorem of Fan and Xu [FX02, Theorem 1.1]:

Corollary 7.18. *Let G be an Eulerian graph of pathwidth at most 6 that does not satisfy Hajós' conjecture. Then there is a graph G' of pathwidth at most 6 with $|V(G')| \leq |V(G)|$ that contains at most one vertex of degree less than 6 and does not satisfy Hajós' conjecture.*

Based on (7.4) and on Lemma 7.6, 7.7 and 7.8 we finally show that Hajós' conjecture is satisfied for all Eulerian graphs of pathwidth 6.

Proof of Theorem 7.3. Assume that the class of graphs with pathwidth at most 6 does not satisfy Hajós' conjecture. Let G be a member of the class that does not satisfy the conjecture with minimal number of vertices. By Theorem 7.2, G has at least 13 vertices. By Corollary 7.18

$$G \text{ contains at most one vertex of degree 2 or 4.} \quad (7.5)$$

Thus, the three vertices $v(i, i+1)$ with $i = 1, 2, 3$ or the three vertices $v(i, i-1)$ with $i = n', n'-1, n'-2$ all have degree at least 6. Without loss of generality, suppose that

$$\deg(u), \deg(v), \deg(w) \geq 6 \text{ for } u := v(1, 2), v := v(2, 3), w := v(3, 4) \quad (7.6)$$

As u and v are both of degree 6, there are three possibilities.

- (i) u and v have common neighbourhood $N = \{x_1, \dots, x_6\}$, or
- (ii) u and v are adjacent with common neighbourhood $N = \{x_1, \dots, x_5\}$, or
- (iii) u and v are adjacent with common neighbourhood $N = \{x_1, \dots, x_4\}$ and private neighbours x_u and x_v .

We will now always delete u and v and optionally some edges. Further, we optionally add some edges in the neighbourhood of the two vertices. The obtained graph is still of pathwidth 6 and consequently has a legal colouring.

First assume (i) or (ii). By Lemma 7.8 and Lemma 7.6, N is an independent set and there is no vertex in $G - \{u, v\}$ that has at least three neighbours in N . This is not possible as w must have at least six neighbours in $N \cup \{u, v\}$ by (7.6).

Last assume (iii) and define $u' = v(n', n'-1)$, $v' = v(n'-1, n'-2)$ and $w' = v(n'-2, n'-3)$. By symmetry of the two sides of the path \mathcal{P} of G 's path decomposition, we can suppose that

- (I) u' and v' are two adjacent degree-6 vertices with common neighbourhood $N' = \{x'_1, \dots, x'_4\}$ and private neighbours x'_u and x'_v and $\deg(w') \geq 6$, or
- (II) there is a vertex y of degree less than 6 among u', v', w' .

Our aim is now to find a path between x_u and x_v in $G - R$ with $R = N \cup \{u, v\}$ (respectively a path between $x_{u'}$ and $x_{v'}$ in $G - R'$ with $R' = N' \cup \{u', v'\}$). Then we can use Lemma 7.7 to see that N is an independent set and there is no vertex in $G - \{u, v\}$ that has at least three neighbours in N . This is not possible as w must have at least six neighbours in R by (7.6).

We assume now that there is no path between x_u and x_v in $G - R$ with $R = N \cup \{u, v\}$ and denote the set of vertices in the component of x_u in $G - R$ by V_u . Similarly, we define V_v . The vertex z of V_u respectively V_v that maximises $s(z)$ is denoted by z_u respectively z_v . Note that the neighbourhood of z_a (for $a = u$ and $a = v$) satisfies

$$N(z_a) \subseteq (B_{s(z_a)} \cap V_a). \quad (7.7)$$

First assume that $t(V_u) = t(V_v)$. Then, as (P, \mathcal{B}) is smooth, $t(V_u) = t(V_v) = n'$. By (7.7), the neighbours of z_u and z_v are contained in the sets $B_{s(z_u)} \cap V_u$ and $B_{s(z_v)} \cap V_v$ that are both of size at most 5. This contradicts (7.5). Thus we can suppose that

$$t(V_u) < t(V_v) (\leq n'). \quad (7.8)$$

Then, z_v might have degree 6, but

z_u has degree less than 6.

Now, we split up the proof. First assume that (I) holds. By symmetry, we can apply the previous part of the proof and find a vertex $z'_{u'}$ in the component $V'_{u'} \neq V'_{v'}$ in $G - R'$ (with $R' = \{u', v'\} \cup N'$). By (7.5), z_u equals $z'_{u'}$.

As $z_u \in V_u$, there is a path P_{x_u, z_u} from x_u to z_u in $G - R$. Further, by (7.8), there is no x_u - $x'_{u'}$ -path $P_{x_u, x'_{u'}}$ in $G - R - R'$. This means that the path P_{x_u, z_u} contains a vertex r' of $R' \subset B_{n'}$ which contradicts (7.8).

Now assume that (II) holds. As we have seen before, we can assume (7.8) and obtain from (7.5) that $y = z_u$.

If $z_u = y = u'$ or $z_u = y = v'$, then all neighbours of z_u (ie particularly a vertex of V_u) are contained in $B_{n'}$. This contradicts (7.8).

Thus suppose that $z_u = y = w'$. Then u' and v' must be of degree 6. If u' and v' have four common neighbours, then (I) holds and we are done. If u' and v' have five common neighbours then note that $v(n' - 3, n' - 4)$ must have degree at least 6 by (7.5). Thus we can apply Lemma 7.6 to get a legal colouring of G . Therefore, it remains to consider that u' and v' are non-adjacent with common neighbourhood N' of size 6. Then all neighbours of z_u (ie a vertex of V_u) are contained in $N' \subset B_{n'}$ which contradicts (7.8). \square

Conclusion

In this thesis we have approached various prominent problems of graph theory including the list-edge-colouring conjecture, the overfull conjecture, Hajós’ conjecture, and the structural characterisation of t -perfect graphs. In each case, we went another step on the long road to solving the problem.

First, we provided structural characterisations for t -perfect triangulations and quadrangulations of the projective plane. While the proof for triangulations turned out to be rather technical towards the end leaving room for future improvement, the proof for quadrangulations was quite beautiful.

We devised a novel method that transforms every quadrangulation of the sphere into a quadrangle. This method is based on t -contractions and deletions of degree-2 vertices, ie on simple minor operations — in contrast to most other transformation techniques.

Similar to our approach, one can try to characterise other classes of t -perfect, embeddable graphs. Our method gives a first clue on how to exploit the information from an embedding for this purpose.

We verified the list-edge-colouring conjecture for two additional classes of graphs, namely the generalised Petersen graphs $GP(3k, k)$ and $GP(4k, k)$. In the process, we discovered an interesting connection between Dürer-type graphs and Jacobsthal numbers.

Our fundamental idea was to show that a colouring of the outer cycle determines the colouring of the whole graph. This insight allowed us to count the (signed) 1-factorisations in order to verify the conjecture with the method of Alon and Tarsi.

While we may still be a long way from a generic proof of the list-edge-colouring conjecture for the class of generalised Petersen graphs, one should be able to use our approach to apply the method of Alon and Tarsi for other classes of generalised Petersen graphs as well.

For graphs with fixed maximum degree and bounded treewidth we found an improved upper bound for the number of edges. Based on this finding and the overfull conjecture, we came up with our own, stronger conjecture for this particular class of graphs. Proving our conjecture would imply proving the overfull conjecture for this class.

Even though we were unable to prove that the conjecture holds for the proposed bound of $\Delta \geq k + \sqrt{k}$, we did improve the bound from $\Delta \geq 2k$ to $\Delta \geq 2k - 1$. The approach taken by our proof can probably be extended to show $\Delta \geq 2k - c$ for a given small constant c . However, it is rather technical and extensive even for the simplest case $c = 1$. Furthermore, it seems impossible to attain the desired bound of

$\Delta \geq k + \sqrt{k}$ this way. A fresh, completely different technique is probably required for this.

Nevertheless, we did at least prove a weakened version of our conjecture and showed that the involved bound is tight, if the conjecture holds true.

Finally, for the first time, Hajós' conjecture was proved for graphs that do not necessarily contain a vertex of degree 2 or 4. To this end, we devised a novel technique to construct cycle decompositions for graphs with degree-6 vertices. Our technique may be generalised and its basic ideas can be transferred to tackle further classes of graphs with vertices of degree 6 or even higher. Thus, we have paved the way to extend Hajós' conjecture to a new range of graph classes.

In sum, we provided valuable insights by contributing to a broad range of colouring problems touching on different areas of graph theory.

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Lebenslauf

Der Lebenslauf ist aus Gründen des Datenschutzes nicht enthalten.

