Involutive Reductions and Solutions of Differential Equations

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Für Karin

We dance around in a ring and suppose,
But the Secret sits in the middle and knows.

Robert Frost

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Chapter 1

Introduction

The Norwegian mathematician Marius Sophus Lie pioneered the study of transformation groups that leave differential equations invariant. He founded the theory of continuous transformation groups and Lie groups [1, 2, 3, 4, 5, 6, 7, 8]. Although Lie started with geometry, his group theoretic investigations lead him to the study of differential equations. His goal was to establish a broad theory of integration of differential equations that would incorporate the diverse and ad hoc integration methods for solving special classes of differential equations under a common concept.
From this the concept of **symmetry** evolved. This concept has lead to many developments throughout the twentieth century. The theory of Lie groups and Lie algebras is now applied to diverse fields of mathematics and to nearly any area of theoretical physics, in particular classical and quantum mechanics, fluid dynamics, relativity or particle physics.
Lie's method of infinitesimal transformations provides the most widely applicable technique to find closed form solutions of ordinary and partial differential equations.

For nonlinear ordinary differential equations Lie's method provides a means of reducing the solution to a series of quadratures. Through group classification Lie succeeded in identifying all ordinary differential equations that can either be reduced to lower-order ones or completely integrated via group theoretic techniques.
Applied to partial differential equations the symmetry method leads to group invariant solutions and conservation laws. Exploiting the symmetries new solutions can be derived from old ones and partial differential equations can be classified into equivalence classes. The group-invariant solutions obtained by Lie's approach provide insight into physical models themselves and serve as benchmarks in the design, accuracy testing and comparison of numerical algorithms.
The ideas of Lie influenced the study of physically important systems of differential equations for example in classical and celestial mechanics, fluid dynamics,

elasticity and many other applied areas. In these fields the theory of transformation groups led to new important solutions. In [9] and the references therein numerous examples and solutions are treated starting from classical field theory, on to the Dirac and the Klein-Gordon equation, models of incompressible and non-Newtonian fluids, boundary layer problems, magneto-hydrodynamics, nonlinear optics and acoustics.

As mentioned in section 2.9 there are various generalizations of Lie's original method, where the transformations act on the dependent and the independent variables. Lie himself already studied transformations which also depend upon the first-order derivatives of the dependent variables, which results in the so-called **contact transformations**. This concept of generalization by transforming in a larger space surmounted in the theory of **Lie-Bäcklund transformations**. The famous theory by Emmy Noether demonstrates [11, 12] that if such groups exist, they have applications in the theory of conservation laws, where the connection between symmetries and conservation laws is given by Noether's theorem.
When speaking of Lie-Bäcklund transformations the so-called **recursion operators** also have to be mentioned. These are operators which typically depend upon derivatives of the dependent variables and which commute with the symmetry operator [39]. In this way they generate infinitely many symmetries and solutions of a differential equation. Such differential equations are then called completely integrable. These sorts of equations are also solvable with the inverse scattering method. One famous example of such equations admitting recursion operators is the Korteweg-de-Vries equation.
Another source of generalizations was the inclusion of conditions to the differential equations. This leads to the so-called **nonclassical** or **conditional symmetries**. They were first introduced by Bluman and Cole [13] who studied the general solution of the heat equation. The nonclassical method, as it is also called, leads to the solution of physically significant nonlinear partial differential equations, such as the nonlinear Schrödinger equation [14], the Boussinesq equation [15], the Kadomtsev-Petviashvili equation [16] or the Fitzhugh-Nagumo equation [17]. For an introduction on nonclassical symmetries we refer to Clarkson and Mansfield [18].
If the differential equations include small perturbations the theory of **approximate transformation groups** applies [19]. Such transformation groups then lead to approximate symmetry Lie algebras and to approximate invariant solutions and conservation laws.
Another generalization leads to **potential symmetries**. In contrast to the symmetries mentioned above potential symmetries are *nonlocal* symmetries. They no longer depend only on the independent variables, the dependent variables and their derivatives. In fact if the differential equation under consideration can be written in conserved form the ordinary symmetries of this conserved quantity lead to transformations of the original differential equation which depend on the

potential of the dependent variables, hence the name.

The application of the method of Lie groups to concrete physical systems of differential equations involves tedious and messy computations. Even for relatively simple equations the calculations are bound to fail if done with pencil and paper. In such a situation computer algebra systems such as *Mathematica*, `MAPLE` or `REDUCE` are very useful. Today there exist a variety of symbolic packages which are able to calculate the defining or determining equations of the Lie symmetry group of a differential equation. The more sophisticated packages then reduce the determining system into an equivalent but more suitable system, solve that system and calculate the infinitesimal generators that span the Lie algebra of symmetries.
The `REDUCE` package SPDE developed by Schwarz [20] automatically derives and often successfully solves the determining equations for Lie point symmetries with the intervention by the user. The package CRACK by Wolf and Brand [21] solves overdetermined systems of differential equations with polynomial terms. Based on CRACK the `REDUCE`-packages LIEPDE and APPLYSYM by Wolf [19, 22] calculate Lie point and contact symmetries by deriving and solving a few simple determining equations before continuing with the computation of the more complicated ones. The solutions obtained are then used by APPLYSYM to reduce the differential equation. This package is applicable only for point symmetries where the generators are at worst rational.
The MACSYMA-package `SYMMGRP.MAX` by Champagne, Hereman and Winternitz [23] has been widely used and tested in hundreds of equations. It has the possibility of using it interactively to allow the user to find symmetry groups.
The *Mathematica*-package **MathLie** by Baumann [24] allows the computation of point, approximate, nonclassical, potential and Lie-Bäcklund symmetries. Furthermore the package is able to use the symmetries obtained to reach a reduction of the original differential equation. This package also allows operations concerning Lie algebras.

Another important aspect of calculating symmetries of differential equations deals with the simplification of the determining system. There are various implementations of such simplifications. Perhaps the earliest go back to 1974 where Arais et al. implemented Cartan's exterior form approach to involutive systems [25]. For such systems all integrability conditions are identically satisfied. Early implementations of the method of Riquier and Janet to transform an arbitrary system of differential equations to an involutive system already appear in [26, 27, 28]. The implementation by Topunov in `REFAL` can only reduce linear systems of partial differential equations. In the 1990's the works of Schwarz [29] and Reid and collaborators [30, 31, 32, 33, 34] lead to sophisticated implementations of such simplification algorithms. Schwarz described an algorithm based on the theory

of Riquier and Janet [35, 36] to transform a linear system of partial differential equations into involutive form. In modern language the involutive form is a differential Groebner basis with respect to a selected term ordering. The purpose of this implementation was to determine the size of the Lie symmetry group of a given system of partial differential equations without having to integrate the determining equations.
The implementations of Reid et al. in `MACSYMA` and later `MAPLE` reduces systems of partial differential equations to a simplified standard form as it is called. This procedure can be seen as a generalization of the Gaussian reduction method for matrices or linear systems. It is applicable to linear systems and reduces them to an equivalent simplified ordered triangular system with all integrability conditions included and all redundancies eliminated.
The `MAPLE`-program **`diffgrob2`** by Mansfield [37] is designed to calculate the differential Groebner basis of a finitely generated ideal of partial differential equations with polynomial terms. It allows the computation of elimination ideals, integrability conditions and compatibility conditions of a system of polynomially nonlinear partial differential equations, up to certain constraints which are explained in [37].
In 1996 Wittkopf developed an algorithm to reduce polynomially nonlinear systems of partial differential equations to the form of a reduced differential Groebner basis [38]. In essence the algorithm is a differential analogue of Buchberger's elimination algorithm for polynomial equations, to which it refers.

The above survey of methods and symbolic program packages to calculate certain kinds of symmetries already shows the directions in which todays theory of symmetry analysis is heading. On the one side there are the theoretical aspects. They involve the generalization of the theory to gain more kinds of symmetries and to draw conclusions from them. These conclusions include the solutions arising from the symmetries. But for partial differential equations there seem to be only two methods to get solutions. These are the **method of characteristics** and the **direct insertion**. The solutions obtained are so-called **invariant solutions**. To calculate invariant solutions the invariant surface condition is used. This condition ensures that solutions of the equation are transformed into themselves, and not into other solutions. The invariant surface condition is a first order linear partial differential equation.
For the method of characteristics this first order differential equation has to be solved first and the solutions have to be recognized as new dependent and independent variables. Inserting these invariants in the original differential equation a reduced equation is obtained. The problem for this method consists in the solution of the invariant surface condition. For simple partial differential equations of first order a solution is obtained very easy. But for more complicated equations the method runs into trouble. The calculation of the invariants can be

very difficult for some problems.
The direct method consists in solving the invariant surface condition with respect to one partial derivation. This derivation is then inserted in the original differential equation and a symmetry analysis of this new equation is performed. The problem here is the fast growth of the number of terms and complexity in the new differential equation because of the insertion. This leads to a considerable more time-consuming calculations to reach the new symmetry analysis.

Other problems concerning symmetry analysis are concerned with implementations of the existing theory into computer algebra packages. As mentioned above there exist two sorts of implementations. First there are the packages which solve coupled systems of partial differential equations, like `CRACK`, **MathLie** or `SYMMGRP.MAX`. Second, the packages which try to simplify systems of coupled partial differential equations. Such packages, like `rif`, `diffgrob2` use involutive methods incorporating integrability and compatibility conditions to reach a simplification of the coupled system. So both sorts of implementations are used to make the computations and solutions more easy.
But there is no implementation which uses the advantages of both kinds of implementations. There is no single package which simplifies **and** solves systems of coupled partial differential equations. Without doubt the coupling of both procedures could reach simplifications and solutions the single algorithms are not capable of. The two procedures could help each other in finding additional simplifications not possible by one alone. The involutive simplifier puts the system in a form which is easier to solve and the solutions obtained by the solver and their integrability conditions can be used to simplify the system even further.
One reason that there is no such implementation lies in the fact that the single procedures are written in different computer algebra packages by different people. To create the necessary interfaces the source codes of the various packages may have to be changed in part to allow such a coupling and some parts may have to be rewritten completely in another computer algebra system. For this to happen the source codes would have to be published.

Another problem which one has to face to reach solutions of differential equations by using symmetry analysis is concerned with the infinitesimal generators. If linear, these generators can be split into single generators which span a Lie algebra. The solutions are then found by using each subalgebra for the construction of a so-called **optimal system** of infinitesimal generators [39], from which the solutions corresponding to each optimized generator can be calculated.
But already in an example in [40] appear infinitesimal generators with products of group constants. If this is the case the division of the full generator into simpler ones by setting one group constant to one and the others to zero does not lead to linear independent generators and therefore to no Lie algebra. In [40]

the problem is solved by taking only some constants as group constants and the other ones as parameters of the solution which have to satisfy certain conditions. But there is no way one can tell which constants are group constants and which are just parameters.

The above mentioned drawbacks can be summarized as follows:

- no practical or automated method to gain invariant solutions automatically

- no coupling of involutive and solution methods to simplify **and** solve coupled systems of partial differential equations

- group structure constants appear nonlinearly and therefore analysis on the basis of Lie algebras is not possible for these cases.

So there is a clear need for other methods to reach for invariant solutions for differential equations, for a coupling of involutive and solution methods for systems of partial differential equations and for the problem of the nonlinear appearing group constants.
One solution out of this dilemma concerning the first item is to take a closer look at invariant solutions. As said above they are obtained by the solution of an additional differential equation, the invariant surface condition. So the invariant solutions are nothing but solutions of a coupled system of partial differential equations. For this reason the first and the second item above are related. If there is a useful tool for the simultaneous simplification and solution of a coupled system of partial differential equations, this tool directly leads to the automated construction of invariant solutions. The development, implementation and application of such an involutive solution procedure forms one part of this thesis.
The third problem, the nonlinear appearance of the group structure constants is solved in this work in the following way. There is simply no division of the original infinitesimal generator into a Lie algebra. For this reason we do not mention the methods concerning Lie algebras in this thesis and refer to [39, 40]. The full generator of the infinitesimal transformations is used to find invariant solutions. During the calculation of the solutions different cases for the constants are generated, simplified and reduced separately in the form of a case distinction. We will see that this leads to new solutions which cannot be found by using the division of the generator.
This is similar to a group classification problem occurring when trying to find equivalence transformations of a differential equation involving an arbitrary function. But here the different functional forms of the arbitrary function evolve very

naturally as prefactors which have to be assumed unequal to zero to reach a simplification and/or solution.

The following work is divided into nine parts. The second chapter introduces the necessary notions and criteria to perform a symmetry analysis and how to calculate the determining equations and invariant solutions.
The third chapter does the same for the concept of involutivity. Here the necessary steps to transform a system of coupled partial differential equations to an involutive form are listed and the single steps are explained and illustrated on examples.
The next part describes the solution steps incorporated into the heuristic solver. These steps form the skeleton of the solution part of the implementation of the involutive solution algorithm introduced here.
In the fifth chapter appear some notes on the implementation of the involutivity concept and the solution procedure. We illustrate how a new calculus dealing with lists is introduced to simplify the occurring calculations. This calculus, which we call **discrete involutive calculus**, covers the representation of terms, equations and systems with conditions as well as an own implementation of multiplications and differentiations of these list representations.
The involutive solver is then illustrated on simple examples in chapter six. On a step by step basis we will see which calculations are performed and how the solutions are built up.
Finally, the next three chapters deal with the application of the implemented procedures on more interesting and difficult examples. A full classification involving solutions is performed in chapter seven for a class of nonlinear diffusion equations involving an arbitrary diffusion function.
Chapter eight deals with equations appearing in financial mathematics, which also include boundary values. The most important of these boundary value problems is the Black-Scholes equation. The involutive reduction procedure is applied to this equation and some generalizations.
Chapter nine then features some solutions which were obtained by applying the involutive solution procedure to some physical problems, for example the diffusion of a chemically reactive species.

Chapter 2

The Symmetry Concept

Before studying the invariance properties of differential equations we need to introduce the basic concept. This concept, the symmetry concept, contains the notions of groups, groups of transformations, Lie groups of transformations and infinitesimal transformations. On the basis of these notions we study the invariance of surfaces under the infinitesimal transformations of Lie groups. Such transformations are then called **symmetries**.
We will see that Lie groups of transformations are completely determined by their infinitesimal behavior. Given this infinitesimal Lie group the global group of transformations can be recovered. But for our purposes the infinitesimal transformations, or infinitesimals as they are called, is all we need.
Furthermore, Lie groups of transformations which act on the space of dependent and independent variables are naturally extended to Lie groups of transformations acting on any enlarged space which includes all derivatives of the dependent variables up to a fixed finite order. To do this preservation of the derivative relations or contact conditions which connect higher order differentials has to be required. This preservation induces a unique extended group action in any enlarged space. In this enlarged space the derivatives of the dependent variables are treated as coordinates too. The group structure is also enlarged naturally to this prolonged space.
As a consequence of the enlargement of the group structure, extended Lie groups of transformations are characterized completely by their infinitesimals. These extended infinitesimals are determined completely by the infinitesimals of the unprolonged group which acts on the space of independent and dependent variables. It is then very natural to look at differential equations as hypersurfaces in this extended or prolonged space. Consequently, the invariance principles of surfaces extend to the invariance principles of differential equations. This allows the construction of an algorithm to determine the infinitesimal transformations admitted by a given differential equation.
Note that during this work we treat only point transformations. These are transformations on the space of the dependent and independent variables. This means

that the unprolonged transformations and with them the infinitesimal transformations only depend on the dependent and independent variables, and not on their derivatives. If the unprolonged transformations do depend on derivatives of the dependent variables these transformations are generally called Lie-Bäcklund or Bäcklund transformations. These are treated for example in [39, 24, 40]. Other types of transformations are mentioned in section 2.9.
The first object which is needed to understand the notion of symmetry is a group.

2.1 Group

Definition 2.1.0.1 *Definition: Group*
A set G of elements with a law composition ϕ which satisfies

i) **Closure***:* $\phi(a, b) \in G \quad \forall a, b \in G$

ii) **Associativity***:* $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c) \quad \forall a, b, c \in G$

iii) **Identity***:* $\exists\ e \in G$ *unique such that* $\phi(a, e) = \phi(e, a) = a \quad \forall a \in G$

iv) **Inverse***:* $\forall a \in G \ \exists\ a^{-1}$ *unique such that* $\phi\left(a, a^{-1}\right) = \phi\left(a^{-1}, a\right) = e$

is called a group.

Note that a group G is called **Abelian** if $\phi(a, b) = \phi(b, a)$ for all elements a, b in G. A subgroup of G is a group formed by a subset of elements of G with the same law of composition ϕ.
Some examples of groups are given by:

a) The set of all reals with the law of composition given by $\phi(a, b) = a + b$ is a group. The identity element is given by $e = 0$ and the inverse of an element a is $a^{-1} = \frac{1}{a}$. This group is Abelian.

b) The set $GL(n, R)$ of invertible $n \times n$ matrices with rational entries forms a group. The law of composition is given by matrix multiplication, the identity is the identity matrix and the inverse element is the inverse of a given matrix. Note that this group is not Abelian since the matrix multiplication is not commutative.

c) The set of all integers with the law of composition given by $\phi(a, b) = a + b$ is also a group. In fact it is a subgroup of all reals with the same law of composition mentioned in a).

In the following we will concentrate on special groups. Since we want to study the invariance of some differential equation under certain transformations the groups considered further on are groups of transformations.

2.2 Groups and Lie Groups of Transformations

Definition 2.2.0.2 *Definition: Group of Transformation:*
For $x \in D \subset \mathbb{R}^n$, ε, $\delta \in S \subset \mathbb{R}$, ϕ a law of composition of parameters ε, δ

$$x^* = X(x;\varepsilon) \tag{2.1}$$

is a group of transformations in D if

i) $\forall \varepsilon \in S$ the transformations (2.1) are one-to-one and onto D, in particular $x^* \in D$

ii) S with law of composition ϕ is a group

iii) $x^* = x$ for $\varepsilon = e$ (identity: $X(x;e) = x$)

iv) if $x^* = X(x;\varepsilon)$, $x^{**} = X(x^*;\delta)$ then $x^{**} = X(x;(\phi(\varepsilon,\delta))$.

Note that the law of composition in this definition is an arbitrary one. The parameter ε is an element of a set $S \subset \mathbb{R}$. A special group of transformations is obtained if this parameter is continuous in $\mathbb{R}$. These transformations then form a so-called Lie group of transformations:

Definition 2.2.0.3 *Definition: Lie Group of Transformations:*
A Lie group of transformations is a group of transformations satisfying i) - iv) above and additionally:

v) the parameter of the group ε is continuous on $S \subset \mathbb{R}$. Without loss of generality $\varepsilon = 0$ can be chosen to be the identity e

vi) X is differentiable to the order necessary in the following calculations with respect to $x \in D$ and an analytic function of ε in S

vii) the law of composition $\phi(\varepsilon,\delta)$ is analytic in ε, δ $\forall \varepsilon$, $\delta \in S$

Note that these conditions assure that the transformations (2.1) are smooth enough such that the operations needed to calculate symmetries are permitted.
Consider for example the group of **translations** along the x-axis in a plane. If the coordinates of the plane are x and y this Lie group of transformations is given by

$$\begin{aligned} x^* &= x + \varepsilon, \\ y^* &= y \end{aligned} \tag{2.2}$$

for $\varepsilon \in \mathbb{R}$. Here the law of composition is $\phi(\varepsilon,\delta) = \varepsilon + \delta$. This group corresponds to a motion parallel to the x-axis.
Another important Lie group of transformations are the **scalings**. Considering

the three coordinates x, y and z, a scaling of the x and y coordinates is given for example by

$$\begin{aligned} x^* &= \alpha x, \\ y^* &= \alpha^2 y, \\ z^* &= z. \end{aligned} \tag{2.3}$$

Here the law of composition is $\phi(\alpha, \beta) = \alpha\beta$ and the identity element is $e = 1$. Note that this group of transformations can be re-parametrized by introducing the new parameter $\varepsilon = \alpha - 1$. In terms of this new group parameter (2.3) can be rewritten as

$$\begin{aligned} x^* &= (1+\varepsilon)x, \\ y^* &= (1+\varepsilon)^2 y, \\ z^* &= z \end{aligned}$$

so that the new identity element is $e = 0$.
As a last example for Lie groups of transformations we mention the **rotations**. In two dimensions this group is represented by $G = SO(2)$ or in coordinates:

$$\begin{aligned} x^* &= x\cos\theta - y\sin\theta, \\ y^* &= x\sin\theta + y\cos\theta \end{aligned} \tag{2.4}$$

where $0 \le \theta < 2\pi$ is the angle of rotation.

2.3 Infinitesimal Transformations

The analytic dependence of (2.1) on ε in the neighborhood of the identity $\varepsilon = 0$ implies the existence of the infinitesimal transformations.
Consider a Lie group of transformations $x^* = X(x;\varepsilon)$ with the identity $\varepsilon = 0$ and a law of composition ϕ. Expanding this transformation around $\varepsilon = 0$ leads to:

$$\begin{aligned} x^* &= X(x;0) + \varepsilon \cdot \left.\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left.\frac{\partial^2 X(x;\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} + 0(\varepsilon^3) \\ &= x + \varepsilon \cdot \left.\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + 0(\varepsilon^2). \end{aligned}$$

This expansion around $\varepsilon = 0$ is called the **infinitesimal transformation** of the Lie group of transformations (2.1). Hereby the elements

$$\xi(x) = \left.\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

are called the **infinitesimals** of the transformation.
These infinitesimal transformations prove to be very important in the construction of solutions of differential equations. They define the transformations locally. In fact the global transformation (2.1) can be reconstructed from the infinitesimals. This is the content of the first fundamental theorem of Lie (see [40])

Theorem 2.3.0.1 *Lie's first Fundamental Theorem:*
There is a parameterization $\tau(\varepsilon)$ such that the Lie group of transformations

$$x^* = X(x;\varepsilon)$$

is equivalent to the solution of the initial value problem for the system of first order differential equations

$$\frac{dx^*}{d\tau} = \xi(x^*) \quad \textit{with} \quad x^* = x \quad \textit{when} \quad \tau = 0. \tag{2.5}$$

This parameterization is given by

$$\tau(\varepsilon) = \int_0^{\varepsilon} \Gamma(\varepsilon')d\varepsilon'$$

where

$$\Gamma(\varepsilon) = \left.\frac{\partial\phi(a,b)}{\partial b}\right|_{(a,b)=(\varepsilon^{-1},\varepsilon)}$$

and

$$\Gamma(0) = 1.$$

The law of composition for this new parameterization is given by $\phi(\tau_1,\tau_2) = \tau_1+\tau_2$ so that $\varepsilon^{-1} = -\varepsilon$.

This theorem shows that the infinitesimal transformations contain the essential information determining a Lie group of transformations. The global group is fully determined by the infinitesimals, i.e. by the local behavior of the group of transformations. The global transformation can be recovered by solving the initial value problem (2.5). Additionally it shows that a Lie group of transformations (2.1) defines a stationary flow.
To illustrate Lie's first fundamental Theorem we go back to the examples of Lie groups of transformations above. For the group of translations (2.2) $\varepsilon^{-1} = -\varepsilon$, $\frac{\partial\phi(a,b)}{\partial b} = 1$ and hence $\Gamma(\varepsilon) = 1$. Since $X(x;\varepsilon) = (x+\varepsilon, y)$ we get $\frac{\partial X(x;\varepsilon)}{\partial\varepsilon} = (1,0)$ and the infinitesimals are given by

$$\xi(\varepsilon) = \left.\frac{\partial X(x;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = (1,0).$$

Therefore the system of first-order ordinary differential equations (2.5) becomes

$$\frac{dx^*}{d\varepsilon} = 1, \quad \frac{dy^*}{d\varepsilon} = 0 \quad \text{with} \quad x^*(0) = x, \ y^*(0) = y.$$

The solution to this initial value problem is seen to be (2.2).
For the group of scalings (2.3) $\varepsilon^{-1} = -\frac{\varepsilon}{1+\varepsilon}$ and $\frac{\partial\phi(a,b)}{\partial b} = 1 + a$. Thus

$$\Gamma(\varepsilon) = \left.\frac{\partial\phi(a,b)}{\partial b}\right|_{(a,b)=(\varepsilon^{-1},\varepsilon)} = 1 + \varepsilon^{-1} = \frac{1}{1+\varepsilon}.$$

Furthermore,

$$\xi(x,y,z) = \left.\frac{\partial X(x;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = (x, 2y, 0).$$

Inserting these results in (2.5) leads to

$$\frac{dx^*}{d\varepsilon} = \frac{x^*}{1+\varepsilon}, \quad \frac{dy^*}{d\varepsilon} = \frac{2y^*}{1+\varepsilon} \quad \text{with} \quad x^*(0) = x, \ y^*(0) = y$$

whose solution is (2.3). The new parameter is given by

$$\tau = \int_0^\varepsilon \Gamma(\varepsilon')d\varepsilon' = \int_0^\varepsilon \frac{d\varepsilon'}{1+\varepsilon'} = \log(1+\varepsilon)$$

and the group (2.3) becomes

$$\begin{aligned} x^* &= e^\tau x, \\ y^* &= e^{2\tau} y, \\ z^* &= z \end{aligned}$$

with new law of composition $\phi(\tau_1, \tau_2) = \tau_1 + \tau_2$.
For the group of two-dimensional rotations (2.4) we get

$$\xi = \left.\frac{\partial X(x;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = (-y, x)$$

and thus

$$\frac{dx^*}{d\varepsilon} = -y^*, \quad \frac{dy^*}{d\varepsilon} = x^* \quad \text{with} \quad x^*(0) = x, \ y^*(0) = y.$$

The solution to this initial value problem is again (2.4).

2.4 Infinitesimal Generator

Now we want to look at a function $F(x^i)$ under the action of a Lie group of transformations. This group of transformations is given by

$$x^{i*} = X^i(x;\varepsilon) \quad \text{with} \quad \left.\frac{dx^{i*}}{d\varepsilon}\right|_{\varepsilon=0} = \xi^i(x). \tag{2.6}$$

This transformation naturally implies a transformation of an arbitrary function $F(x^i)$:

$$\begin{aligned} F(x^{i*}) &= F(x^i + \varepsilon\,\xi^i + 0(\varepsilon^2)) \\ &= F(x^i) + \left.\frac{\partial F}{\partial x^{i*}}\right|_{\varepsilon=0} \varepsilon\,\xi^i + 0(\varepsilon^2) \\ &= F(x^i) + \frac{\partial F}{\partial x^i}\,\xi^i\varepsilon + 0(\varepsilon^2). \end{aligned}$$

Thus the infinitesimal transformation (2.6) leads to an infinitesimal transformation of the function $F(x^i)$. This infinitesimal transformation depends upon the infinitesimals ξ^i of the infinitesimal transformation of the independent variables. This fact can be represented by introducing an operator, called **infinitesimal generator**, which is defined as

$$X := \xi^i(x)\frac{\partial}{\partial x^i}$$

where summation over repeated indices is assumed. With this the infinitesimal transformation of the function $F(x^i)$ under the infinitesimal transformation (2.6) of the independent variables is expressed as

$$\begin{aligned} F(x^{i*}) &= F(x^i) + \varepsilon X F(x^i) + \frac{\varepsilon^2}{2}X^2 F(x^i) + O(\varepsilon^3) && (2.7) \\ &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k F(x^i). && (2.8) \end{aligned}$$

This shows that a Lie group of transformations of the coordinates of some space induces a Lie group of transformations of an object which is expressed in these coordinates. This infinitesimal transformation is completely given, or generated, by the infinitesimal transformations of the coordinates. The global transformation of an object $F(x^i)$ is again completely given by its infinitesimal counterpart.

2.5 Invariant Functions, Surfaces and Curves

Since we now know how a function changes under a Lie group of transformations we are able to introduce the notion of an invariant function:

Definition 2.5.0.4 *Definition: Invariant Function:*
An infinitely differentiable function $F(x)$ is an invariant function of a Lie group of transformations

$$x^* = X(x;\varepsilon) \tag{2.9}$$

if and only if

$$F(x^*) = F(x)$$

for any group transformation (2.9). If $F(x)$ is an invariant function of (2.9) then $F(x)$ is called an **invariant** *of (2.9) and $F(x)$ is said to be invariant under (2.9). The transformation (2.9) is then called a* **symmetry** *of $F(x)$. The procedure of finding the transformation (2.9) which leaves $F(x)$ invariant is called* **symmetry analysis**.

Using (2.8) it is apparent that a function is invariant under (2.9) if and only if (for a prove see [40])

$$XF(x) = 0.$$

The next step towards the invariance of differential equations is the invariance of surfaces $F(x) = 0$, and curves $F(x, y) = 0$.

Definition 2.5.0.5 *Definition: Invariant Surface:*
A surface $F(x) = 0$ is an invariant surface under a Lie group of transformations (2.9) if and only if

$$F(x^*) = 0 \quad \textit{when} \quad F(x) = 0.$$

To check the invariance of a given surface under the Lie group of transformations

$$x^* = X(x; \varepsilon) = x + \varepsilon\xi + 0(\varepsilon^2) \tag{2.10}$$

the following theorem is used:

Theorem 2.5.0.2 *Theorem:*
A surface $F(x) = 0$ is an invariant surface for the transformation (2.10) if and only if

$$XF(x) = 0 \quad \textit{when} \quad F(x) = 0,$$

where

$$X = \xi^i \frac{\partial}{\partial x^i}$$

is the infinitesimal generator of (2.10).

This theorem is seen to be true on the background of the action of a Lie group of transformation (2.9) on functions. The same is obviously also true for invariant curves:

Definition 2.5.0.6 *Definition: Invariant Curve*
A curve $F(x, y) = 0$ is an invariant curve for a Lie group of transformations

$$\begin{aligned} x^* &= X(x, y; \varepsilon) = x + \varepsilon\xi(x, y) + 0(\varepsilon^2), \\ y^* &= Y(x, y; \varepsilon) = y + \varepsilon\eta(x, y) + 0(\varepsilon^2) \end{aligned} \tag{2.11}$$

with infinitesimal generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.12}$$

if and only if

$$F(x^*, y^*) = 0 \quad \textit{when} \quad F(x, y) = 0.$$

Theorem 2.5.0.3 *Theorem:*
A curve $F(x,y) = 0$ is an invariant curve of (2.11) if and only if

$$XF(x,y) = \xi^i(x,y)\frac{\partial F}{\partial x^i} + \eta(x,y)\frac{\partial F}{\partial y} = 0 \quad \textit{when} \quad F(x,y) = 0. \tag{2.13}$$

This theorem can be used to construct an invariant curve when the infinitesimals or symmetries ξ and η of the Lie group of transformations are known. Then (2.13) is a first order linear partial differential equation for $F(x, y)$.

2.6 Prolongations

Since we are interested in solutions of differential equations using symmetry analysis we first need to construct the Lie group of transformations which leaves differential equations invariant. Such groups of transformations are of the form

$$\begin{aligned} x^* &= X(x, u; \varepsilon), \\ u^* &= U(x, u; \varepsilon) \end{aligned} \tag{2.14}$$

and act on the space of independent and dependent variables $x = (x^1, \ldots, x^n)$ and $u = (u_1, \ldots, u_m)$. A Lie group of the form (2.14) admitted by some differential equation Δ has the equivalent properties of mapping any solution $u = \theta(x)$ of Δ into another solution and leaving Δ invariant in the sense that Δ is unchanged in terms of the transformed variables for any solution $u = \theta(x)$ of Δ. For such a group the derivatives of the dependent variables with respect to the independent variables are transformed appropriately such that the contact conditions are preserved.

This natural extension of Lie groups of transformations of the dependent and independent variables to Lie groups of transformations acting on the dependent variables, the independent variables and the derivatives of the dependent variables with respect to the independent variables is called **prolongation**. This is done in the following way.

Consider a Lie group of point transformations (2.14) and let

$$\begin{aligned} u_{\sigma,i} &:= \frac{\partial u_\sigma}{\partial x^i}, \\ u^*_{\sigma,i} &:= \frac{\partial U^*_\sigma}{\partial x^{i*}} \end{aligned}$$

and so on. Furthermore let

$$D_i := \frac{\partial}{\partial x^i} + u_{\sigma,i}\frac{\partial}{\partial u_\sigma} + u_{\sigma,ii_1}\frac{\partial}{\partial u_{\sigma,i_1}} + \ldots + u_{\sigma,ii_1\ldots i_n}\frac{\partial}{\partial u_{\sigma,i_1\ldots i_n}} + \ldots$$

be the total derivative operator where summation over repeated indices is assumed.
The transformations (2.14) are assumed to be one-to-one in some domain D in the space of the dependent and independent variables x and u and k-times differentiable in D. The transformations preserve the contact conditions

$$\begin{aligned} du_\sigma &= u_{\sigma,i_1} dx^{i_1}, \\ &\;\;\vdots \\ du_{\sigma,i_1\ldots i_{k-1}} &= u_{\sigma,\,i_1\ldots i_k} dx^{i_k} \end{aligned}$$

if and only if

$$\begin{aligned} du^*_\sigma &= u^*_{\sigma,\,i_1} dx^{i_1 *}, \\ &\;\;\vdots \\ du^*_{\sigma,\,i_1\ldots i_{k-1}} &= u^*_{\sigma,\,i_1\ldots i_k} dx^{i_k *}. \end{aligned} \tag{2.15}$$

From the transformations (2.14) we obtain

$$\begin{aligned} du^*_{\sigma,\,i_1\ldots i_{k-1}} &= D_j U_{\sigma,\,i_1\ldots i_{k-1}} dx^j, \\ dx^{i_k *} &= D_j X^{i_k} dx^j. \end{aligned}$$

Inserting this in (2.15) we get

$$D_j U_{\sigma,\,i_1\ldots i_{k-1}} = u^*_{\sigma,\,i_1\ldots i_k} D_j X^{i_k}. \tag{2.16}$$

Since the transformations (2.14) define a Lie group we can expand around $\varepsilon = 0$:

$$\begin{aligned} U_{\sigma,\,i_1\ldots i_{k-1}} &= u_{\sigma,\,i_1\ldots i_{k-1}} + \varepsilon \eta_{(\sigma,\,i_1\ldots i_{k-1})} + O(\varepsilon^2), \\ X^{i_k} &= x^{i_k} + \varepsilon \xi^{i_k} + O(\varepsilon^2), \end{aligned}$$

where the lower indices of η are enclosed in brackets since they do not represent derivatives, only indices. Using this in (2.16) we get

$$D_j(u_{\sigma,\,i_1\ldots i_{k-1}} + \varepsilon \eta_{(\sigma,\,i_1\ldots i_{k-1})} + O(\varepsilon^2)) = u^*_{\sigma,\,i_1\ldots i_k} D_j(x^{i_k} + \varepsilon \xi^{i_k} + O(\varepsilon^2)).$$

Threading the total derivative operator D_j leads to

$$u_{\sigma,\,i_1\ldots i_{k-1},j} + \varepsilon D_j\, \eta_{(\sigma,\,i_1\ldots i_{k-1})} + O(\varepsilon^2) = u^*_{\sigma,\,i_1\ldots i_k}(\delta^{i_k}_j + \varepsilon D_j\, \xi^{i_k} + O(\varepsilon^2)). \tag{2.17}$$

Since the extension of the Lie group of transformations to the derivatives is again to be a Lie group of transformations we can expand $u^*_{\sigma, i_1 \dots i_k}$ around $\varepsilon = 0$:

$$u^*_{\sigma, i_1 \dots i_k} = u_{\sigma, i_1 \dots i_k} + \varepsilon \eta_{(\sigma, i_1 \dots i_k)} + O(\varepsilon^2).$$

This leads to the so-called **prolongation formula**

$$\begin{aligned} & u_{\sigma, i_1 \dots i_{k-1}, j} + \varepsilon D_j \, \eta_{(\sigma, i_1 \dots i_{k-1})} + O(\varepsilon^2) = \\ & \quad (u_{\sigma, i_1 \dots i_k} + \varepsilon \eta_{(\sigma, i_1 \dots i_k)} + O(\varepsilon^2))(\delta_j^{i_k} + \varepsilon D_j \, \xi^{i_k} + O(\varepsilon^2)) \\ & \quad u_{\sigma, i_1 \dots i_{k-1}, j} + \varepsilon (\eta_{(\sigma, i_1 \dots i_{k-1}, j)} + u_{\sigma, i_1 \dots i_k} D_j \, \xi^{i_k}) + O(\varepsilon^2) \end{aligned}$$

or

$$\eta_{(\sigma, i_1 \dots i_{k-1}, j)} = D_j \, \eta_{(\sigma, i_1 \dots i_{k-1})} - u_{\sigma, i_1 \dots i_k} \cdot D_j \, \xi^{i_k}. \tag{2.18}$$

This prolongation formula shows that the k-th extended transformation depends upon derivatives of the dependent variables up to k-th order when starting with a point transformation, which means a transformation of the form (2.14) on the space of dependent and independent variables.
As an example consider the case of one dependent variable u and two independent variables, x and t. Since we are looking for point transformations the unprolonged infinitesimal generator looks like

$$X = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u},$$

where ξ is the infinitesimal transformation of x, τ the infinitesimal transformation of t and η the infinitesimal transformation of u. With this the infinitesimal transformation of $u_{x,t}$ reads

$$\begin{aligned} \eta_{(x,t)} = \; & \eta_{x,t} + (\eta_{x,u} - \xi_{x,t}) u_t + (\eta_{t,u} - \xi_{x,t}) u_x - \tau_x u_{t,t} + (\eta_u - \xi_x - \tau_t) u_{x,t} \\ & - \xi_t u_{x,x} - \tau_{x,u} u_t^2 + (\eta_{u,u} - \xi_{x,u} - \tau_{y,u}) u_x u_t - \xi_{y,u} u_x^2 - \tau_{u,u} u_t^2 - \xi_{u,u} u_x^2 u_t \\ & - 2\tau_u u_t u_{x,t} - 2\xi_u u_x u_{x,t} - \xi_u u_t u_{x,x} - \tau_u u_x u_{t,t}, \end{aligned}$$

where lower indices not enclosed in brackets are again differentiations with respect to the corresponding variable.
Having thus defined a Lie group of transformations acting on any enlarged space incorporating derivatives it is relatively easy to write down the corresponding k-th extended infinitesimal generator:

$$\begin{aligned} X^{(k)} = \; & \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta_{(\sigma)}(x, u) \frac{\partial}{\partial u_\sigma} + \eta_{(\sigma, i_1)}(x, u, u_{j_1}) \frac{\partial}{\partial u_{\sigma, i_1}} + \dots \\ & \dots + \eta_{(\sigma, i_1 \dots i_k)}(x, u, u_{j_1}, \dots, u_{j_1 \dots j_k}) \frac{\partial}{\partial u_{\sigma, i_1 \dots i_k}}. \end{aligned}$$

Furthermore we recognize that the higher order infinitesimals $\eta_{(\sigma, i_1 \dots i_k)}$ are completely determined by the infinitesimals of the Lie group of transformations which act on the space of the dependent and independent variables. The relation to build the higher order infinitesimals from the infinitesimals of the space of dependent and independent variables is given by the prolongation formula (2.18).

2.7 The Symmetry Criterion

Since we want to apply the prolonged infinitesimal transformations to the construction of solutions to partial differential equations we need a criterion of invariance. As we will see this criterion will also be in infinitesimal form and will lead directly to an algorithm to determine the infinitesimal generator X admitted by the differential equation under consideration. The invariant surfaces of the corresponding Lie group of point transformations then lead us to invariant solutions.

But first we start with the invariance criterion. Consider a partial differential equation of k-th order

$$\Delta(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k}) = 0 \tag{2.19}$$

where x^i denote the n independent variables, u denotes the dependent variable and $u_{i_1,\dots,i_j}$ denotes the set of coordinates of the j-th order partial derivatives of u with respect to the x^i:

$$u_{i_1,\dots,i_j} = \frac{\partial^j u}{\partial x^{1 i_1} \cdots x^{n i_n}}, \quad i_1 + \dots i_n = j.$$

In terms of all these coordinates, the independent variables, the dependent variables and the derivatives, (2.19) is an algebraic equation defining a hypersurface in the k-th prolonged space $(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k})$.

For any solution $u = \theta(x)$ of (2.19) the equations

$$u_{i_1,\dots,i_j} = \frac{\partial^j \theta(x)}{\partial x^{1 i_1} \dots x^{n i_n}}$$

define a solution surface which lies on (2.19).

Having thus recognized a differential equation as a hypersurface in an appropriately prolonged space we can define its invariance.

Definition 2.7.0.7 *Invariance of Differential Equations:*
The Lie group of transformations

$$\begin{aligned} x^{i*} &= X^i(x, u; \varepsilon), \\ u^* &= U(x, u; \varepsilon) \end{aligned} \tag{2.20}$$

leaves the differential equation (2.19) invariant if and only if its k-th prolongation leaves the surface (2.19) invariant.

Invariance of (2.19) under the k-th prolongation of (2.20) means that any solution $u = \theta(x)$ of (2.19) maps into some other solution $u = \varphi(x;\varepsilon)$ of (2.19) under the action of the Lie group of transformations (2.20). Moreover, if a transformation (2.20) maps any solution $u = \theta(x)$ of (2.19) into another solution $u = \varphi(x;\varepsilon)$ of (2.20) than the surface (2.19) is invariant with respect to (2.20) with

$$u_{i_1,\dots,i_j} = \frac{\partial^j \varphi(x;\varepsilon)}{\partial x^{1^{i_1}} \dots x^{n^{i_n}}}.$$

As a consequence, the family of all solutions of (2.19) is invariant under (2.20) if and only if (2.19) admits (2.20).
This definition readily leads to an infinitesimal criterion for the invariance of a differential equation:

Theorem 2.7.0.4 *Theorem:*
The differential equation

$$\Delta(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k}) = 0$$

admits the Lie group of transformations

$$\begin{aligned} x^{i*} &= X^i(x,u;\varepsilon), \\ u^* &= U(x,u;\varepsilon) \end{aligned}$$

if and only if

$$X^{(k)}\Delta(x^i, \dots, u_{i_1,\dots,i_k}) = 0 \quad \textit{when} \quad \Delta(x^i, \dots, u_{i_1,\dots,i_k}) = 0,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

is the infinitesimal generator of $\Delta(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k}) = 0$*,*

$$X^{(k)} = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \dots + \eta_{(i_1,\dots,i_k)}\frac{\partial}{\partial u_{i_1,\dots,i_k}}$$

is the k-th extended infinitesimal generator and

$$\eta_{(i_1,\dots,i_l,j)} = D_j\,\eta_{(i_1,\dots,i_l)} - u_{i_1,\dots,i_l} \cdot D_j\,\xi^{i_l}$$

are the corresponding extended infinitesimals.

This theorem provides an effective computational procedure to find the Lie group of transformations admitted by a given differential equation. This group is denoted the **symmetry group** of the differential equation.
To do this let the infinitesimals ξ^i, η be unknown functions of the dependent and independent variables. The extended infinitesimals of the prolonged infinitesimal generator are then certain differential expressions involving the partial derivatives of the infinitesimals ξ^i and η with respect to the dependent and independent variables. After eliminating any dependencies among the derivatives of the dependent variable by inserting the differential equation to restrict the transformations to the solution surface, the values of all the remaining derivatives of the dependent variable can be arbitrary.
Since these derivatives appear polynomially the coefficients of this polynomial have to vanish separately. From this separation results an overdetermined system of partial differential equations which is linear (except for nonclassical symmetries) and overdetermined. This system of partial differential equations is called the determining system or the **determining equations**.
In most instances these determining equations can be solved by elementary methods for partial differential equations, such as integration of pseudo-ordinary differential equations.
Summarizing, we are directly lead to an algorithm to determine the infinitesimals ξ^i, η and therefore the infinitesimal generator X admitted by a given differential equation:

i) Start with the unknown infinitesimals and build up the extended infinitesimal generator.

ii) Apply the extended infinitesimal generator to the differential equation under consideration.

iii) Insert the differential equation to use the dependencies of the partial derivatives given by the differential equation and to restrict the transformations to the solution surface.

iv) Split the resulting polynomial of partial derivatives of the dependent variable on which the infinitesimals do not depend.

v) Solve the determining system.

To illustrate the symmetry criterion and the separation of the equation which is polynomial in the derivatives we consider the nonlinear diffusion equation

$$u_t = (K(u)u_x)_x \tag{2.21}$$

where u is a function of x and t or in expanded form:

$$u_t = K'(u)u_x + K(u)u_{x,x}.$$

To use the above symmetry criterion we need the extended infinitesimal transformations $\eta_{(t)}$, $\eta_{(x)}$ and $\eta_{(x,x)}$ of the unprolonged infinitesimal generator

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

The corresponding prolonged infinitesimal generator reads

$$X^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta_{(x)}\frac{\partial}{\partial u_x} + \eta_{(t)}\frac{\partial}{\partial u_t} + \eta_{(x,x)}\frac{\partial}{\partial u_{x,x}}$$

where

$$\begin{aligned}
\eta_{(t)} &= \eta_t + u_t(\eta_u - \tau_t) - u_x\xi_t + u_x u_t \xi_u - u_t^2\tau_u, \\
\eta_{(x)} &= \eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u - u_t\tau_x - u_x u_t \tau_u
\end{aligned}$$

and

$$\begin{aligned}
\eta_{(x,x)} = \; & \eta_{x,x} + u_x(2\eta_{x,u} - \xi_{x,x}) + u_x^2(\eta_{u,u} - 2\xi_{x,u}) + u_{x,x}(\eta_u - 2\xi_x) - 3u_x u_{x,x}\xi_u \\
& -u_x^3\xi_{u,u} - 2u_{t,x}\tau_x - 2u_x u_{t,x}\tau_u - u_t\tau_{x,x} - 2u_t u_x \tau_{x,u} - u_t u_{x,x}\tau_u - u_t u_x^2 \tau_{u,u}
\end{aligned}$$

where the prolongation formula (2.18) was used. According to the theorem above the symmetry criterion is given by

$$X^{(2)}(u_t - K'(u)u_x^2 - K(u)u_{x,x})\big|_{u_t = K'(u)u_x + K(u)u_{x,x}} = 0.$$

Putting it all together we get an equation which is a multivariate polynomial in the variables $u_{x,x}$, $u_{x,t}$, $u_{t,t}$, u_x and u_t. Since ξ, τ and η do not depend on any of these variables and the polynomial equation has to be true for all values of these variables the coefficients have to vanish separately. This gives the equations

$$\begin{aligned}
\tau_x &= 0, \\
K(u)\xi_u &= 0, \\
K(u)\tau_u &= 0, \\
\xi_t + 2K'(u)\eta_x + K(u)(2\eta_{x,u} - \xi_{x,x}) &= 0, \\
K(u)(\tau_t - 2\xi_x) + K'(u)\eta &= 0, \\
K(u)\eta_{u,u} + K'(u)(\tau_u - 2\xi_x + \eta_u) + K''(u)\eta &= 0, \\
\eta_t - K(u)\eta_{x,x} &= 0.
\end{aligned} \tag{2.22}$$

These are the determining equations for the equation (2.21).

2.8 Invariant Solutions

Having calculated the infinitesimals of the symmetry generator the question is how to use these symmetries to get a solution of the differential equation under consideration. In this thesis we use the symmetries to construct so-called invariant solutions. This notion is connected with the invariance of a surface:

Definition 2.8.0.8 *Definition: Invariant Solution:*
A surface $u = \theta(x)$ in the space of dependent and independent variables is an **invariant solution** *of*

$$\Delta(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k}) = 0$$

under the action of the infinitesimal generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

of the Lie group of transformations

$$\begin{aligned} x^{i*} &= X^i(x,u;\varepsilon), \\ u^* &= U(x,u;\varepsilon) \end{aligned}$$

if and only if

i) $u = \theta(x)$ is an invariant surface of the generator,

ii) $u = \theta(x)$ solves $\Delta(x^i, u, u_{i_1}, \dots, u_{i_1,\dots,i_k}) = 0$.

Using the condition for the invariance of a surface we see that $u = \theta(x)$ is an **invariant solution if and only if**

i) $X(u - \theta(x)) = 0$ when $u = \theta(x)$

ii) $u = \theta(x)$ solves $\Delta(x^i, u, u_{i_1}, \dots, u_{i_k}) = 0$.

The condition i) is called the **invariant surface condition**.
Therefore, to get an invariant solution of a given differential equation we have to solve the linear first order partial differential equation i) together with the differential equation ii) under consideration.
One way to construct invariant solutions would be to insert the infinitesimals in the invariant surface condition and solve the resulting equation. The solution to this equation is then recognized as new dependent and independent variables. Inserting these invariants in the original differential equation a reduced equation is obtained. This resulting differential equation is then to be solved and an invariant solution is obtained. This method is also known as the **method of characteristics** [40].
The methods of characteristics is well known and generally applicable. But it relies on the solution of an explicit set of invariants. If these invariants are not found this method is of no use.
Therefore we use another method to get invariant solutions. The idea goes back to Olver and Rosenau [41, 42]. Their approach consists in adding the integrability conditions to the original equations of the invariant surface condition and

the differential equation under consideration. The result is a combined system of partial differential equations. Unlike the nonclassical method, where a symmetry analysis is performed on this coupled system, in [41] the system is directly reduced by using some integrability conditions, with no symmetry background at all.
We generalized this procedure in such a way, that not only some integrability conditions are calculated, but a whole involutive reduction procedure incorporating heuristics to solve simple partial differential equations is implied. Unlike other computer algebra packages like `CRACK` [21] or **PDESolve** [24], which "just" aim at solving a system of overdetermined polynomially nonlinear partial differential equations, or `rif` [38] or `diffalg` [58], which "just" perform a reduction of such a system to a simplified form, our procedure implemented in *Mathematica* is able to simplify and solve systems as above in one step, as long as the heuristic solver is able to solve the corresponding differential equations. That means that by starting the calculation simplifications by all integrability conditions and solutions of partial differential equations by heuristics are done. This procedure is explained in more detail in [59].
But before explaining this procedure in detail, we introduce in the next chapter the notion and concept of involutivity.

2.9 Notes

We already said that in this work we deal only with point symmetries. These are symmetries where the infinitesimals depend only on the dependent and the independent variables. Of course this dependency of the infinitesimals can be generalized. The invariance of some differential equation is then again some symmetry, but in a more general sense.
If the infinitesimal transformations depend additionally on the first order derivatives of the dependent variables the corresponding transformations are called **contact transformations**. Like point transformations contact transformations act on a finite-dimensional space. Such transformations are used to construct a wider class of solutions [40] or to construct transformations between classes of differential equations, for example when transforming nonlinear differential equations to linear ones [40].
Generalizing even further we arrive at so-called **generalized symmetries** or **Lie-Bäcklund symmetries**. These are symmetries where the unprolonged infinitesimal transformations depend not only on the dependent and independent variables, but also on higher-order derivatives. When requiring an additional condition on generalized symmetries they are called **variational symmetries**. These are symmetries which leave an action integral invariant. This then leads to the invariance of Euler-Lagrange equations.
Unlike the contact transformations the Lie-Bäcklund transformations do not act

upon a finite space. Lie-Bäcklund transformations are used to construct integrals of motion via the famous Noether theorem [11, 12, 39]. Furthermore they can be used to construct so-called **recursion operators** which generate an infinite class of solutions, for example in the case of the Korteweg-de-Vries equation [39, 40].
Another class of transformations are the **equivalence transformations** [9, 10]. These are transformations of a differential equation which involves an arbitrary function. Equivalence transformations are transformations which leave a specific class of differential equations invariant. The equivalence relation then divides the set of all differential equations of a given family of differential equations into disjoint classes of equivalent equations. For each of these classes a representative is chosen. Equivalence transformations are used to classify differential equations according to its arbitrary function. This procedure is called a **group classification problem.**
Later on we use another method to achieve a group classification problem. We use this other method because of two reasons. The first reason is that for the construction of equivalence transformations the construction of optimal systems and discrete symmetries are needed. But until today there is no complete implementation for the calculation of optimal systems and discrete symmetries. The second reason is the choice of the representative. This choice is not unique and if an application involves a different representative a transformation is needed to the differential equation under consideration.

Chapter 3

The Concept of Involutivity

The notion of involutivity goes back to the French mathematicians Charles Riquier [35] and Maurice Janet [36]. They inspected the problem of transforming a system of coupled partial differential equations to a simpler form by adding and inserting all integrability conditions of the system to the system. This simpler form is called involutive and has the same set of solutions as the original system and is obtained in a finite number of steps, or in modern language, algorithmically.
There are two main purposes of this procedure. First, and this is the one we are interested in, the transformed system is much simpler than the original system and therefore the solution of the system is much easier to accomplish. The second purpose is the fact that the dimension of the solution space of the system is easily read off from the transformed system. This is used for example in the computer programs by Schwarz [29] and Reid [31].
The reason for the fact that the dimension of the solution space is very easy to determine for an involutive system is based upon a term ordering and the addition of all integrability conditions to the system. Because of this ordering certain leading derivatives with respect to the given term ordering are expressed as functions of other derivatives, which are called parametric. Parametric derivatives are derivatives that cannot be obtained by differentiation of the leading derivatives.

3.1 The Term Ordering

The dimension of the solution space is given by the number of parametric derivatives. Of course the division of derivatives into leading ones and parametric ones does depend on the ordering procedure. Therefore, a system which is involutive with respect to one term ordering is usually not involutive with respect to another ordering.
Note that the ordering of the terms involved cannot be chosen arbitrarily. There is one restriction. The term ordering has to be invariant under arbitrary differ-

entiations. Denoting the ordering by ">" this condition is expressed as follows: Consider two terms u and v which are ordered as

$$u > v.$$

Then the ordering has to satisfy

$$\partial u > \partial v,$$

where ∂ represents an arbitrary differentiation. This relation expresses the compatibility of differentiation with the term ordering. For some examples of admissible terms orderings see [54, 55, 56].
We work exclusively with the ordering which uses the following criteria. First there is an ordering of the dependent variables. For two dependent variables, for example

$$u > v.$$

Then we have to choose an ordering of the independent variables, for example

$$x > y$$

for the two independent variables x and y. Now consider two terms

$$\left(\frac{\partial^{i_1+i_2}u}{\partial x^{i_1}\partial y^{i_2}}\right)^{k_1}, \quad \left(\frac{\partial^{j_1+j_2}v}{\partial x^{j_1}\partial y^{j_2}}\right)^{k_2}.$$

Then

$$\left(\frac{\partial^{i_1+i_2}u}{\partial x^{i_1}\partial y^{i_2}}\right)^{k_1} > \left(\frac{\partial^{j_1+j_2}v}{\partial x^{j_1}\partial y^{j_2}}\right)^{k_2}$$

if :

i) $i_1 + i_2 > j_1 + j_2$

ii) if $i_1 + i_2 = j_1 + j_2$ then $u > v$

iii) if $i_1 + i_2 = j_1 + j_2$ and $u = v$ then $i_1 > j_1$

iv) if $i_1 + i_2 = j_1 + j_2$, $u = v$ and $i_1 = j_1$ then $k_1 > k_2$

For the above mentioned example this produces the following relations among all second order derivatives:

$$u_{x,x} > u_{x,y} > u_{y,y} > v_{x,x} > v_{x,y} > v_{y,y} > u_x > u_y > v_x > v_y > u > v,$$

where as usual we use the notation with the lower indices for differentiation, i. e. $u_{x,y}$ for $\frac{\partial^2 u}{\partial x \partial y}$.

3.2 The Involutive Algorithm

As said above the transformation to involutive form is an algorithm. This algorithm is build up of three major steps. The first step is called **autoreduction**. Hereby each equation of a coupled system of partial differential equations is solved with respect to the leading derivative under the given term ordering and back-inserted into the system. This reduction of the system with respect to itself ends when no insertions of the system into itself for the ordering given are possible.
In the next step, the **completion**, the system is enlarged by equations which ensure the calculation of integrability conditions. These equations are derivatives of some equations with respect to certain independent variables, which are called non-multiplicative, and guarantee that integrability conditions can be calculated.
Finally, in the last step the **integrability conditions** are calculated and are reduced with respect to the system. If these new equations are not identically zero they are appended to the system to reenter the autoreduction step. This loop structure is entered over and over again until the integrability conditions are identically satisfied when inserting the system. It is illustrated in figure 3.1. To get an idea of how to manage these tasks we explain them now in a more detailed way.

3.3 Autoreduction

Consider a polynomially nonlinear coupled system of partial differential equations. The problem we want to solve is to insert all equations of the system into itself to reach a simplification, but without losing any solutions. We do this in the following way.
First we order the system with respect to the term ordering. Now the equation with the lowest derivative order and lowest in order for the dependent and independent variables is last. We start with this last equation. What could happen now is that this equation factors. If this is the case new systems of equations are build. These new systems incorporate the old system, where the equation that factors is replaced by each of the factors. If the equation is build up of two factors for example, the original system is replaced by two new systems, one for each factor. These new systems are then again sorted with respect to the term ordering and the last equation is inspected.
Consider for example the system

$$
\begin{aligned}
\{\text{Equation } 1 &= 0, \\
\text{Equation } 2 &= 0, \\
\text{Equation } 3 &= 0, \\
3x^2 u_{x,x,y} \cdot v \cdot u &= 0,
\end{aligned}
$$

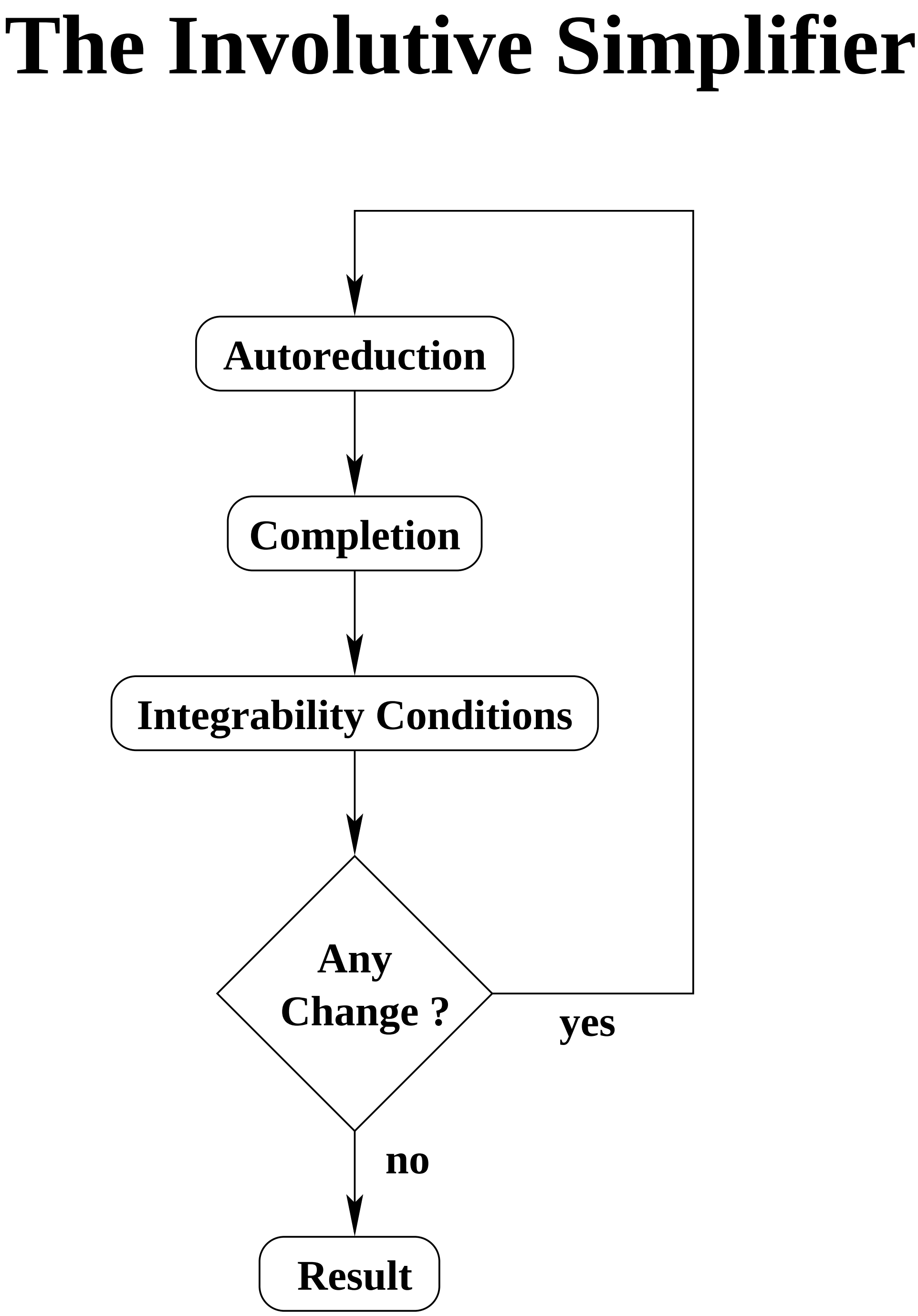


Figure 3.1: The loop structure of the involutive simplifier.

$$\begin{aligned} \text{Equation 5} &= 0, \\ \text{Equation 6} &= 0\} \end{aligned}$$

in the dependent variables u and v. The fourth equation is a product of four factors. The factor $3x^2$ does not involve the dependent variables at all. The other three factors involve u and v. Thus the system is replaced by the three systems

$$\begin{aligned} \{\text{Equation 1} &= 0, \\ \text{Equation 2} &= 0, \\ \text{Equation 3} &= 0, \\ u_{x,x,y} &= 0, \\ \text{Equation 5} &= 0, \\ \text{Equation 6} &= 0\}, \end{aligned}$$

$$\begin{aligned} \{\text{Equation 1} &= 0, \\ \text{Equation 2} &= 0, \\ \text{Equation 3} &= 0, \\ v &= 0, \\ \text{Equation 5} &= 0, \\ \text{Equation 6} &= 0\} \end{aligned}$$

and

$$\begin{aligned} \{\text{Equation 1} &= 0, \\ \text{Equation 2} &= 0, \\ \text{Equation 3} &= 0, \\ u &= 0, \\ \text{Equation 5} &= 0, \\ \text{Equation 6} &= 0\} \end{aligned}$$

which are then sorted with respect to the given term ordering.

Now consider the case when the last equation does not factor. Then the second-to-last equation is inspected for terms where the last equation can be inserted. If there are none the third-to-last equation is inspected, and so on. If the last equation cannot be inserted into the system this procedure starts all over again with the second-to-last equation. If this equation factors corresponding new systems are built, sorted with respect to the ordering given and the procedure starts again with the last equation of each new system.

If the second-to-last equation does not factor the third-to-last equation is inspected if the second-to-last equation can be inserted and so on. If no insertions

can be made at all the autoreduction procedure stops and returns all the systems it has produced.
Suppose that at some stage the $(-n)$-th equation can be inserted in the $(-m)$-th equation, with $m > n$. Since we are dealing with polynomially nonlinear systems it may happen that the leading derivative of the $(-n)$-th equation has a prefactor which incorporates the dependent variables or derivatives of them. Since we don't want to loose solutions we now have to consider different cases.
First there is the case where the prefactor is unequal to zero. Then the $(-n)$-th equation can be inserted in the $(-m)$-th equation, but under the condition that the prefactor is not equal to zero. This means that the simplification of the $(-m)$-th equation with respect to the $(-n)$-th equation is only valid for this condition not being zero. Therefore this condition has to be remembered somehow.
We do this by prepending the conditions which should be unequal to zero to the system. The system is only valid if these conditions are satisfied. That means that a "case" is built up of two parts. First the conditions which are unequal to zero, and second the system under these conditions. Of course the factoring and the case distinctions mentioned above have to take these conditions into account. The conditions which have to be unequal to zero are canceled out for the case distinctions which appear in the factoring and in the prefactors.
To illustrate this we again use the above example. Suppose this system has the additional condition $u \neq 0$:

$$\begin{aligned} \{\{u &\neq 0\}, \\ \{\text{Equation } 1 &= 0, \\ \text{Equation } 2 &= 0, \\ \text{Equation } 3 &= 0, \\ 3x^2 u_{x,x,y} \cdot v \cdot u &= 0, \\ \text{Equation } 5 &= 0, \\ \text{Equation } 6 &= 0\}\}. \end{aligned}$$

Like above the fourth equation factors. But since the system is only valid for $u \neq 0$ there are only two new systems:

$$\begin{aligned} \{\{u &\neq 0\}, \\ \{\text{Equation } 1 &= 0, \\ \text{Equation } 2 &= 0, \\ \text{Equation } 3 &= 0, \\ u_{x,x,y} &= 0, \\ \text{Equation } 5 &= 0, \\ \text{Equation } 6 &= 0\}\}. \end{aligned}$$

and

$$\begin{aligned}\{\{u &\neq 0\},\\ \{\text{Equation }1 &= 0,\\ \text{Equation }2 &= 0,\\ \text{Equation }3 &= 0,\\ v &= 0,\\ \text{Equation }5 &= 0,\\ \text{Equation }6 &= 0\}\}.\end{aligned}$$

The third system would contain a contradiction and is therefore left out.
Now consider again the case where a prefactor occurs in the $(-n)$-th equation. Before checking if the whole prefactor is new or already appearing in the list of the conditions unequal to zero the prefactor is factorized. If new prefactors occur a case distinction is performed. There exists one case where the whole prefactor is unequal to zero. Then the $(-n)$-th equation can be inserted in the $(-m)$-th equation. On the other hand there are the cases where each factor of the prefactor is equal to zero. Here the condition to be zero is appended to the system. This new system is then sorted according to the term ordering and the new bunch of systems enters again the autoreduction step.
As an example consider the system

$$\begin{aligned}\{\{u &\neq 0\},\\ \{\text{Equation }1 &= 0,\\ \text{Equation }2 &= 0,\\ \text{Equation }3 &= 0,\\ 3x^2u_{x,x,y}\cdot v\cdot u &= 0,\\ 3xyuv_yu_{x,x}+6xyv_yu_{x,x}+u_y\cdot v_x &= 0,\\ \text{Equation }6 &= 0\}\}.\end{aligned}$$

The (-2)-nd equation can be inserted in the (-3)-rd, but only if the prefactor of $u_{x,x}$ is unequal to zero. However this prefactor is a product of three terms: $3xy$, $u+2$ and v_y. The first prefactor is unequal to zero since the system has to be true for all values of the independent variables. For the other two factors a case distinction is performed. There is the case where $u+2$ and v_y are unequal to zero, the case where $u=-2$ and the case where $v_y=0$:

$$\begin{aligned}\{\{u &\neq 0,\\ u+2 &\neq 0,\\ v_y &\neq 0\},\\ \{\text{Equation }1 &= 0,\end{aligned}$$

$$
\begin{aligned}
\text{Equation 2} &= 0,\\
\text{Equation 3} &= 0,\\
3x^2 u_{x,x,y}\cdot v\cdot u &= 0,\\
3xyuv_y u_{x,x} + 6xyv_y u_{x,x} + u_y\cdot v_x &= 0,\\
\text{Equation 6} &= 0\}\},
\end{aligned}
$$

where the (-2)-nd equation is then inserted in the (-3)-rd one,

$$
\begin{aligned}
\{\{u &\neq 0\},\\
\{\text{Equation 1} &= 0,\\
\text{Equation 2} &= 0,\\
\text{Equation 3} &= 0,\\
3x^2 u_{x,x,y}\cdot v\cdot u &= 0,\\
3xyuv_y u_{x,x} + 6xyv_y u_{x,x} + u_y\cdot v_x &= 0,\\
\text{Equation 6} &= 0,\\
u+2 &= 0\}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\{\{u &\neq 0,\\
u+2 &\neq 0,\\
v_y &\neq 0\},\\
\{\text{Equation 1} &= 0,\\
\text{Equation 2} &= 0,\\
\text{Equation 3} &= 0,\\
3x^2 u_{x,x,y}\cdot v\cdot u &= 0,\\
3xyuv_y u_{x,x} + 6xyv_y u_{x,x} + u_y\cdot v_x &= 0,\\
\text{Equation 6} &= 0,\\
v_y &= 0\}\}.
\end{aligned}
$$

If all insertions of the $(-n)$-th equation into the $(-m)$-th equation are made it is observed if the $(-n)$-th equation can be inserted in the $(-n-1)$-th equation and so on. Following this procedure all simplifications of the system with respect to its own equations are made where different cases are considered for equations which factorize or for the prefactors of equations which can be inserted into the system.
An overview of the autoreduction procedure is given in figure 3.2.

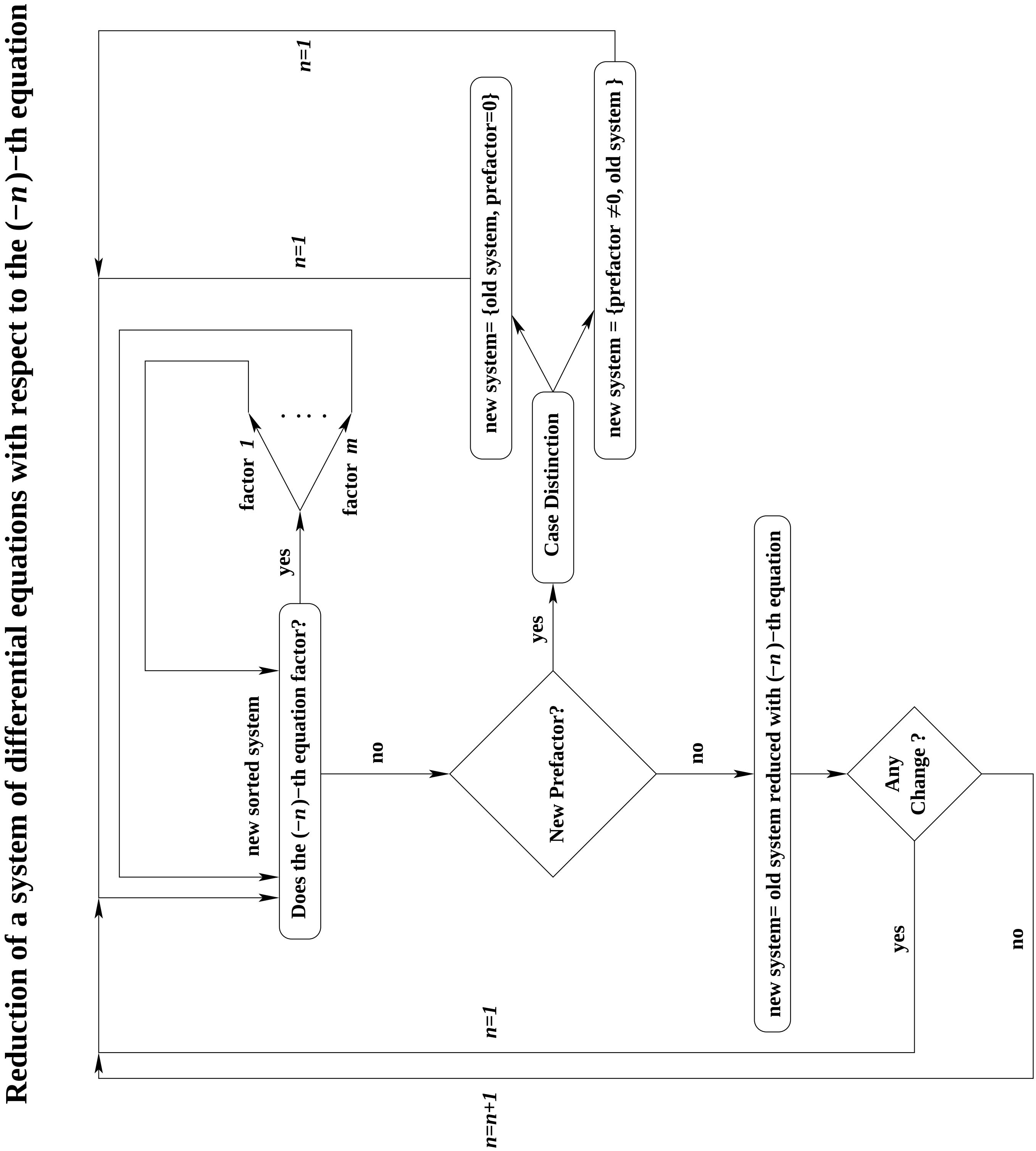


Figure 3.2: Flow chart of the autoreduction algorithm.

3.4 Completion

If the system is reduced with respect to itself it enters the completion step. In this step equations are constructed by differentiation with respect to certain independent variables in such a way that all integrability conditions can be calculated by cross-differentiation. To explain what is done in detail we need the notion of **multiplicative** and **non-multiplicative** variables.

Consider some derivatives of the dependent variables with respect to the independent variables. Now pick one of these terms. This term now involves derivatives with respect to the independent variables. An independent variable of this derivation is called **multiplicative** when the order of derivation with respect to this variable is equal or larger than the highest order of derivation of the subsystem incorporating derivatives of the same dependent variable with respect to this independent variable. If this is not the case the independent variable is non-multiplicative.

To illustrate the notions of multiplicative and non-multiplicative variables consider three dependent variables u, v and w with the ordering $u > v > w$ and the three independent variables x, y, z with $x > y > z$. For the following set of derivatives the multiplicative variables are given:

Derivative	multiplicative variable(s)
$u_{x,x}$	x, y, z
$u_{x,z}$	y, z
$u_{y,y}$	y, z
$u_{z,z}$	z
v_x	x, y, z
v_y	y, z
v_z	z
w_x	x, y, z

Coming back to the autoreduced system we want to construct all equations which lead to the calculation of integrability conditions by cross-differentiation. This is done in the following way.

We start with the last equation with respect to the term ordering. The leading term of this equation is differentiated with respect to its non-multiplicative variables. If the resulting terms cannot be written as the derivation of the leading terms of other equations with respect to their multiplicative variables, the equations is differentiated with respect to this non-multiplicative variable and is appended to the system. If this is done for all non-multiplicative variables of the leading term of the last equation with respect to the term ordering the same procedure is applied to the second-to-last equation and so on.

If all equations have undergone this procedure a new system is formed. The above mentioned procedure is again applied on this new system and so on until every differentiation of a leading term with respect to its non-multiplicative variables can be expressed as derivatives of other leading terms of the other equations with respect to their multiplicative variables.

3.5 Integrability Conditions

When the system under consideration is completed the next step is the calculation of the integrability conditions themselves. Here one equation after another is picked and the leading derivative is differentiated with respect to its non-multiplicative variables. If such a derivation can be expressed as a derivative of another leading term with respect to its multiplicative variables those derivations have equal leading terms and can be equated to form an integrability condition. This is done for all equations in the system. After that all the integrability conditions are reduced with respect to the system. That means that every equation is inserted into the integrability conditions until no more simplifications occur. The result of this insertion are reduced integrability conditions. If these are not identically satisfied they are appended to the system and again enter the autoreduction step and so on.
This loop structure of autoreduction, completion and integrability conditions is performed over and over again until every integrability condition is satisfied identically when reducing it with respect to the system. The result of this loop are systems whose integrability conditions do not lead to new information or reductions because of the system itself. Such a system is called an **involutive system**.

3.6 Notes

There are several implementations to turn a given system of coupled partial differential equation to involutive form. There is a `REDUCE`-package of F. Schwarz [29] which transforms linear systems of partial differential equations to involutive form. This implementation does not consider different cases for prefactors of equations which are inserted into others. All prefactors are assumed to be unequal to zero.
The `REFAL`-package of Topunov [28] and the `MAPLE`-package of Reid [32] transform a linear system of coupled differential equations to involutive form (or standard form as it is called in Reid) and at the end of the calculation give a list of coefficients which are assumed to be nonzero throughout the calculation.
The `MAPLE`-package `rif` [38] also performs case distinctions when transforming to a simpler form. It refers to Groebner-basis techniques and is built in `MAPLE`.

Chapter 4

The Solution Procedure

As was said before, an involutive system is easier to solve than the original system. So the next step towards solution of a system of partial differential equations is a tool to solve partial differential equations. This tool uses simple heuristics to solve different kinds of partial differential equations and to simplify or separate them. In fact the tool used is an adaptation of **MathLie**'s **PDESolve** [24] which additionally considers that equations can factorize and features a built-in interface to couple to the involutive form procedure.
The solution procedure itself consists of four individual solution steps. These steps solve monomial equations, try a direct and an indirect separation of partial differential equations and try to solve pseudo-ordinary differential equations. These four solution steps are arranged in a loop. The system of partial differential equations enters this loop over and over again until no more changes occur. This loop structure is represented in figure 4.1. We will see that many of the partial differential equations which appear in intermediate steps are solved with this tool. In the following we want to describe the single steps of this solution procedure in more detail. Note that before trying to solve each equation of the system it is observed if this equation factorizes, that means if it can be rewritten as a product. If this is the case this equation is not altered in the solution steps.

4.1 The Monomial Solver

The first heuristic used in this solution procedure solves monomial equations. These are equations of the form

$$\frac{\partial^{i_1+\ldots+i_n} f(x^1,\ldots,x^n)}{\partial {x^1}^{i_1}\ldots\partial {x^n}^{i_n}} = 0. \tag{4.1}$$

This sort of equations can be easily solved to give

$$f(x^1,\ldots,x^n) = \sum_{k=1}^{n}\sum_{j=0}^{i_k-1} c_{j,k}(x^1,\ldots x^{k-1},x^{k+1},\ldots,x^n)x^{kj}. \tag{4.2}$$

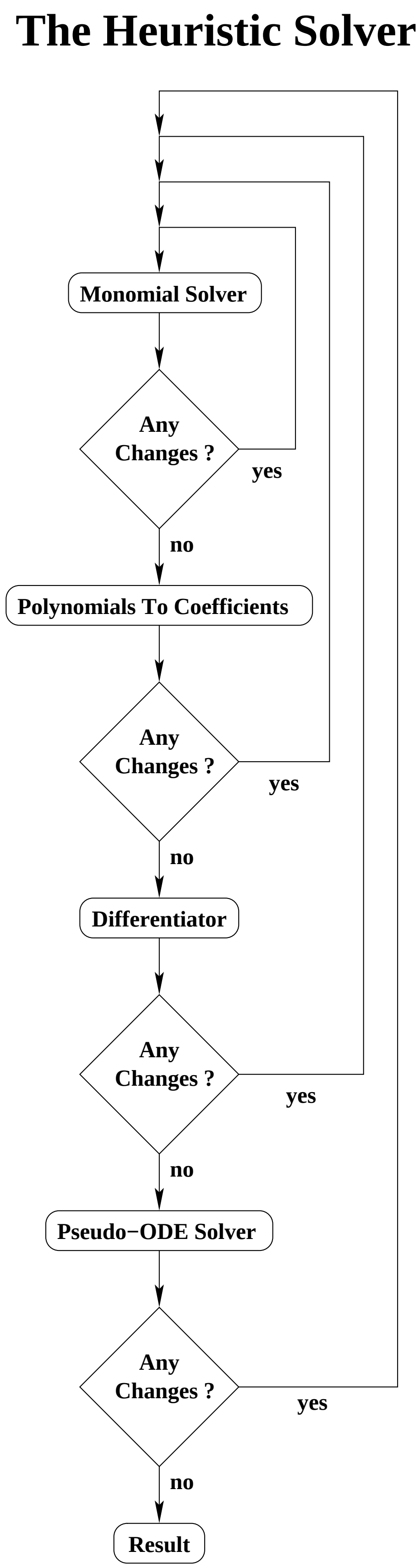


Figure 4.1: The heuristic solution algorithm.

If an equation is of the form (4.1) the dependent variable f is substituted into the whole system and in the conditions unequal to zero as the right-hand side of (4.2) and this solution is then appended as a rule to the system. The new appearing functions $c_{j,k}$ are further on treated as additional dependent variables. The monomial solver is applied to every equation in the system, except the ones that factorize.

4.2 Direct Separation

The next heuristic searches for equations which are real polynomials in an independent variable. This means that no dependent variable which appears in this equation depends on the polynomial variable, for example like

$$f_y(x,y) + z(f(x,y) + g_x(x,y)) + z^2(g(x,y) + yg_y(x,y)) = 0,$$

where f and g depend on x and y, but not on z. Since this equation has to be true for all values of z it separates into the three equations

$$\begin{aligned} f_y(x,y) &= 0, \\ f(x,y) + g_x(x,y) &= 0, \\ g(x,y) + yg_y(x,y) &= 0. \end{aligned}$$

Thus each coefficient for the various powers of the variable which appears polynomially is equal to zero. This simplification step, which is called **direct separation** or "Polynomial To Coefficients" in figure 4.1, then appends the coefficients of the polynomial variable to the system.

If every equation has undergone this procedure it is checked if the system of partial differential equations has changed. If this is the case the system together with the conditions which have to be unequal to zero and possible rules from previous solution steps reenter the monomial solver, since it can happen that monomial equations appear because of this separation, like in the example above.

4.3 Indirect Separation

If the system has not changed in the direct separation step the system enters the indirect separation. Again, if some equation factorizes it is not treated with this procedure. The **indirect separation** searches for separations as a result of differentiations. In figure 4.1 it appears as "Differentiator". This is achieved in the following way.

Say some equation involves an independent variable polynomially, like in the direct separation, but additionally some of the dependent variables depend on this independent variable. Let the highest exponent of this independent variable be k. Then this equation is differentiated $k+1$ times with respect to this variable to reach a simplification. Take for example the equation

$$f(x,y,z)+y^3g(x,z)+y^2h(x,z)+yl(x,z)+m(x,z)=0,$$

where f depends on x, y and z and g, h, l and m only depend on x and z. In this case the highest exponent of the polynomial variable y is three. So this equation is differentiated four times with respect to y to give

$$f_{y,y,y,y}=0,$$

which is a monomial equation and is solved by the monomial solver and appended to the system. So a simpler form of the equation is reached. This example shows the importance of the $(k+1)$-time differentiation. Additionally it is seen that such operations only lead to reductions if the generated equations can be solved by the solution algorithm, in this case the monomial solver.

4.4 Pseudo Ordinary Differential Equations

If every indirect separation, if possible, has been appended to the system it is observed if anything has changed. If so, the new system reenters the monomial solution step to take advantage of the simplifications. When the system does not incorporate equations with polynomial dependence of a dependent variable the system enters the solver for pseudo ordinary differential equations.
This procedure searches for partial differential equations where only differentiations of a single dependent variable with respect to one independent variable occurs. Such an equation is for example of the form

$$b(x^1,\ldots,x^n)+a(x^1,\ldots,x^n)f_{x^i}(x^1,\ldots,x^n)+c(x^1,\ldots,x^n)f_{x^i,x^i}(x^1,\ldots,x^n)=0.$$

Here only derivatives of f with respect to x^i occur. Every equation which is of such a form enters the standard *Mathematica* solver for differential equations, **DSolve**. But before entering **DSolve**, the equation has to be rewritten as an ordinary differential equation.
If **DSolve** is able to solve this equation it produces constants of integration. These constants of integration do not depend upon any independent variables. The coefficients of the dependent variable and the constants of integration have to be transformed to be functions of all of the other independent variables. In the example above the constants of integration then depend on $x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n$.
If a solution can be found by **DSolve** it is inserted into the system, in the conditions which have to be unequal to zero and in the solutions obtained so far. The new appearing “constants” of integration are appended to the set of dependent variables.

Chapter 5

Implementation Notes

As was already said above the French mathematicians Charles Riquier and Maurice Janet already gave a recipe to calculate the involutive form of a system of partial differential equations. This algorithm is the basis of todays implementations in many computer algebra systems. So the basic features of all involutive packages are all the same. First the system is reduced with respect to itself, then it is completed and the integrability conditions are calculated and reduced with respect to the system. But this only provides the skeleton for the implementation in a computer algebra system. The rest is up to the programmer.
One problem the programmer faces when trying to implement an involutive form algorithm is that he cannot rely on the implementation methods of other authors, simply because they are not published. If there are any descriptions at all, they limit themselves to what has to be done when doing the calculations with pencil and paper. Nothing is said how to check if some equation can be inserted into another one, or how to insert it, when and how to do case distinctions and so on. Todays implementations are merely black boxes.
In this chapter we give an overview of how the involutive algorithm is implemented, how a system of partial differential equations is represented in this implementation and the operations which act upon this representation.
The reader may wonder what this is all about. *Mathematica* already offers a front end where mathematical objects are represented in a very intuitive way, very close to mathematics done with pencil and paper. But there are some problems concerning the speed and effectiveness with this representation. We will see that some operations, such as checking if some equation can be inserted in another one, is not very simple to implement using the standard representation of *Mathematica*. Note that this is not only the problem of *Mathematica*, any computer algebra system with a wide variety of applications deals with these kind of problems.
But as we will see there are ways to get around these difficulties. We concentrated upon the strengths of *Mathematica* by mainly using representations and operations which are relatively fast, such as operations on lists. To do this a

whole calculus had to be implemented using lists. This calculus, the **discrete involutive calculus**, includes addition of terms and equations, simplification of terms and equations, the multiplication of equations with terms, how to find prefactors of terms, the operation of derivation and so on.

5.1 Representing Derivatives

Consider a derivative like

$$u_{\sigma, {x^1}^{i_1},\ldots,{x^n}^{i_n}}, \tag{5.1}$$

which incorporates some dependent variable, here u_σ, and the independent variables $x^1, \ldots, x^n$. What is the important information in this construction? Well, the derivative can be reconstructed if the dependent variable which appears in (5.1) is known and how often this dependent variable is differentiated with respect to each independent variable, i. e. the $(n+1)$-tuple

$$u_\sigma, i_1, \ldots, i_n$$

where i_j corresponds to x^j, so the ordering of the i_j's is important. Keeping in mind that there may be several dependent variables and a given ordering of the dependent variables and the independent variables we need only the position of u_σ in the list of the m dependent variables, since the ordering of the dependent and the independent variables is fixed. So all that is needed to recover (5.1) is

$$\{0, \ldots, 1, \ldots, 0, i_1, \ldots, i_n\},$$

where the 1 is on the σ-th position.
But the term ordering used in the involutive form algorithm also features the total order of differentiation, see section 3.1. This additional number is the first criterion when ordering derivatives. The second criterion is the order of the dependent variables and the third the order of the independent variables. So we have to rewrite the above construction to

$$\{i_1 + \cdots + i_n, 0, \ldots, 1, \ldots, 0, i_1, \ldots, i_n\}.$$

But we are not finished yet. Keeping in mind that we deal with polynomially nonlinear systems of partial differential equations we somehow have to express the power of the derivative as an additional index. We append this index to the list above and finally receive the unique representation of (5.1):

$$\{i_1 + \cdots + i_n, 0, \ldots, 1, \ldots, 0, i_1, \ldots, i_n, 1\}.$$

Using this method we can express all derivatives as lists of such a form. As an example consider the derivatives $u_{x,x,y}$, v_y^3 and u itself where u and v are the only dependent variables and x and y the independent ones. The dependent variables are ordered such that $u > v$ and the independent ones as $x > y$. These derivatives are then expressed as

$$\begin{gathered}\{3,1,0,2,1,1\},\\ \{1,0,1,0,1,3\},\\ \{0,1,0,0,0,1\}.\end{gathered}$$

5.2 Terms, Equations and Systems

Since terms are built up from derivatives we simply list the single derivatives to form a term. Hereby we distinguish between "real" derivatives, which are derivatives containing dependent variables, and prefactors, such as numbers or the set of independent variables. These prefactors are appended to the derivatives. The whole term is then ordered according to the term ordering, i. e. derivatives which are greater with respect to the term ordering precede those who are lower. Then a term like $5x^2yu_{x,x,y}v_y^3u$ for the same ordering as above looks like

$$\{\{3,1,0,2,1,1\},\{1,0,1,0,1,3\},\{0,1,0,0,0,1\},5x^2y\}$$

and a term with no dependent variable at all, an inhomogeneity, e.g. $37x$ is represented by

$$\{37x\}.$$

To represent an equation we transform it to the form

$$\text{left-hand-side} - \text{right-hand-side} = 0,$$

list the single terms and sort them with respect to the term ordering. The equation

$$5x^2y\,u_{x,x,y}v_y^3\,u = 37x$$

is then represented by

$$\{\{\{3,1,0,2,1,1\},\{1,0,1,0,1,3\},\{0,1,0,0,0,1\},5x^2y\},\{-37x\}\}.$$

Finally, a system of polynomially nonlinear partial differential equations with conditions unequal to zero is just a list of these conditions and the system. This system itself is just a list of the equations.
These are all the functions needed to transform a system of partial differential equations into the new representation. Before this is done it is observed if the system is polynomially nonlinear in the dependent variables which appear in the system itself, not in the conditions. This is very important, because later on in the solution step it may happen that dependent variables appear no more polynomially nonlinear, but "more" nonlinear, for example in some exponent.
If the system is not polynomially nonlinear in the dependent variables appearing in the system it does not enter the involutive form algorithm at all.

5.3 Multiplication of Terms, Collecting Derivatives and Terms

During some steps in the involutive form algorithm it happens that equations are multiplied with another derivative or term. This is done by appending the term which is multiplied with the multiplication factor. Therefore a function which collects derivatives which only differ in the exponential and a function which collects terms which only differ in their prefactors is needed.
If two derivatives in a single term only differ by their exponentials they are replaced by the same derivative, but with an exponential which is the sum of the single exponentials. Then the result is sorted with respect to the term ordering. Take for example the term

$$\{\{2,1,0,2,0,1\},\{1,0,1,0,1,2\},3,\{1,0,1,0,1,1\},2\}$$

which is the result of multiplying $\{\{2,1,0,2,0,1\},\{1,0,1,0,1,2\},3\}$ with the term $\{\{1,0,1,0,1,1\},2\}$. The second and the fourth derivative only differ by the exponential so they are replaced by $\{1,0,1,0,1,3\}$. Then there are the two number 3 and 2. They are multiplied to 6. So the result of this collection of derivatives is

$$\{\{2,1,0,2,0,1\},\{1,0,1,0,1,3\},6\}.$$

In a similar way the collection of terms which only differ by the prefactor works. Here the prefactors are replaced by the sum of the single prefactors and the resulting equation is sorted with respect to the term ordering.
Note that the sum is of the prefactors is simplified with the *Mathematica*-function **Simplify**. This is done because it is the only *Mathematica*-function which tests if the result is zero. If it is zero **Simplify** is very fast. But if the result is not zero and involves many fractions or is threaded very deep it may take some time and/or memory to get the result. This is a potential bottle-neck in this involutive form algorithm.

5.4 Derivations

Another basic operation when inserting some equations into others is differentiation. But using the list notation this operation is very simple and very fast. Simply add a corresponding "derivative list" to the expression which is to be differentiated. Take for example $u_{x,y}$ which is represented by $\{2,1,0,1,1,1\}$ when dealing with the dependent and independent variables appearing in the above examples. Differentiation with respect to x results in raising the first and the fourth number in this list by one or by adding a derivative to the list expression number by number:

$$\partial_x u_{x,y} = u_{x,x,y} \cong \{1,0,0,1,0,0\} + \{2,1,0,1,1,1\} = \{3,1,0,2,1,1\}.$$

Of course the derivation of products and higher derivations with respect to different independent variables have to be taken into account. The derivation of products is implemented analogous to the product rule. Higher derivatives with respect to different independent variables are handled as follows. To use the speed and effectiveness of the derivative list we first separate a higher order derivative list into single derivative lists. Each derivative list is then applied to the expression to be differentiated one after another. Take for example the derivation $\partial^3_{x,x,y}$, whose derivative list is given by $\{3,0,0,2,1,0\}$. This derivative list is separated into three simple derivative lists:

$$\{3,0,0,2,1,0\} \cong \{\{1,0,0,1,0,0\},\{1,0,0,1,0,0\},\{1,0,0,0,1,0\}\}.$$

Each of these derivative lists is then added to the list representation of the expression to be differentiated, eventually by using the product rule.

5.5 Inserting Equations into other Equations

The list representation presented above is also useful when observing if an equation can be inserted into another one. To do this we take the leading derivative of the equation which is to be inserted. From each list representation of a derivative of the equation in which to insert the other equation the leading derivative is subtracted. If a negative number occurs in the difference the leading derivative cannot be inserted. If all numbers stay positive the leading derivative can be inserted in this derivative of the equation. Take for example the term $5x^2yu^2_{x,x,y}v^3_yu$ and the leading derivative u_x. The corresponding list representations are $\{\{3,1,0,2,1,2\},\{1,0,1,0,1,3\},\{0,1,0,0,0,1\},5x^2y\}$ and $\{1,1,0,1,0,1\}$. Subtracting the last from the former termwise we get

$$\{\{2,0,0,1,1,1\},\{0,-1,0,-1,1,2\},\{-1,0,0,-1,0,0\}\}.$$

The second and the third list contain negative numbers so the leading derivative cannot be inserted. But in the first list only positive numbers occur, so the leading derivative can be inserted.

Note that the difference already contains the operator which is necessary to insert the leading derivative into the first list. The first number tells us that the leading derivative has to be differentiated two times. The fourth and the fifth list entries show that we have to differentiate once with respect to x and once with respect to y. The last entry shows that by inserting the leading derivative one time the remaining power of the first derivative is one. This information is used to insert the equation with the leading derivative into the equation with the above term.

5.6 Calling the Involutive Solution Procedure and Interpreting the Results

In the previous chapters we talked about the algorithms which are used and how they are implemented in *Mathematica*. This section features the interpretation of the results of the involutive solution procedure and the calling of its implemented counterpart **InvolutePDESolve**.
The involutive solution/reduction algorithm needs three input parameters. First the systems of differential equations with the corresponding conditions is needed. The other two things left to start the evaluation are the dependent and independent variables. Here it is important to know which dependent variable depends on which independent variable, because it may happen that a dependent variable is independent of some or even all independent variables. Therefore it is mandatory that the dependent variables in the system of differential equations is given with all the variables on which it depends.
In *Mathematica* dependencies are written with square brackets. Take for example the dependent variable u which depends on x, y, z and t, the dependent variable v which depends only on t and the dependent variable K which does not depend on any of these independent variables, so actually it is a constant. For the function **InvolutePDESolve** these dependent variables in the system of differential equations and its conditions have to be written as

$$u[x, y, z, t], \quad v[t] \text{ and } K[].$$

To calculate the solution of the determining equations (2.22) for the nonlinear diffusion equation (7.1) with the condition $K(u) \neq 0$ for example, the user has to enter:

$$\mathbf{InvolutePDESolve}[\{\{\{K[u]\}, dets\}\}, \{\xi^x, \xi^t, \eta, K\}, \{x, t, u\}],$$

where *dets* stands for the determining system, which can be calculated for example with the **MathLie**-function **DeterminingEquations**.
The result of this calculation is given by a list of different cases. Hereby each list is again given by two lists, the first for the conditions and solutions obtained so far and the second for the rest of the equations which could not be solved. Note that the result may contain expressions with the head **free** and a unique numbering involving the $-sign which is used to designate the new constants or functions introduced by the heuristic solver, for example **free[$2]**[].
To make the result of this call more readable to the unexperienced user the function **FormatOutput** can be used. The result of this function applied to one single case involves three headers named *Solutions*, *Conditions* and *Equations*, each followed by the results obtained.

5.7 Notes

We already saw that the new representation of derivatives, terms and equations as lists containing their basic information was very useful for multiplications, derivations and for inserting equations into other ones. All these operations occur in the autoreduction step which inserts and simplifies a system of partial differential equations with respect to itself while taking care of possible case distinctions which may occur during the calculations.

But the list representation is also useful in the other two major steps towards an involutive form, the completion and the integrability conditions. All the necessary steps in the calculation of the involutive form, such as the calculation of the multiplicative and non-multiplicative variables and with them the computation of the integrability conditions are carried out in the list representation. This is done because this representation is very effective and fast.

Chapter 6

Involutive Reductions

We already mentioned in section 2.8 that we use another method to find invariant solutions. This method uses an involutive algorithm to simplify a system of partial differential equations coupled with a heuristic solver to solve or reduce the system. Therefore we call it the method of **involutive reductions**. The system under consideration is built up from the equation(s) which is/are to be solved and their invariant surface conditions. In this way the same procedure which is able to solve the system of determining equations is also capable of solving or reducing a differential equation when its symmetries are given. The problem is indeed the same whether considering the determining system or the coupled system of differential equation and invariant surface condition. It is all about case distinction and solution of a coupled system of partial differential equations.
In this chapter we illustrate the algorithm on several simple examples. We will see that the solutions of these examples is achieved by the touch of a button.

6.1 Methodology Applied to the Diffusion Equation

The standard diffusion equation in 1+1 dimensions is given by

$$u_t = u_{x,x}, \tag{6.1}$$

where lower indices represent partial differentiations with respect to time t and spatial coordinate x. In the following we discuss the invariance of (6.1) under two infinitesimal transformations leading to two different kinds of solutions.

6.1.1 Error-Function Solution

The standard diffusion equation allows among other transformations a scaling with the infinitesimal generator

$$X = x\partial_x + 2t\partial_t,$$

so a function $u = f(x,t)$ is invariant under this transformation if it satisfies the first order partial differential equation

$$xu_x + 2tu_t = 0, \tag{6.2}$$

which is nothing but the invariant surface condition. To reduce equation (6.1) with the condition (6.2) the coupled system first enters the involutive simplifier. Calling our involutive reduction method on the combined system (6.1,6.2) we get the same result as is achieved by the direct method, but completely automatic and in a direct way. When tracing the intermediate results to take a look inside the "black box", we see that the involution algorithm reduces (6.1) with (6.2). First the involutive simplifier is called. Since the integrability conditions of the resulting system are all satisfied the involutive algorithm ends with the system

$$\begin{aligned} xu_x + 2tu_{x,x} &= 0, \\ xu_x + 2tu_t &= 0. \end{aligned} \tag{6.3}$$

The following solution step first tries to solve monomial equations, that means equations consisting of just one term. Since (6.3) does not contain such equations the solution algorithm enters the next step. If an equation is a polynomial in one variable the coefficients of the various powers of this variable are identically zero. Again the system does not change, just like in the next step, which differentiates equations with respect to variables appearing polynomially.
Next, the system enters a solver for ordinary differential equations. Hereby, the first equation of (6.3) has the solution

$$u = f_3(t) + \sqrt{\pi t}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) f_2(t)$$

with the new arbitrary functions $f_2(t)$ and $f_3(t)$. Inserting this in (6.3), we get

$$\sqrt{\pi t}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) f_2(t) + 2t\sqrt{\pi t}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) f_2'(t) + 2tf_3'(t) = 0 \tag{6.4}$$

as the only equation which has to be fulfilled. With this the solver ends and we again enter the involutive part of the algorithm. Since f_2 and f_3 only depend on t differentiation with respect to x and back-insertion of the result in (6.4) delivers

$$f_3'(t) = 0. \tag{6.5}$$

The reduction step inserts this in the system. We get

$$f_2(t) + 2tf_2'(t) = 0.$$

With these results the involutive part ends. Entering the solution algorithm the monomial solver integrates (6.5), so f_3 is a constant. The ODE-solver then gives

$$f_2(t) = \frac{C}{\sqrt{t}}.$$

Combining these results the algorithm **InvolutivePDESolve** ends with

$$u = C_1 + C_2\sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

6.1.2 Airy Solution

The standard diffusion equation is also invariant under the infinitesimal generator

$$X = \partial_t - 2t\partial_x + xu\partial_u.$$

Looking for invariant solutions under this transformation, we get the additional equation

$$xu + 2tu_x - u_t = 0. \tag{6.6}$$

Combining equations (6.1,6.6) and following the steps in which changes happen we call the involutive solver.
In the first part of the involutive step equation (6.1) is reduced by (6.6) to

$$-xu - 2tu_x + u_{x,x}.$$

The following steps, the computation and reduction of the integrability conditions do not change anything. So the involutive procedure delivers the system

$$\begin{aligned} -xu - 2tu_x + u_{x,x} &= 0, \\ -xu - 2tu_x + u_t &= 0. \end{aligned} \tag{6.7}$$

Entering the solution step the algorithm first looks for monomial equations. Since there are none the program checks for polynomial variables. Again the system does not change under this operation as well as under a differentiation. Next (6.7) enters the solver for ordinary differential equations. In this step the first equation is solved to give

$$u = e^{tx}(\mathrm{Ai}(x + t^2)f_2(t) + \mathrm{Bi}(x + t^2)f_3(t))$$

with new free functions f_2 and f_3. Hereby Ai and Bi denote Airy-functions [57]. Inserting this in (6.7) we get an identity and the relation

$$-\mathrm{Ai}(x+t^2)f_2'(t) - \mathrm{Bi}(x+t^2)f_3'(t) + 2t^2\mathrm{Ai}(x+t^2)f_2(t) + 2t^2\mathrm{Bi}(x+t^2)f_3(t) = 0. \tag{6.8}$$

Since f_2 and f_3 only depend on t, differentiation of this condition with respect to x and inserting this in (6.8) delivers the following ODEs:

$$\begin{aligned} f_2'(t) - 2t^2 f_2(t) &= 0, \\ f_3'(t) - 2t^2 f_3(t) &= 0, \end{aligned}$$

with which the involutive part ends. These two equations are then integrated by the ad-hoc solver to

$$\begin{aligned} f_2(t) &= C_1 e^{\frac{2}{3}t^2}, \\ f_3(t) &= C_2 e^{\frac{2}{3}t^2}. \end{aligned}$$

With these solutions the involutive solver ends with the invariant solution

$$u = e^{\frac{2}{3}t^2+tx}(C_1 \mathrm{Ai}(x+t^2) + C_2 \mathrm{Bi}(x+t^2)).$$

Again this solution is obtained completely automatic.

6.2 A Nonlinear Diffusion Equation

As a second example we consider the nonlinear diffusion equation

$$u_t = u_x^2 + u u_{x,x}. \tag{6.9}$$

This equation is invariant under the infinitesimal generator

$$X = x\partial_x + 2u\partial_u,$$

meaning that the corresponding invariant surface condition

$$2u - xu_x = 0 \tag{6.10}$$

has to be satisfied. The method of invariants just works the same way as for the standard diffusion equation. We just state the result:

$$u = \frac{x^2}{C - 6t} \tag{6.11}$$

where C is a constant.

To treat the problem of finding invariant solutions with the involutive method we again discuss those steps in which important changes occur. First the combined system (6.9,6.10) enters the involution step. The first part of the involutive algorithm reduces the system with itself. In our example (6.10) is inserted in (6.9) to give

$$\begin{aligned} -2u + xu_x &= 0, \\ -6u^2 + x^2 u_t &= 0. \end{aligned} \tag{6.12}$$

The following completion step does not change anything which is also true for the computation and reduction of the integrability conditions. So the involutive procedure ends with (6.12).

Now the system enters the heuristic solver. The first step of the solver searches for monomial equations. Since there are none and also no simplifications are possible the system enters the differentiator. Hereby the first equation of (6.12) is differentiated two times with respect to x leading to the equation

$$xu_{x,x,x} = 0.$$

The following monomial solver provides

$$u = f_1(t) + xf_2(t) + x^2 f_3(t).$$

Inserting this in (6.12) gives

$$\begin{aligned} xf_3(t) + 2x^2 f_4(t) - 2f_2(t) - 2xf_3(t) - 2x^2 f_4(t) &= 0, \\ -6(f_2(t) + xf_3(t) + x^2 f_4(t))^2 + x^2 f_2'(t) + x^3 f_3'(t) + x^4 f_4'(t) &= 0. \end{aligned}$$

Since each equation has to be satisfied for all values of x the polynomial simplifier delivers

$$\begin{aligned} f_2(t) &= 0, \\ f_3(t) &= 0, \\ -6f_3(t)^2 - 12f_2(t)f_4(t) + f_2'(t) &= 0, \\ -12f_3(t)f_4(t) + f_3'(t) &= 0, \\ -6f_4(t)^2 + f_4'(t) &= 0 \end{aligned}$$

which results in

$$u = f_4(t)x^2, \tag{6.13}$$

where f_4 has to solve the ODE

$$-6f_4(t)^2 + f_4'(t) = 0. \tag{6.14}$$

The single remaining condition (6.14) now enters the solver for ordinary differential equations. The result is

$$f_4(t) = \frac{1}{C_1 - 6t}.$$

Inserting this in (6.13) we get one solution of (6.9):

$$u = \frac{x^2}{C - 6t}.$$

With this result the involutive solver stops. This is exactly the same solution as (6.11) derived by the method of invariants, but in a much more convenient way. It is done automatically!

These examples show that the involutive reduction method is able to solve simple differential equations when a symmetry transformation is given. In the next chapters we use the involutive reduction method to physical applications to see that the case distinction and solution capabilities also provide solutions in these circumstances.

Chapter 7

Involutive Reductions and Solutions of a Nonlinear Diffusion Equation

In this chapter we deal with the 1+1 dimensional nonlinear diffusion equation

$$u_t = (K(u)\, u_x)_x \tag{7.1}$$

for a single function u of the two independent variables x and t representing space and time respectively. We are going to examine equation (7.1) for an arbitrary diffusivity $K(u)$. We also identify different kinds of functions of $K(u)$ in a group classification problem.

This class of equations has many applications, among others it is used in plasma physics [43], in describing convectionless transport of fluids in homogeneous, non-deformable porous media [44] or in dissipative nonlinear media [45]. See also the applications discussed in [46, 47].

Exact solutions of (7.1) with $K = const.$ were found in [13]. In [48] exact separable solutions are found for this class of equations with an additional source term. In [43] a symmetry classification is given also including a source term. In [49] some functional forms of exact solutions of equation (7.1) with $K = u^{-4/3}$ and $u^{-2/3}$ are derived. In [45] an overview of a quasilinear heat equation with a source is given. In [50] appears a group classification including optimal systems of (7.1).

The classification and some solutions of this equation using symmetry analysis were presented in [40, 51], or [44, 45, 48, 52] for related equations. All the solutions known were found by using a single element of the symmetry algebra or the optimal system of the infinitesimal transformations. In contrast, we searched for invariant solutions under the full generator of the symmetry group in connection with an involutive procedure. Applying this method to (7.1) we were able to find solutions which depend on up to six group constants appearing in the generator of the group.

7.1 Classification with Respect to the Diffusivity

Equivalence transformations are used in [45, 50, 51] to approach the group classification problem of equation (7.1). The problem with this method is the use of optimal systems of subalgebras. Their computation can be quite difficult. Also, discrete symmetry groups have to be taken into account to obtain a fully reduced optimal system.
Another lack of this method are the results obtained. They incorporate only simple representatives of classes of functions. For example we will see below that $K(u) = C_1(C_2 + uC_3)^{\frac{C_4}{C_3}}$ appears among the classified diffusivities. This diffusivity is clearly more general than its analogue $K(u) = u^\sigma$, which is obtained by equivalence transformations.
We resolve these difficulties by applying the involutive reduction method. With this procedure we performed a symmetry analysis of (7.1). Our examinations are focused on point symmetries. We resolved the following cases from the classification problem concerning the diffusivity K:

- K arbitrary:

$$\begin{aligned} \xi^x &= C_1 + xC_2, \\ \xi^t &= C_3 + 2tC_2, \\ \eta &= 0 \end{aligned}$$

- $K = const.$

$$\begin{aligned} \xi^x &= C_1 + x(C_2 + tC_4) + tC_5, \\ \xi^t &= C_3 + t(2C_2 + tC_4), \\ \eta &= f(x,t) + u\left(-\frac{C_4}{2}t - \frac{x(C_4x + 2C_5)}{4K} + C_6\right), \end{aligned}$$

where f has to satisfy the equation

$$f_t - Kf_{x,x} = 0.$$

Note that f represents the linear form of equation (7.1) with $K = const.$

- $K = C_1(C_2 + uC_3)^{\frac{C_4}{C_3}}$

$$\begin{aligned} \xi^x &= C_5 + \frac{x}{2}(C_4 + C_6), \\ \xi^t &= C_7 + tC_6, \\ \eta &= C_2 + uC_3 \end{aligned}$$

- $K = e^{C_1(u-C_2)}$

$$\begin{aligned} \xi^x &= C_3 + \frac{x}{2}(C_1C_4 + C_6), \\ \xi^t &= C_5 + tC_6, \\ \eta &= C_4 \end{aligned}$$

- $K = \frac{C_1}{(u-C_2)^{4/3}}$

$$\begin{aligned} \xi^x &= C_3 + x(C_4 + xC_5), \\ \xi^t &= C_6 + 2t(C_4 + \frac{2}{3}C_7), \\ \eta &= (u - C_2)(-3xC_5 + C_7) \end{aligned}$$

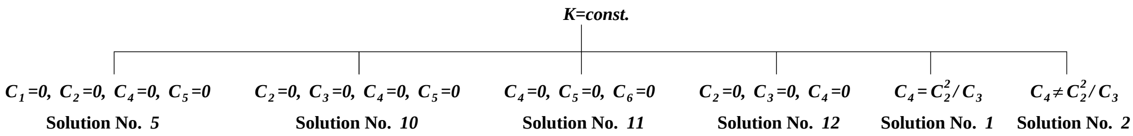
where the C_i's denote the arbitrary group constants and ξ^x, ξ^t and η are the infinitesimals for the independent variables x, t and the dependent variable u. We note that we get the same classification as in [40]. But in contrast to [40] we treat each appearing constant as a group constant. This means that the above symmetries are not longer built up from linear independent simpler symmetries, namely the coefficients of the C_i's. Therefore we do not split the symmetries to form an algebra and therefore cannot calculate equivalence transformations as is done in [51].

7.2 Reductions and Solutions

In this section we construct exact solutions and reductions for the cases in section 7.1. We insert each symmetry into the coupled system of (7.1) and its invariant surface condition

$$\begin{aligned} u_t - (K(u)\, u_x)_x &= 0, \\ \eta - \xi^x u_x - \xi^t u_t &= 0 \end{aligned} \tag{7.2}$$

and use the method explained in the previous section to reduce the resulting coupled system of partial differential equations. For arbitrary K this procedure does not work because the functional form of u is not given. This case is not considered here.



7.2.1 $K = const.$

The most general case when $K = const.$ is given by the coupled system

$$\begin{array}{rcl} -Ku_{x,x} + u_t &=& 0, \\ -(C_1 + x(C_2 + tC_4) + tC_5)u_x - & & \\ (C_3 + t(2C_2 + tC_4))u_t + & & \\ u\left(-\frac{C_4}{2}t - \frac{x(C_4x+2C_5)}{4K} + C_6\right) &=& 0. \end{array} \tag{7.3}$$

The function f mentioned in section 7.1 for this case is not considered here because it represents the linearity of the diffusion equation.
Applying the involutive reduction procedure to (7.3) we get the following cases: We will show how the results of the involutive reduction procedure lead to the mentioned solutions.

Case $C_1 = 0$, $C_2 = 0$, $C_4 = 0$, $C_5 = 0$

Besides the conditions for the C_i's the involutive solver delivered

$$u(x,t) = f_8(t)e^{-\frac{x\sqrt{C_6}}{\sqrt{K}\sqrt{C_3}}} + f_9(t)e^{\frac{x\sqrt{C_6}}{\sqrt{K}\sqrt{C_3}}}, \tag{7.4}$$

where the functions f_8 and f_9 have to satisfy the condition

$$-C_6\, f_8(t) - e^{\frac{2\,x\sqrt{C_6}}{\sqrt{K}\sqrt{C_3}}}\, C_6\, f_9(t) + C_3 f_8'(t) + e^{\frac{2\,x\sqrt{C_6}}{\sqrt{K}\sqrt{C_3}}}\, C_3\, f_9'(t) = 0.$$

Since f_8 and f_9 only depend on t the above condition simplifies to the following two equations:

$$\begin{array}{rcl} -C_6 f_8(t) + C_3 f_8'(t) &=& 0, \\ -C_6 f_9(t) + C_3 f_9'(t) &=& 0. \end{array}$$

These are easily solved:

$$\begin{array}{rcl} f_8(t) &=& Ae^{\frac{C_6 t}{C_3}}, \\ f_9(t) &=& Be^{\frac{C_6 t}{C_3}}. \end{array}$$

Inserting this into equation (7.4) we get the solution no. 5 in table 1:

$$u(x,t) = Ae^{\frac{C_6 t}{C_3} - \sqrt{\frac{C_6}{KC_3}}x} + Be^{\frac{C_6 t}{C_3} + \sqrt{\frac{C_6}{KC_3}}x}, \tag{7.5}$$

where here and in the following A and B represent constants of integration. Note that in [13] the special case $A = 0$ occurs (see below).

Case $C_2 = 0$**,** $C_3 = 0$**,** $C_4 = 0$**,** $C_5 = 0$

This case is almost identical to the case above. The only difference is that here C_3 is identically zero, and not C_1. The reduction delivers

$$u(x,t) = f_8(t) e^{\frac{C_6}{C_1} x}$$

with the condition

$$K C_6^2 f_8(t) - C_1^2 f_8'(t) = 0.$$

The solution to this equation is

$$f_8(t) = A e^{\frac{K C_6^2}{C_1^2} t}$$

and so the solution for u reads (no. 10 in table 1)

$$u(x,t) = A e^{\frac{C_6}{C_1^2}(x C_1 + K C_6 t)}.$$

Case $C_4 = 0$**,** $C_5 = 0$**,** $C_6 = 0$

This choice leads to three solutions. The first one has to satisfy the additional conditions $C_1 = 0$, $C_2 = 0$ and $C_3 = 0$. The only remaining equation is the diffusion equation itself. This corresponds to the case where there are no symmetries at all.
The second case solves the diffusion equation for the additional conditions $C_2 = 0$ and $C_3 = 0$. The resulting solution is a constant. In the third case we get

$$\begin{aligned} u(x,t) &= \sqrt{\frac{\pi K (C_3 + 2tC_2)}{2C_2}} e^{\frac{C_1^2}{2KC_2(C_3+2tC_2)}} \, erf\left(\frac{C_1 + xC_2}{\sqrt{2KC_2(C_3 + 2tC_2)}}\right) f_1(t) \\ &\quad + f_2(t), \end{aligned}$$

with the additional condition

$$\begin{aligned} \sqrt{2\pi} e^{\frac{C_1^2}{2KC_2(C_3+2tC_2)}} \, erf\left(\frac{C_1 + xC_2}{\sqrt{2KC_2(C_3 + 2tC_2)}}\right) (K(C_3 + 2tC_2)^2 f_1'(t) \\ + (KC_2(C_3 + 2tC_2) - C_1^2) f_1(t)) + 2\sqrt{KC_2}(C_3 + 2tC_2)^{3/2} f_2'(t) &= 0. \end{aligned}$$

Since this condition has to be true for all x the equation separates into

$$\begin{aligned} (K(C_3 + 2tC_2)^2 f_1'(t) + (KC_2(C_3 + 2tC_2) - C_1^2) f_1(t)) &= 0 \\ f_2'(t) &= 0. \end{aligned}$$

Solving these equations the solution of the diffusion equation in this case reads (no. 11 table 1)

$$u(x,t) = A + B\sqrt{\frac{K\pi}{2C_2}} \, erf\left(\frac{C_1 + xC_2}{\sqrt{2KC_2(C_3 + 2tC_2)}}\right).$$

Hereby A and B are two constants of integration. Comparing this with the solution $u = C_1\, erf(x/2\sqrt{t}) + C_2$ given in [13] we see that our solution implies three more nontrivial constants.

Case $C_2 = 0$**,** $C_3 = 0$**,** $C_4 = 0$

The involutive solver delivered besides the conditions for C_2, C_3 and C_4 the reduction

$$u(x,t) = f(t)e^{-\frac{x(xC_5-4KC_6)}{4K(C_1+tC_5)}}, \tag{7.6}$$

where f has to satisfy

$$K((-C_1C_5 - tC_5^2 + 2KC_6^2)f(t) - 2(C_1+tC_5)^2 f'(t) = 0.$$

This equation is now solved and the result inserted in (7.6) to give the well-known Gaussian solution (no. 12 table 1)

$$u(x,t) = \frac{A}{\sqrt{C_1+tC_5}}e^{-\frac{(xC_5-2KC_6)^2}{4KC_5(C_1+tC_5)}}.$$

Case $C_4 = \frac{C_2^2}{C_3}$

The calculations for this case requires a lot of space. So we simply mention the results here (solution no. 1 table 1):

$$u(x,t) = \frac{e^{-\frac{\xi_2(x,t)}{12KC_2{}^3(tC_2+C_3)^3}}}{\sqrt{tC_2+C_3}}(A\cdot Ai\,(\xi_3(x,t)) + B\cdot Bi\,(\xi_3(x,t))),$$

with the Airy functions Ai and Bi [57] and the abbreviations

$$\begin{aligned}\xi_1(x,t) &= C_3(C_1+tC_5)^2 + 2x\,(tC_2+C_3)\,(C_1C_2 - C_3C_5)\\ &\quad +2K(tC_2+C_3)^2\,(C_2+2C_6)\,,\\[1em] \xi_2(x,t) &= 3(tC_2+C_3)^2\left(C_3{}^2C_5{}^2 + \left(xC_2{}^2 + C_3C_5\right)^2 +\right.\\ &\quad \left.2KC_2{}^2C_3\,(C_2+2C_6)\right) + 2C_3{}^2(C_1C_2 - C_3C_5)^2 +\\ &\quad 6C_3\,(tC_2+C_3)\,(C_1C_2-C_3C_5)\left(xC_2{}^2 + C_3C_5\right),\end{aligned}$$

and

$$\xi_3(x,t) = \frac{K^{\frac{4}{3}}C_3{}^{\frac{1}{3}}\xi_1(x,t)}{2^{\frac{4}{3}}(tC_2+C_3)^2(C_1C_2-C_3C_5)^{\frac{2}{3}}}.$$

This solution is new compared to [13, 39, 51], where ζ_3 is given by $\zeta_3 = x + t^2$, the square-root does not appear and the argument of the exponential function reads $xt + \frac{2}{3}t^3$.

Case $C_4 \neq \frac{C_2^2}{C_3}$

Again the calculations are rather messy and we state only the results (solution no. 2 table 1):

$$u(x,t) = \frac{K^2 e^{-\frac{\xi_3(x,t)}{4K}} \xi_1(x,t)}{2\xi_4(t)^{\frac{3}{4}} \left(K^2 \tilde{C}_2\right)^{\frac{5}{4}}} \left(A\, {}_1F_1\left(\tilde{C}_5, \frac{3}{2}, \xi_2(x,t)\right) + B\, U\left(\tilde{C}_5, \frac{3}{2}, \xi_2(x,t)\right)\right)$$

with the abbreviations

$$\xi_1(x,t) = C_1C_2 - C_3C_5 + x(C_2^2 - C_3C_4) + t(C_1C_4 - C_3C_5),$$

$$\xi_2(x,t) = \frac{\tilde{C}_3 \xi_1(x,t)^2}{2\xi_4(t)\tilde{C}_2^2},$$

$$\begin{aligned} \xi_3(x,t) &= \frac{t\tilde{C}_4^2}{\tilde{C}_2^2} - \frac{2\tilde{C}_4\xi_1(x,t)}{\tilde{C}_2^2} + \frac{\left(C_2 + tC_4 + K\tilde{C}_3\right)\xi_1(x,t)^2}{\xi_4(t)\tilde{C}_2^2} \\ &\quad + \frac{\tilde{C}_1}{(-\tilde{C}_2)^{\frac{3}{2}}} \arctan\left(\frac{C_2 + tC_4}{\sqrt{-\tilde{C}_2}}\right), \end{aligned}$$

$$\xi_4(t) = 2tC_2 + C_3 + t^2C_4,$$

$$\tilde{C}_1 = -{C_1}^2C_4 + 2C_1C_2C_5 - C_3{C_5}^2 + 2K\left({C_2}^2 - C_3C_4\right)(C_2 + 2C_6),$$

$$\tilde{C}_2 = C_2^2 - C_3C_4,$$

$$\tilde{C}_3 = \sqrt{\frac{\tilde{C}_2}{K^2}},$$

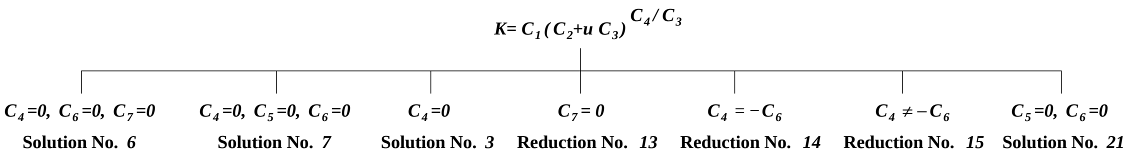
$$\tilde{C}_4 = C_1C_4 - C_2C_5$$

and finally

$$\tilde{C}_5 = \frac{\tilde{C}_3(\tilde{C}_1 + 6K^4\tilde{C}_3^{\frac{3}{2}})}{8\tilde{C}_2^2}.$$

We notice that the hypergeometric functions reduce to other special functions if the parameters are chosen appropriately. For example, the above solution incorporates Laguerre-Polynomials, Weber-Functions or Hermite-Polynomials for ${}_1F_1$ or Spherical Bessel-Functions, Cunningham-Functions or Bateman-Functions for U. A complete overview of the special cases of the hypergeometric functions is found in [57].

Note that our solution also reduces to the ones known when choosing the constants appropriately. To get the solution in [13, 39, 51] we have to choose $C_5 = -\frac{\nu}{2}$ so that the confluent hypergeometric functions reduce to the parabolic cylinder functions cited therein.



7.2.2 $K = C_1(C_2 + uC_3)^{\frac{C_4}{C_3}}$

Inserting the symmetries for this kind of diffusivity in (7.2) we arrive at

$$\begin{aligned} u_t - K_x u_x - K u_{x,x} &= 0, \\ 2C_2 + 2C_3 - 2(C_5 + tC_6)u_t - (2C_7 + x(C_4 + C_6))u_x &= 0, \\ (C_2 + C_3 u)K_x - KC_4 u_x &= 0, \\ (C_2 + uC_3)K_t - KC_4 u_t &= 0, \end{aligned} \tag{7.7}$$

where we have appended the differential equation for K to the system. The automatic solution procedure of (7.7) ends with twelve cases. Four of them have the result that u is a constant. We don't list them here. The other reductions lead to the following classification:

Case $C_7 = 0$**,** $C_4 = 0$**,** $C_6 = 0$

For this choice of the constants the diffusivity K is also constant. Like in the case where K is a constant initially, we get a partial solution with additional free functions:

$$u = -\frac{C_2}{C_3} + f_1(t)e^{-\sqrt{\frac{C_3}{KC_5}}x} + f_2(t)e^{-\sqrt{\frac{C_3}{KC_5}}x},$$

with the condition

$$-C_3 f_1(t) - e^{2\sqrt{\frac{C_3}{KC_5}}x} C_3 f_2(t) + C_5\left(f_1'(t) + e^{2\sqrt{\frac{C_3}{KC_5}}x} f_2'(t)\right) = 0.$$

The solution is given by (no. 6 table 1)

$$u(x,t) = Ae^{\frac{C_3}{C_5}t + \sqrt{\frac{C_3}{C_5 K}}x} + Be^{\frac{C_3}{C_5} - \sqrt{\frac{C_3}{C_5 K}}x} - \frac{C_2}{C_3}.$$

This solution is up to changes of constants identical to (7.5).

Case $C_4 = 0$**,** $C_5 = 0$**,** $C_6 = 0$

Here the involutive reduction arrives at

$$u = -\frac{C_2}{C_3} + f(t)e^{\frac{C_3}{C_7}x}$$

where f has to satisfy the equation

$$KC_3^2 f - C_7^2 f'(t) = 0.$$

Solving this we get

$$u(x,t) = Ae^{\frac{C_3}{C_7}(KC_3t + xC_7)} - \frac{C_2}{C_3},$$

where again A is a constant of integration. This solution is found in table 1 where it has the number 7.

Case $C_4 = 0$

In this case no partial solution is obtained. The reduction stops at

$$\begin{aligned}
-2C_2 - 2C_3u + (xC_6 + 2C_7)u_x + 2(C_5 + tC_6)u_t &= 0, \\
2KC_2(2C_3 - C_6) + 2KC_3(2C_3 - C_6)u - & \\
(x^2C_6^2 + 2K(4C_3 - 3C_6)(C_5 + tC_6) + 4xC_6C_7 + 4C_7^2)u_t & \\
+4K(C_5 + tC_6)^2 u_{t,t} &= 0.
\end{aligned}$$

Solving the first equation leads to

$$u = (C_5 + xC_6)^{\frac{C_3}{C_6}} f\left(\frac{2C_7 + tC_6}{C_6\sqrt{C_5 + tC_6}}\right) - \frac{C_2}{C_3}.$$

Inserting this in the second equation we get an equation for f:

$$-2f(\zeta)C_3 + C_6\zeta f'(\zeta) + 2Kf''(\zeta) = 0$$

where ζ stands for $\frac{2C_7 + xC_6}{C_6\sqrt{C_5 + tC_6}}$. This equation has the solution

$$f(\zeta) = A_1F_1\left(-\frac{C_3}{C_6}, \frac{1}{2}, -\frac{C_6\zeta^2}{4K}\right) + B\zeta_1F_1\left(\frac{1}{2} - \frac{C_3}{C_6}, \frac{1}{2}, -\frac{C_6\zeta^2}{4K}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function [57]. Combining both results the solution to the diffusion equation is given by (no. 3 table 1)

$$u(x,t) = \zeta_1^{\frac{C_3}{C_6}}\left(A_1F_1\left(-\frac{C_3}{C_6}, \frac{1}{2}, -\frac{\zeta_2^2}{4K}\right) + \zeta_2B_1F_1\left(\frac{1}{2} - \frac{C_3}{C_6}, \frac{3}{2}, -\frac{\zeta_2^2}{4K}\right)\right) - \frac{C_2}{C_3}$$

with the abbreviations

$$\begin{aligned}
\zeta_1 &= C_5 + tC_6, \\
\zeta_2 &= \frac{2C_7 + xC_6}{C_6\sqrt{\zeta_1}}
\end{aligned}$$

and the two constants A and B. Again this result is more general than the ones in [13, 39, 51] and reduces to them if the constants are chosen in such a way that the confluent hypergeometric functions reduce to the parabolic cylinder functions.

Case $C_7 = 0$

For this case the involutive solver ends with

$$\begin{aligned} u &= -\frac{C_2}{C_3} + (C_5 + tC_6)^{\frac{C_3}{C_6}} f_1(x), \\ K &= \frac{f_2(x)}{C_5 + tC_6}. \end{aligned}$$

Hereby, $f_1(x)$ and $f_2(x)$ are arbitrary functions which are coupled according to

$$a = f_1(x)^{C_6} f_2(x)^{C_3}$$

where a is a constant and $f_2(x)$ has to satisfy the condition

$$C_3 f_2'(x)^2 - C_6^2 f_2(x) - C_6 f_2(x) f_2''(x) = 0.$$

This equation has the implicit solution

$$\begin{aligned} 2f_2(x)\frac{\sqrt{1-\zeta(f_2(x))}}{\sqrt{-A^{\frac{1}{C_6}} f_2(x)^{\frac{2C_3}{C_6}} + 2f_2(x)C_6^2}}\, {}_2F_1\left(\zeta_1, \frac{1}{2}; 1+\zeta_1; \zeta(f_2(x))\right) &= \\ 2B\frac{\sqrt{1-\zeta(B)}}{\sqrt{-A^{\frac{1}{C_6}} B^{\frac{2C_3}{C_6}} + 2BC_6^2}}\, {}_2F_1\left(\zeta_1, \frac{1}{2}; 1+\zeta_1; \zeta(B)\right) - x & \end{aligned}$$

with the hypergeometric function ${}_2F_1$ [57], the constants A, B and the abbreviations

$$\zeta(h) = \frac{A^{\frac{1}{C_6}} h^{-1+\frac{2C_3}{C_6}}}{2C_6^2}$$

and

$$\zeta_1 = \frac{C_6}{4C_3 - 2C_6}.$$

This solution is listed as no. 13 in table 2.

Case $C_4 = -C_6$

In this case the involutive solver ends with four partial differential equations where K and u occur. Three of them are coupled, but the fourth is a first order partial differential equation just involving K:

$$2KC_6 + 2C_7 K_x + 2(C_5 + tC_6)K_t = 0.$$

This equation is solved to give

$$K = \frac{f\left(x - \frac{C_7}{C_6}\log\left(C_5 + tC_6\right)\right)}{C_5 + tC_6}.$$

Inserting this partial solution into the other equations we arrive at one equation which just involves the arbitrary function f. It reads (reduction no. 14 table 2)

$$C_6C_7f'(\zeta) - C_3f'(\zeta)^2 + f(\zeta)(C_6^2 + C_6f''(\zeta)) = 0.$$

We were not able to solve this equation in general. So in this case no solution is obtained, just a reduction. This is also true for the case when $C_4 \neq -C_6$, as is seen below.

Case $C_4 \neq -C_6$

Like above we arrive at

$$K = (C_5 + tC_6)^{\frac{C_4}{C_6}} f\left((C_5 + tC_6)^{-\frac{C_4+C_6}{2C_6}}\left(x + \frac{2C_7}{C_4 + C_6}\right)\right)$$

and an ordinary differential equation which we were not able to solve in the general case:

$$\zeta C_4(C_4 + C_6)f'(\zeta) + 2C_3f'(\zeta)^2 + f(\zeta)(-2C_4^2 + 2C_4f''(\zeta)) = 0.$$

So again only a reduction was possible (no. 15 table 2).

Case $C_5 = 0$**,** $C_6 = 0$

Besides this form of the constants the involutive reduction procedure delivered the partial solutions

$$\begin{aligned} u &= -\frac{C_2}{C_3} + f_1(t)(xC_4 + 2C_7)^{\frac{2C_3}{C_4}}, \\ K &= f_2(t)(xC_4 + 2C_7)^2. \end{aligned}$$

The functions $f_1(t)$ and $f_2(t)$ have to satisfy the conditions

$$\begin{aligned} 2C_3(2C_3 + C_4)f_1(t)f_2(t) - f_1'(t) &= 0, \\ 2C_4(2C_3 + C_4)f_2(t)^2 - f_2'(t) &= 0. \end{aligned}$$

Solving the second equation and substituting the result in the first equation delivers

$$\begin{aligned} f_2(t) &= \frac{1}{A - 2tC_4(2C_3 + C_4)}, \\ f_1(t) &= B(A - 2tC_4(2C_3 + C_4))^{-\frac{C_3}{C_4}}. \end{aligned}$$

Inserting this in the relation for the diffusivity we obtain

$$K = \frac{(xC_4 + 2C_7)^2}{A - 2tC_4(2C_3 + C_4)}$$

and the solution (no. 21 table 3)

$$u(x,t) = -\frac{C_2}{C_3} + B\left(\frac{(xC_4 + 2C_7)^2}{A + 2tC_4(2C_3 + C_4)}\right)^{\frac{C_3}{C_4}}.$$

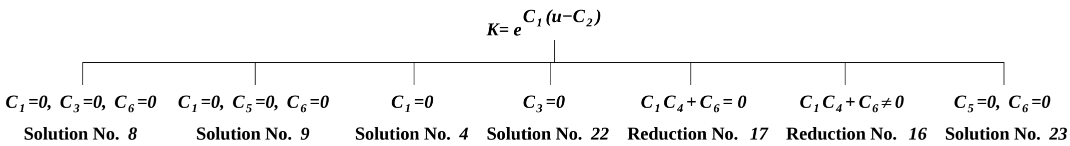
This solution contains besides the integration constants A and B four group parameters which are involved in the symmetry analysis. This solution reduces to the one in [51] if the C_i are chosen appropriately.

7.2.3 $K = e^{C_1(u-C_2)}$

For this class of diffusivities system (7.2) reads

$$\begin{aligned} u_t - K_x u_x - K u_{x,x} &= 0, \\ C_4 - (C_5 + tC_6)u_t - \left(C_3 + \frac{x}{2}(C_1C_4 + C_6)\right)u_x &= 0, \\ K_x - KC_1u_x &= 0, \\ K_t - KC_1u_t &= 0. \end{aligned}$$

As before, we mention only the cases where interesting solutions occur, meaning we don't list the cases where u is a constant or K is zero.
Acting upon this system with the involutive solver delivers seven non-trivial cases:



For the first three cases we only list the results here, because either the solutions where obtained directly, as in the first two cases, or they involve additional space-consuming computations, as in the third case.

Case $C_1 = 0$, $C_3 = 0$, $C_6 = 0$ (solution no. 8 table1)

$$u(x,t) = t\frac{C_4}{C_5} + x^2\frac{C_4}{2KC_5} + A + xB.$$

Case $C_1 = 0$, $C_5 = 0$, $C_6 = 0$ (solution no. 9 table 1)

$$u(x,t) = x\frac{C_4}{C_3} + A.$$

Case $C_1 = 0$ (solution no. 4 table 1)

$$\begin{aligned} u(x,t) &= \frac{C_4(2C_3+xC_6)^2}{2KC_6^2(C_5+tC_6)}\, {}_2F_2\left(1,1;\frac{3}{2},2;-\frac{(2C_3+xC_6)^2}{4C_6(C_5+tC_6)}\right)+ \\ & A\sqrt{\frac{K\pi}{C_6}}\, erf\left(\frac{2C_3+xC_6}{2\sqrt{KC_6(C_5+tC_6)}}\right)+\frac{C_4}{C_6}\log\left(C_5+tC_6\right)+B. \end{aligned}$$

To our knowledge this result is new. It does not appear in [13, 39, 51].

Case $C_3 = 0$

For this choice of the constants we obtain the diffusivity

$$K = \frac{2(A+Bx)-x^2C_6}{2(C_5+tC_6)}.$$

Again the result for u is obtained directly (solution no. 22 table 3):

$$u(x,t) = -\frac{C_4}{C_6}\log\left(\frac{-2(A+Bx)+x^2C_6}{C_5+tC_6}\right)+C.$$

C is an additional constant of integration.

Case $C_1C_4 + C_6 = 0$

In this case we get the system

$$\begin{aligned} K_t - C_1Ku_t &= 0, \\ K(C_6+C_1C_3u_x)+(C_5+tC_6)K_t &= 0, \\ C_6K + C_3K_x + (C_5+tC_6)K_t &= 0, \\ 4K^2C_6^2 - 4C_3^2K_t + 4K(C_5+tC_6)(3C_6K_t+(C_5+tC_6)K_{t,t} &= 0. \end{aligned}$$

Integrating the third equation and re-inserting the result in the system we arrive at the diffusivity (reduction no. 17 table 2)

$$K = \frac{f\left(x-\frac{C_3}{C_6}\log\left(C_5+tC_6\right)\right)}{C_5+tC_6}$$

and the equation

$$C_3f'(\zeta)+f(\zeta)(C_6+f''(\zeta))=0.$$

Again we found no general solution to this ordinary differential equation.

Case $C_1C_4 + C_6 \neq 0$

In this case the involutive solver stops with

$$\begin{aligned}
-2C_4 + (2C_3 + x(C_1C_4 + C_6))u_x + 2(C_5 + tC_6)K_t &= 0, \\
-2C - 1C_4K + (2C_3 + x(C_1C_4 + C_6))K_x + 2(C_5 + tC_6)K_t &= 0, \\
2K^2C_1C_4(C_1C_4 - C_6) - (2C_3 + x(C_1C_4 + C_6))^2K_t & \\
+2K(C_5 + tC_6)(3(-C_1C_4 + C_6)K_t + 2(C_5 + tC_6)K_{t,t}) &= 0.
\end{aligned}$$

Integrating the second equation of this system we get

$$K = (C_5 + tC_6)^{\frac{C_1C_4}{C_6}} f\left((C_5 + tC_6)^{-\frac{C_1C_4+C_6}{2C_6}}\left(x + \frac{2C_3}{C_1C_4 + C_6}\right)\right).$$

Re-inserting this diffusivity into the system we arrive at the following ordinary differential equation for the function f (reduction no. 16 table 2):

$$\zeta(C_1C_4 + C_6)f'(\zeta) + f(\zeta)(-2C_1C_4 + 2f''(\zeta)) = 0.$$

It was not possible for us to solve this equation in general, so no general solution is obtained.

Case $C_5 = 0$**,** $C_6 = 0$

When $C_5 = 0$ and $C_6 = 0$ the involutive solver immediately arrives at the diffusivity

$$K = \frac{(2C_3 + xC_1C_4)^2}{A - 2tC_1^2C_4^2}$$

and the solution (no. 23 table 3)

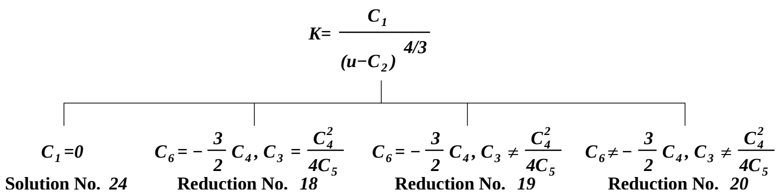
$$u(x,t) = B + \frac{1}{C_1}\log\left(\frac{(2C_3 + xC_1C_4)^2}{A - 2tC_1^2C_4^2}\right).$$

7.2.4 $K = \frac{C_1}{(u-C_2)^{\frac{4}{3}}}$

If K is of this form the system which is given as input to the involutive solver reads

$$\begin{aligned}
u_t - K_xu_x - Ku_{x,x} &= 0, \\
(-3xC_5 + C_6)(-C_2 + u) - \left(C_1 + 2tC_4 + \frac{4}{3}tC_6\right)u_t & \\
-(C_3 + xC_4 + x^2C_5)u_x &= 0, \\
(-C_2 + u)K_t + \frac{4}{3}Ku_t &= 0, \\
(-C_2 + u)K_x + \frac{4}{3}Ku_x &= 0.
\end{aligned}$$

These equations are also given in [49]. Leaving this system to the involutive solver we get, besides some rather boring solutions, four interesting new solutions/reductions:



Case $C_1 = 0$

For this case the diffusivity is given by

$$K = -\frac{(C_3 + xC_4 + x^2C_5)^2}{A - t(C_4^2 + 4C_3C_5)}$$

and solution is immediately obtained (no. 24 table 3):

$$u(x,t) = C_2 + \frac{B(-A + t(C_4^2 - 4C_3C_5))^{\frac{3}{4}}}{(C_3 + xC_4 + x^2C_5)^{\frac{3}{2}}}.$$

If $C_1 \neq 0$ the situation changes rather drastically. In contrast to this case, where a solution is obtained automatically, only reductions are achieved. Since the calculations involved are rather cumbersome we only state the results.

Case $C_6 = -\frac{3}{2}C_4$, $C_3 = \frac{C_4^2}{4C_5}$

For this choice of constants we obtain the diffusivity

$$K = \frac{(C_4 + 2xC_5)^4}{16C_1^4} f\left(\frac{2C_1 + tC_4 + 2txC_5}{C_4 + 2xC_5}\right).$$

Here a solution is possible, but unfortunately only an implicit one (reduction no. 18 table 2):

$$\begin{aligned} \zeta \;=\; & \frac{4f^{\frac{1}{4}}}{A} + B - \frac{4 \cdot 2^{\frac{2}{3}} C_1^{\frac{2}{3}}}{3^{\frac{5}{6}} A^{\frac{4}{3}} C_5^{\frac{2}{3}}} \arctan\left(\frac{1}{\sqrt{3}}\left(1 + \frac{6^{\frac{1}{3}} f^{\frac{1}{4}} A^{\frac{1}{3}} C_5^{\frac{2}{3}}}{C_1^{\frac{2}{3}}}\right)\right) + \\ & \frac{2 \cdot 2^{\frac{2}{3}} C_1^{\frac{2}{3}}}{3 \cdot 3^{\frac{1}{3}} A^{\frac{4}{3}} C_5^{\frac{2}{3}}} \log \frac{\left(-2C_1^{\frac{2}{3}} + 6^{\frac{1}{3}} f^{\frac{1}{4}} A^{\frac{1}{3}} C_5^{\frac{2}{3}}\right)^2}{4C_1^{\frac{4}{3}} + 2 \cdot 6^{\frac{1}{3}} f^{\frac{1}{4}} A^{\frac{1}{3}} C_1^{\frac{2}{3}} C_5^{\frac{2}{3}} + 6^{\frac{2}{3}} \sqrt{f} A^{\frac{2}{3}} C_5^{\frac{4}{3}}}. \end{aligned}$$

Case $C_6 = -\frac{3}{2}C_4$, $C_3 \neq \frac{C_4^2}{4C_5}$

Here the diffusivity is of the functional form

$$K = \left(-1 + \frac{(C_4 + 2xC_5)^2}{C_4^2 - 4C_3C_5}\right)^2 f\left(C_5\left(-\frac{t}{C_1} + \frac{2}{\sqrt{-C_4^2 + 4C_3C_5}} \arctan \frac{C_4 + 2xC_5}{\sqrt{-C_4^2 + 4C_3C_5}}\right)\right)$$

where the function f has to satisfy the equation (reduction no. 19 table 2)

$$\begin{aligned} -16f(\zeta)^2C_5(C_4^2 - 4C_3C_5) + \frac{1}{C_1}f'(\zeta)\left((C_4^2 - 4C_3C_5)^2 - 12C_1C_5^3f'(\zeta)\right) & \\ +16C_5^3f(\zeta)f''(\zeta) &= 0. \end{aligned}$$

Again we did not achieve a solution of this equation for the general case.

Case $C_6 \neq -\frac{3}{2}C_4$, $C_3 \neq \frac{C_4^2}{4C_5}$

Finally, when both conditions are unequal zero K has the form

$$\begin{aligned} K &= (C_3 + xC_4 + x^2C_5)^2 e^{-\frac{4(3C_4+2C_6)}{3\sqrt{-C_4^2+4C_3C_5}} \arctan \frac{C_4+2xC_5}{\sqrt{-C_4^2+4C_3C_5}}} \times \\ & \quad f\left(\frac{C_5}{2}\left(\frac{4\arctan \frac{C_4+2xC_5}{\sqrt{-C_4^2+4C_3C_5}}}{\sqrt{-C_4^2 + 4C_3C_5}} - \frac{3\log\left(3C_1 + 2t(3C_4 + 2C_6)\right)}{3C_4 + 2C_6}\right)\right). \end{aligned}$$

Here the function f is a solution of (reduction no. 20 table 2)

$$\begin{aligned} 16f(\zeta)^2(9C_3C_5 + C_6(3C_4 + C_6)) - 27C_5f'(\zeta)\left(-4e^{\frac{2\zeta(3C_4+2C_6)}{3C_5}} + C_5f'(\zeta)\right) & \\ -12f(\zeta)C_5((3C_4 + 2C_6)f'(\zeta) - 3C_5f''(\zeta)) &= 0. \end{aligned}$$

All the solutions and reductions obtained in this section are summarized in the tables 1, 2 and 3. Table 1 lists the solutions of (7.1) where K is constant. The first column lists the number of the corresponding solution/reduction. In the second column the solution is given, while in the third column the corresponding abbreviations are listed.

Table 2 features the cases when only a reduction or an implicit solution was possible. The second column lists the forms of the diffusivity. The third column contains the implicit solution, as in the first and the sixth case, or the ordinary differential equation to which equation (7.1) was reduced.

The content of table 3 are the nonconstant diffusivities of equation (7.1), which appear in the second column, and the corresponding solutions, which are listed in the third column.

Table 1: Solutions of (1) when K is constant.

No.	Solution	Conditions
1	$u(x,t) = \frac{e^{-\frac{\xi_2(x,t)}{12KC_2^3(tC_2+C_3)^3}}}{\sqrt{tC_2+C_3}}\left(A\,\mathrm{AiryAi}\left(\xi_3(x,t)\right) + B\,\mathrm{AiryBi}\left(\xi_3(x,t)\right)\right)$	$\xi_1(x,t) = C_3(C_1+tC_5)^2 + 2x\,(tC_2+C_3)\,(C_1C_2 - C_3C_5) + 2K(tC_2+C_3)^2\,(C_2+2C_6)$ $\xi_2(x,t) = 3(tC_2+C_3)^2\left(C_3{}^2C_5{}^2 + \left(xC_2{}^2 + C_3C_5\right)^2 + 2KC_2{}^2C_3\,(C_2+2C_6)\right) + 2C_3{}^2(C_1C_2 - C_3C_5)^2 + 6C_3\,(tC_2+C_3)\,(C_1C_2 - C_3C_5)\left(xC_2{}^2 + C_3C_5\right)$ $\xi_3(x,t) = \frac{K^{\frac{4}{3}}C_3{}^{\frac{1}{3}}\xi_1(x,t)}{2^{\frac{4}{3}}(tC_2+C_3)^2(C_1C_2-C_3C_5)^{\frac{2}{3}}}$
2	$u(x,t) = \frac{K^2e^{-\frac{\xi_3(x,t)}{4K}}\xi_1(x,t)}{2\xi_4(t)^{\frac{3}{4}}\left(K^2\tilde{C}_2\right)^{\frac{5}{4}}}\left(A_1\mathrm{F}_1\left(\tilde{C}_5,\frac{3}{2},\xi_2(x,t)\right) + B\,\mathrm{U}\left(\tilde{C}_5,\frac{3}{2},\xi_2(x,t)\right)\right)$	$\xi_1(x,t) = C_1C_2 - C_3C_5 + x(C_2^2 - C_3C_4) + t(C_1C_4 - C_3C_5)$ $\xi_2(x,t) = \frac{\tilde{C}_3\xi_1(x,t)^2}{2\xi_4(t)\tilde{C}_2^2}$ $\xi_3(x,t) = \frac{t\tilde{C}_4^2}{\tilde{C}_2^2} - \frac{2\tilde{C}_4\xi_1(x,t)}{\tilde{C}_2^2} + \frac{\left(C_2+tC_4+K\tilde{C}_3\right)\xi_1(x,t)^2}{\xi_4(t)\tilde{C}_2^2} + \frac{\tilde{C}_1}{(-\tilde{C}_2)^{\frac{3}{2}}}\arctan\left(\frac{C_2+tC_4}{\sqrt{-\tilde{C}_2}}\right),$ $\xi_4(t) = 2tC_2 + C_3 + t^2C_4$ $\tilde{C}_1 = -C_1{}^2C_4 + 2C_1C_2C_5 - C_3C_5{}^2 + 2K\left(C_2{}^2 - C_3C_4\right)(C_2+2C_6)$ $\tilde{C}_2 = C_2^2 - C_3C_4$ $\tilde{C}_3 = \sqrt{\frac{\tilde{C}_2}{K^2}}$ $\tilde{C}_4 = C_1C_4 - C_2C_5$ $\tilde{C}_5 = \frac{\tilde{C}_3(\tilde{C}_1+6K^4\tilde{C}_3^{\frac{3}{2}})}{8\tilde{C}_2^2}$
3	$u(x,t) = \zeta_1^{\frac{C_3}{C_6}}\left(A_1F_1\left(-\frac{C_3}{C_6},\frac{1}{2},-\frac{\zeta_2^2}{4K}\right) + \zeta_2B_1F_1\left(\frac{1}{2} - \frac{C_3}{C_6},\frac{3}{2},-\frac{\zeta_2^2}{4K}\right)\right) - \frac{C_2}{C_3}$	$\zeta_1 = C_5 + tC_6$ $\zeta_2 = \frac{2C_7+xC_6}{C_6\sqrt{\zeta_1}}$
4	$u(x,t) = \frac{C_4(2C_3+xC_6)^2}{2KC_6^2(C_5+tC_6)}\,{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{(2C_3+xC_6)^2}{4C_6(C_5+tC_6)}\right) + A\sqrt{\frac{K\pi}{C_6}}\mathrm{erf}\left(\frac{2C_3+xC_6}{2\sqrt{KC_6(C_5+tC_6)}}\right) + \frac{C_4}{C_6}\log\left(C_5+tC_6\right) + B$	
5	$u(x,t) = Ae^{\frac{C_6t}{C_3} - \sqrt{\frac{C_6}{KC_3}}x} + Be^{\frac{C_6t}{C_3} + \sqrt{\frac{C_6}{KC_3}}x}$	
6	$u(x,t) = Ae^{\frac{C_3}{C_5}t + \sqrt{\frac{C_3}{C_5K}}x} + Be^{\frac{C_3}{C_5} - \sqrt{\frac{C_3}{C_5K}}x} - \frac{C_2}{C_3}$	
7	$u(x,t) = Ae^{\frac{C_3}{C_7}(KC_3t+xC_7)} - \frac{C_2}{C_3}$	
8	$u(x,t) = t\frac{C_4}{C_5} + x^2\frac{C_4}{2KC_5} + A + xB$	
9	$u(x,t) = x\frac{C_4}{C_3} + A$	
10	$u(x,t) = Ae^{\frac{C_6}{C_1^2}(xC_1+KC_6t)}$	
11	$u(x,t) = A + B\sqrt{\frac{K\pi}{2C_2}}\mathrm{erf}\left(\frac{C_1+xC_2}{\sqrt{2KC_2(C_3+2tC_2)}}\right)$	
12	$u(x,t) = \frac{A}{\sqrt{C_1+tC_5}}e^{-\frac{(xC_5-2KC_6)^2}{4KC_5(C_1+tC_5)}}$	

Table 2: Reductions of (1) for various diffusivities K.

No.	K	Conditions
13	$K = \frac{f(x)}{C_5+tC_6}$	$2f_2(x)\frac{\sqrt{1-\zeta(f_2(x))}}{\sqrt{-A^{\frac{1}{C_6}}f_2(x)^{\frac{2C_3}{C_6}}+2f_2(x)C_6^2}}\,{}_2F_1\left(\zeta_1,\frac{1}{2};1+\zeta_1;\zeta(f_2(x))\right) =$ $2B\frac{\sqrt{1-\zeta(B)}}{\sqrt{-A^{\frac{1}{C_6}}B^{\frac{2C_3}{C_6}}+2BC_6^2}}\,{}_2F_1\left(\zeta_1,\frac{1}{2};1+\zeta_1;\zeta(B)\right) - x.$ $\zeta(h) = \frac{A^{\frac{1}{C_6}}h^{-1+\frac{2C_3}{C_6}}}{2C_6^2}$ $\zeta_1 = \frac{C_6}{4C_3-2C_6}$
14	$K = \frac{f\left(x-\frac{C_7}{C_6}\log(C_5+tC_6)\right)}{C_5+tC_6}$	$C_6C_7f'(\zeta) - C_3f'(\zeta)^2 + f(\zeta)(C_6^2 + C_6f''(\zeta)) = 0$
15	$K = (C_5+tC_6)^{\frac{C_4}{C_6}} f\left((C_5+tC_6)^{-\frac{C_4+C_6}{2C_6}}\left(x+\frac{2C_7}{C_4+C_6}\right)\right)$	$\zeta C_4(C_4+C_6)f'(\zeta) + 2C_3f'(\zeta)^2 + f(\zeta)(-2C_4^2 + 2C_4f''(\zeta)) = 0$
16	$K = (C_5+tC_6)^{\frac{C_1C_4}{C_6}} f\left((C_5+tC_6)^{-\frac{C_1C_4+C_6}{2C_6}}\left(x+\frac{2C_3}{C_1C_4+C_6}\right)\right)$	$\zeta(C_1C_4+C_6)f'(\zeta) + f(\zeta)(-2C_1C_4 + 2f''(\zeta)) = 0$
17	$K = \frac{f\left(x-\frac{C_3}{C_6}\log(C_5+tC_6)\right)}{C_5+tC_6}$	$C_3f'(\zeta) + f(\zeta)(C_6 + f''(\zeta)) = 0$
18	$K = \frac{(C_4+2xC_5)^4}{16C_1^4} f\left(\frac{2C_1+tC_4+2txC_5}{C_4+2xC_5}\right)$	$\zeta = \frac{4f^{\frac{1}{4}}}{A} + B - \frac{4\cdot 2^{\frac{2}{3}}C_1^{\frac{2}{3}}}{3^{\frac{5}{6}}A^{\frac{4}{3}}C_5^{\frac{2}{3}}}\arctan\left(\frac{1}{\sqrt{3}}\left(1+\frac{6^{\frac{1}{3}}f^{\frac{1}{4}}A^{\frac{1}{3}}C_5^{\frac{2}{3}}}{C_1^{\frac{2}{3}}}\right)\right)+$ $\frac{2\cdot 2^{\frac{2}{3}}C_1^{\frac{2}{3}}}{3\cdot 3^{\frac{1}{3}}A^{\frac{4}{3}}C_5^{\frac{2}{3}}}\log\frac{\left(-2C_1^{\frac{2}{3}}+6^{\frac{1}{3}}f^{\frac{1}{4}}A^{\frac{1}{3}}C_5^{\frac{2}{3}}\right)^2}{4C_1^{\frac{4}{3}}+2\cdot 6^{\frac{1}{3}}f^{\frac{1}{4}}A^{\frac{1}{3}}C_1^{\frac{2}{3}}C_5^{\frac{2}{3}}+6^{\frac{2}{3}}\sqrt{f}A^{\frac{2}{3}}C_5^{\frac{4}{3}}}$
19	$K = \left(-1+\frac{(C_4+2xC_5)^2}{C_4^2-4C_3C_5}\right)^2 f\left(C_5\left(-\frac{t}{C_1}+\frac{2}{\sqrt{-C_4^2+4C_3C_5}}\arctan\frac{C_4+2xC_5}{\sqrt{-C_4^2+4C_3C_5}}\right)\right)$	$-16f(\zeta)^2C_5(C_4^2-4C_3C_5) + \frac{1}{C_1}f'(\zeta)\left((C_4^2-4C_3C_5)^2 - 12C_1C_5^3f'(\zeta)\right)$ $+16C_5^3f(\zeta)f''(\zeta) = 0$
20	$K = (C_3+xC_4+x^2C_5)^2 e^{-\frac{4(3C_4+2C_6)}{2\sqrt{-C_4^2+4C_3C_5}}\arctan\frac{C_4+2xC_5}{\sqrt{-C_4^2+4C_3C_5}}} f\left(\frac{C_5}{2}\left(\frac{4\arctan\frac{C_4+2xC_5}{\sqrt{-C_4^2+4C_3C_5}}}{\sqrt{-C_4^2+4C_3C_5}} - \frac{3\log(3C_1+2t(3C_4+2C_6))}{3C_4+2C_6}\right)\right)$	$16f(\zeta)^2(9C_3C_5 + C_6(3C_4+C_6)) - 27C_5f'(\zeta)\left(-4e^{\frac{2\zeta(3C_4+2C_6)}{3C_5}} + C_5f'(\zeta)\right)$ $-12f(\zeta)C_5((3C_4+2C_6)f'(\zeta) - 3C_5f''(\zeta)) = 0$

Table 3: Solutions of (1) for various K.

No.	K	*Solution*
21	$K = \frac{(xC_4+2C_7)^2}{A-2tC_4(2C_3+C_4)}$	$u(x,t) = -\frac{C_2}{C_3} + B\left(\frac{(xC_4+2C_7)^2}{A+2tC_4(2C_3+C_4)}\right)^{\frac{C_3}{C_4}}$
22	$K = \frac{2(A+Bx)-x^2C_6}{2(C_5+tC_6)}$	$u(x,t) = -\frac{C_4}{C_6}\log\left(\frac{-2(A+Bx)+x^2C_6}{C_5+tC_6}\right) + C$
23	$K = \frac{(2C_3+xC_1C_4)^2}{A-2tC_1^2C_4^2}$	$u(x,t) = B + \frac{1}{C_1}\log\left(\frac{(2C_3+xC_1C_4)^2}{A-2tC_1^2C_4^2}\right)$
24	$K = -\frac{(C_3+xC_4+x^2C_5)^2}{A-t(C_4^2+4C_3C_5)}$	$u(x,t) = C_2 + \frac{B(-A+t(C_4^2-4C_3C_5))^{\frac{3}{4}}}{(C_3+xC_4+x^2C_5)^{\frac{3}{2}}}$

7.3 Notes

We demonstrated that an involutive solution procedure is able to classify the different diffusivities of the nonlinear diffusion equation (7.1). Moreover it was possible to deduce in a straightforward way new solutions (tables 1, 3). If no explicit solutions were found we obtained implicit ones or at least reductions (table 2).

In conclusion we examined thirteen different models with the involutive solver. Overall 16 solutions were found. In eight cases we found no general solution. However, we derived at least an implicit solution or an ordinary differential equation.

The involutive solution method was able to detect new solutions which incorporate up to six additional constants. These solutions do not appear in the literature (to our knowledge).

Chapter 8

Black-Scholes Type Equations

In this chapter we will apply the method of involutive reduction to equations of financial mathematics, in particular to the Black-Scholes equation [60] and generalizations of it. But first we start with the original problem of Black and Scholes.

8.1 The Black-Scholes Equation

In 1973 Fischer Black and Merton Scholes published their famous paper concerning the pricing of European options on underlying assets [60]. This model allows to rate the price of an option when the price of the stock of the option is known. To do this it uses the so-called *equivalent martingale procedure* which states that in the absence of arbitrage there exist unique probabilities for an option to rise and fall. This amounts to a risk neutral evaluation procedure in which the value of an option can be determined by calculating the expected future value of the option by using these equivalent martingale probabilities and then discounting at the risk-free rate of interest [61].
Before discussing the solutions to this equation we first derive the Black-Scholes model for the pricing of a European option. This derivation closely follows the one given in [61]. Let $u(x(t), t)$ denote the value of a derivative security at date t which additionally depends on the value of the corresponding stock. The value of the derivative security at a date $t + \Delta t$ is then $u(x(t + \Delta t), t + \Delta t)$. Using a risk neutral evaluation procedure and a derivative security with payout $x(T)$ at date T, its value at date t is given by

$$u(t) = E_t^Q \left[\frac{u(t + \Delta t)}{A(t + \Delta t)} \right] A(t). \tag{8.1}$$

Hereby E_t^Q means the expected value of the asset at date T using equivalent martingale probabilities and $A(t)$ is the value of the money market account at date t. Expanding around t we get

$$u(x(t+\Delta t), t+\Delta t) = u(x(t),t)+\Delta t\frac{\partial u}{\partial t}+\Delta x\frac{\partial u}{\partial x}+\frac{1}{2}(\Delta x)^2\frac{\partial^2 u}{\partial x^2}+O((\Delta x)^3,(\Delta t)^2).$$

Doing the same for the money market account A leads to [61]

$$\frac{1}{A(t+\Delta t)} = \frac{1}{A(t)}\left(1 - r\Delta t + \frac{1}{2}(r\Delta t)^2 + O((\Delta t)^3)\right),$$

where r denotes the continuously compounded spot interest rate. Inserting these expansions in (8.1) and simplifying gives

$$-u(x(t),t)r\Delta t+\Delta t\frac{\partial u}{\partial t}+E_t^Q[\Delta x]\frac{\partial u}{\partial x}+\frac{1}{2}E_t^Q[(\Delta x)^2]\frac{\partial^2 u}{\partial x^2}+O((\Delta t)^2,(\Delta x)^3). \quad (8.2)$$

This equation is the starting point for the Black-Scholes model. For this model three assumptions are made at this point. The first one is that terms of the order $(\Delta t)^2$ and higher and terms of order $(\Delta x)^3$ and higher are neglected. The second assumption concerns the form of $E_t^Q[\Delta x]$. It is assumed that it is of the form

$$E_t^Q[\Delta x] = rx(t)\Delta t.$$

This assumption means that the instantaneous expected return on the stock per unit time is constant and equals r. The third assumption states that for small Δt the change in the stock price Δx is also small. This assumption leads to a volatility of the form [61]

$$E_t^Q[(\Delta x)^2] = \sigma_0^2 x(t)^2\Delta t,$$

where σ_0 is called the volatility of the stock. Inserting these conditions in equation (8.2) and dividing by Δt leads to the well-known Black-Scholes partial differential equation

$$\frac{\partial u}{\partial t} + \frac{x^2\sigma_0^2}{2}\frac{\partial^2 u}{\partial x^2} + r\left(x\frac{\partial u}{\partial x} - u\right) = 0, \quad (8.3)$$

where u is the value of the option which depends upon the time t and the value of the underlying stock x. σ_0 is the volatility of the stock, which is the square root of the stock return's instantaneous variance. It measures the oscillations of the stock price. r is the risk free interest rate.
Hereby a European call option gives the holder the right, but not the obligation to buy the underlying asset at the strike price K at the expiration date T. These considerations lead to the boundary condition or terminal payoff-function

$$u(x,T) = (x-K)\theta(x-K) = \begin{cases} x-K, & x \geq K, \\ 0, & x < K. \end{cases} \quad (8.4)$$

Equation (8.3) together with the boundary condition (8.4) was the original derivative security priced in [60].
Performing a symmetry analysis for (8.3) with the conditions $\sigma \neq 0$ and $r \neq 0$ leads to the infinitesimal transformations

$$\begin{aligned}
\xi^x &= C_1 + C_2 t + C_3 t\text{'} \\
\xi^t &= C_4 x + \frac{C_2}{2} x \log x + t(C_5 x + C_3 x \log x), \\
\eta &= \frac{u}{8\sigma_0^2}\left[4C_3 \log^2 x + 8C_6\sigma_0^2 + C_3 t^2 (2r+\sigma_0^2)^2\right. \\
&\quad +2\log x(4C - 5 - (C_2 + 2tC_3(2r+\sigma_0^2)) \\
&\quad \left.+t(-4C_3\sigma_0^2 2 + C_2(2r+\sigma_0^2)^2 + C_5(-8r+4\sigma_0^2))\right].
\end{aligned} \tag{8.5}$$

Note that these symmetries differ from the ones in [62]. But we will see that they will lead to the same fundamental solution.
According to the solution procedure introduced in this thesis we have to append the invariant surface condition to the differential equation (8.3):

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{x^2\sigma_0^2}{2}\frac{\partial^2 u}{\partial x^2} + r\left(x\frac{\partial u}{\partial x} - u\right) &= 0, \\
\eta(x,t,u) - \xi^x(x,t,u)\frac{\partial u}{\partial x} - \xi^2(x,t,u)\frac{\partial u}{\partial t} &= 0
\end{aligned}$$

with the infinitesimal transformations (8.5). Applying the involutive reduction/solution procedure to this coupled system of partial differential equations leads to at least six cases for the appearing constants C_1 to C_6. In one case the calculation involves huge expressions requiring more than one gigabyte of memory. Here the calculations were stopped. Three other cases lead either to contradictions or the symmetry constants are all zero, which does not lead to any simplification at all.
The other two cases lead to the solutions in the next two sections.

8.1.1 $C_2 = 0,\ C_3 = 0,\ C_4 = 0,\ C_5 = 0$

For these values of the parameters and $\sigma_0 \neq 0$ and $r \neq 0$ arbitrary the involutive solution procedure ends with the result

$$u(x,t) = x^{-\frac{r}{\sigma_0^2} - \frac{1}{\sigma_0}\sqrt{\frac{(2r+\sigma_0^2)^2}{\sigma_0^2} - \frac{8C_6}{C_1}}}\left(\sqrt{x} f^1(t) + x^{\frac{1}{2} + \frac{1}{\sigma_0}\sqrt{\frac{(2r+\sigma_0^2)^2}{\sigma_0^2} - \frac{8C_6}{C_1}}} f^2(t)\right), \tag{8.6}$$

where f^1 and f^2 are functions of t which have to satisfy the condition

$$C_6 f^1(t) - C_1 f^{1'}(t) + x^{\frac{1}{\sigma_0}\sqrt{\frac{(2r+\sigma_0^2)^2}{\sigma_0^2} - \frac{8C_6}{C_1}}}(C_6 f^2(t) - C_1 f^{2\prime}(t)). \tag{8.7}$$

Since this condition has to be true for all values of x and t (8.7) splits into the conditions

$$C_6 f^i(t) - C_1 f^{i'}(t) = 0$$

for $i = 1, 2$ whose solutions are

$$f^i(t) = K_i e^{\frac{C_6}{C_1} t}.$$

Inserting this in (8.6) leads to the solution

$$u(x,t) = e^{\frac{C_6}{C_1} t} x^{-\frac{r}{\sigma_0^2} - \frac{1}{2\sigma_0}\sqrt{\frac{(2r+\sigma_0^2)^2}{\sigma_0^2} - \frac{8C_6}{C_1}}} \left(K_1 \sqrt{x} + K_2 x^{\frac{1}{2} + \frac{1}{\sigma_0}\sqrt{\frac{(2r+\sigma_0^2)^2}{\sigma_0^2} - \frac{8C_6}{C_1}}} \right).$$

Note that as $t \to T$ this solution only satisfies the boundary value (8.4) if $r = 0$.

8.1.2 $C_1 = 0,\ C_2 = 0,\ C_3 = 0$

In this case we get

$$u(x,t) = f(t) x^{\frac{C_5 \log x + 2C_6\sigma_0^2 + tC_5(-2r+\sigma_0^2)}{2(C_4+tC_5)\sigma_0^2}} \tag{8.8}$$

and the condition

$$\begin{aligned} -8(C_4+tC_5)^2\sigma_0^2 f'(t) + \big[t^2 C_5^2(2r+\sigma_0^2)^2 + 2tC_5(-2C_5\sigma_0^2 + C_4(2r+\sigma_0^2)^2) + \\ 4\sigma_0^2(2C_4^2 r - C_6^2\sigma_0^2 - C_4(C_5 + 2C_6 r - C_6\sigma_0^2))\big] f(t) &= 0 \end{aligned}$$

which the function f has to satisfy. This first-order ordinary differential equation is solved by **DSolve** to

$$f(t) = \frac{K_1}{\sqrt{C_4 + tC_5}} e^{\frac{t(2r+\sigma_0^2)^2}{8\sigma_0^2} + \frac{(-2C_4 r + (C_4 - 2C_6)\sigma_0^2)^2}{8\sigma_0^2 C_5 (C_4+tC_5)}}.$$

Inserting this result in (8.8) leads to the solution

$$u(x,t) = \frac{K_1}{\sqrt{C_4+tC_5}} x^{\frac{C_5 \log x + 2C_6\sigma_0^2 + tC_5(-2r+\sigma_0^2)}{2(C_4+tC_5)\sigma_0^2}} e^{\frac{t(2r+\sigma_0^2)^2}{8\sigma_0^2} + \frac{(-2C_4 r + (C_4-2C_6)\sigma_0^2)^2}{8\sigma_0^2 C_5(C_4+tC_5)}}.$$

Rewriting this with $x^a = e^{a \log x}$ leads to

$$u(x,t) = \frac{K_1}{\sqrt{C_4+tC_5}} e^{\frac{1}{8\sigma_0^2}\left(t(2r+\sigma_0^2)^2 + \frac{(-2C_4 r + (C_4-2C_6)\sigma_0^2)^2}{C_5(C_4+tC_5)} + \frac{4\log x (C_5 \log x + 2C_6\sigma_0^2 + tC_5(-2r+\sigma_0^2))}{C_4+tC_5} \right)}. \tag{8.9}$$

Analyzing the solution (8.9) we observe that as t varies it can happen that the denominators are zero as $t \to T$. This is the case for $C_5 = -\frac{C_4}{T}$. At first thought it seems that this behavior leads to the conclusion that (8.9) does not satisfy the boundary condition (8.4). But keep in mind that we are dealing with a linear differential equation. That means that not a single solution has to satisfy the boundary value problem, but a superposition. Furthermore, the general solution can be written as

$$u(x,t) = \int_{-\infty}^{\infty} u_\lambda(x,t) u_T(\lambda) d\lambda \tag{8.10}$$

where $u_\lambda(x,t)$ tends to the delta function $\delta(x-\lambda)$ for $t \to T$ and $u_T(\lambda)$ is the boundary value (8.4).
So we see that if the solution (8.9) can be written as a delta series the superposition of the solutions (8.9) does indeed satisfy the boundary value problem (8.4) for some suitably chosen parameter λ.
For this purpose we set $C_5 = -\frac{C_4}{T}$. Doing this the right hand side of (8.9) gives (after some calculation)

$$K_1 \sqrt{\frac{T}{C_4}} \frac{e^{-\frac{\left(\frac{\log x}{\sigma_0\sqrt{2}} + \frac{T}{2\sqrt{2}C_4\sigma_0}(-2\sigma_0^2 C_6 + C_4(-2r+\sigma_0^2))\right)^2}{T-t} + \frac{-2r+\sigma_0^2}{2\sigma_0^2}\log x + \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2} t}}{\sqrt{T-t}}.$$

If t tends to T we have

$$\lim_{t\to T} K_1 \sqrt{\frac{T}{C_4}} \frac{e^{-\frac{\left(\frac{\log x}{\sigma_0\sqrt{2}} + \frac{T}{2\sqrt{2}C_4\sigma_0}(-2\sigma_0^2 C_6 + C_4(-2r+\sigma_0^2))\right)^2}{T-t} + \frac{-2r+\sigma_0^2}{2\sigma_0^2}\log x + \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2} t}}{\sqrt{T-t}} =$$

$$K_1 \sqrt{\frac{T}{C_4}}\, e^{\frac{-2r+\sigma_0^2}{2\sigma_0^2}\log x + \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2} T} \lim_{t\to T} \frac{e^{-\frac{\left(\frac{\log x}{\sigma_0\sqrt{2}} + \frac{T}{2\sqrt{2}C_4\sigma_0}(-2\sigma_0^2 C_6 + C_4(-2r+\sigma_0^2))\right)^2}{T-t}}}{T-t}.$$

Comparing this with

$$\lim_{\varepsilon\to 0} \frac{e^{-\frac{x^2}{\varepsilon}}}{\sqrt{\pi\varepsilon}} = \delta(x) \tag{8.11}$$

we see that in the limit $t \to T$ the solution (8.9) is indeed a delta function:

$$\sqrt{\frac{\pi T}{C_4}} K_1\, e^{\frac{-2r+\sigma_0^2}{2\sigma_0^2}\log x + \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2} T} \delta\left(\frac{\log x}{\sigma_0\sqrt{2}} + \frac{T}{2\sqrt{2}C_4\sigma_0}(-2\sigma_0^2 C_6 + C_4(-2r+\sigma_0^2))\right).$$

Introducing the new constant λ as

$$\log\lambda = -\frac{T}{2C_4}(-2\sigma_0^2 C_6 + C_4(-2r+\sigma_0^2))$$

leads to

$$u_\lambda(x,T) = K_1\sqrt{\frac{\pi T}{C_4}}\, e^{\frac{-2r+\sigma_0^2}{2\sigma_0^2}\log x + \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2}}\,\delta\left(\frac{\log x}{\sigma_0\sqrt{2}} - \frac{\log\lambda}{\sigma_0\sqrt{2}}\right).$$

According to the formula

$$\delta(z-z_0) = \frac{1}{\left|\frac{\partial z}{\partial x}\right|}\delta(x-x_0), \tag{8.12}$$

which connects the delta function of $z(x)$ to the delta function of x we have

$$u_\lambda(x,T) = K_1\sigma_0\sqrt{\frac{2\pi T}{C_4}}\, e^{\frac{-2r+\sigma_0^2}{2\sigma_0^2}\log\lambda + \frac{(2r+\sigma_0^2)^2 T}{8\sigma_0^2}}\,\lambda\delta(x-\lambda).$$

Finally, the condition

$$u_\lambda(x,T) = \delta(x-\lambda)$$

fixes the constant of integration

$$K_1 = \frac{1}{\sigma_0}\sqrt{\frac{C_4}{2\pi T}} e^{-\frac{-2r+\sigma_0^2}{2\sigma_0^2}\log\lambda - \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2}T}.$$

Putting it all together results in

$$u_\lambda(x,t) = \frac{1}{\lambda\sigma_0\sqrt{2\pi(T-t)}}\, e^{-\frac{\left(\frac{\log x}{\sigma_0\sqrt{2}} - \frac{\log\lambda}{\sigma_0\sqrt{2}}\right)}{T-t} + \frac{-2r+\sigma_0^2}{2\sigma_0^2}(\log x - \log\lambda) - \frac{(2r+\sigma_0^2)^2}{8\sigma_0^2}(T-t)}. \tag{8.13}$$

The overall solution is obtained by inserting (8.13) and (8.4) in (8.10):

$$u(x,t) = \int_{-\infty}^{\infty} u_\lambda(x,t)(\lambda-K)\theta(\lambda-K)d\lambda = \int_K^\infty \lambda u_\lambda(x,t)d\lambda - K\int_K^\infty u_\lambda(x,t)d\lambda.$$

Solving the corresponding integrals we arrive at the famous solution of Black and Scholes [60]:

$$u_{BS} = xN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{8.14}$$

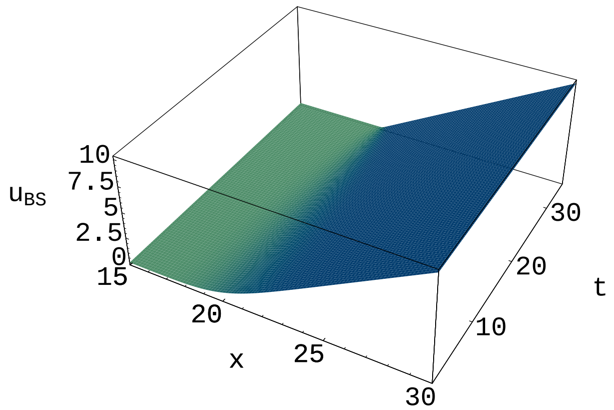


Figure 8.1: The Black-Scholes solution for $T = 35/365$, $K = 20$, $r = 0.0425$, $\sigma = 0.25$.

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma_0\sqrt{2(T-t)}}\left(\log\frac{x}{K} + \frac{2r+\sigma_0^2}{2}(T-t)\right), \\ d_2 &= \frac{1}{\sigma_0\sqrt{2(T-t)}}\left(\log\frac{x}{K} - \frac{(-2r+\sigma_0^2)}{2}(T-t)\right) \end{aligned}$$

and

$$N(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z} e^{-u^2}\,du.$$

Note that Black and Scholes derived their solution by transforming equation (8.3) to the heat equation. The solution in ([62]) was obtained by similar considerations. But they used a symmetry generator where the constants λ and T are built in.
This result is illustrated in figure 8.1 as a reference for a 35-day European call option ($T = 35/365$) with a strike price of 20 dollars ($K = 20$), a constant spot interest rate of 4.25% ($r = 0.0425$) and a volatility of 25% per year ($\sigma_0 = 0.25$).

In this section we solved the Black-Scholes equation (8.3) for two parameter combinations. In both cases we found the corresponding solutions by using the method of involutive reduction. The first solution does not appear in the literature. It does not satisfy the necessary boundary conditions for options, especially European call options.
The second solution is the famous solution of Black and Scholes [60]. However,

in contrast to the original solution procedure, which used a transformation to the heat equation, we solved the differential equation in a straightforward way.
In the next sections we are going to consider generalizations of equation (8.3).

8.2 A Time-Dependent Volatility

In this section we release the assumptions which have lead to the model of Black and Scholes. The first and second assumptions made above, the neglection of terms of order $(\Delta t)^2$, $(\Delta x)^3$ and higher and the form of $E_t^Q[\Delta x]$ however are still made, so that the general form of equation (8.3) stays more or less the same. The only change we are making is the form of $E_t^Q[(\Delta x)^2]$. In contrast to the Black-Scholes model we assume that it is of the form

$$E_t^Q[(\Delta x)^2] = \sigma(t)\Delta t,$$

where $\sigma(t)$ denotes a general volatility function, depending only on the time t. Remember that $E_t^Q[(\Delta x)^2]$ for the Black-Scholes model is of the form $\sigma_0^2 x^2$ which follows from an assumption made for small t and a Brownian motion, so an arbitrary time-dependent volatility is clearly a generalization. Inserting this in (8.2) we arrive at the following generalized Black-Scholes type equation:

$$\frac{\partial u}{\partial t} + \frac{\sigma^2(t)}{2}\frac{\partial^2 u}{\partial x^2} + r\left(x\frac{\partial u}{\partial x} - u\right) = 0 \tag{8.15}$$

together with the boundary condition

$$u(x,T) = \begin{cases} x-K, & x \geq K \\ 0, & x < K \end{cases}, \tag{8.16}$$

which is not changed either. Applying the involutive solution procedure to the determining equations for Lie point symmetries of (8.15) we arrive at three cases.

8.2.1 General Solutions For a Time-Dependent Volatility

In the first case the symmetries of (8.15) are given by (the symmetry part representing the linearity of equation (8.15) is not considered here)

$$\begin{aligned} \xi^x &= f^1(t), \\ \xi^t &= 0, \\ \eta &= u(x,t)\left(f^3(x) - \frac{e^{-rt}}{r}f^2(x)\right), \end{aligned}$$

where the unknown functions $f^1(r)$, $f^2(x)$ and $f^3(x)$ have to satisfy the conditions

$$\begin{aligned} xr^2 f^1(t) - e^{-rt}\sigma^2(t) f^2(x) - xr {f^1}'(t) &= 0, \quad (8.17) \\ f^2(x) - x {f^2}'(x) + e^{rt} xr {f^3}'(x) &= 0. \quad (8.18) \end{aligned}$$

Differentiating the first condition (8.17) twice with respect to x leads to

$$f^2(x) = k_1 + k_2 x.$$

Inserting this in the conditions (8.17) and (8.18) leads to

$$\begin{aligned} xr^2 f^1(t) - \sigma^2(t) e^{-rt}(k_1 + k_2 x) - xr {f^1}'(t) &= 0, \quad (8.19) \\ k_1 + e^{rt} xr {f^3}'(x) &= 0. \quad (8.20) \end{aligned}$$

Equation (8.20) is solved to give

$$f^3(x) = k_3 - \frac{k_1 e^{-rt}}{r} \log x.$$

Inserting this in (8.19) gives

$$-k_2\sigma^2(t)e^{-rt} + x\left(-k_1\sigma^2(t)e^{-rt} + r(rf^1(t) - {f^1}'(t)\right) = 0.$$

Since this condition has to be true for all values of x k_2 must be zero. Inserting all this information in the infinitesimal transformations and appending the invariant surface condition to the differential equation (8.15) leads to the system:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\sigma^2(t)}{2}\frac{\partial^2 u}{\partial x^2} + r\left(x\frac{\partial u}{\partial x} - u\right) &= 0, \\ u\left(k_3 - \frac{xk_2 g(t)}{r}\right) - f^1(t) u_x &= 0, \\ -g(t)a\sigma^2(t) + r(rf^1(t) - r{f^1}'(t)) &== 0, \\ g'(t) + rg(t) &= 0, \end{aligned}$$

where e^{-rt} has been replaced by $g(t)$ and the corresponding differential equation has been appended to the system. This has to be done since we are looking for solutions classified also by r, which appears in the exponent. But remember that

the function **InvolutePDESolve** can only handle polynomially nonlinear functions in the corresponding dependent variables, so the replacement of e^{-rt} with $g(t)$ is a must.
Trying to solve the above system of partial differential equations with the involutive method we arrive again at three cases. The first case is not very interesting since here $f^1(t) = 0$, $k_2 = 0$ and $k_3 = 0$, so the invariant surface condition vanishes and does not help in the solution of the problem. The other two cases are treated separately in the following.

$k_2 = 0$

For $k_2 = 0$ we obtained the solution for the differential equation for $g(t)$. Besides that we got

$$f^1(t) = C_1 e^{rt}$$

and

$$u(x,t) = F^1(t) e^{\frac{e^{-rt} x k_2}{C_1}}, \tag{8.21}$$

where $F^1(t)$ has to satisfy

$$-k_2^2 \sigma^2(t) F^1(t) + 2C_1^2 e^{2rt}(rF^1(t) - F^{1\prime}(t)) = 0.$$

The solution for this linear first-order differential equation is given by

$$F^1(t) = C_2 e^{r(t-C_3) - \frac{k_3^2}{2C_1^2}\int_{C_3}^t e^{-2\tau r}\sigma^2(\tau)d\tau}.$$

Inserting this result in (8.21) leads to the solution for the value of an option:

$$u(x,t) = C_2 e^{\frac{e^{-rt} x k_2}{C_1} + r(t-C_3) - \frac{k_3^2}{2C_1^2}\int_{C_3}^t e^{-2\tau r}\sigma^2(\tau)d\tau}.$$

Note that this solution does not satisfy the boundary condition (8.4). It is not considered in the literature.

The General Solution

If none of the constants or appearing functions is zero the involutive solution procedure ends with two differential equations for $u(x,t)$. Solving the first one leads to

$$u(x,t) = F^1(t)\ e^{\frac{x\left(2k_3 - \frac{xk_2 e^{-rt}}{r}\right)}{2f^1(t)}} \tag{8.22}$$

and a rather lengthy equation which, when inserting (8.22), is a polynomial in x. Equating to zero the coefficients of this polynomial, which has to be true for all values of x, results in two differential equations for $f^1(t)$ and and $F^1(t)$. They are solved to

$$f^1(t) = \frac{e^{rt}}{r}\left(C_1 r - a\int_{K_1}^t e^{-2r\tau}\sigma^2(\tau)d\tau\right)$$

and

$$F^1(t) = K_2 e^{r(t-K_4) - \frac{k_3^2 r}{2}\int_{K_4}^t \frac{\sigma^2(\tau)}{f^{1\,2}(\tau)}d\tau + \frac{k_2}{2r}\int_{K_4}^t \frac{e^{-r\tau}\sigma^2(\tau)}{f^1(\tau)}d\tau}.$$

Inserting the last result in (8.22) gives

$$u(x,t) = K_3 e^{r(t-K_4) - \frac{k_3^2 r}{2}\int_{K_4}^t \frac{\sigma^2(\tau)}{f^{1\,2}(\tau)}d\tau + \frac{k_2}{2r}\int_{K_4}^t \frac{e^{-r\tau}\sigma^2(\tau)}{f^1(\tau)}d\tau + \frac{1}{re^{rt}f^1(t)}\left(xk_3re^{rt} - \frac{x^2k_2}{2}\right)}. \tag{8.23}$$

Like in the case of the classical Black-Scholes differential equation the exponent has a denominator which does depend on t. So there is the possibility that (8.23) tends to the delta function as $t \to T$. For this to happen the condition

$$f^1(T) = 0$$

has to be satisfied. This leads to

$$C_1 r = k_2 \int_{K_1}^T e^{-2r\tau}\sigma^2(\tau)d\tau$$

which results in

$$f^1(t) = \frac{k_2 e^{rt}}{r}\int_t^T e^{-2r\tau}\sigma^2(\tau)d\tau.$$

Now we have to see if we can rewrite equation (8.22) in the form of the left-hand-side of (8.11). To do this we rewrite the integrals which appear in (8.22). The first integral can be rewritten to

$$\int_{K_4}^{t} d\tau \frac{\sigma^2(\tau)}{f^{1\,2}(\tau)} = \frac{r^2}{k_2^2} \int_{K_4}^{t} d\tau \frac{e^{-2r\tau}\sigma^2(\tau)}{\left(\int_\tau^T dt' e^{-2rt'}\sigma^2(t')\right)^2}.$$

The substitution

$$v = \int_\tau^T dt' e^{-2rt'}\sigma^2(t')$$

leads to

$$\frac{r^2}{k_2^2}\left(\frac{1}{\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)} - \frac{1}{\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}\right).$$

Applying the same substitution to the second integral in (8.22) gives

$$\int_{K_4}^{t} d\tau \frac{e^{-r\tau}\sigma^2(\tau)}{f^1(\tau)} = -\frac{r}{k_2}\log\frac{\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}{\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}.$$

Inserting these integrals in (8.22) and simplifying results in

$$u(x,t) = K_3\sqrt{\frac{\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}{\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}}\; e^{r(t-K_4)+\frac{k_3^2 r^2}{2k_2^2\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)} - \frac{\left(\frac{e^{-rt}}{\sqrt{2}}x - \frac{k_3 r}{k_2\sqrt{2}}\right)^2}{\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}}. \quad (8.24)$$

Taking the limit $t \to T$ and using the formulas (8.11) and (8.12) like we did in the case of the classical Black-Scholes equation, we arrive at

$$\lim_{t\to T} u(x,t) = K_3\sqrt{2\pi\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}\; e^{rT}\delta\left(x - \frac{k_3 r}{k_2}e^{rT}\right).$$

The normalization condition leads to

$$K_3 = \frac{1}{\sqrt{2\pi\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}}\; e^{rK_4 - 2rT - \frac{k_3^2 r^2}{2k_2^2\int_{K_4}^T d\tau e^{-2r\tau}\sigma^2(\tau)}}.$$

Defining a new constant λ as

$$\lambda = \frac{k_3 r}{k_2}e^{rT}$$

and inserting in (8.24), we finally get the solution

$$u_\lambda(x,t) = \frac{e^{r(t-2T)}}{\sqrt{2\pi \int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}} e^{-\left(\frac{xe^{-rt}}{\sqrt{2\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}} - \frac{\lambda e^{-rT}}{\sqrt{2\int_t^T d\tau e^{-2r\tau}\sigma^2(\tau)}}\right)^2}. \quad (8.25)$$

The general solution satisfying the boundary condition (8.4) is then given by

$$u(x,t) = \int_{-\infty}^{\infty} u_\lambda(x,t)(\lambda - K)\theta(\lambda - K)d\lambda = \int_K^{\infty} \lambda u_\lambda(x,t)d\lambda - K\int_K^{\infty} u_\lambda(x,t)d\lambda$$

which evaluates to

$$u_{GBS}(x,t) = \sqrt{\frac{f(t)}{2\pi}} e^{-d^2} + \frac{x-K}{\sqrt{\pi}}\tilde{N}(d) \quad (8.26)$$

with

$$\tilde{N}(s) = \int_s^{\infty} e^{-y^2}dy,$$

$$d = \frac{Ke^{-r(T-t)} - x}{\sqrt{2f(t)}}$$

and

$$f(t) = \int_t^T e^{-2r(\tau-t)}\sigma^2(\tau)d\tau.$$

Analyzing the above solution, which is new to our knowledge, we recognize that the volatility is damped exponentially. So long as the volatility does not rise exponentially the exact form of the volatility is not important. The exponential function determines the integral and with it the solution. As time goes on the damping gets weaker and weaker until it reaches a minimum at striking time.
Moreover, for volatilities in the range of 0-1 the above solution and the classical solution of Black and Scholes are very similar. The time-dependence of the solution (8.26) is only visible when subtracting (8.26) from the classical solution (8.14). For very large volatilities above say 5 the solutions differ greatly and the time-dependence of (8.26) is clearly visible. Such an example is given in figure 8.2 for an oscillating volatility of the form $\sigma(t) = \sigma_0 + \sigma_1 \sin\omega t$ for $K = 20$, $\omega = 500$, $\sigma_0 = 50$ and $\sigma_1 = 40$. For more examples of time-dependent solutions see [63].

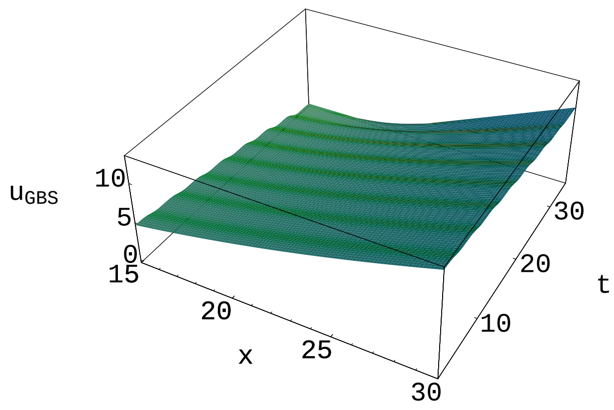


Figure 8.2: Solution for an oscillating volatility with the parameters $\sigma_0 = 50$ and $\sigma_1 = 40$.

In this section we found two new solutions to a generalized Black-Scholes type equation with an arbitrary time-dependent volatility. The first one does not satisfy the boundary value of European call options and therefore does not seem to be of great importance. The second solution however allows to price the value of an option when the time-dependence of the volatility of the underlying asset is known.

8.3 Time- and Asset Price-Dependent Volatility

In this section we generalize the form of the volatility mentioned in the section before. There we analyzed a purely time-dependent volatility. Here we consider a volatility which additionally depends on the value x of the stock. This means that we are looking for solutions of the differential equation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2 u}{\partial x^2} + r\left(x\frac{\partial u}{\partial x} - u\right) = 0. \tag{8.27}$$

Applying an involutive solution procedure to the determining equations of (8.27) with the conditions $r \neq 0$ and $\lambda(x,t) \neq 0$ we get four different cases. However, three of them do not lead to solutions either because the symmetry constants vanish identically or the symmetries are so general that the following reduction does not lead to a solution.
The fourth case is the one we will consider in a more detailed way. The involutive reduction procedure ends with

$$
\begin{aligned}
\sigma(x,t) &= x^{C_1} f^1(t), \\
\xi^x &= 0, \\
\xi^t &= f^2(t), \\
\eta &= u f^3(x,t),
\end{aligned}
$$

where the functions $f^1(t)$, $f^2(t)$ and $f^3(x,t)$ have to satisfy

$$
\begin{aligned}
f^1 f^{2\prime} + 2 f^2 f^{1\prime} &= 0, \\
5 f^{1\prime 2} + f^1(2r(C_1-1) f^{1\prime} - f^{1\prime\prime}) &= 0, \\
-2xr f^2 f^{1\prime} + x^{2C_1} f^{1^3} f^3_x &= 0, \\
4r f^2 f^{1\prime} + f^1(2xr f^3_x + x^{2C_1} f^{1^2} f^3_{x,x} + 2 f^3_t) &= 0, \\
-r f^2(2x^2 r - x^{2C_1}(-3+2C_1) f^{1^2}) f^{1\prime} - x^{2C_1} f^{1^3} f^3_t &= 0.
\end{aligned}
\tag{8.28}
$$

Hereby the underscript denotes differentiation with respect to the corresponding variable. Note that in the second and the last equation appear expressions which involve the parameter C_1. If these expressions are zero before the solution of the corresponding other solutions evolve than the ones considered here. These will be mentioned separately. We start with the problem to find the most general solution.

8.3.1 Search for the Most General Solution

Since the symmetries are built up of functions which have to satisfy further differential equations we append these differential equations to ones given as input to the involutive solver. That means that this input is given by

$$
\begin{aligned}
u_t + \frac{g(x) f^1(t)}{2} u_{x,x} + r(x u_x - u) &= 0, \\
u f^3(x,t) - f^2(t) u_t &= 0, \\
f^1 f^{2\prime} + 2 f^2 f^{1\prime} &= 0, \\
5 f^{1\prime 2} + f^1(2r(C_1-1) f^{1\prime} - f^{1\prime\prime}) &= 0, \\
-2xr f^2 f^{1\prime} + x^{2C_1} f^{1^3} f^3_x &= 0, \\
4r f^2 f^{1\prime} + f^1(2xr f^3_x + x^{2C_1} f^{1^2} f^3_{x,x} + 2 f^3_t) &= 0, \\
-r f^2(2x^2 r - x^{2C_1}(-3+2C_1) f^{1^2}) f^{1\prime} - x^{2C_1} f^{1^3} f^3_t &= 0, \\
x g'(x) - C_1 g(x) &= 0,
\end{aligned}
$$

where $\sigma(x,t)$ has been replaced by $g(x) f^1(t)$. The substitution $x^{C_1} = g(x)$ has

been used since we are looking for solutions in dependence of the constant C_1. But since the solver can only handle polynomially nonlinear differential equations this has to be done (but it is not really a drawback since the information is stored in the form of the corresponding differential equation).
Applying the involutive solution method to the above system with the conditions $r \neq 0$, $g(x) \neq 0$ and $f^1(t) \neq 0$ results in two cases. The first one does not lead to any solution since the appearing functions f^2 and f^3 are identically zero. In this case the symmetries are identically zero and no solution is obtained.
The second case ends with five differential equations involving u, f^1 and f^3. f^2 was found to be

$$f^2(t) = K_2\ e^{-rt}\sqrt{e^{r(2tC_1+K_1)} - e^{r(2t-C_1K_1)}}.$$

Solving the differential equation for $f^1(t)$ leads to

$$f^1(t) = \frac{K_3 e^{\frac{rt}{2}}}{\left(e^{r(2tC_1+K_1)} - e^{r(2t+C_1K_1)}\right)^{\frac{1}{4}}}.$$

Inserting this result in the other differential equations leads to

$$f^3(x,t) = K_4 - \frac{rK_2(2C_1-3)}{2} e^{-rt}\sqrt{e^{r(2tC_1+K_1)} - e^{r(2t+C_1K_1)}} + \frac{x^{2-2C_1}r^2K_2e^{r(2t(C_1-1)+K_1)}}{2K_3^2},$$

$$\begin{aligned} u(x,t) &= f^4(x) e^{\frac{r(3-2C_1)}{2}t + \frac{K_4 \arctan e^{-rt-\frac{rC_1K_1}{2}}\sqrt{e^{r(2tC_1+K_1)}-e^{r(2t+C_1K_1)}}}{K_2 e^{\frac{rC_1K_1}{2}} r(C_1-1)}} \\ & \quad e^{\frac{rx^{2-2C_1} e^{-rt}\sqrt{e^{r(2tC_1+K_1)}-e^{r(2t+C_1K_1)}}}{2K_3^2(C_1-1)}} \end{aligned}$$

and the differential equation

$$f^4(x)\left(e^{rC_1K_1}x^2r^2K_2 + 2x^{2C_1}K_3^2K_4\right) + x^{4C_1}K_3^4K_2 f^{4\prime\prime}(x) = 0 \tag{8.29}$$

for the function f^4. At this point we are stuck concerning the general solution of this equation. But for certain values of the parameter C_1 a solution is possible. For $C_1 = 3/2$ for example we get

$$f^4(x) = \frac{e^{-\frac{\sqrt{-\frac{e^{\frac{3}{2}rK_1}r^2}{K_3^4}}}{x}}}{K_3^4K_2}\left(c_1\ {}_1F_1\left(1 - \frac{K_4}{K_3^2K_2\sqrt{-\frac{e^{\frac{3}{2}rK_1}r^2}{K_3^4}}}, 2, \frac{2\sqrt{-\frac{e^{\frac{3}{2}rK_1}r^2}{K_3^4}}}{x}\right)\right.$$

$$+c_2 U\left(1-\frac{K_4}{K_3^2 K_2\sqrt{-\frac{e^{\frac{3}{2}rK_1}r^2}{K_3^4}}}, 2, \frac{2\sqrt{-\frac{e^{\frac{3}{2}rK_1}r^2}{K_3^4}}}{x}\right)\Bigg).$$

For $C_1 = 1/2$ f^4 is given by

$$\begin{aligned} f^4(x) &= \frac{xe^{-x\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}}}{K_3^4K_2}\left(c_1\,{}_1F_1\left(1-\frac{K_4}{K_3^2K_2\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}}, 2, 2x\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}\right)\right. \\ &\quad \left. +c_2U\left(1-\frac{K_4}{K_3^2K_2\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}}, 2, 2x\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}\right)\right). \end{aligned}$$

Hereby ${}_1F_1$ and U are hypergeometric functions (see [57]). Note that we did not find any other value for C_1 that allows the solution of (8.29).

Despite the fact that we found solutions for $C_1 = 3/2$ and $C_1 = 1/2$ these solutions suffer an important drawback: they involve imaginary parts. The parameters appearing in the square roots

$$\sqrt{-\frac{r^2e^{\frac{1}{2}rK_1}}{K_3^4}}, \quad \sqrt{-\frac{r^2e^{\frac{3}{2}rK_1}}{K_3^4}}$$

are all positive and so the expression under the square root is negative. This results in a non-vanishing imaginary part of the solution, so they may not be of any interest.

However, we mentioned above that by setting certain expressions involving C_1 to zero other solutions are obtained. They will be discussed in the next sections.

8.3.2 Power Law Solution

In this section we first solve the second equation of (8.28) and then set C_1 equal to 1. This results in

$$f^1(t) = C_3 e^{-\frac{1}{4}C_2 r}.$$

Inserting this in (8.28) end setting $C_1 = 1$ here leads to the new system

$$\begin{aligned} f^{2\prime}(t) &= 0, \\ f^3_x(x,t) &= 0, \\ f^3_t(x,t) &= 0. \end{aligned}$$

whose solutions are

$$\begin{aligned} f^2(t) &= C_4, \\ f^3(x,t) &= C_5. \end{aligned}$$

Thus the involutive solution algorithm starts with

$$\begin{aligned} u_t + \frac{C_3^2 k^2 x^2}{2} u_{x,x} + r(x u_x - u) &= 0, \\ C_5 u - C_4 u_t &= 0, \end{aligned}$$

where the parameter combination $e^{-\frac{1}{4}C_2 r}$ has been replaced by k to bring the system into a polynomially nonlinear form. The solution procedure ends with

$$u(x,t) = x^{-\frac{r}{C_3^2 k^2} - \frac{1}{C_3 k}\sqrt{\frac{(C_3^2 k^2 + 2r)^2}{C_3^2 k^2} - \frac{8C_5}{C_4}}} \left(\sqrt{x} g^1(t) + x^{\frac{1}{2} + \frac{1}{C_3 k}\sqrt{\frac{(C_3^2 k^2 + 2r)^2}{C_3^2 k^2} - \frac{8C_5}{C_4}}} g^2(t) \right),$$

where the functions $g^1(t)$ and $g^2(t)$ have to satisfy the equation

$$C_5 g^1(t) + C_5 g^2(t) x^{\frac{\sqrt{\frac{(C_3^2 k^2 + 2r)^2}{C_3^2 k^2} - \frac{8C_5}{C_4}}}{C_3 k}} - C_4 g^{1\prime}(t) + x^{\frac{\sqrt{\frac{(C_3^2 k^2 + 2r)^2}{C_3^2 k^2} - \frac{8C_5}{C_4}}}{C_3 k}} g^{2\prime}(t) = 0.$$

This equation has to be true for all values of x and t. For this reason the equation splits into the equations

$$C_5 g^i(t) - C_4 g^{i\prime}(t) = 0$$

for $i = 1, 2$, whose solution is

$$g^i(t) = K_1 e^{\frac{C_5}{C_4} t}.$$

Inserting this in the above equation for u leads to the solution

$$\begin{aligned}
u(x,t) &= e^{\frac{C_5}{C_4}t} x^{-\frac{re^{\frac{1}{2}C_2 r}}{C_3^2} - \frac{e^{\frac{1}{4}C_2 r}}{2C_3}\sqrt{4r + \frac{C_3^4 + 4e^{C_2 r} r^2}{C_3^2} e^{-\frac{1}{2}C_2 r} - \frac{8C_5}{C_4}}} \\
&\left(K_1\sqrt{x} + K_2 x^{\frac{1}{2} + \frac{e^{\frac{1}{4}C_2 r}}{C_3}\sqrt{4r + \frac{C_3^4 + 4e^{C_2 r} r^2}{C_3^2} e^{-\frac{1}{2}C_2 r} - \frac{8C_5}{C_4}}} \right).
\end{aligned}$$

Like in the case for the classical Black-Scholes problem, where we obtained a similar result, this solution does not satisfy the boundary condition (8.4) and is therefore of no interest concerning the pricing of European options.
In this section we set $C_1 = 1$ after the solution of the second equation of (8.28). In the next section we do this before the solution of the system (8.28). This leads to another class of solutions.

8.3.3 Airy Solution

Setting $C_1 = 1$ in system (8.28) leads to

$$\sigma(x,t) = x f^1(t)$$

and

$$\begin{aligned}
f^1 f^{2\prime} + 2 f^2 f^{1\prime} &= 0, \\
5 f^{1\prime 2} - f^1 f^{1\prime\prime}) &= 0, \\
-2xr f^2 f^{1\prime} + x {f^1}^3 f_x^3 &= 0, \\
4r f^2 f^{1\prime} + f^1(2xr f_x^3 + x^2 {f^1}^2 f_{x,x}^3 + 2 f_t^3) &= 0, \\
-r f^2 (2r + {f^1}^2) f^{1\prime} - {f^1}^3 f_t^3 &= 0.
\end{aligned}$$

Solving the second equation, inserting it back into the system leads to

$$\begin{aligned}
f^1(t) &= \frac{C_3}{(4t + C_2)^{1/4}}, \\
f^2(t) &= C_4\sqrt{4t + C_2}, \\
f^3(x,t) &= C_6 + \frac{C_4 r}{2}\sqrt{4t + C_2} + \frac{C_4 r^2}{2C_3^2}(4t + C_2) - \frac{2C_4 r}{C_3^2}\log x.
\end{aligned}$$

Constructing the invariant surface condition, appending it to equation (8.27) where $\sigma(x,t)$ has been replaced by $\frac{xC_3}{(4t+C_2)^{1/4}}$, and applying the function **InvolutePDESolve** to it gets

$$u(x,t) = g^1(x)x^{-\frac{r}{C_3^2}\sqrt{4t+C_2}}e^{\frac{rt}{2}+\frac{C_6}{2C_4}\sqrt{4t+C_2}+\frac{r^2}{12C_3^2}(4t+C_2)^{3/2}}.$$

Hereby the function $g(t)$ has to satisfy the condition

$$2\left(C_3^2C_6+\log\left(x^{-2C_4r}\right)\right)g(x)+x^2C_3^4C_4g''(x)=0.$$

The solution to this equation reads

$$\begin{aligned} g(x) &= \sqrt{x}\left(K_1Ai\left(\left(\frac{r}{2C_3^4}\right)^{1/3}\frac{C_3^4C_4-8C_3^2C_6+16C_4r\log x}{8C_4r}\right)\right. \\ &\quad \left.+K_2Bi\left(\left(\frac{r}{2C_3^4}\right)^{1/3}\frac{C_3^4C_4-8C_3^2C_6+16C_4r\log x}{8C_4r}\right)\right), \end{aligned}$$

where Ai and Bi are Airy-functions (see [57]). Finally, inserting the result in the preliminary solution for u leads to

$$\begin{aligned} u(x,t) &= x^{\frac{1}{2}-\frac{r}{C_3^2}\sqrt{4t+C_2}}e^{\frac{rt}{2}+\frac{C_6}{2C_4}\sqrt{4t+C_2}+\frac{r^2}{12C_3^2}(4t+C_2)^{3/2}}\cdot \\ &\quad \left(K_1Ai\left(\left(\frac{r}{2C_3^4}\right)^{1/3}\frac{C_3^4C_4-8C_3^2C_6+16C_4r\log x}{8C_4r}\right)\right. \\ &\quad \left.+K_2Bi\left(\left(\frac{r}{2C_3^4}\right)^{1/3}\frac{C_3^4C_4-8C_3^2C_6+16C_4r\log x}{8C_4r}\right)\right). \end{aligned}$$

Unfortunately this solution does not satisfy the boundary condition (8.4) either.

In this chapter we found known as well as new solutions to applications of the world of financial mathematics. With the method of involutive reduction we calculated the well-known solution of Black and Scholes (8.3) which allows to price European call options which a specified boundary condition. This was also done for a generalized Black-Scholes-type equation which involves an arbitrary time-dependent volatility.
Besides the solutions to these boundary-value problems we found other solutions to the considered applications. To our knowledge they do not appear in the literature and are therefore new.

Chapter 9

Further Applications

In the last three chapters we saw how the method of involutive reductions and solutions works. The proceeding is always the same. First, calculate the determining equations for the infinitesimal generators of the differential equation under consideration, in dependence of any appearing parameters or arbitrary functions. If this is done use the invariant surface condition to build coupled systems of differential equations and solve these systems. This leads to invariant reductions or solutions. After illustrating this procedure many times, in this chapter we only state the solutions and the notation for constants and arbitrary functions found for a bunch of differential equations with the method of involutive reductions/solutions. Note that the majority of these solutions are new and do not appear in the literature.
In the following examples appear different solutions. Each solution is preceded by a declaration of the constants and/or the form of certain arbitrary functions which are involved in the differential equation. The constants named C_i are the group constants of the symmetry group of the original differential equation, while the other constants named k_i, K_i or c_i are either constants coming from the solution of ordinary differential equations or are symmetry constants of a reduced equation.

9.1 Incompressible Laminar Boundary Layer Equations

In this section we consider the incompressible laminar boundary layer equations for a radial flow in the absence of a pressure gradient with an axially symmetric swirling component of velocity w. The governing equations of this model are (lower indices represent derivatives with respect to the corresponding variable)

$$
\begin{aligned}
uu_x + vu_y - \frac{w^2}{x} &= \nu u_{y,y} \\
uw_x + vw_y + \frac{uw}{x} &= \nu w_{y,y} \\
(xu)_x + (xv)_y &= 0.
\end{aligned}
\tag{9.1}
$$

Hereby x is the radial distance from the symmetry axis, y is the coordinate parallel to this axis, while u and v are the velocity components in these directions. The details of the derivation are found in [64]. The symmetry group of this problem is built up of a two-dimensional discrete symmetry group and a continuous symmetry group. The involutive reduction procedure delivered the following solutions (we state only the non-trivial solutions we found):

- K, k, C_6 constant, $f(x)$ arbitrary function:

$$
\begin{aligned}
u(x,y) &= -\frac{K\sqrt{e^{2k}-x^2} + 2\,C_6^2\left(e^{2k}-x^2\right)^2}{\left(x^3 - xe^{2k}\right)\left(f(x) - y\,C_6\right)^2}, \\
v(x,y) &= \frac{2\left(e^{2k}-x^2\right)^2 \nu\, C_6^2\left(2xf(x) - 2x\,y\,C_6 + f'(x)\left(e^{2k}-x^2\right)\right)}{C_6 x\left(e^{2k}-x^2\right)\left(f(x) - yC_6\right)^2} \\
&\quad + \frac{\sqrt{e^{2k}-x^2}K\left(-xf(x) + xyC_6 + f'(x)\left(e^{2k}-x^2\right)\right)}{C_6 x\left(e^{2k}-x^2\right)\left(f(x) - y\,C_6\right)^2}, \\
w(x,y) &= -\frac{e^k\left(\sqrt{e^{2k}-x^2}K + 2\nu\,C_6^2\left(e^{2k}-x^2\right)^2\right)}{x\left(e^{2k}+x^2\right)^{3/2}\left(f(x) - y\,C_6\right)^2}
\end{aligned}
$$

- c_2, c_3 constant:

$$
\begin{aligned}
u(x,y) &= -\frac{2x\nu c_3^2}{(c_2 + y\,c_3)^2}, \\
v(x,y) &= -\frac{4\nu c_3}{c_2 + y\,c_3}, \\
w(x,y) &= 0
\end{aligned}
$$

- c_2, c_3 constant:

$$\begin{aligned} u(x,y) &= -\frac{3x\nu c_3^2}{(c_2+y\,c_3)^2}, \\ v(x,y) &= -\frac{6\nu c_3}{c_2+y\,c_3}, \\ w(x,y) &= \pm\frac{3x\nu c_3^2}{(c_2+y\,c_3)^2} \end{aligned}$$

- C_1, C_2 constant, $f(x)$ arbitrary:

$$\begin{aligned} u(x,y) &= -\sqrt{\frac{2x^2C_2-C_1^2}{x^2}}, \\ v(x,y) &= \frac{2y\,C_2}{x\sqrt{\frac{2x^2C_2-C_1^2}{x^2}}}-f(x), \\ w(x,y) &= \frac{C_1}{x} \end{aligned}$$

9.2 A Driven Single Flux Line in Superconductors

The dynamics of a driven single flux line in a bulk type-II superconductor at low temperatures, where the driving force and the velocity of the string is very large, is modeled by the two-dimensional equation (see [65])

$$u_t(1+u_x^2)-ku_{x,x}=0, \tag{9.2}$$

where $u(x,t)$ is the shape function, u is the direction of the driving Lorentz force and x and t are space and time coordinates respectively. Note that the symmetry group of (9.2) is five-dimensional.
The following solutions were found by the involutive reduction procedure:

- C_7, C_8 constant:

$$u(x,t)=\frac{\pm C_8\,x-C_7}{C_8}$$

- C_2, C_7, C_8 constant:

$$u(x,t) = -\frac{C_7C_8 \pm \sqrt{-C_8^2(C_2 - xC_8)^2}}{C_8^2}$$

- C_1, C_2, C_7, C_8 constant:

$$u(x,t) = \frac{1}{C_8}\left(-C_7 - \sqrt{C_7^2 + 2xC_2C_8 - x^2C_8^2 + 2C_8(C_1 + ktC_8)}\right)$$

- K, c_1, C_2, C_4 constant:

$$u(x,t) = K + \frac{C_2}{C_4}t - \frac{kC_4}{C_2}\log\left(\cos\left(c_1 + \frac{C_2x}{kC_4}\right)\right)$$

9.3 A Burgers Type Equation

Next we state the solutions of a type of Burgers equation [40] involving an arbitrary function $f(u)$ which depends upon u, which is a function of space x and time t:

$$\frac{\partial u}{\partial t} - u\,\frac{\partial u}{\partial x} - \alpha\,\frac{\partial^2 u}{\partial x^2} + f(u) = 0.$$

The following solutions were found by using the two-dimensional, three-dimensional and the continuous symmetry groups (note that for each functional form of f a different symmetry group is received):

- C_1, C_3, C_4 constant, $f(u) = C_4(u - 3C_3)^3$:

$$u(x,t) = \frac{6C_3C_4t - \sqrt{2(C_4t - C_1)} - 6C_1C_3}{2(C_4t - C_1)}$$

- C_1, C_2, C_3 constant, $f(u) = C_1 + uC_2$:

$$u(x,t) = C_1e^{-C_2t} - \frac{C_1}{C_2}$$

- K_1, K_2 constant, $f(u) = K_1$:

$$u(x,t) = K_2 - K_1 t$$

- K_1, k_1, c_1, C_1, C_2, C_5 constant, $f(u) = C_1 + uC_2$:

$$u(x,t) = \frac{K_1 - c_1 C_1 e^{C_2 t} + k_1 C_5 (C_2 x - C_1 t)}{c_1 C_2 e^{C_2 t} + k_1 C_5}$$

- C_2, C_3, C_4 constant, $f(u) = C_4 u^3$:

$$u(x,t) = -\frac{C_3 \left(1 + \sqrt{1 + 8C_4\alpha}\right)}{2C_4(C_2 + C_3 x)}$$

- C_4, C_5 constant, $f(u) = -\frac{u^3}{8\alpha}$:

$$u(x,t) = \frac{4C_5\alpha}{C_4 + C_5 x}$$

- k_4, k_5, C_4 constant, $f(u) = C_4 u^3$:

$$u(x,t) = -\frac{1 + k_5\sqrt{1 + 8\alpha C_4}}{2C_4(k_4 + k_5 x)}$$

- C_4, C_5, C_6 constant, $f(u) = -\frac{u^3}{9\alpha}$:

$$u(x,t) = \frac{3\alpha(C_5 + 2xC_6)}{C_4 + x(C_5 + xC_6)}$$

- C_4, C_5, C_6 constant, $f(u) = -\frac{u^3}{9\alpha}$:

$$u(x,t) = 3\alpha\frac{C_5 + 2xC_6 + \sqrt{C_5^2 - 4C_4C_6}}{C_4 + x(C_5 + xC_6)}$$

- C_5, C_6 constant, $f(u) = -\frac{u^3}{9\alpha}$:

$$u(x,t) = \frac{12C_6\alpha}{C_5 + 2xC_6}$$

9.4 The Hasegawa-Mima Equation

The Hasegawa-Mima equation describes low-frequency drift waves in magnetized plasmas or flows in the atmosphere or oceans [66]. It reads

$$\frac{\partial}{\partial t}\left(\nabla^2\phi - \phi\right) + \left\{\phi, \nabla^2\phi\right\} + \beta\frac{\partial \phi}{\partial x} = 0,$$

where $\{f,g\} := f_x g_y - f_y g_x$ is the Poisson bracket, β is proportional to the density gradient in the plasma case or to the Coriolis parameter in the geophysical case. In both cases $\phi(x,y,t)$ is the stream function.
The symmetry group of this problem is nine-dimensional. Solutions to this equation are:

- C_1, C_2, C_4, C_6, C_7, C_9 constant:

$$\phi(x,y,t) = C_1 + \frac{C_7}{C_2}t - \frac{\beta C_2 C_6 - C_7 C_4}{C_4(C_9 \pm i\beta C_2)}\left(y - \frac{C_9}{C_2}t\right) + \frac{C_9 C_6 \pm iC_7 C_4}{C_4(C_9 \pm i\beta C_2)}$$

- C_1, C_4, C_6, C_7, C_9 constant:

$$\phi(x,y,t) = C_1 + \frac{C_6}{C_4}x + y\left(\frac{C_7}{C_9} - \beta\right) + \frac{\beta C_6}{C_4}t$$

- C_1, C_2, C_4, C_6, C_9 constant:

$$\phi(x,y,t) = C_1 \mp i\frac{C_6 C_9}{C_4 C_2}t \mp i\frac{C_6}{C_4}\left(y - \frac{C_9}{C_2}t\right)$$

- C_1, C_2, C_4, C_6, C_7, C_9 constant:

$$\phi(x,y,t) = C_1 + \frac{C_6C_7}{C_4C_9}t + \frac{C_6(C_6C_9 - C_2C_7)}{C_4(C_6C_9 - \beta C_2)}x + \frac{C_6(\beta C_9 - C_7)}{C_9(\beta C_2 - C_6)}\left(y - \frac{C_6}{C_4}t\right)$$

- K_1, K_2, K_3, k_2, k_6, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3 + \sqrt{\frac{k_6}{k_2\beta}}\left(K_2e^{\sqrt{\frac{k_2\beta}{k_6}}x} - K_1e^{-\sqrt{\frac{k_2\beta}{k_6}}x}\right) \\ &\quad + \frac{C_6}{k_2C_2}\beta(C_2y - C_9t) + x\frac{k_2C_7 - k_6C_9}{k_2C_2\beta} + \frac{C_7}{C_2}t\end{aligned}$$

- C_4, C_6 constant, f arbitrary function of its argument:

$$\phi(x,y,t) = \frac{C_6(x+\beta t) + f(\mp i(x+\beta t) + y)}{C_4}$$

- C_1, C_2, C_4, C_6 constant:

$$\phi(x,y,t) = C_2 + C_1(y \mp i(x + t\beta)) + \frac{C_6}{C_4}x + \frac{C_6\beta}{C_4}t$$

- C_4, C_6 constant, f arbitrary function of its argument:

$$\phi(x,y,t) = \frac{C_6}{C_4}(x+\beta t) + f\left(y - \frac{C_6}{C_4\beta}(x+\beta t)\right)$$

- K_1, K_2, C_2, C_4 constant:

$$\phi(x,y,t) = K_2 + \frac{C_2\beta^2}{C_4}t + \frac{C_2(\beta - K_1)}{C_4}x + K_1\left(y - \frac{C_2\beta}{C_4}t\right)$$

- K_1, k_1, k_2, C_2, C_4, C_6, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= \frac{C_7}{C_2}t + K_1 + \frac{C_6}{C_4}x + \left(\frac{C_7}{C_9} - \frac{C_2C_6\beta}{C_4C_9}\right)\left(y - \frac{C_9}{C_2}t\right) + \\ &\sqrt{\frac{C_4C_9 - C_2C_6}{C_4C_9}}\left(k_2 e^{\sqrt{\frac{C_4C_9}{C_4C_9-C_2C_6}}\left(y-\frac{C_9}{C_2}t\right)} - k_1 e^{-\sqrt{\frac{C_4C_9}{C_4C_9-C_2C_6}}\left(y-\frac{C_9}{C_2}t\right)}\right)\end{aligned}$$

- K_1, K_2, k_3, k_2, k_4, k_6, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= \sqrt{k_2^2+k_4^2}\cos(Ax)\left(K_1\sqrt{\frac{k_6C_2-k_4C_9}{k_4^2(k_4C_9-\beta k_2C_2)}}\cos\left(B\left(y-\frac{C_9}{C_2}t\right)\right)\right. \\ &\left. -\frac{K_2}{k_2k_4(k_4C_9-\beta k_2C_2)}\sqrt{\frac{k_6C_2-k_4C_9}{k_4C_9-\beta k_2C_2}}\sin\left(B\left(y-\frac{C_9}{C_2}t\right)\right)\right) \\ &+\sqrt{k_2^2+k_4^2}\sin(Ax)\left(K_2\sqrt{\frac{k_6C_2-k_4C_9}{k_4^2(k_4C_9-\beta k_2C_2)}}\cos\left(B\left(y-\frac{C_9}{C_2}t\right)\right)\right. \\ &\left. -\frac{K_1}{k_2k_4}\sqrt{\frac{k_2^2(k_6C_2-k_4C_9)}{k_4C_9-\beta k_2C_2}}\sin\left(B\left(y-\frac{C_9}{C_2}t\right)\right)\right) \\ &+x\frac{k_6C_9-k_2C_7}{k_4C_9-\beta k_2C_2} + y\frac{k_4C_7-\beta k_6C_2}{k_4C_9-\beta k_2C_2} + K_3\beta t(k_6C_9-k_2C_7),\end{aligned}$$

with

$$A = \sqrt{\frac{k_2^2(k_4C_9-\beta k_2C_2)}{(k_2^2+k_4^2)(k_6C_2-k_4C_9)}}x,$$

$$B = \sqrt{\frac{k_4^2(\beta k_2C_2-k_4C_9)}{(k_2^2+k_4^2)(k_4C_9-k_6C_2)}}$$

- K_1, K_2, K_3, k_4, k_6, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3 + \frac{k_6}{k_4}x + \frac{\beta k_6}{k_4}t + \frac{y}{C_9}\left(C_7 - \frac{\beta k_6C_2}{k_4}\right) - \\ &\sqrt{\frac{k_4C_9-k_6C_2}{k_4C_9}}\left(K_1 e^{-\sqrt{\frac{k_4C_9}{k_4C_9-k_6C_2}}\left(y-\frac{C_9}{C_2}t\right)} - K_2 e^{\sqrt{\frac{k_4C_9}{k_4C_9-k_6C_2}}\left(y-\frac{C_9}{C_2}t\right)}\right)\end{aligned}$$

- K_1, K_2, K_3, k_3, k_5, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3 + \frac{k_3}{k_5}y \pm i\beta t\left(\frac{k_3}{k_5} - \frac{C_7}{C_9}\right) \pm i\frac{k_5C_7 - k_3C_9}{k_5C_9}x \\ &\quad + \sqrt{\frac{k_3}{\beta k_5}}\left(K_2 e^{\sqrt{\frac{\beta k_5}{k_3}}x} - K_1 e^{-\sqrt{\frac{\beta k_5}{k_3}}x}\right)\end{aligned}$$

- K_1, k_1, k_5 constant:

$$\phi(x,y,t) = K_1 + \frac{k_1}{k_5}y$$

- K_1, K_2, K_3 k_1, k_3, C_2, C_7 constant:

$$\phi(x,y,t) = K_1 + \frac{k_3}{k_1}x + \frac{C_7}{C_2}t + K_2y + K_3y^2 + \frac{y^3}{6}\left(\frac{k_1C_7}{k_3C_2} - \beta\right)$$

- K_1, K_2, K_3, k_1, k_2, k_3, C_2, C_7 constant:

$$\begin{aligned}\phi(x,y,t) &= \frac{1}{k_2\beta}\left(\frac{\sqrt{\beta k_2 k_3(k_1^2+k_2^2)}}{k_1}\left(K_2 e^{\sqrt{\frac{k_2\beta}{k_3(k_1^2+k_2^2)}}(k_1y-k_2x)} - K_1 e^{-\sqrt{\frac{k_2\beta}{k_3(k_1^2+k_2^2)}}(k_1y-k_2x)}\right)\right. \\ &\quad \left. + \frac{k_2C_7(x+\beta t) + (k_3C_2\beta - k_1C_7)y}{C_2}\right) + K_3\end{aligned}$$

- k_1, k_2, k_3, C_2, C_7, C_9 constant:

$$\phi(x,y,t) = \frac{(k_3C_2\beta \pm ik_2C_7)y + (k_2C_7 - k_3C_9)(x+\beta t) + K_1k_2(C_2\beta \pm iC_9)}{k_2C_2\beta \pm ik_2C_9}$$

- K_1, K_2, K_3, k_2, k_3, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3+\frac{k_3}{k_2}y+\frac{t}{C_2}\left(C_7-\frac{k_3C_9}{k_2}\right)+\frac{k_2C_7-k_3C_9}{\beta k_2C_2}x+\\ &\sqrt{\frac{k_3}{k_2\beta}}\left(K_2e^{\sqrt{\frac{k_2\beta}{k_3}}x}-K_1e^{-\sqrt{\frac{k_2\beta}{k_3}}x}\right)\end{aligned}$$

- K_1, K_2, K_3, k_1, k_3, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3+\frac{k_3\beta}{k_1}t+\frac{y}{C_9}\left(C_7-\frac{\beta k_3C_2}{k_1}\right)+\\ &\sqrt{\frac{k_1C_9-k_3C_2}{k_1C_9}}\left(K_2e^{\sqrt{\frac{k_1C_9}{k_1C_9-k_3C_2}}\left(y-\frac{C_9}{C_2}t\right)}+K_1e^{-\sqrt{\frac{k_1C_9}{k_1C_9-k_3C_2}}\left(y-\frac{C_9}{C_2}t\right)}\right)\end{aligned}$$

- C_2, C_7 constant, f arbitrary function of its argument:

$$\phi(x,y,t)=\frac{C_7}{C_2\beta}(x+\beta t)+f(y\pm i(x+\beta t))$$

- C_2, C_9 constant, f arbitrary function of its argument:

$$\phi(x,y,t)=\frac{C_9}{C_2}(x+\beta t)+f\left(y-\frac{C_9}{C_2\beta}(x+\beta t)\right)$$

- K_1, k_2, k_3, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= \frac{1}{C_2C_9(\beta C_2-C_3)}\left(k_3C_2(C_9\beta-C_7)y+C_9(k_2C_7-k_3C_9)x\right.\\ &\left.+K_1C_9(\beta k_2C_2-k_3C_2)+C_9\beta(k_2C_7-k_3C_9)t\right)\end{aligned}$$

- K_1, K_2, C_2, C_7 constant:

$$\phi(x,y,t) = K_2 + \left(\frac{C_7}{C_2\beta} \mp iK_1\right)(x+\beta t) + K_1 y$$

- K_1, K_2, C_2, C_9 constant:

$$\phi(x,y,t) = K_2 + \frac{C_9}{C_2}\left(x - \frac{K_1}{\beta}\right) + \frac{C_9}{C_2}(\beta - K_1)t + K_1 y$$

- K_1, K_2, K_3, k_1, k_2, k_3, C_2, C_7, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= \frac{C_7}{C_2}t + \sqrt{\frac{(k_1^2+k_2^2)(k_3C_2-k_1C_9)}{k_1^2(k_1C_9-\beta k_2C_2)}}\cos\left(A\left(y - \frac{C_9}{C_2}t\right)\right) \\ &\quad (K_1\sin(Bx) + K_2\cos(Bx)) \\ &\quad + \frac{k_1}{k_1k_2}\sqrt{\frac{k_2^2(k_1^2+k_2^2)(k_3C_2-k_9C_2)}{k_1C_9-\beta k_2C_2}}\sin\left(A\left(y - \frac{C_9}{C_2}t\right)\right) \\ &\quad (K_2\sin(Bx) - K_1\cos(Bx)) \\ &\quad + \frac{k_3C_9-k_2C_7}{k_1C_9-\beta k_2C_2}x + \frac{k_1C_7-\beta k_3C_2}{k_1C_9-\beta k_2C_2}y + \\ &\quad \frac{C_9}{k_1C_9-\beta k_2C_2}\left(\beta k_3C_9 - \frac{k_1C_7C_9}{C_2}\right) + K_3,\end{aligned}$$

where

$$A = \sqrt{\frac{k_1^2(\beta k_2C_2 - k_1C_9)}{(k_1^2+k_2^2)(k_1C_9-k_3C_2)}},$$

$$B = \sqrt{\frac{k_2^2(\beta k_2C_2 - k_1C_9)}{(k_1^2+k_2^2)(k_1C_9-k_3C_2)}}$$

- f arbitrary function of its argument:

$$\phi(x,y,t) = f(y \pm i(x+\beta t))$$

- K_1, k_1, k_2, C_2, C_9 constant:

$$\phi(x,y,t) = K_1 + \frac{C_7}{\beta C_2}\left(x + \beta t - \frac{k_1}{\beta k_2} y\right)$$

- K_1, K_2 constant:

$$\phi(x,y,t) = K_2 + K_1(y \pm i(x+\beta t))$$

- K_1, K_2, K_3, k_3, k_5 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3 \pm i\beta\left(1 - \frac{k_3}{k_5}\right)(x+\beta t) + \frac{k_3}{k_5} y + \\ &\sqrt{\frac{k_3}{\beta k_5}}\left(K_2 e^{\sqrt{\frac{\beta k_5}{k_3}} x} - K_1 e^{-\sqrt{\frac{\beta k_5}{k_3}} x}\right)\end{aligned}$$

- K_1, K_2, K_3, k_3, k_5, C_2, C_9 constant:

$$\begin{aligned}\phi(x,y,t) &= K_3 + \frac{k_3}{k_5}\left(y - \frac{C_9}{C_2} t\right) + \frac{C_9(\beta k_5 - k_3)}{\beta k_5 C_2} x + \frac{\beta C_9}{C_2} t + \\ &\sqrt{\frac{k_3}{\beta k_5}}\left(K_2 e^{\sqrt{\frac{\beta k_5}{k_3}} x} - K_1 e^{-\sqrt{\frac{\beta k_5}{k_3}} x}\right)\end{aligned}$$

- K_1, K_2, K_3, k_5, k_7, C_2, C_7 constant:

$$\phi(x,y,t) = K_2 + \left(\frac{C_7}{\beta C_2} \mp i\frac{k_7}{k_5}\right)(x+\beta t) + \frac{k_7}{k_5} y + \sqrt{\frac{k_7}{\beta k_5}}\left(K_1 e^{\sqrt{\frac{\beta k_5}{k_7}} x} - K_2 e^{-\sqrt{\frac{\beta k_5}{k_7}} x}\right)$$

- K_1, K_2, K_3, k_5, k_7 constant:

$$\phi(x,y,t) = K_3 + \frac{k_7}{k_5}(y \mp i(x+\beta t)) + \sqrt{\frac{k_7}{\beta k_5}}\left(K_2 e^{\sqrt{\frac{\beta k_5}{k_7}}x} - K_1 e^{-\sqrt{\frac{\beta k_5}{k_7}}x}\right)$$

9.5 Diffusion of a Chemically Reactive Species in a Laminar Boundary Layer Flow

In the last section we are dealing with the two-dimensional equations of motion of a chemically reactive species in a laminar boundary layer flow. The chemically reactive species is hereby emitted from a surface of a body which is located in a hydrodynamic flow field. The details can be found in [67].
The equations of motion for this model are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9.3}$$

which describes the conservation of the total mass, the kinetic equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\varrho}p' - \nu\frac{\partial^2 u}{\partial y^2} = 0, \tag{9.4}$$

and the equation of the concentration field for the critical component given off by the surface

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} - d\frac{\partial^2 c}{\partial y^2} + \alpha W(c) = 0. \tag{9.5}$$

Hereby $u(x,y)$ and $v(x,y) =$ are the velocity components of the fluid in the x, y-direction parallel and orthogonal to the surface, $p = p(x)$ is the known pressure distribution at the edge of the boundary layer, ϱ is the density and ν is the kinematic viscosity, which are assumed to be constant, as well as the temperature. $c(x,y)$ is the dimensionless concentration based on the surface value, d is the coefficient of diffusion in the mixture, α is a reaction rate constant and $W(c)$ is a dimensionless reaction rate which depends only on the concentration of the critical component, where a simple one-stage reaction mechanism in an isothermal flow field is considered. In this reaction the reactive component given off by the surface, which is the critical component, and the reaction products form a three-component mixture together with either an inert or reactive carrier fluid, which is present as the major component.
The solutions which we found by using the involutive solution method and the two-dimensional, three-dimensional and two-fold continuous symmetry goups (for

different functional forms of $W(c)$) are:

- K constant, f^1, f^2, f^3, f^4 arbitrary functions of x:

$$\begin{aligned}
p(x) &= K - \frac{\varrho}{2} f^1(x)^2, \\
W(x,y) &= -\frac{1}{\alpha} f^1(x) f^{2\prime}(x), \\
u(x,y) &= f^1(x), \\
v(x,y) &= f^4(x) - y f^{1\prime}(x), \\
c(x,y) &= f^2(x)
\end{aligned}$$

- K_1, K_2, k_1, C_1, C_2, C_5 constant, f^1, f^2 arbitrary functions of x:

$$\begin{aligned}
p(x) &= K_2 - \frac{\varrho}{2} f^1(x)^2, \\
W(c(x,y)) &= C_1 + C_2 c(x,y), \\
u(x,y) &= f^1(x), \\
v(x,y) &= f^2(x) - y f^{1\prime}(x), \\
c(x,y) &= K_1 e^{\frac{C_2 C_5}{k_1} y f^1(x)} e^{\frac{C_2 A(x)}{k_1^2}} - \frac{C_1}{C_2},
\end{aligned}$$

where

$$A(x) = \int_{x_0}^{x} dx' \left(-\frac{k_1^2 \alpha}{f^1(x')} + d C_2 C_5^2 f^1(x') - k_1 C_5 f^2(x') \right)$$

- k_1, k_3, k_5, C_1, C_2, C_9 constant:

$$\begin{aligned}
p(x) &= C_1, \\
W(x,y) &= k_1, \\
u(x,y) &= \frac{k_3 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= \frac{C_2}{2C_9} + y^2 \left(\frac{k_1 \alpha}{2d + 12\nu} + k_5 (k_3 - 6x\nu)^{-2-\frac{d}{3\nu}} \right)
\end{aligned}$$

- k_2, K_1, K_2, C_1, C_2, C_3, C_9 constant:

$$
\begin{aligned}
p(x) &= C_1, \\
W(c(x,y)) &= (C_2 + C_3 c(x,y))^{1+\frac{2C_9}{C_3}}, \\
u(x,y) &= \frac{k_2 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= -\frac{C_2}{C_3} + K_2 y^{-\frac{C_3}{C_9}} \cdot \\
&\quad \left(C_9^2 (6x\nu - k_2)^{\frac{d(C_3+C_9)}{3\nu C_9} - 2} - K_1 C_3 (d(C_3 + C_9) - 6\nu C_9) \right)^{-\frac{C_3}{2C_9}}
\end{aligned}
$$

- C_4, C_7 constant, f^1, f^2 arbitrary function of x:

$$
\begin{aligned}
p(x) &= C_7, \\
W(x,y) &= -\frac{C_4}{\alpha} f^{2\prime}(x), \\
u(x,y) &= C_4, \\
v(x,y) &= f^1(x), \\
c(x,y) &= f^2(x)
\end{aligned}
$$

- k_1, k_4, k_5, C_2, C_6, C_7 constant, f^2, f^3 arbitryary function of y:

$$
\begin{aligned}
p(x) &= C_7 + C_2 C_6, \\
W(x,y) &= \frac{1}{\alpha}(d f^{2\prime\prime}(y) - k_1 f^{2\prime}(y)), \\
u(x,y) &= k_5 - \frac{C_2 C_6}{\varrho k_1} y + \frac{k_4 \nu}{k_1} e^{\frac{k_1}{\nu} y}, \\
v(x,y) &= k_1, \\
c(x,y) &= f^2(y)
\end{aligned}
$$

- k_1, k_2, k_3, K_1, K_2, C_2, C_4, C_7 constant, f arbitrary function of its argument:

$$
\begin{aligned}
p(x) &= C_7, \\
W(x,y) &= f\left(y - \frac{k_1}{C_4}\log\left(C_2 + C_4 x\right)\right), \\
u(x,y) &= -\frac{6k_3^2\nu(C_2 + C_4 x)}{C_4\left(k_2 + k_3\left(y - \frac{k_1}{C_4}\log\left(C_2 + C_4 x\right)\right)\right)^2}, \\
v(x,y) &= -\frac{6k_3\nu\left(k_1 k_3 + C_4\left(k_2 + k_3\left(y - \frac{k_1}{C_4}\log\left(C_2 + C_4 x\right)\right)\right)\right)}{C_4\left(k_2 + k_3\left(y - \frac{k_1}{C_4}\log\left(C_2 + C_4 x\right)\right)\right)^2}, \\
c(x,y) &= K_2 + \frac{1}{d}\int_{\zeta_0}^{y - \frac{k_1}{C_4}\log\left(C_2 + C_4 x\right)} d\zeta'(k_2 + k_3\zeta')^{-\frac{6\nu}{d}} \cdot \\
&\quad \left(K_1 d + \alpha\int_{\zeta_0'}^{\zeta'} d\zeta''(k_2 + k_3\zeta'')^{\frac{6\nu}{d}} f(\zeta'')\right)
\end{aligned}
$$

- K_1, K_2, k_2, k_3, C_2, C_4, C_7 constant, f arbitrary function of y:

$$
\begin{aligned}
p(x) &= C_7, \\
W(x,y) &= f(y), \\
u(x,y) &= -\frac{6\nu k_3^2(C_2 + C_4 x)}{C_4(k_2 + k_3 y)^2}, \\
v(x,y) &= -\frac{6k_3\nu}{k_2 + k_3 y}, \\
c(x,y) &= K_2 + \frac{1}{d}\int_{y_0}^{y} dy'(k_2 + k_3 y')^{-\frac{6\nu}{d}}\left(K_1 d + \alpha\int_{y_0'}^{y'} dy''(k_2 + k_3 y'')^{\frac{6\nu}{d}} f(y'')\right)
\end{aligned}
$$

- K_1, K_2, k_1, k_4, k_5, C_7 constant, f arbitrary function of y:

$$
\begin{aligned}
p(x) &= C_7, \\
W(x,y) &= f(y),
\end{aligned}
$$

$$
\begin{aligned}
u(x,y) &= k_5 + \frac{k_4\nu}{k_1} e^{\frac{k_1}{\nu} y}, \\
v(x,y) &= k_1, \\
c(x,y) &= K_2 + \frac{1}{d}\int_{y_0}^{y} dy' e^{\frac{k_1}{d}y'} \left(K_1 d + \alpha \int_{y_0'}^{y'} dy'' e^{-\frac{k_1}{d}y''} \right)
\end{aligned}
$$

- K_1, k_1, k_3, C_4, C_7 constant, f arbitrary function of x:

$$
\begin{aligned}
p(x) &= k_1, \\
W(x,y) &= \frac{f(x)}{y}, \\
u(x,y) &= \frac{k_3 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= \frac{1}{6x\nu - k_3}\left(K_1 + \alpha \int_{x_0}^{x} dx' f(x') \right)
\end{aligned}
$$

- k_1, k_2, k_3, k_5, C_4, C_6 constant:

$$
\begin{aligned}
p(x) &= k_1, \\
W(x,y) &= k_3, \\
u(x,y) &= \frac{k_5 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= \frac{C_4}{C_6} + y^2 \left(\frac{k_3\alpha}{2d + 12\nu} + k_2 (k_5 - 6x\nu)^{-2-\frac{d}{3\nu}} \right)
\end{aligned}
$$

- K_1, K_2, k_1, k_3, C_2, C_3, C_4, C_5, C_6 constant:

$$
\begin{aligned}
p(x) &= k_3, \\
W(x,y) &= k_1,
\end{aligned}
$$

$$
\begin{aligned}
u(x,y) &= -\frac{6\nu C_3^2 x^{\frac{C_6}{C_5+C_6}}}{\left(C_2 + C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}\right)^2}, \\
v(x,y) &= 3\nu C_3 x^{-\frac{C_5}{2(C_5+C_6)}} \frac{C_2\left(\frac{C_5}{C_5+C_6} - 2\right) - 2C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}}{\left(C_2 + C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}\right)^2}, \\
c(x,y) &= -\frac{C_4}{C_5} + \frac{k_1 \alpha x^{\frac{C_5}{C_5+C_6}}}{2C_3^2(d+6\nu)}\left(C_2 + C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}\right)^2 + \\
&\quad K_1 x^{\frac{C_5}{C_5+C_6}}\left(C_2 + C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}\right)^{\lambda_1} + \\
&\quad x^{\frac{C_5}{C_5+C_6}} K_2 \left(C_2 + C_3 y x^{-\frac{C_5}{2(C_5+C_6)}}\right)^{\lambda_2}
\end{aligned}
$$

with

$$
\lambda_1 = \frac{1}{2} + \frac{-3\nu(C_5+2C_6) + \sqrt{dC_5(C_5+C_6)(d - 30\nu + \frac{C_6(d-6\nu)^2}{dC_5} + \frac{9C_5\nu^2}{d(C_5+C_6)})}}{2d(C_5+C_6)},
$$

$$
\lambda_2 = \frac{1}{2} + \frac{-3\nu(C_5+2C_6) - \sqrt{dC_5(C_5+C_6)(d - 30\nu + \frac{C_6(d-6\nu)^2}{dC_5} + \frac{9C_5\nu^2}{d(C_5+C_6)})}}{2d(C_5+C_6)}
$$

- K_1, K_2, C_2, C_3, C_4 constant, f arbitrary function of y:

$$
\begin{aligned}
p(x) &= C_4, \\
W(x,y) &= f(y), \\
u(x,y) &= -\frac{6x\nu C_3^2}{(C_2 + C_3 y)^1}, \\
v(x,y) &= -\frac{6\nu C_3}{C_2 + C_3 y}, \\
c(x,y) &= K_2 + \frac{1}{d}\int_{y_0}^{y} dy'(C_2 + C_3 y')^{-\frac{6\nu}{d}}\left(K_1 d + \alpha \int_{y_0'}^{y'} dy''(C_2 + C_3 y'')^{\frac{6\nu}{d}} f(y'')\right)
\end{aligned}
$$

- K_1, K_2, k_2, k_3, C_2, C_3, C_4, C_6 constant:

$$\begin{aligned}
p(x) &= k_3, \\
W(x,y) &= k_2, \\
u(x,y) &= -\frac{6x\nu C_3^2}{(C_2+C_3y)^2}, \\
v(x,y) &= -\frac{6\nu C_3}{C_2+C_3y}, \\
c(x,y) &= K_2+\log\left(x^{\frac{C_4}{C_6}}(C_2+C_3y)^{\frac{6\nu C_4}{C_6(d-6\nu)}}\right)+\frac{k_2\alpha y^2}{2(d+6\nu)}+ \\
&\quad \frac{k_2C_2\alpha y}{C_3(d+6\nu)}+\frac{K_1 d}{C_3(d-6\nu)}(C_2+C_3y)^{1-\frac{6\nu}{d}}
\end{aligned}$$

- K_1, K_2, k_2, k_3, C_4, C_6, C_7, C_8 constant, J, Y Bessel functions:

$$\begin{aligned}
p(x) &= k_2, \\
W(c(x,y)) &= C_8(C_4+C_5c(x,y)), \\
u(x,y) &= -\frac{6x\nu k_3^2}{(k_2+k_3y)^2}, \\
v(x,y) &= -\frac{6\nu k_3}{k_2+k_3y}, \\
c(x,y) &= -\frac{C_4}{C_7}+x^{\frac{C_7}{C_6}}(k_2+k_3y)^{\frac{1}{2}-\frac{3\nu}{d}}\left(K_1\mathrm{J}_\lambda(\xi)+K_2\mathrm{Y}_\lambda(\xi)\right),
\end{aligned}$$

with

$$\lambda=\frac{1}{2}\sqrt{1+\frac{36\nu^2}{d^2}-\frac{12(C_6+2C_7)}{dC_6}},$$

$$\xi=-i\sqrt{\frac{C_7C_8\alpha}{d}\frac{k_2+k_3y}{k_3}}$$

- k_1, k_2, k_4, k_5, C_4, C_6 constant:

$$p(x) = k_1,$$

$$
\begin{aligned}
W(x,y) &= k_2, \\
u(x,y) &= \frac{k_5 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= \frac{C_4}{C_6} + y^2 \left(\frac{k_2 \alpha}{2d + 12\nu} + k_4 (k_5 - 6x\nu)^{-2-\frac{d}{3\nu}} \right)
\end{aligned}
$$

- k_1, k_3, K_2, C_4, C_5, C_6, C_8 constant:

$$
\begin{aligned}
p(x) &= k_1, \\
W(c(x,y)) &= C_8 (C_4 + C_5 c(x,y))^{\frac{C_5 + C_6}{C_5}}, \\
u(x,y) &= \frac{k_3 - 6x\nu}{y^2}, \\
v(x,y) &= -\frac{6\nu}{y}, \\
c(x,y) &= -\frac{C_4}{C_5} + K_2 2^{\frac{C_5}{C_6}} y^{-\frac{2C_5}{C_6}} (6x\nu - k_3)^{\frac{2C_5}{C_6}} \left(\frac{C_6^2}{d(2C_5 + C_6) - 6\nu C_6} \right. \\
& \left(36x^2 K_2^{\frac{C_6}{C_5}} \alpha \nu^2 C_5^{\frac{C_6}{C_5}} C_8 + (2d(2C_5 + C_6) - 12\nu C_6)(6x\nu - k_3)^{\frac{d(2C_5 + C_6)}{3\nu C_6}} \right. \\
& \left. \left. -12x K_2^{\frac{C_6}{C_5}} \alpha \nu C_5^{\frac{C_6}{C_5}} C_8 k_3 + K_2^{\frac{C_6}{C_5}} \alpha C_5^{\frac{C_6}{C_5}} C_8 k_3^2 \right) \right)^{-\frac{C_5}{C_6}}
\end{aligned}
$$

Chapter 10

Conclusion

In this work we found new solutions to physically important differential equations. We did this by introducing a new method to find invariant solutions. We call this method the method of **involutive reduction** or **solution**. The basic idea for this new method appeared in the work of Olver and Rosenau [41, 42]. In this work the invariant surface condition was appended to the differential equation under consideration and this coupled system was then solved by using some integrability conditions to simplify the calculations.
We generalized this procedure by using a whole involutive algorithm to calculate all integrability conditions according to Riquier and Janet. To use all these integrability conditions effectively for a reduction or even a solution to differential equations, we coupled this algorithm with a solution tool based on **MathLie** [24] to solve simple partial differential equations by heuristic methods.
The involutive reduction procedure automatically applies the involutive algorithm and the heuristic solver alternately to the coupled system of differential equations. In this way the involutive algorithm helps to simplify the involved differential equations which more easily leads to solutions by the heuristic solver. In reverse, the solutions found by the heuristic solver lead to simplifications by the insertion of these solutions, which is done by the involutive algorithm automatically. Thus, each single method helps the other method in simplifying and reducing the coupled system.
To implement the involutive solution procedure, that is the automated coupling of involutive and heuristic methods, in an effective way, we used a unique representation of the equations under consideration. In this new representation a single part of a term of an equation is built up as a list of numbers. That means that a term, an equation and a system of equations is nothing more than several nested lists. For such nested lists we developped and implemented the discrete involutive calculus which reflects the computations with the usual representation of terms in equations.
Another important aspect of the involutive reduction method is concerned with case distinctions. Case distinctions are very important when keeping in mind

that an equation can only be inserted into another one if its prefactor is unequal to zero. Since we are dealing with polynomially nonlinear differential equations these prefactors also depend upon the dependent variables which appear in the system of differential equations. To reach such a case distinction each prefactor is checked if it involves expressions containing dependent variables. If so, the prefactor is compared against a list of previously obtained prefactors which are unequal to zero. If it is not contained there a case distinction is performed and new systems are built up, one in which the new prefactor is unequal to zero and one in which it is zero. To remember the previous prefactors unequal to zero a list of these prefactors is added to each system so the user is able to see on which conditions the result obtained depends.
With the involutive reduction procedure, which is implemented in *Mathematica*, we obtained solutions to physical problems, such as the diffusion of a chemically reactive species or low-frequency drift-waves in magnetized plamas, as well as solutions to differential equations which play an important role in financial mathematics. For a generalized Black-Scholes equation in which the volatility depends on time we even solved a boundary value problem. To obtain such solutions we used symmetry analysis and searched for invariant solutions. To get rid of the problem concerning the nonlinearity of the group constants we used the full infinitesimal generator for the invariant solutions.
Summarizing, with this work we showed that this new method, the involotive solution procedure, and with it the inplementation of it are able to automatically find solutions to differential equations.

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Berechnung von invarianten Lösungen von Differentialgleichungen. Dazu wird eine neue Methode benutzt, die sog. **Involutive Reduktion**. Die Grundlage dieser Methode findet sich in der Arbeit von Olver und Rosenau [41, 42]. Hierin wird bemerkt, daß durch Ergänzen der ursprünglichen Differentialgleichungen mit der Invarianzbedingung für Lösungsflächen und der Ausnutzung von einigen Integrabilitätsbedingungen dieses gekoppelte System gelöst werden kann.
Diese Vorgehensweise wurde dahingehend verallgemeinert, daß anstelle einiger wahlloser Integrabilitätsbedingungen ein involutiver Algorithmus nach Riquier und Janet verwendet wird, um systematisch alle Integrabilitätsbedingungen zu berechnen und diese zur Vereinfachung des Systems zu verwenden. Darüberhinaus wird dieser involutive Algorithmus zum ersten mal überhaupt mit einem heuristischen Differentialgleichungslöser gekoppelt, der auf **MathLie** zurückgeht.
Diese automatische Kopplung von Vereinfachung und Lösung von gekoppelten Differentialgleichungen bildet die Grundlage der involutiven Reduktion. Durch das Wechselspiel beider Komponenten ergänzen sich beide und führen so zu einer Reduktion bzw. Lösung des ursprünglichen Systems von polynomial nichtlinearen Differentialgleichungen. Der Vereinfachungsalgorithmus nutzt das System und seine Integrabilitätsbedingungen um dieses zu vereinfachen und dadurch leichter zu lösen, wohingegen der Lösungsalgorithmus Lösungen hervorbringt, die mit dem Vereinfachungsalgorithmus wieder in das System eingesetzt werden usw. Auf diese Weise ergänzen sich beide Algorithmen bei der Reduktion bzw. Lösung von Systemen von Differentialgleichungen.
Um die Methode der involutiven Reduktion in *Mathematica* effektiv umzusetzen werden die Differentialgleichungen auf eindeutige Weise in eine Listendarstellung transformiert. Um mit dieser Listendarstellung Berechnungen durchzuführen wird ein eigener Calculus (discrete involutive caclulus) implementiert der die nötigen Berechnungen erlaubt.
Des weiteren wurde ein Mechanismus entwickelt und implementiert der es erlaubt anhand von Vorfaktoren eine Fallunterscheidung durchzuführen. Dies ist nötig um keine Lösungen zu verlieren. Dazu wird untersucht ob ein Vorfaktor einer Differentialgleichung ungleich Null ist oder nicht. Dies geschieht durch Vergleich des Vorfaktors mit einer bereits existierenden Liste von Vorfaktoren die ungleich Null

sind. Falls der zu untersuchende Vorfaktor darin auftaucht wird die Gleichung eingesetzt. Falls er nicht auftaucht wird eine Fallunterscheidung durchgeführt. Der bestehende Fall wird ersetzt durch zwei andere Fälle, einer in dem der Vorfaktor als ungleich Null angenommen wird, die Gleichung eingesetzt wird und die Vorfaktorliste modifiziert wird, und ein anderer in dem der Vorfaktor identisch Null ist und somit keine Einsetzung möglich ist.
Mit Hilfe dieser eben kurz beschriebenen involutiven Reduktion werden Lösungen zu physikalischen Problemen berechnet, so zum Beispiel für die Diffusion einer chemisch reaktiven Substanz, für magnetisierte Plasmen oder für grundlegende Gleichungen der Finanzmathematik, wie der Black-Scholes Gleichung. Hierbei wird mit Hilfe der Symmetrieanalyse und der involutiven Reduktion das Randwertproblem für eine verallgemeinerte Black-Scholes Gleichung mit einer beliebig zeitabhängigen Volatilität gelöst. Hierzu sei bemerkt, daß bei der Aufstellung der Invarianzbedingung für die Lösungsfläche stets der vollständige infinitesimale Generator benutzt wird, um Probleme mit nichtlinear auftretenden Gruppenkonstanten zu vermeiden und möglichst allgemeine Lösungen zu finden.
Abschliessend läßt sich feststellen, daß die hier entwickelte und implementierte Lösungsmethode der involutiven Reduktion, wie hier gezeigt, fähig ist vollautomatisch Lösungen für gekoppelte polynomial-nichtlineare Systeme von Differentialgleichungen zu berechnen.

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Erklärung

Ich erkläre hiermit, daß ich die vorliegende Arbeit selbständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche erkenntlich gemacht habe.

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