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CARMA Models with Random Coefficients and Inference for Renewal Sampled Lévy Driven Moving Average Processes

Dissertation

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It's the questions we can't answer
that teach us the most.
They teach us how to think.
If you give a man an answer,
all he gains is a little fact.
But give him a question
and he'll look for his own answers.

– Patrick Rothfuss
The Name of the Wind

My mother and my dearest friends

Abstract

This thesis covers miscellaneous topics in the field of time series analysis and stochastic processes and consists of four topics where the first two are connected by the appearance of random coefficients and the last two by inference of Lévy driven continuous time moving average processes.

In **Chapter 2**, we consider a random recurrence equation of the form $X_n = M_n X_{n-1} + Q_n$, $n \in \mathbb{N}$, where $(M_n, Q_n)_{n\in\mathbb{N}}$ is assumed to be an i.i.d. sequence in $\mathbb{R}^2$. Much attention has been paid to causal strictly stationary solutions of that random recurrence equation, i.e. to strictly stationary solutions of this equation when X_0 is assumed to be independent of $(M_n, Q_n)_{n\in\mathbb{N}}$. For this case, a complete characterization when such causal solutions exist can be found in literature. We shall dispose of the independence assumption of X_0 and $(M_n, Q_n)_{n\in\mathbb{N}}$ and derive necessary and sufficient conditions for a strictly stationary, not necessarily causal solution of this equation to exist.

In **Chapter 3**, we introduce a continuous time autoregressive moving average process with random Lévy coefficients, termed RC-CARMA(p,q) process, of order p and $q < p$ via a subclass of multivariate generalized Ornstein-Uhlenbeck processes. Sufficient conditions for the existence of a strictly stationary solution and the existence of moments are obtained. We further investigate second order stationarity properties, calculate the autocovariance function and spectral density, and give sufficient conditions for their existence.

In **Chapter 4**, we study a Lévy driven continuous time moving average process X sampled at random times which follow a renewal structure independent of X. Asymptotic normality of the sample mean, the sample autocovariance, and the sample autocorrelation is established under certain conditions on the kernel and the random times. We compare our results to a classical non-random equidistant sampling method and give an application to parameter estimation of the Lévy driven Ornstein-Uhlenbeck process.

As an extension of the results in Chapter 4, we consider in **Chapter 5** multivariate Lévy driven continuous time moving average processes $X = (X_t)_{t\in\mathbb{R}}$, given by

$$X_t = \mu + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,,$$

where $\mu \in \mathbb{R}^d$, $f\colon \mathbb{R} \to \mathbb{R}^{d\times m}$ and $L = (L_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^m$-valued Lévy process. We first sample the process X at a non-random equidistant sequence $(\Delta n)_{n\in\mathbb{Z}}$ for some $\Delta > 0$ and establish the asymptotic normality of the sample mean. Secondly, we use a renewal sampling sequence independent of X and derive also in this case the asymptotic normality of the sample mean.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit verschiedenen Themen der Zeitreihenanalyse und stochastischer Prozesse und besteht aus vier Teilen, von denen die ersten beiden durch die Betrachtung von zufälligen Koeffizienten und die letzten beiden durch Inferenz von Lévy getriebenen Moving Average Prozessen als jeweils miteinander verknüpft betrachtet werden können.

In **Kapitel 2** betrachten wir die zufällige Rekurrenzgleichung $X_n = M_n X_{n-1} + Q_n$, $n \in \mathbb{N}$, wobei $(M_n, Q_n)_{n\in\mathbb{N}}$ als unabhängige und gleichverteilte Folge in $\mathbb{R}^2$ angenommen wird. Kausale, strikt stationäre Lösungen jener zufälligen Rekurrenzgleichung, d.h. strikt stationäre Lösungen dieser Gleichung wenn X_0 als unabhängig von $(M_n, Q_n)_{n\in\mathbb{N}}$ angenommen wird, sind weitreichend untersucht worden. Eine vollständige Charakterisierung solcher kausalen, strikt stationären Lösungen können in der Literatur gefunden werden. Wir werden die Unabhängigkeitsannahme von X_0 und $(M_n, Q_n)_{n\in\mathbb{N}}$ beiseite lassen und leiten notwendige und hinreichenden Bedingen für eine strikt stationäre, nicht notwendigerweise kausale Lösung obiger Gleichung her.

In **Kapitel 3** stellen wir einen zeitstetigen Autoregressiven Moving Average Prozess mit zufälligen Lévykoeffizienten, genannt RC-CARMA(p,q) Prozess, der Ordnung p und $q < p$ als eine Unterklassen mehrdimensionaler verallgemeinerter Ornstein-Uhlenbeck Prozesse vor. Wir geben hinreichende Bedingungen für eine strikt stationäre Lösung und für die Existenz seiner Momente. Weiterhin untersuchen wir Eigenschaften zweiter Ordnung, berechnen die Autokovarianzfunktion und die Sprektraldichte und geben hinreichende Bedingungen für die Existenz derselben.

In **Kapitel 4** studieren wir den Lévy getriebenen zeitstetigen Moving Average Prozess X, welcher an zufälligen Zeitpunkten, die bezüglich einer Erneuerungsstruktur unabhängig von X definiert sind, abgegriffen wird. Asymptotische Normalität des Stichprobenmittelwertes, der Stichprobenautokovarianz und der Stichprobenautokorrelation werden unter bestimmten Voraussetzungen an den Kern und den zufälligen Zeiten nachgewiesen. Wir vergleichen unsere Ergebnisse mit denen einer klassischen nicht zufälligen und äquidistanten Stichprobenentnahme und wenden unsere Resultate zur Parameterschätzung eines Lévy getriebenen Ornstein-Uhlenbeck Prozesses an.

Als Erweiterung unserer Ergebnisse aus Kapitel 4 betrachten wir in **Kapitel 5** mehrdimensionale Lévy getriebene zeitstetige Moving Average Prozesse $X = (X_t)_{t\in\mathbb{R}}$, welche durch

$$X_t = \mu + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,,$$

wobei $\mu \in \mathbb{R}^d$, $f\colon \mathbb{R} \to \mathbb{R}^{d\times m}$ und $L = (L_t)_{t\in\mathbb{R}}$ ein $\mathbb{R}^m$-wertiger Lévy Prozess sind, definiert sind. Als Erstes tasten wir den Prozess X an einer nicht zufälligen äquidistanten Folge $(\Delta n)_{n\in\mathbb{Z}}$ für ein $\Delta > 0$ ab und weisen die asymptotische Normalität des Stichprobenmittelwertes nach. Nachfolgend nutzen wir zur Stichprobenentnahme zufälligen Zeiten, die einer Erneuerungsstruktur unabhängig von X genügen, und weisen auch in diesem Fall die asymptotische Normalität des Stichprobenmittelwertes nach.

Contents

1 Introduction

Time series describe phenomena which are regularly or irregularly observed such as the annual birth rate, the monthly unemployment rate, or high frequency data. Nowadays, the increasing availability of data, also at high frequency, in finance, economics, and physics has highlighted on the one hand the need of more complex models to describe their dynamics and on the other hand the necessity of versatile statistical methods capable of dealing also with irregularly spaced data.

To account for these two sides, we focus in this thesis on three classes of models to which has been paid considerable attention in the last decades. We first study the class of AR(1) processes with random coefficients and extend known results. Secondly, we generalize the class of CARMA processes to account for random coefficients and examine their properties.

In the third place, we propose a renewal sampling scheme for continuous time moving average processes, which embrace irregularly spaced data and comprise high frequency, and analyze the asymptotics of its sample mean and sample autocovariance.

In Section 1.1, we start with a collection of preliminary theory needed to introduce the results presented in the chapters to come, whereas Section 1.2 is concerned with a summary of the main results of this thesis.

1.1 Preliminaries and Notations

In this section we present some preliminary results and used notations. We start with some basic concepts before we dive more deeply into the subject matter in the subsections afterwards.

Throughout, when it comes to stochastic integration, we will always assume as given a *complete* probability space $(\Omega, \mathcal{F}, P)$, i.e. the σ-algebra $\mathcal{F}$ contains additionally all subsets of nullsets. In addition we have given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. By a filtration we mean a family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. Our filtration satisfies, if not stated otherwise, the usual hypotheses, i.e. $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$, and the filtration is right-continuous.

We will use $\mathbb{N}$ for the set of all strictly positive integers and write $\mathbb{N}_0$ to include zero. Further, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}$ the complex plane. For real-valued vector or matrix spaces of dimension n or $n \times m$, respectively, we use $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, and we write A' to denote the transpose of a vector or matrix A. With "$\stackrel{d}{=}$" we denote equality in distribution, $\stackrel{d}{\to}$ convergence in distribution, and similar for convergence in probability, $\xrightarrow{P}$, and almost sure (abbreviated a.s.) convergence, $\xrightarrow{\text{a.s.}}$. The term "càdlàg" describes stochastic processes with right-continuous sample paths and finite left limits, similar does "càglàd" for processes with left-continuous sample paths and finite right limits. With $\mathcal{L}(X)$ we denote the *law* of a random variable X.

In time series analysis, a subfield of stochastics, to which this thesis belongs to, one of the most important and commonly examined properties of the processes under consideration is *stationarity*. We differ between two concepts, *strict* stationarity and *weak* stationarity, and give the definition of both.

Definition 1.1. Let $T \subset \mathbb{R}$ be an arbitrary index set and $X = (X_t)_{t\in T}$ a stochastic process.

(a) X is called *strictly stationary* if its finite-dimensional distributions are shift-invariant, i.e. for all $n \in \mathbb{N}$, for all $t_1, \dots, t_n \in T$ and all $h \in \mathbb{R}$ such that $t_1 + h, \dots, t_n + h \in T$ it holds

$$(X_{t_1+h}, \dots, X_{t_n+h}) \stackrel{d}{=} (X_{t_1}, \dots, X_{t_n}) .$$

(b) X is called *weakly stationary* if $\mathbf{E}(X_t^2) < \infty$ for all $t \in T$, the mean $\mathbf{E}(X_t)$ is constant over time, i.e. does not depend on t, and the covariances $\mathbf{Cov}(X_{t+h}, X_t) = \mathbf{E}(X_{t+h} - \mathbf{E}(X_{t+h}))\mathbf{E}(X_t - \mathbf{E}(X_t))$ do not depend on t for all $h \in \mathbb{R}$ such that $t + h \in T$.

Observe that any strictly stationary process $X = (X_t)_{t\in T}$ with $\mathbf{E}(X_t^2) < \infty$ for all $t \in T$ is also weakly stationary whereas the converse is not true in general. An example for a strictly stationary time series is an i.i.d. sequence $(X_t)_{t\in T}$, which is, if $\mathbf{E}(X_1^2) < \infty$, also weakly stationary. A weakly stationary but not necessarily strictly stationary time series is, for example, a white noise, i.e. a sequence $(X_t)_{t\in T}$ such that $\mathbf{E}(X_t^2) < \infty$, $\mathbf{E}(X_t) = 0$, $\mathbf{Var}(X_t) = \sigma^2$ for all $t \in T$, where $\sigma^2 > 0$, and $\mathbf{Cov}(X_t, X_s) = 0$ for all $s, t \in T$ such that $t \neq s$. We also write $(X_t)_{t\in T} \sim \mathrm{WN}(0, \sigma^2)$. Most often we use the term stationary instead of weakly stationary. If a sequence $X = (X_t)_{t\in T}$ is stationary, we call $\gamma(h) = \mathbf{Cov}(X_{t+h}, X_t)$, $h \in \mathbb{Z}$, the *autocovariance function* (*of* X) *at lag* h.

1.1.1 Random recurrence equations

Crucial ingredients in time series are moving average and autoregressive processes. In this section, we especially examine the discrete time AR(1) process. For a polynomial $a(z) = 1 - a_1 z - a_2 z^2 - \cdots - a_p z^p$, we call $X = (X_t)_{t\in\mathbb{Z}}$ an *AR*(p) *process* or *autoregressive process of order* p if

$$a(B)X_t = Z_t , \quad t \in \mathbb{Z} ,$$

where B denotes the *backshift operator*, i.e. $BX_t = X_{t-1}$, and $(Z_t)_{t\in\mathbb{Z}} \sim \mathrm{WN}(0,\sigma^2)$. Cleary, if $a(z) = 1 - az$, then the process $X = (X_t)_{t\in\mathbb{Z}}$ which satisfies

$$X_t = aX_{t-1} + Z_t\,, \quad t \in \mathbb{Z}\,, \tag{1.1}$$

is an AR(1) process. By iterating (1.1) it can be shown, cf. Example 3.1.2 in Brockwell [23] that

$$X_t = \sum_{j=0}^{\infty} a^j Z_{t-j} \quad P\text{-a.s.}\,, \quad t \in \mathbb{Z}\,, \tag{1.2}$$

is the unique stationary solution to (1.1) if $|a| < 1$, and

$$X_t = -\sum_{j=1}^{\infty} a^{-j} Z_{t+j} \quad P\text{-a.s.}\,, \quad t \in \mathbb{Z}\,, \tag{1.3}$$

is the unique stationary solution to (1.1) if instead $|a| > 1$, whereas, if $|a| = 1$, it can be shown that there does not exist a stationary solution. Clearly, if we assume that $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence with finite second moment, (1.2) and (1.3) become now strictly stationary solutions.

If we relax the finite variance condition on the i.i.d. sequence $(Z_t)_{t\in\mathbb{Z}}$ and assume, in case of $|a| < 1$, that $\mathbf{E}(\log^+|Z_1|) < \infty$, then the series in (1.2) converges almost surely absolutely and is the unique strictly stationary solution, cf. Lemma 1 in Yohai and Maronna [68]. Analogously, by the same arguments as in Yohai and Maronna [68], one can show that, if $|a| > 1$ and $\mathbf{E}(\log^+|Z_1|) < \infty$, the series in (1.3) converges almost surely absolutely and is the unique strictly stationary solution to (1.1).

We turn our attention now to the so called *random recurrence equation*

$$X_n = M_n X_{n-1} + Q_n\,, \quad n \in \mathbb{N}\,, \tag{1.4}$$

where $(M_n, Q_n)_{n\in\mathbb{N}}$ is an $\mathbb{R}^2$-valued i.i.d sequence and the starting random variables X_0 is assumed to be independent of $(M_n, Q_n)_{n\in\mathbb{N}}$. (1.4) can be recognized as a discrete time AR(1) process with random coefficients. In time series analysis, the assumption that X_0 is independent of the sequence $(M_n, Q_n)_{n\in\mathbb{N}}$ is termed a *causality-assumption* or also a *non-anticipativity* assumption, and a corresponding solution a *causal solution.* If we drop this causality-assumption, i.e. X_0 depends on the future, we call a corresponding solution a *non-causal solution.*

Iterating (1.4) leads to

$$\begin{aligned} X_{n+h} &= M_{n+h}M_{n+h-1}X_{n+h-2} + M_{n+h}Q_{n+h-1} + Q_{n+h} \\ &= \cdots = \left(\prod_{i=h+1}^{n+h} M_i\right)X_h + \sum_{i=h+1}^{n+h}\left(\prod_{j=i+1}^{n+h} M_j\right)Q_i \quad \forall\, h, n \in \mathbb{N}_0\,, \end{aligned}$$

where we set $\prod_{i=0}^{-1} M_i := 1$. If $\prod_{i=1}^{n-1} M_i$ converges almost surely absolutely to 0 for $n \to \infty$ and $\sum_{n=1}^{\infty}(\prod_{i=1}^{n-1} M_i)Q_n$ converges almost surely absolutely, a unique (in distribution) causal strictly stationary solution to (1.4) is given by

$$X_n = \sum_{k=1}^{n}\left(\prod_{i=1}^{k-1} M_i\right)Q_k + \left(\prod_{i=1}^{n} M_i\right)X_0\,, \quad n \in \mathbb{N}_0\,, \tag{1.5}$$

where X_0 is chosen independent of $(M_n, Q_n)_{n\in\mathbb{N}}$ and such that

$$\mathcal{L}(X_0) = \mathcal{L}\left(\sum_{n=1}^{\infty}\prod_{i=1}^{n-1} M_i Q_n\right).$$

In the following theorem, necessary and sufficient conditions for the sum in (1.5) to converge almost surely absolutely are given. If $P(X > 0) > 0$ we denote the truncated mean $A_X(y)$ with

$$A_X(y) := \mathbf{E}(X^+ \wedge y) = \int_0^y P(X > x)\,\mathrm{d}x\,, \quad y > 0\,. \tag{1.6}$$

Theorem 1.2. (Theorem 2.1 of Goldie and Maller [41])
Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q) such that it holds $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then the following are equivalent:

(i) $\prod_{i=1}^{n} M_i \to 0$ *a.s. as* $n \to \infty$ *and* $\int_1^{\infty} \frac{\log q}{A_{-\log|M|}(\log q)} P_{|Q|}(\mathrm{d}q) < \infty$.

(ii) The infinite sum

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n-1} M_i\right)Q_n$$

converges almost surely absolutely.

The theorem above is a reduced version of Theorem 2.1 in Goldie and Maller [41]. When an additional nondegeneracy condition holds, convergence of (1.6) occurs under certain moment assumptions on the i.i.d. sequence $(M_n, Q_n)_{n\in\mathbb{N}}$.

Theorem 1.3. (Corollary 4.1 of Goldie and Maller [41])
Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q) and suppose that $P(M = 0) = 0$ and $P(Q + Mc = c) < 1$ and $-\infty < \mathbf{E}(\log|M|) < 0$. Then the infinite sum $\sum_{n=1}^{\infty}(\prod_{i=1}^{n-1} M_i)Q_n$ converges almost surely absolutely if and only if $\mathbf{E}(\log^+|Q|) < \infty$.

The random variable $\sum_{n=1}^{\infty}(\prod_{i=1}^{n-1} M_i)Q_n$ is called *perpetuity* in actuarial sciences, which describes the actual value of a permanent commitment to make a payment at regular intervals into an infinite future. $(Q_n)_{n\in\mathbb{N}}$ describes these regular payments and $(M_n)_{n\in\mathbb{N}}$ the cumulative discount factors which are both subject to random fluctuations.

A complete characterization of perpetuities and of causal strictly stationary solutions to the random recurrence equation (1.4) under various assumptions can be found in Goldie and Maller [41] and references therein.

1.1.2 Dependence of random variables and central limit theorems

To develop valid asymptotic inference, it is necessary to have central limit theorems for stochastic processes with various dependence structures at hand. This means, tools which describe for a stochastic process $(X_t)_{t\in T}$ with index sets $T \subset \mathbb{R}$ whether and at which rate for example the sample mean $\overline{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ or the sample autocovariance at lag h, i.e. $\widehat{\gamma}(h) = \frac{1}{n}\sum_{k=1}^{n-h}(X_k - \overline{X}_n)(X_{k+h} - \overline{X}_n)$, $h \in \{0, \dots, n-1\}$, converge to a limit distribution (assuming $T \supset \mathbb{N}$). We give in here a short overview on central limit theorems which are used in Chapter 4 and 5.

We start with the most well-known central limit theorem by Lindeberg and Lévy for a sequence where convergence of the sample mean towards a normal distribution is achieved at rate $\sqrt{n}$.

Theorem 1.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathbf{E}(X_1) = \mu$ and $\mathbf{Var}(X_1) = \sigma^2$, where $\sigma^2 \in [0, \infty)$. Let $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$, then*

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)\,, \quad n \to \infty\,.$$

Proof. Theorem 27.1 in Billingsley [12]. □

The i.i.d. assumption in Theorem 1.4 can be relaxed and replaced for example by the *Lindeberg condition* or the *Lyapunov condition*, cf. Billingsley [12]. But we want to relax it even further which is the purpose of the following definition and the theorem thereafter.

Definition 1.5. (m-Dependence)
A strictly stationary sequence of random variables $(X_t)_{t\in\mathbb{Z}}$ is called *m-dependent*, where $m \in \mathbb{N}_0$, if for each $t \in \mathbb{Z}$ the two sets of random variables $(X_j)_{j\le t}$ and $(X_j)_{j>t+m}$ are independent.

Observe that, if $m = 0$, a 0-dependent strictly stationary sequence is an i.i.d. sequence. The following theorem is due to Hoeffding and Robbins [45] where the form given here can be found in Brockwell and Davis [23].

Theorem 1.6. *Let $(X_t)_{t\in\mathbb{Z}}$ be a strictly stationary m-dependent sequence of random variables with mean zero and autocovariance function $\gamma(\cdot)$. If $v_m = \gamma(0) + 2\sum_{j=1}^{m}\gamma(j) \neq 0$, then*

(i) $\lim_{n\to\infty} \mathbf{Var}(\overline{X}_n) = v_m$ *and*

(ii) $\sqrt{n}\,\overline{X}_n \xrightarrow{d} N(0, v_m)$ *as* $n \to \infty$.

Proof. Theorem 6.4.2 in Brockwell and Davis [23]. □

Sometimes, different notions of asymptotic independence are needed and therefore, we introduce the concept of strong mixing. There are various mixing coefficients, like α, β, ρ, ψ, et. al, but for the present elaboration α- and ρ-mixing are sufficient.

Definition 1.7. (Mixing)
On a probability space $(\Omega, \mathcal{F}, P)$ for any two σ-algebras $\mathcal{A}, \mathcal{C} \subset \mathcal{F}$ the following measures of dependence can be defined

$$\alpha(\mathcal{A}, \mathcal{C}, P) := \sup |P(A \cap C) - P(A)P(C)| , \quad A \in \mathcal{A} , \ C \in \mathcal{C} ,$$
$$\rho(\mathcal{A}, \mathcal{C}, P) := \sup |\mathbf{Corr}(f, g)| , \quad f \in L^2(\Omega, \mathcal{A}, P) , \ g \in L^2(\Omega, \mathcal{C}, P) .$$

We say that a strictly stationary sequence of $\mathbb{R}^d$-valued random variables $Z = (Z_n)_{n\in\mathbb{Z}}$ is

$$\textit{strongly mixing} \text{ if } \alpha_n := \alpha(\mathcal{A}, \mathcal{C}_n; P) \to 0 \quad \text{as } n \to \infty ,$$
$$\rho\textit{-mixing} \text{ if } \rho_n := \rho(\mathcal{A}, \mathcal{C}_n; P) \to 0 \quad \text{as } n \to \infty ,$$

for the σ-algebra of the past $\mathcal{A} = \sigma(Z_0, Z_{-1}, Z_{-2}, \dots)$ and the σ-algebra of the future $\mathcal{C}_n = \sigma(Z_n, Z_{n+1}, Z_{n+2}, \dots)$.

More general, the σ-algebras $\mathcal{A}$ and $\mathcal{C}_n$ for a non-stationary time series $X = (X_n)_{n\in\mathbb{N}}$ are defined as $\mathcal{A}^J := \sigma(X_k , -\infty < k \le J)$ and $\mathcal{C}_{J+n} := \sigma(X_k , J + n \le k < \infty)$ and the corresponding strong mixing coefficient as

$$\alpha_n := \sup_{J\in\mathbb{Z}} \alpha(\mathcal{A}^J, \mathcal{C}_{J+n}) ,$$

see also Bradley [20] for more information on strong mixing coefficients as well as for the subsequent remark.

Remark 1.8. *Suppose that $X = (X_k)_{k\in\mathbb{Z}}$ is a strictly stationary, strongly mixing sequence of random variables.*

(a) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a Borel function. Define the random sequence $Y = (Y_k)_{k\in\mathbb{Z}}$ by $Y_k = f(X_k)$, $k \in \mathbb{Z}$. Then the sequence Y is strictly stationary and strongly mixing with $\alpha_n^Y \le \alpha_n^X$ for each $n \in \mathbb{N}$.

(b) If d is a nonnegative integer and $g\colon \mathbb{R}^{d+1} \to \mathbb{R}$ is a Borel function and $Z = (Z_k)_{k\in\mathbb{Z}}$ is given by $Z_k := g(X_{k-d}, \dots, X_k)$, $k \in \mathbb{Z}$, then Z is strictly stationary and strongly mixing with $\alpha_n^Z \le \alpha_{n-d}^X$ for each $n \ge d + 1$.

There is a certain connection between α- and ρ-mixing which we need and which is due to Bradley [19].

Remark 1.9. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\varepsilon > 0$ and $\mathcal{A}, \mathcal{C} \subset \mathcal{F}$ two sub-σ-algebras. Assume that $D \in \mathcal{F}$ such that $P(D) \ge 1 - \varepsilon$ and $\rho(\mathcal{A}, \mathcal{C}, P(\cdot|D)) \le \varepsilon$, then $\alpha(\mathcal{A}, \mathcal{C}, P) \le 4\varepsilon$.*

The following theorem gives asymptotic normality of the sample mean under the assumption of existing $2 + \delta$ moment for some $\delta > 0$.

Theorem 1.10. *Let $(X_k)_{k\in\mathbb{Z}}$ be a strictly stationary strongly mixing sequence of random variables such that $\mathbf{E}(X_0) = 0$. Suppose that for some $\delta > 0$ it holds $\mathbf{E}(|X_0|^{2+\delta}) < \infty$ and $\sum_{n=1}^{\infty} (\alpha_n)^{\delta/(2+\delta)} < \infty$. Then*

(a) $\sigma_{\overline{X}}^2 := \mathbf{E}(X_0^2) + 2\sum_{n=1}^{\infty} \mathbf{E}(X_0 X_n)$ *exists in* $[0,\infty)$ *and the sum is absolutely convergent.*

(b) $\sqrt{n}\,\overline{X}_n \xrightarrow{d} N(0,\sigma_{\overline{X}}^2)$ *as* $n \to \infty$.

Proof. Theorem 10.7 in Bradley [20]. □

The so called *growth condition* $\sum_{n=1}^{\infty}(\alpha_n)^{\delta/(2+\delta)} < \infty$ can be strengthened such that the moment assumption can be relaxed. This is the contents of the next theorem.

Theorem 1.11. *Let* $(X_k)_{k\in\mathbb{Z}}$ *be a strictly stationary strongly mixing sequence of random variables such that* $\mathbf{E}(X_0) = 0$ *and* $\mathbf{E}(X_0^2) < \infty$. *Further, suppose that the mixing coefficients* α_n *are exponentially decreasing as* $n \to \infty$ *and* $\mathbf{E}(|X_0|^2 \log^+ |X_0|) < \infty$.
Then

(a) $\sigma_{\overline{X}}^2 := \mathbf{E}(X_0^2) + 2\sum_{n=1}^{\infty} \mathbf{E}(X_0 X_n)$ *exists in* $[0,\infty)$ *and the sum is absolutely convergent.*

(b) $\sqrt{n}\,\overline{X}_n \xrightarrow{d} N(0,\sigma_{\overline{X}}^2)$ *as* $n \to \infty$.

Proof. Corollary 10.20 (c) in Bradley [20]. □

An always useful result is the following variant of Slutsky's Lemma.

Theorem 1.12. *Let* $(X_n)_{n\in\mathbb{N}}$ *and* $(Y_{nj})_{n\in\mathbb{N}}$ *for each* $j \in \mathbb{N}$ *be sequences of* $\mathbb{R}^d$*-valued random variables, and* Y *and* $\mathbb{R}^d$*-valued random variable. Suppose that*

(i) $Y_{nj} \xrightarrow{d} Y_j$ *as* $n \to \infty$ *for each* $j \in \mathbb{N}$,

(ii) $Y_j \xrightarrow{d} Y$ *as* $j \to \infty$, *and*

(iii) $\lim_{j\to\infty} \limsup_{n\to\infty} P(|X_n - Y_{nj}| > \varepsilon) = 0$ *for every* $\varepsilon > 0$.

Then

$$X_n \xrightarrow{d} Y\,, \quad n \to \infty\,.$$

Proof. Theorem 6.3.9 in Brockwell and Davis [23]. □

1.1.3 Lévy processes

In this subsection, we give the definition of Lévy processes, state some of their important properties and give a short overview of so called *multiplicative* Lévy processes.

Definition 1.13. (Lévy process)
An $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t\geq 0}$ is called a *Lévy process* if it satisfies the following conditions.

(i) X has *independent increments*, i.e. for all $n \in \mathbb{N}$ and all $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) X has *stationary increments*, i.e. $X_{t+s} - X_s \stackrel{d}{=} X_t$ for all $t, s \geq 0$.

(iii) $X_0 = 0$ a.s.

(iv) X is *stochastically continuous*, i.e. for every $t \geq 0$ and $\varepsilon > 0$, it holds

$$\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0 \,.$$

(v) X has almost surely càdlàg paths.

We call an $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t \geq 0}$ an *additive Lévy process*, if X satisfies Definition 1.13. If we drop the Assumption (v) in Definition 1.13, we call L a *Lévy process in law.* Lévy processes in law have a strong connection with infinitely divisible distributions. In what follows, we also consider Lévy processes with index set $\mathbb{R}$, so called *two-sided Lévy processes*, i.e. $L = (L_t)_{t \in \mathbb{R}}$ which can be constructed by taking two independent copies $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ of a Lévy process $(X_t)_{t \geq 0}$ and setting

$$L_t = \begin{cases} X'_t \,, & \text{if } t \geq 0 \,, \\ -X''_{-(t-)} \,, & \text{if } t < 0 \,, \end{cases}$$

where X''_{t-} denotes the left-limit of X''_t.

Definition 1.14. (Infinitely Divisible Distributions)
A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$, where

$$\mu_n^n = \underbrace{\mu_n * \cdots * \mu_n}_{n\text{-times}}$$

denotes the n-fold convolution.

Examples of infinitely divisible distributions are among others the Gaussian, Poisson, exponential, and gamma distributions. If $X = (X_t)_{t \in \mathbb{R}}$ is a Lévy process and μ denotes the distribution of X_1, then μ is infinitely divisible. Conversely, if μ is an infinitely divisible distribution, then there is a Lévy process in law $X = (X_t)_{t \in \mathbb{R}}$ such that $P_{X_1} = \mu$, where we denote with P_Z the distribution of a random variable Z.

Lemma 1.15. (Properties/Implications of Infinitely Divisible Distributions)
The following statements hold:

(i) If μ_1 and μ_2 are infinitely divisible, then so is $\mu_1 * \mu_2$.

(ii) If $(\mu_k)_{k \in \mathbb{N}}$ is a sequence of infinitely divisible distributions and $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, then μ is infinitely divisible.

(iii) If μ is infinitely divisible, then μ^t is well-defined for every $t \in [0, \infty)$ and infinitely divisible.

(iv) If $(X_t)_{t \geq 0}$ is an $\mathbb{R}^d$-valued Lévy process in law, then P_{X_t} is infinitely divisible for each $t \geq 0$. Further, letting $P_{X_1} = \mu$, we have $P_{X_t} = \mu^t$.

(v) Conversely, if μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a unique Lévy process in law $(X_t)_{t \geq 0}$ such that $P_{X_1} = \mu$.

(vi) Every Lévy process in law has a modification which is a Lévy process.

Proof. Lemma 7.4, Lemma 7.5, Lemma 7.8, and Lemma 7.9, Theorem 7.10 together with Theorem 11.5 and Corollary 11.6 in Sato [61]. □

An infinitely divisible distribution can be characterized in terms of its characteristic triplet which is the contents of the following theorem.

Theorem 1.16. (Lévy-Khintchine formula)
Let $D := \{x \in \mathbb{R}^d \colon |x| \leq 1\}$, the closed unit ball. Then

(i) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then for $z \in \mathbb{R}^d$

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x)\right) \nu(\mathrm{d}x)\right], \quad (1.7)$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$, called Lévy measure*, satisfying*

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty\,. \quad (1.8)$$

(ii) The representation of $\widehat{\mu}(z)$ in (1.7) by A, γ, and ν is unique.

(iii) Conversely, if A is a symmetric nonnegative-definite $d \times d$ matrix, ν a measure on $\mathbb{R}^d$ satisfying (1.8), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.7).

Proof. Theorem 8.1 in Sato [61]. □

Definition 1.17. (Characteristic Triplet of a Lévy process)
We call (A, ν, γ) in Theorem 1.16 the *generating* or *characteristic triplet* of μ or, because of Lemma 1.15 (v), of the Lévy process $X = (X_t)_{t\geq 0}$ with $P_{X_1} = \mu$ in which case we write (A_X, ν_X, γ_X). A or A_X, respectively, is also called the *Gaussian covariance matrix* and we write σ_X^2 instead of A_X if the Lévy process X is univariate. If $A = 0$, μ is called *purely non-Gaussian.* Sometimes, the Lévy measure is also denoted by Π_X.

$\mathrm{GL}(\mathbb{R}, m)$ denotes the general linear group, i.e. the set of all $m \times m$ invertible matrices associated with the ordinary matrix multiplication. The group structure therefore allows us to define *left increments* $X_t X_s^{-1}$ and *right increments* $X_s^{-1} X_t$ for $0 \leq s < t < \infty$ of a $\mathrm{GL}(\mathbb{R}, m)$-valued process.

Definition 1.18. (Multiplicative Lévy process)
Let $m \in \mathbb{N}$. A $\mathrm{GL}(\mathbb{R}, m)$-valued stochastic process $X = (X_t)_{t\geq 0}$ is a *(multiplicative) right Lévy process*, if the following conditions are satisfied:

(1) For any $n \geq 1$ and $0 < t_1 < \cdots < t_n$, the random variables $X_0, X_{t_1} X_0^{-1}, \ldots, X_{t_n} X_{t_{n-1}}^{-1}$ are independent.
(2) $X_0 = I$ a.s.
(3) The distribution of $X_t X_s^{-1}$ for $s < t$ depends only on $t - s$, i.e. $X_t X_s^{-1} \stackrel{d}{=} X_{t-s} X_0^{-1} = X_{t-s}$.
(4) It is stochastically continuous.
(5) It has càdlàg paths.

X is called a *(multiplicative) left Lévy process*, if X satisfies the condition (2), (4), (5) and

(1') For any $n \geq 1$ and $0 < t_1 < \cdots < t_n$, the random variables $X_0, X_0^{-1}X_{t_1}, \ldots, X_{t_{n-1}}^{-1}X_{t_n}$ are independent.

(3') The distribution of $X_s^{-1}X_t$ for $s < t$ depends only on $t-s$, i.e. $X_s^{-1}X_t \overset{d}{=} X_0^{-1}X_{t-s} = X_{t-s}$.

Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, a right Lévy process $(X_t)_{t\geq 0}$ is called a *right* $\mathbb{F}$*-Lévy process* if it is adapted to $\mathbb{F}$ and for any $s < t$ the left increment $X_t X_s^{-1}$ is independent of $\mathcal{F}_s$. *Left* $\mathbb{F}$*-Lévy processes* and *(additive)* $\mathbb{F}$*-Lévy processes* are defined similarly.

1.1.4 Stochastic Integration

A crucial ingredient for Chapters 3-5 is the theory of stochastic integration whose review is the contents of this section. We start with the definition of a semimartingale for which we first need the definition of the total variation of a real-valued function. More on the theory of stochastic integration can be found in the books of Protter [58] and Medvegyev [55].

Definition 1.19. (Total Variation)
Let $g\colon [0,\infty) \to \mathbb{R}$. For every $t \geq 0$ the *(total) variation* of g on $[0,t]$ is defined by

$$V_t(g) := \sup\left\{\sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)|\colon 0 = t_0 \leq t_1 \leq \cdots \leq t_n = t\,,\, n \in \mathbb{N}\right\}.$$

We say that g is of *finite variation on compacts* if $V_t(g) < \infty$ for all $t \geq 0$.

Definition 1.20. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis. A stochastic process $M = (M_t)_{t\geq 0}$ is called a *martingale* if

(i) M is adapted to the filtration $\mathbb{F}$.

(ii) $\mathbf{E}|M_t| < \infty$ for all $t \geq 0$.

(iii) $\mathbf{E}(M_t|\mathcal{F}_s) = M_s$ P-a.s. for all $s, t \geq 0$ such that $s \leq t$.

Definition 1.21. (Semimartingale)
A process $X = (X_t)_{t\geq 0}$ is a *semimartingale with respect to the filtration* $\mathbb{F} = (F_t)_{t\geq 0}$, if there exist processes M and A with $M_0 = 0$ and $A_0 = 0$ such that

$$X_t = X_0 + M_t + A_t$$

where

(i) M is a *local martingale*, i.e. M is adapted to the filtration $\mathbb{F}$, càdlàg and there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to \infty$ almost surely for $n \to \infty$ such that the stopped process $(M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n>0\}})_{t\geq 0}$ is a uniformly integrable martingale for every $n \in \mathbb{N}$.

(ii) A is a finite variation process, i.e. A is adapted to $\mathbb{F}$, càdlàg and has paths of finite variation on compacts.

Theorem 1.22. *An additive Lévy process is a semimartingale.*

Proof. P. 55 in Protter [58]. □

Definition 1.23. (Matrix-valued Semimartingales)
A matrix-valued stochastic process $X = (X_t)_{t\geq 0}$ is called an $\mathbb{F}$-*semimartingale* or simply a *semimartingale* if every component $(X_t^{(i,j)})_{t\geq 0}$ is a semimartingale with respect to the filtration $\mathbb{F}$.

Definition 1.24. (Locally Bounded Processes)
A stochastic process $H = (H_t)_{t\geq 0}$ with values in $\mathbb{R}^{m\times l}$ is said to be *locally bounded* if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to \infty$ almost surely for $n \to \infty$ such that for each $n \in \mathbb{N}$ the stopped process $(H_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n>0\}})_{t\geq 0}$ is bounded.

Definition 1.25. (Predictable Processes)
Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis. The *predictable σ-algebra $\mathcal{P}$* on $\mathbb{R}_+ \times \Omega$ is the smallest σ-algebra making all $\mathbb{F}$-adapted, càglàd processes measurable. Therefore a stochastic process $X = (X_t)_{t\geq 0}$ is called *predictable* if it is, considered as a mapping $(t,\omega) \to X_t(\omega)$ of $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}$, measurable with respect to $\mathcal{P}$.

The following definitions and properties are stated here for matrix-valued stochastic integrals, but are also valid in an univariate setting with the obvious simplifications, cf. Karandikar [46] and Protter [58].

Definition 1.26. (Matrix-valued Stochastic Integrals)
For a semimartingale X in $\mathbb{R}^{m\times n}$ and a locally bounded predictable process H in $\mathbb{R}^{l\times m}$ the $\mathbb{R}^{l\times n}$-valued *(left) stochastic integral* $J_1 = \int H \,\mathrm{d}X$ is defined by is components

$$J_1^{(i,j)} = \sum_{k=1}^{m} \int H^{(i,k)} \,\mathrm{d}X^{(k,j)} .$$

Similiar, for $X \in \mathbb{R}^{l\times m}$, $H \in \mathbb{R}^{m\times n}$, the $\mathbb{R}^{l\times n}$-valued *(right) stochastic integral* $J_2 = \int \mathrm{d}X H$ is defined by its components

$$J_2^{(i,j)} = \sum_{k=1}^{m} \int H^{(k,j)} \,\mathrm{d}X^{(i,k)} ,$$

and for $X \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{l\times m}$, and $H' \in \mathbb{R}^{n\times p}$ the $\mathbb{R}^{l\times p}$-valued stochastic integral $J_3 = \int H \,\mathrm{d}X H'$ is defined by its components via

$$J_3^{(i,j)} = \sum_{k=1}^{n} \sum_{h=1}^{m} \int H^{(i,h)} H'^{(k,j)} \,\mathrm{d}X^{(h,k)} .$$

Remark 1.27. *By the previous definitions, it can be easily seen that also in the multivariate case the stochastic integration preserves the semimartingale property as stated for example in the one-dimensional case in Protter [58] (Definition III.3 and Theorem IV.29), i.e. if H and H' are locally bounded predictable processes and X a semimartingale then also, in the notation of the previous definition, J_1, J_2, and J_3 are semimartingales.*

Definition 1.28. (Matrix-valued Quadratic Covariation)
For two semimartingales $X \in \mathbb{R}^{l\times m}$ and $Y \in \mathbb{R}^{m\times n}$ the *quadratic covariation* $[X, Y]$ is defined by its components via

$$[X,Y]^{(i,j)} = \sum_{k=1}^{m} [X^{(i,k)}, Y^{(k,j)}]$$

and similar its continuous part $[X, Y]^c$ via

$$([X,Y]^c)^{(i,j)} = \sum_{k=1}^{m} [X^{(i,k)}, Y^{(k,j)}]^c$$

such that it also holds true for matrix-valued semimartingales

$$[X,Y]_t = [X,Y]_t^c + X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s\,, \quad t \geq 0\,.$$

Proposition 1.29. (Properties of the Matrix-valued Stochastic Integral)
For two semimartingales $X, Y \in \mathbb{R}^{m\times m}$ and two locally bounded predictable processes $G, H \in \mathbb{R}^{m\times m}$ the following two equalities hold almost surely

$$\begin{aligned}
\left[\int_{(0,\cdot]} G_s \,\mathrm{d}X_s, \int_{(0,\cdot]} \mathrm{d}Y_s H_s\right]_t &= \int_{(0,t]} G_s \,\mathrm{d}[X,Y] H_s\,, \quad t \geq 0\,,\\
\left[X_\cdot, \int_{(0,\cdot]} G_s \,\mathrm{d}Y_s\right]_t &= \left[\int_{(0,\cdot]} \mathrm{d}X_s G_s, Y_\cdot\right]_t\,, \quad t \geq 0\,,
\end{aligned}$$

and the integration by parts formula takes the form

$$(XY)_t = \int_{0+}^{t} X_{s-} \,\mathrm{d}Y_s + \int_{0+}^{t} \mathrm{d}X_s Y_{s-} + [X_\cdot, Y_\cdot]_t\,, \quad t \geq 0\,.$$

Proof. Equations (4)-(6) in Karandikar [46]. □

An application of multivariate stochastic integration which is used in Chapter 3 is the multivariate stochastic exponential.

Definition 1.30. (Multivariate Stochastic Exponential)
Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{m\times m}$. Then its *left stochastic exponential* $\overleftarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} Z_{s-} \,\mathrm{d}X_s\,, \quad t \geq 0\,,$$

and its *right stochastic exponential* $\overrightarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} \mathrm{d}X_s Z_{s-}\,, \quad t \geq 0\,.$$

It is clear, by Remark 1.27, that the left and the right stochastic exponential are semimartingales and it can be shown that for its transpose it holds $\overleftarrow{\mathcal{E}}(X)'_t = \overrightarrow{\mathcal{E}}(X')_t$. As observed by Karandikar [46] and stated by Behme and Lindner [8], the right and the left stochastic exponentials of a process X are invertible at time t if and only if

$$\det(I + \Delta X_s) \neq 0 \quad \text{for all } s \leq t\,.$$

1.1.5 Infinite moving average processes

Moving average processes are together with autoregressive processes main ingredients in time series analysis. In this section, we examine the moving average processes in discrete time and continuous time of infinite order. Let $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ be a polynomial. We then call $X = (X_t)_{t\in\mathbb{Z}}$ an *MA(q) process* or *moving average process of order q* if

$$X_t = \theta(B) Z_t\,, \quad t \in \mathbb{Z}\,,$$

where $(Z_t)_{t\in\mathbb{Z}} \sim \mathrm{WN}(0,\sigma^2)$ or an $\mathrm{IID}(0,\sigma^2)$ sequence. X is then a stationary process with mean zero and autocovariance function

$$\gamma(h) = \mathbf{Cov}(X_{t+h}, X_t) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}\,, & \text{if } |h| \leq q\,, \\ 0\,, & \text{if } |h| > q\,. \end{cases}$$

Definition 1.31. (MA(∞) process)
If $(Z_t)_{t\in\mathbb{Z}} \sim \mathrm{WN}(0,\sigma^2)$, and $\mu \in \mathbb{R}$ some constant, then we say that $X = (X_t)_{t\in\mathbb{Z}}$ is a two-sided *moving average of infinite order* (MA(∞) *process*) if there exists a sequence $(\psi_j)_{j\in\mathbb{Z}}$ such that

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\,, \quad t \in \mathbb{Z}\,. \tag{1.9}$$

It can be shown that X is a strictly stationary process, for which a necessary condition is the convergence of the series in (1.9). The series converges almost surely absolutely and in mean square to the same limit if $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and X has the autocovariance function $\gamma(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|h|}$, $h \in \mathbb{Z}$, see Proposition 3.1.2 in Brockwell and Davis [23]. Other conditions for convergence can also be found in Theorem 1.4.1 in Samorodnitsky [60].

In continuous time, a moving average process of infinite order $X = (X_t)_{t\in\mathbb{R}}$ is defined by

$$X_t := \mu + \int_{-\infty}^{\infty} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,, \tag{1.10}$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a mean zero two-sided Lévy process, called the *driving Lévy process*, and $f\colon \mathbb{R} \to \mathbb{R}$ a suitable *kernel function.* Necessary and sufficient conditions for the integral to exist can be found for example in Rajput and Rosinski [59] or in Chapter 57 of Sato [61]. If $f \in L^2(\mathbb{R})$ and $\mathbf{E}(L_1^2) < \infty$, the integral can be defined in the L^2-sense. Observe that X, if the integral exists, is strictly stationary and then its distribution is infinitely divisible, cf. also Rajput and Rosinski [59].

By the Wold decomposition, it is well-known that any discrete time mean zero stationary process $X = (X_t)_{t\in\mathbb{Z}}$ which is not deterministic can be expressed as a sum $X_t = U_t + V_t$, where $U = (U_t)_{t\in\mathbb{Z}}$ is a one-sided MA(∞) process and $(V_t)_{t\in\mathbb{Z}}$ a deterministic process which is uncorrelated with U, cf. Theorem 5.7.1 in Brockwell and Davis [23]. Similarly in a continuous time setting, a weakly stationary process can be represented in terms of a two-sided moving average process (with respect to an orthogonal increment process) if and only if its spectral distribution is absolutely continuous, cf. Doob [36], page 533. A key example is the CARMA process as can be seen in Brockwell [22].

Since a large class of stationary processes permits a moving average representation, there is an interest in inference of moving average processes. In discrete time, the following results are taken from Brockwell and Davis [23]. Theorem 1.32 shows the asymptotic normality of the sample mean of a moving average process of infinite order, and Theorem 1.33 the asymptotic normality of its sample autocorrelation function which is given by $\widehat{\rho}(h) := \widehat{\gamma}(h)/\widehat{\gamma}(0)$, where $\widehat{\gamma}(h) = \frac{1}{n}\sum_{k=1}^{n-h}(X_k - \overline{X}_n)(X_{k+h} - \overline{X}_n)$, $h \in \{0, \dots, n-1\}$.

Theorem 1.32. *Let $X = (X_t)_{t\in\mathbb{Z}}$ be given by*

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\,, \quad t \in \mathbb{Z}\,,$$

where $Z = (Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of random variables with $\mathbf{E}(Z_1^2) = \sigma^2 \in (0,\infty)$ and mean zero, $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$, and $\sum_{j=-\infty}^{\infty}\psi_j \neq 0$. Then

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0,v)\,, \quad n \to \infty\,,$$

where $v = \sum_{h=-\infty}^{\infty}\gamma(h) = \sigma^2(\sum_{j=-\infty}^{\infty}\psi_j)^2$, and $\gamma(\cdot)$ is the autocovariance function of X.

Proof. Theorem 7.1.2 in Brockwell and Davis [23]. □

Theorem 1.33. *Let $X = (X_t)_{t\in\mathbb{Z}}$ be given by*

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\,, \quad t \in \mathbb{Z}\,,$$

where $Z = (Z_t)_{t\in\mathbb{Z}}$ is i.i.d with mean zero and variance $\sigma^2 \in (0,\infty)$ such that $\mathbf{E}(Z_1^4) < \infty$, and $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$. Then, for each $h \in \mathbb{N}$,

$$\sqrt{n}((\widehat{\rho}(1),\dots,\widehat{\rho}(h))' - (\rho(1),\dots,\rho(h))') \xrightarrow{d} N(0,W)\,, \quad n \to \infty\,,$$

where $W = (w_{ij})_{i,j=1,\dots,h}$ is the covariance matrix whose entries are given by Bartlett's formula,

$$w_{ij} = \sum_{k-\infty}^{\infty} \left(\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\right) \cdot \left(\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\right).$$

Proof. Theorem 7.2.1 and Remark 1 thereafter in Brockwell and Davis [23]. □

In the continuous time case, Theorem 1.34 shows the asymptotic normality of the sample mean $\overline{X}_{n;\Delta} = \frac{1}{n}\sum_{k=1}^{n} X_{k\Delta}$ of a continuous time moving average process, and Theorem 1.35 the asymptotic normality of its sample autocorrelation function. They are due to Cohen and Lindner [31].

Theorem 1.34. *Let $X = (X_t)_{t\in\mathbb{Z}}$ be given by (1.10) with $\mu \in \mathbb{R}$ and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process with mean zero and variance $\sigma^2 \in (0,\infty)$. Let $\Delta > 0$. Suppose that*

$$\left(F_\Delta \colon [0,\Delta] \to [0,\infty]\,, \quad u \mapsto F_\Delta(u) = \sum_{j=-\infty}^{\infty} |f(u+j\Delta)|\right) \in L^2([0,\Delta])\,. \tag{1.11}$$

Then $\sum_{h=-\infty}^{\infty} |\gamma(\Delta h)| < \infty$,

$$\sum_{h=-\infty}^{\infty} \gamma(\Delta h) = \sigma^2 \int_0^\Delta \left(\sum_{h=-\infty}^{\infty} f(u+h\Delta)\right)^2 \mathrm{d}u\,,$$

and

$$\sqrt{n}(\overline{X}_{n;\Delta} - \mu) \xrightarrow{d} N\left(0, \sigma^2 \int_0^\Delta \left(\sum_{h=-\infty}^{\infty} f(u+h\Delta)\right)^2 \mathrm{d}u\right), \quad n \to \infty\,.$$

Proof. Theorem 2.1 in Cohen and Lindner [31]. □

Observe that, if $\mu = 0$, further natural estimator for the autocovariance function $\gamma(\cdot)$ and the autocorrelation function $\rho(\cdot)$ are given by

$$\gamma^*_{n;\Delta}(h\Delta) = \frac{1}{n}\sum_{t=1}^{n} X_{t\Delta}X_{(t+h)\Delta}\,, \quad h \in \mathbb{N}\,,$$
$$\rho^*_{n;\Delta}(h\Delta) = \gamma^*_{n;\Delta}(h\Delta)/\gamma^*_{n;\Delta}(0)\,, \quad h \in \mathbb{N}\,.$$

Theorem 1.35. *Let $X = (X_t)_{t\in\mathbb{Z}}$ be given by (1.10) with $\mu \in \mathbb{R}$ and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process with mean zero, variance $\sigma^2 \in (0,\infty)$, and finite fourth moment. Denote $\eta = \sigma^{-4}\mathbf{E}(L_1^4)$ and suppose that $\Delta > 0$, $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$,*

$$\left([0,\Delta] \to [0,\infty]\,, \quad u \mapsto \sum_{k=-\infty}^{\infty} f(u+k\Delta)^2\right) \in L^2([0,\Delta])\,, \tag{1.12}$$

as well as

$$\sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} |f(s)f(s+k\Delta)|\,\mathrm{d}s\right)^2 < \infty\,. \tag{1.13}$$

(a) If $\mu = 0$, then for each $h \in \mathbb{N}$

$$\sqrt{n}((\gamma^*_{n;\Delta}(1), \ldots, \gamma^*_{n;\Delta}(h))' - (\gamma(1), \ldots, \gamma(h))') \xrightarrow{d} N(0, V_\Delta), \quad n \to \infty,$$

where $V_\Delta = (v_{pq;\Delta})_{p,q=0,\ldots,h} \in \mathbb{R}^{(h+1)\times(h+1)}$ can be given explicitly.

(b) Assume $\mu \in \mathbb{R}$, and that in addition to (a) (1.11) holds. Denote by

$$\widehat{\gamma}_{n;\Delta}(h\Delta) = \frac{1}{n}\sum_{t=1}^{n-h}(X_{t\Delta} - \overline{X}_{n;\Delta})(X_{(t+h)\Delta} - \overline{X}_{n;\Delta}), \quad h = 0, 1, \ldots, n-1,$$

the sample autocovariance. Then for each $h \in \mathbb{N}$

$$\sqrt{n}((\widehat{\gamma}_{n;\Delta}(1), \ldots, \widehat{\gamma}_{n;\Delta}(h))' - (\gamma(1), \ldots, \gamma(h))') \xrightarrow{d} N(0, V_\Delta), \quad n \to \infty,$$

where $V_\Delta = (v_{pq;\Delta})_{p,q=0,\ldots,h} \in \mathbb{R}^{(h+1)\times(h+1)}$ can be given explicitly.

(c) Assume that f is not almost everywhere equal to zero and denote with $\widehat{\rho}_n(h\Delta) = \widehat{\gamma}_{n;\Delta}(h\Delta)/\widehat{\gamma}_{n;\Delta}(0)$ the sample autocorrelation. Then, under the assumptions of (a), we have for each $h \in \mathbb{N}$

$$\sqrt{n}((\rho^*_{n;\Delta}(1), \ldots, \rho^*_{n;\Delta}(h))' - (\rho(1), \ldots, \rho(h))') \xrightarrow{d} N(0, W_\Delta), \quad n \to \infty,$$

where $W_\Delta = (w_{ij;\Delta})_{i,j=1,\ldots,h}$ is the covariance matrix whose entries can be given explicitly in terms of Bartlett's formula plus an additional term, cf. Theorem 3.5 in Cohen and Lindner [31]. If additionally (1.11) holds, we have for each $h \in \mathbb{N}$

$$\sqrt{n}((\widehat{\rho}_{n;\Delta}(1), \ldots, \widehat{\rho}_{n;\Delta}(h))' - (\rho(1), \ldots, \rho(h))') \xrightarrow{d} N(0, W_\Delta), \quad n \to \infty.$$

Proof. Theorem 3.5 in Cohen and Lindner [31]. □

Besides the representation of weakly stationary processes as moving averages, there are also very well-known processes inside the class of moving averages. For example, if $f\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto \frac{1}{\Gamma(d+1)}(s_+^d - (s-1)_+^d)$, for some $d \in (0, \frac{1}{2})$, the moving average process becomes a fractional Lévy noise based on increments of length 1 provided L has finite second moment, and if the kernel function is $f\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto \mathrm{e}^{-as}\mathbf{1}_{[0,\infty)}(s)$, $a > 0$, the process obtained is the Lévy driven Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process can be used to model the volatility of a financial asset, see Barndorff-Nielsen and Shepard [1], or the intermittency in a turbulence flow, see Barndorff-Nielsen and Schmiegel [2]. Marquardt [52] showed that fractional Lévy processes in the moving average context permit long memory behavior. Other applications for fractional Lévy processes can be found in Cohen [30] and in the references therein.

For more applications of continuous time moving average processes see also the references in Basse-O'Conner and Pedersen [3].

1.1.6 CARMA processes

In here we present the definition of CARMA processes and restate some of their already discovered properties.

Definition 1.36. (CARMA Process)
Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}$-valued additive Lévy process and $p, q \in \mathbb{N}$. We define a *(complex valued) CARMA process* $Y = (Y_t)_{t\in\mathbb{R}}$ driven by L through

$$Y_t = \mathbf{b}'\mathbf{X}_t\,, \quad t \in \mathbb{R}\,,$$

where $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ is a $\mathbb{C}^p$-valued process which satisfies the stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{X}_t = A\mathbf{X}_t\,\mathrm{d}t + \mathbf{e}\,\mathrm{d}L_t\,, \quad t \in \mathbb{R}\,, \tag{1.14}$$

or equivalently

$$\mathbf{X}_t = \mathrm{e}^{A(t-s)}\mathbf{X}_s + \int_s^t \mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}L_u\,, \quad \forall\, s \le t \in \mathbb{R}\,.$$

Here are

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{bmatrix}, \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}$$

with $a_1, \dots, a_p, b_0, \dots, b_{p-1} \in \mathbb{C}$ such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A is considered as $A = (-a_1)$.

To understand why a definition like this makes sense, first of all recall that we call any solution $Y = (Y_t)_{t\in\mathbb{Z}}$ of

$$Y_t - \phi_1 Y_{t-1} - \dots - \phi_p Y_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}\,, \tag{1.15}$$

where $p, q \in \mathbb{N}$, $q < p$, $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q \in \mathbb{C}$ and $Z = (Z_t)_{t\in\mathbb{Z}}$ i.i.d. or $\mathrm{WN}(0, \sigma^2)$, an *ARMA(p,q) process.*

If we define the polynomials

$$\begin{aligned} \Phi(z) &= 1 - \phi_1 z - \dots - \phi_p z^p\,, \quad z \in \mathbb{C}\,, \quad \text{and} \\ \Theta(z) &= 1 + \theta_1 z + \dots + \theta_q z^q\,, \quad z \in \mathbb{C}\,, \end{aligned}$$

and denote with B the backshift operator, i.e. $BY_t = Y_{t-1}$, we can write (1.15) as

$$\Phi(B)Y_t = \Theta(B)Z_t\,, \quad t \in \mathbb{Z}\,.$$

By defining the *backward difference* as $\Delta = (\mathbf{Id} - B)$, where B denotes the *backshift operator*, i.e. $\Delta Y_t = Y_t - Y_{t-1}$, we are able to find $a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_q \in \mathbb{C}$ with $\sum_{i=0}^p a_i = 1$ and $\sum_{i=0}^q b_i = 1$ such that

$$\begin{aligned}
Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} &= a_0 Y_t + a_1 \Delta Y_t + \cdots + a_p \Delta^p Y_t \\
Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} &= b_0 Z_t + b_1 \Delta Z_t + \cdots + b_q \Delta^q Z_t
\end{aligned}$$

so that (1.15) takes the form

$$\begin{aligned}
(a_0 \Delta^0 + a_1 \Delta^1 + \cdots + a_p \Delta^p) Y_t &= (b_0 \Delta^0 + b_1 \Delta^1 + \cdots + b_q \Delta^q) Z_t \\
&= (b_0 \Delta^0 + b_1 \Delta^1 + \cdots + b_q \Delta^q) \Delta S_t
\end{aligned}$$

with a random walk $S_t = \sum_{j=1}^t Z_j$.

From the above we recognize that the continuous time analogue, i.e. where the difference of time points become infinitesimal small, should be of the form

$$(a_0 D^0 + a_1 D^1 + \cdots + a_p D^p) Y_t = (b_0 D^0 + b_1 D^1 + \cdots + b_q D^q) D L_t\,, \tag{1.16}$$

where D denotes the differentiation operator with respect to t, and the choice of $L = (L_t)_{t \in \mathbb{R}}$ to be a Lévy process reflects the fact that we also want to have an i.i.d. noise in continuous time for what a Lévy process is the natural extension of a random walk $S = (S_t)_{t \in \mathbb{Z}}$. Remark that it is not possible to differentiate in our setting without changing the definition of the ordinary differentiation operator such that the above differentiation is understood informally.

We write $\mathbf{X}_t = (X_t^1, \ldots, X_t^p)'$ and have for $i = 1, \ldots, p-1$

$$\begin{aligned}
(1.14) \quad &\Longrightarrow \quad \mathrm{d}X_t^i = X_t^{i+1}\,\mathrm{d}t \\
&\Longleftrightarrow \quad X_t^i - X_s^i = \int_s^t X_u^{i+1}\,\mathrm{d}u \\
&\Longleftrightarrow \quad X_t^{i+1} = D X_t^i \quad \Longleftrightarrow \quad X_t^i = D^{i-1} X_t^1
\end{aligned} \tag{1.17}$$

and for $i = p$, arguing formally,

$$\begin{aligned}
(1.14) \quad &\Longrightarrow \quad \mathrm{d}X_t^p = -a_p X_t^1\,\mathrm{d}t - \cdots - a_1 X_t^p\,\mathrm{d}t + \mathrm{d}L_t \\
''&\Longleftrightarrow'' \quad D X_t^p + a_1 X_t^p + \cdots + a_p X_t^1 = D L_t \\
&\overset{(1.17)}{\Longrightarrow} \quad D D^{p-1} X_t^1 + a_1 D^{p-1} X_t^1 + \cdots + a_p D^0 X_t^1 = D L_t \\
&\Longrightarrow \quad a(D) X_t^1 = D L_t\,,
\end{aligned}$$

where $a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$. From this

$$a(D) X_t^i = a(D) D^{i-1} X_t^1 = D^{i-1} a(D) X_t^1 = D^{i-1} D L_t$$

such that through $Y_t = \mathbf{b}' \mathbf{X}_t$, $t \geq 0$, and with $b(z) = b_0 + b_1 z + \cdots + b_{p-1} z^{p-1}$ we get

$$a(D) Y_t = a(D)(b_0 X_t^1 + \cdots + b_{p-1} X_t^p)$$

$$= b_0 D^0 DL_t + b_1 D^1 DL_t + \cdots + b_{p-1} D^{p-1} DL_t = b(D)DL_t\,.$$

This formal deviation gives exactly what we wanted to have by formulating the extended difference equation (1.16).

Processes as in Definition 1.36 were first considered for L being a Gaussian process by Doob [35]. Brockwell [21], [22] gave the now commonly used definition with L being a Lévy process and showed when a strictly stationary solution exists assuming that $\mathbf{E}(L_1^2) < \infty$. This work was later completed by Brockwell and Lindner [26] who gave conditions for existence and uniqueness of strictly stationary solutions of CARMA processes.

Proposition 1.37. *Let $Y = (Y_t)_{t\in\mathbb{R}} = (\mathbf{b}'\mathbf{X}_t)_{t\in\mathbb{R}}$ be a CARMA(p,q) process such that $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ fulfills (1.14). Let $L = (L_t)_{t\in\mathbb{R}}$ denote the corresponding Lévy process and A the companion matrix.*

(a) If $\mathbf{X}_0$ is independent of the Lévy process $L = (L_t)_{t\in\mathbb{R}}$ and $\mathbf{E}(L_1^2) < \infty$ then $(Y_t)_{t\geq 0}$ is strictly stationary if and only if the eigenvalues of A all have strictly negative real parts and $\mathbf{X}_0 \stackrel{d}{=} \int_0^\infty \mathrm{e}^{As}\mathbf{e}\,\mathrm{d}L_s$.

(b) Suppose that L is not a deterministic Lévy process and that $a(\cdot)$ and $b(\cdot)$ have no common zeroes. Then there exists a unique strictly stationary CARMA process Y if and only if $\mathbf{E}(\log^+|L_1|) < \infty$ and $a(\cdot)$ is non-zero on the imaginary axis.

Proof. (a) Proposition 2 in Brockwell [24]. (b) Theorem 3.3 in Brockwell and Lindner [26]. □

The condition in Proposition 1.37 (a) that $\mathbf{E}(L_1^2) < \infty$ can be relaxed to $\mathbf{E}(L_1^r) < \infty$ for some $r > 0$, cf. Brockwell [22].

In both cases of Proposition 1.37, the CARMA process permits a moving average representation of the form

$$Y_t = \int_{-\infty}^{\infty} g(t-s)\,\mathrm{d}L_s\,, \quad t \geq 0\,, \tag{1.18}$$

where we can choose $t \in \mathbb{R}$ in case of Proposition 1.37 (b), and if the eigenvalues of A all have strictly negative real parts,

$$g(t) = \begin{cases} \mathbf{b}'\mathrm{e}^{At}\mathbf{e}\,, & \text{if } t \geq 0\,, \\ 0\,, & \text{if } t < 0\,, \end{cases}$$

which is known as the *kernel* of the CARMA process Y and can be written more explicitly as

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{it\lambda}\frac{b(i\lambda)}{a(i\lambda)}\,\mathrm{d}\lambda\,, \quad t \in \mathbb{R}\,.$$

The more general representation of g in case of Proposition 1.37 (b) can be found in Proposition 3.2 of Brockwell and Lindner [26].

CARMA processes are nowadays a quite popular class of processes in time series analysis and modeling. There are extensions to the original definition, like multivariate CARMA processes (MCARMA processes) as in Marquardt and Stelzer [54], fractionally integrated CARMA processes (FICARMA processes) as in Brockwell and Marquardt [29] which in contrast to CARMA processes show long memory behavior, and multivariate fractionally integrated CARMA processes (MFICARMA processes), see Marquardt [53].

Recently, Basse-O'Conner et al. [5], showed that a CARMA process $Y = (Y_t)_{t\in\mathbb{R}}$ permits besides the moving average representation in (1.18) also a continuous time autoregressive (CAR) representation, i.e.

$$R(D)Y_t = \int_0^\infty Y_{t-s} f(s)\,\mathrm{d}s + DL_t\,, \quad t \in \mathbb{R}\,,$$

where R is a polynomial of order $p-q$ and $f\colon \mathbb{R} \to \mathbb{R}$ a deterministic function, both defined through the polynomials $a(\cdot)$ and $b(\cdot)$. Another such representation can be found in Basse-O'Conner et al. [4].

CARMA processes provide a wide field of applications and can therefore serve, for example, as stochastic volatility models, cf. Brockwell [21], Todorov and Tauchen [65], and Todorov [64], as well as temperature models, cf. Benth et. al. [10], and electricity, cf. García et. al [37].

A comprehensive overview on recent results can also be found in Brockwell [25] and the references therein.

1.2 Main Results of this Thesis

In **Chapter 2** we extend the results of Subsection 1.1.1. We consider again the random recurrence equation (1.4). The aim of Theorem 2.3 therefore is to characterize completely when X_0, possibly dependent on $(M_n, Q_n)_{n\in\mathbb{N}}$, can be chosen such that a strictly stationary solution $(X_n)_{n\in\mathbb{N}}$ to (1.4) exists.

If we choose $(M_n, Q_n)_{n\in\mathbb{N}_0}$ as an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q), we obtain the following results regarding (1.4).

(a) Under the assumption of $P(M = 0) > 0$, a random variable X_0 (possibly on a suitably enlarged probability space) can be chosen such that the stochastic process $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary. This stationary solution is unique in distribution and obtained by choosing X_0 independent of $(M_n, Q_n)_{n\in\mathbb{N}}$ with

$$\mathcal{L}(X_0) = \mathcal{L}\left(\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i} M_j\right) Q_{i+1}\right). \tag{1.19}$$

(b) Suppose that $P(M = 0) = 0$ and that $\prod_{i=1}^n M_i$ converges almost surely to 0 as $n \to \infty$. Then the following are equivalent:

(i) A random variable X_0 (possibly on a suitably enlarged probability space) can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary.

(ii) The infinite sum $\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i} M_j\right) Q_{i+1}$ converges almost surely absolutely.

(iii) With $A_{-\log|M|}$ as defined in (1.6), it holds that

$$\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)}\, P_{|Q|}(\mathrm{d}q) < \infty\,.$$

If these equivalent conditions are satisfied, then the stationary solution is unique in distribution, and it is obtained by choosing X_0 independent of $(M_n, Q_n)_{n\in\mathbb{N}}$ and with distribution $\mathcal{L}(X_0)$ given by (1.19).

(c) Suppose that $P(M=0)=0$ and that $\prod_{i=1}^{n} M_i^{-1}$ converges almost surely to 0 as $n\to\infty$. Then the following are equivalent:

(i) A random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary.

(ii) The infinite sum $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ converges almost surely absolutely.

(iii) With $A_{\log|M|}$ as defined in (1.6), it holds

$$\int_1^\infty \frac{\log q}{A_{\log|M|}(\log q)}\, P_{|M^{-1}Q|}(\mathrm{d}q) < \infty\,.$$

If these equivalent conditions are satisfied, then the stationary solution is unique and given by

$$X_n = -\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_{n+j}^{-1}\right) Q_{n+i}\,, \quad n\in\mathbb{N}_0\,.$$

(d) Suppose that $P(M=0)=0$ and that neither $\prod_{i=1}^{n} M_i$ nor $\prod_{i=1}^{n} M_i^{-1}$ converges almost surely to 0 as $n\to\infty$. Then a random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary if and only if there is some $c\in\mathbb{R}$ such that $P(Q+Mc=c)=1$. If this condition is satisfied, a strictly stationary solution is given by the degenerate and constant process $X_n=c$ for all $n\in\mathbb{N}_0$. If additionally $P(|M|=1)<1$, then $(X_n=c)_{n\in\mathbb{N}_0}$ is the only strictly stationary solution of (1.4).

These results can be extended to the index set $\mathbb{Z}$ with the only difference that in case of (a) and (b) above the solution is not only unique in distribution but almost surely and given by $X_t=\sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} M_{t-j}\right) Q_{t-i}$ for all $t\in\mathbb{Z}$.

In **Chapter 3**, Section 3.2, we introduce a CARMA process with random coefficients of order p and $q<p$, termed RC-CARMA(p,q) process. More specifically, let $p\in\mathbb{N}$ and $C=(C_t)_{t\geq 0}=(M_t^{(1)},\ldots,M_t^{(p)},L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\Pi_{M^{(1)}}(\{1\})=0$.

Let $b_0, \dots, b_{p-1} \in \mathbb{R}$. Let $U = (U_t)_{t \geq 0}$ be $\mathbb{R}^{p \times p}$-valued defined by

$$U_t := \begin{bmatrix} 0 & t & 0 & \dots & 0 \\ 0 & 0 & t & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t \\ -M_t^{(p)} & -M_t^{(p-1)} & -M_t^{(p-2)} & \dots & -M_t^{(1)} \end{bmatrix}, \quad \mathbf{e} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \tag{1.20}$$

and $q := \max\{j \in \{0, \dots, p-1\} \colon b_j \neq 0\}$. Then we call any process $R = (R_t)_{t \geq 0}$ which satisfies

$$R_t = \mathbf{b}' V_t \,, \quad t \geq 0 \,, \tag{1.21}$$

where $V = (V_t)_{t \geq 0}$ is a solution to the SDE

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathbf{e}\, \mathrm{d}L_t \,, \quad t \geq 0 \,, \tag{1.22}$$

an *RC-CARMA(p,q) process.*

It is easy to see that when we choose $(M_t^{(1)}, \dots, M_t^{(p)}) = (a_1, \dots, a_p)t$ with $a_1, \dots, a_p \in \mathbb{R}$, we get a classical CARMA(p,q) process $(S_t)_{t \geq 0} = (\mathbf{b}' V_t)_{t \geq 0}$, although on the positive real line.

Brockwell and Lindner [28] gave a rigorous interpretation of (1.16) by showing that a CARMA(p,q) process $(S_t = \mathbf{b}' \mathbf{X}_t)_{t \in \mathbb{R}}$ driven by a Lévy process L satisfies the integral equation

$$a(D) J^p S_t = b(D) J^{p-1}(L_t) + a(D) J^p (\mathbf{b}' \mathrm{e}^{At} \mathbf{X}_0) \,, \quad t \in \mathbb{R} \,,$$

where $a(z)$ and $b(z)$ are as in the discussion of Section 1.1.6, and J denotes the integration operator which associates with any càdlàg function $f = (f_t)_{t \in \mathbb{R}} \colon \mathbb{R} \to \mathbb{C} \,, t \mapsto f_t$, the function $J(f)$ defined by $J(f)_t := \int_0^t f_s \, \mathrm{d}s$.

We show that the RC-CARMA(p,0) process can be interpreted to satisfy a similar p^{th} order differential equation, as the CARMA process does, which then can be done thoroughly by showing that the RC-CARMA(p,0) process satisfies a certain integral-differential equation. Conversely, every process satisfying this certain integral-differential equation is an RC-CARMA(p,0) process.

That the state vector process $V = (V_t)_{t \geq 0}$ satisfying (1.22) is an MGOU (see Appendix A.2 for its definition and properties) driven by a Lévy process (X, Y) which can be given explicitly in terms of the Lévy process (U, L) is proved as well as the existence of a strictly stationary solution of (1.22) and hence of (1.21) if $\mathbf{E}\left[\log^+ \|U_1\|\right] < \infty$, $\mathbf{E}\left[\log^+ |L_1|\right] < \infty$, there exists a $t_0 > 0$ such that

$$\mathbf{E}\left[\log \left\| \overleftarrow{\mathcal{E}}(U)_{t_0} \right\|\right] < 0 \,, \tag{1.23}$$

and V_0 is chosen independent C and such that $V_0 \overset{d}{=} \int_0^\infty \overleftarrow{\mathcal{E}}(U)_{s-} \, \mathbf{e} \, \mathrm{d}L_s$ which exists if (1.23) holds for some $t_0 > 0$. Conversely, if V_0 can be chosen independent of C such that V is strictly stationary, $M = (M_t^{(1)}, \ldots, M_t^{(p)})_{t \geq 0}$ is independent of L, and L not deterministic, then there exists a $t_0 > 0$ such that (1.23) holds.

It is shown in Section 3.3 that if

$$\mathbf{E} \left\| C_1 \right\|^2 < \infty \text{ and all eigenvalues of } D \text{ have strictly negative real parts}, \tag{1.24}$$

where $D = \mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1] + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]$, then the state vector V is strictly and weakly stationary and, if further $\mathbf{E}[L_1] = 0$, the autocovariance function of the RC-CARMA(p,q) process can be given explicitly as

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}' \mathrm{e}^{h\mathbf{E}[U_1]} \operatorname{vec}^{-1}(-D^{-1}\mathbf{e}_{p^2}) \, \mathbf{E}(L_1^2) \mathbf{b} \, ,$$

where $\otimes$ denotes the Kronecker product and vec the vectorizing operator on which more information are provided in Section A.1. Moreover, we give conditions for the existence of higher moments in terms of the moments of C.

Further, we give the spectral density of the RC-CARMA process and show that we can associate to every RC-CARMA process $R = (R_t)_{t \geq 0}$ a certain CARMA process $S = (S_t)_{t \geq 0}$ where A is chosen to be $\mathbf{E}[U_1]$. In this case, the autocovariance functions of R and S differ only by a constant.

In **Chapter 4**, we consider a continuous time moving average process $X = (X_t)_{t \in \mathbb{R}}$ as defined in (1.10) for $f \in L^2(\mathbb{R})$ and $L = (L_t)_{t \in \mathbb{R}}$ a Lévy process with zero mean and finite second moment σ_L^2 sampled at a renewal sequence. This means, we consider a sequence of increasing random times $(T_n)_{n \in \mathbb{Z}}$ defined by

$$T_0 := 0 \quad \text{and} \quad T_n := \begin{cases} \sum_{i=1}^n W_i \, , & n \in \mathbb{N} \, , \\ -\sum_{i=n}^{-1} W_i \, , & -n \in \mathbb{N} \, , \end{cases}$$

where $W = (W_n)_{n \in \mathbb{Z} \setminus \{0\}}$ is an i.i.d. sequence of positive supported random variables independent of the driving Lévy process L and such that $P(W_1 > 0) > 0$. Moreover, we define the sampled process $Y = (Y_n)_{n \in \mathbb{Z}}$ by

$$Y_n := X_{T_n} \, , \quad n \in \mathbb{Z} \, , \tag{1.25}$$

and study the behavior of its sample mean $\overline{Y}_n = \frac{1}{n} \sum_{k=1}^n Y_k$, its sample autocovariance $\widehat{\gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n-h} (Y_k - \overline{Y}_n)(Y_{k+h} - \overline{Y}_n)$, $h = 0, 1, \ldots, n-1$, and its sample autocorrelation $\widehat{\rho}_n(h) = \widehat{\gamma}_n(h) / \widehat{\gamma}_n(0)$. Throughout, we compare our results to the non-random equidistant sampling of Theorem 1.34 and 1.35.

Under the condition that $\mathbf{E}(|L_1|^2 \log^+ |L_1|) < \infty$, $f \in L^2(\mathbb{R})$, $\int_{\mathbb{R}} |f(s)|^2 \log^+ |f(s)| \, \mathrm{d}s < \infty$, and

$$\int_{\mathbb{R}} |f(u)| \sum_{k \in \mathbb{Z}} \mathbf{E} |f(T_k + u)| \, \mathrm{d}u < \infty \, , \tag{1.26}$$

we show in Section 4.2 that $\sigma^2_{\overline{Y}} := \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0, Y_k)$ exists in $[0,\infty)$, is absolutely convergent,

$$\sigma^2_{\overline{Y}} = \mathbf{E}(L_1^2) \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} f(u)\mathbf{E}(f(T_k+u))\,\mathrm{d}u\,,$$

and

$$\sqrt{n}\,(\overline{Y}_n - \mu) \xrightarrow{d} N(0, \sigma^2_{\overline{Y}}) \quad \text{as } n\to\infty\,.$$

The proof of this result relies on the foundations of Section 4.1 where we show that the sampled process Y is strictly stationary and that every m-dependent process $X^{(m)}$ sampled in the above way (1.25) is strongly mixing with exponentially decreasing mixing coefficients. We then consider the "truncated" moving average

$$X_t^{(m)} := \mu + \int_{\mathbb{R}} f_m(t-s)\,\mathrm{d}L_s\,, \quad t\in\mathbb{R}\,,$$

where $f_m = f\mathbf{1}_{[-m/2,m/2]}$, and prove that its sampled version $Y^{(m)}$ converges, by a central limit theorem for strictly stationary strongly mixing sequences with exponentially decreasing mixing coefficients, cf. Theorem 1.11, towards a normal distribution. Showing the other conditions of Theorem 1.12, we establish the asymptotic normality of the sample mean $\overline{Y}_n$.

In Section 4.3 we exhibit that assuming for the Lévy process $L = (L_t)_{t\in\mathbb{R}}$ to have expectation zero and $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$, $f \in L^2(\mathbb{R})\cap L^4(\mathbb{R})$, $f\neq 0$ λ-a.e, (1.26), $\int_{\mathbb{R}} |f(s)|^4(\log^+|f(s)|)^2\,\mathrm{d}s < \infty$, and for an $h\in\mathbb{N}$

$$\int_{\mathbb{R}} |f(u)| \sum_{k=1}^{\infty} \mathbf{E}|f(u+T_p)f(u+T_k)f(u+T_{k+q})|\,\mathrm{d}u < \infty\,, \quad \forall p,q\in\{0,\ldots,h\}\,,$$

as well as

$$\sum_{k=1}^{\infty} \mathbf{E}\left[\left(\int_{\mathbb{R}} |f(u)f(u+T_k)|\,\mathrm{d}u\right)^2\right] < \infty\,,$$

that it holds

$$\sqrt{n}(\widehat{\rho}_n(1)-\rho(1),\ldots,\widehat{\rho}_n(h)-\rho(h))' \xrightarrow{d} N(0,\mathbf{W})\,, \quad n\to\infty\,.$$

Here $\mathbf{W} = (\mathbf{W}_{pq})_{p,q=1,\ldots,h} \in \mathbb{R}^{h\times h}$ is given by

$$\mathbf{W}_{pq} = (\mathbf{Z}_{pq} - \rho(p)\mathbf{Z}_{0q} - \rho(q)\mathbf{Z}_{p0} + \rho(p)\rho(q)\mathbf{Z}_{00})/\gamma(0)^2\,,$$

where $\mathbf{Z} = (\mathbf{Z}_{pq})_{p,q=0,\ldots,h} \in \mathbb{R}^{h+1\times h+1}$ is the covariance matrix obtained in the limit normal distribution of the autocovariance function. An explicit expression for $\mathbf{Z}$ is also available and given. The proof of the asymptotic normality result of the autocorrelation function follows in a similar manner as the one for the sample mean.

In Section 4.4, we propose an estimator for the parameter of an Ornstein-Uhlenbeck process for which we apply the results of Section 4.3.

In **Chapter 5**, Section 5.1, we extend the results of Cohen and Lindner stated in Theorem 1.34 to a multivariate setting. More precisely, we consider an $\mathbb{R}^d$-valued continuous time moving average process $X = (X_t)_{t\in\mathbb{R}}$ defined by

$$X_t := \mu + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,, \tag{1.27}$$

where $\mu \in \mathbb{R}^d$, $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided $\mathbb{R}^m$-valued Lévy process with mean zero and finite second moment, and $f\colon \mathbb{R} \to \mathbb{R}^{d\times m}$ is in $L^2(\mathbb{R}^{d\times m})$, i.e.

$$L^2(\mathbb{R}^{d\times m}) := \left\{ f\colon \mathbb{R} \to \mathbb{R}^{d\times m} \text{ measurable: } \int_{\mathbb{R}} \|f(s)\|^2 \,\mathrm{d}s < \infty \right\}$$

for some norm on $\mathbb{R}^{d\times m}$.

We see that the integrals on the right-hand side of (1.27) exist since $f \in L^2(\mathbb{R}^{d\times m})$ and $\mathbf{E}\,\|L_1\|^2 < \infty$. X is then strictly stationary. We show that the autocovariance function of X is given by

$$\Gamma(\Delta h) = \int_{\mathbb{R}} f(s)\Sigma_L f(s)'\,\mathrm{d}s$$

where Σ_L denotes the covariance matrix of the Lévy process L.

Further, when X is sampled equidistantly at a sequence $(\Delta n)_{n\in\mathbb{N}}$ for some $\Delta > 0$, then the asymptotic normality of the sample mean ${}^{\Delta}\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_{k\Delta}$ can be established under the assumption that

$$\left(\widetilde{F}_\Delta\colon [0,\Delta] \to [0,\infty]\,, \quad u \mapsto \widetilde{F}_\Delta(u) = \sum_{h=-\infty}^{\infty} \|f(u+h\Delta)\| \right) \in L^2([0,\Delta])\,. \tag{1.28}$$

More precisely, if (1.28) holds, $f \in L^2(\mathbb{R}^{d\times m})$, and $L = (L_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^m$-valued Lévy process with zero mean and finite second moment, we have that $\sum_{h=-\infty}^{\infty} \|\Gamma(\Delta h)\| < \infty$,

$$\sum_{h=-\infty}^{\infty} \Gamma(\Delta h) = \int_0^\Delta F(s)\Sigma_L F(s)'\,\mathrm{d}s\,,$$

where $F_\Delta(u) = \sum_{h=-\infty}^{\infty} f(u+h\Delta)$, and

$$\sqrt{n}({}^{\Delta}\overline{X}_n - \mu) \xrightarrow{d} N\left(0, \int_0^\Delta F(s)\Sigma_L F(s)'\,\mathrm{d}s\right) \quad \text{as } n\to\infty\,.$$

Finally in Section 5.2, we extend the results of Section 4.2 on the sample mean of a renewal sampled moving average process to a multivariate setting. More precisely, we show that

$$\sqrt{n}\,(\overline{Y}_n - \mu) \xrightarrow{d} N(0, \Sigma_{\overline{Y}}) \quad \text{as } n \to \infty\,,$$

where

$$\Sigma_{\overline{Y}} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} f(u)\Sigma_L \mathbf{E}(f(T_k+u)')\,\mathrm{d}u\,,$$

assuming that $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$, $\int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\| \ \mathrm{d}s < \infty$, and

$$\int_{\mathbb{R}} \|f(u)\| \sum_{k=1}^{\infty} \mathbf{E}\, \|f(T_k+u)\| \ \mathrm{d}u < \infty\,.$$

In both Sections 5.1 and 5.2 we see that the conditions of the univariate cases extend naturally towards a multivariate setting.

Some known results on the Kronecker product are summarized in Appendix A.1. Appendix A.2 provides a collections of results on multivariate generalized Ornstein-Uhlenbeck (MGOU) processes which are due to Behme and Lindner [8] and Behme [9], while Appendix A.3 contains some detailed calculations for the results of Section 4.4.

Chapter 2-4 are based on research articles, namely Chapter 2 is based on Brandes and Lindner [16] (published), Chapter 3 is based on Brandes [14] (published), and Chapter 4 is based on Brandes and Curato [15] (submitted).

2 Non-causal strictly stationary solutions of random recurrence equations

This chapter is based on the published article by Brandes and Lindner [16] "Non-causal strictly stationary solutions of random recurrence equations". Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$, (M, Q) a generic copy of it, and let the real-valued process $(X_n)_{n\in\mathbb{N}_0}$ be defined recursively by

$$X_n = M_n X_{n-1} + Q_n\,, \quad n \in \mathbb{N}\,, \tag{2.1}$$

where X_0 is some starting random variable, defined on the same probability space. Our goal is to characterize when the starting random variable X_0 can be chosen such that the derived process $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary, meaning that for all $n \in \mathbb{N}_0, m \in \mathbb{N}$ and $h_1, \dots, h_m \in \mathbb{N}_0$,

$$\mathcal{L}(X_{h_1}, \dots, X_{h_m}) = \mathcal{L}(X_{h_1+n}, \dots, X_{h_m+n})$$

where $\mathcal{L}(Y)$ denotes the law of a random vector Y. Much attention has been paid to this question when X_0 is assumed to be *independent* of $(M_n, Q_n)_{n\in\mathbb{N}}$, in which case $(X_n)_{n\in\mathbb{N}_0}$ becomes a time-homogeneous Markov process. In this case, an independent X_0 can be chosen such that the process becomes stationary if and only if the Markov process admits an invariant probability measure μ, in which case X_0 and μ are related by $\mu = \mathcal{L}(X_0)$. By the definition of the invariant measure, this is further equivalent to saying that the distributional fixed point equation

$$\mathcal{L}(X) = \mathcal{L}(Q + MX)\,, \quad \text{with } X \text{ independent of } (M, Q),$$

has a solution. A complete solution of when such a distributional fixed point and hence a choice of an independent X_0 exists making $(X_n)_{n\in\mathbb{N}_0}$ strictly stationary has been achieved by Goldie and Maller [41, Theorem 3.1], while necessary and sufficient conditions under some extra conditions had been obtained earlier by Vervaat [67, Theorems 1.5 and 1.6]. We also mention Brandt [17, Theorem 1], who gave sufficient conditions when $(M_n, Q_n)_{n\in\mathbb{N}_0}$ was allowed to be stationary and ergodic rather than i.i.d., the book by Brandt et al. [18], where this equation and more general recursive equations with stationary and ergodic input are treated, and Bougerol and Picard [13] who consider a multivariate extension of Vervaat's result. We refer to the paper by Goldie and Maller [41] for further references when X_0 is assumed to be independent of $(M_n, Q_n)_{n\in\mathbb{N}}$.

In time series analysis, the assumption that X_0 is independent of the sequence $(M_n, Q_n)_{n\in\mathbb{N}}$ is termed a *causality-assumption* or also a *non-anticipativity assumption*, and a corresponding solution a *causal* solution. The aim of this chapter is to dispose of this causality assumption and to characterize completely when X_0, possibly dependent on $(M_n, Q_n)_{n\in\mathbb{N}}$, can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ becomes strictly stationary. It will turn out that non-causal solutions which depend on the future may indeed exist. This chapter can then be seen as a discrete time analogue of Behme et al. [6], who consider strictly stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. Note also that related questions for ARMA processes (with deterministic coefficients) have been dealt with in Brockwell and Davis [23, Theorem 3.1.3 and Problem 4.28] for the second order stationary case, and in Brockwell and Lindner [27, Theorem 1] for the strictly stationary case. A discussion of non-causal autoregressive models in economic time series can be found in Lanne and Saikkonen [50].

2.1 Preliminaries

Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an $\mathbb{R}^2$-valued i.i.d. sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, let X_0 be a random variable on the same probability space, and define $(X_n)_{n\in\mathbb{N}_0}$ by (2.1). Denote

$$\Pi_n := \prod_{i=1}^{n} M_i, \quad n \in \mathbb{N}_0 ,$$

with the usual convention that the empty product is 1. By successive iteration, it is easy to see that

$$X_{n+h} = \left(\prod_{i=h+1}^{n+h} M_i\right) X_h + \sum_{i=h+1}^{n+h} \left(\prod_{j=i+1}^{n+h} M_j\right) Q_i \quad \forall\, h, n \in \mathbb{N}_0 . \tag{2.2}$$

By Theorem 3.1 (c) of Goldie and Maller [41], if $P(M = 0) = 0$ and $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$, then a *causal* strictly stationary solution of (2.1) exists if and only if $\sum_{n=1}^{\infty} \Pi_{n-1} Q_n$ converges almost surely absolutely, in which case $\mathcal{L}(\sum_{n=1}^{\infty} \Pi_{n-1} Q_n)$ is the unique invariant measure. In Goldie and Maller [41, Theorem 2.1], of which a reduced version is given already in Chapter 1 as Theorem 1.2, they also give a necessary and sufficient condition for this sum to converge almost surely absolutely. It will be also an important tool for the proof of our characterization of all (not-necessarily causal) solutions we give in Theorem 2.3 below. For a random variable X, we denote its distribution by P_X, and if $P(X > 0) > 0$ we denote

$$A_X(y) := \mathbf{E}(X^+ \wedge y) = \int_0^y P(X > x)\, \mathrm{d}x , \quad y > 0 . \tag{2.3}$$

Then the function $(0, \infty) \to (0, 1]$, $y \mapsto \frac{A_X(y)}{y}$ is nonincreasing, cf. [41, Remark 2.2]. We can now state those parts of Theorem 2.1 of [41] which are relevant for our further investigations. In the formulation below, the equivalence of (ii) and (iii) and the last assertions follow from Theorem 2.1 together with Lemma 5.5 (applied with $Z_0 := 0$) in Goldie and Maller [41].

Theorem 2.1. [41, Theorem 2.1]
Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q) such that $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then the following are equivalent:

(i) $\Pi_n \to 0$ *a.s. as* $n \to \infty$ *and* $\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)} P_{|Q|}(\mathrm{d}q) < \infty$.

(ii) The infinite sum $\sum_{n=1}^\infty \Pi_{n-1} Q_n$ *converges almost surely absolutely.*

(iii) $\Pi_n \to 0$ *a.s. as* $n \to \infty$ *and* $\sum_{i=1}^n \Pi_{i-1} Q_i$ *converges in distribution to a finite random variable as* $n \to \infty$.

If $\Pi_n \to 0$ *a.s.* $(n \to \infty)$ *but* $\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)} P_{|Q|}(\mathrm{d}q) = \infty$, *then* $|\sum_{i=1}^n \Pi_{i-1} Q_i|$ *converges in probability to* ∞ *as* $n \to \infty$. *Further, if* Π_n *does not converge almost surely to 0 as* $n \to \infty$ *and* $P(Q + Mc = c) < 1$ *for all* $c \in \mathbb{R}$, *then* $|\sum_{i=1}^n \Pi_{i-1} Q_i|$ *converges in probability to* ∞ *as* $n \to \infty$.

Conditions for the almost sure convergence of Π_n to 0 have been obtained by Kesten and Maller [48, Lemma 1.1]. To state their results, let $(M_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of real valued random variables such that $P(M_1 = 0) = 0$. Consider the random walk

$$S_n := \sum_{i=1}^n (-\log M_i)\,, \quad n \in \mathbb{N}\,.$$

Then $\prod_{i=1}^n M_i$ converges almost surely to 0 if and only if S_n drifts almost surely to $+\infty$ as $n \to \infty$, and $\prod_{i=1}^n M_i^{-1}$ converges almost surely to 0 if and only if S_n converges almost surely to $-\infty$ as $n \to \infty$. Further, it is well known that $(S_n)_{n\in\mathbb{N}}$ either converges almost surely to $+\infty$, or converges almost surely to $-\infty$, or oscillates in the sense that $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = +\infty$ almost surely. Then by Lemma 1.1 in Kesten and Maller [48], we have $S_n \to \infty$ a.s. as $n \to \infty$ if and only if either $0 < \mathbf{E}(-\log|M|) \le \mathbf{E}\Big|\log|M|\Big| < \infty$, or $\mathbf{E}(\log|M|)^- = \infty$ and $\int_1^\infty \frac{y}{A_{-\log|M|}(y)} P_{\log|M|}(\mathrm{d}y) < \infty$ with $A_{-\log|M|}$ as defined in (2.3). Similarly, $S_n \to -\infty$ a.s. as $n \to \infty$ if and only if either

$$0 < \mathbf{E}(\log|M|) \le \mathbf{E}\Big|\log|M|\Big| < \infty\,, \tag{2.4}$$

or

$$\mathbf{E}(\log|M|)^+ = \infty \quad \text{and} \quad \int_1^\infty \frac{y}{A_{\log|M|}(y)} P_{-\log|M|}(\mathrm{d}y) < \infty\,. \tag{2.5}$$

Since $\lim_{y\to\infty} A_{\log|M|}(y)$ is finite if and only if $\mathbf{E}(\log|M|)^+ < \infty$, we see that

$$\int_1^\infty \frac{y}{A_{\log|M|}(y)} P_{-\log|M|}(\mathrm{d}y) < \infty \tag{2.6}$$

is implied by both (2.4) and (2.5), hence (2.6) always holds whenever $\prod_{i=1}^n M_i^{-1} \to 0$ a.s. $(n \to \infty)$.

Remark 2.2. *If $P(M=0)=0$ and $\mathbf{E}\Big|\log|M|\Big|<\infty$, then $\Pi_n\to 0$ a.s. $(n\to\infty)$ if and only $\mathbf{E}(\log|M|)<0$, in which case $A_{-\log|M|}(x)$ converges to $\mathbf{E}((-\log|M|)^+)<\infty$ as $x\to\infty$. Hence, provided that $\mathbf{E}\Big|\log|M|\Big|<\infty$, condition (i) of Theorem 2.1 can be replaced by*

$$\mathbf{E}(\log|M|)<0 \quad \text{and} \quad \mathbf{E}(\log^+|Q|)<\infty\,,$$

where $\log^+(x)=\log(\max\{1,x\})$ for $x\in\mathbb{R}$, cf. Theorem 1.3.

2.2 Results

The following is our main result and characterizes when X_0 can be chosen for (2.1) to have a strictly stationary, not necessarily causal, solution.

Theorem 2.3. *Let $(M_n,Q_n)_{n\in\mathbb{N}_0}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M,Q). Consider the random recurrence equation (2.1).*
(a) Suppose that $P(M=0)>0$. Then a random variable X_0 (possibly on a suitably enlarged probability space) can be chosen such that the stochastic process $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary. This stationary solution is unique in distribution and obtained by choosing X_0 independent of $(M_n,Q_n)_{n\in\mathbb{N}}$ with

$$\mathcal{L}(X_0)=\mathcal{L}\left(\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i}M_j\right)Q_{i+1}\right)\,. \tag{2.7}$$

(b) Suppose that $P(M=0)=0$ and that $\prod_{i=1}^n M_i$ converges almost surely to 0 as $n\to\infty$, i.e. that $\sum_{i=1}^n\log|M_i|\to-\infty$ a.s. as $n\to\infty$. Then the following are equivalent:

- *(i) A random variable X_0 (possibly on a suitably enlarged probability space) can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary.*
- *(ii) The infinite sum $\sum_{i=0}^\infty\left(\prod_{j=1}^i M_j\right)Q_{i+1}$ converges almost surely absolutely.*
- *(iii) With $A_{-\log|M|}$ as defined in (2.3), it holds that*

$$\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)}\,P_{|Q|}(\mathrm{d}q)<\infty\,.$$

If these equivalent conditions are satisfied, then the stationary solution is unique in distribution, and it is obtained by choosing X_0 independent of $(M_n,Q_n)_{n\in\mathbb{N}}$ and with distribution $\mathcal{L}(X_0)$ given by (2.7).
(c) Suppose that $P(M=0)=0$ and that $\prod_{i=1}^n M_i^{-1}$ converges almost surely to 0 as $n\to\infty$, i.e. that $\sum_{i=1}^n\log|M_i|\to+\infty$ a.s. as $n\to\infty$. Then the following are equivalent:

- *(i) A random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary.*
- *(ii) The infinite sum $\sum_{i=1}^\infty\left(\prod_{j=1}^i M_j^{-1}\right)Q_i$ converges almost surely absolutely.*

(iii) With $A_{\log|M|}$ as defined in (2.3), it holds

$$\int_1^\infty \frac{\log q}{A_{\log|M|}(\log q)}\, P_{|M^{-1}Q|}(\mathrm{d}q) < \infty\,.$$

If these equivalent conditions are satisfied, then the stationary solution is unique and given by

$$X_n = -\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_{n+j}^{-1}\right) Q_{n+i}\,,\quad n\in\mathbb{N}_0\,. \tag{2.8}$$

(d) Suppose that $P(M=0)=0$ and that neither $\prod_{i=1}^n M_i$ nor $\prod_{i=1}^n M_i^{-1}$ converges almost surely to 0 as $n\to\infty$. Then a random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary if and only if there is some $c\in\mathbb{R}$ such that $P(Q+Mc=c)=1$. If this condition is satisfied, a strictly stationary solution is given by the degenerate and constant process $X_n=c$ for all $n\in\mathbb{N}_0$. If additionally $P(|M|=1)<1$, then $(X_n=c)_{n\in\mathbb{N}_0}$ is the only strictly stationary solution of (2.1).

Observe that the solution given by (2.8) depends on the future and is a non-causal solution.

Proof. (b) Suppose that $P(M=0)=0$ and that $\prod_{i=1}^n M_i\to 0$ a.s. as $n\to\infty$. The equivalence of (ii) and (iii) is clear from Theorem 2.1. Now assume (i) and let $(X_n)_{n\in\mathbb{N}_0}$ be a strictly stationary solution of (2.1). Since $\mathcal{L}(X_{n+h_1},\dots,X_{n+h_m})=\mathcal{L}(X_{h_1},\dots,X_{h_m})$ for all $m\in\mathbb{N}$ and $h_1,\dots,h_m\in\mathbb{N}_0$ by strict stationarity, and since $\prod_{i=1+h_k}^{n+h_k} M_i\to 0$ a.s. as $n\to\infty$ for each $k\in\{1,\dots,m\}$, it follows from (2.2) and Slutsky's lemma that

$$\left(\sum_{i=1+h_1}^{n+h_1}\left(\prod_{j=i+1}^{n+h_1} M_j\right)Q_i,\dots,\sum_{i=1+h_m}^{n+h_m}\left(\prod_{j=1+h_m}^{n+h_m} M_j\right)Q_i\right)$$

converges in distribution as $n\to\infty$ to $\mathcal{L}(X_{h_1},\dots,X_{h_m})$. Since this limit does not depend on X_0, we see that the stationary solution must be unique in distribution. Further, setting $m=1$ and $h_1=0$, we get convergence in distribution of $\sum_{i=1}^n\left(\prod_{j=i+1}^n M_j\right)Q_i$, and since

$$\mathcal{L}\left(\sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} M_j\right)Q_i\right)=\mathcal{L}\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} M_j\right)Q_i\right) \tag{2.9}$$

as a consequence of the i.i.d. assumption on $(M_n,Q_n)_{n\in\mathbb{N}}$, we see that also $\sum_{i=1}^n \Pi_{i-1}Q_i$ converges in distribution to a finite random variable as $n\to\infty$. Hence (ii) follows from Theorem 2.1.

For the converse, assume (ii), and choose X_0 independent of $(M_n,Q_n)_{n\in\mathbb{N}}$ with distribution given by (2.7). Then $(X_n)_{n\in\mathbb{N}_0}$ is a time–homogeneous Markov process, and it is easy to check that $\mathcal{L}(M_1X_0+Q_1)=\mathcal{L}(X_0)$. Hence $\mathcal{L}(X_0)$ is an invariant probability measure and the Markov process $(X_n)_{n\in\mathbb{N}_0}$ consequently strictly stationary.

(a) If $P(M = 0) > 0$, for each $h_k \in \mathbb{N}_0$ we automatically have $\prod_{i=1+h_k}^{n+h_k} M_i \to 0$ a.s. and almost sure absolute convergence of $\sum_{i=1}^n \left(\prod_{j=1}^{i-1} M_j\right) Q_i$ as $n \to \infty$. The existence of a stationary solution and the uniqueness assertion is then in complete analogy to the corresponding proof in (b).

(c) Suppose that $P(M = 0) = 0$ and that $\prod_{i=1}^n M_i^{-1} \to 0$ almost surely as $n \to \infty$. Since

$$\sum_{i=1}^n \left(\prod_{j=1}^i M_j^{-1}\right) Q_i = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} M_j^{-1}\right) M_i^{-1} Q_i$$

for $n \in \mathbb{N}$ and since $(M_n, M_n^{-1}Q_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence, the equivalence of (ii) and (iii) follows from Theorem 2.1. Now assume (i) and let X_0 be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary. Rewriting (2.2) we have

$$X_h = \left(\prod_{i=h+1}^{n+h} M_i^{-1}\right) X_{n+h} - \sum_{i=h+1}^{n+h} \left(\prod_{j=h+1}^{i} M_j^{-1}\right) Q_i \tag{2.10}$$

for every $h \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Since $\mathcal{L}(X_{h+n}) = \mathcal{L}(X_0)$ by strict stationarity, and since $\prod_{i=h+1}^{n+h} M_i^{-1}$ converges almost surely to 0 as $n \to \infty$, we conclude from Slutsky's lemma that $\left(\prod_{i=h+1}^{n+h} M_i^{-1}\right) X_{h+n}$ converges in probability to 0 as $n \to \infty$, hence $-\sum_{i=1}^n \left(\prod_{j=1}^i M_{h+j}^{-1}\right) Q_{h+i}$ must converge in probability to X_h as $n \to \infty$. This shows uniqueness of the solution and the given form, and from the discussion above and Theorem 2.1 we see that the convergence must be almost surely absolutely, hence we obtain (ii). Conversely, if (ii) is satisfied, define X_n by (2.8). Then it is easy to see that $(X_n)_{n\in\mathbb{N}_0}$ is a strictly stationary solution of (2.1).

(d) Suppose that $P(M = 0) = 0$ and that neither Π_n nor Π_n^{-1} converges to 0 a.s. as $n \to \infty$. Suppose that $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$. Then $P(Q = 0) < 1$ and $|\sum_{i=1}^n \Pi_{i-1} Q_i|$ converges in probability to ∞ as $n \to \infty$ by Theorem 2.1, hence so does $\left|\sum_{i=1}^n \left(\prod_{j=i+1}^n M_j\right) Q_i\right|$ by (2.9). Assume that a stationary version $(X_n)_{n\in\mathbb{N}_0}$ exists. By (2.2) for $h = 0$ this implies that $|\Pi_n X_0|$ converges in probability to ∞ as $n \to \infty$, hence so does $|\Pi_n|$. By stationarity, we conclude that $\Pi_n^{-1} X_n$ converges in probability to 0 as $n \to \infty$, and hence we conclude from (2.10) for $h = 0$ that $\sum_{i=1}^n \Pi_{i-1}^{-1} M_i^{-1} Q_i$ converges in probability to $-X_0$. Since $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$, we also have $P(M^{-1}Q + M^{-1}d = d) < 1$ for all $d \in \mathbb{R}$, and since Π_n^{-1} does not converge to 0 a.s. by assumption it follows again from Theorem 2.1 that $|\sum_{i=1}^n \Pi_{i-1}^{-1} M_i^{-1} Q_i|$ converges in probability to ∞, a contradiction. Hence no strictly stationary solution can exist unless $P(Q + Mc = c) = 1$ for some $c \in \mathbb{R}$.

Now if there is some $c \in \mathbb{R}$ such that $P(Q + Mc = c) = 1$, then $Q_n = (c - M_n c)$ a.s., and (2.1) is equivalent to

$$X_n - c = M_n(X_{n-1} - c), \quad n \in \mathbb{N}. \tag{2.11}$$

Hence $X_n = c$ for each $n \in \mathbb{N}_0$ is obviously a strictly stationary solution. To show uniqueness if $P(|M| = 1) < 1$, let $(X_n)_{n\in\mathbb{N}_0}$ be some strictly stationary solution of (2.1).

From (2.11) we obtain $|X_n - c| = |\Pi_n|\,|X_0 - c|$, hence

$$\log|X_n - c| = \log|X_0 - c| + \sum_{i=1}^{n} \log|M_i|\,, \quad n \in \mathbb{N}\,,$$

with the convention that $\log 0 = -\infty$. But as the modulus of a random walk with $P(\log|M_i| = 0) < 1$, $\left|\sum_{i=1}^n \log|M_i|\right|$ converges in probability to $+\infty$ as $n \to \infty$ (this is well known; for instance it is an immediate consequence of Theorem III.9 in Petrov [57]), hence $\left|\log|X_n - c|\right|$ converges in probability to ∞ as $n \to \infty$. But since $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary, this is only possible if $\left|\log|X_n - c|\right| = \infty$ a.s., i.e. if $X_n = c$ a.s. $\square$

Remark 2.4. *It follows from Theorem 2.3 that the strictly stationary solution to (2.1), provided it exists, is unique in distribution unless $P(|M| = 1) = 1$ and $Q = (1 - M)c$ a.s. for some $c \in \mathbb{R}$. If $P(|M| = 1) = P(Q = (1 - M)c) = 1$ for some $c \in \mathbb{R}$, then the strictly stationary solution is indeed no longer unique in distribution, as follows from Theorem 3.1 (b) (i)–(iii) in Goldie and Maller [41], where moreover all causal solutions in this case are characterized.*

Remark 2.5. *Let $(M_n, Q_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q). Then the same characterization as in Theorem 2.3 also holds for the existence of strictly stationary solutions to the equation $X_n = M_n X_{n-1} + Q_n$ indexed by $n \in \mathbb{Z}$. The only difference is now that, in cases (a) and (b), the strictly stationary solution (if existent) is not only unique in distribution, but unique almost surely, and given by $X_t = \sum_{i=0}^\infty \left(\prod_{j=0}^{i-1} M_{t-j}\right) Q_{t-i}$ for all $t \in \mathbb{Z}$, with convergence almost surely absolutely. This follows from (2.2) by fixing $t = n + h$ and letting $h \to -\infty$.*

In light of part (c) of Theorem 2.3, in comparison with part (b) of Theorem 2.3, it is natural to ask for the relationship between the almost sure absolute convergence of $\sum_{i=1}^\infty \left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ and that of $\sum_{i=1}^\infty \left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$, or in other words, the relationship between the convergence of the integrals $\int_1^\infty \frac{\log q}{A_{\log|M|}(\log q)} P_{|M^{-1}Q|}(\mathrm{d}q)$ and $\int_1^\infty \frac{\log q}{A_{\log|M|}(\log q)} P_{|Q|}(\mathrm{d}q)$. We have the following result:

Proposition 2.6. *Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q) such that $P(M = 0) = 0$.*

(a) If $\sum_{i=1}^\infty \left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$ converges almost surely absolutely, then so does $\sum_{i=1}^\infty \left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$.

(b) Conversely, if additionally $\mathbf{E}\left|\log|M|\right| < \infty$, then almost sure absolute convergence of $\sum_{i=1}^\infty \left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ implies that of $\sum_{i=1}^\infty \left(\sum_{j=1}^{i-1} M_j^{-1}\right) Q_i$.

(c) If (M, Q) are such that $P(M > 1) = 1$, $\mathbf{E}(\log M) = \infty$ and $Q = M$, then $\sum_{i=1}^\infty \left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ converges almost surely absolutely, while $\sum_{i=1}^\infty \left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$ does not.

Proof. The proof of (a) and (b) is in complete analogy to the proof of Theorem 3.1 in Lindner and Maller [51] for convergence of Lévy integrals and hence omitted. We only remark that for the proof of (a), Equation (7.1) in Lindner and Maller [51] has to be replaced by

$$P(|M^{-1}Q| > q) \leq P(|M^{-1}| > \sqrt{q}) + P(|Q| > \sqrt{q})$$

for $q \geq 1$ and that (2.6) is used to show convergence of the corresponding integral involving $P(|M^{-1}| > \sqrt{q})$. The proof of (b) is similar to that in Lindner and Maller [51], using

$$P(|Q| > q) \leq P(|M| > \sqrt{q}) + P(|M^{-1}Q| > \sqrt{q}) .$$

The convergence statement in (c) is trivial from Theorem 2.1 since $P_{|M^{-1}Q|}$ is the Dirac measure at 1, while the divergence assertion follows as in Lindner and Maller [51, Theorem 3.1 (c)]. □

Similar to Brandt [17] or Brandt et al. [18], it would be interesting to know if some of the results of this chapter can be extended to inputs that are strictly stationary and ergodic rather than i.i.d., but we leave this for future research.

3 Continuous time autoregressive moving average processes with Lévy coefficients

This chapter is grounded on the published article by Brandes [14] "Continuous time autoregressive moving average processes with random Lévy coefficients". Let $q < p$ be non-negative integers and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process, i.e. a process with stationary and independent increments, càdlàg sample paths and $L_0 = 0$ almost surely, which is continuous in probability. A CARMA(p,q) process $S = (S_t)_{t\in\mathbb{R}}$ driven by L is defined via

$$S_t = \mathbf{b}'\mathbf{X}_t\,, \quad t \in \mathbb{R}\,, \tag{3.1}$$

with $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ a $\mathbb{C}^p$-valued process which is a solution to the stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{X}_t = A\mathbf{X}_t\,\mathrm{d}t + \mathbf{e}\,\mathrm{d}L_t\,, \quad t \in \mathbb{R}\,, \tag{3.2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{bmatrix}, \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}$$

with $a_1, \dots, a_p, b_0, \dots, b_{p-1} \in \mathbb{C}$ such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A is interpreted as $A = (-a_1)$.

It is well-known that the solution of (3.2) is unique for any $\mathbf{X}_0$ and given by

$$\mathbf{X}_t = \mathrm{e}^{At}\left(\mathbf{X}_0 + \int_{(0,t]} \mathrm{e}^{-As}\mathbf{e}\,\mathrm{d}L_s\right), \quad t \geq 0\,.$$

Processes of this kind were first considered for L being a Gaussian process by Doob [35]. Brockwell [22] gave the now commonly used definition with L being a Lévy process.

A CARMA process $S = (S_t)_{t\in\mathbb{R}}$ as defined in (3.1) and (3.2) can be interpreted as a solution of the p^{th}-order linear differential equation

$$a(D)S_t = b(D)DL_t\,,$$

where $a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$, $b(z) = b_0 + b_1 z + \cdots + b_{p-1}z^{p-1}$ and D denotes the differentiation operator. In this sense, CARMA processes are a natural continuous time analog of discrete time ARMA processes. Similar to ARMA processes, CARMA processes provide a tractable but rich class of stochastic processes. Their possible autocovariance functions $h \mapsto \mathbf{Cov}(S_t, S_{t+h})$ are linear combinations of (complex) exponentials and thus provide a wide variety of possible models when modeling empirical data.

In discrete time, ARMA processes with random coefficients (RC-ARMA) have attracted a lot of interest recently, in particular, AR processes with random coefficients, see e.g. Nicholls and Quinn [56]. They have applications as non-linear models for various processes, e.g. bilinear GARCH processes introduced by Storti and Vitale [63]. RC-ARMA processes also arise as a special case of conditional heteroscedastic ARMA (CHARMA) models proposed by Tsay [66] and are used for financial volatility processes, see e.g. He and Teräsvirta [44], to name just a few.

As CARMA processes constitute the natural continuous time analog of ARMA processes, it is, therefore, natural to ask for CARMA processes with random coefficients. The CARMA$(1,0)$ process with random (Lévy) coefficients has already been studied. It is known as the *generalized Ornstein-Uhlenbeck (GOU) process*, which is obtained as the solution to the SDE

$$\mathrm{d}X_t = X_{t-}\,\mathrm{d}\xi_t + \mathrm{d}L_t\,, \quad t \geq 0\,,$$

where $(\xi, L) = (\xi_t, L_t)_{t\geq 0}$ is a bivariate Lévy process. It has been shown by de Haan and Karandikar [33] that GOU processes arise as the natural continuous time analog of the AR(1) process with random i.i.d. coefficients. By choosing $(\xi_t)_{t\geq 0} = (-a_1 t)_{t\geq 0}$, the GOU process reduces to the classical Lévy-driven Ornstein-Uhlenbeck process, which is a CAR(1), i.e. CARMA(1,0) process.

Both the Ornstein-Uhlenbeck process as well as the generalized Ornstein-Uhlenbeck process have various applications in insurance and financial mathematics, see e.g. Barndorff-Nielsen and Shepard [1] and Klüppelberg et al. [49].

The aim of this chapter is to introduce CARMA processes with random Lévy autoregressive coefficients of higher orders, $p \geq 1$, and to study stationarity and other natural properties. The definition of our process is done in such a way that it includes the generalized Ornstein-Uhlenbeck process for order $(1,0)$ as a special case and that it reduces to the usual CARMA process when the autoregressive Lévy coefficients are chosen to be deterministic Lévy processes, i.e. pure drift and henceforth linear functions.

The chapter is organized as follows. In Section 3.1 we give some preparative results regarding multivariate stochastic integration, the multivariate stochastic exponential, and multivariate generalized Ornstein-Uhlenbeck processes. In Section 3.2 we define a CARMA process with random coefficients (RC-CARMA) and present some basic properties as well as sufficient conditions for the existence of a strictly stationary solution. Similar to Brockwell and Lindner [28] for CARMA processes, we further show that the RC-CARMA$(p, 0)$ process satisfies an integral-differential equation and examine its path properties. Section 3.3 is concerned with the existence of moments, the autocovariance function and spectral density, whereby it turns out that the latter two have an interesting connection to those of CARMA processes. We end Section 3.3 by investigating an RC-CARMA$(2, 1)$ process in more detail. Conclusively, in Section 3.4 we present some simulations.

3.1 Preliminaries

Throughout we will always assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. By a *filtration* we mean a family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. Our filtration satisfies, if not stated otherwise, the *usual hypotheses*, i.e. $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and the filtration is right-continuous.

$\mathrm{GL}(\mathbb{R}, m)$ denotes the general linear group of order m, i.e. the set of all $m \times m$ invertible matrices associated with the ordinary matrix multiplication. If $A \in \mathrm{GL}(\mathbb{R}, m)$, we denote with A' its transpose and with A^{-1} its inverse.

For càdlàg processes $X = (X_t)_{t\geq 0}$ we denote with X_{t-} and $\Delta X_t := X_t - X_{t-}$ the left-limit and the jump at time t, respectively. A d-dimensional Lévy processes $L = (L_t)_{t\geq 0}$ can be identified by its characteristic exponent (A_L, γ_L, Π_L) due to the Lévy-Khintchine formula, cf. Theorem 1.16, i.e. if μ denotes the distribution of L_1, then its characteristic function is given by

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, A_L z\rangle + i\langle \gamma_L, z\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{\{|x|\leq 1\}}(x))\, \Pi_L(\,\mathrm{d}x)\right],\ z \in \mathbb{R}^d.$$

Here, A_L is the Gaussian covariance matrix which is in one dimension denoted by σ_L^2, Π_L a measure on $\mathbb{R}^d$ which satisfies $\Pi_L(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\, \Pi_L(\,\mathrm{d}x) < \infty$, called the *Lévy measure*, and $\gamma_L \in \mathbb{R}^d$ a constant. Further, $|x|$ denotes the Euclidean norm of x.

For a detailed account of Lévy processes we refer to the book of Sato [61].

Stochastic Integration

A matrix-valued stochastic process $X = (X_t)_{t\geq 0}$ is called an $\mathbb{F}$-*semimartingale* or simply a *semimartingale* if every component $(X_t^{(i,j)})_{t\geq 0}$ is a semimartingale with respect to the filtration $\mathbb{F}$.

For a semimartingale $X \in \mathbb{R}^{m\times n}$, and $H \in \mathbb{R}^{l\times m}$ and $G \in \mathbb{R}^{n\times p}$ two locally bounded predictable processes, the $\mathbb{R}^{l\times p}$-valued stochastic integral $J = \int H \,\mathrm{d}XG$ is defined, cf.

Definition 1.26, by its components via

$$J^{(i,j)} = \sum_{k=1}^{n} \sum_{h=1}^{m} \int H^{(i,h)} G^{(k,j)} \,\mathrm{d}X^{(h,k)} .$$

It can easily be seen that also in the multivariate case the stochastic integration preserves the semimartingale property, cf. Remark 1.27.

For two semimartingales $X \in \mathbb{R}^{l\times m}$ and $Y \in \mathbb{R}^{m\times n}$ the $\mathbb{R}^{l\times n}$-valued *quadratic covariation* $[X, Y]$ is defined, cf. Definition 1.28, by its components via

$$[X, Y]^{(i,j)} = \sum_{k=1}^{m} [X^{(i,k)}, Y^{(k,j)}] \tag{3.3}$$

and similar its continuous part $[X, Y]^c$ such that it also holds true for matrix-valued semimartingales

$$[X, Y]_t = [X, Y]^c_t + X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s \,, \quad t \geq 0 \,. \tag{3.4}$$

Finally, as stated in Theorem 1.29, for two semimartingales $X, Y \in \mathbb{R}^{m\times m}$ the integration by parts formula takes the form

$$(XY)_t = \int_{0+}^{t} X_{s-} \,\mathrm{d}Y_s + \int_{0+}^{t} \mathrm{d}X_s Y_{s-} + [X., Y.]_t \,, \quad t \geq 0 \,.$$

The Multivariate Stochastic Exponential

Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{m\times m}$ with $X_0 = 0$. Due to Definition 1.30, its *left stochastic exponential* $\overleftarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} Z_{s-} \,\mathrm{d}X_s \,, \quad t \geq 0 \,,$$

where $I \in \mathbb{R}^{m\times m}$ denotes the identity matrix. The *right stochastic exponential* of X, denoted as $\overrightarrow{\mathcal{E}}(X)_t$, is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} \mathrm{d}X_s Z_{s-} \,, \quad t \geq 0 \,.$$

It can be shown that both the left and the right stochastic exponential are semimartingales and that for its transpose it holds $\overleftarrow{\mathcal{E}}(X)'_t = \overrightarrow{\mathcal{E}}(X')_t$. As observed by Karandikar [46], the right and the left stochastic exponentials of a semimartingale X are invertible at all times $t \geq 0$ if and only if

$$\det(I + \Delta X_t) \neq 0 \quad \forall\, t \geq 0 \,. \tag{3.5}$$

We also need the following result of Karandikar [46].

Proposition 3.1. (Inverse of the Stochastic Exponential)
Let $X = (X_t)_{t\geq 0}$ be a semimartingale with $X_0 = 0$ such that (3.5) holds. Define the semimartingale

$$U_t := -X_t + [X,X]_t^c + \sum_{0<s\leq t} \Big((I+\Delta X_s)^{-1} - I + \Delta X_s \Big) , \quad t \geq 0 . \tag{3.6}$$

Then

$$\overleftarrow{\mathcal{E}}\,(X)_t^{-1} = \left[\overleftarrow{\mathcal{E}}\,(U')_t \right]' = \overrightarrow{\mathcal{E}}\,(U)_t \quad \forall\, t \geq 0 , \tag{3.7}$$

and

$$U_t = -X_t - [X,U]_t \quad \forall\, t \geq 0 . \tag{3.8}$$

Further,

$$\det(I + \Delta U_t) \neq 0 \quad \forall\, t \geq 0 ,$$

and X can be represented by

$$X_t = -U_t + [U,U]_t^c + \sum_{0<s\leq t} \Big((I+\Delta U_s)^{-1} - I + \Delta U_s \Big) , \quad t \geq 0 . \tag{3.9}$$

Proof. For (3.7) and (3.8) see Karandikar [46], Theorem 1. For the remaining assertions, observe that $\Delta U_t = (I + \Delta X_t)^{-1} - I$ from (3.6), so that $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$. Further, from (3.6) we obtain $[U,U]_t^c = [X,X]_t^c$. Inserting this, $\Delta U_t = (I + \Delta X_t)^{-1} - I$, and the form of U_t from (3.6) into the right-hand side of (3.9) gives X_t so that (3.9) is true. □

Multivariate Generalized Ornstein-Uhlenbeck processes

We give a short overview of results regarding multivariate generalized Ornstein-Uhlenbeck (MGOU) processes which are used throughout. MGOU processes were introduced by Behme and Lindner [8] and further investigated in Behme [9].

Definition 3.2. Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (3.5), and let V_0 be a random variable in $\mathbb{R}^m$. Then the $\mathbb{R}^m$-valued process $V = (V_t)_{t\geq 0}$ given by

$$V_t = \overleftarrow{\mathcal{E}}\,(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}\,(X)_{s-}\, \mathrm{d}Y_s \right) , \quad t \geq 0 ,$$

is called a *multivariate generalized Ornstein-Uhlenbeck (MGOU) process* driven by (X,Y). The underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is such that it satisfies the usual hypotheses and such that (X,Y) is a semimartingale.

The MGOU process will be called *causal* or *non-anticipative*, if V_0 is independent of (X,Y), and *strictly non-causal* if V_t is independent of $(X_s, Y_s)_{0\leq s<t}$ for all $t \geq 0$.

Remark 3.3. *It follows from Behme and Lindner [8], Theorem 3.4, that an MGOU process $V = (V_t)_{t\geq 0}$ with an $\mathcal{F}_0$-measurable V_0 solves the SDE*

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathrm{d}Z_t\,, \quad t \geq 0\,, \tag{3.10}$$

where $U = (U_t)_{t\geq 0}$ is another $\mathbb{R}^{m\times m}$-valued Lévy process defined by (3.6) so that $\overleftarrow{\mathcal{E}}(X)_t^{-1} = \overrightarrow{\mathcal{E}}(U)_t$, and $Z = (Z_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^m$ given by

$$Z_t = Y_t + \sum_{0<s\leq t} \Big((I + \Delta X_s)^{-1} - I\Big)\,\Delta Y_s - [X, Y]_t^c\,, \quad t \geq 0\,.$$

With these U and Z the MGOU process can also be written as

$$V_t = \overrightarrow{\mathcal{E}}(U)_t \left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1}\,\mathrm{d}Y_s\right), \quad t \geq 0\,. \tag{3.11}$$

Conversely, if $(U, Z) = (U_t, Z_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that it holds $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, then for every $\mathcal{F}_0$-measurable random vector V_0 the solution to (3.10) is an MGOU process driven by (X, Y), where X is given by (3.9) and $Y_t = Z_t + [X, Z]_t$.

Convention 3.4. Observe that an MGOU process and similarly the process given by (3.11) is well-defined for any starting random vector V_0, regardless if it is $\mathcal{F}_0$-measurable or not. We shall hence speak of (3.11) as a *solution to (3.10)*, regardless if V_0 is $\mathcal{F}_0$-measurable or not. Observe that if V_0 is chosen to be independent of (U, Z) or equivalently (X, Y), then the natural augmented filtration of (U, Z) may be enlarged by $\sigma(V_0)$ such that (U, Z) still remains a semimartingale, see Protter [58], Theorem VI.2, and with this enlarged filtration, V_0 is measurable.

To investigate the strict stationarity property of RC-CARMA processes later, we introduce the property of irreducibility of a class of MGOU processes as it has been done in Section 4 of Behme and Lindner [8].

Definition 3.5. Suppose that $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (3.5). Then an affine subspace H of $\mathbb{R}^m$ is called *invariant* for the class of MGOU processes $V = (V_t)_{t\geq 0}$ driven by (X, Y) if for all $x \in H$ the choice $V_0 = x$ implies $V_t \in H$ a.s. for all $t \geq 0$.

If there exists no proper affine subspace H such that, for all $x \in H$, $V_0 = x$ implies $V_t \in H$ a.s. for all $t \geq 0$, we call the class of MGOU processes *irreducible*.

Irreducibility is thus a property of the considered model. By abuse of language, we will call an MGOU process irreducible if the corresponding class satisfies Definition 3.5.

A more comprehensive overview on MGOU processes can be found in the Appendix A.2, or in the paper of Behme and Lidner [8].

3.2 The RC-CARMA process

Let $p \in \mathbb{N}$ and $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \dots, M_t^{(p)}, L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\Pi_{M^{(1)}}(\{1\}) = 0$. Let $b_0, \dots, b_{p-1} \in \mathbb{R}$. Let $U = (U_t)_{t\geq 0}$ be $\mathbb{R}^{p\times p}$-valued defined by

$$U_t := \begin{bmatrix} 0 & t & 0 & \dots & 0 \\ 0 & 0 & t & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t \\ -M_t^{(p)} & -M_t^{(p-1)} & -M_t^{(p-2)} & \dots & -M_t^{(1)} \end{bmatrix}, \quad \mathbf{e} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \tag{3.12}$$

and $q := \max\{j \in \{0, \dots, p-1\} \colon b_j \neq 0\}$. Then we call any process $R = (R_t)_{t\geq 0}$ which satisfies

$$R_t = \mathbf{b}' V_t\,, \quad t \geq 0\,, \tag{3.13}$$

where $V = (V_t)_{t\geq 0}$ is a solution to the SDE

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathbf{e}\,\mathrm{d}L_t\,, \quad t \geq 0\,, \tag{3.14}$$

an *RC-CARMA(p,q) process*, i.e. a CARMA process with random Lévy coefficients. We speak of C and $\mathbf{b}$ as the *parameters* of the RC-CARMA process.

As will be seen in Proposition 3.9 below, the assumption $\Pi_{M^{(1)}}(\{1\}) = 0$ implies $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, so that V is an MGOU process as in (3.10) and, as in Convention 3.4, by a solution of (3.14) we mean a process of the form (3.11) with starting random variable V_0 not necessarily $\mathcal{F}_0$-measurable. We shall call the process V a *state vector process* of the RC-CARMA process R.

Observe that we get a classical CARMA(p,q) process $(S_t)_{t\geq 0} = (\mathbf{b}'V_t)_{t\geq 0}$, although on the positive real line, by choosing $(M_t^{(1)}, \dots, M_t^{(p)}) = (a_1, \dots, a_p)t$ with $a_1, \dots, a_p \in \mathbb{R}$. Further, we recognize that there is less sense in choosing the coefficients of the moving average side to be random since they are just defining the weights of the components of V to form S.

Recall that a CARMA(p,q) process $S = (S_t)_{t\in\mathbb{R}}$ satisfies the formal p^{th}-order linear differential equation

$$a(D)S_t = b(D)DL_t\,, \tag{3.15}$$

where $a(z) = z^p + a_1 z^{p-1} + \dots + a_p$, $b(z) = b_0 + b_1 z + \dots + b_{p-1} z^{p-1}$ and D denotes the differentiation with respect to t. When we consider an RC-CARMA(p,0) process $(R_t)_{t\geq 0} = (b_0 V_t^1)_{t\geq 0}$ for $(V_t)_{t\geq 0} = (V_t^1, \dots, V_t^p)_{t\geq 0}$ solving (3.14), we formally find that

$$a_M(D)R_t = a_M(D)\mathbf{b}'V_t = a_M(D)b_0 V_t^1 = b_0 DL_t = b(D)DL_t\,, \tag{3.16}$$

where
$$a_M(z) = z^p + \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} z^{p-1} + \cdots + \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} z^0 \,.$$

Observe that $\frac{\mathrm{d}M_t^{(i)}}{\mathrm{d}t}$ is not defined in a rigorous way but just an intuitive way of writing. Thus, it is possible to interpret also the RC-CARMA(p,0) process as a solution to a formal p^{th}-order linear differential equation with random coefficients.

To justify (3.16), look at the first $p-1$ components of V

$$\mathrm{d}V_t^i = V_t^{i+1}\,\mathrm{d}t \quad \Leftrightarrow \quad \frac{\mathrm{d}V_t^i}{\mathrm{d}t} = V_t^{i+1} \quad \Leftrightarrow \quad D^i V_t^1 = V_t^{i+1}\,, \quad i = 1, \ldots, p-1\,. \tag{3.17}$$

Formal division by $\mathrm{d}t$ yields for the p^{th} component

$$\begin{aligned}
& \mathrm{d}V_t^p = -V_t^1\,\mathrm{d}M_t^{(p)} - \cdots - V_t^p\,\mathrm{d}M_t^{(1)} + \mathrm{d}L_t \\
\text{“}\Leftrightarrow\text{”} \quad & \frac{\mathrm{d}V_t^p}{\mathrm{d}t} = -V_t^1 \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} - \cdots - V_t^p \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} + \frac{\mathrm{d}L_t}{\mathrm{d}t} \\
\overset{(3.17)}{\Leftrightarrow} \quad & DV_t^p = -V_t^1 \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} - \cdots - D^{p-1}V_t^1 \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} + DL_t \\
\Leftrightarrow \quad & DL_t = D^p V_t^1 + \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} D^{p-1} V_t^1 + \cdots + \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} V_t^1 \\
\Leftrightarrow \quad & DL_t = a_M(D) V_t^1 \,.
\end{aligned}$$

Brockwell and Lindner [28] gave a rigorous interpretation of (3.15) by showing that a CARMA(p,q) process $(S_t = \mathbf{b}'\mathbf{X}_t)_{t\in\mathbb{R}}$ driven by a Lévy process L satisfies the integral equation

$$a(D)J^p S_t = b(D)J^{p-1}(L_t) + a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0)\,, \quad t \in \mathbb{R}\,,$$

where $a(z)$ and $b(z)$ are as before, and J denotes the integration operator which associates with any càdlàg function $f = (f_t)_{t\in\mathbb{R}} \colon \mathbb{R} \to \mathbb{C}\,, t \mapsto f_t$, the function $J(f)$ defined by

$$J(f)_t := \int_0^t f_s\,\mathrm{d}s\,.$$

Similarly, we give a rigorous interpretation of (3.16) as an integral-differential equation and show that the RC-CARMA($p,0$) process solves this equation, hence making the formal deviation of (3.16) above thoroughly.

We call a function $g\colon [0,\infty) \to \mathbb{R}$ *differentiable with càdlàg derivative* Dg, if g is continuous and there exists a càdlàg function Dg such that g is at every point $t \in \mathbb{R}$ right- and left-differentiable with right-derivative Dg_t and left-derivative $Dg_{t-} = \lim_{\varepsilon\downarrow 0,\varepsilon\neq 0} Dg_{t-\varepsilon}$, respectively. In other words, g is absolutely continuous and has càdlàg density Dg (see the discussion in [28] at the beginning of Section 2).

We call a function $g\colon [0,\infty) \to \mathbb{R}$ *p-times continuously differentiable with càdlàg derivative* $D^p g$, if g is $p-1$-times differentiable in the usual sense and the $(p-1)^{st}$ derivative $D^{(p-1)}g$ is differentiable with càdlàg derivative $D^p g = D(D^{p-1}g)$ as defined above.

Theorem 3.6. *Let $C = (C_t)_{t\geq 0} = (M_t^{(1)},\dots,M_t^{(p)},L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ and an $\mathbb{F}$-semimartingale with $\Pi_{M^{(1)}}(\{1\}) = 0$. Let $\mathbf{b}' = [b_0,\dots,b_{p-1}] \in \mathbb{R}^p$ with $b_0 \neq 0$ and $b_1 = \cdots = b_{p-1} = 0$, and consider the RC-CARMA$(p,0)$ process $R = (R_t)_{t\geq 0}$ defined by (3.13) and (3.14), where $V = (V_t)_{t\geq 0}$ is the state vector process (with V_0 not necessarily $\mathcal{F}_0$-measurable). Denote by $\mathcal{D}^{p-1}$ the set of all $\mathbb{F}$-adapted, $\mathbb{R}$-valued processes $G = (G_t)_{t\geq 0}$ which are $p-1$ times differentiable with càdlàg derivative $D^{p-1}G$.*
(a) Define $W = (W_t)_{t\geq 0}$ by

$$W_t := R_t - \mathbf{b}'\overrightarrow{\mathcal{E}}(U)_t V_0 , \quad t \geq 0 .$$

Then $W \in \mathcal{D}^{p-1}$, it is an RC-CARMA process with parameters C, $\mathbf{b}$, and initial state vector $\mathbf{0}$, and it satisfies the integral-differential equation

$$D^{p-1}W_t + \left(\sum_{i=1}^{p}\int_{(0,t]} D^{i-1}W_{s-}\,\mathrm{d}M_s^{(p-i+1)}\right) = b_0 L_t , \quad t \geq 0 . \tag{3.18}$$

If V_0 is additionally $\mathcal{F}_0$-measurable, then also $R \in \mathcal{D}^{p-1}$ and there exists an $\mathcal{F}_0$-measurable random variable Z_0 such that

$$D^{p-1}R_t + \left(\sum_{i=1}^{p}\int_{(0,t]} D^{i-1}R_{s-}\,\mathrm{d}M_s^{(p-i+1)}\right) = b_0 L_t + Z_0 , \quad t \geq 0 . \tag{3.19}$$

(b) Conversely, if $\widetilde{R} = (\widetilde{R}_t)_{t\geq 0} \in \mathcal{D}^{p-1}$ satisfies

$$D^{p-1}\widetilde{R}_t + \left(\sum_{i=1}^{p}\int_{(0,t]} D^{i-1}\widetilde{R}_{s-}\,\mathrm{d}M_s^{(p-i+1)}\right) = b_0 L_t + Z_0 \tag{3.20}$$

for some $\mathcal{F}_0$-measurable Z_0, then $\widetilde{R}$ is an RC-CARMA process with parameters C, $\mathbf{b}$, and state vector process $\widetilde{V} = (\widetilde{V}_t)_{t\geq 0} := (b_0^{-1}(\widetilde{R}_t, D\widetilde{R}_t,\dots,D^{p-1}\widetilde{R}_t))_{t\geq 0}$. Especially, $\widetilde{V}_0$ is $\mathcal{F}_0$-measurable.

Proof. (a) As already observed (and to be shown in Proposition 3.9 (a) below), the condition $\Pi_{M^{(1)}}(\{1\}) = 0$ implies that V is an MGOU process. So

$$V_t = \overrightarrow{\mathcal{E}}(U)_t \left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1}\,\mathrm{d}Y_s\right)$$

for some Lévy process Y as specified in Remark 3.3. Hence, $(V_t - \overrightarrow{\mathcal{E}}(U)_t V_0)_{t\geq 0}$ is adapted and consequently so is $(R_t - \mathbf{b}'\overrightarrow{\mathcal{E}}(U)_t V_0)_{t\geq 0}$, and it is obviously an RC-CARMA process with initial state vector $\mathbf{0}$. Hence, for the proof of (3.18) it suffices to assume that $V_0 = \mathbf{0}$.

Denote $V = (V_t)_{t\geq 0} = (V_t^1, \ldots, V_t^p)_{t\geq 0}$. By (3.17), we have $D^i V_t^1 = V_t^{i+1}$, $i = 1, \ldots, p-1$. Hence, V_t^1 is $p-1$-times differentiable with $(p-1)^{st}$ càdlàg derivative V_t^p. By the defining SDE of the RC-CARMA process (3.14) and the form of the matrix $U = (U_t)_{t\geq 0}$ given in (3.12), we also have

$$V_t^p = V_0^p - \sum_{i=1}^{p} \int_{(0,t]} V_{s-}^i \, \mathrm{d}M_s^{(p-i+1)} + L_t \,, \tag{3.21}$$

and since $V_0 = 0$, that

$$L_t = D^{p-1} V_t^1 + \sum_{i=1}^{p} \int_{(0,t]} D^{i-1} V_{s-}^1 \, \mathrm{d}M_s^{(p-i+1)} \,. \tag{3.22}$$

Multiplying (3.22) by b_0 gives (3.18) when $V_0 = \mathbf{0}$ and hence (3.18) in general.

For (3.19) let

$$K_t = (K_t^1, \ldots, K_t^p)' := \overrightarrow{\mathcal{E}}(U)_t V_0 \,.$$

When we consider the SDE which defines the right stochastic exponential $\mathrm{d}\overrightarrow{\mathcal{E}}(U)_t = \mathrm{d}U_t \overrightarrow{\mathcal{E}}(U)_{t-}$, we obtain

$$\mathrm{d}K_t = \mathrm{d}U_t K_{t-} \quad \text{with} \quad K_0 = V_0 \,, \tag{3.23}$$

and due to the form of the process U

$$\mathrm{d}K_t^i = K_t^{i+1} \, \mathrm{d}t \quad \Leftrightarrow \quad \frac{\mathrm{d}K_t^i}{\mathrm{d}t} = K_t^{i+1} \quad \Leftrightarrow \quad D^i K_t^1 = K_t^{i+1} \,, \quad i = 1, \ldots, p-1 \,.$$

Further, from the last component of (3.23)

$$\begin{aligned} K_t^p = D^{p-1} K_t^1 &= V_0^p - \int_{(0,t]} K_{s-}^1 \, \mathrm{d}M_s^p - \cdots - \int_{(0,t]} K_{s-}^p \, \mathrm{d}M_s^1 \\ &= V_0^p - \sum_{i=1}^{p} \int_{(0,t]} D^{i-1} K_{s-}^1 \, \mathrm{d}M_s^{(p-i+1)} \,. \end{aligned}$$

Hence,

$$D^{p-1} K_t^1 + \sum_{i=1}^{p} \int_{(0,t]} D^{i-1} K_{s-}^1 \, \mathrm{d}M_s^{(p-i+1)} = V_0^p \,,$$

where V_0^p is $\mathcal{F}_0$-measurable such that we obtain (3.19) via $R_t = W_t + \mathbf{b}' K_t$, where W_t satisfies (3.18).

(b) For the converse, let $\widetilde{R} \in \mathcal{D}^{p-1}$ satisfy (3.20), and denote $\widetilde{V}_t = (\widetilde{V}_t^1, \ldots, \widetilde{V}_t^p) = (b_0^{-1}(\widetilde{R}_t, D\widetilde{R}_t, \ldots, D^{p-1}\widetilde{R}_t))$, $t \geq 0$. By the fundamental theorem of calculus

$$\widetilde{V}_t^i = \widetilde{V}_0^i + \int_{(0,t]} \widetilde{V}_s^{i+1} \, \mathrm{d}s \,, \quad i = 1, \ldots, p-1 \,, \tag{3.24}$$

and from (3.20) we obtain

$$\begin{aligned}\widetilde{V}_t^p &= b_0^{-1} Z_0 - \sum_{i=1}^{p} \int_{(0,t]} D^{i-1} \widetilde{V}_{s-}^1 \,\mathrm{d}M_s^{(p-i+1)} + L_t \\ &= b_0^{-1} Z_0 - \sum_{i=1}^{p} \int_{(0,t]} \widetilde{V}_{s-}^i \,\mathrm{d}M_s^{(p-i+1)} + L_t\,, \quad t \geq 0\,. \end{aligned} \tag{3.25}$$

But (3.24) and (3.25) mean that $\widetilde{V}_0^p = b_0^{-1} Z_0$ and that $\widetilde{V} = (\widetilde{V}_t)_{t\geq 0}$ satisfies

$$\widetilde{V}_t = \widetilde{V}_0 + \int_{(0,t]} \mathrm{d}U_s \widetilde{V}_{s-} + \mathbf{e} L_t\,.$$

Since obviously $\mathbf{b}'\widetilde{V}_t = b_0 b_0^{-1} \widetilde{R}_t = \widetilde{R}_t$, it follows that $\widetilde{R}$ is an RC-CARMA$(p,0)$ process with parameters C, $\mathbf{b}$, and state vector process $\widetilde{V}$. □

Remark 3.7. *Differentiating (3.19) formally gives*

$$D^p R_t + \sum_{i=1}^{p} D^{i-1} R_{s-} D M_s^{(p-i+1)} = b_0 D L_t\,,$$

hence (3.16) and the RC-CARMA$(p,0)$ process can be interpreted as a solution to a formal p^{th}-order linear differential equation with random coefficients. To obtain a similar equation and hence interpretation for RC-CARMA(p,q) processes with $q>0$ seems not so easy since it is in general not possible to interchange the stochastic integration with the differentiation operator D.

Remark 3.8. *Similar as in case of CARMA(p,q) processes, we easily see for an RC-CARMA(p,q) process $R=(R_t)_{t\geq 0}$ with $q<p$, $b_q \neq 0$, and $b_j = 0$ for $j>q$ with*

$$R_t = \mathbf{b}' V_t = b_0 V_t^1 + \cdots + b_q V_t^{q+1}$$

that, by (3.17), R is $(p-q-1)$-times differentiable with $(p-q-1)^{st}$ càdlàg derivative

$$D^{p-q-1} R_t = b_0 D^{p-q-1} V_t^1 + \cdots + b_q D^{p-q-1} V_t^{q+1} = b_0 V_t^{p-q} + \cdots + b_q V_t^p\,.$$

The following proposition shows that the state vector process $V=(V_t)_{t\geq 0}$ is an MGOU process and gives its specific driving Lévy process (X,Y). Further, V is an irreducible MGOU process if we assume that U is independent of L and L is not deterministic.

Proposition 3.9. *Let $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq 0}$ be an $\mathbb{R}^{p+1}$-valued Lévy process and an $\mathbb{F}$-semimartingale. Let $\Pi_{M^{(1)}}$ denote the Lévy measure of $M^{(1)}$ and let U be defined as in (3.12).*

(a) Then $\overleftarrow{\mathcal{E}}(U)_t \in \mathrm{GL}(\mathbb{R},p)$ for all $t\geq 0$ if and only if $\Pi_{M^{(1)}}(\{1\}) = 0$. In this case, for any starting random vector V_0 the solution to

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathbf{e}\,\mathrm{d}L_t$$

is an MGOU process driven by (X, Y), i.e. $V = (V_t)_{t\geq 0}$ takes the form

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}\, \mathrm{d}Y_s\right) = \overrightarrow{\mathcal{E}}(U)_t\left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1}\, \mathrm{d}Y_s\right), \quad t \geq 0.$$

Here, $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a Lévy process defined by

$$X_t = \begin{bmatrix} 0 & -t & 0 & \dots & 0 \\ 0 & 0 & -t & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -t \\ N_t^{(p)} & N_t^{(p-1)} & N_t^{(p-2)} & \dots & N_t^{(1)} \end{bmatrix}$$

with

$$N_t^{(i)} = M_t^{(i)} + t\,\sigma_{M^{(1)},M^{(i)}} + \sum_{0<s\leq t} \frac{\Delta M_s^{(i)} \Delta M_s^{(1)}}{1 - \Delta M_s^{(1)}}, \quad i = 1, \dots, p,$$

where $\sigma_{M^{(1)},M^{(i)}}$ denotes the Gaussian covariance of $M^{(1)}$ and $M^{(i)}$. X satisfies $\det(I + \Delta X_t) \neq 0$ for all $t \geq 0$, and $Y_t = \mathbf{e}\widetilde{Y}_t$ with

$$\widetilde{Y}_t = L_t + t\sigma_{M^{(1)},L} + \sum_{0<s\leq t} \frac{\Delta M_s^{(1)} \Delta L_s}{1 - \Delta M_s^{(1)}}.$$

(b) Denote $M = (M_t)_{t\geq 0} = (M_t^{(1)}, \dots, M_t^{(p)})_{t\geq 0}$. Assume that M is independent of L and L not deterministic. Then the MGOU process V obtained in (a) is irreducible.

Proof. (a) Since C is a Lévy process and a semimartingale with respect to $\mathbb{F}$, it is clear that also U as defined in (3.12) is a semimartingale and therefore also $\overleftarrow{\mathcal{E}}(U)$ is a semimartingale with respect to $\mathbb{F}$. Since $\overleftarrow{\mathcal{E}}(U)$ is non-singular if and only if $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, we calculate

$$\begin{aligned} \det(I + \Delta U_t) &= \det \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -\Delta M_t^{(p)} & -\Delta M_t^{(p-1)} & \dots & -\Delta M_t^{(2)} & 1 - \Delta M_t^{(1)} \end{pmatrix} \\ &= (1 - \Delta M_t^{(1)}) \end{aligned} \tag{3.26}$$

which shows that $\overleftarrow{\mathcal{E}}(U)_t$ is non-singular if and only if $\Delta M_t^{(1)} \neq 1$ for all $t \geq 0$. Since $M^{(1)}$ is a Lévy process, the latter is equivalent to $\Pi_{M^{(1)}}(\{1\}) = 0$.

By Remark 3.3, V is an MGOU process driven by (X, Y), where X is given by (3.9) and satisfies $\det(I + \Delta X_t) \neq 0$ for all $t \geq 0$, and $Y_t = \mathbf{e}L_t + [X, \mathbf{e}L]_t$.

From (3.3) we obtain for the components of $[U,U]^c$ due to the form of U in (3.12)

$$([U,U]^c_t)^{(i,j)} = \sum_{k=1}^{p} [U^{(i,k)}, U^{(k,j)}]^c = \begin{cases} 0\,, & i = 1, \ldots, p-1\,, \\ [U^{(p,p)}, U^{(p,j)}]^c_t = t\sigma_{M^{(1)}, M^{(p-j+1)}}\,, & i = p\,. \end{cases}$$

The form of $I + \Delta U_t$ is implicitly given in (3.26) such that

$$(I + \Delta U_t)^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \frac{\Delta M_t^{(p)}}{1-\Delta M_t^{(1)}} & \frac{\Delta M_t^{(p-1)}}{1-\Delta M_t^{(1)}} & \ldots & \frac{\Delta M_t^{(2)}}{1-\Delta M_t^{(1)}} & \frac{1}{1-\Delta M_t^{(1)}} \end{pmatrix}$$

which is well-defined since $\Pi_{M^{(1)}}(\{1\}) = 0$. Summing up the terms according to (3.9) leads to the stated form of the processes $(N_t^{(1)}, \ldots, N_t^{(p)})_{t\geq 0}$ and X.

For $Y_t = \mathbf{e}L_t + [X, \mathbf{e}L]_t$ we obtain with (3.3) componentwise

$$[X, \mathbf{e}L]_t^{(i)} = \sum_{k=1}^{p} [X^{(i,k)}, (\mathbf{e}L)^{(k)}]_t = [X^{(i,p)}, L]_t = \begin{cases} 0\,, & i = 1, \ldots, p-1\,, \\ [N^{(1)}, L]_t\,, & i = p\,. \end{cases}$$

Since

$$\Delta N_t^{(1)} = \frac{\Delta M_t^{(1)}}{1 - \Delta M_t^{(1)}}\,,$$

and $[N^{(1)}, L]^c_t = [M^{(1)}, L]^c_t = \sigma_{M^{(1)}, L}\, t$ we get the stated form of Y by (3.4).

(b) Suppose that $V = (V_t)_{t\geq 0} = (V_t^1, \ldots, V_t^p)_{t\geq 0}$ is not irreducible. Hence, there exists an invariant affine subspace H with $\dim H \in \{0, \ldots, p-1\}$. Then for all $x \in H$ it holds $P(V_t \in H | V_0 = x) = 1$ for all $t \geq 0$. Since V is càdlàg, we obtain $P(V_t \in H\ \forall\, t \geq 0 | V_0 = x) = 1$. Further, if $H' \supset H$ with $\dim H' = p-1$, then

$$P(V_t \in H'\ \forall\, t \geq 0 \,|\, V_0 = x) = 1 \quad \forall\, x \in H\,. \tag{3.27}$$

We shall show that $P(V_t \in H'\ \forall\, t \geq 0 \,|\, V_0 = x) < 1$ for all $x \in \mathbb{R}^p$, hence contradicting (3.27). So assume w.l.o.g. that $\dim H = p-1$. Since then H is a hyperplane, there exists $\lambda \in \mathbb{R}^p$ with $\lambda \neq 0$ and $a \in \mathbb{R}$ such that $H = \{y \in \mathbb{R}^p \colon \lambda' y = a\}$.

Let $V_0 = x \in H$. Let $i_1, \ldots, i_k \in \{1, \ldots, p\}$ with $i_n \neq i_m$ for all $n \neq m$, $\lambda_{i_n} \neq 0$ for all $n \in \{1, \ldots, k\}$ and $\lambda_j = 0$ for $j \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_k\}$. W.l.o.g. assume $i_1 < i_2 < \cdots < i_k$. By the existence of an invariant affine subspace, this yields $\lambda_{i_1} V_t^{i_1} + \cdots + \lambda_{i_k} V_t^{i_k} = a$. This is, since $D^{i_n - i_1} V_t^{i_1} = V_t^{i_n}$, $n = 1, \ldots, k$, equivalent to

$$\lambda_{i_1} V_t^{i_1} + \lambda_{i_2} D^{i_2 - i_1} V_t^{i_1} + \cdots + \lambda_{i_k} D^{i_k - i_1} V_t^{i_1} = a\,. \tag{3.28}$$

But (3.28) is an inhomogeneous ordinary linear differential equation of order $i_k - i_1$. Hence, $V_t^{i_1}$ is a smooth deterministic function in t and so are V_t^j for all $j \in \{1, \dots, p\}$ since $V_t^j = D^{j-1} V_t^1$. When we consider the last component of V, we obtain by (3.21) with $V_0^p = x^p$

$$L_t = V_t^p + \sum_{k=1}^{p} \int_{(0,t]} V_{s-}^k \, \mathrm{d}M_s^{(p-k+1)} - x_p \,. \tag{3.29}$$

But under the assumption that M and L are independent and L is not deterministic, (3.29) cannot hold (observe that V_t is deterministic by (3.28) when $V_0 = x$). This gives the wanted contradiction and therefore that V is irreducible. □

Remark 3.10. *We can write (3.29) using partial integration as*

$$L_t = V_t^p + \sum_{k=1}^{p} V_t^k M_t^{(p-k+1)} - \sum_{k=1}^{p} \int_{(0,t]} M_{s-}^{(p-k+1)} \, \mathrm{d}V_s^k - x_p \,,$$

hence L_t is a functional of M. It would therefore be enough assuming that L is not measurable with respect to the filtration generated by M to ensure irreducibility of the state vector process V.

Remark 3.11. *When the components $M^{(1)}, \dots, M^{(p)}, L$ of the Lévy process C in Proposition 3.9 (a) are additionally independent, then the components do not jump together almost surely and the Gaussian covariances vanish for different components so that the formulas for $N^{(1)}, \dots, N^{(p)}$ simplify to*

$$N_t^{(1)} = M_t^{(1)} + t\,\sigma^2_{M_t^{(1)}} + \sum_{0<s\le t} \frac{(\Delta M_s^{(1)})^2}{1 - \Delta M_s^{(1)}} \quad \textit{and} \quad N_t^{(i)} = M_t^{(i)} \text{ a.s.} \quad \textit{for } i = 2, \dots, p \,.$$

Further, $Y_t = \mathbf{e} L_t$ and X are independent in that case (the latter is already true when just L is independent of $(M^{(1)}, \dots, M^{(p)})$).

Recall that a process is strictly stationary if its finite-dimensional distributions are shift-invariant. As in the case of the MGOU process, we can find sufficient conditions for the existence of a strictly stationary solution of the RC-CARMA SDE (3.14) and therefore a strictly stationary RC-CARMA process. Assume throughout that the used norm $\|\cdot\|$ on $\mathbb{R}^{p\times p}$ is submultiplicative.

Theorem 3.12. *Let $C = (C_t)_{t\ge0} = (M_t^{(1)}, \dots, M_t^{(p)}, L_t)_{t\ge0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\Pi_{M^{(1)}}(\{1\}) = 0$ and a semimartingale with respect to the given filtration $\mathbb{F}$. Let $\mathbf{b}' := [b_0, \dots, b_{p-1}] \in \mathbb{R}^p$, $q := \max\{j \in \{0, \dots, p-1\} \colon b_j \neq 0\}$ and $U = (U_t)_{t\ge0}$ be given as in (3.12). Assume that $\mathbf{E}\left[\log^+ \|U_1\|\right] < \infty$ and $\mathbf{E}\left[\log^+ |L_1|\right] < \infty$.*

(a) Suppose there exists a $t_0 > 0$ such that

$$\mathbf{E}\left[\log \left\| \overleftarrow{\mathcal{E}}(U)_{t_0} \right\| \right] < 0 \,. \tag{3.30}$$

Then $\overleftarrow{\mathcal{E}}(U)_t$ *converges a.s. to* 0 *as* $t \to \infty$, *and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-}\, \mathbf{e}\, \mathrm{d}L_s$ *converges a.s. to a finite random vector as* $t \to \infty$, *denoted by* $\int_0^\infty \overleftarrow{\mathcal{E}}(U)_{s-}\, \mathbf{e}\, \mathrm{d}L_s$. *Further, (3.14) admits a strictly stationary solution which is causal and unique in distribution, and is achieved by choosing* V_0 *to be independent of* C *and such that* $V_0 \stackrel{d}{=} \int_0^\infty \overleftarrow{\mathcal{E}}(U)_{s-}\, \mathbf{e}\, \mathrm{d}L_s$.

(b) Conversely, if V_0 *can be chosen independent of* C *such that* V *is strictly stationary,* $M = (M_t^{(1)}, \ldots, M_t^{(p)})_{t\geq 0}$ *is independent of* L *and* L *not deterministic, then there exists a* $t_0 > 0$ *such that (3.30) holds.*

In both cases with this choice of V_0, *the RC-CARMA process given by* $R_t = \mathbf{b}' V_t$, $t \geq 0$, *is strictly stationary, too.*

Proof. (a) The assertions regarding V follow from Theorem 5.4 with Remark 5.5 (b) and Theorem 5.2 (a) in Behme and Lindner [8].
(b) Under the assumptions made, V is irreducible by Proposition 3.9 (b). Therefore Theorem 5.4 of Behme and Lindner [8] applies.

That R is strictly stationary if V is, is obvious. □

3.3 Existence of moments and second order properties

In this section, we calculate the autocovariance function (ACVF) of an RC-CARMA process and give a connection to the autocovariance function of a specific CARMA process obtained by choosing $A = \mathbf{E}[U_1]$. Further, we give sufficient conditions for the existence of the ACVF and the spectral density. We end this section with an exemplary investigation of the RC-CARMA$(2,1)$ process. But we start with a result which guarantees the existence of higher moments. Assume that the used norm $\|\cdot\|$ on $\mathbb{R}^{p\times p}$ is submultiplicative.

Proposition 3.13. *Let* $R = (R_t)_{t\geq 0}$ *be an RC-CARMA*(p,q) *process with parameters* $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq 0}$, $\mathbf{b}$ *and strictly stationary state vector process* $V = (V_t)_{t\geq 0}$ *with* V_0 *independent of* C *and* C *a semimartingale with respect to the given filtration* $\mathbb{F}$. *Assume that for* $\kappa > 0$ *we have for some* $t_0 > 0$

$$\mathbf{E}\, \|C_1\|^{\max\{\kappa,1\}} < \infty \quad \textit{and} \quad \mathbf{E} \left\| \overleftarrow{\mathcal{E}}(U)_{t_0} \right\|^{\kappa} < 1\,. \tag{3.31}$$

Then $\mathbf{E}|R_0|^\kappa < \infty$, *and if (3.31) holds for* $\kappa = 1$,

$$\mathbf{E}[R_0] = b_0 \frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\,.$$

Remark 3.14. *The assumption (3.31) in the previous proposition actually already implies the existence of a strictly stationary state vector process* $V = (V_t)_{t\geq 0}$, *that is unique in distribution, since* $\mathbf{E}\, \|C_1\|^{\max\{\kappa,1\}} < \infty$ *obviously implies* $\mathbf{E}[\log^+ \|U_1\|] < \infty$ *and* $\mathbf{E}[\log^+ |L_1|] <$

∞*, and by Jensen's inequality and (3.31) we further have*

$$\kappa\mathbf{E}\left[\log\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|\right] \leq \log\mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|^{\kappa} < 0\,.$$

Then Theorem 3.12 applies.

Proof of Proposition 3.13. By Remark 3.14 the strictly stationary state vector process V is unique in distribution. By Proposition 3.3 in Behme [9] we then have $\mathbf{E}\left\|V_0\right\|^{\kappa} < \infty$ and hence $\mathbf{E}|R_0|^{\kappa} < \infty$.

Now let $\kappa = 1$. Again from [9], Proposition 3.3, we know that $\mathbf{E}[U_1]$ is invertible and

$$\mathbf{E}[V_0] = -\mathbf{E}[U_1]^{-1}\mathbf{e}\,\mathbf{E}[L_1]\,.$$

Observe that $\mathbf{E}[U_1]$ is a companion matrix and it is well-known that the inverse of this is of the form

$$\mathbf{E}[U_1]^{-1} = \begin{bmatrix} -\frac{\mathbf{E}[M_1^{(p-1)}]}{\mathbf{E}[M_1^{(p)}]} & -\frac{\mathbf{E}[M_1^{(p-2)}]}{\mathbf{E}[M_1^{(p)}]} & \dots & -\frac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(p)}]} & -\frac{1}{\mathbf{E}[M_1^{(p)}]} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{3.32}$$

such that

$$\mathbf{E}[V_0] = \mathbf{e}_1\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\,,$$

where $\mathbf{e}_1$ denotes the first unit vector in $\mathbb{R}^p$, and

$$\mathbf{E}[R_0] = \mathbf{E}[\mathbf{b}'V_0] = \mathbf{b}'\mathbf{e}_1\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]} = b_0\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\,.$$

□

The following proposition gives sufficient conditions for the existence of the autocovariance function of an RC-CARMA process and states its form. We denote with $\otimes$ the Kronecker product and by vec the vectorizing operator which maps a matrix H form $\mathbb{R}^{p\times m}$ into $\mathbb{R}^{pm}$ stacking its columns one under another. vec^{-1} means the inverse operation such that $\mathrm{vec}^{-1}(\mathrm{vec}(H))$ yields H.

Proposition 3.15. *Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA(p,q) process with parameters $C = (C_t)_{t\geq 0}$, $\mathbf{b}$ and state vector process $V = (V_t)_{t\geq 0}$ with V_0 independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Suppose that it holds $\mathbf{E}\left\|C_1\right\|^2, \mathbf{E}\left\|V_s\right\|^2 < \infty$, then for $t \geq 0$ we have*

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\,\mathbf{Cov}(V_t)\mathbf{b} \quad \forall\, h \geq 0\,,$$

where $\mathbf{Cov}(V_t) = \mathbf{E}[V_t V_t'] - \mathbf{E}[V_t]\mathbf{E}[V_t']$ *denotes the covariance matrix of* V_t.
In particular, if V *is strictly stationary, (3.31) holds for* $\kappa = 2$ *and we denote*

$$D = \mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1] + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]\,, \tag{3.33}$$

then all eigenvalues of D *have strictly negative real parts and the matrix*

$$F = \int_0^\infty \int_0^s \mathrm{e}^{uD}(\mathrm{e}^{(s-u)(\mathbf{E}[U_1]\otimes I)} + \mathrm{e}^{(s-u)(I\otimes\mathbf{E}[U_1])})\,\mathrm{d}u\,\mathrm{d}s$$

is finite. Further, if $\mathbf{E}[L_1] = 0$ *we obtain*

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\,\mathrm{vec}^{-1}(-D^{-1}\mathbf{e}_{p^2})\,\mathbf{E}(L_1^2)\mathbf{b}\,,$$

where $\mathbf{e}_{p^2}$ *denotes the* $(p^2)^{th}$*-unit vector in* $\mathbb{R}^{p^2}$, *and if* $M = (M_t^{(1)}, \dots, M_t^{(p)})_{t\geq 0}$ *and* L *are independent, we obtain*

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\left(\mathrm{vec}^{-1}\left(-D^{-1}\mathbf{e}_{p^2}\mathbf{Var}(L_1) + F\mathbf{e}_{p^2}(\mathbf{E}[L_1])^2\right) - \mathbf{e}_1\mathbf{e}_1'\left(\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\right)^2\right)\mathbf{b}\,.$$

Proof. This follows immediately from Proposition 3.4 and the subsequent remark in Behme [9], by observing that

$$\begin{aligned}\mathrm{vec}^{-1}((\mathbf{E}[U_1]\otimes\mathbf{E}[U_1])^{-1}\mathrm{vec}(\mathbf{E}[\mathbf{e}L_1]\mathbf{E}[\mathbf{e}L_1]')) &= \mathbf{E}[U_1]^{-1}\mathbf{E}[\mathbf{e}L_1]\mathbf{E}[\mathbf{e}L_1]'(\mathbf{E}[U_1]^{-1})' \\ &= \mathbf{e}_1\mathbf{e}_1'\left(\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\right)^2\end{aligned}$$

is obtained by (3.32), and the properties of the vectorizing and the Kronecker product operations. ☐

The following is Remark 3.5 (a) in Behme [9] for our purposes.

Remark 3.16. *Let* $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \dots, M_t^{(p)}, L_t)_{t\geq 0}$ *be a Lévy process in* $\mathbb{R}^{p+1}$ *with* $\mathbf{E}\,\|C_1\|^2 < \infty$ *satisfying* $\Pi_{M^{(1)}}(\{1\}) = 0$ *and* $U = (U_t)_{t\geq 0}$ *a Lévy process in* $\mathbb{R}^{p\times p}$ *of the form (3.12). Let* D *be as in (3.33). Then*

$$\mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|^2 < 1 \quad \textit{for some } t_0 > 0$$

if and only if

all eigenvalues of D *have strictly negative real parts.*

Therefore, that condition (3.31) holds for $\kappa = 2$ *can be replaced by*

$$\mathbf{E}\,\|C_1\|^2 < \infty \textit{ and all eigenvalues of } D \textit{ have strictly negative real parts.} \tag{3.34}$$

Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA process with parameters $C = (M^{(1)}, \ldots, M^{(p)}, L)$ and $\mathbf{b}$. If $\mathbf{E}|M^{(1)}|, \ldots, \mathbf{E}|M^{(p)}| < \infty$, we can associate to R and each vector $\mathbf{X}_0$ a CARMA process $S = (S_t)_{t\geq 0}$, given by (3.1) and (3.2) with $A = \mathbf{E}[U_1]$.

Each of these processes, i.e. when $\mathbf{X}_0$ varies over all random variables, will be called a *CARMA process associated with the given RC-CARMA process* with state vector process $(\mathbf{X}_t)_{t\geq 0}$. It is then interesting to compare the autocovariance function of R with that of S provided both are strictly stationary with finite variance.

We denote with $\oplus$ the Kronecker sum, i.e. $A \oplus A = A \otimes I + I \otimes A$.

Theorem 3.17. *Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA(p, q) process with parameters $C = (C_t)_{t\geq 0}$, $\mathbf{b}$ and strictly stationary state vector process $V = (V_t)_{t\geq 0}$ with V_0 independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that (3.31) holds for $\kappa = 2$, that $\mathbf{E}[L_1] = 0$ and denote $\widetilde{D} := \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$. Then $\mathbf{E}[U_1]$ and $\widetilde{D}$ have only eigenvalues with strictly negative real parts. Further, $\mathbf{X}_0$ can be chosen independent of C and unique in distribution such that the state vector process $(\mathbf{X}_t)_{t\geq 0}$ of the associated CARMA process $S = (S_t)_{t\geq 0}$ becomes strictly stationary with finite variance. Its autocovariance function can be expressed for all $t \geq 0$ as*

$$\mathbf{Cov}(S_{t+h}, S_t) = \mathbf{b}' \mathrm{e}^{h\mathbf{E}[U_1]} \operatorname{vec}^{-1}(-\widetilde{D}^{-1}\mathbf{e}_{p^2}) \, \mathbf{E}(L_1^2)\mathbf{b} \,, \quad \forall\, h \geq 0\,. \tag{3.35}$$

Then the autocovariance function of S and R differ only by a multiplicative constant. More precisely,

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{Cov}(S_{t+h}, S_t) \cdot \varrho_{RC} \quad \forall\, t\,, h \geq 0\,, \tag{3.36}$$

where

$$\varrho_{RC} = 1 - \mathbf{e}_{p^2}' B D^{-1} \mathbf{e}_{p^2} \tag{3.37}$$

with $B = \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]$. Furthermore, if $\mathbf{Var}(R_0) > 0$, the autocorrelation functions of both R and S agree, i.e.

$$\mathbf{Corr}[S_{t+h}, S_t] = \mathbf{Corr}[R_{t+h}, R_t]\,, \quad \forall\, t\,, h \geq 0\,. \tag{3.38}$$

Proof. That $\mathbf{E}[U_1]$ has only eigenvalues with strictly negative real parts follows since (3.31) by Jensen's inequality implies $\left\|\mathbf{E}[\overleftarrow{\mathcal{E}}(U)_{t_0}]\right\| \leq \mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\| \leq \left(\mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|^2\right)^{1/2} < 1$. Hence, since $\mathbf{E}[\overrightarrow{\mathcal{E}}(U)_{t_0}] = \mathrm{e}^{t_0\mathbf{E}[U_1]}$ by Proposition 3.1 in Behme [9] we obtain by the submultiplicativity of the norm

$$\left\|e^{nt_0\mathbf{E}[U_1]}\right\| \leq \left\|e^{t_0\mathbf{E}[U_1]}\right\|^n \leq \left(\mathbf{E}\left\|\overrightarrow{\mathcal{E}}(U)_{t_0}\right\|^2\right)^{n/2} \to 0\,, \quad n \to \infty\,,$$

so that all eigenvalues of $\mathbf{E}[U_1]$ have strictly negative real parts (e.g. Proposition 11.8.2 in Bernstein [11]). That then also $\widetilde{D}$ has only eigenvalues with strictly negative real parts follows by Fact 11.17.11 of Bernstein [11].

That $\mathbf{X}$ admits a strictly stationary solution which is unique in distribution with finite variance and $\mathbf{X}_0$ independent of C under the given conditions, is well-known (e.g. Brockwell [21]) or alternatively follows from Remark 3.14. (3.35) follows from Proposition 3.15. (3.38) is clearly true as long as (3.36) holds and $\mathbf{Var}(R_0) \neq 0$.

To show that (3.36) is indeed true, we recognize first that the covariance of the CARMA process S and the RC-CARMA process R differ only in the matrices D as defined in (3.33) and $\widetilde{D}$. So, it is enough to show that

$$x = \widetilde{x}\varrho_{RC}$$

where $x := D^{-1}\mathbf{e}_{p^2}$ and $\widetilde{x} := \widetilde{D}^{-1}\mathbf{e}_{p^2}$ and ϱ_{RC} is defined by (3.37).

Since both D and $\widetilde{D}$, under the assumptions made, are invertible, x and $\widetilde{x}$ are well-defined. Further, we have that

$$D = \widetilde{D} + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1] =: \widetilde{D} + B\,,$$

where the matrix $B = (b_{i,j})_{i,j=1,\dots,p^2}$ does only have values different from zero in the last row. The latter can be seen due to the form of the matrix U_1 by

$$\mathbf{E}[U_1 \otimes U_1] = \begin{bmatrix} \mathbf{0}_p & \mathbf{E}[U_1] & \mathbf{0}_p & \dots & \mathbf{0}_p \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{E}[U_1] & \dots & \mathbf{0}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \dots & \mathbf{E}[U_1] \\ -\mathbf{E}[M_1^{(p)}U_1] & -\mathbf{E}[M_1^{(p-1)}U_1] & -\mathbf{E}[M_1^{(p-2)}U_1] & \dots & -\mathbf{E}[M_1^{(1)}U_1] \end{bmatrix},$$

and

$$\mathbf{E}[U_1] \otimes \mathbf{E}[U_1] = \begin{bmatrix} \mathbf{0}_p & \mathbf{E}[U_1] & \mathbf{0}_p & \dots & \mathbf{0}_p \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{E}[U_1] & \dots & \mathbf{0}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \dots & \mathbf{E}[U_1] \\ -\mathbf{E}[M_1^{(p)}]\mathbf{E}[U_1] & -\mathbf{E}[M_1^{(p-1)}]\mathbf{E}[U_1] & -\mathbf{E}[M_1^{(p-2)}]\mathbf{E}[U_1] & \dots & -\mathbf{E}[M_1^{(1)}]\mathbf{E}[U_1] \end{bmatrix}.$$

Then

$$\mathbf{e}_{p^2} = Dx = (\widetilde{D} + B)x = \widetilde{D}x + \sum_{i=1}^{p^2} b_{p^2,i}x_i\mathbf{e}_{p^2}$$

such that

$$\widetilde{D}x = \left(1 - \sum_{i=1}^{p^2} b_{p^2,i} x_i\right) \mathbf{e}_{p^2} = (1 - (\mathbf{e}'_{p^2} B) D^{-1} \mathbf{e}_{p^2}) \mathbf{e}_{p^2} = \varrho_{RC} \mathbf{e}_{p^2} .$$

Hence, $x = \varrho_{RC} \widetilde{D}^{-1} \mathbf{e}_{p^2} = \varrho_{RC} \widetilde{x}$. ☐

The following two propositions give handy sufficient conditions for (3.34) and therefore also for the existence of a strictly stationary solution by Remark 3.14.

Proposition 3.18. *Suppose that $\mathbf{E}\|C_1\|^2 < \infty$ and that $\mathbf{E}[U_1]$ has only eigenvalues with strictly negative real parts. Denote $\widetilde{D} := \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$, and $B := \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1] = (b_{i,j})_{i,j=1,\dots,p^2}$ with $b_{i,j} = 0$ for all $i \neq p^2$ and all $j = 1, \dots, p^2$ and $b_{p^2,p(k-1)+j} = \mathbf{Cov}(M_1^{(p-k+1)}, M_1^{(p-j+1)})$ for $k, j = 1, \dots, p$. Then the minimal singular value $\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})$, which is the square root of the minimal eigenvalue of $(\widetilde{D} \oplus \widetilde{D})(\widetilde{D} \oplus \widetilde{D})'$, is strictly positive, and if*

$$\sum_{i,j=1}^{p} [\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})]^2 < \frac{1}{4} \sigma_{\min}(\widetilde{D} \oplus \widetilde{D})^2 , \tag{3.39}$$

then (3.34) applies.

Proof. Assume that $\mathbf{E}[U_1]$ has only eigenvalues with strictly negative real parts. Then so does $\widetilde{D} = \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$ by Fact 11.17.11 of Bernstein [11] and hence also $\widetilde{D} \oplus \widetilde{D}$. In particular, $\widetilde{D} \oplus \widetilde{D}$ is invertible so that its minimal singular value is strictly positive. By Fact 11.18.17 of [11], the sum $D = \widetilde{D} + B$ has only eigenvalues with strictly negative real parts if $\|B\|_{\mathrm{F}} < 1/2 \sigma_{\min}(\widetilde{D} \oplus \widetilde{D})$, where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Due to the form of B, we see immediately that $\|B\|_{\mathrm{F}}^2 = \sum_{j=1}^{p^2} b_{p^2,j}^2$, hence (3.39). ☐

Let us denote with $\|A\|_1$ the column sum and with $\|A\|_\infty$ the row sum norm of a matrix A, respectively. Further, $\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$ and $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$ denote the condition number of an invertible A with respect to $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

Proposition 3.19. *Suppose that $\mathbf{E}\|C_1\|^2 < \infty$ and that $\mathbf{E}[U_1]$ has only pairwise distinct eigenvalues with strictly negative real parts, which we denote by $\mu_1, \dots, \mu_p$. Let $\widetilde{D}$ and B be as in Proposition 3.18, denote*

$$S := \begin{bmatrix} 1 & \dots & 1 \\ \mu_1 & \dots & \mu_p \\ \vdots & \ddots & \vdots \\ \mu_1^{p-1} & \dots & \mu_p^{p-1} \end{bmatrix} , \quad \Lambda := \mathrm{diag}(\mu_1, \dots, \mu_p) ,$$

and by $\mathrm{spec}(\widetilde{D})$ the set of all eigenvalues of $\widetilde{D}$. Then $\mathbf{E}[U_1]$ and $\widetilde{D}$ are diagonalizable, more precisely $S^{-1}\mathbf{E}[U_1]S = \Lambda$ and $(S^{-1} \otimes S^{-1})\widetilde{D}(S \otimes S) = \Lambda \oplus \Lambda$. Further, if *(a)* $\kappa_1(S)^2 \cdot \max_{i,j=1,\dots,p} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min_{\lambda \in \operatorname{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|$, *or*

(b) $\kappa_\infty(S)^2 \cdot \sum_{i,j=1}^{p} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min_{\lambda \in \operatorname{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|$,

then (3.34) applies.

Proof. That $\mathbf{E}[U_1]$ is diagonalizable and that $S^{-1}\mathbf{E}[U_1]S = \Lambda$ under the assumption of pairwise distinct eigenvalues, is well-known, see e.g. Fact 5.16.4 in Bernstein [11]. Then

$$
\begin{aligned}
(S^{-1} \otimes S^{-1})\widetilde{D}(S \otimes S) &= (S^{-1} \otimes S^{-1})(\mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1])(S \otimes S) \\
&= S^{-1}\mathbf{E}[U_1]S \otimes S^{-1}S + S^{-1}S \otimes S^{-1}\mathbf{E}[U_1]S \\
&= \Lambda \otimes I + I \otimes \Lambda = \Lambda \oplus \Lambda,
\end{aligned}
$$

so that $\widetilde{D}$ is diagonalizable and has only eigenvalues with strictly negative real parts. Let $r \in \{1, \infty\}$. By the Theorem of Bauer-Fike (e.g. Theorem 7.2.2 in Golub and van Loan [42]) we have for μ being an eigenvalue of $D = \widetilde{D} + B$ that $\min_{\lambda \in \operatorname{spec}(\widetilde{D})} |\lambda - \mu| \leq \kappa_r(S \otimes S) \|B\|_r$. In particular, if

$$
\kappa_r(S \otimes S) \|B\|_r < \min_{\lambda \in \operatorname{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|,
$$

then D can only have eigenvalues with strictly negative real parts. Observe that by Fact 9.9.61 in Bernstein [11] it holds $\kappa_r(S \otimes S) = \|S \otimes S\|_r \|S^{-1} \otimes S^{-1}\|_r = \|S\|_r^2 \|S^{-1}\|_r^2 = \kappa_r(S)^2$ so that the statement follows. $\square$

Remark 3.20. *(a) Proposition 3.19 can also be formulated for other natural matrix norms corresponding to the r-norms with $r \in [1, \infty]$, i.e.*

$$
\|A\|_r = \sup_{x \neq 0} \frac{\|Ax\|_r}{\|x\|_r}.
$$

(b) For $r = \infty$ observe that Theorem 1 in Gautschi [38] gives estimates for the condition number $\kappa_\infty(S)$ when S has the form as in Proposition 3.19 which then gives practicable conditions for (3.34) to hold.

(c) Both Proposition 3.18 and 3.19 state that, if a strictly stationary CARMA process with finite second moments and matrix A whose eigenvalues have only strictly negative real parts is given, then an RC-CARMA process with $\mathbf{E}[U_1] = A$ can be chosen to be strictly stationary with finite second moments, provided the variances of the $M^{(i)}$ are sufficiently small. In other words, the CARMA matrix may be slightly perturbed and still give a strictly stationary RC-CARMA process with finite second moments.

Example 3.21. Consider an RC-CARMA(2, 1) process under the assumption of Proposition 3.19. Denote with $\lambda_1 \neq \lambda_2$ the eigenvalues of $\mathbf{E}[U_1]$ with strictly negative real parts. If

$$
\kappa \cdot \max_{i,j=1,2} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min_{\lambda \in \operatorname{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|,
$$

where

$$\kappa = \left(\frac{(1+\max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|} \right)^2 ,$$

then $\widetilde{D} + B$ has only eigenvalues with strictly negative real parts.

Proof. Since $\mathbf{E}[U_1]$ is a companion matrix, we have

$$S = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \frac{\lambda_2}{\lambda_2-\lambda_1} & -\frac{1}{\lambda_2-\lambda_1} \\ -\frac{\lambda_1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} \end{bmatrix} .$$

Hence, straightforward calculations yield

$$\kappa_1(S) = \frac{(1+\max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|} .$$

Observe that $\kappa_\infty(S) = \kappa_1(S)$ such that Proposition 3.19 (b) gives a weaker sufficient condition. □

Let $X = (X_t)_{t\in\mathbb{R}}$ be a weakly stationary real-valued stochastic process with $\mathbf{E}|X_t|^2 < \infty$ for each $t \in \mathbb{R}$, then the *autocovariance function* of X with lag h is defined by

$$\gamma_X(h) := \mathbf{Cov}(X_{t+h}, X_t) = \mathbf{E}[(X_{t+h} - \mathbf{E}[X_{t+h}])(X_t - \mathbf{E}[X_t])] , \quad h \in \mathbb{R} .$$

and the *autocorrelation function* of X is

$$\rho_X(h) := \frac{\gamma_X(h)}{\gamma_X(0)} = \mathbf{Corr}(X_{t+h}, X_t) , \quad h \in \mathbb{R} .$$

If $\gamma_X \colon \mathbb{R} \to \mathbb{R}$ is the autocovariance function of such a process $X = (X_t)_{t\in\mathbb{R}}$ with $\int_\mathbb{R} |\gamma_X(h)| \,\mathrm{d}h < \infty$, then its Fourier transform

$$f_X(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega h} \gamma_X(h) \,\mathrm{d}h , \quad \omega \in \mathbb{R} ,$$

is called the *spectral density* if the integral exists.

It is well-known (e.g. Brockwell [21]) that the spectral density of a CARMA(p,q)-process $S = (S_t)_{t\geq 0}$ of order $q < p$ is given by

$$f_S(\omega) = \frac{\sigma^2}{2\pi} \frac{|b(i\omega)|^2}{|a(i\omega)|^2} , \quad \omega \in \mathbb{R} ,$$

with σ^2 being the variance of the driving Lévy process. Under the stated conditions of Theorem 3.17 we see that the spectral density of an RC-CARMA(p,q) process $R = (R_t)_{t\geq 0}$ with parameters C and $\mathbf{b}$ is given by

$$f_R(\omega) = f_S(\omega) \varrho_{RC} , \quad \omega \in \mathbb{R} ,$$

where $f_S(\omega)$ denotes the spectral density of the associated CARMA process with the matrix $A = \mathbf{E}[U_1]$ as in Theorem 3.17 and the constant ϱ_{RC} is as in (3.37).

Remark 3.22. *It is well-known that if $S = (S_t)_{t\in\mathbb{R}}$ is a weakly stationary CARMA(p, q) process, then the equidistantly sampled process $S^\Delta = (S_{n\Delta})_{n\in\mathbb{N}_0}$, for some $\Delta > 0$, is a weak ARMA(p, q') process for some $q' < p$, see e.g. Section 3 in Brockwell [24].*

Since under the conditions of Theorem 3.17 the autocovariance function of an RC-CARMA(p, q) process $R = (R_t)_{t\geq 0}$ differs from that of the associated CARMA(p, q) process only by a multiplicative constant, it follows immediately that also $R^\Delta = (R_{\Delta n})_{n\in\mathbb{N}_0}$ is a weakly stationary ARMA(p, q') process for some $q' < p$.

Next, we evaluate exemplarily the covariance structure of an RC-CARMA$(2, 1)$ process under the assumption $\mathbf{E}[L_1] = 0$.

Example 3.23. (Covariance of RC-CARMA(2,1))
Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA$(2, 1)$ process with parameters $C = (C_t)_{t\geq 0} = (M_t^{(1)}, M_t^{(2)}, L_t)_{t\geq 0}$, $\mathbf{b}$ and strictly stationary state vector process $V = (V_t)_{t\geq 0}$. Let V_0 be independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that $\mathbf{E}[L_1] = 0$, that (3.31) holds for $\kappa = 2$, and denote with $S = (S_t)_{t\geq 0}$ the associated CARMA process characterized by Theorem 3.17. Then, for all $t \geq 0$,

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{Cov}(S_{t+h}, S_t)\varrho_{RC} = \mathbf{b}' \mathrm{e}^{h\mathbf{E}[U_1]} \begin{bmatrix} \frac{b_0}{\mathbf{E}[M_1^{(2)}]} \\ b_1 \end{bmatrix} \frac{\mathbf{E}[L_1^2]}{2\mathbf{E}[M_1^{(1)}]} \varrho_{RC} \quad \forall\, h \geq 0\,, \tag{3.40}$$

where

$$\varrho_{RC} = \frac{2\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}]}{(2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]}\,.$$

Proof. Under the assumptions made, an application of Theorem 3.17 yields an associated CARMA process S and the first equality in (3.40). Clearly,

$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -\mathbf{E}[M_1^{(2)}] & -\mathbf{E}[M_1^{(1)}] \end{bmatrix},$$

and the general form of $\mathbf{Cov}(R_{t+h}, R_t)$ is obtained from Proposition 3.15, i.e.

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}' \mathrm{e}^{h\mathbf{E}[U_1]}\, \mathrm{vec}^{-1}(-D^{-1}\mathbf{e}_4)\, \mathbf{E}(L_1^2)\mathbf{b}\,. \tag{3.41}$$

Easy calculations show that

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -\mathbf{E}[M_1^{(2)}] & -\mathbf{E}[M_1^{(1)}] & 0 & 1 \\ -\mathbf{E}[M_1^{(2)}] & 0 & -\mathbf{E}[M_1^{(1)}] & 1 \\ \mathbf{Var}[M_1^{(2)}] & \mathbf{Cov}[M_1^{(2)}, M_1^{(1)}] - \mathbf{E}[M_1^{(2)}] & \mathbf{Cov}[M_1^{(2)}, M_1^{(1)}] - \mathbf{E}[M_1^{(2)}] & \mathbf{Var}[M_1^{(1)}] - 2\mathbf{E}[M_1^{(1)}] \end{bmatrix}.$$

Let

$$\varrho = \frac{1}{(2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]}\,,$$

and denote $y = \begin{bmatrix} -\varrho & 0 & 0 & -\varrho\mathbf{E}[M_1^{(2)}] \end{bmatrix}'$. Then $Dy = \mathbf{e}_4$ such that

$$\operatorname{vec}^{-1}(-D^{-1}\mathbf{e}_4) = \operatorname{vec}^{-1}\left(- \begin{bmatrix} -\varrho \\ 0 \\ 0 \\ -\varrho\mathbf{E}[M_1^{(2)}] \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{E}[M_1^{(2)}] \end{bmatrix} \varrho\,.$$

Summarizing,

$$\operatorname{vec}^{-1}(-D^{-1}\mathbf{e}_4)\mathbf{b} = \begin{bmatrix} \frac{1}{\mathbf{E}[M_1^{(2)}]} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \frac{2\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}]}{2\mathbf{E}[M_1^{(1)}]} \varrho = \begin{bmatrix} \frac{b_0}{\mathbf{E}[M_1^{(2)}]} \\ b_1 \end{bmatrix} \frac{1}{2\mathbf{E}[M_1^{(1)}]} \varrho_{RC}$$

and with (3.41), we get the stated shape of (3.40). □

Observe that for $M^{(1)}$ and $M^{(2)}$ being deterministic, $\varrho_{RC} = 1$, hence dividing (3.40) by ϱ_{RC} gives the autocovariance function of a CARMA$(2,1)$ process.

Example 3.24. (Covariance of RC-CARMA(3,2))
Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA$(3,2)$ process with parameters $C = (C_t)_{t\geq 0}$, $\mathbf{b}$ and strictly stationary state vector process $V = (V_t)_{t\geq 0}$. Let V_0 be independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that $\mathbf{E}[L_1] = 0$ and (3.31) hold for $\kappa = 2$. Then, for all $t, h \geq 0$,

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}' \mathrm{e}^{h\mathbf{E}[U_1]} \begin{bmatrix} \frac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(2)}]} b_0 - b_2 \\ b_1 \\ \mathbf{E}[M_1^{(2)}] b_2 - b_0 \end{bmatrix} \frac{\mathbf{E}[L_1^2]}{2(\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}] - \mathbf{E}[M_1^{(3)}])} \cdot \varrho_{RC}\,,$$

where

$$\varrho_{RC} = \frac{2(\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}] - \mathbf{E}[M_1^{(3)}])}{\xi + 2(\mathbf{Cov}(M_1^{(3)}, M_1^{(1)}) - \mathbf{E}[M_1^{(3)}]) - \frac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(3)}]}\mathbf{Var}[M_1^{(3)}]}$$

with $\xi = (2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]$.

The proof follows by similar calculations as in Example 3.23.

3.4 Simulations

We compare in this section two simulations of an RC-CARMA$(2,1)$ process, one when $\mathbf{E}[U_1]$ has only real strictly negative eigenvalues and the other when $\mathbf{E}[U_1]$ has complex eigenvalues with strictly negative real parts.

For our simulations we have chosen as random coefficient processes $M^{(1)}$, $M^{(2)}$ two independent compound poisson processes, i.e. with depiction

$$M_t^{(1)} = \sum_{i=1}^{N_t^{(1)}} X_i^{(1)} \quad \text{and} \quad M_t^{(2)} = \sum_{i=1}^{N_t^{(2)}} X_i^{(2)}\,, \quad t \geq 0\,.$$

In the case of real eigenvalues we have chosen $\mathbf{E}[N_1^{(1)}] = 1.5$, $\mathbf{E}[N_1^{(2)}] = 2$, $X_i^{(1)} \sim \mathcal{N}(1, 0.3^2)$, and $X_i^{(2)} \sim \mathcal{N}(0.25, 0.2^2)$, and as driving process a standard Brownian motion $B = (B_t)_{t\geq 0}$. Hence, for $\widetilde{D} = \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$ and $D = \widetilde{D} + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]$ we have

$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -1/2 & -3/2 \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1/2 & -3/2 & 0 & 1 \\ -1/2 & 0 & -3/2 & 1 \\ 0 & -1/2 & -1/2 & -3 \end{bmatrix}, \quad \text{and}$$

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1/2 & -3/2 & 0 & 1 \\ -1/2 & 0 & -3/2 & 1 \\ 0.205 & -1/2 & -1/2 & -1.365 \end{bmatrix}.$$

Hence, we have for $\mathbf{E}[U_1]$ the eigenvalues $\mu_1 = -1/2$ and $\mu_2 = -1$, and for D the eigenvalues $\lambda_1 \approx -0.29$, $\lambda_2 \approx -1.29 + 1.28i$, $\lambda_3 = -1.29 - 1.28i$, and $\lambda_4 = -1.5$. Since all eigenvalues of D have strictly negative real parts, we obtain the existence of a strictly stationary solution by Remark 3.14 and Remark 3.16.

Nevertheless, observe that

$$\mathbf{Var}(M_1^{(1)}) + \mathbf{Var}(M_1^{(2)}) = 1.84 \not< \frac{1}{4}\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})^2 \approx 0.0799$$

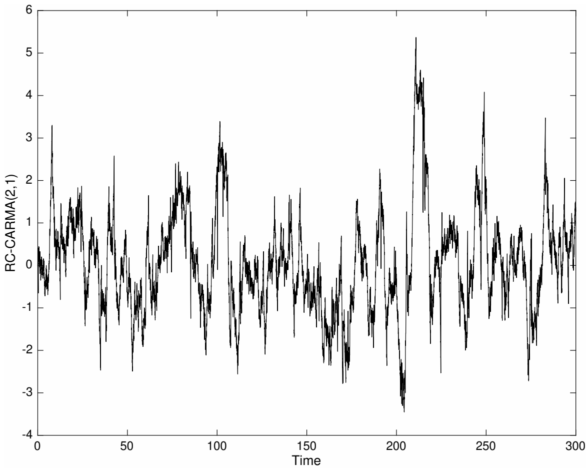
showing that the condition in Proposition 3.18 is not necessary. Moreover,

$$\kappa = \left(\frac{(1 + \max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|} \right)^2 = 25$$

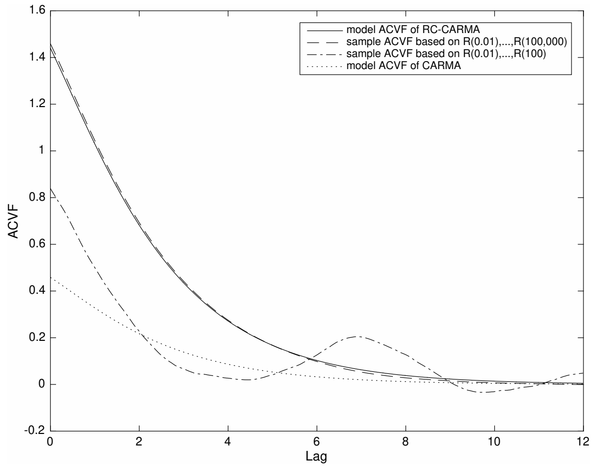
so that

$$\kappa \cdot \max_{i=1,2} |\mathbf{Var}(M_1^{(i)})| = 40.875 \not< \min_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)| = 1\,,$$

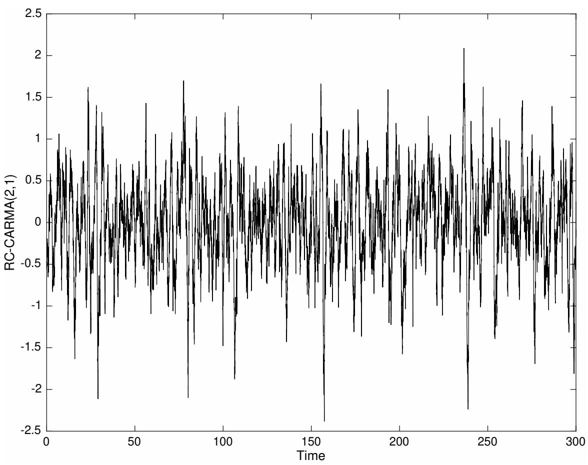
showing that also the condition in Example 3.23 and hence in Proposition 3.19 is not necessary.



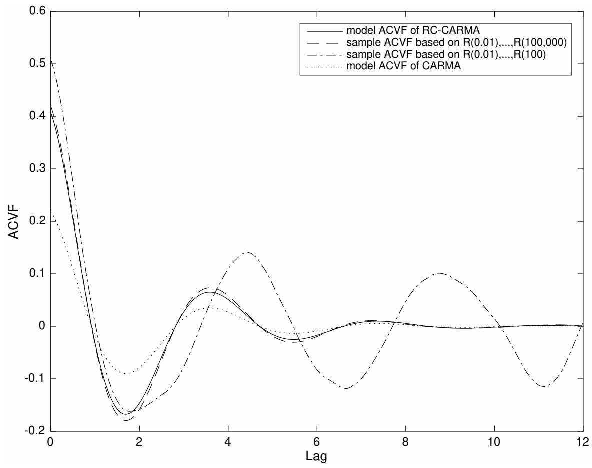
(a) Simulation with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.



(b) ACVFs with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.



(c) Simulation with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$.



(d) ACVFs with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$.

Figure 3.1: Simulated RC-CARMA(2,1) process and ACVFs.

In case of complex eigenvalues we have chosen $\mathbf{E}[N_1^{(1)}] = 2$, $\mathbf{E}[N_1^{(2)}] = 7.5$, $X_i^{(1)} \sim \mathcal{N}(0.5, 0.1^2)$, and $X_i^{(2)} \sim \mathcal{N}(0.4, 0.05^2)$, and have left the driving process unchanged. Then

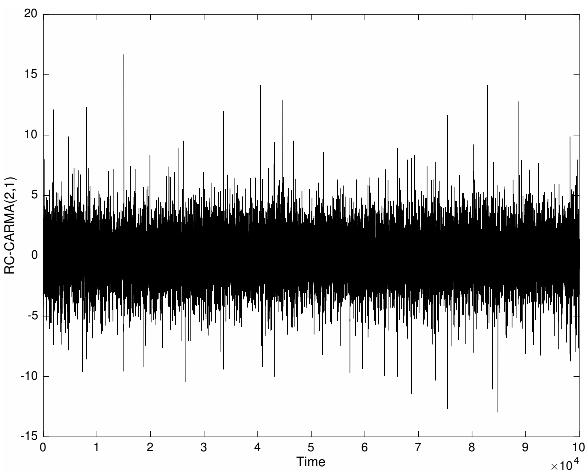
$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}$$

gives two complex-valued eigenvalues $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$. Also, D has just eigenvalues with strictly negative real parts. Furthermore, an observation similar to the one above showing non-necessity of the conditions in Proposition 3.18 and 3.19 can be made.

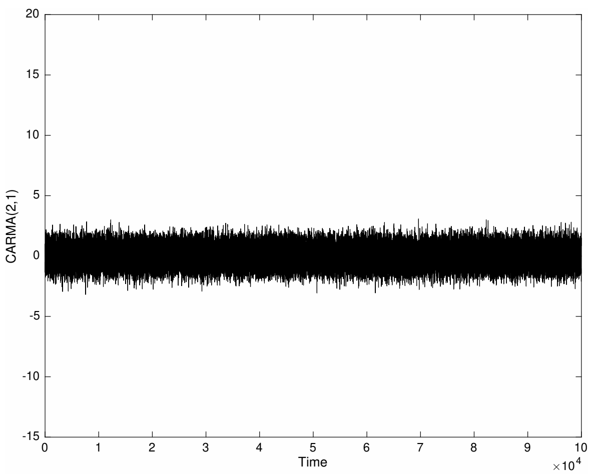
For both the complex and the real eigenvalues case, we have simulated $10,000,000$ observations with a mesh size $k = 0.01$, i.e. $R_{0.01}, R_{0.02}, \ldots, R_{100,000}$. In Figure 1 (a), (c) we see the corresponding plots until time 300. Plots 3.1(b) and 3.1(d) show the corresponding autocovariance functions (ACVF).

The solid line corresponds to the model autocovariance function, the dashed one to the sampled ACVF based on the data $R_{0.01}, \ldots, R_{100,000}$, and the dotted shows the model ACVF of the corresponding CARMA$(2,1)$ process. The dashed-dotted line shows the sample ACVF based on $R_{0.01}, \ldots, R_{100}$.

We see that using $10,000,000$ observations to calculate the sample autocovariance function, it nearly agrees in both cases with the model autocovariance function. Plot 3.1(d) shows a sinusoidal oscillation which is also in the CARMA case characteristic for allowing complex eigenvalues and visualizes the variety of possible autocovariance functions.



(a) Simulation of RC-CARMA$(2,1)$ with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.



(b) Simulation of associated CARMA$(2,1)$ with $A = \mathbf{E}[U_1]$.

Figure 3.2: Simulated RC-CARMA(2,1) and CARMA(2,1) processes.

The first plot in Figure 3.2 shows all simulated observations of the RC-CARMA$(2,1)$ process where $\mathbf{E}[U_1]$ has real eigenvalues. The second plot shows an equally sized simulation of the associated CARMA$(2,1)$ process, i.e. with the choice $A = \mathbf{E}[U_1]$. It can be seen that the RC-CARMA$(2,1)$ process provides larger outliers around the between -5 and 5 concentrated band than the CARMA$(2,1)$ process around its band. This may indicate possible heavy tails of RC-CARMA processes.

We justify these observations intuitively in the following remark recalling results on random recurrence equations.

Remark 3.25. *Observe that a stationary CARMA process driven by a Brownian motion has a normal marginal stationary distribution, in particular, it has light tails.*

On the other hand, as a consequence of results of Kesten [47] and Goldie [40], it is known that a generalized Ornstein-Uhlenbeck process and so an RC-CARMA$(1,0)$ *process will have Pareto tails under wide conditions, even if the driving process is a Brownian motion, see e.g. Lindner and Maller [51] (Theorem 4.5) or Behme [7] (Theorem 4.1).*

We henceforth expect using the multivariate results of Kesten [47] that under wide conditions the RC-CARMA process will also have Pareto tails for higher orders. However, we leave a thorough investigation of this for forthcoming research.

4 Lévy driven moving average process sampled at a renewal sequence

This chapter is an extension of the submitted article by Brandes and Curato [15] "On the sample autocovariance of a Lévy driven moving average process when sampled at a renewal sequence". Time series models cover a wide field of applications in finance, insurance, and meteorology. In particular, moving average processes became a standard tool as many time series provide a moving average representation. Nowadays, it is often preferred to use continuous time models over discrete one not only for theoretical reasons, but also, for example, due to their applicability to high-frequency data.

In this chapter we analyze the distributional limit of sample mean and sample autocovariance function of a Lévy driven continuous time moving average process when sampled at a renewal sequence. More in general, let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous time moving average process of the form

$$X_t = \mu + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,, \tag{4.1}$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided $\mathbb{R}$-valued Lévy process, i.e. a stochastic process with independent and stationary increments, càdlàg sample paths and $L_0 = 0$ almost surely, which is continuous in probability, $\mu \in \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ a deterministic function, called *kernel*, for which the integral exists. We call processes of the form (4.1) a *continuous time moving average process with mean μ and kernel function f driven by L.*

The process X is infinitely divisible, as seen in Rajput and Rosinski [59], and strictly stationary meaning that its finite dimensional joint distributions are shift-invariant, i.e. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in \mathbb{R}$ it holds

$$\mathcal{L}(X_{t_1+h}, \ldots, X_{t_n+h}) = \mathcal{L}(X_{t_1}, \ldots, X_{t_n}) \quad \forall\, h \in \mathbb{R}\,.$$

A popular example of a Lévy driven moving average process is given by the Ornstein-Uhlenbeck (OU) process used to model the volatility of a financial asset, see [1], or the intermittency in a turbulence flow, see [2]. The OU process is in fact a tractable mathematical model that can adequately describe the price of an asset as well as the volatility fluctuations on different time scales.

Continuous time moving average processes as in (4.1) are the natural continuous time analogue of discrete time moving average processes

$$\widetilde{X}_t = \mu + \sum_{k\in\mathbb{Z}} \psi_k Z_{t-k}\,, \quad t \in \mathbb{Z}\,,$$

where $(\psi_k)_{k\in\mathbb{Z}}$ is a square summable sequence of real coefficients, and $(Z_t)_{t\in\mathbb{Z}}$ an independent and identically distributed (i.i.d.) sequence with zero mean and finite second moment. These processes and the asymptotic behavior of their sample mean and their sample autocorrelation function have been widely studied (cf. Theorem 1.32, Theorem 1.33, Brockwell and Davis [23], Davis and Mikosch [32], and Hannan [43]).

We study a renewal sampling of the process X in (4.1). We select a sequence of increasing random times $(T_n)_{n\in\mathbb{Z}}$ such that $T_n \to \infty$ almost surely. More in detail, we assume that $W = (W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ is an i.i.d. sequence of positive supported random variables independent of the driving Lévy process L and such that $P(W_1 > 0) > 0$. We then define $(T_n)_{n\in\mathbb{Z}}$ by

$$T_0 := 0 \quad \text{and} \quad T_n := \begin{cases} \sum_{i=1}^{n} W_i\,, & n \in \mathbb{N}\,, \\ -\sum_{i=n}^{-1} W_i\,, & -n \in \mathbb{N}\,, \end{cases} \tag{4.2}$$

and the sampled process $Y = (Y_n)_{n\in\mathbb{Z}}$ via

$$Y_n := X_{T_n}\,, \quad n \in \mathbb{Z}\,. \tag{4.3}$$

We are interested in studying the sample moments of the process Y. We do this for different reasons. First of all, continuous processes are often used in time series analysis because they can be sampled at non-equidistant points in time and therefore provide a model for non-equidistant data which are often available for statistical inference.

Secondly, results for non-equidistant sampling schemes have not yet been shown. But when X in (4.1) is observed on a lattice $\{\Delta t \colon t = 0, 1, 2, \dots\}$, the asymptotic behavior of the sample mean and the sample autocorrelation has been studied in various cases. In particular, Cohen and Lindner [31] proved asymptotic normality of the sample mean and the sample autocorrelation under $\mathbf{E}(L_1^2) < \infty$ and $f \in L^2(\mathbb{R})$, and $\mathbf{E}(L_1^4) < \infty$ and $f \in L^4(\mathbb{R})$ plus some extra assumptions, respectively, cf. also Theorem 1.34 and 1.35. Spangenberg [62] showed in the long memory case under the assumption of $\mathbf{E}(L_1^4) < \infty$ for $f(t) \sim C_d t^{d-1}$ for $d \in (0, 1)$ and some constant C_d a central limit theorem where the limit distribution is Rosenblatt, and in the case of a slowly varying Lévy process with index $\alpha \in (2, 4)$ that the limit distribution is either Rosenblatt or a stable distribution, depending on the interplay of d and α. Drapatz [34] proved that the sample autocovariance is asymptotically stable distributed when the Lévy process has infinite variance with regularly varying tails with index $\alpha \in (0, 2)$.

The central limit theorems presented here generalize the results of Theorem 1.34 and 1.35 at the costs of slightly more restrictive moment conditions. We compare throughout this chapter our results with the ones of Theorem 1.34 and 1.35.

Moreover, we present a parameter estimation of the mean reverting parameter of a Lévy driven OU process

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,, \tag{4.4}$$

sampled at a Poisson rate, i.e. a sequence $(T_n)_{n\in\mathbb{Z}}$ where W is a sequence of i.i.d. exponentially distributed random variables. We then compare the efficiency of our estimator with an estimator based on the results of Theorem 1.35 for an equidistant sampling.

This chapter is structured as follows. In Section 4.1 we give some preliminary results regarding strict stationarity of a process sampled at a renewal sequence and the mixing property that it fulfills. Section 4.2 is concerned with establishing a central limit theorem for the sample mean of a randomly sampled continuous time moving average process as is Section 4.3 for the sample autocovariance and sample autocorrelation function. Finally in Section 4.4, we show the parameter estimation of a Lévy driven OU process.

4.1 Preliminaries

In this section, we provide some results on properties of continuous time moving average processes and their renewal sampled processes used in the upcoming sections. These results are set in a slightly more general framework than needed.

As a first result we prove that a strictly stationary process sampled at a renewal sequence inherits the strict stationarity. In particular, this shows that the sampled process (4.3) is strictly stationary.

Denote with $\stackrel{d}{=}$ equality in distribution and with A' the transpose of a matrix $A \in \mathbb{R}^{d\times m}$.

Proposition 4.1. *Let $X = (X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary process $X_t = (X_t^{(1)}, \dots, X_t^{(d)})'$, $(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ an i.i.d. sequence supported on $[0,\infty)$ independent of X. Define for $n \in \mathbb{Z}$ a sequence of random times via (4.2). Then the $\mathbb{R}^d$-valued process $Y = (Y_n)_{n\in\mathbb{Z}}$ defined by $Y_n^{(i)} := X_{T_n}^{(i)}$, $i = 1, \dots, d$, is strictly stationary. More generally, the process $(Y_n, T_n - T_{n-1})_{n\in\mathbb{Z}}$ is strictly stationary.*

Proof. Observe that $(Y_n, T_n - T_{n-1})_{n\in\mathbb{Z}}$ is strictly stationary if and only if $(Y_n, T_{n+1} - T_n)_{n\in\mathbb{Z}}$ is strictly stationary, and the latter implies strict stationarity of Y. Hence, it suffices to show that $(Y_n, T_{n+1} - T_n)_{n\in\mathbb{Z}}$ is strictly stationary. For that, let $m \le n$, $B \in \mathcal{B}(\mathbb{R}^{(d+1)(n-m+1)})$, the Borel-$\sigma$-algebra on $\mathbb{R}^{(d+1)(n-m+1)}$, and denote the distribution of the random vector Z by P_Z. Define

$$R_k := T_{k+m} - T_m\,, \quad k = 1, \dots, n-m+1\,.$$

Conditioning and using the strict stationarity of X, we obtain

$$\begin{aligned}
&P((Y_m', \dots, Y_n', T_{m+1} - T_m, \dots, T_{n+1} - T_n)' \in B) \\
&\quad = P((X_{T_m}', \dots, X_{T_n}', R_1, R_2 - R_1, \dots, R_{n-m+1} - R_{n-m})' \in B) \\
&\quad = P((X_{T_m}', X_{T_m+R_1}', \dots, X_{T_m+R_{n-m}}', R_1, R_2 - R_1, \dots, R_{n-m+1} - R_{n-m})' \in B) \\
&\quad = \int_{\mathbb{R}^{n-m+2}} P((X_u', X_{u+v_1}', \dots, X_{u+v_{n-m}}', v_1, v_2 - v_1, \dots, v_{n-m+1} - v_{n-m})' \in B)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{(T_m,R_1,\dots,R_{n-m+1})}(\,\mathrm{d}(u,v_1,\dots,v_{n-m+1}))\\
&= \int_{\mathbb{R}^{n-m+2}} P((X_0',X_{v_1}',\dots,X_{v_{n-m}}',v_1,v_2-v_1,\dots,v_{n-m+1}-v_{n-m})' \in B)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{(T_m,R_1,\dots,R_{n-m+1})}(\,\mathrm{d}(u,v_1,\dots,v_{n-m+1}))\\
&= \int_{\mathbb{R}^{n-m+1}} P((X_0',X_{v_1}',\dots,X_{v_{n-m}}',v_1,v_2-v_1,\dots,v_{n-m+1}-v_{n-m})' \in B)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{(R_1,\dots,R_{n-m+1})}(\,\mathrm{d}(v_1,\dots,v_{n-m+1}))\,,
\end{aligned}
$$

where in the last line we used that the integrand does not depend on u. Using that $(R_1,\dots,R_{n-m+1})' \stackrel{d}{=} (T_1,\dots,T_{n-m+1})'$, the latter is equal to

$$
\begin{aligned}
&= \int_{\mathbb{R}^{n-m}} P((X_0',X_{v_1}',\dots,X_{v_{n-m}}',v_1,v_2-v_1,\dots,v_{n-m+1}-v_{n-m})' \in B)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{(T_1,\dots,T_{n-m+1})}(\,\mathrm{d}(v_1,\dots,v_{n-m+1}))\\
&= P((Y_0',\dots,Y_{n-m}',T_1,T_2-T_1,\dots,T_{n-m+1}-T_{n-m})' \in B)\,,
\end{aligned}
$$

by the same calculation as above, showing the strict stationarity of $(Y_n,T_{n+1}-T_n)_{n\in\mathbb{Z}}$. □

In order to prove central limit theorems, we recall the concept of mixing which was given in Definition 1.7: On a probability space $(\Omega,\mathcal{F},P)$ for any two σ-algebras $\mathcal{A},\mathcal{C}\subset\mathcal{F}$ the following measures of dependence can be defined

$$
\begin{aligned}
\alpha(\mathcal{A},\mathcal{C},P) &:= \sup|P(A\cap C)-P(A)P(C)|\,, \quad A\in\mathcal{A}\,,\ C\in\mathcal{C}\,,\\
\rho(\mathcal{A},\mathcal{C},P) &:= \sup|\mathbf{Corr}(f,g)|\,, \quad f\in L^2(\Omega,\mathcal{A},P)\,,\ g\in L^2(\Omega,\mathcal{C},P)\,.
\end{aligned}
$$

We say that a strictly stationary sequence of random vectors $Z=(Z_n)_{n\in\mathbb{Z}}$ is

$$
\begin{aligned}
\textit{strongly mixing} \text{ if } \alpha_n &:= \alpha(\mathcal{A},\mathcal{C}_n;P)\to 0 \ \text{ as } n\to\infty\,,\\
\rho\textit{-mixing} \text{ if } \rho_n &:= \rho(\mathcal{A},\mathcal{C}_n;P)\to 0 \ \text{ as } n\to\infty\,,
\end{aligned}
$$

for the σ-algebra of the past $\mathcal{A}=\sigma(Z_0,Z_{-1},Z_{-2},\dots)$ and the σ-algebra of the future $\mathcal{C}_n=\sigma(Z_n,Z_{n+1},Z_{n+2},\dots)$. For more information about mixing coefficients see also Bradley [20].

Recall that a process $X=(X_t)_{t\in\mathbb{R}}$ is called an m-dependent process when $(X_t)_{t\le s}$ and $(X_t)_{t>s+m}$ are independent for each s.

Proposition 4.2. *Let $X=(X_t)_{t\in\mathbb{R}}$ be for some $m\in\mathbb{N}$ an $\mathbb{R}^d$-valued m-dependent strictly stationary process and $Y=(Y_n)_{n\in\mathbb{Z}}$ defined by $Y_n:=X_{T_n}$ with $(T_n)_{n\in\mathbb{Z}}$ as in (4.2) for a positive supported sequence of i.i.d. random variables $W=(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ such that $P(W_1>0)>0$ and W is independent of X. Then Y is strongly mixing with exponentially decreasing mixing coefficients α_n. More generally, $(Y_n,T_n-T_{n-1})_{n\in\mathbb{Z}}$ is strongly mixing with exponentially decreasing mixing coefficients.*

Proof. Let $Z_n = (Y_n', T_n - T_{n-1})'$. Let $D_n := \{T_n > m\}$ and $n \geq 1$ so large such that $P(D_n) > 0$. First, we show that the two σ-algebras $\mathcal{A}$ and $\mathcal{C}_n$ defined as above are independent under the conditional probability measure $P(\cdot|D_n)$. Therefore, let $A \in \mathcal{A}$ and $C_j \in \mathcal{C}_n$ be of the form $A = \{X_{T_i}' \in B\,, T_i - T_{i-1} \in F\}$ for some $i \leq 0$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $F \in \mathcal{B}(\mathbb{R})$, and $C_j = \{X_{T_j}' \in B'\,, T_j - T_{j-1} \in F'\}$ for some $j \geq n$, $B' \in \mathcal{B}(\mathbb{R}^d)$, and $F' \in \mathcal{B}(\mathbb{R})$, respectively. Then, by the Doob-Dynkin lemma and the m-dependence of X,

$$\begin{aligned}
P(A \cap C_j|D_n) &= \frac{1}{P(D_n)}\mathbf{E}(\mathbf{1}_{A\cap C_j}\mathbf{1}_{D_n}) = \frac{1}{P(D_n)}\mathbf{E}(\mathbf{1}_{D_n}\mathbf{E}[\mathbf{1}_{A\cap C_j}|\sigma(T_n)])\\
&= \frac{1}{P(D_n)}\int_{(m,\infty)} \mathbf{E}[\mathbf{1}_{A\cap C_j}|T_n = t]P_{T_n}(\mathrm{d}t)\\
&= \frac{1}{P(D_n)}\int_{(m,\infty)} P(X_{T_i}' \in B, T_i - T_{i-1} \in F, X_{T_j}' \in B', T_j - T_{j-1} \in F'|T_n = t)P_{T_n}(\mathrm{d}t)\\
&= \frac{1}{P(D_n)}P(X_{T_i}' \in B, T_i - T_{i-1} \in F)\int_{(m,\infty)} P(X_{T_j}' \in B', T_j - T_{j-1} \in F'|T_n = t)P_{T_n}(\mathrm{d}t)\\
&= P(A|D_n)\frac{1}{P(D_n)}\int_{(m,\infty)} P(X_{T_j}' \in B', T_j - T_{j-1} \in F'|T_n = t)P_{T_n}(\mathrm{d}t)\,.
\end{aligned}$$

Observe that $P(A) = P(A|D_n)$ since X_{T_i}' for $i \leq 0$ and T_n are independent. A calculation like the one above for $B = \mathbb{R}^d$, i.e. $A = \Omega$, gives

$$P(C_j|D_n) = \frac{1}{P(D_n)}\int_{(m,\infty)} P(X_{T_j}' \in B', T_j - T_{j-1} \in F'|T_n = t)P_{T_n}(\mathrm{d}t)$$

such that all together we obtain

$$P(A \cap C_j|D_n) = P(A|D_n)P(C_j|D_n) \quad \text{for } j \geq n\,. \tag{4.5}$$

Similarly we can obtain (4.5) for $A' = \{X_{T_{i_1}}' \in B_1, \dots, X_{T_{i_k}}' \in B_k, T_{i_1} - T_{i_1-1} \in F_1, \dots, T_{i_k} - T_{i_k-1} \in F_k\}$ for $i_1, \dots, i_k \leq 0$, $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R}^d)$, and $F_1, \dots, F_k \in \mathcal{B}(\mathbb{R})$, and $C_n' = \{X_{T_{j_1}}' \in B_1', \dots, X_{T_{j_l}}' \in B_l', T_{j_1} - T_{j_1-1} \in F_1', \dots, T_{j_l} - T_{j_l-1} \in F_l'\}$ for $j_1, \dots, j_l \geq n$, $B_1', \dots, B_l' \in \mathcal{B}(\mathbb{R}^d)$, and $F_1', \dots F_l' \in \mathcal{B}(\mathbb{R})$. Observe that sets of the form A' generate the σ-algebra $\mathcal{A}$ and sets of the form C_n' generate the σ-algebra $\mathcal{C}_n$ and both are $\cap$-stable. Thus, we conclude that (4.5) is true for all $A \in \mathcal{A}$ and $C_n \in \mathcal{C}_n$. Using measure theoretic induction, and

$$\mathbf{Cov}_{P(\cdot|D_n)}(\mathbf{1}_A, \mathbf{1}_{C_n}) = P(A \cap C_n|D_n) - P(A|D_n)P(C_n|D_n) = 0\,,$$

we obtain that $\rho(\mathcal{A}, \mathcal{C}_n, P(\cdot|D_n)) = \sup|\mathbf{Corr}_{P(\cdot|D_n)}(f, g)| = 0$ where the supremum is taken over all $f \in L^2(\Omega, \mathcal{A}, P(\cdot|D_n))$ and $g \in L^2(\Omega, \mathcal{C}_n, P(\cdot|D_n))$.

Since $P(D_n) = 1 - P(D_n^c)$ and $0 = \rho(\mathcal{A}, \mathcal{C}_n; P(\cdot|D_n)) \leq P(D_n^c)$, it follows, from Remark 1.9 that

$$\alpha_n(\mathcal{A}, \mathcal{C}_n; P) \leq 4P(D_n^c) = 4P(T_n \leq m)\,.$$

Since $(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ is supported on $[0,\infty)$ and $P(W_1>0)>0$, there exists an $r>0$ such that $P(W_1>r)>0$ and hence $P(W_1+\cdots+W_{\lceil m/r\rceil}>m)>0$, where $\lceil x\rceil$ for $x\in\mathbb{R}$ denotes the smallest integer $k\in\mathbb{N}$ so that $k\geq x$. Denote $q:=1-P(W_1+\cdots+W_{\lceil m/r\rceil}>m)<1$. Then, as long as $n\leq\lceil m/r\rceil$, we obtain $P(T_n\leq m)\leq q$. For $n>\lceil m/r\rceil$ set $k_n=\lfloor\frac{n}{\lceil m/r\rceil}\rfloor$ for $n\in\mathbb{N}$. Then, by the i.i.d. property of $(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$,

$$\begin{aligned}P(T_n\leq m)&\leq P(W_1+\cdots+W_{\lceil m/r\rceil}\leq m, W_{\lceil m/r\rceil+1}+\cdots+W_{2\lceil m/r\rceil}\leq m,\ldots,\\&\qquad W_{(k_n-1)\lceil m/r\rceil+1}+\cdots+W_{k_n\lceil m/r\rceil}\leq m, W_{k_n\lceil m/r\rceil+1}+\cdots+W_n\leq m)\\&=P(W_1+\cdots+W_{\lceil m/r\rceil}\leq m)^{k_n}P(W_{k_n\lceil m/r\rceil+1}+\cdots+W_n\leq m)\\&\leq q^{k_n}\,,\end{aligned}$$

and

$$\alpha_n(\mathcal{A},\mathcal{C}_n,P)\leq 4q^{k_n}\to 0\quad\text{as}\quad n\to\infty \tag{4.6}$$

showing that Z and hence Y are strongly mixing with exponentially decreasing mixing coefficients. □

We give some results leading to the characterization of finiteness of the moments of a Lévy driven continuous time moving average process.

Recall that an $\mathbb{R}$-valued Lévy processes $L=(L_t)_{t\geq\mathbb{R}}$ can be characterized by its characteristic triplet $(\sigma_L^2,\nu_L,\gamma_L)$ due to the Lévy-Khintchine formula, cf. Theorem 1.16, i.e. if μ denotes the infinitely divisible distribution of L_1, then its characteristic function is given by

$$\widehat{\mu}(z)=\exp\left[i\gamma_L z-\frac{1}{2}\sigma_L^2z^2+\int_{\mathbb{R}}(\mathrm{e}^{izx}-1-izx\,\mathbf{1}_{\{|x|\leq 1\}})\,\nu_L(\mathrm{d}x)\right],\quad z\in\mathbb{R}.$$

Here, σ_L^2 is the Gaussian covariance, ν_L a measure on $\mathbb{R}$ which satisfies $\nu_L(\{0\})=0$ and $\int_{\mathbb{R}}(|x|^2\wedge 1)\,\nu_L(\mathrm{d}x)<\infty$, called the *Lévy measure*, and $\gamma_L\in\mathbb{R}$. If $\int_{|x|>1}|x|\nu_L(\mathrm{d}x)<\infty$, then $\mathbf{E}(L_1)=\gamma_L+\int_{|x|>1}x\,\nu_L(\mathrm{d}x)$.

For a detailed account on Lévy processes we refer to the book of Sato [61].

The next lemma shows, for the continuous time moving average process X as defined in (4.1), finiteness of the r^{th}- and log-moments under certain similar conditions on the driving Lévy process L and the kernel function f

In the following we use the notation $\log^+(x):=\log(\max\{1,x\})$.

Lemma 4.3. *Let $X=(X_t)_{t\geq 0}$ be defined by $X_t:=\mu+\int_{\mathbb{R}}f(t-s)\,\mathrm{d}L_s$, where $f\in L^2(\mathbb{R})$ and $L=(L_t)_{t\geq 0}$ is a one-dimensional Lévy process with mean zero.*

(a) If $f\in L^r(\mathbb{R})$ for some $r>2$ and $\mathbf{E}|L_1|^r<\infty$, then $\mathbf{E}|X_t|^r<\infty$ for all $t\in\mathbb{R}$.

(b) If $\mathbf{E}(|L_1|^2\log^+|L_1|)<\infty$, and

$$\int_{\mathbb{R}}|f(s)|^2\log^+|f(s)|\,\mathrm{d}s<\infty\,,$$

then $\mathbf{E}(|X_t|^2 \log^+ |X_t|) < \infty$ *for all* $t \in \mathbb{R}$.

(c) If $\mathbf{E}(|L_1|^4(\log^+ |L_1|)^2) < \infty$, $f \in L^4(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} |f(s)|^4(\log^+ |f(s)|)^2 \,\mathrm{d}s < \infty\,,$$

then $\mathbf{E}(X_t^4(\log^+ |X_t|)^2) < \infty$ *for all* $t \in \mathbb{R}$ *and, for* $h \in \mathbb{N}$, $\mathbf{E}(|X_t X_{t+h}|^2 \log^+ |X_t X_{t+h}|) < \infty$ *for all* $t \in \mathbb{R}$.

Proof. (a) It is enough to show that $\mathbf{E}|Z|^r < \infty$, where

$$Z = \int_0^\infty f(s)\,\mathrm{d}L_s\,.$$

By the definition of a two-sided Lévy process, this can be easily extended to $\int_{\mathbb{R}} f(s)\,\mathrm{d}L_s$ which is equal in distribution to $\int_{\mathbb{R}} f(-s)\,\mathrm{d}L_s = X_0$. Since X is strictly stationary, we obtain the result.

Since $f \in L^2(\mathbb{R})$, f is locally L-integrable by Corollary 57.11 of Sato [61] and

$$\int_0^\infty f^2(s)\sigma_L^2 \,\mathrm{d}s < \infty\,. \tag{4.7}$$

Next, consider

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} (|f(s)x|^2 \wedge 1)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s &\le \int_0^\infty \int_{\mathbb{R}} |f(s)x|^2\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \\ &\le \int_0^\infty |f(s)|^2\,\mathrm{d}s \int_{\mathbb{R}} |x|^2 \nu_L(\mathrm{d}x) < \infty \end{aligned} \tag{4.8}$$

since $f \in L^2(\mathbb{R})$ and $\mathbf{E}|L_1|^2 < \infty$.

Moreover, choose $c(x) = \mathbf{1}_{\{|x|\le 1\}}(x)$ and observe that $0 = \mathbf{E}(L_1) = \gamma_L + \int_{\{|x|>1\}} x\,\nu_L(\mathrm{d}x)$, then

$$\begin{aligned}&\int_0^\infty \left| f(s)\gamma_L + \int_{\mathbb{R}} f(s)x(c(f(s)x) - c(x))\nu_L(\mathrm{d}x)\right| \mathrm{d}s \\ &= \int_0^\infty \Bigg| f(s)\gamma_L + \int_{\mathbb{R}} f(s)x\mathbf{1}_{\{|x|>1\}}(x)\nu_L(\mathrm{d}x) - \int_{\mathbb{R}} f(s)x\mathbf{1}_{\{|x|>1\}}(x)\nu_L(\mathrm{d}x) \\ &\qquad\qquad + \int_{\mathbb{R}} f(s)x(c(f(s)x) - c(x))\nu_L(\mathrm{d}x)\Bigg| \mathrm{d}s \\ &= \int_0^\infty \left| \int_{\mathbb{R}} f(s)x(\mathbf{1}_{\{|f(s)x|\le 1\}}(f(s)x) - 1)\nu_L(\mathrm{d}x)\right| \mathrm{d}s \\ &= \int_0^\infty \left| - \int_{\mathbb{R}} f(s)x\mathbf{1}_{\{|f(s)x|>1\}}(f(s)x)\nu_L(\mathrm{d}x)\right| \mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\infty \int_{\mathbb{R}} |f(s)x|^2 \mathbf{1}_{\{|f(s)x|>1\}}(f(s)x)\nu_L(\mathrm{d}x)\,\mathrm{d}s \\
&\leq \int_0^\infty \int_{\mathbb{R}} |f(s)x|^2 \nu_L(\mathrm{d}x)\,\mathrm{d}s < \infty\,, \qquad (4.9)
\end{aligned}$$

by (4.8). Then (4.7), (4.8), and (4.9) together with Proposition 57.16 of Sato [61] give that $Z = \int_0^\infty f(s)\,\mathrm{d}L_s$ is definable, and Proposition 57.13 (ii) of Sato [61] yields its infinite divisibility, and its Lévy measure is given by

$$\nu_Z(C) = \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_C(f(s)x)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s\,, \quad C \in \mathcal{B}(\mathbb{R}\setminus\{0\})\,.$$

By Corollary 25.8 of Sato [61], Z then has finite r^{th}-moment, if $\int_{|x|>1} |x|^r\,\nu_Z(\mathrm{d}x) < \infty$. To see that this is indeed true, consider

$$\begin{aligned}
\int_{|x|>1} |x|^r\,\nu_Z(\mathrm{d}x) &= \int_0^\infty \int_{\mathbb{R}} |f(s)x|^r \mathbf{1}_{(1,\infty)}(|f(s)x|)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \\
&\leq \int_0^\infty |f(s)|^r\,\mathrm{d}s \int_{\mathbb{R}} |x|^r \nu_L(\mathrm{d}x) < \infty
\end{aligned}$$

since $f \in L^r(\mathbb{R})$, $\mathbf{E}|L_1|^r < \infty$, and, since $r > 2$,

$$\int_{|x|\leq 1} |x|^r\,\nu_L(\mathrm{d}x) \leq \int_{\mathbb{R}} (|x|^2 \wedge 1)\,\nu_L(\mathrm{d}x) < \infty\,, \qquad (4.10)$$

by the properties of the Lévy measure ν_L.

(b) Observe that $\log^+|ab| \leq \log^+|a| + \log^+|b|$ for $a, b \in \mathbb{R}$. Hence,

$$\begin{aligned}
\int_{|x|>1} |x|^2 \log^+|x|\,\nu_Z(\mathrm{d}x) &= \int_0^\infty \int_{\mathbb{R}} |f(s)x|^2 \log^+|f(s)x|\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \\
&\leq \int_0^\infty \int_{\mathbb{R}} |f(s)x|^2 (\log^+|f(s)| + \log^+|x|)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \\
&\leq \int_0^\infty |f(s)|^2 \log^+|f(s)|\,\mathrm{d}s \int_{\mathbb{R}} |x|^2\,\nu_L(\mathrm{d}x) \\
&\qquad + \int_0^\infty |f(s)|^2\,\mathrm{d}s \int_{\mathbb{R}} |x|^2 \log^+|x|\,\nu_L(\mathrm{d}x) < \infty
\end{aligned}$$

since $\mathbf{E}|L_1|^2 < \infty$, $\int_0^\infty |f(s)|^2 \log^+|f(s)|\,\mathrm{d}s < \infty$, $\int_{\mathbb{R}} x^2 \log^+|x|\,\nu_L(\mathrm{d}x) < \infty$, and $f \in L^2(\mathbb{R})$, by assumption.

(c) Observe that $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$ is equivalent to $\int_{|x|>1} |x|^4(\log^+|x|)^2\,\nu_L(\mathrm{d}x) < \infty$ and that, by Proposition 25.4 and Theorem 25.3 of Sato [61], $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$ implies $\mathbf{E}|L_1|^4 < \infty$, since, for $|x| > 1$,

$$|x|^4 \leq \begin{cases} |x|^4(\log^+|x|)^2\,, & \text{if } \log^+|x| > 1\,, \\ \mathrm{e}^4\,, & \text{if } \log^+|x| \leq 1\,, \end{cases}$$

such that

$$\int_{|x|>1} |x|^4 \nu_L(\mathrm{d}x) \le \int_{|x|>1} |x|^4 (\log^+ |x|)^2 \nu_L(\mathrm{d}x) + \mathrm{e}^4 \int_{|x|>1} \nu_L(\mathrm{d}x) < \infty\,.$$

For $|x| \le 1$, (4.10) implies $\int_{|x|\le 1} |x|^4 \nu_L(\mathrm{d}x) < \infty$. Henceforth we obtain

$$\begin{aligned}\int_{|x|>1} |x|^4 (\log^+ |x|)^2\, \nu_Z(\mathrm{d}x) &= \int_0^\infty \int_{\mathbb{R}} |f(s)x|^4 (\log^+ |f(s)x|)^2\, \nu_L(\mathrm{d}x)\,\mathrm{d}s \\ &\le 2 \int_0^\infty \int_{\mathbb{R}} |f(s)x|^4 ((\log^+ |f(s)|)^2 + (\log^+ |x|)^2)\, \nu_L(\mathrm{d}x)\,\mathrm{d}s \\ &\le \int_0^\infty |f(s)|^4 (\log^+ |f(s)|)^2\,\mathrm{d}s \int_{\mathbb{R}} |x|^4\, \nu_L(\mathrm{d}x) \\ &\quad + \int_0^\infty |f(s)|^4\,\mathrm{d}s \int_{\mathbb{R}} |x|^4 (\log^+ |x|)^2\, \nu_L(\mathrm{d}x) < \infty\end{aligned}$$

since $\mathbf{E}|L_1|^4 < \infty$, $\int_0^\infty |f(s)|^4 (\log^+ |f(s)|)^2\,\mathrm{d}s < \infty$, $\int_{\mathbb{R}} |x|^4 (\log^+ |x|)^2\, \nu_L(\mathrm{d}x) < \infty$, and $f \in L^4(\mathbb{R})$, by assumption, and ν_L is a Lévy measure. This gives $\mathbf{E}(|X_t|^4 (\log^+ |X_t|)^2) < \infty$ for all $t \in \mathbb{R}$. Using the strict stationarity of X and the Cauchy-Schwarz inequality gives

$$\begin{aligned}\mathbf{E}(|X_t X_{t+h}|^2 \log^+ |X_t X_{t+h}|) &\le \mathbf{E}(|X_0 X_h|^2 \log^+ |X_0|) + \mathbf{E}(|X_0 X_h|^2 \log^+ |X_h|) \\ &\le 2\mathbf{E}(|X_0|^4 (\log^+ |X_0|)^2)\mathbf{E}|X_0|^4 < \infty\end{aligned}$$

which gives the result. □

Remark 4.4. *Throughout this chapter, we assume that the Lévy process $L = (L_t)_{t\in\mathbb{R}}$ has expectation zero. This assumption can be dropped in many cases. For example, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we define with $L'_t = L_t - t\mathbf{E}(L_1)$, $t \in \mathbb{R}$, another Lévy process with mean zero and the same variance such that*

$$X_t = \mu + \mathbf{E}(L_1) \int_{\mathbb{R}} f(s)\,\mathrm{d}s + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L'_s\,, \quad t \in \mathbb{R}\,,$$

and X_t has mean $\mu + \mathbf{E}(L_1) \int_{\mathbb{R}} f(s)\,\mathrm{d}s$.

4.2 Sample Mean

The objective of this section is to show the asymptotic normality of the sample mean

$$\overline{Y}_n := \sum_{k=1}^n Y_k = \sum_{k=1}^n X_{T_k}\,, \quad n \in \mathbb{N}\,, \tag{4.11}$$

where $X = (X_t)_{t\in\mathbb{R}}$ and $Y = (Y_n)_{n\in\mathbb{Z}}$ are given in (4.1) and (4.3), respectively.

To do so, we consider a certain truncated continuous time moving average process. Therefore, let $f_m\colon \mathbb{R} \to \mathbb{R}, s \mapsto f(s)\mathbf{1}_{[-m/2,m/2]}$ be a kernel function with compact support, and $X^{(m)} = (X_t^{(m)})_{t\in\mathbb{R}}$ be defined by

$$X_t^{(m)} := \mu + \int_{\mathbb{R}} f_m(t-s)\,\mathrm{d}L_s = \mu + \int_{\mathbb{R}} f(t-s)\mathbf{1}_{[-m/2,m/2]}(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,, \tag{4.12}$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process with zero mean and $\mathbf{E}|L_1|^2 < \infty$, $\mu \in \mathbb{R}$, and $f \in L^2(\mathbb{R})$. Then the process $X^{(m)} = (X_t^{(m)})_{t\in\mathbb{R}}$ is an m-dependent process. Moreover, $X^{(m)}$ is strictly stationary and, by Proposition 4.1, so is the sequence $Y^{(m)} = (Y_n^{(m)})_{n\in\mathbb{Z}}$ defined by

$$Y_n^{(m)} := X_{T_n}^{(m)}, \tag{4.13}$$

where $(T_n)_{n\in\mathbb{Z}}$ is defined as in (4.2) independent of X.

The next proposition gives a result on the asymptotic behavior of the sample mean of $Y^{(m)}$, i.e. $\overline{Y}_n^{(m)} := \frac{1}{n}\sum_{k=1}^n Y_k^{(m)}$, as $n \to \infty$. We denote with $\xrightarrow{d}$ convergence in distribution.

Proposition 4.5. *Let $X_t^{(m)}$ be defined as in (4.12), where $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process with $\mathbf{E}(L_1) = 0$. Assume that $\mathbf{E}(|L_1|^2 \log^+ |L_1|) < \infty$, $f \in L^2(\mathbb{R})$, and*

$$\int_{\mathbb{R}} |f(s)|^2 \log^+ |f(s)| \,\mathrm{d}s < \infty .$$

Let $(T_n)_{n\in\mathbb{Z}}$ be as in (4.2) independent of L, and define $Y^{(m)} = (Y_n^{(m)})_{n\in\mathbb{Z}}$ by (4.13). Then, for $\overline{Y}_n^{(m)} = \frac{1}{n}\sum_{k=1}^n Y_k^{(m)}$, we have
(a) $\sigma^2_{\overline{Y}^{(m)}} := \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0^{(m)}, Y_k^{(m)})$ exists in $[0,\infty)$ and is absolutely convergent.
(b) $\sqrt{n}\left(\overline{Y}_n^{(m)} - \mu\right) \xrightarrow{d} N(0, \sigma^2_{\overline{Y}^{(m)}})$ as $n \to \infty$.

Proof. The assumptions on L and f imply, by Lemma 4.3 (b), $\mathbf{E}(|Y_0^{(m)}|^2 \log^+ |Y_0^{(m)}|) = \mathbf{E}(|X_0^{(m)}|^2 \log^+ |X_0^{(m)}|) < \infty$, since $Y_0^{(m)} = X_0^{(m)}$. Further, define $\widetilde{X}_t^{(m)} = X_t^{(m)} - \mu$ such that with $\widetilde{Y}_n^{(m)} = Y_n^{(m)} - \mu$ due to the strict stationarity of $(Y_n^{(m)})_{n\in\mathbb{Z}}$ and since $Y_0^{(m)} = X_0^{(m)}$, we obtain a sequence with expectation zero. Hence, w.l.o.g. $\mu = 0$.

Observe that $(Y_n^{(m)})_{n\in\mathbb{Z}}$ fulfills the assumptions of Proposition 4.2 and is therefore strongly mixing with exponentially decreasing mixing coefficient $\alpha_n^{Y^{(m)}}$. Hence, due to (4.6), $\alpha_n^{Y^{(m)}} = O(\mathrm{e}^{k_n \log q})$ as $n \to \infty$. This shows that the assumptions of Theorem 1.11 hold and (a) and (b) follow immediately from this. □

The following proposition states a result on the convergence of the covariances of $Y^{(m)}$ towards the ones of Y. By Proposition 4.1, the process Y is strictly stationary.

Proposition 4.6. *Let X be defined by (4.1), $X^{(m)}$ by (4.12), the processes Y and $Y^{(m)}$ by (4.3) and (4.13), respectively, with $(T_n)_{n\in\mathbb{Z}}$ as in (4.2) and assume that $\mu = 0$. Then*

$$\mathbf{E}(|Y_k Y_l - Y_k^{(m)} Y_l^{(m)}|) \to 0 \quad as \quad m \to \infty \quad for \quad k,l \in \mathbb{Z}. \tag{4.14}$$

Further, it holds

$$\mathbf{E}(Y_k Y_l) = \mathbf{E}(L_1^2) \int_{\mathbb{R}} \mathbf{E}(f(u) f(T_{|l-k|} + u)) \,\mathrm{d}u \quad for \quad k,l \in \mathbb{Z}, \tag{4.15}$$

and similar for $\mathbf{E}(Y_k^{(m)} Y_l^{(m)})$ with f replaced by $f_m = f\mathbf{1}_{[-m/2,m/2]}$.

Proof. Let T_k be a random time taken from the sequence $(T_n)_{n\in\mathbb{Z}}$. Let $f_m\colon [-m/2, m/2] \to \mathbb{R}$ be defined by $f_m(u) := f(u)\mathbf{1}_{[-m/2,m/2]}(u)$. Clearly $|f_m(u)| \le |f(u)|$ for all $u \in \mathbb{R}$. We denote with $\sigma(T)$ the σ-algebra generated by some random variable T. Then, by conditioning on T_k, the Itô Isometry, and Fubini's theorem,

$$
\begin{aligned}
\mathbf{E}|Y_k^{(m)}|^2 &= \mathbf{E}\left|\int_{\mathbb{R}} f_m(T_k-u)\,\mathrm{d}L_u\right|^2 = \mathbf{E}\left[\mathbf{E}\left(\left(\int_{\mathbb{R}} f_m(T_k-u)\,\mathrm{d}L_u\right)^2 \middle| \sigma(T_k)\right)\right] \\
&= \int_{\mathbb{R}} \mathbf{E}\left(\left(\int_{\mathbb{R}} f_m(t-u)\,\mathrm{d}L_u\right)^2 \middle| T_k = t\right) P_{T_k}(\mathrm{d}t) \\
&= \int_{\mathbb{R}} \mathbf{E}(L_1^2)\int_{\mathbb{R}} (f_m(t-u))^2\,\mathrm{d}u\, P_{T_k}(\mathrm{d}t) \le \mathbf{E}(L_1^2)\int_{\mathbb{R}} f(u)^2\,\mathrm{d}u < \infty\,.
\end{aligned}
\tag{4.16}
$$

Further, observe that

$$
\begin{aligned}
\mathbf{E}|Y_k - Y_k^{(m)}|^2 &= \mathbf{E}\left|\int_{\mathbb{R}} f(T_k-u)\,\mathrm{d}L_u - \int_{\mathbb{R}} f(T_k-u)\mathbf{1}_{[-m/2,m/2]}(T_k-u)\,\mathrm{d}L_u\right|^2 \\
&= \mathbf{E}\left|\int_{\mathbb{R}\setminus[T_k-m/2,T_k+m/2]} f(T_k-u)\,\mathrm{d}L_u\right|^2 \\
&= \mathbf{E}\left[\mathbf{E}\left(\left(\int_{\mathbb{R}\setminus[T_k-m/2,T_k+m/2]} f(T_k-u)\,\mathrm{d}L_u\right)^2 \middle| \sigma(T_k)\right)\right] =: \mathbf{E}(\mathrm{I})\,.
\end{aligned}
\tag{4.17}
$$

By the Doob-Dynkin Lemma there exists a measurable function $\varphi^{(m)}\colon [0,\infty) \to \mathbb{R}$ such that $\varphi^{(m)} \circ T_k = \mathrm{I}$. Define

$$
\varphi^{(m)}(t) := \mathbf{E}\left(\left(\int_{\mathbb{R}\setminus[t-m/2,t+m/2]} f(T_k-u)\,\mathrm{d}L_u\right)^2 \middle| T_k = t\right),
$$

then obviously $\varphi^{(m)} \circ T_k = \mathrm{I}$. But, since L is independent of $(T_n)_{n\in\mathbb{Z}}$, it holds

$$
\begin{aligned}
\varphi^{(m)}(t) &= \mathbf{E}\left(\left(\int_{\mathbb{R}\setminus[t-m/2,t+m/2]} f(t-u)\,\mathrm{d}L_u\right)^2\right) \\
&= \mathbf{E}(L_1^2)\int_{\mathbb{R}\setminus[t-m/2,t+m/2]} f(t-u)^2\,\mathrm{d}u \to 0 \quad \text{as} \quad m \to \infty\,,
\end{aligned}
$$

since $f \in L^2(\mathbb{R})$. Hence $\varphi^{(m)}(T_k(\omega)) \to 0$ as $m \to \infty$ for all $k \in \mathbb{Z}$ and all $\omega \in \Omega$.

Define

$$
\varphi(t) := \mathbf{E}\left(\left(\int_{\mathbb{R}} f(t-u)\,\mathrm{d}L_u\right)^2\right) = \mathbf{E}(L_1^2)\int_{\mathbb{R}} f(t-u)^2\,\mathrm{d}u\,.
$$

Then $\mathbf{E}(\varphi \circ T_k) = \mathbf{E}(Y_k^2) < \infty$ such that, since $|\varphi^{(m)} \circ T_k| \le |\varphi \circ T_k|$, we obtain by the dominated convergence theorem for (4.17)

$$
\mathbf{E}|Y_k - Y_k^{(m)}|^2 = \mathbf{E}(\mathrm{I}) = \mathbf{E}(\varphi^{(m)} \circ T_k) \to 0 \quad \text{as } m \to \infty\,. \tag{4.18}
$$

Henceforth, by (4.16), (4.18), and the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbf{E}|Y_kY_l - Y_k^{(m)}Y_l^{(m)}| &= \mathbf{E}|Y_kY_l - Y_k^{(m)}Y_l^{(m)} + Y_k^{(m)}Y_l - Y_k^{(m)}Y_l| \\ &\le \mathbf{E}(|Y_k^{(m)}||Y_l - Y_l^{(m)}|) + \mathbf{E}(|Y_l||Y_k - Y_k^{(m)}|) \\ &\le \sqrt{\mathbf{E}|Y_k^{(m)}|^2}\sqrt{\mathbf{E}|Y_l - Y_l^{(m)}|^2} + \sqrt{\mathbf{E}|Y_l|^2}\sqrt{\mathbf{E}|Y_k - Y_k^{(m)}|^2} \to 0\end{aligned}$$

for $m \to \infty$, i.e. (4.14).

For the last statement (4.15), let w.l.o.g. $k, l \in \mathbb{N}_0$ and $k \le l$. Then, by the same arguments as above,

$$\mathbf{E}\left[\int_{\mathbb{R}} f(T_k - u)\,\mathrm{d}L_u \int_{\mathbb{R}} f(T_l - u)\,\mathrm{d}L_u\right] = \mathbf{E}\left[\mathbf{E}\left[\int_{\mathbb{R}} f(T_k - u)\,\mathrm{d}L_u \int_{\mathbb{R}} f(T_l - u)\,\mathrm{d}L_u \middle| \sigma(T_k)\right]\right]$$

using the Doob-Dynkin Lemma

$$= \int_{\mathbb{R}} \mathbf{E}\left[\int_{\mathbb{R}} f(t-u)\,\mathrm{d}L_u \int_{\mathbb{R}} f\left(t + \sum_{i=k+1}^{l} W_i - u\right)\mathrm{d}L_u \middle| T_k = t\right] P_{T_k}(\mathrm{d}t)$$

T_k independent of L and $\sum_{i=k+1}^{l} W_i$, so repeating the first steps and the Itô Isometry give

$$= \int_{[0,\infty)}\int_{[0,\infty)}\int_{\mathbb{R}} f(t-u)f(t+s-u)\,\mathrm{d}u\,\mathbf{E}(L_1^2)\,P_{\sum_{i=k+1}^{l} W_i}(\mathrm{d}s)P_{T_k}(\mathrm{d}t)$$

Substituting $v = t - u$ and using $P_{T_{l-k}} = P_{\sum_{i=k+1}^{l} W_i}$ together with Fubini's theorem shows that this is equal to

$$= \mathbf{E}(L_1^2)\int_{\mathbb{R}} \mathbf{E}(f(v)f(T_{l-k} + v))\,\mathrm{d}u\,.$$

such that the statement follows. □

Now we are in the position to prove asymptotic normality of $\overline{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k$ as $n \to \infty$, which is the objective of the following theorem.

Theorem 4.7. *Let X be defined as in (4.1) such that $\mu \in \mathbb{R}$, L has expectation zero and $\mathbf{E}(|L_1|^2 \log^+ |L_1|) < \infty$, $f \in L^2(\mathbb{R})$, and $\int_{\mathbb{R}} |f(s)|^2 \log^+ |f(s)|\,\mathrm{d}s < \infty$. Let Y be defined by (4.3) with $(T_n)_{n\in\mathbb{Z}}$ as in (4.2) such that $W = (W_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with positive support, $P(W_1 > 0) > 0$, and W is independent of L. Assume that*

$$\int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} \mathbf{E}|f(T_k + u)|\,\mathrm{d}u < \infty\,. \tag{4.19}$$

Then
(a) $\sigma_Y^2 := \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0, Y_k)$ exists in $[0,\infty)$, is absolutely convergent, and

$$\sigma_Y^2 = \mathbf{E}(L_1^2)\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} f(u)\mathbf{E}(f(T_k + u))\,\mathrm{d}u\,. \tag{4.20}$$

(b) $\sqrt{n}\,(\overline{Y}_n - \mu) \xrightarrow{d} N(0, \sigma_Y^2)$ as $n \to \infty$.

Proof. (a) As in Proposition 4.5, we set w.l.o.g. $\mu = 0$. Observe that $\mathbf{E}|Y_0^2| = \mathbf{E}|X_0|^2 < \infty$ since $f \in L^2(\mathbb{R})$ and L has finite second moment. Further, by (4.15) and (4.19) together with the dominated convergence theorem

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\mathbf{E}(Y_0 Y_k)| &= \sum_{k\in\mathbb{Z}} \left| \mathbf{E}(L_1^2) \int_{\mathbb{R}} f(u) \mathbf{E}(f(T_k+u))\,\mathrm{d}u \right| \\ &\le \mathbf{E}(L_1^2) \int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} \mathbf{E}|f(T_k+u)|\,\mathrm{d}u < \infty\,. \end{aligned} \tag{4.21}$$

This gives the absolute summability of $\sigma_{\overline{Y}}^2$ and the same calculation without the modulus and without the last line gives (4.20).

(b) By Proposition 4.5, we have that for the sequence $(Y_n^{(m)})_{n\in\mathbb{Z}}$ as in (4.13) defined via the m-dependent process $(X_t^{(m)})_{t\in\mathbb{R}}$ as in (4.12) its sample mean is asymptotically normal, i.e.

$$\sqrt{n}\,\overline{Y}_n^{(m)} \xrightarrow{d} Z^{(m)} \quad \text{with} \quad Z^{(m)} \stackrel{d}{=} N(0, \sigma^2_{\overline{Y}^{(m)}})\,. \tag{4.22}$$

By Proposition 4.6, we have that $\mathbf{E}(Y_0^{(m)} Y_k^{(m)}) \to \mathbf{E}(Y_0 Y_k)$ as $m \to \infty$ and, since

$$\sum_{k\in\mathbb{Z}} |\mathbf{E}(Y_0^{(m)} Y_k^{(m)})| \le \mathbf{E}(L_1^2) \int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} \mathbf{E}|f(T_k+u)|\,\mathrm{d}u < \infty\,,$$

by (4.21), it follows from the dominated convergence theorem that $\lim_{m\to\infty} \sigma^2_{\overline{Y}^{(m)}} = \sigma^2_{\overline{Y}}$. Hence,

$$Z^{(m)} \xrightarrow{d} Z \quad \text{as} \quad m \to \infty \quad \text{with} \quad Z \stackrel{d}{=} N(0, \sigma^2_{\overline{Y}})\,. \tag{4.23}$$

Define for $k \in \mathbb{Z}$

$$\begin{aligned} Y_k^{f-f_m} &:= \int_{\mathbb{R}} f(T_k - u) - f(T_k - u)\mathbf{1}_{[-m/2,m/2]}(T_k - u)\,\mathrm{d}L_u \\ &= \int_{\mathbb{R}\setminus[T_k - m/2, T_k + m/2]} f(T_k - u)\,\mathrm{d}L_u\,. \end{aligned}$$

Then $(Y_n^{f-f_m})_{n\in\mathbb{Z}}$ is strictly stationary, by Proposition 4.1. Further, by Cauchy-Schwarz's inequality,

$$\begin{aligned} &\mathbf{E}(Y_0^{f-f_m} Y_k^{f-f_m}) \\ &\le \left(\mathbf{E}\left(\int_{\mathbb{R}\setminus[-m/2,m/2]} f(-u)\,\mathrm{d}L_u \right)^2 \mathbf{E}\left(\int_{\mathbb{R}\setminus[T_k - m/2, T_k + m/2]} f(T_k - u)\,\mathrm{d}L_u \right)^2 \right)^{1/2} \to 0 \end{aligned}$$

as $m \to \infty$ since $f \in L^2(\mathbb{R})$. Since, by (4.21),

$$\sum_{k\in\mathbb{Z}} |\mathbf{E}(Y_0^{f-f_m} Y_k^{f-f_m})| \le \mathbf{E}(L_1^2) \int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} \mathbf{E}|f(T_k+u)|\,\mathrm{d}u \quad \forall\, m \in \mathbb{N}\,,$$

the dominated convergence theorem yields $\lim_{m\to\infty}\sum_{k\in\mathbb{Z}}|\mathbf{E}(Y_0^{f-f_m}Y_k^{f-f_m})|=0$.

Hence, by Theorem 7.1.1 in Brockwell and Davis [23],

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}\mathbf{Var}(n^{1/2}(\overline{Y}_n-\overline{Y}_n^{(m)})) &= \lim_{m\to\infty}\lim_{n\to\infty} n\mathbf{Var}\left(\frac{1}{n}\sum_{k=1}^{n}Y_k^{f-f_m}\right)\\ &= \lim_{m\to\infty}\sum_{k\in\mathbb{Z}}\mathbf{E}(Y_0^{f-f_m}Y_k^{f-f_m})=0\,.\end{aligned}$$

Using Chebychef's inequality gives then

$$\lim_{m\to\infty}\limsup_{n\to\infty}P(n^{1/2}|\overline{Y}_n-\overline{Y}_n^{(m)}|>\varepsilon)=0\quad\forall\,\varepsilon>0\,.$$

Together with (4.22) and (4.23), the claim follows by a variant of Slutsky's Lemma, cf. Theorem 1.12. □

Remark 4.8. *When $(T_n)_{n\in\mathbb{Z}}$ is deterministic, i.e. $T_n=\Delta n$ for $n\in\mathbb{Z}$ and some $\Delta>0$, then Theorem 1.34 established the asymptotic normality of the sample mean under the conditions $\mathbf{E}(L_1^2)<\infty$, $\mathbf{E}(L_1)=0$, $f\in L^2(\mathbb{R})$, and*

$$\left(u\mapsto\sum_{j=-\infty}^{\infty}|f(u+\Delta j)|\right)\in L^2([0,\Delta])\,. \tag{4.24}$$

Observe that (4.24) implies (4.19) since

$$\begin{aligned}\int_{\mathbb{R}}|f(u)|\sum_{k=1}^{\infty}|f(\Delta k+u)|\,\mathrm{d}u &= \sum_{j=-\infty}^{\infty}\int_0^{\Delta}|f(u+\Delta j)|\sum_{k=1}^{\infty}|f(u+\Delta k)|\,\mathrm{d}u\\ &\le \int_0^{\Delta}\left(\sum_{j=-\infty}^{\infty}|f(u+\Delta j)|\right)^2\mathrm{d}u\,.\end{aligned}$$

Hence, Theorem 4.7 generalizes Theorem 1.34 for the case of the renewal sampling sequence $(T_n)_{n\in\mathbb{N}}$, however on the cost of the slightly more restrictive conditions on the Lévy process and the kernel, i.e. $\mathbf{E}(|L_1|^2\log^+|L_1|)<\infty$ and $\int_{\mathbb{R}}|f(s)|^2\log^+|f(s)|\,\mathrm{d}s<\infty$.

Remark 4.9. *To establish the asymptotic normality of the sample mean, we need that (4.19) is satisfied. For example, for $|f(u)|\le K(|u|^{-\alpha}\wedge 1)$ such that $\alpha>1$ and some constant $K>0$, this is true without any further conditions.*

To see this, observe that, due to $|f(u)|\le K(|u|^{-\alpha}\wedge 1)$, decomposing $\int_{\mathbb{R}}$ as $\int_{-\infty}^{-t/2}+\int_{-t/2}^{\infty}$, we obtain for a constant C'_α for $t\ge 0$ that

$$\int_{\mathbb{R}}|f(u)||f(t+u)|\,\mathrm{d}u\le C'_\alpha\mathbf{1}_{\{t\le 2\}}(t)+C'_\alpha t^{-\alpha}\mathbf{1}_{\{t>2\}}(t)\,.$$

and replacing t by $-t$, similarly for $t\le 0$, resulting in

$$\int_{\mathbb{R}}|f(u)||f(t+u)|\,\mathrm{d}u\le C_\alpha(|t|^{-\alpha}\wedge 1)\,. \tag{4.25}$$

for some C_α. Hence,

$$\begin{aligned}\int_{\mathbb{R}} |f(u)| \sum_{k=1}^{\infty} \mathbf{E}|f(T_k+u)|\,\mathrm{d}u &= \sum_{k=1}^{\infty} \mathbf{E}\Big(\int_{\mathbb{R}} |f(u)||f(T_k+u)|\,\mathrm{d}u\Big)\\ &\le \sum_{k=1}^{\infty} \mathbf{E}(C_\alpha \mathbf{1}_{\{T_k\le 1\}} + C_\alpha T_k^{-\alpha}\mathbf{1}_{\{T_k>2\}})\\ &\le C_\alpha \sum_{k=1}^{\infty} P(T_k\le 1) + C_\alpha \sum_{k=1}^{\infty} \mathbf{E}(T_k^{-\alpha}\mathbf{1}_{\{T_k>1\}})\,. \end{aligned} \tag{4.26}$$

The first sum in (4.26) converges since, as it has been shown at the end of the proof of Proposition 4.2, there exists an $r>0$ such that $P(W_1>r)>0$ and hence, for all $m\in\mathbb{N}$, $P(W_1+\cdots+W_{\lceil m/r\rceil}>m)>0$. Denote $q:=1-P(W_1+\cdots+W_{\lceil m/r\rceil}>m)<1$. If $k\le\lceil m/r\rceil$, it holds $P(T_k\le m)\le q$. Otherwise for $k>\lceil m/r\rceil$, set $l_k(m)=\lfloor\frac{k}{\lceil m/r\rceil}\rfloor$. Then, by the i.i.d. property of $(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$, we have $P(T_k\le m)\le q^{l_k(m)}$.

For establishing the convergence of the second sum, observe that

$$\begin{aligned}\mathbf{E}(T_k^{-\alpha}\mathbf{1}_{\{T_k>1\}}) \le \mathbf{E}(T_k^{-\alpha}\mathbf{1}_{\{T_k\ge 1\}}) = \mathbf{E}(T_k^{-\alpha}\mathbf{1}_{\{T_k^{-\alpha}\le 1\}}) &= \int_0^1 P(T_k^{-\alpha}\mathbf{1}_{\{T_k^{-\alpha}\le 1\}}>t)\,\mathrm{d}t\\ = \int_0^1 P(1\le T_k<t^{-1/\alpha})\,\mathrm{d}t &\le \int_1^\infty P(T_k\le v)\alpha v^{-\alpha-1}\,\mathrm{d}v\end{aligned}$$

such that

$$\sum_{k=1}^{\infty} \mathbf{E}(T_k^{-\alpha}\mathbf{1}_{\{T_k>1\}}) \le \int_1^\infty \alpha v^{-\alpha-1}\sum_{k=1}^{\infty} P(T_k\le v)\,\mathrm{d}v\,. \tag{4.27}$$

Since $P(T_k\le v)\le P(T_k\le\lceil v\rceil)\le q^{l_k(\lceil v\rceil)}$ and

$$l_k(\lceil v\rceil)=\left\lfloor\frac{k}{\lceil\lceil v\rceil/r\rceil}\right\rfloor \ge \frac{k}{\lceil\lceil v\rceil/r\rceil}-1\ge C\frac{k}{v/r}-1 \quad \forall\, k\in\mathbb{N},\ v\ge 1\,,$$

for some $C>0$, we obtain with $\tilde q:=q^C$ that

$$\sum_{k=1}^{\infty} P(T_k\le v)\le q^{-1}\sum_{k=1}^{\infty} q^{C\frac{k}{v/r}} \le \frac{q^{-1}}{1-\tilde q^{r/v}}\,. \tag{4.28}$$

Let $\tilde q(x)=\tilde q^x$ for $x\ge 0$. Then for $x\in[0,1]$, by the mean value theorem, there exists $\xi\in(0,1)$ such that

$$1-\tilde q^x=\tilde q(0)-\tilde q(x)=(-x)\tilde q'(\xi)=(-x)\tilde q^\xi\log(\tilde q)\ge x\tilde q|\log(\tilde q)|$$

since $\tilde q\in(0,1)$. This yields for $v>r$ that $\frac{1}{1-\tilde q^{r/v}}\le\frac{v}{r\tilde q|\log(\tilde q)|}$. This together with (4.28) gives for (4.27)

$$\int_1^\infty \alpha v^{-\alpha-1}\sum_{k=1}^{\infty} P(T_k\le v)\,\mathrm{d}v\le\int_1^\infty \alpha v^{-\alpha}\frac{q^{-1}}{r\tilde q|\log(\tilde q)|}\,\mathrm{d}v<\infty\,,$$

since $\alpha>1$. □

4.3 Sample Autocovariance

In this section, we present a multivariate central limit theorem for the autocovariance and autocorrelation functions when the process is sampled at a renewal sequence. We start by considering the strictly stationary, mean zero process

$$X_t = \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s\,, \quad t \in \mathbb{R}\,. \tag{4.29}$$

As in the previous section, let $(T_n)_{n\in\mathbb{Z}}$ be a sequence of random times defined by (4.2) with $(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ a nonnegative i.i.d. noise independent of L such that $P(W_1 > 0) > 0$, and the sampled process $Y_n = X_{T_n}$ for $n \in \mathbb{Z}$. Recall that for a mean zero process,

$$\gamma_n^*(h) = \frac{1}{n}\sum_{k=1}^{n} Y_k Y_{k+h}\,, \quad h \in \mathbb{N}\,, \tag{4.30}$$

is a natural estimator for the autocovariance function.

Let $X^{(m)} = (X_t^{(m)})_{t\in\mathbb{R}}$ be defined as in (4.12) with $\mu = 0$, and $Y^{(m)} = (Y_n^{(m)})_{n\in\mathbb{Z}}$ as in (4.13). If, for a fixed $h \in \mathbb{N}$, we set $Z_{h,k}^{(m)} := Y_k^{(m)} Y_{k+h}^{(m)}$, it can be easily seen, that also $Z_h^{(m)} = (Z_{h,k}^{(m)})_{k\in\mathbb{Z}}$ is a strictly stationary, strongly mixing sequence with coefficient $\alpha_n^{Z_h^{(m)}} \le \alpha_{n-h}^{Y^{(m)}}$ for all $n > h$ by Remark 1.8 (b). Hence, to establish the central limit theorem, we first show a central limit theorem for

$$\gamma_n^{*,m}(h) = \frac{1}{n}\sum_{k=1}^{n} Z_{h,k}^{(m)}\,, \quad h \in \mathbb{N}\,.$$

But before we give some preliminary results regarding Lévy processes and their integrals. In the following theorem, we recall the multivariate extension of Theorem 2.7 in Rajput and Rosinski [59], which characterizes the continuous time moving average process. It can also be regarded as an extension to Proposition 57.13 of Sato [61].

Theorem 4.10. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet $(\gamma_L, \sigma^2, \nu_L)$ and $g\colon \mathbb{R} \to \mathbb{R}^d$ be a measurable function. Denote with $D_d := \{x\colon |x| \le 1\}$ the unit ball in $\mathbb{R}^d$. Then*

(a) g is L-integrable (i.e. integrable with respect to the Lévy process L) as a limit in probability in the sense of Rajput und Rosinski [59] if and only if

(i) $\displaystyle\int_{\mathbb{R}} \left\| g(s)\gamma_L + \int_{\mathbb{R}} g(s)x(\mathbf{1}_{D_d}(g(s)x) - \mathbf{1}_{D_1}(x))\,\nu_L(\mathrm{d}x) \right\| \mathrm{d}s < \infty,$

(ii) $\displaystyle\int_{\mathbb{R}} \left\| g(s)\sigma^2 g(s)' \right\| \mathrm{d}s < \infty,$ *and*

(iii) $\displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} (\|g(s)x\|^2 \wedge 1)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s < \infty.$

(b) If g is L-integrable, the distribution of $\int_{\mathbb{R}} g(s)\,\mathrm{d}L_s$ is infinitely divisible with characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ given by

$$
\begin{aligned}
\gamma_{\text{int}} &= \int_{\mathbb{R}} g(s)\gamma_L + \int_{\mathbb{R}} g(s)x(\mathbf{1}_{D_d}(g(s)x) - \mathbf{1}_{D_1}(x))\,\nu_L(\mathrm{d}x)\,\mathrm{d}s\,,\\
\Sigma_{\text{int}} &= \sigma^2 \int_{\mathbb{R}} g(s)g(s)'\,\mathrm{d}s\,, \quad \textit{and}\\
\nu_{\text{int}}(B) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_B(g(s)x)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \quad \textit{for all Borel sets } B \subset \mathbb{R}^d \setminus \{0\}.
\end{aligned}
$$

Corollary 4.11. *By a similar calculation as for the univariate case in the proof of Lemma 4.3, we deduce that, if L has expectation zero and finite second moment and $g \in L^2(\mathbb{R}^d)$, then the conditions (i), (ii), and (iii) of Theorem 4.10 (a) are satisfied and $\int_{\mathbb{R}} g(s)\,\mathrm{d}L_s$ is infinitely divisible with characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ as given in Theorem 4.10 (b).*

Point (a) in the lemma below generalizes expression (3.5) in Cohen and Lindner [31] to non-lattice times and presents a different and quicker proof of that fact even for integer-times.

Lemma 4.12. *Let $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, and $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process with expectation zero and finite fourth moment. Denote $\sigma^2 := \mathbf{E}(L_1^2)$, $\eta := \sigma^{-4}\mathbf{E}(L_1^4)$, and $f_m := f\mathbf{1}_{[-m/2,m/2]}$. Then the following statements hold:*
(a) For $X_t := \int_{\mathbb{R}} f(t-u)\,\mathrm{d}L_u$, we have for all $r,s,t,v \in \mathbb{R}$

$$
\begin{aligned}
\mathbf{E}(X_rX_sX_tX_v) &= (\eta-3)\sigma^4 \int_{\mathbb{R}} f(u+r)f(u+s)f(u+t)f(u+v)\,\mathrm{d}u\\
&\quad + \mathbf{E}(X_rX_s)\mathbf{E}(X_tX_v) + \mathbf{E}(X_rX_t)\mathbf{E}(X_sX_v) + \mathbf{E}(X_rX_v)\mathbf{E}(X_sX_t)\,. \quad (4.31)
\end{aligned}
$$

(b) Let additionally $X_t^{(m)} := \int_{\mathbb{R}} f_m(t-u)\,\mathrm{d}L_u$, then we have for all $r,s,t,v \in \mathbb{R}$

$$
\begin{aligned}
\mathbf{E}(X_rX_sX_t^{(m)}X_v^{(m)}) &= (\eta-3)\sigma^4 \int_{\mathbb{R}} f(u+r)f(u+s)f_m(u+t)f_m(u+v)\,\mathrm{d}u\\
&\quad + \mathbf{E}(X_rX_s)\mathbf{E}(X_t^{(m)}X_v^{(m)}) + \mathbf{E}(X_rX_t^{(m)})\mathbf{E}(X_sX_v^{(m)}) + \mathbf{E}(X_rX_v^{(m)})\mathbf{E}(X_sX_t^{(m)})\,.
\end{aligned}
$$

Proof. Since X_r, X_s, X_t, and X_v all have expectation zero, the 4^{th} order joint cumulant $\mathbf{Cum}(\mathbf{X})$ of $\mathbf{X} := (X_r, X_s, X_t, X_v)$ is given by

$$
\begin{aligned}
\mathbf{Cum}(\mathbf{X}) &= \mathbf{E}(X_rX_sX_tX_v) - \mathbf{E}(X_rX_s)\mathbf{E}(X_tX_v)\\
&\quad - \mathbf{E}(X_rX_t)\mathbf{E}(X_sX_v) - \mathbf{E}(X_rX_v)\mathbf{E}(X_sX_t)\,, \quad (4.32)
\end{aligned}
$$

see Proposition 4.2.2 in Giraitis et al. [39]. On the other hand,

$$
\mathbf{Cum}(\mathbf{X}) = \frac{\partial^4}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} \log \mathbf{E}(\mathrm{e}^{i\langle u, \mathbf{X}\rangle})\bigg|_{u_1=u_2=u_3=u_4=0}\,,
$$

cf. Definition 4.2.1 of Giraitis et al. [39]. Let $g(u) = (f(r-u), f(s-u), f(t-u), f(v-u))$, then it holds $\mathbf{X} = \int_{\mathbb{R}} g(u)\,\mathrm{d}L_u$, and, since $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, we obtain $g \in L^2(\mathbb{R}^4) \cap L^4(\mathbb{R}^4)$ which yields by Corollary 4.11 that $\mathbf{X}$ is infinitely divisible with characteristic exponent

$$\log \mathbf{E}(\mathrm{e}^{i\langle u, \mathbf{X}\rangle}) = i\langle \gamma_{\mathrm{int}}, u\rangle - \frac{1}{2}\langle u, \Sigma_{\mathrm{int}} u\rangle + \int_{\mathbb{R}^4} (\mathrm{e}^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle \mathbf{1}_{D_4}(x))\, \nu_{\mathrm{int}}(\mathrm{d}x)\,.$$

Hence, by Theorem 4.10 (b) and the well-known fact $\int_{\mathbb{R}} x^4 \nu_L(\mathrm{d}x) = (\eta - 3)\sigma^4$,

$$\mathbf{Cum}(\mathbf{X}) = \int_{\mathbb{R}^4} x_1 x_2 x_3 x_4\, \nu_{\mathrm{int}}(\mathrm{d}x) = (\eta - 3)\sigma^4 \int_{\mathbb{R}} f(r-u) f(s-u) f(t-u) f(v-u)\,\mathrm{d}s\,.$$

Then this together with (4.32) yields (a).

For (b) just observe that also $f_m \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ such that with $h(u) = (f(r-u), f(s-u), f_m(t-u), f_m(v-u))$ we obtain that also $\mathbf{Z} = \int_{\mathbb{R}} h(u)\,\mathrm{d}L_u$ is infinitely divisible by Corollary 4.11. Similar argumentations as above give the demanded result. $\square$

A straightforward consequence is now the following result which has already been obtained in Cohen and Lindner [31, Lemma 3.2].

Corollary 4.13. *Let $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process with expectation zero and finite fourth moment. Denote $\sigma^2 := \mathbf{E}(L_1^2)$ and $\eta := \sigma^{-4}\mathbf{E}(L_1^4)$. Then*

$$\mathbf{E}\bigg(\int_{\mathbb{R}} f(s)\,\mathrm{d}L_s\bigg)^4 = (\eta - 3)\sigma^4 \int_{\mathbb{R}} f^4(s)\,\mathrm{d}s + 3\sigma^4 \bigg(\int_{\mathbb{R}} f^2(s)\,\mathrm{d}s\bigg)^2\,.$$

In the following lemma we give a similar expression as (4.31) when the deterministic times r, s, t, and v are replaced by random times.

Lemma 4.14. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process with expectation zero and finite fourth moment, X be defined by (4.29), with $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, and $X_t^{(m)}$ by (4.29) with f replaced by f_m. The processes Y and $Y^{(m)}$ are defined by (4.3) and (4.13), respectively, with $(T_n)_{n\in\mathbb{Z}}$ as in (4.2), where $W = (W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ is as usual. Denote $\sigma^2 := \mathbf{E}(L_1^2)$ and $\eta := \sigma^{-4}\mathbf{E}(L_1^4)$, and let $l, m, n \in \mathbb{Z}$. Let $F(s,t) := \int_{\mathbb{R}} f(u+s) f(u+t)\,\mathrm{d}u$, then*

(a)
$$\begin{aligned}\mathbf{E}(Y_0 Y_l Y_m Y_n) &= (\eta - 3)\sigma^4 \int_{\mathbb{R}} f(u)\mathbf{E}(f(u+T_l) f(u+T_m) f(u+T_n))\,\mathrm{d}u \\ &\quad + \sigma^4 \mathbf{E}(F(0,T_l)) F(T_m, T_n)) + \sigma^4 \mathbf{E}(F(0,T_m) F(T_l, T_n)) \\ &\quad + \sigma^4 \mathbf{E}(F(0,T_n) F(T_l, T_m))\,.\end{aligned}$$

(b) If $0 \le l \le m \le n$, then $\mathbf{E}(F(0,T_l)) F(T_m, T_n)) = \mathbf{E}(F(0,T_l))\mathbf{E}(F(0,T_{n-m}))$.

Proof. (a) Due to the definition of $(T_n)_{n\in\mathbb{Z}}$ it follows that $T_l \le T_m \le T_n$. Conditioning on the random times yields, by the independence of L and W and Lemma 4.12 (a),

$$\mathbf{E}(Y_0 Y_l Y_m Y_n) = \mathbf{E}\Big(\mathbf{E}\Big[X_0 X_{T_l} X_{T_m} X_{T_n} | \sigma(T_l, T_m, T_n)\Big]\Big)$$

$$
\begin{aligned}
&= \int_{[0,\infty)^3} \mathbf{E}\Big[X_0 X_s X_t X_v | (T_l, T_m, T_n)' = (s,t,v)'\Big] P_{(T_l,T_m,T_n)}(\mathrm{d}(s,t,v)) \\
&= \int_{[0,\infty)^3} \mathbf{E}\Big(X_0 X_s X_t X_v) P_{(T_l,T_m,T_n)}(\mathrm{d}(s,t,v)) \\
&=: \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}\,,
\end{aligned}
$$

say, where A, B, C, and D corresponds to the parts arising from the decomposition in (4.31). Then, by Fubini's theorem,

$$
\begin{aligned}
\mathrm{A} &= (\eta-3)\sigma^4 \int_{[0,\infty)^3} \int_{\mathbb{R}} f(u) f(u+s) f(u+t) f(u+v)\,\mathrm{d}u\, P_{(T_l,T_m,T_n)}(\mathrm{d}(s,t,v)) \\
&= (\eta-3)\sigma^4 \int_{\mathbb{R}} f(u)\,\mathbf{E}\Big[f(u+T_l) f(u+T_m) f(u+T_n)\Big]\,\mathrm{d}u\,.
\end{aligned}
$$

Since $\mathbf{E}(X_s X_t) = \sigma^2 \int_{\mathbb{R}} f(u+s) f(u+t)\,\mathrm{d}u$ for all $s,t \in \mathbb{R}$, we obtain

$$
\begin{aligned}
\mathrm{B} &= \int_{[0,\infty)^3} \mathbf{E}(X_0 X_s) \mathbf{E}(X_t X_v)\, P_{(T_l,T_m,T_n)}(\,\mathrm{d}(s,t,v)) \\
&= \sigma^4 \int_{[0,\infty)^3} \int_{\mathbb{R}} f(u) f(u+s)\,\mathrm{d}u \int_{\mathbb{R}} f(w+t) f(w+v)\,\mathrm{d}w\, P_{(T_l,T_m,T_n)}(\,\mathrm{d}(s,t,v)) \\
&= \sigma^4\ \mathbf{E}\Bigg[\int_{\mathbb{R}} f(u) f(u+T_l)\,\mathrm{d}u \int_{\mathbb{R}} f(u+T_m) f(u+T_n)\,\mathrm{d}u\Bigg]\,.
\end{aligned}
$$

Likewise

$$
\begin{aligned}
\mathrm{C} &= \sigma^4\ \mathbf{E}\Bigg[\int_{\mathbb{R}} f(u) f(u+T_m)\,\mathrm{d}u \int_{\mathbb{R}} f(w+T_l) f(w+T_n)\,\mathrm{d}w\Bigg]\,, \quad \text{and} \\
\mathrm{D} &= \sigma^4\ \mathbf{E}\Bigg[\int_{\mathbb{R}} f(u) f(u+T_n)\,\mathrm{d}u \int_{\mathbb{R}} f(w+T_l) f(w+T_m)\,\mathrm{d}w\Bigg]\,.
\end{aligned}
$$

With the definition of $F(s,t)$, the assertion follows.

(b) Observe that, since $P_{\sum_{i=m+1}^n W_i} = P_{T_{n-m}}$, by independence of the sequence W,

$$
\begin{aligned}
&\mathbf{E}[F(0,T_l) F(T_m,T_n)] \\
&\quad= \int_{[0,\infty)^3} \int_{\mathbb{R}} f(u) f(u+s)\,\mathrm{d}u \int_{\mathbb{R}} f(w+s+t) f(w+s+t+v)\,\mathrm{d}w \\
&\qquad\qquad P_{\sum_{i=m+1}^n W_i}(\mathrm{d}v) P_{\sum_{i=l+1}^m W_i}(\mathrm{d}t) P_{T_l}(\mathrm{d}s) \\
&\quad= \int_{[0,\infty)} \int_{\mathbb{R}} f(u) f(u+s)\,\mathrm{d}u\, P_{T_l}(\mathrm{d}s) \int_{[0,\infty)} \int_{\mathbb{R}} f(w) f(w+v)\,\mathrm{d}w\, P_{T_{n-m}}(\mathrm{d}v) \\
&\quad= \mathbf{E}(F(0,T_l)) \mathbf{E}(F(0,T_{n-m}))\,,
\end{aligned}
$$

which gives the result. □

From Lemma 4.14, the following proposition gives the expression of $n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^*(q))$ as $n \to \infty$ needed in the upcoming central limit theorem.

Proposition 4.15. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process with expectation zero and finite fourth moment, and denote $\sigma^2 := \mathbf{E}(L_1^2)$ and $\eta := \sigma^{-4}\mathbf{E}(L_1^4)$. Suppose that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, and let X and Y be defined by (4.29) and (4.3) with $(T_n)_{n\in\mathbb{Z}}$ by (4.2). Denote*

$$
\begin{aligned}
F(s,t) &:= \int_{\mathbb{R}} f(u+s)f(u+t)\,\mathrm{d}u\,, \quad s,t\in\mathbb{R}\,, \quad \text{and} \\
\kappa_f(k,l,m) &:= (\eta-3)\sigma^4\int_{\mathbb{R}} f(u)\mathbf{E}(f(u+T_k)f(u+T_l)f(u+T_m))\,\mathrm{d}u \\
&\quad + \sigma^4\mathbf{E}(F(0,T_l)F(T_k,T_m)) + \sigma^4\mathbf{E}(F(0,T_m)F(T_k,T_l))\,, \quad \text{for } k,l,m\in\mathbb{Z}\,.
\end{aligned}
$$

Let $p,q\in\mathbb{N}$, denote $Z_{p,i} := Y_iY_{i+p}$, $Z_{q,i} := Y_jY_{j+q}$ for $i,j\in\mathbb{Z}$ and assume that

$$
\int_{\mathbb{R}} |f(u)|\sum_{k=1}^{\infty}\mathbf{E}|f(u+T_p)f(u+T_k)f(u+T_{k+q})|\,\mathrm{d}u < \infty\,, \tag{4.33}
$$

and

$$
\sum_{k=1}^{\infty}\mathbf{E}\left[\left(\int_{\mathbb{R}}|f(u)f(u+T_k)|\,\mathrm{d}u\right)^2\right] < \infty\,. \tag{4.34}
$$

Then

$$
\mathbf{Cov}(Z_{p,i},Z_{q,i}) = \kappa(p,j-i,j-i+q) + \sigma^4\mathbf{Cov}(F(0,T_p),F(T_{j-i},T_{j-i+q}))\,, \tag{4.35}
$$

$$
\mathbf{Cov}(F(0,T_p),F(T_{j-i},T_{j-i+q})) = 0 \quad \text{for } j-i\le p \quad \text{or} \quad j-i\le q\,, \tag{4.36}
$$

$$
\sum_{k\in\mathbb{Z}}|\mathbf{Cov}(Z_{p,0},Z_{q,k})| < \infty\,, \qquad \sum_{k\in\mathbb{Z}}|\kappa(p,k,k+q)| < \infty\,, \tag{4.37}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} n\,\mathbf{Cov}(\gamma_n^*(p),\gamma_n^*(q)) &= \sum_{k\in\mathbb{Z}}\mathbf{Cov}(Z_{p,0},Z_{q,k}) = \\
&\sum_{k\in\mathbb{Z}}\kappa(p,k,k+q) + \sigma^4\sum_{k=-q+1}^{p-1}\mathbf{Cov}(F(0,T_p),F(T_k,T_{k+q}))\,.
\end{aligned} \tag{4.38}
$$

where $\gamma_n^*(p)$ and γ_n^*q are defined in (4.30).

Proof. From Lemma 4.14 (a), since $\mathbf{E}(Y_iY_{i+p}) = \sigma^4\mathbf{E}(F(T_i,T_{i+p})) = \sigma^4\mathbf{E}(F(0,T_p))$, and by the stationarity of Y, we have

$$
\begin{aligned}
\mathbf{Cov}(Z_{p,i},Z_{q,j}) &= \mathbf{E}(Y_iY_{i+p}Y_jY_{j+q}) - \mathbf{E}(Y_iY_{i+p})\mathbf{E}(Y_jY_{j+q}) \\
&= (\eta-3)\sigma^4\mathbf{E}\left(\int_{\mathbb{R}} f(u)f(u+T_p)f(u+T_{j-i})f(u+T_{j-i+q})\,\mathrm{d}u\right) \\
&\quad + \sigma^4\mathbf{E}(F(0,T_p)F(T_{j-i},T_{j-i+q})) + \sigma^4\mathbf{E}(F(0,T_{j-i})F(T_p,T_{j-i+q})) \\
&\quad + \sigma^4\mathbf{E}(F(0,T_{j-i+q})F(T_p,T_{j-i})) - \sigma^4\mathbf{E}(F(0,T_p))\mathbf{E}(F(T_{j-i},T_{j-i+q})) \\
&= \kappa(p,j-i,j-i+q) + \sigma^4\mathbf{Cov}(F(0,T_p),F(T_{j-i},F_{j-i+q}))
\end{aligned}
$$

which is (4.35). Equation (4.36) is an immediate consequence of Lemma 4.14 (b). For the proof of (4.37), by (4.35) and (4.36), it is enough to show that $\sum_{k\in\mathbb{Z}} |\kappa(p,k,k+q)| < \infty$. To see this, observe that

$$\sum_{k\in\mathbb{Z}} |\kappa(p,k,k+q)| \le (\eta-3)\sigma^4 \int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} |\mathbf{E}(f(u+T_p)f(u+T_k)f(u+T_{k+q}))|\,\mathrm{d}u$$
$$\sigma^4 \sum_{k\in\mathbb{Z}} |\mathbf{E}(F(0,T_k)F(T_p,T_{k+q}))| + \sigma^4 \sum_{k\in\mathbb{Z}} |\mathbf{E}(F(0,T_{k+q})F(T_p,T_k))|\,.$$

The first of these summands is finite by (4.33) and the second is finite since, by Cauchy-Schwarz's inequality,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\mathbf{E}(F(0,T_k)F(T_p,T_{k+q}))| &\le \sum_{k\in\mathbb{Z}} (\mathbf{E}(F(0,T_k))^2)^{1/2}(\mathbf{E}(F(T_p,T_{k+q}))^2)^{1/2} \\ &\le \left(\sum_{k\in\mathbb{Z}} \mathbf{E}(F(0,T_k))^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}} \mathbf{E}(F(T_p,T_{k+q}))^2\right)^{1/2}\end{aligned}$$

which is finite by (4.34). The same argument yields finiteness of the third summand, showing (4.37).

To see (4.38), observe that by the stationarity of Y, with $k = j - i$,

$$\begin{aligned} n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^*(q)) &= \frac{1}{n}\sum_{i,j=1}^{n} \mathbf{Cov}(Z_{p,i}, Z_{q,j}) = \sum_{i,j=1}^{n} \frac{1}{n}\mathbf{Cov}(Z_{p,0}, Z_{q,j-i}) \\ &= \sum_{k=-n+1}^{n-1} \frac{n-|k|}{n}\mathbf{Cov}(Z_{p,0}, Z_{q,k})\,. \end{aligned}$$

Since $\sum_{k\in\mathbb{Z}} |\mathbf{Cov}(Z_{p,0}, Z_{q,k})| < \infty$ by (4.37), the latter converges to $\sum_{k\in\mathbb{Z}} \mathbf{Cov}(Z_{p,0}, Z_{q,k})$ as $n\to\infty$ by the dominated convergence theorem, which together with (4.35) and (4.36) finishes the proof of (4.38). □

Remark 4.16. *(a) If $q=0$ or $p=0$, it is easy to see that $\mathbf{Cov}(F(0,T_p)F(T_k,T_{k+q})) = 0$ for all $k \in \{-q+1,\dots,p-1\}$. Hence, the second summand in (4.38) disappears.*

(b) It is easy to check, by the Cauchy-Schwarz inequality, that (4.33) holds for example under the assumption of $f \in L^2(\mathbb{R})$ and

$$\left(u \mapsto \sum_{k\in\mathbb{Z}} \mathbf{E}|f(u+T_p)f(u+T_k)f(u+T_{k+q})|\right) \in L^2(\mathbb{R}).$$

(c) If we choose $(T_n)_{n\in\mathbb{Z}}$ to be deterministic, i.e. $T_n = n\Delta$ for $n\in\mathbb{Z}$ and some $\Delta > 0$, it is easy to see that (4.33) and (4.34) are implied by (1.12) and (1.13) in Theorem 1.35 for establishing the asymptotic normality of the sample autocovariance of the moving average process observed on a lattice. (4.33) then reduces to (1.13), which in Cohen and Linder [31] was shown to be implied by (3.3) of [31].

Remark 4.17. *Similarly to Remark 4.9, a sufficient condition for the validity of (4.33) and (4.34) is that* $|f(u)| \le K(|u|^{-\alpha} \wedge 1)$ *for some* $K > 0$ *and* $\alpha > 1/2$*, instead of* $\alpha > 1$ *as in Remark 4.9.*

To see this, observe that, by (4.25),

$$\sum_{k\in\mathbb{Z}} \mathbf{E}\bigg(\int_{\mathbb{R}} |f(u)f(u+T_k)| \bigg)^2 \le \sum_{k\in\mathbb{Z}} C_\alpha^2 \mathbf{E}(|T_k|^{-2\alpha} \wedge 1) < \infty\,,$$

by the the same calculations as in Remark 4.9. Hence, (4.34) is true. To establish (4.33), observe that

$$\begin{aligned}
&\int_{\mathbb{R}} |f(u)f(u+T_k)||f(u+T_p)f(u+T_{k+q})| \,\mathrm{d}u \\
&\quad \le \bigg(\int_{\mathbb{R}} f^2(u)f^2(u+T_k) \bigg)^{1/2} \bigg(\int_{\mathbb{R}} f^2(u+T_p)f^2(u+T_{k+q}) \bigg)^{1/2} \\
&\quad \le (C_{2\alpha}(|T_k|^{-2\alpha} \wedge 1))^{1/2} (C_{2\alpha}(|T_{k+q}-T_p|^{-2\alpha} \wedge 1))^{1/2}
\end{aligned}$$

for some $C_{2\alpha}$*, by (4.25). Applying the Cauchy-Schwarz inequality twice then gives*

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \mathbf{E}\bigg(\int_{\mathbb{R}} |f(u)f(u+T_k)||f(u+T_p)f(u+T_{k+q})| \,\mathrm{d}u \bigg) \\
&\quad \le \sum_{k\in\mathbb{Z}} C_{2\alpha} (\mathbf{E}(|T_k|^{-2\alpha} \wedge 1))^{1/2} (\mathbf{E}(|T_{k+q}-T_p|^{-2\alpha} \wedge 1))^{1/2} \\
&\quad \le C_{2\alpha} \bigg(\sum_{k\in\mathbb{Z}} \mathbf{E}(|T_k|^{-2\alpha} \wedge 1) \bigg)^{1/2} \bigg(\sum_{k\in\mathbb{Z}} \mathbf{E}(|T_{k+q}-T_p|^{-2\alpha} \wedge 1) \bigg)^{1/2} \\
&\quad = C_{2\alpha} \sum_{k\in\mathbb{Z}} \mathbf{E}(|T_k|^{-2\alpha} \wedge 1)\,,
\end{aligned}$$

and the latter is finite by the calculation in Remark 4.9.

The next proposition shows that similar results as obtained in Proposition 4.15 are valid for the truncated sequence $Y^{(m)}$.

Proposition 4.18. *Let the assumption and notations of Proposition 4.15 be satisfied. For* $m \in \mathbb{N}$*, define* $f_m := f\mathbf{1}_{[-m/2,m/2]}$*,* $F_m := \int_{\mathbb{R}} f_m(u+s)f_m(u+t)$ *for* $s,t \in \mathbb{R}$*,* $X_t^{(m)} := \int_{\mathbb{R}} f_m(t-u)\,\mathrm{d}L_u$*,* $Y_n^{(m)} := X_{T_n}^{(m)}$*. Let* $p,q \in \mathbb{N}$*, and define* $Z_{p,i}^{(m)} := Y_i^{(m)} Y_{i+p}^{(m)}$*,* $Z_{q,j}^{(m)} := Y_j^{(m)} Y_{j+q}^{(m)}$*, and*

$$\gamma_n^{*,m}(h) := \frac{1}{n} \sum_{k=1}^{n} Y_k^{(m)} Y_{k+h}^{(m)}\,, \quad h = p,q\,.$$

Then (4.33) and (4.34) also hold for f_m*, and for all* $k \in \mathbb{Z}$

$$|\kappa_{f_m}(p,k,k+q)| \le \kappa_{|f|}(p,k,k+q)\,,$$

$$\kappa_{f_m}(p,k,k+q) \to \kappa_f(p,k,k+q) \quad as \quad m \to \infty\,, \tag{4.39}$$

$$\mathbf{Cov}(F_m(0,T_p)F_m(T_k,T_{k+q})) \to \mathbf{Cov}(F(0,T_p)F(T_k,T_{k+q})) \quad as \quad m \to \infty\,, \tag{4.40}$$

and

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty} n\mathbf{Cov}(\gamma_n^{*,m}(p),\gamma_n^{*,m}(q)) \\ &= \sum_{k\in\mathbb{Z}} \kappa(p,k,k+q) + \sigma^4 \sum_{k=-q+1}^{p-1} \mathbf{Cov}(F(0,T_p),F(T_k,T_{k+q}))\,.\end{aligned} \tag{4.41}$$

Proof. That (4.33) and (4.34) also hold for f_m is clear since $|f_m| \le |f|$, as is $|\kappa_{f_m}| \le \kappa_{|f|}$. Since $|f_m| \le |f|$ and $f_m \to f$ as $m \to \infty$, the dominated convergence theorem shows (4.39) and (4.40). And (4.41) then follows from (4.39), (4.40), and (4.38) again by the dominated convergence theorem. □

The following proposition proof the demanded central limit theorem for the truncated sequence $Y^{(m)}$.

Proposition 4.19. *Let $X_t^{(m)} = \int_{\mathbb{R}} f_m(t-u)\,\mathrm{d}L_u$, where $f_m = f\mathbf{1}_{[-m/2,m/2]}$ and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process with $\mathbf{E}(L_1) = 0$. Assume that $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$, $f \in L^2(\mathbb{R})\cap L^4(\mathbb{R})$, and*

$$\int_{\mathbb{R}} |f(s)|^4(\log^+|f(s)|)^2\,\mathrm{d}s < \infty\,.$$

With the same notations for $F_m(s,t)$, $Y^{(m)} = (Y_n^{(m)})_{n\in\mathbb{Z}}$, and $Z_{h,k}^{(m)}$ as in Proposition 4.18, we then have

(a) $\mathbf{Z}_{p,q}^m$ given by

$$\mathbf{Z}_{p,q}^m = \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Z_{p,0}^{(m)},Z_{q,k}^{(m)}) = \sum_{k\in\mathbb{Z}} \kappa_{f_m}(p,k,k+q) + \sigma^4 \sum_{k=-q+1}^{p-1} \mathbf{Cov}(F_m(0,T_p),F_m(T_k,T_{k+q}))$$

exists in $[0,\infty)$, and is absolutely convergent, for each $p,q \in \mathbb{N}_0$.

(b) $\sqrt{n}(\gamma_n^{*,m}(0)-\gamma^m(0),\dots,\gamma_n^{*,m}(h)-\gamma^m(h))' \xrightarrow{d} N(0,\mathbf{Z}^m)$ as $n\to\infty$, where the covariance matrix is $\mathbf{Z}^m = (\mathbf{Z}_{pq}^m)_{p,q=0,\dots,h} \in \mathbb{R}^{h+1\times h+1}$ with entries given by (a).

Proof. Define $Q_k := (Z_{0,k}^{(m)}, Z_{1,k}^{(m)},\dots,Z_{h,k}^{(m)})' \in \mathbb{R}^{h+1}$. Then $(Q_k)_{k\in\mathbb{Z}}$ is obviously strictly stationary and we have

$$\frac{1}{n}\sum_{k=1}^{n} Q_k = (\gamma_n^{*,m}(0),\dots,\gamma_n^{*,m}(h))'\,.$$

If we can show that

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}\lambda' Q_k - \lambda'(\gamma_n^m(0),\dots,\gamma_n^m(h))'\right) \xrightarrow{d} N(0,\lambda'\mathbf{Z}^m\lambda) \quad \forall\,\lambda\in\mathbb{R}^{h+1}\,,$$

we obtain, by the Cramér-Wold theorem, the assertion of (b).

By the assumptions on L and f, we obtain, by Lemma 4.3 (c), $\mathbf{E}(|Y_0^{(m)}|^4(\log^+|Y_0^{(m)}|)^2) = \mathbf{E}(|X_0^{(m)}|^4(\log^+|X_0^{(m)}|)^2) < \infty$, since $Y_0^{(m)} = X_0^{(m)}$, and so $\mathbf{E}(|Z_{h,0}|^2\log^+|Z_{h,0}|) < \infty$, by the Cauchy-Schwarz inequality. Therefore also $\mathbf{E}(|\lambda' Q_0|^2\log^+|\lambda' Q_0|) < \infty$ for all $\lambda \in \mathbb{R}^{h+1}$.

Observe that $(\lambda' Q_n)_{n\in\mathbb{Z}}$ is strongly mixing for each $\lambda \in \mathbb{R}^{h+1}$ with $\alpha_n^{\lambda' Q} \le \alpha_{n-h}^{Y(m)}$ for all $n > h$, by Remark 1.8 (b), such that $(\alpha_n^{\lambda' Q})$ is exponentially decreasing. Hence, the assumptions of Theorem 1.11 hold for $\lambda' Q_k - \mathbf{E}(\lambda' Q_0)$ and (a) and (b) follow immediately. Observe that the assertion there also holds when $\lambda'\mathbf{Z}^m\lambda = 0$, in which case we have L^2-convergence to 0 by Bradley [20], Proposition 8.3. □

Now, we can establish the multivariate asymptotic normality of the sample autocovariance and sample autocorrelation.

Theorem 4.20. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process with expectation zero such that $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$, and denote $\sigma^2 := \mathbf{E}(L_1^2)$ and $\eta := \sigma^{-4}\mathbf{E}(L_1^4)$. Let $h \in \mathbb{N}_0$, suppose that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, $\int_{\mathbb{R}} |f(s)|^4(\log^+|f(s)|)^2\,\mathrm{d}s < \infty$, and assume that (4.33) and (4.34) hold for all $p, q \in \{0, \ldots, h\}$.*

(a) Then

$$\sqrt{n}(\gamma_n^*(0) - \gamma(0), \ldots, \gamma_n^*(h) - \gamma(h))' \xrightarrow{d} N(0, \mathbf{Z}), \quad n \to \infty,$$

where $\gamma(h) = \mathbf{E}(Y_0 Y_h)$ and $\mathbf{Z} = (\mathbf{Z}_{pq})_{p,q=0,\ldots,h} \in \mathbb{R}^{h+1\times h+1}$ is the covariance matrix defined by

$$\mathbf{Z}_{pq} = \sigma^4 \sum_{k=-q+1}^{p-1} \mathbf{Cov}(F(0, T_p), F(T_k, T_{k+q})) + \sum_{k\in\mathbb{Z}} \kappa(p, k, k+q)$$

with $\kappa(p, k, k+q)$ for $k, p, q \in \mathbb{Z}$ and $F(s, t)$ given as in Proposition 4.15.

(b) If additionally

$$\int_{\mathbb{R}} |f(u)| \sum_{k\in\mathbb{Z}} \mathbf{E}|f(T_k + u)|\,\mathrm{d}u < \infty \tag{4.42}$$

hold, and we denote by

$$\widehat{\gamma}_n(j) = \frac{1}{n}\sum_{k=1}^{n-j}(Y_k - \overline{Y}_n)(Y_{k+j} - \overline{Y}_n), \quad j = 0, 1, \ldots, n-1,$$

the sample autocovariance, then we have for each $h \in \mathbb{N}$

$$\sqrt{n}(\widehat{\gamma}_n(0) - \gamma(0), \ldots, \widehat{\gamma}_n(h) - \gamma(h))' \xrightarrow{d} N(0, \mathbf{Z}), \quad n \to \infty,$$

where $\mathbf{Z}$ as defined in (a).

(c) Let $\rho_n^*(p) = \gamma_n^*(p)/\gamma_n^*(0)$ and $\widehat{\rho}_n(p) = \widehat{\gamma}_n(p)/\widehat{\gamma}_n(0)$ for $p \in \mathbb{N}$. Suppose that $f \neq 0$ λ-a.e. Then, under the assumptions of (a), we have for each $h \in \mathbb{N}$

$$\sqrt{n}(\rho_n^*(1) - \rho(1), \ldots, \rho_n^*(h) - \rho(h))' \xrightarrow{d} N(0, \mathbf{W}), \quad n \to \infty,$$

where $\mathbf{W} = (\mathbf{W}_{pq})_{p,q=1,\ldots,h} \in \mathbb{R}^{h \times h}$ is given by

$$\mathbf{W}_{pq} = (\mathbf{Z}_{pq} - \rho(p)\mathbf{Z}_{0q} - \rho(q)\mathbf{Z}_{p0} + \rho(p)\rho(q)\mathbf{Z}_{00})/\gamma(0)^2.$$

If additionally (4.42) is satisfied, then it also holds

$$\sqrt{n}(\widehat{\rho}_n(1) - \rho(1), \ldots, \widehat{\rho}_n(h) - \rho(h))' \xrightarrow{d} N(0, \mathbf{W}), \quad n \to \infty.$$

Proof. (a) By Proposition 4.19, we have

$$\sqrt{n}(\gamma_n^{*,m}(0) - \gamma^m(0), \ldots, \gamma_n^{*,m}(h) - \gamma^m(h))' \xrightarrow{d} \mathbf{V}_m \quad \text{as } n \to \infty.$$

Here $\mathbf{V}_m \stackrel{\mathrm{d}}{=} N(0, \mathbf{Z}^m)$, where $\mathbf{Z}^m = (\mathbf{Z}^m_{pq})_{p,q=0,\ldots,h} \in \mathbb{R}^{h+1 \times h+1}$ is given as in Proposition 4.19.

By Proposition 4.18, $\lim_{m\to\infty} \mathbf{Z}^m = \mathbf{Z}$, where Proposition 4.15 gives the form and finiteness of $\mathbf{Z}_{pq}$, the entries of $\mathbf{Z}$. Henceforth,

$$\mathbf{V}_m \xrightarrow{d} \mathbf{V}, \quad m \to \infty,$$

where $V \stackrel{\mathrm{d}}{=} N(0, \mathbf{Z})$.

By Theorem 1.12, the claim will follow if we can show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(n^{1/2}|\gamma_n^{*,m}(p) - \gamma^m(p) - \gamma_n^*(p) + \gamma(p)| > \varepsilon) = 0 \quad \forall \varepsilon > 0, \quad p \in \{0, \ldots, h\}.$$

Since $\mathbf{E}(\gamma_n^{*,m}(p)) = \gamma^m(p)$ and $\mathbf{E}(\gamma_n^*(p)) = \gamma(p)$, this will follow from Chebychef's inequality if we can show that

$$\begin{aligned}\lim_{m\to\infty} \lim_{n\to\infty} \mathbf{Var}(n^{1/2}(\gamma_n^*(p) - \gamma_n^{*,m}(p)) &= \lim_{m\to\infty} \lim_{n\to\infty} \Big[n\mathbf{Var}(\gamma_n^*(p)) + n\mathbf{Var}(\gamma_n^{*,m}(p)) \\ &\quad - 2n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^{*,m}(p))\Big] = 0 \quad \forall\, p \in \{0, \ldots, h\}.\end{aligned}$$

But since

$$\lim_{m\to\infty} \lim_{n\to\infty} n\mathbf{Var}(\gamma_n^{*,m}(p)) = \lim_{n\to\infty} n\mathbf{Var}(\gamma_n^*(p)) = \mathbf{Z}_{pp},$$

by Proposition 4.18, it remains only to show that

$$\lim_{m\to\infty} \lim_{n\to\infty} n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^{*,m}(p)) = \mathbf{Z}_{pp} \quad \forall\, p \in \{0, \ldots, h\}. \tag{4.43}$$

In doing so, denote $G_m(s,t) := \int_{\mathbb{R}} f(u+s)f_m(u+t)\,\mathrm{d}u$ and $F_m(s,t) := \int_{\mathbb{R}} f_m(u+s)f_m(u+t)\,\mathrm{d}u$. Observe first that from Lemma 4.12 (b), similar to the proof of Lemma 4.14 (a), by conditioning on T_p, T_k, and T_{k+p}, that for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\mathbf{Cov}(Z_{p,0}, Z_{p,k}^{(m)}) &= \mathbf{E}(Y_0 Y_p Y_k^{(m)} Y_{k+p}^{(m)}) - \mathbf{E}(Y_0 Y_p)\mathbf{E}(Y_k^{(m)} Y_{k+p}^{(m)}) \\ &= (\eta-3)\sigma^4 \mathbf{E}\bigg(\int_{\mathbb{R}} f(u) f(u+T_p) f_m(u+T_k) f_m(u+T_{k+p})\,\mathrm{d}u \bigg) \\ &\quad + \sigma^4 \mathbf{E}(F(0,T_p)F_m(T_k,T_{k+p})) + \sigma^4 \mathbf{E}(G_m(0,T_k)G_m(T_p,T_{k+p})) \\ &\quad + \sigma^4 \mathbf{E}(G_m(0,T_{k+p})G_m(T_p,T_k)) - \sigma^4 \mathbf{E}(F(0,T_p))\mathbf{E}(F_m(T_k,T_{k+p}))\end{aligned}$$

Further, as in the proof of Lemma 4.14 (b), it follows that

$$\mathbf{E}(F(0,T_p)F_m(T_k,T_{k+p})) = \mathbf{E}(F(0,T_p))\mathbf{E}(F_m(T_k,T_{k+p})) \quad \text{when } |k| \geq p\,.$$

Denoting

$$\begin{aligned}\kappa_{f,f_m}(p,k,k+p) &:= (\eta-3)\sigma^4 \mathbf{E}\bigg(\int_{\mathbb{R}} f(u) f(u+T_p) f_m(u+T_k) f_m(u+T_{k+p})\,\mathrm{d}u \bigg) \\ &\quad + \sigma^4 \mathbf{E}(G_m(0,T_k)G_m(T_p,T_{k+p})) + \sigma^4 \mathbf{E}(G_m(0,T_{k+p})G_m(T_p,T_k))\,,\end{aligned}$$

we hence have

$$\mathbf{Cov}(Z_{p,0}, Z_{p,k}^{(m)}) = \begin{cases} \kappa_{f,f_m}(p,k,k+p)\,, & |k| \geq p\,, \\ \kappa_{f,f_m}(p,k,k+p) + \sigma^4 \mathbf{Cov}(F(0,T_p)F_m(T_k,T_{k+p}))\,, & |k| < p\,. \end{cases}$$

Next, observe that as in the proof of Proposition 4.18, since $|f_m| \leq |f|$, for all $k \in \mathbb{Z}$,

$$\begin{aligned}|\kappa_{f,f_m}(p,k,k+p)| &\leq \kappa_{|f|}(p,k,k+p) \quad \forall\, m \in \mathbb{N}\,, \\ \kappa_{f,f_m}(p,k,k+p) &\to \kappa_f(p,k,k+p) \quad \text{as } \; m \to \infty\,, \\ \mathbf{Cov}(F(0,T_p)F_m(T_k,T_{k+p})) &\to \mathbf{Cov}(F(0,T_p)F(T_k,T_{k+p})) \quad \text{as } \; m \to \infty\,.\end{aligned}$$

By stationarity, we obtain for $n \geq p$

$$\begin{aligned}n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^{*,m}(p)) &= \frac{1}{n}\sum_{i,j=1}^{n} \mathbf{Cov}(Z_{p,0}, Z_{p,j-i}^{(m)}) = \sum_{k=-n+1}^{n-1} \frac{n-|k|}{n} \mathbf{Cov}(Z_{p,0}, Z_{p,k}^{(m)}) \\ &= \sum_{k=-n+1}^{n-1} \frac{n-|k|}{n} \kappa_{f,f_m}(p,k,k+p) + \sigma^4 \sum_{k=-p+1}^{p-1} \frac{n-|p|}{n} \mathbf{Cov}(F(0,T_p)F_m(T_k,T_{k+p}))\,.\end{aligned}$$

Applying Lebesgues dominated convergence theorem once then gives

$$\begin{aligned}&\lim_{n\to\infty} n\mathbf{Cov}(\gamma_n^*(p), \gamma_n^{*,m}(p)) \\ &\qquad = \sum_{k=-\infty}^{\infty} \kappa_{f,f_m}(p,k,k+p) + \sigma^4 \sum_{k=-p+1}^{p-1} \mathbf{Cov}(F(0,T_p)F_m(T_k,T_{k+p}))\,,\end{aligned}$$

and applying it a second time gives

$$\lim_{m\to\infty}\lim_{n\to\infty} n\mathbf{Cov}(\gamma_n^*(p),\gamma_n^{*,m}(p))$$
$$= \sum_{k=-\infty}^{\infty}\kappa_f(p,k,k+p)+\sigma^4\sum_{k=-p+1}^{p-1}\mathbf{Cov}(F(0,T_p)F(T_k,T_{k+p}))\,,$$

which is (4.43). This finishes the proof of (a).

(b) This follows if we can show that $\sqrt{n}|\gamma_n^*(p)-\widehat{\gamma}_n(p)|\to 0$ in probability for $n\to\infty$ and $p\in\{0,\dots,h\}$. The latter can be done in exactly the same way as in the proof of Proposition 7.3.4 in Brockwell and Davis [23] with X replaced by Y in connection with the observation that, by Theorem 4.7, $\sqrt{n}\overline{Y}_n$ converges in distribution to a normal random variable as $n\to\infty$, and hence $\overline{Y}_n$ must converges to 0 in probability as $n\to\infty$.

(c) Follows readily as in the proof of Theorem 7.2.1 in Brockwell and Davis [23]. □

Remark 4.21. *(a) Due to the form of* $\mathbf{Z}$*, there seems to be no simplification for* $\mathbf{W}$ *possible. Also observe that* $\mathbf{W}$ *in general depends on* η *as seen in Theorem 3.5 (c) of Cohen and Lindner [31].*
(b) Part (a) of Theorem 4.20 in particular applies if $|f(u)|\le K(|u|^{-\alpha}\wedge 1)$ *for some* $K>0$ *and* $\alpha>1/2$ *which can be seen by Remark 4.17.*
(c) Similarly, part (b) of Theorem 4.20 applies if $|f(u)|\le K(|u|^{-\alpha}\wedge 1)$ *for some* $K>0$ *and* $\alpha>1$ *as shown in Remark 4.9.*

4.4 An Application to Parameter Estimation of the Ornstein-Uhlenbeck Process

In this section, we present a parameter estimation of a Lévy driven Ornstein-Uhlenbeck (OU) process sampled by a Poisson process. An OU process is a moving average process $X=(X_t)_{t\in\mathbb{R}}$ with kernel function $f\colon\mathbb{R}\to\mathbb{R}, s\mapsto \mathrm{e}^{-as}\mathbf{1}_{[0,\infty)}(s)$ where the parameter $a>0$. This yields

$$X_t=\int_{-\infty}^t \mathrm{e}^{-a(t-s)}\,\mathrm{d}L_s\,,$$

where $L=(L_t)_{t\in\mathbb{R}}$ is a Lévy process with zero mean and $\sigma^2=\mathbf{E}(L_1^2)<\infty$. We define $Y_n:=X_{T_n}$, $n\in\mathbb{Z}$, where $(T_n)_{n\in\mathbb{Z}}$ is given by (4.2) with $W=(W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ a sequence of i.i.d. random variables independent of L and such that $W_1\sim\mathrm{Exp}(\lambda)$, $\lambda>0$.

By Proposition 4.1, $Y=(Y_n)_{n\in\mathbb{Z}}$ is strictly stationary, $\mathbf{E}(Y_0)=0$, and $\mathbf{E}(Y_0^2)<\infty$. Then we obtain, by Proposition 4.6,

$$\begin{aligned}\gamma(h)=\mathbf{E}(Y_0Y_h)&=\sigma^2\int_{\mathbb{R}}f(u)\mathbf{E}(f(T_h+u))\,\mathrm{d}u\\&=\sigma^2\int_0^\infty \mathrm{e}^{-au}\int_0^\infty \mathrm{e}^{-a(t+u)}\frac{\lambda^h}{\Gamma(h)}t^{h-1}\mathrm{e}^{-\lambda t}\,\mathrm{d}t\,\mathrm{d}u\end{aligned}$$

$$= \sigma^2 \int_0^\infty \mathrm{e}^{-2au}\,\mathrm{d}u \frac{\lambda^h}{(a+\lambda)^h} \int_0^\infty \frac{(a+\lambda)^h}{\Gamma(h)} t^{h-1}\mathrm{e}^{-(a+\lambda)t}\,\mathrm{d}t = \frac{\sigma^2}{2a}\left(\frac{\lambda}{a+\lambda}\right)^h$$

as the autocovariance function of the process Y. Further $\gamma(0) = \sigma^2/2a$, and henceforth for the autocorrelation function $\rho(h) = \left(\frac{\lambda}{a+\lambda}\right)^h$. In particular, $\rho(1) = \frac{\lambda}{\lambda+a}$ and hence

$$a = \lambda\left(\frac{1}{\rho(1)} - 1\right). \tag{4.44}$$

Since we have given a central limit theorem for the autocorrelation function in Section 4.3, we define an estimator a^* for a for a known parameter λ of the distribution of W by means of

$$a^* = \lambda\left(\frac{1}{\rho^*(1)} - 1\right),$$

where $\rho^*(1) = \gamma^*(1)/\gamma^*(0)$ with $\gamma^*(h) = \frac{1}{n}\sum_{k=1}^n Y_k Y_{k+h}$. We can then give the following theorem.

Theorem 4.22. *Let $L = (L_t)_{t\geq 0}$ be a Lévy process with mean zero, $\sigma^2 = \mathbf{E}(L_1^2)$ and $\eta = \sigma^{-4}\mathbf{E}(L_1^4)$, and $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$, $(T_n)_{n\in Z}$ be defined as in (4.2) with $W = (W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ an i.i.d. sequence such that $W_1 \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$, and $X_t = \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s$ with $f(s) := \mathbf{1}_{[0,\infty)}(s)\mathrm{e}^{-as}$. Let $a^* = \lambda\left(\frac{1}{\rho^*(1)} - 1\right)$. Then*

$$\sqrt{n}(a^* - a) \xrightarrow{d} N\left(0, \frac{(\lambda+a)^4}{\lambda^2}\mathbf{W}_{11}\right), \quad n \to \infty,$$

where

$$\mathbf{W}_{11} = \left(\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2}\right)((\eta-3)a+3) + \frac{2a}{\lambda+2a}. \tag{4.45}$$

Proof. As usual, $Y_k := X_{T_k}$, $k \in \mathbb{Z}$, and $Z_{k,h} := Y_k Y_{k+h}$, $k \in \mathbb{Z}$, $\gamma_n^*(h) = \frac{1}{n}\sum_{k=1}^n Z_{k,h}$, $h = 0,\dots,n-1$ and hence $\rho_n^*(1) = \gamma_n^*(1)/\gamma_n^*(0)$. Observe that $P(W_1 > 0) > 0$, since W is exponentially distributed and it has positive support. Further, $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ is obvious as is $\int_{\mathbb{R}} |f(s)|^4(\log^+|f(s)|)^2\,\mathrm{d}s < \infty$, and, since clearly $f(s) \leq K(|s|^{-\alpha} \wedge 1)$ for some $K > 0$ and $\alpha > 1/2$, it follows, by Remark 4.17, that (4.33) and (4.34) are satisfied. Therefore, by Theorem 4.20 (c), we have

$$\sqrt{n}(\rho_n^*(1) - \rho(1)) \xrightarrow{d} N(0, \mathbf{W}_{11}), \quad n \to \infty,$$

where

$$\mathbf{W}_{11} = (\mathbf{Z}_{11} - 2\rho(1)\mathbf{Z}_{01} + \rho(1)^2\mathbf{Z}_{00})/\gamma(0)^2 = \frac{4a^2}{\sigma^4}\left(\mathbf{Z}_{11} - 2\frac{\lambda}{a+\lambda}\mathbf{Z}_{01} + \left(\frac{\lambda}{a+\lambda}\right)^2\mathbf{Z}_{00}\right)$$

with

$$\mathbf{Z}_{pq} = \sigma^4 \sum_{k=-q+1}^{p-1} \mathbf{Cov}(F(0,T_p), F(T_k, T_{k+q})) + \sum_{k\in\mathbb{Z}} \kappa(p,k,k+q)\,.$$

Thus, under our assumptions on the distribution of W, an easy but tedious calculation yields that $\mathbf{W}_{11}$ is given by (4.45), cf. Lemma A.16.

To complete the proof, define $g\colon \mathbb{R} \to \mathbb{R}, x \mapsto \lambda\left(\frac{1}{x} - 1\right)$ with $g'(x) = -\frac{\lambda}{x^2}$ such that $g(\rho_n^*(1)) = a^*$ and the delta-method, cf. Proposition 6.4.3 in Brockwell and Davis [23], yields

$$\sqrt{n}(a^* - a) \xrightarrow{d} N(0, g'(\rho(1))\mathbf{W}_{11} g'(\rho(1)))\,, \quad n \to \infty\,,$$

where

$$g'(\rho(1)) = -\frac{\lambda}{\rho(1)^2} = -\frac{(\lambda+a)^2}{\lambda}\,.$$

□

Let us now consider the case when the parameter λ of $W_1 \sim \mathrm{Exp}(\lambda)$ is additionally unknown. Since, in addition to the observations $Y_1, \dots, Y_{n+1}$, we also have the observation times $T_1, \dots, T_{n+1}$, we also observe the waiting times $W_i = T_i - T_{i-1}$, $i = 1, \dots, n+1$, and hence can define

$$\widehat{\lambda} := \left(\frac{1}{n}\sum_{k=1}^{n} W_{k+1}\right)^{-1},$$

which by the strong law of large numbers is a strongly consistent estimator for λ, since $\mathbf{E}(W_1) = \lambda^{-1}$.

By (4.44), this suggests the estimator

$$\widehat{a} = \widehat{\lambda}\left(\frac{1}{\rho^*(1)} - 1\right).$$

Since $\rho^*(1)$ and $\widehat{\lambda}$ are consistent estimators, so is $\widehat{a}$. The asymptotic normality of $\widehat{a}$ is given in the following theorem.

Theorem 4.23. *Let $L = (L_t)_{t\ge 0}$ be a Lévy process with mean zero, $\sigma^2 = \mathbf{E}(L_1^2)$, $\eta = \sigma^{-4}\mathbf{E}(L_1^4)$, and $\mathbf{E}(|L_1|^4(\log^+|L_1|)^2) < \infty$. Assume that $(T_n)_{n\in Z}$ is defined as in (4.2) with $W = (W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ an i.i.d. sequence such that $W_1 \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$, and $X_t = \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s$ with $f(s) := \mathbf{1}_{[0,\infty)}(s)\mathrm{e}^{-as}$. Set $Y_k := X_{T_k}$, $k \in \mathbb{Z}$, and let $\rho^*(1)$ as before. Then*

$$\sqrt{n}(\widehat{a} - a) \xrightarrow{d} N\left(0, \frac{(\lambda+a)^4}{\lambda^2}\mathbf{W}_{11} - a^2\right), \quad n \to \infty\,,$$

where $\mathbf{W}_{11}$ is given by (4.45).

Proof. For $m \in \mathbb{N}$ define $f_m := f\mathbf{1}_{[-m/2,m/2]}$ and $Y_n^{(m)} := \int_{\mathbb{R}} f_m(t-s)\,\mathrm{d}L_s$. Then the sequences $(Y_n^2, Y_nY_{n+1}, T_{n+1}-T_n)_{n\in\mathbb{Z}}$ and $((Y_n^{(m)})^2, Y_n^{(m)}Y_{n+1}^{(m)}, T_{n+1}-T_n)_{n\in\mathbb{Z}}$ are both strictly stationary by Proposition 4.1 and the latter is also strongly mixing with exponentially decreasing mixing coefficients by Proposition 4.2.

Proceeding then exactly as in the proof of Proposition 4.19 and Theorem 4.20, i.e. establishing first a central limit theorem for the by m truncated quantities and then letting m tend to infinity, shows that

$$\sqrt{n}\left(\left(\gamma_n^*(0), \gamma_n^*(1), \frac{1}{n}\sum_{k=1}^{n} W_{k+1}\right) - \left(\gamma(0), \gamma(1), \frac{1}{\lambda}\right)\right) \xrightarrow{d} N(0, \Sigma)\,, \quad n \to \infty\,,$$

where

$$\Sigma = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \mathbf{Cov}(Y_0^2, Y_k^2) & \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) & \mathbf{Cov}(Y_0^2, T_{k+1}-T_k) \\ \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) & \mathbf{Cov}(Y_0Y_1, Y_kY_{k+1}) & \mathbf{Cov}(Y_0Y_1, T_{k+1}-T_k) \\ \mathbf{Cov}(Y_0^2, T_{k+1}-T_k) & \mathbf{Cov}(Y_0Y_1, T_{k+1}-T_k) & \mathbf{Cov}(T_1, T_{k+1}-T_k) \end{pmatrix}. \quad (4.46)$$

An easy but tedious calculation, cf. Lemma A.17, then shows that

$$\Sigma = \begin{pmatrix} \mathbf{Z}_{00} & \mathbf{Z}_{01} & 0 \\ \mathbf{Z}_{10} & \mathbf{Z}_{11} & -\frac{\sigma^2}{2(\lambda+a)^2} \\ 0 & -\frac{\sigma^2}{2(\lambda+a)^2} & \frac{1}{\lambda^2} \end{pmatrix} \quad (4.47)$$

with $\mathbf{Z}_{00}$, $\mathbf{Z}_{01}$, and $\mathbf{Z}_{11}$ as in the proof of Theorem 4.22.

To complete, we define $g : \mathbb{R}^3 \to \mathbb{R}, (x_1, x_2, x_3) \mapsto \frac{1}{x_3}(\frac{x_1}{x_2} - 1)$ such that

$$\widehat{a} = g\left(\gamma_n^*(0), \gamma_n^*(1), \frac{1}{n}\sum_{k=1}^{n} W_{k+1}\right) = \widehat{\lambda}\left(\frac{1}{\rho_n^*(1)} - 1\right).$$

Henceforth, by the delta-method, cf. Proposition 6.4.3 in Brockwell and Davis [23], we obtain with $\mu := (\gamma(0), \gamma(1), \frac{1}{\lambda})$

$$\sqrt{n}(\widehat{a} - a) \xrightarrow{d} N(0, (\nabla g(\mu))\Sigma(\nabla g(\mu))')\,, \quad n \to \infty\,,$$

where

$$\nabla g(x) = (\partial g/\partial x_1, \partial g/\partial x_2, \partial g/\partial x_3) = (1/(x_2x_3), -x_1/(x_2^2x_3), 1/x_3^2(1 - x_1/x_2))$$

such that, by a straightforward calculation,

$$(\nabla g(\mu))\Sigma(\nabla g(\mu))' = \frac{(\lambda+a)^4}{\lambda^2}\mathbf{W}_{11} - a^2$$

and the result follows. □

Remark 4.24. *Note that the shrinking phenomenon observed in the asymptotic variance of the estimator $\widehat{a}$ with respect to the asymptotic variance of a^* depends on the non zero asymptotic covariance between the sample autocovariance $\gamma_n^*(1)$ and the estimator $\widehat{\lambda}$.*

Next, we compare our results to an equidistant sampling method, more precisely to the one of Theorem 1.35. Sampling at equidistant times $\Delta, 2\Delta, \ldots, n\Delta$ for $\Delta > 0$, leads to an autocovariance function

$$\gamma_{eq}(h) = \mathbf{E}(X_0 X_h) = \sigma^2 \int_{\mathbb{R}} f(u) f(u+h)\,\mathrm{d}u = \frac{\sigma^2}{2a}\mathrm{e}^{-ah}\,, \quad h > 0\,,$$

from which we conclude that $\rho_{eq}(\Delta) = \gamma_{eq}(\Delta)/\gamma(0) = \mathrm{e}^{-a\Delta}$ and hence

$$a = -\frac{\log(\rho_{eq}(\Delta))}{\Delta}\,. \tag{4.48}$$

For an estimator of $\rho_{eq}(\Delta)$, i.e. for $\rho_{eq}^*(\Delta) = \gamma_{eq;n;\Delta}^*(\Delta)/\gamma_{eq;n;\Delta}^*(0)$, where

$$\gamma_{eq;n;\Delta}^*(h\Delta) = \frac{1}{n}\sum_{t=1}^{n} X_{t\Delta} X_{(t+h)\Delta}\,, \quad h \in \mathbb{N}\,,$$

we suggest, given a central limit theorem for $\rho_{eq}^*(\Delta)$, as an estimator for a from (4.48)

$$\widehat{a}_{eq} := -\frac{\log(\rho_{eq}^*(\Delta))}{\Delta}\,.$$

By Theorem 3.5 of Cohen and Lindner [31], cf. Theorem 1.35, we have

$$\sqrt{n}(\rho_{eq}^*(\Delta) - \rho_{eq}(\Delta)) \xrightarrow{d} N(0, V)\,, \quad n \to \infty\,,$$

where

$$\begin{aligned} V = &\frac{(\eta-3)\sigma^4}{\gamma_{eq}(0)^2}\int_0^{\Delta}(g_{1;\Delta}(u) - \rho(\Delta) g_{0;\Delta})^2\,\mathrm{d}u \\ &+ \sum_{k=1}^{\infty}(\rho((k+1)\Delta) + \rho((k-1)\Delta) - 2\rho(\Delta)\rho(k\Delta))^2 \end{aligned}$$

with

$$g_{q;\Delta}\colon [0,\Delta] \to \mathbb{R}\,,\ u \mapsto \sum_{k=-\infty}^{\infty} f(u+k\Delta) f(u+(k+q)\Delta)$$

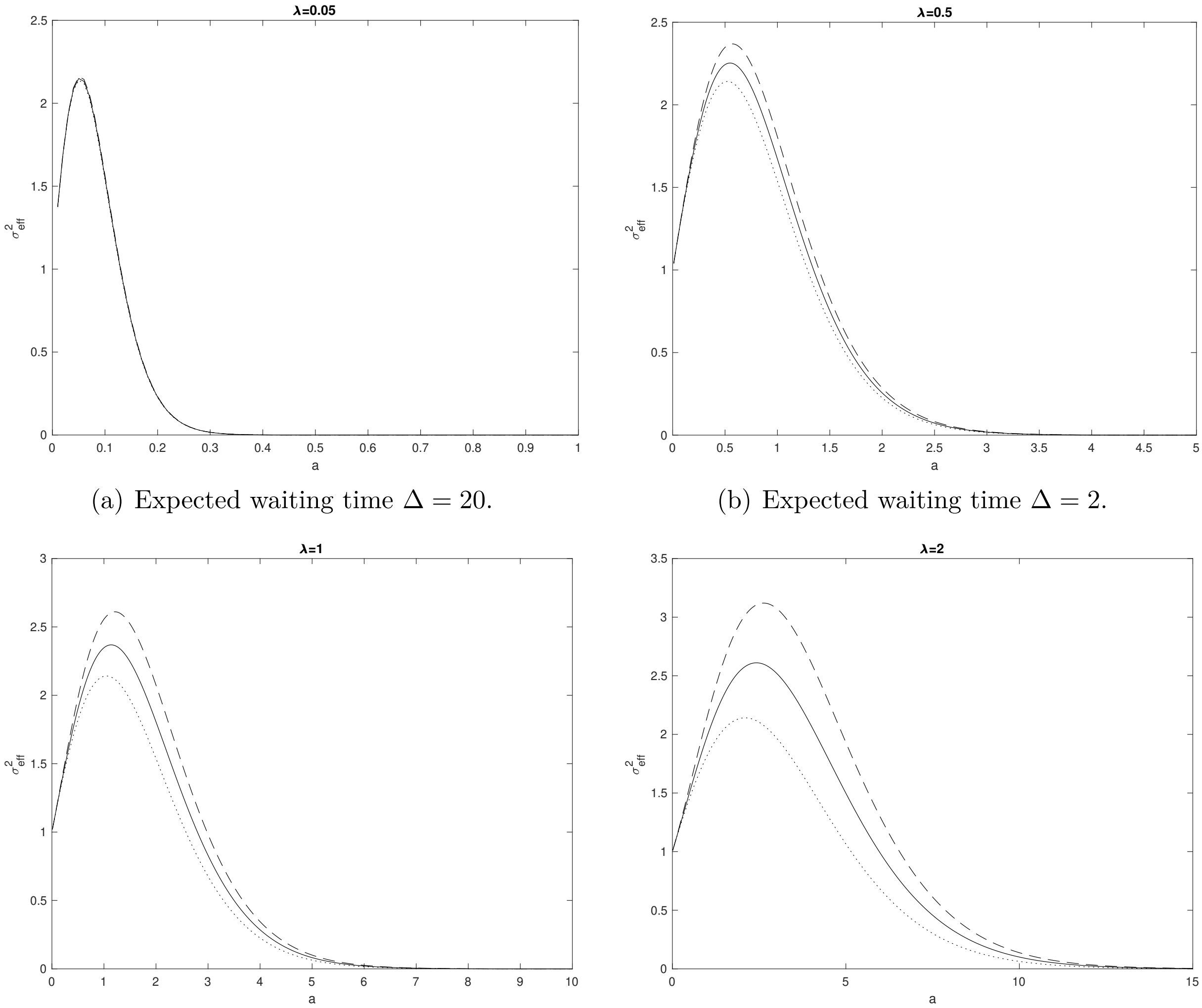
given as in Proposition 3.1 of Cohen and Lindner [31]. A simple calculation of V given the Ornstein-Uhlenbeck kernel and its autocorrelation function and an application of the delta-method, cf. Lemma A.18, leads to

$$\sqrt{n}(\widehat{a}_{eq} - a) \xrightarrow{d} N(0, \Delta^{-2}(\mathrm{e}^{2a\Delta} - 1))\,, \quad n \to \infty\,.$$

Having $W_1 \sim \text{Exp}(\lambda)$, we have an expected waiting time of $\mathbf{E}(T_i - T_{i-1}) = \frac{1}{\lambda}$ in the random sampling in comparison to an deterministic waiting time of Δ in the equidistant sampling. This suggest to compare the variance of the asymptotic distributions of the estimators $\widehat{a}$ and $\widehat{a}_{eq}$ by choosing $\lambda = \frac{1}{\Delta}$ or $\Delta = \frac{1}{\lambda}$, respectively, such that the expected waiting time of the random sampling and of the deterministic sampling agree. In doing so, we compare the efficiency of $\widehat{a}$ and $\widehat{a}_{eq}$ depending on λ and a by plotting relative variance σ^2_{eff} given by

$$\sigma^2_{eff} = \frac{\frac{(\lambda+a)^4}{\lambda^2}\mathbf{W}_{11} - a^2}{\lambda^2(\mathrm{e}^{2a\frac{1}{\lambda}} - 1)},$$

where $\mathbf{W}_{11}$ given as in (4.45).



(a) Expected waiting time $\Delta = 20$.

(b) Expected waiting time $\Delta = 2$.

(c) Expected waiting time $\Delta = 1$.

(d) Expected waiting time $\Delta = 0.5$.

Figure 4.1: σ^2_{eff} depending on a and λ in case of $\eta = 3, 4, 5$.

We plot in Figure 4.1 the relative efficiency σ^2_{eff} with respect to the mean reverting parameter a. The dotted line belongs to $\eta = 3$, the solid line to $\eta = 4$ and the dashed line to $\eta = 5$.

From the analytic form of σ^2_{eff}, we see that for $a \to \infty$, the denominator becomes big and therefore the ratio tends to zero. Depending on λ, we see this behavior in the graphs already for small a.

Considering the Taylor expansions of order 2 of $f(a) = \frac{(\lambda+a)^4}{\lambda^2}\mathbf{W}_{11} - a^2$ and $g(a) = \lambda^2(\mathrm{e}^{2a\frac{1}{\lambda}} - 1)$ at 0, we obtain

$$f(a) \approx 2\lambda a + 6a^2 + o(a^2)$$
$$g(a) \approx 2\lambda a + 2a^2 + o(a^2)$$

showing that the denominator tends faster to 0 for $a \downarrow 0$ as the numerator, which is reflected in the graph as $\sigma^2_{eff} \downarrow 1$ for $a \downarrow 0$.

Summarizing, for small a, the equidistant estimator $\widehat{a}_{eq}$ is more efficient unless λ is not chosen to be small. By the relation $\Delta = \frac{1}{\lambda}$, we see that the estimator $\widehat{a}$ becomes more efficient the lower the sampling frequency is.

λ \ η	3	4	5
0.05	0.1288	0.1294	0.1300
0.5	1.2878	1.3455	1.3983
1	2.5755	2.7965	2.9814
2	5.1509	5.9627	6.5465

Table 4.1: Values of a for which $\widehat{a}$ becomes more efficient depending on λ and η.

Table 4.1 shows, depending on λ and η, the smallest value of a for which $\sigma^2_{eff} \leq 1$. For values of a less than 2, the estimator based on an equidistant sampling is more efficient than $\widehat{a}$ unless the sampling frequency Δ is greater than 1.

We see that the non-equidistant sampling performs worse as the kurtosis of the driving Lévy process increases. The best scenario across all time scales is observed for $\eta = 3$ which corresponds to the Brownian motion case.

5 On sample mean central limit theorems of multivariate Lévy driven moving averages

Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided $\mathbb{R}^m$-valued Lévy process, i.e. a stochastic process with independent and stationary increments, càdlàg sample paths and $L_0 = 0$ almost surely, which is continuous in probability. We further assume that L has expectation zero and finite second moment, and let $f\colon \mathbb{R} \to \mathbb{R}^{d\times m}$ in $L^2(\mathbb{R}^{d\times m})$, i.e.

$$L^2(\mathbb{R}^{d\times m}) := \left\{ f\colon \mathbb{R} \to \mathbb{R}^{d\times m} \text{ measurable:} \int_{\mathbb{R}} \|f(s)\|^2 \, \mathrm{d}s < \infty \right\}$$

for some norm on $\mathbb{R}^{d\times m}$. For fixed $\mu \in \mathbb{R}^d$ the *multivariate continuous time moving average process with mean* μ *and kernel function* f *driven by* L denoted by $X = (X_t)_{t\in\mathbb{R}}$ can be defined in the L^2-sense by

$$X_t := \mu + \int_{\mathbb{R}} f(t-s)\, \mathrm{d}L_s \,, \quad t \in \mathbb{R} \,, \tag{5.1}$$

more precisely,

$$X_t = \mu + \int_{\mathbb{R}} f(t-s)\, \mathrm{d}L_s := \mu + \begin{pmatrix} \sum_{j=1}^m \int_{\mathbb{R}} f^{(1,j)}(t-s)\, \mathrm{d}L_s^{(j)} \\ \vdots \\ \sum_{j=1}^m \int_{\mathbb{R}} f^{(d,j)}(t-s)\, \mathrm{d}L_s^{(j)} \end{pmatrix} ,$$

where the integrals on the right-hand side exist since $f \in L^2(\mathbb{R}^{d\times m})$ and $\mathbf{E}\,\|L_1\|^2 < \infty$. X is then as its univariate counterpart strictly stationary.

(5.1) can be considered as continuous time analogue of the discrete time multivariate moving average process

$$\widetilde{X}_t = \mu + \sum_{k\in\mathbb{Z}} C_{t-k} Z_k \,, \quad t \in \mathbb{Z} \,, \tag{5.2}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an $\mathbb{R}^m$-valued independent and identically distributed (i.i.d.) noise with expectation zero and covariance matrix Σ, and $(C_k = (C_k^{(i,j)})_{i,j=1,\dots,d})_{k\in\mathbb{Z}}$ is a sequence of $\mathbb{R}^{d\times m}$-valued matrices such that $\sum_{k\in\mathbb{Z}} |C_k^{(i,j)}| < \infty$ for all $i = 1,\dots,d$ and $j = 1,\dots,m$. The asymptotic behavior of the sample mean of $\widetilde{X}_t$ in (5.2) and the autocorrelation in the special case of $\widetilde{X}_t$ in (5.2) being bivariate have been studied for example in Section 11 of Brockwell and Davis [23].

When X in (5.1) is observed on a lattice $\{\Delta t\colon t = 0, 1, 2, \dots\}$, the asymptotic behavior of the sample mean and the sample autocorrelation has been studied in various cases when L and f were assumed to be univariate. In particular, Cohen and Lindner [31], cf. Theorem 1.34 and 1.35 proved asymptotic normality of the sample mean and the sample autocorrelation under $\mathbf{E}(L_1^2) < \infty$ and $f \in L^2(\mathbb{R})$, and $\mathbf{E}(L_1^4) < \infty$ and $f \in L^4(\mathbb{R})$ plus some extra assumptions, respectively. Spangenberg [62] showed in the long memory case that under the assumption of $\mathbf{E}(L_1^4) < \infty$ for $f(t) \sim C_t t^{d-1}$ for $d \in (0, 1)$ and some constant C_d a central limit theorem where the limit distribution is Rosenblatt, and in case of a slowly varying Lévy process with index $\alpha \in (2, 4)$ that the limit distribution is either Rosenblatt or a stable distribution, depending on the interplay of d and α. Drapatz [34] proved for the sample autocovariance function when the Lévy process has infinite variance with regularly varying tails with index $\alpha \in (0, 2)$ that its limit distribution is a stable distribution whose parameters can be given in terms of the characteristics of the driving Lévy process.

In this chapter on the one hand we want to study the asymptotic behavior of the sample mean

$$
{}^{\Delta}\overline{X}_n := \frac{1}{n} \sum_{k=1}^{n} X_{k\Delta}\,, \quad n \to \infty\,,
$$

of the process X as defined in (5.1) when sampled at $(\Delta n)_{n\in\mathbb{N}}$, where $\Delta > 0$ is fixed, thus extending the results of Theorem 1.34 to a multivariate setting. And on the other hand, we study a renewal sampling of the process X, i.e. we select a sequence of increasing random times $(T_n)_{n\in\mathbb{Z}}$ such that $T_n \to \infty$ almost surely (abbreviated a.s.). More in detail, we assume that $W = (W_n)_{n\in\mathbb{Z}\setminus\{0\}}$ is an i.i.d. sequence of positive supported random variables independent of the driving Lévy process L and such that $P(W_1 > 0) > 0$. We then define $(T_n)_{n\in\mathbb{Z}}$ by

$$
T_0 := 0 \quad \text{and} \quad T_n := \begin{cases} \sum_{i=1}^{n} W_i\,, & n \in \mathbb{N}\,, \\ -\sum_{i=n}^{-1} W_i\,, & -n \in \mathbb{N}\,, \end{cases} \tag{5.3}
$$

and the sampled process $Y = (Y_n)_{n\in\mathbb{Z}}$ via

$$
Y_n := X_{T_n}\,, \quad n \in \mathbb{Z}\,. \tag{5.4}
$$

We thus extend the results of Section 4.2 on the asymptotic normality of the sample mean of a renewal sampled continuous time moving average process to a multivariate setting.

The chapter is organized as follows. In Section 5.1 we establish a central limit theorem for the sample mean of a multivariate continuous time moving average process when observed on a lattice. In Section 5.2 we establish a central limit theorem for the sample mean when the multivariate moving average process is sampled at a renewal sequence.

5.1 Asymptotic normality of the sample mean of an equidistant sampled MA

In this section, we establish a central limit theorem for the sample mean, i.e.

$$^{\Delta}\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_{k\Delta}\,,$$

for $n \to \infty$ when X is sampled at $(\Delta n)_{n\in\mathbb{N}}$ where $\Delta > 0$ is fixed. We define $\Gamma(\Delta h) = \mathbf{Cov}(X_0, X_{h\Delta}) = (\gamma^{(i,j)}(\Delta h))_{i=1,\dots,d,j=1,\dots,m}$ the autocovariance function of the $\mathbb{R}^d$-valued process X as in (5.1), where we denote with A' the transpose of a vector or matrix A. It can be shown that $\gamma^{(i,i)}(\Delta h)$ is again an autocovariance function, those of the i^{th}-component of the process X, cf. Section 11 of Brockwell and Davis [23]. Observe that

$$\begin{aligned}\gamma^{(i,j)}(\Delta h) &= \mathbf{E}(X_0^{(i)} X_{\Delta h}^{(j)}) - \mu^{(i)}\mu^{(j)} \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} \mathbf{E}\bigg(\int_{\mathbb{R}} f^{(i,k)}(-s)\,\mathrm{d}L_s^{(k)} \int_{\mathbb{R}} f^{(j,l)}(\Delta h - s)\,\mathrm{d}L_s^{(l)}\bigg) \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} \mathbf{Cov}(L_1^{(k)}, L_1^{(l)}) \int_{\mathbb{R}} f^{(i,k)}(s) f^{(j,l)}(\Delta h + s)\,\mathrm{d}s\,, \end{aligned} \tag{5.5}$$

where the last equality follows from the Itô isometry. Then

$$\Gamma(\Delta h) = \int_{\mathbb{R}} f(s)\Sigma_L f(s)'\,\mathrm{d}s$$

with Σ_L to be explained below.

An $\mathbb{R}^m$-valued Lévy processes $L = (L_t)_{t\geq 0}$ can be identified by its characteristic triplet $(\Sigma_L, \nu_L, \gamma_L)$ due to the Lévy-Khintchine formula, i.e. if μ denotes the infinitely divisible distribution of L_1, then its characteristic function is given by

$$\widehat{\mu}(z) = \exp\left[i\langle \gamma_L, z\rangle - \frac{1}{2}\langle z, \Sigma_L z\rangle + \int_{\mathbb{R}^m} (\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{\{|x|\leq 1\}}(x))\,\nu_L(\mathrm{d}x)\right]\,, \quad z \in \mathbb{R}^m\,.$$

Here, Σ_L is the Gaussian covariance, ν_L a measure on $\mathbb{R}^m$ which satisfies $\nu_L(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\,\nu_L(\mathrm{d}x) < \infty$, where $|\cdot|$ denotes the Euclidean norm, called the *Lévy measure*, and $\gamma_L \in \mathbb{R}$ some constant. We denote with $(\sigma_L^{(k,l)})^2$ the entries of the covariance matrix, i.e. the covariance between the l^{th} and k^{th} component of L.

For a detailed account on Lévy processes we refer to the book of Sato [61].

Theorem 5.1. *Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^m$-valued Lévy process with zero mean and finite covariance matrix Σ_L, let $\mu \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^{d\times m})$ such that $X = (X_t)_{t\in\mathbb{R}}$ is defined as in (5.1). Suppose that $\Delta > 0$ and*

$$\bigg(\widetilde{F}_\Delta\colon [0,\Delta] \to [0,\infty]\,, \quad u \mapsto \widetilde{F}_\Delta(u) = \sum_{h=-\infty}^{\infty} \|f(u + h\Delta)\| \bigg) \in L^2([0,\Delta])\,. \tag{5.6}$$

Then $\sum_{h=-\infty}^{\infty} \|\Gamma(\Delta h)\| < \infty$,

$$\begin{aligned}\sum_{h=-\infty}^{\infty} \Gamma(\Delta h) &= \int_0^{\Delta} F(s)\Sigma_L F(s)' \,\mathrm{d}s \\ &= \left(\sum_{k=1}^{m}\sum_{l=1}^{m} (\sigma_L^{(k,l)})^2 \int_0^{\Delta} F_{\Delta}^{(i,k)}(s) F_{\Delta}^{(j,l)}(s) \,\mathrm{d}s \right)_{i,j=1,\dots,d},\end{aligned} \tag{5.7}$$

where $F_{\Delta}(u) = \sum_{h=-\infty}^{\infty} f(u + h\Delta)$, *and the sample mean is asymptotically normal, i.e.*

$$\sqrt{n}({}^{\Delta}\overline{X}_n - \mu) \xrightarrow{d} N\left(0, \int_0^{\Delta} F(s)\Sigma_L F(s)' \,\mathrm{d}s\right) \quad \text{as } n \to \infty\,.$$

Proof. For convenience we assume that $\Delta = 1$, and write $F = F_1$ and $\widetilde{F} = \widetilde{F}_1$, respectively. By (5.6), we have that the functions defined by

$$\widetilde{F}^{(i,j)}(u) := \sum_{h=-\infty}^{\infty} |f^{(i,j)}(u + h)|$$

are in $L^2([0, \Delta])$ as well for all $i = 1, \dots, d$ and all $j = 1, \dots, m$. We continue F and $\widetilde{F}^{(i,j)}$ for all $i = 1, \dots, d$ and $j = 1, \dots, m$, respectively, periodically on $\mathbb{R}$ by setting

$$F(u) = \sum_{h=-\infty}^{\infty} f(u + h)\,, \quad \text{and} \quad \widetilde{F}^{(i,j)}(u) = \sum_{h=-\infty}^{\infty} |f^{(i,j)}(u + h)|\,, \quad u \in \mathbb{R}\,.$$

By the equivalence of norms, w.l.o.g. we take the 1-norm, i.e. for a matrix $A \in \mathbb{R}^{d\times m}$ we have $\|A\| = \sum_{i=1}^{d}\sum_{j=1}^{m} |a^{(i,j)}|$. Then

$$\begin{aligned}\sum_{h=-\infty}^{\infty} \|\Gamma(h)\| &= \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{h=-\infty}^{\infty} |\gamma^{(i,j)}(h)| \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{h=-\infty}^{\infty} \left| \sum_{k=1}^{m}\sum_{l=1}^{m} (\sigma_L^{(k,l)})^2 \int_{\mathbb{R}} f^{(i,k)}(s) f^{(j,l)}(h+s) \,\mathrm{d}s \right| \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} |\sigma_L^{(k,l)}|^2 \int_{\mathbb{R}} |f^{(i,k)}(s)| \sum_{h=-\infty}^{\infty} |f^{(j,l)}(h+s)| \,\mathrm{d}s \\ &= \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} |\sigma_L^{(k,l)}|^2 \int_0^1 \sum_{h=-\infty}^{\infty} |f^{(i,k)}(s+h)| \widetilde{F}^{(j,l)}(s) \,\mathrm{d}s \\ &= \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} |\sigma_L^{(k,l)}|^2 \int_0^1 \widetilde{F}^{(i,k)}(s) \widetilde{F}^{(j,l)}(s) \,\mathrm{d}s \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} |\sigma_L^{(k,l)}|^2 \left(\int_0^1 (\widetilde{F}^{(i,k)})^2 \,\mathrm{d}s \int_0^1 (\widetilde{F}^{(j,l)})^2 \,\mathrm{d}s \right)^{1/2} \mathrm{d}s < \infty\,,\end{aligned} \tag{5.8}$$

where the second last inequality follows form the Cauchy-Schwarz inequality. Similar, we find for each component, by (5.5),

$$\begin{aligned}\sum_{h=-\infty}^{\infty}\gamma^{(i,j)}(h) &= \sum_{h=-\infty}^{\infty}\sum_{k=1}^{m}\sum_{l=1}^{m}(\sigma_L^{(k,l)})^2\int_{\mathbb{R}} f^{(i,k)}(s)f^{(j,l)}(h+s)\,\mathrm{d}s\\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}(\sigma_L^{(k,l)})^2\int_{\mathbb{R}} f^{(i,k)}(s)\sum_{h=-\infty}^{\infty} f^{(j,l)}(h+s)\,\mathrm{d}s\\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}(\sigma_L^{(k,l)})^2\int_0^1 \sum_{h=-\infty}^{\infty} f^{(i,k)}(s+h)F^{(j,l)}(s)\,\mathrm{d}s\\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}(\sigma_L^{(k,l)})^2\int_0^1 F^{(i,k)}(s)F^{(j,l)}(s)\,\mathrm{d}s\end{aligned}$$

such that

$$\sum_{h=-\infty}^{\infty}\Gamma(\Delta h) = \int_0^1 F(s)\Sigma_L F(s)'\,\mathrm{d}s$$

which is (5.7).

For the asymptotic normality, by subtracting the mean, we may assume without loss of generality that $\mu = 0$, and we use the Cramér-Wold theorem, i.e. prove

$$\sqrt{n}\lambda'^{\,1}\overline{X}_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda' X_k = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\sum_{i=1}^{d}\lambda_i X_k^{(i)} \xrightarrow{d} N\left(0, \lambda'\int_0^1 F(s)\Sigma_L F(s)'\,\mathrm{d}s\lambda\right)\ \forall\,\lambda\in\mathbb{R}^d\,,$$

i.e. we are considering

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\sum_{i=1}^{d}\lambda_i X_k^{(i)} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\sum_{i=1}^{d}\lambda_i\sum_{j=1}^{m}\int_{\mathbb{R}} f^{(i,j)}(k-s)\,\mathrm{d}L_s^{(j)}\,.$$

To do so, for $a\in\mathbb{N}$, we define $f_a(s) = (f^{(i,j)}(s)\mathbf{1}_{[-a,a]}(s))_{\substack{i=1,\dots,d\\ j=1,\dots,m}}$ such that

$$X_{t;a} := \int_{\mathbb{R}} f_a(t-s)\,\mathrm{d}L_s = \int_{t-a}^{t+a} f(t-s)\,\mathrm{d}L_s = \begin{pmatrix}\sum_{j=1}^m\int_{t-a}^{t-a} f^{(1,j)}(t-s)\,\mathrm{d}L_s^{(j)}\\ \vdots\\ \sum_{j=1}^m\int_{t-a}^{t-a} f^{(d,j)}(t-s)\,\mathrm{d}L_s^{(j)}\end{pmatrix},\quad t\in\mathbb{R}\,,$$

is a $2a$-dependent process in the sense that $(X_{t;a})_{t\le s}$ and $(X_{t;a})_{t\ge s+2a}$ are independent. Then also the sequence $(\lambda' X_{t;a})_{t\in\mathbb{R}}$ is strictly stationary and $2a$-dependent with zero mean and autocovariance function $\lambda'\Gamma_{f_a}(h)\lambda$ such that we can use a central limit theorem for $2a$-dependent sequences, cf. Brockwell and Davis [23] Theorem 6.4.2, to obtain for $\lambda'\overline{X}_n^{(a)} := \frac{1}{n}\sum_{k=1}^{n}\lambda' X_{k;a}$ that

$$\sqrt{n}\lambda'\overline{X}_n^{(a)} \xrightarrow{d} Y^{(a)}\,,\quad n\to\infty\,,\quad \text{with } Y^{(a)} \overset{d}{=} N(0,\nu_a)\,, \tag{5.9}$$

where

$$\nu_a = \sum_{h=-2a}^{2a} \lambda' \Gamma_{f_a}(h)\lambda = \sum_{h=-2a}^{2a} \sum_{i=1}^{d}\sum_{j=1}^{d} \lambda_i\lambda_j \gamma_{f_a}^{(i,j)}(h)$$
$$= \sum_{h=-2a}^{2a} \sum_{i=1}^{d}\sum_{j=1}^{d} \lambda_i\lambda_j \sum_{k=1}^{m}\sum_{l=1}^{m} (\sigma_L^{(k,l)})^2 \int_{\mathbb{R}} f_a^{(i,k)}(s) f_a^{(j,l)}(h+s)\,\mathrm{d}s\,.$$

Since for all $i = 1, \dots, d$ and $j = 1, \dots, m$ we have $|f_a^{(i,j)}(s)| \leq |f^{(i,j)}(s)|$ for all $a \in \mathbb{N}$, it follows by Lebesgue's dominated convergence theorem that

$$\lim_{a\to\infty} \int_{\mathbb{R}} f_a^{(i,k)}(s) f_a^{(j,l)}(h+s)\,\mathrm{d}s = \int_{\mathbb{R}} f^{(i,k)}(s) f^{(j,l)}(h+s)\,\mathrm{d}s \quad \forall i,j,k,l\,.$$

From this we conclude that, for all $i, j \in \{1, \dots, d\}$, $\lim_{a\to\infty} \gamma_{f_a}^{(i,j)}(h) = \gamma^{(i,j)}(h)$ for each $h \in \mathbb{Z}$, and hence, since $\sum_{h=-\infty}^{\infty} |\lambda'\Gamma(h)\lambda| < \infty$, by the calculation that led to (5.8), it follows again from Lebesgue's dominated convergence theorem that $\lim_{a\to\infty} v_a = \sum_{h=-\infty}^{\infty} \lambda'\Gamma(h)\lambda$. Hence, by (5.7),

$$Y^{(a)} \xrightarrow{d} Y\,, \quad a \to \infty\,, \quad \text{where} \quad Y \stackrel{d}{=} N\left(0, \lambda' \int_0^1 F(s)\Sigma_L F(s)'\,\mathrm{d}s\lambda\right). \tag{5.10}$$

Next since $|f^{(i,j)} - f_a^{(i,j)}| \leq |f^{(i,j)}|$ almost everywhere and $\lim_{a\to\infty} |f^{(i,j)} - f_a^{(i,j)}| = 0$ for all $i = 1, \dots, d$, $j = 1, \dots, m$, we obtain, by the dominated convergence theorem and a similar calculation as for ν_a, that $\lim_{a\to\infty} \gamma_{f-f_a}^{(i,j)}(h) = 0$ for each $h \in \mathbb{Z}$ and all $i = 1, \dots, d$, and $j = 1, \dots, m$. Hence, $\lim_{a\to\infty} \sum_{h=-\infty}^{\infty} \gamma_{f-f_a}^{(i,j)}(h) = 0$ for all $i = 1, \dots, d$, $j = 1, \dots, m$, by the calculation that led to (5.8) and the dominated convergence theorem. Then

$$\lim_{a\to\infty}\lim_{n\to\infty} \mathbf{Var}(\sqrt{n}(\lambda'\,{}^1\overline{X}_n - \lambda'\overline{X}_n^{(a)}))$$
$$= \lim_{a\to\infty}\lim_{n\to\infty} n\mathbf{Var}\left(\sum_{i=1}^{d} \lambda_i \frac{1}{n}\sum_{t=1}^{n}\sum_{k=1}^{m} \int_{\mathbb{R}} f^{(i,k)}(t-s) - f_a^{(i,k)}(t-s)\,\mathrm{d}L_s^{(k)}\right)$$
$$= \lim_{a\to\infty} \sum_{i=1}^{d}\sum_{j=1}^{d} \lambda_i\lambda_j \sum_{h=-\infty}^{\infty} \gamma_{f-f_a}^{(i,j)}(h) = 0\,,$$

where we have used Theorem 7.1.1 in Brockwell and Davis [23] for the second equality. Then we obtain

$$\lim_{a\to\infty}\limsup_{n\to\infty} P(\sqrt{n}|\lambda'\,{}^1\overline{X}_n - \lambda'\overline{X}_n^{(a)}| > 0) = 0 \quad \forall\varepsilon > 0\,,$$

by an application of Chebychef's inequality. This together with (5.9) and (5.10) gives the claim by an application of a variant of Slutsky's theorem, cf. Theorem 1.12. □

5.2 Sample mean of renewal sampled multivariate moving average processes

In this section, we show the asymptotic normality of the sample mean

$$\overline{Y}_n := \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} X_{T_k}, \quad n \in \mathbb{N}, \tag{5.11}$$

where $X = (X_t)_{t\in\mathbb{R}}$ and $Y = (Y_n)_{n\in\mathbb{Z}}$ are given in (5.1) and (5.3), respectively. Observe that, by Proposition 4.1, the process Y is strictly stationary.

To do so, we consider a certain truncated continuous time moving average process. Therefore, for $a \in \mathbb{N}$, let $f_a \colon \mathbb{R} \to \mathbb{R}^{d\times m}$, $s \mapsto f(s)\mathbf{1}_{[-a/2,a/2]}$ be a kernel function with compact support, and $X^{(a)} = (X_{t;a})_{t\in\mathbb{R}}$ be defined by

$$X_{t;a} := \mu + \int_{\mathbb{R}} f_a(t-s)\,\mathrm{d}L_s = \mu + \int_{\mathbb{R}} f(t-s)\mathbf{1}_{[-a/2,a/2]}(t-s)\,\mathrm{d}L_s, \quad t \in \mathbb{R}, \tag{5.12}$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process with zero mean and $\mathbf{E}\,\|L_1\|^2 < \infty$, $\mu \in \mathbb{R}$, and $f \in L^2(\mathbb{R}^{d\times m})$. Then the process $X^{(a)} = (X_{t;a})_{t\in\mathbb{R}}$ is an a-dependent process. Moreover, $X^{(a)}$ is strictly stationary and, by Proposition 4.1, so is the sequence $Y^{(a)} = (Y_{k;a})_{k\in\mathbb{Z}}$ defined by

$$Y_{k;a} = X_{T_k;a}, \tag{5.13}$$

where $(T_n)_{n\in\mathbb{Z}}$ is defined as in (5.2) independent of X.

Throughout this section, if not stated otherwise, we denote with $\|A\|$ the Euclidean norm of a matrix or vector A. Observe that $\|A\|$ is also called the Frobenius norm of a matrix $A \in \mathbb{R}^{d\times m}$. Observe further that then $\|AB\| \le \|A\|\,\|B\|$ for all $A \in \mathbb{R}^{d\times m}$ and $B \in \mathbb{R}^{m\times k}$, cf. Proposition 9.3.5 of Bernstein [11].

In the following theorem, we recall the multivariate extension of Theorem 2.7 in Rajput and Rosinski [59], which characterizes the continuous time moving average process.

Theorem 5.2. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process on $\mathbb{R}^m$ with characteristic triplet $(\gamma_L, \Sigma_L, \nu_L)$ and $f \colon \mathbb{R} \to \mathbb{R}^{d\times m}$ be a measurable function. Denote with $D_m := \{x \colon |x| \le 1\}$ the unit ball in $\mathbb{R}^m$. Then*

(a) f is L-integrable (i.e. integrable with respect to the Lévy process L) as a limit in probability in the sense of Rajput und Rosinski [59] if and only if

(i) $\displaystyle\int_{\mathbb{R}} \left\| f(s)\gamma_L + \int_{\mathbb{R}^m} f(s)x(\mathbf{1}_{D_d}(f(s)x) - \mathbf{1}_{D_m}(x))\,\nu_L(\mathrm{d}x) \right\| \mathrm{d}s < \infty$*,*

(ii) $\displaystyle\int_{\mathbb{R}} \|f(s)\Sigma_L f(s)'\|\ \mathrm{d}s < \infty$*, and*

(iii) $\displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}^m} (\|f(s)x\|^2 \wedge 1)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s < \infty$*.*

(b) If f is L-integrable, the distribution of $\int_{\mathbb{R}} f(s)\,\mathrm{d}L_s$ is infinitely divisible with characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ given by

$$\begin{aligned}
\gamma_{\text{int}} &= \int_{\mathbb{R}} f(s)\gamma_L + \int_{\mathbb{R}^m} f(s)x(\mathbf{1}_{D_d}(f(s)x) - \mathbf{1}_{D_m}(x))\,\nu_L(\mathrm{d}x)\,\mathrm{d}s\,, \\
\Sigma_{\text{int}} &= \int_{\mathbb{R}} f(s)\Sigma_L f(s)'\,\mathrm{d}s\,, \quad \textit{and} \\
\nu_{\text{int}}(B) &= \int_{\mathbb{R}} \int_{\mathbb{R}^m} \mathbf{1}_B(f(s)x)\,\nu_L(\mathrm{d}x)\,\mathrm{d}s \quad \textit{for all Borel sets } B \subset \mathbb{R}^d \setminus \{0\}.
\end{aligned}$$

Corollary 5.3. *With a simple calculation, one can show that, if L has expectation zero and finite second moment and $g \in L^2(\mathbb{R}^{d\times m})$, then the conditions (i), (ii), and (iii) of Theorem 5.2 (a) are satisfied and $\int_{\mathbb{R}} g(s)\,\mathrm{d}L_s$ is infinitely divisible with characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ as given in Theorem 5.2 (b).*

The next lemma shows that $\mathbf{E}(\|X_t\|^2 \log^+ \|X_t\|) < \infty$ for all $t \in \mathbb{R}$ when we impose certain conditions on the Lévy process L and the kernel f.

Lemma 5.4. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a multivariate moving average process, i.e. $X_t := \mu + \int_{\mathbb{R}} f(t-s)\,\mathrm{d}L_s$, where $f \in L^2(\mathbb{R}^{d\times m})$ and $L = (L_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^m$-valued Lévy process with zero mean. If $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$, and $\int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\|\,\mathrm{d}s < \infty$, then $\mathbf{E}(\|X_t\|^2 \log^+ \|X_t\|) < \infty$ for all $t \in \mathbb{R}$.*

Proof. W.l.o.g. $\mu = 0$. It is enough to show the assertions for $Z = \int_{\mathbb{R}} f(s)\,\mathrm{d}L_s$, for which $Z \stackrel{d}{=} \int_{\mathbb{R}} f(-s)\,\mathrm{d}L_s = X_0$. By the strict stationarity of X, we obtain the result.

By Corollary 5.3, Z is infinitely divisible with triplet $(\gamma_Z, \Sigma_Z, \nu_Z)$ given by Theorem 5.2 (b). By Theorem 25.3 and Proposition 25.4 of Sato [61], we know that $\mathbf{E}(\|Z\|^2 \log^+ \|Z\|) < \infty$, if it holds $\int_{\|x\|>1} \|x\|^2 \log^+ \|x\|\ \nu_Z(\mathrm{d}x) < \infty$. To see that this is indeed true, observe that $\log^+ |ab| \le \log^+ |a| + \log^+ |b|$ for $a, b \in \mathbb{R}$. Hence,

$$\begin{aligned}
\int_{\|x\|>1} \|x\|^2 \log^+ \|x\|\ \nu_Z(\mathrm{d}x) &\le \int_{\mathbb{R}} \int_{\mathbb{R}^m} \|f(s)x\|^2 \log^+ \|f(s)x\|\ \nu_L(\mathrm{d}x)\,\mathrm{d}s \\
&\le \int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\|\ \mathrm{d}s \int_{\mathbb{R}^m} \|x\|^2\ \nu_L(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}} \|f(s)\|^2\ \mathrm{d}s \int_{\mathbb{R}^m} \|x\|^2 \log^+ \|x\|\ \nu_L(\mathrm{d}x) < \infty\,,
\end{aligned}$$

since $\mathbf{E}\,\|L_1\|^2 < \infty$, $\int_0^\infty \|f(s)\|^2 \log^+ \|f(s)\|\ \mathrm{d}s < \infty$, $\int_{\mathbb{R}^m} \|x\|^2 \log^+ \|x\|\ \nu_L(\mathrm{d}x) < \infty$ (as a consequence of $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$, cf. Theorem 25.4 of Sato [61]), and $f \in L^2(\mathbb{R}^{d\times m})$, by assumption. □

The next proposition gives the asymptotic normality of the sample mean of $Y^{(a)}$, i.e. $\overline{Y}_n^{(a)} := \frac{1}{n}\sum_{k=1}^n Y_{k;a}$, as $n \to \infty$. We denote with $\mathbf{Cov}(X,Y) = \mathbf{E}(XY') - \mathbf{E}(X)\mathbf{E}(Y')$ the covariance of $\mathbb{R}^d$-valued, square-integrable random variables X and Y.

Proposition 5.5. *Let $X_{t;a}$ be defined as in (5.12), where $L = (L_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^m$-valued Lévy process with $\mathbf{E}(L_1) = 0$. Assume that $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$, $f \in L^2(\mathbb{R}^{d\times m})$, and*

$$\int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\| \,\mathrm{d}s < \infty .$$

Let $(T_n)_{n\in\mathbb{Z}}$ be as in (5.3) independent of L, and define $Y^{(a)} = (Y_{k;a})_{k\in\mathbb{Z}}$ by (5.13). Then, for $\overline{Y}^{(a)}_n = \frac{1}{n}\sum_{k=1}^n Y_{k;a}$, we have
(a) $\Sigma_{\overline{Y}^{(a)}} := \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_{0;a}, Y_{k;a})$ exists in $[0,\infty)^{d\times d}$ and is absolutely convergent.
(b) $\sqrt{n}\left(\overline{Y}^{(a)}_n - \mu\right) \xrightarrow{d} N(0, \Sigma_{\overline{Y}^{(a)}})$ as $n\to\infty$.

Proof. Observe that $(Y_{k;a})_{k\in\mathbb{Z}}$ is strictly stationary, by Proposition 4.1, and strongly mixing with exponentially decreasing mixing coefficients, by Proposition 4.2. If we can show that

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^n \lambda' Y_{k;a} - \lambda'\mu\right) \xrightarrow{d} N(0, \lambda'\Sigma_{\overline{Y}^{(a)}}\lambda) \quad \forall\, \lambda \in \mathbb{R}^h ,$$

we obtain, by the Cramér-Wold theorem, the assertion of (b).

We obtain, by the assumptions on the Lévy process L, the kernel f, and by Lemma 5.4 that $\mathbf{E}(\|Y_{0;a}\|^2 \log^+ \|Y_{0;a}\|) = \mathbf{E}(\|X_{0;a}\|^2 \log^+ \|X_{0;a}\|) < \infty$, since $Y_{0;a} = X_{0;a}$. Therefore also $\mathbf{E}(|\lambda' Y_{0;a}|^2 \log^+ |\lambda' Y_{0;a}|) < \infty$ for all $\lambda\in\mathbb{R}^d$. Further, define $\widetilde{X}_{t;a} = X_{t;a} - \mu$ such that with $\widetilde{Y}_{k;a} = Y_{k;a} - \mu$ due to the strict stationarity of $(Y_{k;a})_{k\in\mathbb{Z}}$ and since $Y_{0;a} = X_{0;a}$, we obtain a sequence with expectation zero. Hence, w.l.o.g. $\mu = 0$.

Observe that $(\lambda' Y_{k;a})_{k\in\mathbb{Z}}$ is strongly mixing for each $\lambda\in\mathbb{R}^d$ with $\alpha_k^{\lambda' Y^{(a)}} \le \alpha_k^{Y^{(a)}}$ for all $k\in\mathbb{N}$, by Remark 1.8 (a), such that $(\alpha_k^{\lambda' Y^{(a)}})$ is exponentially decreasing. Hence, the assumptions of Theorem 1.11 hold for $\lambda' Y_{k;a}$ and (a) and (b) follow immediately. Observe that the assertion there also holds when $\lambda'\Sigma_{\overline{Y}^{(a)}}\lambda = 0$, in which case we have L^2-convergence to 0 by Bradley [20], Proposition 8.3. □

The following proposition states a result on the convergence of the covariances of $Y^{(a)}$ towards the ones of Y.

Proposition 5.6. *Let X be defined by (5.1) and $X^{(a)}$ by (5.12) such that $f\in L^2(\mathbb{R}^{d\times m})$ and $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process with zero mean and $\mathbf{E}\|L_1\|^2 < \infty$. The processes Y and $Y^{(a)}$ are given by (5.4) and (5.13), respectively, with $(T_n)_{n\in\mathbb{Z}}$ as in (5.3) and assume that $\mu = 0$. Then*

$$\mathbf{E}(|Y_k Y_l' - Y_{k;a} Y_{l;a}'|) \to 0 \quad \textit{as} \;\; a\to\infty \quad \textit{for} \;\; k,l\in\mathbb{Z} . \tag{5.14}$$

Further, it holds

$$\mathbf{E}(Y_k Y_l') = \int_{\mathbb{R}} \mathbf{E}(f(u)\Sigma_L f(T_{|l-k|} + u)')\,\mathrm{d}u \quad \textit{for} \;\; k,l\in\mathbb{Z} , \tag{5.15}$$

and similar for $\mathbf{E}(Y_{k;a} Y_{l;a}')$ with f replaced by $f_a = f\mathbf{1}_{[-a/2,a/2]}$.

Proof. We denote with $Y_{k;a} := Y_k^{(a)}$, and with $Y_{k;a}^{(i)}$, $i = 1, \ldots, d$, the i^{th} component of the vector $Y_{k;a}$. Let T_k be a random time taken from the sequence $(T_n)_{n\in\mathbb{Z}}$ and denote by $\sigma(T_k)$ the σ-algebra generated by T_k. Observe that $\|f_a(u)\| \le \|f(u)\|$ for all $u \in \mathbb{R}$. Denote with $f^{(i,\cdot)}$ the i^{th} row of the matrix valued function f. Then, by conditioning on T_k, the independence of L and T_k, the Itô Isometry, and Fubini's theorem,

$$
\begin{aligned}
\mathbf{E}\|Y_{k;a}\|^2 &= \sum_{i=1}^d \mathbf{E}((Y_{k;a}^{(i)})^2) = \sum_{i=1}^d \mathbf{E}\bigg(\sum_{j=1}^m \int_{\mathbb{R}} f_a^{(i,j)}(T_k - u)\,\mathrm{d}L_u^{(j)}\bigg)^2 \\
&= \sum_{i=1}^d \mathbf{E}\Bigg[\mathbf{E}\Bigg(\bigg(\sum_{j=1}^m \int_{\mathbb{R}} f_a^{(i,j)}(T_k - u)\,\mathrm{d}L_u^{(j)}\bigg)^2 \bigg| \sigma(T_k)\Bigg)\Bigg] \\
&= \sum_{i=1}^d \int_{\mathbb{R}} \mathbf{E}\Bigg(\bigg(\sum_{j=1}^m \int_{\mathbb{R}} f_a^{(i,j)}(t - u)\,\mathrm{d}L_u^{(j)}\bigg)^2 \bigg| T_k = t\Bigg) P_{T_k}(\mathrm{d}t) \\
&= \sum_{i=1}^d \int_{\mathbb{R}} \sum_{j=1}^m \sum_{l=1}^m \int_{\mathbb{R}} f_a^{(i,j)}(t-u)\Sigma_L^{(j,l)} f_a^{(i,l)}(t-u)\,\mathrm{d}u P_{T_k}(\mathrm{d}t) \\
&= \sum_{i=1}^d \int_{\mathbb{R}} f_a^{(i,\cdot)}(u)\Sigma_L f_a^{(i,\cdot)}(u)'\,\mathrm{d}u \\
&\le \sqrt{d} \int_{\mathbb{R}} \|f_a(u)\Sigma_L f_a(u)'\|\,\mathrm{d}u \le \sqrt{d}\,\|\Sigma_L\| \int_{\mathbb{R}} \|f(u)\|^2\,\mathrm{d}u < \infty\,.
\end{aligned} \tag{5.16}
$$

Further, observe that

$$
\begin{aligned}
\mathbf{E}\,\|Y_k - Y_{k;a}\|^2 &= \mathbf{E}\bigg\| \int_{\mathbb{R}} f(T_k - u)\,\mathrm{d}L_u - \int_{\mathbb{R}} f(T_k - u)\mathbf{1}_{[-a/2,a/2]}(T_k - u)\,\mathrm{d}L_u \bigg\|^2 \\
&= \mathbf{E}\bigg\| \int_{\mathbb{R}\setminus[T_k - a/2, T_k + a/2]} f(T_k - u)\,\mathrm{d}L_u \bigg\|^2 \\
&= \mathbf{E}\Bigg[\mathbf{E}\Bigg(\bigg\| \int_{\mathbb{R}\setminus[T_k - a/2, T_k + a/2]} f(T_k - u)\,\mathrm{d}L_u \bigg\|^2 \bigg| \sigma(T_k)\Bigg)\Bigg] =: \mathbf{E}(\mathrm{I})\,.
\end{aligned} \tag{5.17}
$$

By the Doob-Dynkin Lemma there exists a measurable function $\varphi_a \colon [0,\infty) \to \mathbb{R}$ such that $\varphi_a \circ T_k = \mathrm{I}$. Define

$$
\varphi_a(t) := \mathbf{E}\Bigg[\mathbf{E}\Bigg(\bigg\| \int_{\mathbb{R}\setminus[t - a/2, t + a/2]} f(t - u)\,\mathrm{d}L_u \bigg\|^2 \bigg| T_k = t\Bigg)\Bigg]\,,
$$

then obviously $\varphi_a \circ T_k = \mathrm{I}$. But, since L is independent of $(T_n)_{n\in\mathbb{Z}}$, we obtain, by a similar calculation as for (5.16),

$$
\begin{aligned}
0 \le \varphi_a(t) &= \mathbf{E}\Bigg(\bigg\| \int_{\mathbb{R}\setminus[t - a/2, t + a/2]} f(t - u)\,\mathrm{d}L_u \bigg\|^2\Bigg) \\
&\le \sqrt{d}\,\|\Sigma_L\| \int_{\mathbb{R}\setminus[t - a/2, t + a/2]} \|f(t-u)\|^2\,\mathrm{d}u \to 0 \quad \text{as} \quad a \to \infty\,,
\end{aligned}
$$

since $f \in L^2(\mathbb{R}^{d\times m})$. Hence $\varphi_a(T_k(\omega)) \to 0$ as $a \to \infty$ for all $k \in \mathbb{Z}$ and all $\omega \in \Omega$.

Define
$$\varphi(t) := \sqrt{d}\,\|\Sigma_L\| \int_{\mathbb{R}} \|f(t-u)\|^2 \,\mathrm{d}u = \sqrt{d}\,\|\Sigma_L\| \int_{\mathbb{R}} \|f(u)\|^2 \,\mathrm{d}u\,.$$
Then $\mathbf{E}(\varphi \circ T_k) < \infty$ such that, since $|\varphi_a \circ T_k| \le |\varphi \circ T_k|$, we obtain by the dominated convergence theorem for (5.17)
$$\mathbf{E}\,\|Y_k - Y_{k;a}\|^2 = \mathbf{E}(\mathrm{I}) = \mathbf{E}(\varphi_a \circ T_k) \to 0 \quad \text{as } a \to \infty\,. \tag{5.18}$$

Henceforth, by (5.16), (5.18), and the Cauchy-Schwarz inequality,
$$\begin{aligned}
\mathbf{E}\left\|Y_kY_l' - Y_{k;a}Y_{l;a}'\right\| &= \mathbf{E}\left\|Y_kY_l' - Y_{k;a}Y_{l;a}' + Y_{k;a}Y_l' - Y_{k;a}Y_l'\right\| \\
&\le \ \mathbf{E}(\|Y_{k;a}\|\,\|Y_l - Y_{l;a}\|) + \mathbf{E}(\|Y_l\|\,\|Y_k - Y_{k;a}\|) \\
&\le \sqrt{\mathbf{E}\,\|Y_{k;a}\|^2}\sqrt{\mathbf{E}\,\|Y_l - Y_{l;a}\|^2} + \sqrt{\mathbf{E}\,\|Y_l\|^2}\sqrt{\mathbf{E}\,\|Y_k - Y_{k;a}\|^2} \to 0
\end{aligned}$$
for $a \to \infty$, i.e. (5.14).

For the last statement (5.15), let w.l.o.g. $k, l \in \mathbb{N}_0$ and $k \le l$. First observe that, due to the strict stationarity of Y, we have $\mathbf{E}(Y_kY_l) = \mathbf{E}(Y_0Y_{k-l})$ and, by the independence of T_{k-l} and L and the same calculation that lead to (5.5),
$$\begin{aligned}
\mathbf{E}(X_0, X_{T_{k-l}}') &= \mathbf{E}(\mathbf{E}(X_0X_{T_{k-l}}'|\sigma(T_{k-l}))) \\
&= \int_{\mathbb{R}} \mathbf{E}(X_0X_t'|T_{k-l} = t)P_{T_{k-l}}(\mathrm{d}t) \\
&= \int_{\mathbb{R}} \mathbf{E}(X_0X_t')P_{T_{k-l}}(\mathrm{d}t) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)\Sigma_L f(u+t)'\,\mathrm{d}u P_{T_{k-l}}(\mathrm{d}t) = \int_{\mathbb{R}} \mathbf{E}(f(u)\Sigma_L f(u+T_{k-l})')\,\mathrm{d}u
\end{aligned}$$
which gives (5.15). □

Now we are in the position to prove the asymptotic normality of $\overline{Y}_n$ in (5.11). We denote with $\xrightarrow{d}$ convergence in distribution.

Theorem 5.7. *Let X be defined as in (5.1) such that $\mu \in \mathbb{R}$, L has expectation zero and $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$, $f \in L^2(\mathbb{R}^{d\times m})$, and $\int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\| \,\mathrm{d}s < \infty$. Let Y be defined by (5.4) with $(T_n)_{n\in\mathbb{Z}}$ as in (5.3) independent of L. Assume that*
$$\int_{\mathbb{R}} \|f(u)\| \sum_{k=1}^{\infty} \mathbf{E}\,\|f(T_k + u)\| \,\mathrm{d}u < \infty\,. \tag{5.19}$$

Then

(a) $\Sigma_{\overline{Y}}^2 := \sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0, Y_k)$ exists in $[0,\infty)^{d\times d}$, is absolutely convergent, and
$$\Sigma_{\overline{Y}} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} f(u)\Sigma_L \mathbf{E}(f(T_k + u)')\,\mathrm{d}u\,. \tag{5.20}$$

(b) $\sqrt{n}\,(\overline{Y}_n - \mu) \xrightarrow{d} N(0, \Sigma_{\overline{Y}})$ as $n \to \infty$.

Proof. Define $\widetilde{X}_t = X_t - \mu$ such that with $\widetilde{Y}_k = Y_k - \mu$ due to the strict stationarity of $(Y_k)_{k\in\mathbb{Z}}$ and since $Y_0 = X_0$, we obtain a sequence with expectation zero. Hence, w.l.o.g. $\mu = 0$.

(a) Observe that $\mathbf{E}\|Y_0\|^2 = \mathbf{E}\|X_0\|^2 < \infty$ since $f \in L^2(\mathbb{R}^{d\times m})$ and L has finite second moment. Further, by (5.15) and (5.19) together with the dominated convergence theorem

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|\mathbf{E}(Y_0Y_k')\| &= \sum_{k\in\mathbb{Z}} \left\| \int_{\mathbb{R}} f(u)\Sigma_L \mathbf{E}(f(T_k+u)')\,\mathrm{d}u \right\| \\ &\leq \|\Sigma_L\| \int_{\mathbb{R}} \|f(u)\| \sum_{k\in\mathbb{Z}} \mathbf{E}\,\|f(T_k+u)\|\,\mathrm{d}u < \infty\,. \end{aligned} \tag{5.21}$$

This gives the absolute summability of $\Sigma^2_{\overline{Y}}$ and a similar calculation without the modulus gives (5.20).

(b) Using the Cramér-Wold theorem, it is enough to show that

$$\sqrt{n}\,\lambda'\overline{Y}_n \xrightarrow{d} N(0, \lambda'\Sigma_{\overline{Y}}\lambda) \quad \text{as } n\to\infty \quad \forall \lambda \in \mathbb{R}^d\,. \tag{5.22}$$

By Proposition 5.5, we have that the sample mean of the sequence $(Y_{k;a})_{k\in\mathbb{Z}}$ as in (5.13) defined via the a-dependent process $(X_{t;a})_{t\in\mathbb{R}}$ as in (5.12) is asymptotically normal, i.e. we obtain

$$\sqrt{n}\,\lambda'\overline{Y}_n^{(a)} \xrightarrow{d} Z^{(a)} \quad \text{with} \quad Z^{(a)} \stackrel{d}{=} N(0, \lambda'\Sigma_{\overline{Y}^{(a)}}\lambda) \quad \forall \lambda \in \mathbb{R}^d\,. \tag{5.23}$$

By Proposition 5.6, we have that $\mathbf{E}(Y_{0;a}Y_{k;a}') \to \mathbf{E}(Y_0Y_k')$ as $a\to\infty$ and, since

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\mathbf{E}(\lambda' Y_{0;a}Y_{k;a}'\lambda)| &\leq \|\lambda\|^2 \sum_{k\in\mathbb{Z}} \left\|\mathbf{E}(Y_{0;a}Y_{k;a}')\right\| \\ &\leq \|\lambda\|^2\,\|\Sigma_L\| \int_{\mathbb{R}} \|f(u)\| \sum_{k\in\mathbb{Z}} \mathbf{E}\,\|f(T_k+u)\|\,\mathrm{d}u < \infty\,,\end{aligned}$$

by (5.21) and $\|f_a(u)\| \leq \|f(u)\|$ for all $u\in\mathbb{R}$, it follows from the dominated convergence theorem that $\lim_{a\to\infty} \lambda'\Sigma_{\overline{Y}^{(a)}}\lambda = \lambda'\Sigma_{\overline{Y}}\lambda$. Hence,

$$Z^{(a)} \xrightarrow{d} Z \quad \text{as } a\to\infty \quad \text{with} \quad Z \stackrel{d}{=} N(0, \lambda'\Sigma_{\overline{Y}}\lambda) \quad \forall \lambda\in\mathbb{R}^d\,. \tag{5.24}$$

Define for $k\in\mathbb{Z}$

$$\begin{aligned} Y_{k;f-f_a} &:= \int_{\mathbb{R}} f(T_k-u) - f(T_k-u)\mathbf{1}_{[-a/2,a/2]}(T_k-u)\,\mathrm{d}L_u \\ &= \int_{\mathbb{R}\setminus[T_k-a/2,T_k+a/2]} f(T_k-u)\,\mathrm{d}L_u\,.\end{aligned}$$

Then $(Y_{k;f-f_a})_{k\in\mathbb{Z}}$ is strictly stationary, by Proposition 4.1. Further,

$$\begin{aligned}|\mathbf{E}(\lambda' Y_{0;f-f_a}Y_{k;f-f_a}'\lambda)| &\leq \|\lambda\|^2\,\|\mathbf{E}(Y_{0;f-f_a}Y_{k;f-f_a}')\| \\ &\leq \|\lambda\|^2\,\|\Sigma\| \int_{\mathbb{R}\setminus[-a/2,a/2]} \|f(u)\|^2\,\mathrm{d}u \to 0\end{aligned}$$

as $a \to \infty$ since $f \in L^2(\mathbb{R}^{d\times m})$. Since, by (5.21),

$$\sum_{k\in\mathbb{Z}} |\mathbf{E}(\lambda' Y_{0;f-f_a} Y'_{k;f-f_a}\lambda)| \le \|\lambda\|^2 \|\Sigma_L\| \int_{\mathbb{R}} \|f(u)\| \sum_{k\in\mathbb{Z}} \mathbf{E}\, \|f(T_k+u)\|\, \mathrm{d}u < \infty\,,$$

the dominated convergence theorem yields $\lim_{a\to\infty} \sum_{k\in\mathbb{Z}} |\mathbf{E}(\lambda' Y_{0;f-f_a} Y'_{k;f-f_a}\lambda)| = 0$. Hence, by Theorem 7.1.1 in Brockwell and Davis [23],

$$\begin{aligned}\lim_{a\to\infty}\lim_{n\to\infty} \mathbf{Var}(n^{1/2}(\lambda'\overline{Y}_n - \lambda'\overline{Y}^{(a)}_n)) &= \lim_{a\to\infty}\lim_{n\to\infty} n\mathbf{Var}\left(\frac{1}{n}\sum_{k=1}^{n} \lambda' Y_{k;f-f_a}\right)\\ &= \lim_{a\to\infty} \sum_{k\in\mathbb{Z}} \mathbf{E}(\lambda' Y_{0;f-f_a} Y'_{k;f-f_a}\lambda) = 0\,.\end{aligned}$$

An application of Chebychef's inequality yields then

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(n^{1/2}|\lambda'\overline{Y}_n - \lambda'\overline{Y}^{(a)}_n| > \varepsilon) = 0 \quad \forall \varepsilon > 0\,.$$

Together with (5.23) and (5.24), the claim follows by a variant of Slutsky's Lemma, cf. Theorem 1.12. □

Remark 5.8. *(a) When $(T_n)_{n\in\mathbb{Z}}$ is deterministic, i.e. $T_n = \Delta n$ for $n \in \mathbb{Z}$ and some $\Delta > 0$, we established the asymptotic normality in Theorem 5.1 under the condition (5.6). Observe that (5.6) implies (5.19) since*

$$\begin{aligned}\int_{\mathbb{R}} \|f(u)\| \sum_{k=1}^{\infty} \|f(\Delta k + u)\|\, \mathrm{d}u &= \sum_{j=-\infty}^{\infty} \int_0^{\Delta} \|f(u+\Delta j)\| \sum_{k=-1}^{\infty} \|f(u+\Delta k)\|\, \mathrm{d}u\\ &\le \int_0^{\Delta} |\widetilde{F}_\Delta(u)|^2\, \mathrm{d}u\,.\end{aligned}$$

So, Theorem 5.7 generalizes Theorem 5.1 to the case of a renewal sampling sequence $(T_n)_{n\in\mathbb{Z}}$ at the cost of the slightly more restrictive conditions $\mathbf{E}(\|L_1\|^2 \log^+ \|L_1\|) < \infty$ and $\int_{\mathbb{R}} \|f(s)\|^2 \log^+ \|f(s)\|\, \mathrm{d}s < \infty$.

(b) Proposition 5.5, Proposition 5.6 and Theorem 5.7 generalize Proposition 4.5, Proposition 4.6 and Theorem 4.7 of Section 4.2 to a multivariate setting. Choosing in Theorem 5.7 the kernel function $f \in L^2(\mathbb{R})$ and L as a univariate Lévy process with zero mean and second finite moment, we see that condition (5.19) reduces to (4.19) of Theorem 4.7.

Remark 5.9. *Condition (5.19) is satisfied, for example, for $\|f(u)\| \le K(|u|^{-\alpha} \wedge 1)$ with $\alpha > 1$ and $K > 0$.*

To see this, observe that for some C_α

$$\int_{\mathbb{R}} \|f(u)\|\, \|f(t+u)\|\, \mathrm{d}u \le C_\alpha(|t|^{-\alpha} \wedge 1)\,.$$

Hence,

$$\int_{\mathbb{R}} \|f(u)\| \sum_{k=1}^{\infty} \mathbf{E}\, \|f(T_k+u)\|\, \mathrm{d}u \le C_\alpha \sum_{k=1}^{\infty} P(T_k \le 1) + C_\alpha \sum_{k=1}^{\infty} \mathbf{E}(T_k^{-\alpha} \mathbf{1}_{\{T_k>1\}})\,, \qquad (5.25)$$

and the two sums on the right-hand side of (5.25) converges as seen in Remark 4.9.

A Appendix

A.1 Kronecker Product, Vectorizing, and the Stochastic Integral

Definition A.1. (Kronecker Product and Vectorizing Operator)
Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{l\times k}$. Then the *Kronecker product* $A \otimes B \in \mathbb{R}^{nl\times mk}$ of A and B is the partitioned matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix}.$$

Let $A \in \mathbb{R}^{n\times m}$ and denote with A_j the j-th column of A. The *vectorizing operator* is defined as

$$\operatorname{vec}(A) := \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \in \mathbb{R}^{nm}.$$

More informations of the properties of the Kronecker product and the vectorizing operator can be found in Bernstein [11]. The following properties in conjunction with the multivariate stochastic integral hold, see also Lemma 2.1 in Behme [9]. For a matrix $A \in \mathbb{R}^{n\times m}$ we denote by $A' \in \mathbb{R}^{m\times n}$ its transposed.

Lemma A.2. *Let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$, and $Z = (Z_t)_{t\geq 0}$ be $\mathbb{R}^{d\times d}$-valued semimartingales. Then it holds for all $t \geq 0$*

(i) $\displaystyle\int_{(0,t]} (I \otimes Y_{s-})\,\mathrm{d}(I \otimes X_s) = I \otimes \left(\int_{(0,t]} Y_{s-}\,\mathrm{d}X_s\right)$,

(ii) $\displaystyle\int_{(0,t]} \mathrm{d}(X_s \otimes I)(Y_{s-} \otimes I) = \left(\int_{(0,t]} \mathrm{d}X_s Y_{s-}\right) \otimes I$,

(iii) $\displaystyle\int_{(0,t]} \mathrm{d}(I \otimes X_s)(I \otimes Y_{s-}) = I \otimes \left(\int_{(0,t]} \mathrm{d}X_s Y_{s-}\right)$,

(iv) $\displaystyle\int_{(0,t]} (I \otimes Y_{s-})\,\mathrm{d}(X_s \otimes I) = \int_{(0,t]} \mathrm{d}(X_s \otimes I)(I \otimes Y_{s-})$, *and*

(v) $[I \otimes X_., X_. \otimes I]_t = [X_. \otimes I, I \otimes X_.]_t$.

Further, it holds together with the vec*-operator*

(vi) $\operatorname{vec}\left(\int_{(0,t]} X_{s-}\,\mathrm{d}Y_s\,Z_{s-}\right) = \int_{(0,t]} (Z'_{s-} \otimes X_{s-})\,\mathrm{d}(\operatorname{vec}(Y_s))$

(vii) $\operatorname{vec}\left(\int_{(0,t]} \mathrm{d}Y_s X_{s-}\right) = \int_{(0,t]} \mathrm{d}(I \otimes Y_s)(\operatorname{vec}(X_{s-}))$

(viii) $\operatorname{vec}\left(\int_{(0,t]} X_{s-}\,\mathrm{d}Y'_s\right) = \int_{(0,t]} \mathrm{d}(Y_s \otimes I)(\operatorname{vec}(X_{s-}))$

(ix) $\operatorname{vec}\left[X_\cdot, \int_{(0,\cdot]} Y_{s-}\,\mathrm{d}X'_s\right]_t = \int_{(0,t]} \mathrm{d}[I \otimes X_\cdot, X_\cdot \otimes I]_s \operatorname{vec}(Y_{s-})$

(x) $\operatorname{vec}[X_\cdot, Y_\cdot]_t = [I \otimes X_\cdot, \operatorname{vec}(\mathrm{Y}_\cdot)]_t$

(xi) $\operatorname{vec}[Y_\cdot, X'_\cdot]_t = [X_\cdot \otimes I, \operatorname{vec}(\mathrm{Y}_\cdot)]_t$

(xii) $\operatorname{vec}\left[X_\cdot, [Y, X']_\cdot\right]_t = [[I \otimes X, X \otimes I]_\cdot, \operatorname{vec}(\mathrm{Y}_\cdot)]_t = \operatorname{vec}\left[[X, Y]_\cdot, X'_\cdot\right]_t$.

Proof. Follows from the properties of the multivariate stochastic integral and the definition of the vectorizing operator and the Kronecker product. □

A.2 MGOU Processes

Here we give a short overview of multivariate generalized Ornstein-Uhlenbeck processes needed in Chapter 3.

Stochastic Logarithm

Definition A.3. (Multivariate Stochastic Logarithm)
Let $Z = (Z_t)_{t\geq 0}$ be a $\mathrm{GL}(\mathbb{R}, m)$-valued semimartingale with $Z_0 = I$ and $Z_{t-} \in \mathrm{GL}(\mathbb{R}, m)$, $t > 0$. Then the *left stochastic logarithm* $\overleftarrow{\mathrm{Log}}\,(Z)$ and the *right stochastic logarithm* $\overrightarrow{\mathrm{Log}}\,(Z)$ of Z are defined by

$$\overleftarrow{\mathrm{Log}}\,(Z)_t = \int_{(0,t]} Z_{s-}^{-1}\,\mathrm{d}Z_s\,, \quad \text{and} \quad \overrightarrow{\mathrm{Log}}\,(Z)_t = \int_{(0,t]} \mathrm{d}Z_s Z_{s-}^{-1}\,, \quad t \geq 0\,, \tag{A.1}$$

respectively.

Proposition A.4. *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual hypotheses. Then for every $\mathbb{F}$-Lévy process $X = (X_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m}$ satisfying (3.5), the stochastic exponential $Z_t = \overleftarrow{\mathcal{E}}\,(X)_t$ (resp. $\overrightarrow{\mathcal{E}}\,(X)_t$) is a left (resp. right) $\mathbb{F}$-Lévy process in* $\mathrm{GL}(\mathbb{R}, m)$.
Conversely, if $Z = (Z_t)_{t\geq 0}$ is a left (resp. right) $\mathbb{F}$-Lévy process in $\mathrm{GL}(\mathbb{R}, m)$, then Z is an $\mathbb{F}$-semimartingale and $\overleftarrow{\mathrm{Log}}\,(Z)$ (resp. $\overrightarrow{\mathrm{Log}}\,(Z)$) is an additive Lévy process in $\mathbb{R}^{m\times m}$ satisfying (3.5).

Proof. Proposition 2.4 in Behme and Lindner [8]. □

Remark A.5. *Under the assumption of the previous proposition it holds*

$$\overleftarrow{\mathrm{Log}}\left(\overleftarrow{\mathcal{E}}(X)\right)_t = X_t \quad \textit{and} \quad \overrightarrow{\mathrm{Log}}\left(\overrightarrow{\mathcal{E}}(X)\right)_t = X_t \,.$$

Definition and Properties of MGOU processes

As in Behme and Lindner [8] we define

Definition A.6. Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$, such that X satisfies

$$\det(I + \Delta X_t) \neq 0 \quad \forall\, t \geq 0\,, \tag{A.2}$$

and let V_0 be a random variable in $\mathbb{R}^m$. Then the $\mathbb{R}^m$-valued process $V = (V_t)_{t\geq 0}$ given by

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}\, \mathrm{d}Y_s\right), \quad t \geq 0\,, \tag{A.3}$$

is called a *multivariate generalized Ornstein-Uhlenbeck (MGOU) process* driven by (X,Y). The MGOU process will be called *casual* or *non-anticipative*, if V_0 is independent of (X,Y), and *strictly non-causal* if V_t is independent of $(X_s, Y_s)_{0\leq s<t}$ for all $t \geq 0$.

Behme and Lindner [8] defined the MGOU process to satisfy the random recurrence equation

$$V_t = A_{s,t}V_s + B_{s,t} \quad \text{a.s.}\,, \quad 0 \leq s \leq t\,, \tag{A.4}$$

for random functionals $(A_{s,t})_{0\leq s\leq t}, (B_{s,t})_{0\leq s\leq t}$ satisfying the following Assumption A.7. This was motivated by what de Haan and Karandikar did in there paper *Embedding a stochastic difference equation into a continuous-times process* [33] to define the *generalized Ornstein-Uhlenbeck* (GOU) *process.*

Assumption A.7. *Suppose that the* $\mathrm{GL}(\mathbb{R}, m) \times \mathbb{R}^m$*-valued random functional denoted by* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *with* $A_{t,t} = I$ *and* $B_{t,t} = 0$ *a.s. for all* $t \geq 0$ *satisfies the following four conditions.*
(a) For all $0 \leq u \leq s \leq t$ *almost surely*

$$A_{u,t} = A_{s,t}A_{u,s} \quad \textit{and} \quad B_{u,t} = A_{s,t}B_{u,s} + B_{s,t}\,.$$

(b) For all $0 \leq a \leq b \leq c \leq d$ *the families of random matrices* $\{(A_{s,t}, B_{s,t}), a \leq s \leq t \leq b\}$ *and* $\{(A_{s,t}, B_{s,t}), c \leq s \leq t \leq d\}$ *are independent.*
(c) For all $0 \leq s \leq t$

$$(A_{s,t}, B_{s,t}) \overset{d}{=} (A_{0,t-s}, B_{0,t-s})\,.$$

(d) It holds

$$\lim_{t\downarrow 0} A_{0,t} = I \quad \text{and} \quad \lim_{t\downarrow 0} B_{0,t} = 0 \quad \text{in probability}.$$

With Definition A.6, it is then easy to see that V_t with $(A_{s,t}, B_{s,t})_{0\le s\le t}$ defined by

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-} \, \mathrm{d}Y_u \end{pmatrix} \quad \text{a.s.}, \ 0 \le s \le t,$$

where $(X_t, Y_t)_{t\ge 0}$ are as in Definition A.6, satisfies the stated random recurrence equation (A.4), V_t and $(A_{s,t}, B_{s,t})_{0\le s\le t}$ are independent, and $(A_{s,t}, B_{s,t})_{0\le s\le t}$ satisfies Assumption A.7. This was shown in Behme and Lindner [8] as well as the following two results

Theorem A.8. *Let a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\ge 0}, P)$ with $\mathbb{F}$ satisfying the usual hypotheses be given.*
(a) Let $(X, Y) = (X_t, Y_t)_{t\ge 0}$ be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$, such that (X, Y) is a semimartingale with respect to $\mathbb{F}$ and X satisfies (A.2), and let $V = (V_t)_{t\ge 0}$ be the MGOU process driven by (X, Y) with $\mathcal{F}_0$-measurable starting random variable V_0. Then V solves the stochastic differential equation (SDE)

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathrm{d}L_t, \quad t \ge 0, \tag{A.5}$$

where $(U, L) = (U_t, L_t)_{t\ge 0}$ is another Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ with

$$U_t := -X_t + [X, X]_t^c + \sum_{0<s\le t} \Big((I + \Delta X_s)^{-1} - I + \Delta X_s \Big), \quad t \ge 0, \tag{A.6}$$

i.e. it holds

$$\overleftarrow{\mathcal{E}}(X)_t^{-1} = \overrightarrow{\mathcal{E}}(U)_t, \quad t \ge 0,$$

and L given by

$$L_t = Y_t + \sum_{0<s\le t} \Big((I + \Delta X_s)^{-1} - I \Big) \Delta Y_s - [X, Y]_t^c, \quad t \ge 0. \tag{A.7}$$

The process U satisfies

$$\det(I + \Delta U_t) \ne 0 \quad \forall\, t \ge 0. \tag{A.8}$$

(b) Conversely, if (U, L) is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$, such that (U, L) is a semimartingale with respect to $\mathbb{F}$ and U satisfies (A.8), and V_0 is an $\mathbb{R}^m$-valued $\mathcal{F}_0$-measurable starting random variable, then the solution to (A.5) is an MGOU process driven by (X, Y), where (X, Y) is a Lévy process defined by

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \overleftarrow{\operatorname{Log}} \left(\overrightarrow{\mathcal{E}}(U)_t^{-1} \right) \\ L_t + \left[\overleftarrow{\operatorname{Log}} \left(\overrightarrow{\mathcal{E}}(U)^{-1} \right), L \right]_t \end{pmatrix}, \quad t \ge 0,$$

and X satisfies (A.2).

Proposition A.9. *Let a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ with $\mathbb{F}$ satisfying the usual hypotheses be given. Let (X, Y) be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (A.2) and let (U, L) be defined by (A.6) and (A.7). Then*

$$L_t = Y_t + [U, Y]_t\,, \quad t \geq 0\,,$$

and

$$Y_t = L_t + [X, L]_t\,, \quad t \geq 0\,.$$

Moments of MGOU Processes

Lemma A.10. *Let $L = (L_t)_{t\geq 0}$ be a Lévy process, $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis with $\mathbb{F}$ the natural filtration of L, and $H = (H_t)_{t\geq 0}$ an adapted, càdlàg process. Suppose there exists a $\kappa \geq 1$ such that $\mathbf{E}|L_1|^\kappa < \infty$ and $\mathbf{E}\sup_{0<t\leq 1}|H_t|^\kappa < \infty$. Then*

$$\mathbf{E}\left[\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}\,\mathrm{d}L_s\right|^\kappa\right] < \infty\,.$$

In particular, if $\mathbf{E}|L_1| < \infty$ and $\mathbf{E}\sup_{0<t\leq 1}|H_t| < \infty$, for $t > 0$ it holds

$$\mathbf{E}\left[\int_{(0,t]} H_{s-}\,\mathrm{d}L_s\right] = \mathbf{E}[L_1]\int_{(0,t]} \mathbf{E}[H_{s-}]\,\mathrm{d}s\,.$$

Proof. Lemma 6.1 in Behme [7]. □

A multivariate extension yields

Lemma A.11. *Let $(L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{m\times m}$ and $(H_t)_{t\geq 0}$ an adapted, càdlàg process in $\mathbb{R}^{m\times m}$. If $\mathbf{E}\,\|L_1\| < \infty$ and $\mathbf{E}\sup_{0<t\leq 1}\|H_t\| < \infty$, then it holds for $t > 0$*

$$\mathbf{E}\left[\int_{(0,t]} H_{s-}\,\mathrm{d}L_s\right] = \int_{(0,t]} \mathbf{E}[H_{s-}]\,\mathrm{d}(s\mathbf{E}[L_1])\,.$$

This allows us to formulate the following

Proposition A.12. *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{d\times d}$ and suppose for some fixed $\kappa > 0$ that $\mathbf{E}\,\|X_1\|^\kappa < \infty$. Then it holds*

$$\mathbf{E}\left[\sup_{0\leq s\leq t}\left\|\overleftarrow{\mathcal{E}}(X)_s\right\|^\kappa\right] < \infty \quad \text{and} \quad \mathbf{E}\left[\sup_{0\leq s\leq t}\left\|\overrightarrow{\mathcal{E}}(X)_s\right\|^\kappa\right] < \infty \quad \text{for all } t \geq 0\,.$$

Especially for $\kappa = 1$ we get

$$\mathbf{E}\left[\overleftarrow{\mathcal{E}}(X)_t\right] = \mathbf{E}\left[\overrightarrow{\mathcal{E}}(X)_t\right] = \exp(t\mathbf{E}[X_1]) \quad \text{for all } t \geq 0\,.$$

Proof. Proposition 3.1 in Behme [9]. □

Proposition A.13. *Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times n}$ such that U satisfies (A.8). Let $V = (V_t)_{t\geq 0}$ be a strictly stationary solution of the SDE (A.5) with starting value V_0 independent of (U, L). Assume that for $\kappa > 0$ we have for some $t_0 > 0$*

$$\mathbf{E}\,\|U_1\|^{\max\{\kappa,1\}} < \infty\,, \quad \mathbf{E}\,\|L_1\|^{\max\{\kappa,1\}} < \infty \quad \text{and} \quad \mathbf{E}\left\|\overleftarrow{\mathcal{E}}\,(U)_{t_0}\right\|^{\kappa} < 1\,. \tag{A.9}$$

Then $\mathbf{E}\,\|V_0\|^{\kappa} < \infty$. Further if (A.9) holds for $\kappa = 1$, then $\mathbf{E}[U_1]$ is invertible and in particular it holds

$$\mathbf{E}[V_0] = -\mathbf{E}[U_1]^{-1}\mathbf{E}[L_1]\,.$$

Proof. Proposition 3.3 in Behme [9]. □

Proposition A.14. *Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{d\times d} \times \mathbb{R}^{d}$ such that U satisfies (A.8). Let $V = (V_t)_{t\geq 0}$ be a strictly stationary solution of the SDE (A.5) with starting value V_0 independent of (U, L). Suppose that it holds $\mathbf{E}\,\|U_1\|\,, \mathbf{E}\,\|L_1\|\,, \mathbf{E}\,\|V_s\|^2 < \infty$, then for $0 \leq s \leq t$ we have*

$$\mathbf{Cov}(V_t, V_s) = \mathrm{e}^{(t-s)\mathbf{E}[U_1]}\,\mathbf{Cov}(V_s)\,,$$

where $\mathbf{Cov}(V_t, V_s) = \mathbf{E}[V_t V_s'] - \mathbf{E}[V_t]\mathbf{E}[V_s']$ and $\mathbf{Cov}(V_s) = \mathbf{E}[V_s V_s'] - \mathbf{E}[V_s]\mathbf{E}[V_s']$ denoting the covariance matrix of V_s.
In particular, if V is strictly stationary, (A.9) holds for $\kappa = 2$ and we denote

$$C = \mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1] + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]\,,$$

then the matrix

$$D = \int_0^\infty \int_0^s \mathrm{e}^{uC}\big(\mathrm{e}^{(s-u)(\mathbf{E}[U_1]\otimes I)} + \mathrm{e}^{(s-u)(I\otimes \mathbf{E}[U_1])}\big)\,\mathrm{d}u\,\mathrm{d}s$$

is finite. Now, if either $\mathbf{E}[L_1] = 0$ or U and L are independent, we obtain

$$\begin{aligned}\mathbf{Cov}(V_t, V_s) = \mathrm{e}^{(t-s)\mathbf{E}[U_1]}\cdot&\\ \cdot\,\mathrm{vec}^{-1}(-C^{-1}\mathrm{vec}(\mathbf{Cov}(L_1)) + (D - (\mathbf{E}[U_1]\otimes\mathbf{E}[U_1])^{-1})\mathrm{vec}(\mathbf{E}[L_1]\mathbf{E}[L_1']))\,.&\end{aligned}$$

Proof. Proposition 3.4 in Behme [9]. □

Stationary solutions of MGOU processes

Theorem A.15. *Suppose that $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^{m}$ such that X satisfies (A.2). Let $V = (V_t)_{t\geq 0}$ be the MGOU process driven by (X, Y) and let $(U, L) = (U_t, L_t)_{t\geq 0}$ be the Lévy processes defined in (A.6) and (A.7). Suppose $\|\cdot\|$*

is a submultiplicative matrix norm and that $\mathbf{E}\left[\log^+ \|U_1\|\right] < \infty$ *and* $\mathbf{E}\left[\log^+ \|L_1\|\right] < \infty$*. Assume further that there exists a* $t_0 > 0$ *such that*

$$\mathbf{E}\left[\log\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|\right] < 0\,.$$

Then $\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(U)_t = 0$ *almost surely and the integral* $\int_{(0,t]}\overleftarrow{\mathcal{E}}(U)_{s-}\,\mathrm{d}L_s$ *converges almost surely for* $t \to \infty$ *to a finite random variable. Further, a finite* $\mathbb{R}^m$*-valued random variable* V_0 *independent of* (X,Y) *can be chosen with*

$$V_0 \overset{d}{=} \mathrm{d} - \lim_{t\to\infty}\int_{(0,t]}\overleftarrow{\mathcal{E}}(U)_{s-}\,\mathrm{d}L_s\,,$$

such that V *is uniquely determined and strictly stationary.*

Proof. Remark 5.5 (b) together with Theorem 5.4 (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) and Theorem 5.2 (a) of Behme and Lindner [8]. $\square$

A.3 Detailed Calculations on the Results of Section 4.4

This section is dedicated to the detailed calculation of some expressions in Section 4.1, which have been left out there for a more fluent reading experience.

Lemma A.16. *The distributional variance of*

$$\sqrt{n}(\rho_n^*(1) - \rho(1)) \xrightarrow{d} N(0, \mathbf{W}_{11})\,, \quad n \to \infty\,,$$

appearing in the proof of Theorem 4.22, is of the form (4.45), which is

$$\mathbf{W}_{11} = \left(\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2}\right)((\eta-3)a+3) + \frac{2a}{\lambda+2a}\,, \tag{A.10}$$

Proof. Recall that, by Theorem 4.20 (c),

$$\mathbf{W}_{11} = (\mathbf{Z}_{11} - 2\rho(1)\mathbf{Z}_{01} + \rho(1)^2\mathbf{Z}_{00})/\gamma(0)^2 = \frac{4a^2}{\sigma^4}\left(\mathbf{Z}_{11} - 2\frac{\lambda}{a+\lambda}\mathbf{Z}_{01} + \left(\frac{\lambda}{a+\lambda}\right)^2\mathbf{Z}_{00}\right)$$

with

$$\mathbf{Z}_{pq} = \sigma^4\sum_{k=-q+1}^{p-1}\mathbf{Cov}(F(0,T_p), F(T_k, T_{k+q})) + \sum_{k\in\mathbb{Z}}\kappa_f(p,k,k+q)\,,$$

and

$$\begin{aligned}\kappa_f(p,k,k+q) := (\eta-3)\sigma^4\int_{\mathbb{R}} f(u)\mathbf{E}(f(u+T_p)f(u+T_k)f(u+T_{k+q}))\,\mathrm{d}u \\ + \sigma^4\mathbf{E}(F(0,T_k)F(T_p,T_{k+q})) + \sigma^4\mathbf{E}(F(0,T_{k+q})F(T_p,T_k))\,,\end{aligned}$$

for $k, p, q \in \mathbb{Z}$, given as in Proposition 3.19, where also $F(s,t) = \int_{\mathbb{R}} f(s+u)f(t+u)\,\mathrm{d}u$. Here we have $f(u) = \mathrm{e}^{-au}\mathbf{1}_{[0,\infty)}(u)$. Recall that by Remark 4.16, if $p = 0$ or $q = 0$, the first sum in $\mathbf{Z}_{pq}$ vanishes.

It therefore remains to calculate

$$\mathbf{Z}_{11} = \sigma^4 \mathbf{Var}(F(0,T_1)) + \sum_{k\in\mathbb{Z}} \kappa_f(1,k,k+1)\,, \tag{A.11}$$

$$\mathbf{Z}_{01} = \sum_{k\in\mathbb{Z}} \kappa_f(0,k,k+1)\,, \quad \text{and} \tag{A.12}$$

$$\mathbf{Z}_{00} = \sum_{k\in\mathbb{Z}} \kappa_f(0,k,k)\,. \tag{A.13}$$

Recall that $W_1 \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$, and hence $T_n \sim \Gamma(n,\lambda)$, i.e. with density $g_\Gamma(u) = \frac{\lambda^n}{\Gamma(n)} x^{n-1}\mathrm{e}^{-\lambda x}$. We then start with (A.11), more precisely

$$\begin{aligned}
\sigma^4\mathbf{Var}(F(0,T_1)) &= \mathbf{E}(F(0,T_1)^2) - \mathbf{E}(F(0,T_1))^2 \\
&= \sigma^4\mathbf{E}\bigg(\Big(\int_0^\infty \mathrm{e}^{-au}\mathrm{e}^{-a(T_1+u)}\,\mathrm{d}u\Big)^2\bigg) - \mathbf{E}\bigg(\Big(\int_0^\infty \mathrm{e}^{-au}\mathrm{e}^{-a(T_1+u)}\,\mathrm{d}u\Big)\bigg)^2 \\
&= \frac{\sigma^4}{4a^2}\Big(\mathbf{E}(\mathrm{e}^{-2aT_1}) - \mathbf{E}(\mathrm{e}^{-aT_1})^2\Big) \\
&= \frac{\sigma^4}{4a^2}\bigg(\int_0^\infty \mathrm{e}^{-2at}\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t - \Big(\int_0^\infty \mathrm{e}^{-at}\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t\Big)^2\bigg) \\
&= \frac{\sigma^4}{4a^2}\bigg(\frac{\lambda}{\lambda+2a}\int_0^\infty (\lambda+2a)\mathrm{e}^{-(\lambda+2a)t}\,\mathrm{d}t - \frac{\lambda^2}{(\lambda+a)^2}\Big(\int_0^\infty (\lambda+a)\mathrm{e}^{-(\lambda+a)t}\,\mathrm{d}t\Big)^2\bigg) \\
&= \frac{\sigma^4}{4a^2}\bigg(\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2}\bigg)
\end{aligned} \tag{A.14}$$

From this we see that $\mathbf{E}(\mathrm{e}^{-aT_1}) = \frac{\lambda}{\lambda+2a}$. For the sum in (A.11), we differentiate the following three cases. First, consider $k = 0$, then

$$\begin{aligned}
\kappa_f(1,0,1) &= (\eta-3)\sigma^4\int_0^\infty \mathrm{e}^{-au}\mathbf{E}(\mathrm{e}^{-a(u+T_1)}\mathrm{e}^{-au}\mathrm{e}^{-a(u+T_1)})\,\mathrm{d}u \\
&\quad + \sigma^4\mathbf{E}\bigg(\int_0^\infty \mathrm{e}^{-2au}\,\mathrm{d}u\int_{-T_1}^\infty \mathrm{e}^{-2a(u+T_1)}\,\mathrm{d}u\bigg) \\
&\quad + \sigma^4\mathbf{E}\bigg(\int_0^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_1}\,\mathrm{d}u\int_{-T_1}^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_1}\,\mathrm{d}u\bigg) \\
&= (\eta-3)\sigma^4\int_0^\infty \mathrm{e}^{-4au}\,\mathrm{d}u\mathbf{E}(\mathrm{e}^{-2aT_1}) + \frac{\sigma^4}{4a^2} + \frac{\sigma^4}{4a^2}\mathbf{E}(\mathrm{e}^{-2aT_1}) \\
&= \frac{\sigma^4}{4a^2}\frac{\lambda}{\lambda+2a}\bigg((\eta-3)a + 2 + \frac{2a}{\lambda}\bigg)\,.
\end{aligned} \tag{A.15}$$

Next, let $k \geq 1$, then, since

$$\begin{aligned}\mathbf{E}(\mathrm{e}^{-2aT_{k-1}}) &= \int_0^\infty \mathrm{e}^{-2at} \frac{\lambda^{k-1}}{\Gamma(k-1)} t^{k-2} \mathrm{e}^{-\lambda t}\, \mathrm{d}t \\ &= \left(\frac{\lambda}{\lambda+2a}\right)^{k-1} \int_0^\infty \frac{(\lambda+2a)^{k-1}}{\Gamma(k-1)} t^{(k-1)-1} \mathrm{e}^{-(\lambda+2a)t}\, \mathrm{d}t = \left(\frac{\lambda}{\lambda+2a}\right)^{k-1},\end{aligned}$$

for $k \geq 1$, we obtain

$$\begin{aligned}\kappa_f(1,k,k+1) &= (\eta-3)\sigma^4 \int_0^\infty \mathrm{e}^{-au} \mathbf{E}(\mathrm{e}^{-a(u+T_1)} \mathrm{e}^{-a(u+T_k)} \mathrm{e}^{-a(u+T_{k+1})})\, \mathrm{d}u \\ &\quad + \sigma^4 \mathbf{E}\left(\int_0^\infty \mathrm{e}^{-2au} \mathrm{e}^{-aT_k}\, \mathrm{d}u \int_{-T_1}^\infty \mathrm{e}^{-2au} \mathrm{e}^{-2a(T_1+T_{k+1})}\, \mathrm{d}u \right) \\ &\quad + \sigma^4 \mathbf{E}\left(\int_0^\infty \mathrm{e}^{-2au} \mathrm{e}^{-aT_{k+1}}\, \mathrm{d}u \int_{-T_1}^\infty \mathrm{e}^{-2au} \mathrm{e}^{-2a(T_1+T_k)}\, \mathrm{d}u \right) \\ &= (\eta-3)\frac{\sigma^4}{4a} \mathbf{E}(\mathrm{e}^{-a(T_1+T_k+T_{k+1})}) + \frac{2\sigma^4}{4a^2} \mathbf{E}(\mathrm{e}^{-aT_k} \mathrm{e}^{2aT_1} \mathrm{e}^{-a(T_1+T_{k+1})}) \\ &= \frac{\sigma^4}{4a^2} \Bigg((\eta-3)a\mathbf{E}(\mathrm{e}^{-3aT_1}) \mathbf{E}(\mathrm{e}^{-2a\sum_{i=2}^k W_i}) \mathbf{E}(\mathrm{e}^{-aW_{k+1}}) \\ &\qquad\qquad + 2\mathbf{E}(\mathrm{e}^{-aT_k}) \mathbf{E}(\mathrm{e}^{2a\sum_{i=2}^k W_i}) \mathbf{E}(\mathrm{e}^{-aW_{k+1}}) \Bigg) \\ &= \frac{\sigma^4}{4a^2} \Bigg((\eta-3)a\mathbf{E}(\mathrm{e}^{-3aT_1}) \mathbf{E}(\mathrm{e}^{-2aT_{k-1}}) \mathbf{E}(\mathrm{e}^{-aT_1}) \\ &\qquad\qquad + 2\mathbf{E}(\mathrm{e}^{-aT_k}) \mathbf{E}(\mathrm{e}^{2aT_{k-1}}) \mathbf{E}(\mathrm{e}^{-aT_{k+1}}) \Bigg) \\ &= \frac{\sigma^4}{4a^2} \left((\eta-3)a \frac{\lambda}{\lambda+3a} \frac{\lambda}{\lambda+a} \left(\frac{\lambda}{\lambda+2a}\right)^{k-1} + 2\frac{\lambda}{\lambda+a} \frac{\lambda}{\lambda+a} \left(\frac{\lambda}{\lambda+2a}\right)^{k-1} \right) \\ &= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \left(\frac{\lambda}{\lambda+2a}\right)^{k-1} \left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right).\end{aligned}$$

From the latter, we conclude

$$\begin{aligned}\sum_{k=1}^\infty \kappa_f(1,k,k+1) &= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \sum_{k=1}^\infty \left(\frac{\lambda}{\lambda+2a}\right)^{k-1} \left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right) \\ &= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \frac{\lambda+2a}{2a} \left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right). \qquad \text{(A.16)}\end{aligned}$$

Now, if $-k \geq 1$, we obtain, since $T_{-k} \stackrel{d}{=} -T_k$

$$\kappa_f(1,-k,-k+1) = (\eta-3)\sigma^4 \int_{-T_{-k}}^\infty \mathrm{e}^{-au} \mathbf{E}(\mathrm{e}^{-a(u+T_1)} \mathrm{e}^{-a(u+T_{-k})} \mathrm{e}^{-a(u+T_{-k+1})})\, \mathrm{d}u$$

$$
\begin{aligned}
&+ \sigma^4 \mathbf{E}\biggl(\int_{T_{-k}}^{\infty} \mathrm{e}^{-2au} \mathrm{e}^{-aT_{-k}} \,\mathrm{d}u \int_{-T_{-k+1}}^{\infty} \mathrm{e}^{-2au} \mathrm{e}^{-2a(T_1+T_{-k+1})} \,\mathrm{d}u \biggr) \\
&+ \sigma^4 \mathbf{E}\biggl(\int_{T_{-k+1}}^{\infty} \mathrm{e}^{-2au} \mathrm{e}^{-aT_{-k+1}} \,\mathrm{d}u \int_{-T_{-k}}^{\infty} \mathrm{e}^{-2au} \mathrm{e}^{-2a(T_1+T_{-k})} \,\mathrm{d}u \biggr) \\
&= \frac{\sigma^4}{4a^2} \mathbf{E}(\mathrm{e}^{-aT_1}) \Bigl((\eta-3) a \mathbf{E}(\mathrm{e}^{3aT_{-k}} \mathrm{e}^{-aT_{-k+1}}) + 2\mathbf{E}(\mathrm{e}^{aT_{-k}} \mathrm{e}^{aT_{-k+1}}) \Bigr) \\
&= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \Bigl((\eta-3) a \mathbf{E}(\mathrm{e}^{2aT_{-k+1}} \mathrm{e}^{-3aW_{-k}}) + 2\mathbf{E}(\mathrm{e}^{a(T_{-k}+T_{-k+1})}) \Bigr) \\
&= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \mathbf{E}(\mathrm{e}^{-2aT_{k-1}}) \Bigl((\eta-3) a \mathbf{E}(\mathrm{e}^{-3aW_1}) + 2\mathbf{E}(\mathrm{e}^{-aW_1}) \Bigr) \\
&= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \left(\frac{\lambda}{\lambda+2a} \right)^{k-1} \left((\eta-3) \frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right).
\end{aligned}
$$

such that

$$
\sum_{k=1}^{\infty} \kappa_f(1,-k,-k+1) = \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} \frac{\lambda+2a}{2a} \left((\eta-3) \frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right). \tag{A.17}
$$

From (A.14) together (A.15), (A.16), and (A.17) we derive for (A.11)

$$
\begin{aligned}
\mathbf{Z}_{11} = \frac{\sigma^4}{4a^2} \biggl(&\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2} + \frac{2\lambda}{\lambda+a} \frac{\lambda+2a}{2a} \left((\eta-3) \frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right) \\
&+ \frac{\lambda}{\lambda+2a} \left((\eta-3)a + 2 + \frac{2a}{\lambda} \right) \biggr),
\end{aligned}
$$

and hence, since $\gamma(0) = \frac{\sigma^4}{4a^2}$,

$$
\begin{aligned}
\frac{\mathbf{Z}_{11}}{\gamma(0)^2} = &\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2} + \frac{2\lambda}{\lambda+a} \frac{\lambda+2a}{2a} \left((\eta-3) \frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right) \\
&+ \frac{\lambda}{\lambda+2a} \left((\eta-3)a + 2 + \frac{2a}{\lambda} \right).
\end{aligned} \tag{A.18}
$$

We turn our attention to (A.12). First, the case $k=0$

$$
\begin{aligned}
\kappa_f(0,0,1) &= (\eta-3)\sigma^4 \int_0^{\infty} \mathrm{e}^{-4au} \mathbf{E}(\mathrm{e}^{-aT_1}) \,\mathrm{d}u + 2\sigma^4 \mathbf{E}\biggl(\int_0^{\infty} \mathrm{e}^{-2au} \,\mathrm{d}u \int_0^{\infty} \mathrm{e}^{-2au} \mathrm{e}^{-aT_1} \,\mathrm{d}u \biggr) \\
&= \frac{\sigma^4}{4a^2} \frac{\lambda}{\lambda+a} ((\eta-3)a + 2).
\end{aligned} \tag{A.19}
$$

Next, when $k \geq 1$, we obtain

$$
\kappa_f(0,k,k+1) = (\eta-3)\sigma^4 \int_0^{\infty} \mathrm{e}^{-4au} \mathbf{E}(\mathrm{e}^{-aT_k} \mathrm{e}^{-aT_{k+1}}) \,\mathrm{d}u
$$

$$
\begin{aligned}
&+ 2\sigma^4 \mathbf{E}\biggl(\int_0^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_k}\,\mathrm{d}u \int_0^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_{k+1}}\,\mathrm{d}u \biggr) \\
&= \frac{\sigma^4}{4a^2}\Bigl((\eta-3)a\mathbf{E}(\mathrm{e}^{-2aT_k})\mathbf{E}(\mathrm{e}^{-aW_1}) + 2\mathbf{E}(\mathrm{e}^{-2aT_k})\mathbf{E}(\mathrm{e}^{-aW_1}) \Bigr) \\
&= \frac{\sigma^4}{4a^2}\frac{\lambda}{\lambda+a}\left(\frac{\lambda}{\lambda+2a}\right)^k ((\eta-3)a+2)
\end{aligned}
$$

such that together with (A.19)

$$
\begin{aligned}
\sum_{k=0}^\infty \kappa_f(0,k,k+1) &= \frac{\sigma^4}{4a^2}\frac{\lambda}{\lambda+a}\sum_{k=0}^\infty \left(\frac{\lambda}{\lambda+2a}\right)^k ((\eta-3)a+2) \\
&= \frac{\sigma^4}{4a^2}\frac{\lambda}{\lambda+a}\frac{\lambda+2a}{2a}((\eta-3)a+2)\,. \qquad \text{(A.20)}
\end{aligned}
$$

When $-k \geq 1$,

$$
\begin{aligned}
\kappa_f(0,-k,-k+1) &= (\eta-3)\sigma^4 \int_{-T_{-k}}^\infty \mathrm{e}^{-4au}\mathbf{E}(\mathrm{e}^{-aT_{-k}}\mathrm{e}^{-aT_{-k+1}})\,\mathrm{d}u \\
&\quad + 2\sigma^4 \mathbf{E}\biggl(\int_{-T_{-k}}^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_{-k}}\,\mathrm{d}u \int_{-T_{-k+1}}^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_{-k+1}}\,\mathrm{d}u \biggr) \\
&= \frac{\sigma^4}{4a^2}\Bigl((\eta-3)a\mathbf{E}(\mathrm{e}^{-2aT_{k-1}})\mathbf{E}(\mathrm{e}^{-3aW_1}) + 2\mathbf{E}(\mathrm{e}^{-2aT_{k-1}})\mathbf{E}(\mathrm{e}^{-aW_1}) \Bigr) \\
&= \frac{\sigma^4}{4a^2}\left(\frac{\lambda}{\lambda+2a}\right)^{k-1}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right)
\end{aligned}
$$

and so

$$
\sum_{k=1}^\infty \kappa_f(0,-k,-k+1) = \frac{\sigma^4}{4a^2}\frac{\lambda+2a}{2a}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} \right). \qquad \text{(A.21)}
$$

This gives for (A.12), by (A.20) and (A.21),

$$
\mathbf{Z}_{01} = \frac{\sigma^4}{4a^2}\frac{\lambda+2a}{2a}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} + \frac{\lambda}{\lambda+a}((\eta-3)a+2) \right),
$$

and hence, since $\rho(1) = \frac{\lambda}{\lambda+a}$,

$$
\frac{2\rho(1)\mathbf{Z}_{01}}{\gamma(0)^2} = \frac{2\lambda}{\lambda+a}\frac{\lambda+2a}{2a}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a} + \frac{\lambda}{\lambda+a}((\eta-3)a+2) \right). \qquad \text{(A.22)}
$$

Last but not least, we consider (A.13). We start with the case $k = 0$

$$
\begin{aligned}
\kappa_f(0,0,0) &= (\eta-3)\sigma^4 \int_0^\infty \mathrm{e}^{-4au}\,\mathrm{d}u + 2\sigma^4\mathbf{E}\biggl(\int_0^\infty \mathrm{e}^{-2au}\,\mathrm{d}u \int_0^\infty \mathrm{e}^{-2au}\,\mathrm{d}u \biggr) \\
&= \frac{\sigma^4}{4a^2}((\eta-3)a+2)\,. \qquad \text{(A.23)}
\end{aligned}
$$

Next, when $k \geq 1$, we obtain

$$\begin{aligned}\kappa_f(0,k,k) &= (\eta-3)\sigma^4 \int_0^\infty \mathrm{e}^{-4au}\mathbf{E}(\mathrm{e}^{-2aT_k})\,\mathrm{d}u + 2\sigma^4\mathbf{E}\left(\left(\int_0^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_k}\,\mathrm{d}u\right)^2\right)\\ &= \frac{\sigma^4}{4a^2}\left(\frac{\lambda}{\lambda+2a}\right)^k((\eta-3)a+2)\end{aligned}$$

such that together with (A.23)

$$\sum_{k=0}^{\infty}\kappa_f(0,k,k) = \frac{\sigma^4}{4a^2}\frac{\lambda+2a}{2a}((\eta-3)a+2)\,. \tag{A.24}$$

When $-k \geq 1$,

$$\begin{aligned}\kappa_f(0,-k,-k) &= (\eta-3)\sigma^4 \int_{-T_{-k}}^\infty \mathrm{e}^{-4au}\mathbf{E}(\mathrm{e}^{-2aT_{-k}})\,\mathrm{d}u + 2\sigma^4\mathbf{E}\left(\left(\int_{-T_{-k}}^\infty \mathrm{e}^{-2au}\mathrm{e}^{-aT_{-k}}\,\mathrm{d}u\right)^2\right)\\ &= \frac{\sigma^4}{4a^2}\Big((\eta-3)a\mathbf{E}(\mathrm{e}^{-2aT_k}) + 2\mathbf{E}(\mathrm{e}^{-2aT_k})\Big)\\ &= \frac{\sigma^4}{4a^2}\left(\frac{\lambda}{\lambda+2a}\right)^k((\eta-3)a+2)\end{aligned}$$

and so

$$\sum_{k=1}^{\infty}\kappa_f(0,-k,-k) = \frac{\sigma^4}{4a^2}\frac{\lambda}{2a}((\eta-3)a+2)\,. \tag{A.25}$$

This gives for (A.13), by (A.24) and (A.25),

$$\mathbf{Z}_{00} = \frac{\sigma^4}{4a^2}\frac{\lambda+a}{a}((\eta-3)a+2)\,,$$

and hence

$$\frac{2\rho(1)^2\mathbf{Z}_{01}}{\gamma(0)^2} = \left(\frac{\lambda}{\lambda+a}\right)^2\frac{\lambda+a}{a}((\eta-3)a+2)\,. \tag{A.26}$$

Overall, using in the following expression (A.18), (A.22), and (A.26), we obtain

$$\begin{aligned}\mathbf{W}_{11} &= (\mathbf{Z}_{11} - 2\rho(1)\mathbf{Z}_{01} + \rho(1)^2\mathbf{Z}_{00})/\gamma(0)^2\\ &= \frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2} + \frac{2\lambda}{\lambda+a}\frac{\lambda+2a}{2a}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a}\right)\\ &\quad + \frac{\lambda}{\lambda+2a}\left((\eta-3)a+2+\frac{2a}{\lambda}\right) - \frac{2\lambda}{\lambda+a}\frac{\lambda+2a}{2a}\left((\eta-3)\frac{a\lambda}{\lambda+3a} + \frac{2\lambda}{\lambda+a}\right)\end{aligned}$$

$$
\begin{aligned}
&- 2\left(\frac{\lambda}{\lambda+a}\right)^2 \frac{\lambda+2a}{2a}((\eta-3)a+2) + \left(\frac{\lambda}{\lambda+a}\right)^2 \frac{\lambda+a}{a}((\eta-3)a+2) \\
&= \frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2} + \frac{\lambda}{\lambda+2a}((\eta-3)a+2) + \frac{2a}{\lambda+2a} - \left(\frac{\lambda}{\lambda+a}\right)^2((\eta-3)a+2) \\
&= \left(\frac{\lambda}{\lambda+2a} - \frac{\lambda^2}{(\lambda+a)^2}\right)((\eta-3)a+3) + \frac{2a}{\lambda+2a}\,,
\end{aligned}
$$

which is (A.10). □

Lemma A.17. *The asymptotic variance of*

$$
\sqrt{n}\left(\left(\gamma_n^*(0), \gamma_n^*(1), \frac{1}{n}\sum_{k=1}^{n} W_{k+1}\right) - \left(\gamma(0), \gamma(1), \frac{1}{\lambda}\right)\right) \xrightarrow{d} N(0, \Sigma)\,, \quad n \to \infty\,,
$$

appearing in the proof of Theorem 4.23 is given by (4.46), i.e. of the form

$$
\begin{aligned}
\Sigma &= \sum_{k\in\mathbb{Z}} \begin{pmatrix} \mathbf{Cov}(Y_0^2, Y_k^2) & \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) & \mathbf{Cov}(Y_0^2, T_{k+1}-T_k) \\ \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) & \mathbf{Cov}(Y_0Y_1, Y_kY_{k+1}) & \mathbf{Cov}(Y_0Y_1, T_{k+1}-T_k) \\ \mathbf{Cov}(Y_0^2, T_{k+1}-T_k) & \mathbf{Cov}(Y_0Y_1, T_{k+1}-T_k) & \mathbf{Cov}(T_1, T_{k+1}-T_k) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{Z}_{00} & \mathbf{Z}_{01} & 0 \\ \mathbf{Z}_{10} & \mathbf{Z}_{11} & -\frac{\sigma^2}{2(\lambda+a)^2} \\ 0 & -\frac{\sigma^2}{2(\lambda+a)^2} & \frac{1}{\lambda^2} \end{pmatrix}
\end{aligned}
\tag{A.27}
$$

Proof. Observe that

$$
\Sigma = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \mathbf{Cov}(Y_0^2, Y_k^2) & \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) \\ \mathbf{Cov}(Y_0^2, Y_kY_{k+1}) & \mathbf{Cov}(Y_0Y_1, Y_kY_{k+1}) \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{00} & \mathbf{Z}_{01} \\ \mathbf{Z}_{10} & \mathbf{Z}_{11} \end{pmatrix}
$$

follows directly from Theorem 4.20 (a). Since $T_{k+1} - T_k \perp\!\!\!\perp X_0^2$ for all $k \in \mathbb{Z}$,

$$
\sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0^2, T_{k+1}-T_k) = \sum_{k\in\mathbb{Z}} \mathbf{Cov}(X_0^2, T_{k+1}-T_k) = 0\,.
$$

By the i.i.d. property of $(T_{k+1} - T_k)_{k\in\mathbb{Z}}$ and the assumption that $T_{k+1} - T_k \sim \mathrm{Exp}(\lambda)$, we obtain

$$
\sum_{k\in\mathbb{Z}} \mathbf{Cov}(T_1, T_{k+1}-T_k) = \mathbf{Var}(T_1, T_1) = \frac{1}{\lambda^2}\,.
$$

Further, since $T_{k+1} - T_k \perp\!\!\!\perp T_1$ for all $k \in \mathbb{Z}\setminus\{0\}$, $\mathbf{E}(X_0X_{T_1}) = \frac{\sigma^2}{2a}\frac{\lambda}{\lambda+a}$, $\mathbf{E}(X_0, X_t) = \frac{\sigma^2}{2a}\mathrm{e}^{-at}$, and $\mathbf{E}(T_1) = \frac{1}{\lambda}$, we calculate

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathbf{Cov}(Y_0Y_1, T_{k+1}-T_k) &= \mathbf{Cov}(Y_0Y_1, T_1) \\
&= \mathbf{E}(X_0X_{T_1}T_1) - \mathbf{E}(X_0X_{T_1})\mathbf{E}(T_1)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{[0,\infty)} \mathbf{E}(X_0 X_t t) P_{T_1}(\mathrm{d}t) - \frac{\sigma^2}{2a}\frac{1}{\lambda+a} \\
&= \int_{[0,\infty)} t\frac{\sigma^2}{2a}\mathrm{e}^{-at}\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t - \frac{\sigma^2}{2a}\frac{1}{\lambda+a} \\
&= \frac{\sigma^2}{2a}\frac{\lambda}{(\lambda+a)^2}\int_{[0,\infty)} \frac{t^{2-1}}{\gamma(2)}\lambda\mathrm{e}^{-(\lambda+a)t}\,\mathrm{d}t - \frac{\sigma^2}{2a}\frac{1}{\lambda+a} \\
&= \frac{\sigma^2}{2a}\left(\frac{\lambda}{(\lambda+a)^2} - \frac{1}{\lambda+a}\right) \\
&= \frac{\sigma^2}{2a}\frac{-a}{(\lambda+a)^2} = -\frac{\sigma^2}{2(\lambda+a)^2}\,.
\end{aligned}
$$

So, we obtain the form given in (A.27). □

Lemma A.18. *The mean-reverting parameter of the OU process $X = (X_t)_{t\in\mathbb{R}}$ can be given in terms of the autocorrelation function as in (4.48) such that we suggest as an estimator*

$$\widehat{a}_{eq} := -\frac{\log(\rho^*_{eq}(\Delta))}{\Delta}\,,$$

where $\rho^*_{eq}(\Delta) = \gamma^*_{eq;n;\Delta}(\Delta)/\gamma^*_{eq;n;\Delta}(0)$ with $\gamma^*_{eq;n;\Delta}(h\Delta) = \frac{1}{n}\sum_{t=1}^{n} X_{t\Delta}X_{(t+h)\Delta}$, for $h \in \mathbb{N}$. Then $\widehat{a}_{eq}$ satisfies

$$\sqrt{n}(\widehat{a}_{eq} - a) \xrightarrow{d} N(0, \Delta^{-2}(\mathrm{e}^{2a\Delta} - 1))\,, \quad n \to \infty\,.$$

Proof. By Theorem 3.5 of Cohen and Lindner [31], cf. also Theorem 1.35, we have

$$\sqrt{n}(\rho^*_{eq}(\Delta) - \rho_{eq}(\Delta)) \xrightarrow{d} N(0, V)\,, \quad n \to \infty\,,$$

where

$$
\begin{aligned}
V = {} & \frac{(\eta-3)\sigma^4}{\gamma_{eq}(0)^2}\int_0^{\Delta}(g_{1;\Delta}(u) - \rho(\Delta)g_{0;\Delta})^2\,\mathrm{d}u \\
& + \sum_{k=1}^{\infty}(\rho((k+1)\Delta) + \rho((k-1)\Delta) - 2\rho(\Delta)\rho(k\Delta))^2
\end{aligned}
$$

with

$$g_{q;\Delta}\colon [0,\Delta] \to \mathbb{R}\,,\ u \mapsto \sum_{k=-\infty}^{\infty} f(u+k\Delta)f(u+(k+q)\Delta)$$

given as in Proposition 3.1 of Cohen and Lindner [31].

Observe that

$$g_{0;\Delta}(u) = \sum_{k=-\infty}^{\infty} f(u+k\Delta)^2 = \sum_{k=-\infty}^{\infty} \mathbf{1}_{[-k\Delta,\infty)}(u)\mathrm{e}^{-2a(u+k\Delta)}$$

and

$$
\begin{aligned}
g_{1;\Delta}(u) &= \sum_{k=-\infty}^{\infty} f(u+k\Delta)f(u+(k+1)\Delta) \\
&= \sum_{k=-\infty}^{\infty} \mathbf{1}_{[-k\Delta,\infty)}(u)\mathbf{1}_{[-(k+1)\Delta,\infty)}(u)\mathrm{e}^{-a(u+k\Delta)}\mathrm{e}^{-a(u+(k+1)\Delta)} \\
&= \sum_{k=-\infty}^{\infty} \mathbf{1}_{[-k\Delta,\infty)}(u)\mathrm{e}^{-2a(u+k\Delta)}\mathrm{e}^{-a\Delta}
\end{aligned}
$$

such that, since $\rho(\Delta) = \mathrm{e}^{-a\Delta}$,

$$
\frac{(\eta-3)\sigma^4}{\gamma(0)^2}\int_0^{\Delta} g_{1;\Delta}(u) - \rho(\Delta)g_{0;\Delta}(u)\,\mathrm{d}u = 0\,.
$$

Henceforth, we obtain

$$
\begin{aligned}
V &= \sum_{k=1}^{\infty}(\rho((k+1)\Delta) + \rho((k-1)\Delta) - 2\rho(\Delta)\rho(k\Delta))^2 \\
&= \sum_{k=1}^{\infty}(\mathrm{e}^{-a(k+1)\Delta} + \mathrm{e}^{-a(k-1)\Delta} - 2\mathrm{e}^{-a\Delta}\mathrm{e}^{-ak\Delta})^2 \\
&= \sum_{k=1}^{\infty}(\mathrm{e}^{-a(k-1)\Delta} - \mathrm{e}^{-a(k+1)\Delta})^2 \\
&= \sum_{k=0}^{\infty}(\mathrm{e}^{-2a\Delta})^k - 2\sum_{k=1}^{\infty}(\mathrm{e}^{-2a\Delta})^k + \sum_{k=1}^{\infty}(\mathrm{e}^{-2a\Delta})^{k+1} \\
&= \frac{1}{1-\mathrm{e}^{-2a\Delta}} - 2\left(\frac{1}{1-\mathrm{e}^{-2a\Delta}} - 1\right) + \left(\frac{1}{1-\mathrm{e}^{-2a\Delta}} - 1 - \mathrm{e}^{-2a\Delta}\right) \\
&= 1 - \mathrm{e}^{-2a\Delta}\,.
\end{aligned}
$$

To complete the proof, define $h_\Delta\colon (0,\infty) \to \mathbb{R}, x \mapsto -\frac{\log(x)}{\Delta}$ with $h'_\Delta(x) = -\frac{1}{\Delta x}$ such that $h_\Delta(\rho^*_{eq}(\Delta)) = \widehat{a}_{eq}$ and the delta-method, cf. Proposition 6.4.3 in Brockwell and Davis [23], yields

$$
\sqrt{n}(\widehat{a}_{eq} - a) \xrightarrow{d} N(0, h'_\Delta(\rho(\Delta))Vh'_\Delta(\rho(\Delta)))\,, \quad n \to \infty\,,
$$

where

$$
h'_\Delta(\rho(\Delta)) = -\frac{1}{\Delta}\mathrm{e}^{a\Delta}\,,
$$

which then gives the result. □

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Erklärung

Hiermit versichere ich, Dirk-Philip Brandes, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ich erkläre außerdem, dass diese Arbeit weder im In- noch im Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

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