# Semiclassics for quantum systems with internal degrees of freedom

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# Preface

This thesis is based on the results that were developed during my time in the Quantum Chaos group of Frank Steiner and Jens Bolte, Abteilung Theoretische Physik, Universität Ulm, from October 2000 to April 2004. A part of the investigations also have been performed while I took part in the research programme "Semi-Classical Analysis" at the Mathematical Sciences Research Institute (MSRI), Berkeley, California, USA, from February to May 2003.

I could enjoy the friendly and pleasant atmosphere in the Ulm group, and therefore I have to thank all the people who made this group as it is. I am particularly indebted to Jens Bolte, for sharing his broad knowledge with me and allowing me to benefit in a huge amount from the human and amicable collaboration with him. His personal encouragement already started with the beginning of my diploma thesis in 1999. Furthermore, the various discussions with Roman Schubert, Stefan Keppeler and Grischa Haag provided my work with a lot of inspiration and progress, and I thank Jon Harrison for proof-reading a first draft of this work. I also want to thank Prof. Frank Steiner for his support and the possibility to work in his group.

Over several years I could enjoy the task to give exercise classes for mathematics courses and I am very grateful to Dr. Uwe Pittelkow for opening this opportunity to me and for the collaboration in this field.

Financial support for this work was provided by the Deutsche Forschungsgemeinschaft (DFG) within the project "Semiklassik für Observablen und Zustände von Quantensystemen mit Spin" (DFG Ste 241/15-1 & 241/15-2). The stay at the MSRI was also financed by Deutscher Akademischer Austauschdienst (DAAD) under Doktorandenstipendium D/02/47460.

Moreover, I want to thank all my friends, in- and outside of science, and also my brother Johannes for support in personal matters of any kind. Particular thanks also go to Tobias Schwaibold for his companionship and close friendship over the last years. In addition, I thank my host family in Oakland, Kevin and Deborah Blackburn, for taking me in and introducing me to their family while I stayed in Berkeley.

Moreover, a lot of thank belongs to my partner Simone, in particular for her endulgance, understanding and the joy she brings to my life. Finally, I gratefully acknowledge the support by my parents without which this thesis wouldn't have come into existence.

# Vorwort

Die vorliegende Arbeit basiert auf Resultaten, die während meiner Zeit in der Quanten-Chaos Gruppe um Frank Steiner und Jens Bolte in der Abteilung Theoretische Physik, Universität Ulm, im Zeitraum von Oktober 2000 bis April 2004 entstanden sind. Ein Teil der Ergebnisse wurden erzielt während meiner Teilnahme am Programm "Semi-Classical Analysis" am Mathematical Sciences Research Institute (MSRI), Berkeley, California, USA, von Februar bis Mai 2003.

In meiner Zeit bei der Ulmer Gruppe durfte ich die freundliche und angenehme Atmosphäre dort genießen und möchte daher allen Mitgliedern danken, die die Gruppe zu dem machen und machten, was sie ist. Insbesondere bin ich Jens Bolte zu Dank verpflichtet, welcher sein umfassendes Wissen mit mir teilte und es mir ermöglichte, von der freundschaftlichen und menschlichen Zusammenarbeit, welche bereits mit meiner Diplomarbeit im Jahre 1999 begann, als auch von seiner persönlichen Ermutigung zu profitieren. Desweiteren konnte ich durch zahlreiche Diskussionen mit Roman Schubert, Stefan Keppeler und Grischa Haag wichtige Inspirationen und Anregungen sammeln. Außerdem danke ich Jon Harrison, der einen ersten Entwurf dieser Arbeit korrekturgelesen hat und Herrn Prof. Frank Steiner, für seine Unterstützung und die Möglichkeit in seiner Gruppe zu arbeiten.

Über mehrere Jahre hinweg konnte ich zusammen mit Herrn Dr. Uwe Pittelkow die Übungen zur *Höheren Mathematik für Physiker* leiten und danke ihm für die Zusammenarbeit und dafür, mir diese Möglichkeit zu eröffnen.

Finanziell unterstützt wurde diese Arbeit durch die Deutsche Forschungsgemeinschaft (DFG) im Rahmen des Projektes "Semiklassik für Observablen und Zustände von Quantensystemen mit Spin" (DFG Ste 241/15-1 & 241/15-2), wobei der Aufenthalt am MSRI außerdem durch den Deutschen Akademischen Austauschdienst (DAAD) im Rahmen eines Doktorandenstipendiums (D/02/47460) gefördert wurde.

Ich möchte an dieser Stelle auch all meinen Freunden, sowohl innerhalb als auch außerhalb des wissenschaftlichen Umfeldes, und meinem Bruder Johannes dafür danken, daß sie stets ein offenes Ohr für mich hatten. Insbesondere danke ich Tobias Schwaibold für seine Begleitung in den letzten Jahren und die enge Freundschaft, die sich in dieser Zeit entwickelt hat. Desweiteren möchte ich mich bei meinen Gastgebern in Oakland, Kevin und Deborah Blackburn, für die freundliche und herzliche Aufnahme in ihre Familie während meiner Zeit in Berkeley bedanken.

Schließlich gebührt meiner Lebensgefährtin Simone eine ganze Menge Dank für ihre Geduld, ihr Verständnis und dafür, daß sie mein Leben mit soviel Freude erfüllt. Sicherlich soll auch die Unterstützung meiner Eltern nicht unerwähnt bleiben, ohne welche diese Arbeit nicht entstanden wäre.

Ulm, 16. Juni 2004 Rainer Glaser

# Abstract

The main focus of this thesis lies in the field of quantum systems with internal degrees of freedom; we particularly emphasize the semiclassical properties of these systems. The translational degrees of freedom of one or several point particles are usually described classically on cotangent bundles of their configuration manifolds. On the other hand, internal degrees of freedom, mostly due to symmetries or constraints, find their classical description on more general symplectic manifolds, such as coadjoint orbits of Lie groups or Kähler manifolds. The quantum mechanical representation space corresponding to the internal degrees of freedom, in many cases finite-dimensional, has to be incorporated into the Hilbert space of the total system. E.g., in the case of a symmetry the internal degrees of freedom have to be implemented by (anti-)unitary (projective) representations of the corresponding symmetry group. The structure of the quantum mechanical description of the translational degrees of freedom is different in nature; the associated Hilbert space is usually given by the square integrable functions on the configuration space. Thus, the total quantum space, describing the translational as well as the internal degrees of freedom, is given by square integrable functions on the configuration space that take values in the representation space of the internal degrees of freedom. Consequently, quantum observables are matrix-valued operators. In order to study the connection between the quantum and the classical realization of certain properties of the system on a mathematical level, the language of microlocal analysis is used. While microlocal (or semiclassical) analysis is a well-established method in the case where internal degrees of freedom are absent, for the present setting the corresponding techniques have to be checked for their validity and a number of results have to be established. Thus one part of this work is concerned with providing the necessary tools in matrix-valued semiclassical analysis. In particular, we will be interested in the question how the quantum time evolution is reflected on the classical level: in the context of microlocal analysis observables are represented as pseudodifferential operators; but only a certain class of those turn out to have a semiclassically well-defined time evolution. This is due to the fact that the time evolution generated by the quantum Hamiltonian leaves certain subspaces of the Hilbert space semiclassically invariant, and transitions between these subspaces do not allow for a classical interpretation. We will construct semiclassical projection operators onto these subspaces. The off-diagonal contributions of observables with respect to these projections correspond to transitions between the different subspaces, whose time evolution is no longer recordable in terms of pseudodifferential operators but are given as Fourier integral operators. A rigorous formulation of this fact is given by a generalized Egorov Theorem. As a further indication of the semiclassical decoupling of the subspaces will occur when we consider the spectral mean of quantum expectation values. These, in the semiclassical limit, can be expressed as a sum over classical quantities that are uniquely associated with a certain subspace, and the observable's off-diagonal terms do not contribute.

All of these results will be achieved in a hybrid formulation: while the translational degrees of freedom are treated semiclassically the internal degrees of freedom are still quantum mechanical in nature; a fact that is reflected by the non-scalar quantities. In order

to obtain also a semiclassical description for the internal degrees of freedom, a quantization procedure which is not restricted to cotangent bundles is required. Such a method is provided by geometric quantization, which we briefly describe, in particular with respect to coherent states and quantization of coadjoint orbits. We will consider the time evolution of coherent states for general Lie groups and provide a quantization procedure for the internal degrees of freedom that allows to map the quantum description in a one-to-one manner to a classical system. In addition, we will define combined coherent states for both the translational and the internal degrees of freedom. Using this formalism we are then in the position to also perform the semiclassical limit for the internal degrees of freedom: while this limit for the translational degrees of freedom is associated with the parameter $\hbar$, such a parameter for the internal degrees of freedom is given by (or related to) the dimension of the representation space of the internal degrees of freedom. For the total system, incorporating both types of degrees of freedom, we will consider two different semiclassical scenarios: in the first one, only the translational degrees of freedom are treated semiclassically, while the semiclassical parameter for the internal degrees of freedom is fixed. In this context we use the recipes of geometric quantization to map their quantum mechanical description to a classical model in a one-to-one manner. In the second setting both semiclassical parameters are used and a genuine semiclassical limit is performed also for the internal degrees of freedom. For both scenarios we will show that the quantum time evolution of the combined coherent states can be semiclassically approximated in the respective limit. In particular, in leading order the coherent state is propagated according to a classical dynamics.

Furthermore, we will use the possibility to map the quantum mechanical description of the internal degrees of freedom to a classical model in order to reformulate the results given by the Egorov Theorem for general matrix-valued operators. This result yields an important ingredient to clarify the problem of how ergodicity of a classical system is reflected in its quantum mechanical description. This will be achieved by proving a quantum ergodicity theorem for matrix-valued operators.

Finally, the results obtained are applied to the Dirac equation, where it turns out that there is a close connection between the so-called Zitterbewegung and the semiclassical time evolution of observables: we will show that, from a semiclassical point of view, Zitterbewegung can be seen as being caused by the presence of non-diagonal contributions of observables, whose time evolution shows a characteristically different behaviour compared with that of the diagonal terms.

# Zusammenfassung

Der Schwerpunkt der vorliegenden Arbeit ist im Themenbereich der Quantensysteme mit inneren Freiheitsgraden zu sehen, wobei eine besondere Betonung auf die semiklassischen Eigenschaften dieser Systeme gelegt ist. Im Vergleich zu den Freiheitsgraden eines oder mehrerer Punktteilchen, deren Phasenräume durch Kotangentialbündel gegeben sind, finden innere Freiheitsgrade, welche meist durch Symmetrien oder Zwangsbedingungen verursacht sind, ihre klassische Beschreibung auf allgemeineren symplektischen Mannigfaltigkeiten, wie z.B. koadjungierten Orbits bestimmter Lie-Gruppen oder Kähler-Mannigfaltigkeiten. Der quantenmechanische Darstellungsraum der inneren Freiheitsgrade ist in den Hilbertraum des Gesamtsystems einzubauen. So sind z.B. im Falle von Symmetrien die inneren Freiheitsgrade durch (anti-)unitäre (projektive) Darstellungen zu realisieren, während die translatorischen Freiheitsgrade der Punktteilchen durch den Hilbertraum der quadratintegrablen Funktionen auf dem Konfigurationsraum modelliert werden. Dies hat zur Folge, daß als quantenmechanischer Zustandsraum vektorwertige Funktionen und als quantenmechanische Observablen matrixwertige Operatoren zu betrachten sind. Um Untersuchungen anzustellen, welche den Zusammenhang zwischen der klassischen und der quantenmechanischen Realisierung bestimmter Eigenschaften des physikalischen Systems betreffen, wird das mathematische Werkzeug der mikrolokalen Analysis verwendet. Diese Methode ist für den Fall ohne innere Freiheitsgrade, d.h. skalare Operatoren und Wellenfunktionen, wohl etabliert; bei Anwesenheit von inneren Freiheitsgraden sind jedoch die bekannten Resultate auf den nicht-skalaren Fall zu verallgemeinern und auf ihre Gültigkeit zu untersuchen. Ein Teil der Arbeit ist demnach der Bereitstellung von Resultaten des matrixwertigen semiklassischen Kalküls gewidmet. Von besonderem Interesse hierbei ist die Fragestellung, wie sich die quantenmechanische Zeitentwicklung auf der klassischen Seite niederschlägt: es wird die zeitliche Evolution von Observablen untersucht, welche im Rahmen der mikrolokalen Analysis durch Pseudodifferentialoperatoren dargestellt werden. Hierbei zeigt sich, daß es nur für eine bestimmte Klasse von Observablen eine Zeitentwicklung gibt, die sich semiklassisch im Sinne von Pseudodifferentialoperatoren interpretieren läßt. Die Ursache dieses Sachverhalts liegt in der Tatsache begründet, daß es Unterräume des quantenmechanischen Zustandsraum gibt, welche semiklassisch invariant sind unter der Zeitentwicklung, die vom quantenmechanischen Hamiltonoperator erzeugt wird. Ferner erlauben Übergänge zwischen diesen Unterräumen keine direkte klassische Interpretation. Wir werden semiklassische Projektionsoperatoren auf diese Unterräume konstruieren. Die nicht-Diagonalblöcke von Observablen bezüglich dieser Projektoren korrespondieren somit zu Übergängen zwischen den einzelnen Unterräumen und ihre Zeitentwicklung läßt sich nicht mehr durch Pseudodifferentialoperatoren sondern nur noch durch Fourier-Integraloperatoren beschreiben. Daher werden nur Observablen, welche blockdiagonal bezüglich der Aufspaltung in invariante Unterräume sind, eine semiklassische Zeitentwicklung im Sinne von Pseudodifferentialoperatoren besitzen. Eine mathematisch rigorose Formulierung dieser Tatsache werden wir in einem verallgemeinerten Egorov-Theorem treffen. Eine weitere Instanz der semiklassischen Entkoppelung der Unterräume ist zu finden, wenn man spektral gemittelte Erwartungswerte von allgemeinen Observablen betrachtet:

diese lassen sich im semiklassischen Limes durch eine Summe klassischer Objekte ausdrücken, die eindeutig zu einem bestimmten Unterraum assoziiert sind und nicht von nicht-diagonal Beiträgen der betrachteten Observablen abhängen.

Diese Resultate werden im Rahmen einer hybriden Formulierung erreicht: die translatorischen Freiheitsgrade erfahren eine (semi-)klassische Beschreibung, wobei die inneren Freiheitsgrade immer noch einen quantenmechanischen Charakter tragen, welcher sich durch die vektor- bzw. matrixwertigen Größen ausdrückt. Um eine (semi-)klassische Beschreibung auch der inneren Freiheitsgrade möglich zu machen, ist eine Quantisierungsvorschrift für symplektische Mannigfaltigkeiten vonnöten, welche nicht auf Kotangentialbündel eingeschränkt ist. Eine solche Methode ist durch die geometrische Quantisierung gegeben, deren Hauptelemente wir kurz diskutieren und vor allem in Hinsicht auf kohärente Zustände und Quantisierung von koadjungierten Orbits näher betrachten. Wir untersuchen kohärente Zustände für Lie-Gruppen und deren Zeitentwicklung und erhalten gleichzeitig Quantisierungsmethoden für die inneren Freiheitsgrade sowie kombinierte kohärente Zustände für die inneren und translatorischen Freiheitsgrade. Desweiteren sind wir nunmehr in der Lage, auch den semiklassischen Limes für die inneren Freiheitsgrade zu vollziehen: während für die translatorischen Freiheitsgrade der semiklassische Parameter durch $\hbar$ gegeben ist, ist ein semiklassischer Parameter für die inneren Freiheitsgrade in der Dimension des zugehörigen quantenmechanischen Darstellungsraumes zu finden. Für das gesamte System betrachten wir dann sowohl ein Szenario, in dem nur der translatorische Parameter benutzt wird, wobei die inneren quantenmechanischen Freiheitsgrade nur auf ein klassisches System abgebildet werden, als auch den Fall, daß beide semiklassische Parameter verwendet werden. Für diese Szenarien untersuchen wir die Zeitentwicklung der kombinierten kohärenten Zustände und zeigen, daß diese klassisch approximiert werden kann und in führender Ordnung sogar durch die klassische Zeitentwicklung eines kohärenten Zustandes gegeben ist.

Schließlich wenden wir die Möglichkeit, das quantenmechanische Modell der inneren Freiheitsgrade auf ein klassisches abzubilden, auf die bereits untersuchte semiklassische Zeitentwicklung von matrixwertigen Operatoren an und reformulieren das Egorov Theorem in dieser Sprache. Desweiteren werden wir innerhalb dieses Formalismus klären, wie sich ein chaotisches Verhalten des klassischen Systems auf quantenmechanischer Ebene widerspiegelt und beweisen ein Quantenergodizitätstheorem für matrixwertige Observablen.

Zum Abschluß werden wir insbesondere die semiklassischen Projektionsoperatoren und die semiklassische Zeitentwicklung im Zusammenhang mit der Dirac-Gleichung interpretieren und den engen Zusammenhang mit dem Phänomen der sogenannten Zitterbewegung aufzeigen: es wird sich zeigen, daß in einer semiklassischen Betrachtungsweise die bereits erwähnten nicht-diagonal Blöcke von Observablen als die Ursache für Zitterbewegung zu interpretieren sind.

# Contents

# Chapter 1

# Introduction

The main topics of this thesis are concerned with semiclassical studies of systems that possess different types of degrees of freedom. Therefore, we should start with an explanation of what is meant by “semiclassics” and “semiclassical”, and we would like to begin this introduction with

> “Semiclassics is ... ”.

However, a statement of the above type in general is not that easy, since one certainly needs some more information about the underlying physical system in order to give a correct answer. What is true in general is that

> “Semiclassics is concerned with the relationship between quantum mechanics and classical mechanics, in particular, the transition from a quantum mechanical to a classical behaviour.”

But this is just synonymous with the phrase *semiclassical*, and the transition alluded to above crucially depends on the method it is performed with. In order to be more precise, we have to specify the quantum mechanical system and a classical system as the result under the transition from a quantum mechanical to a classical description, i.e. in the *semiclassical limit*. Therefore, the most general question we can answer is

> “What is the semiclassical limit?”

Of course, this question can neither be answered in general. Let us start by considering some general situations: a first class of examples is given by a non-relativistic point-particle moving in euclidean space, whose behaviour is described by the Schrödinger equation. In this case we would say

> “It is the limit in which all relevant actions are small compared with Planck’s constant, i.e. it can be represented by $\hbar \to 0$.”

Another typical example is the eigenvalue problem for the Laplacian on a Riemannian manifold, in which case we would say

> "The semiclassical limit is the limit where energies (eigenvalues) become large, i.e. the limit $E \to \infty$."

In some special cases, e.g. where the Schrödinger operator is connected with the Laplace operator via a scaling transformation, the two situations given above coincide and therefore describe the same physical limit. In general, however, they don't.

A further example is given by a spin system, where an answer could be

> "It is the limit where (spin-)quantum numbers become large, i.e. $s \to \infty$."

For a pure spin system this is equivalent to the limit $\hbar \to 0$. But in a general situation all the limits described above correspond to different physical situations and are not obviously related with each other. For example, if we consider a non-relativistic particle with spin moving in euclidean space we have two obvious semiclassical parameters: Planck's constant and the spin quantum number. While the spin quantum number is associated with the spin degrees of freedom, Planck's constant is related to the translational degrees of freedom. In this case we can imagine various semiclassical limits, the extremal ones being given by:

- The spin quantum number is not considered as becoming semiclassical, i.e. is kept fixed, while Planck's constant tends to zero.
- Only the spin quantum number is treated as semiclassical parameter.

And in between:

- Both the spin quantum number and Planck's constant are used as semiclassical parameters. Different realizations of this limit are specified by the behaviour of the product $\hbar s$ when $\hbar \to 0$ and $s \to \infty$.

This suggests as a general recipe that we first have to identify the relevant types of degrees of freedom together with suitable candidates for semiclassical parameters. Then the semiclassical limit can be considered as being described as the limit with respect to these parameters. In particular, we have to specify the relationship between these semiclassical parameters, which together with their choice will in general yield different classical systems as limit.

Thus, one problem we are faced with is that neither the relevant physical degrees of freedom nor the corresponding semiclassical parameters are obvious a priori. Before we can start with any analysis, we have to find physical justifications for a certain choice of the semiclassical parameter: a frequently encountered situation is when one considers a system that in some sense is macroscopic in which case we expect to recover signs of classical behaviour in the quantum description of some objects. E.g. one finds that good approximate solutions to Schrödinger's equation can be generated from classical information when $\hbar$ is small enough and we use $\hbar \to 0$ as the semiclassical limit. In general, for any physical system there is a characteristic size for the relevant quantities such as distances, velocities, energies, actions,... from which we can derive characteristic units adapted to the system. If we consider a certain quantity of the system as a candidate of a semiclassical parameter, then we have to measure this quantity in the characteristic units. Thus, as a criterion for a semiclassical parameter the ratio between the corresponding physical quantity and the characteristic unit for this quantity has to become very small (or very large). If we can identify several quantities as semiclassical parameters also the ratio between them becomes important and describes different physical settings. Situations of this type occur if one considers particles with internal degrees of freedom: While the translational degrees of freedom are modeled classically on the cotangent bundle of the configuration manifold, the phase space of the internal degrees of freedom is given by more general symplectic manifolds, such as coadjoint orbits of Lie groups or Kähler manifolds. The corresponding quantum space will incorporate both the representation of the translational and the internal degrees of freedom. An element of the quantum space therefore indicates the translational configuration of a point particle together with the configuration of its internal degrees of freedom. Therefore, quantum states will take values in the representation space for the internal degrees of freedom[1].

Connected with this is the question how to implement symmetries (and constraints) on the classical and quantum level. In particular, one has to consider the question if there is a connection between classical and quantum mechanical symmetries and how they can be implemented. This immediately leads to another problem: How can we model the quantum as well as the classical system on a mathematical level and in what mathematical terms can we describe the semiclassical limit.

The following example carries the main characteristics that we will be concerned with in a more general setting. We consider a rigid body, which obviously possesses two different types of degrees of freedom: The center of mass can move by translations in $\mathbb{R}^3$ while the internal degrees of freedom are given by the rotations of the body. As we will see shortly, the rotational freedom is the reason for the quantum mechanical Hilbert space to carry an additional structure: The wave functions will take values in a vector space associated with the internal degrees of freedom.

Let us start by considering only the internal degrees of freedom: Consider $A \in \mathrm{SO}(3)$ as giving the configuration of the body in the sense that it defines a map of a reference configuration $K \subset \mathbb{R}^3$ to the current configuration $A(K) \subset \mathbb{R}^3$. The map $A$ takes a reference point $b \in K$ to a current point $Ab$. For the motion of the rigid body $A = A(t)$ becomes time dependent and the velocity of a point $x(t) = A(t)x$ is $\dot{x} = \dot{A}b = \dot{A}A^{-1}x$. Since $A$ is orthogonal, we can write

$$\dot{x} = \dot{A}A^{-1}x = \omega \times x,$$

where we have used that the Lie algebra $(\mathfrak{so}(3), [\cdot,\cdot])$ is isomorphic to $(\mathbb{R}^3, \times)$ such that the infinitesimal action of $\xi \in \mathfrak{so}(3)$ on $x \in \mathbb{R}^3$ is given by $\xi x = \tilde{\xi} \times x$. Here $\tilde{\xi} \in \mathbb{R}^3$ is the image of $\xi \in \mathfrak{so}(3)$ under the above Lie algebra isomorphism. Now $\dot{x}$ defines the *spatial angular velocity vector* $\omega = \widetilde{\dot{A}A^{-1}}$. The corresponding *body angular velocity* is given by

$$\Omega = A^{-1}\omega,$$

[1] At this point we want to remark that we will use the phrases “internal” and “intrinsic” interchangeably.

which is the angular velocity seen in a body fixed frame. The kinetic energy of the rigid body reads

$$T = \frac{1}{2}\int_K \rho(b)\|\dot{A}b\|\,\mathrm{d}b,$$

where $\rho$ is the mass density, and since

$$\|\dot{A}b\| = \|\omega \times x\| = \|A^{-1}(\omega \times x)\| = \|\Omega \times b\|,$$

the kinetic energy is a quadratic function of $\Omega$, which may be rewritten as

$$T = \frac{1}{2}\langle \Omega, \Theta\Omega\rangle$$

with $\Theta$ the *moment of inertia tensor*, defining the kinetic energy quadratic form on $\mathbb{R}^3 \simeq \mathfrak{so}(3)$. This quadratic form can be diagonalized, the corresponding eigenvalues $(I_1, I_2, I_3)$ are called the principal moments of inertia and the associated basis is given by the principal axes. Now the Lagrangian for the rigid body is a function on $\mathrm{T}\,\mathrm{SO}(3) \simeq \mathrm{SO}(3) \times \mathfrak{so}(3)$ given by

$$L(A, \dot{A}) = \frac{1}{2}\int_K \rho(b)\|\dot{A}b\|^2\,\mathrm{d}b,$$

and $A(t)$ satisfies the corresponding Euler-Lagrange equations if and only if $\Omega(t)$, defined by $A^{-1}(t)\dot{A}(t)v = \Omega(t) \times v$ for all $v \in \mathbb{R}^3$, satisfies Euler's equations

$$\Theta\dot{\Omega} = \Theta\Omega \times \Omega,$$

that can be written by introducing the *body angular momentum* $\Pi := \Theta\Omega$ as

$$\dot{\Pi} = \Pi \times \Omega.$$

Moreover, this equation is equivalent to a conservation of the spatial angular momentum $\pi := A\Theta\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi = 0.$$

In terms of $\Pi$, Euler's equations read (using principal axes as coordinates)

$$\begin{aligned}
\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3\\
\dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1\\
\dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2.
\end{aligned}$$

From these equations it follows immediately that $\|\Pi\|$ is conserved, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi\|^2 = 0$. Because of this conservation the evolution in time of any point $\Pi(0)$ is constrained to $\|\Pi(t)\| = \|\Pi(0)\|$. Thus Euler's equations describe a two dimensional dynamical system on an invariant sphere, which is the *reduced* phase space for the rigid body equations, that derive from the Hamiltonian

$$H = T = \frac{1}{2}\langle \Theta^{-1}\Pi, \Theta\Theta^{-1}\Pi\rangle = \frac{1}{2}\langle \Pi, \Theta^{-1}\Pi\rangle. \tag{1.1}$$

In this way the conservation of angular momentum leads to the reduction of the six-dimensional phase space $\mathrm{T}^*\,\mathrm{SO}(3)$ to a phase space with a dimension that is lowered by 4.

Let us look at this reduction mechanism more closely: Recall that the left action

$$L_g:\ \mathrm{SO}(3) \to \mathrm{SO}(3), \quad h \mapsto gh$$

and the right action

$$R_g:\ \mathrm{SO}(3) \to \mathrm{SO}(3), \quad h \mapsto hg^{-1}$$

of $\mathrm{SO}(3)$ on itself yield isomorphisms of the tangent bundle $\mathrm{T}\,\mathrm{SO}(3)$ with $\mathrm{SO}(3) \times \mathfrak{so}(3)$ as follows. Let $v_g \in \mathrm{T}_g\,\mathrm{SO}(3)$ then $L_g^{-1}$ maps $g$ to the identity $e \in \mathrm{SO}(3)$ such that its differential $(L_g^{-1})_*$ can be used to assign $(L_g^{-1})_* v_g \in \mathrm{T}_e\,\mathrm{SO}(3) \simeq \mathfrak{so}(3)$ to $v_g$ in a one-to-one manner. Hence

$$\lambda':\ \mathrm{T}\,\mathrm{SO}(3) \xrightarrow{\sim} \mathrm{SO}(3) \times \mathfrak{so}(3), \quad v_g \mapsto (g, (L_g^{-1})_* v_g)$$

yields a well-defined isomorphism between $\mathrm{T}\,\mathrm{SO}(3)$ and $\mathrm{SO}(3) \times \mathfrak{so}(3)$. Analogously, we can use the right action to obtain an isomorphism

$$\rho':\ \mathrm{T}\,\mathrm{SO}(3) \xrightarrow{\sim} \mathrm{SO}(3) \times \mathfrak{so}(3), \quad v_g \mapsto (g, (R_g)_* v_g).$$

The physical significance of these isomorphisms becomes clear if one notices that

$$\omega(t) = \widetilde{\dot A(t) A(t)^{-1}} = \widetilde{(R_{A(t)})_* \dot A} = \widetilde{\rho'(\dot A(t))},$$

which means that $\rho'$ gives a parameterization in spatial coordinates. Furthermore, since the adjoint action of $\mathrm{SO}(3)$ on its Lie algebra under the identification $\mathfrak{so}(3) \simeq \mathbb{R}^3$ corresponds to the usual action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$, the transformation $\rho' \circ \lambda'^{-1}(g, v) = (g, \mathrm{Ad}_g\, v)$ shows that $\lambda'$ defines a parameterization of $\mathrm{T}\,\mathrm{SO}(3)$ in body coordinates. If we consider the cotangent bundle $\mathrm{T}^*\,\mathrm{SO}(3)$ instead of the tangent bundle, i.e. momenta instead of velocities, we can dualize the above trivializations to obtain isomorphisms

$$\lambda:\ \mathrm{T}^*\,\mathrm{SO}(3) \xrightarrow{\sim} \mathrm{SO}(3) \times \mathfrak{so}(3)^*, \quad p_g \mapsto (g, (L_g)^* p_g)$$

in body coordinates and

$$\rho:\ \mathrm{T}^*\,\mathrm{SO}(3) \xrightarrow{\sim} \mathrm{SO}(3) \times \mathfrak{so}(3)^*, \quad p_g \mapsto (g, (R_g^{-1})^* p_g).$$

in spatial coordinates.

By using these isomorphisms Hamiltonians and Hamiltonian vector fields that are invariant with respect to the group action are mapped to Hamiltonians and Hamiltonian vector fields on $\mathfrak{so}(3)^*$. We say that the original system on $\mathrm{T}^*\,\mathrm{SO}(3)$ has been reduced to $\mathfrak{so}(3)^*$, which has a Lie-Poisson structure defined by the Poisson bracket (under the identification $\mathfrak{so}(3) \simeq \mathbb{R}^3$)

$$\{f, g\}(\Pi) = -\Pi(\operatorname{grad} f \times \operatorname{grad} g),$$

for all $f,\, g \in C^\infty(\mathfrak{so}(3)^*)$. Thus Euler's equations are equivalent to

$$\dot{f} = \{f, H\}$$

when $H$ is the rigid body Hamiltonian (1.1). The action of $\mathrm{SO}(3)$ on itself can be lifted to an action of $\mathrm{SO}(3)$ on its cotangent bundle $\mathrm{T}^*\,\mathrm{SO}(3)$, which is symplectic with respect to the canonical symplectic structure. This cotangent lift can explicitly be calculated in body coordinates as

$$(\lambda \circ (L_g)^* \circ \lambda^{-1})(h, \mu) = (g^{-1}h, \mu)$$

and in spatial coordinates

$$(\rho \circ (L_g)^* \circ \rho^{-1})(h, \mu) = (g^{-1}h, \operatorname{Ad}_g^* \mu).$$

Under the identification $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ we thus recover the rotation of the rigid body, $\widetilde{\operatorname{Ad}_g^* \mu} = g\tilde{\mu}$. The cotangent lift of the action is not just symplectic but also Hamiltonian. In fact, the infinitesimal action of $\dot{A}(t) \in \mathfrak{so}(3)$ at $(g, \mu) \in \mathrm{T}_g^*\,\mathrm{SO}(3)$ in spatial coordinates is given by

$$J(g, \mu)(\dot{A}) = \langle \mu, \dot{A} \rangle.$$

Thus, for any $(g, \mu) \in \mathrm{T}^*\,\mathrm{SO}(3)$ we can define a map

$$J :\ \mathrm{T}^*\,\mathrm{SO}(3) \to \mathfrak{so}(3)^*$$

that in spatial coordinates is given by $J(g, \mu) = \mu$. For any $\dot{A} \in \mathfrak{so}(3)$ the corresponding infinitesimal action is generated by the Hamiltonian $\langle J, \dot{A} \rangle$. The above map is called the *moment map* for the $\mathrm{SO}(3)$ action on $\mathrm{T}^*\,\mathrm{SO}(3)$; the reason for this notion becomes clear, when one realizes that it is directly related with angular momentum in the case of the rigid body. If we have a Hamiltonian $H :\ \mathrm{T}^*\,\mathrm{SO}(3) \to \mathbb{R}$ that is left-invariant, i.e. $H(g^{-1}h, \operatorname{Ad}_g^* \mu) = H(h, \mu)$ for all $g \in \mathrm{SO}(3)$, then, as a consequence of Noether's theorem (see e.g. [AM78, MR94]), $J$ is constant on the orbits of the left-invariant Hamiltonian vector field $\mathrm{X}_H$ corresponding to $H$. Now, the rigid body Hamiltonian (1.1) given in body coordinates is left-invariant, since it only depends on the angular momenta. Thus, conservation of angular momentum is just a restatement of the fact that the moment map is constant on the orbits of a left-invariant vector field. Also the reduction to the invariant spheres can be described in terms of the moment map. It is immediate from its definition that the moment map intertwines the coadjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$ and the action on $\mathrm{T}^*\,\mathrm{SO}(3)$. Thus, if $G_\mu$ denotes the isotropy group of $\mu \in \mathfrak{so}(3)^*$, which we may identify with $\tilde{\mu} \in \mathbb{R}^3$, then $G_\mu$ also leaves invariant $J^{-1}(\mu) \in \mathrm{T}^*\,\mathrm{SO}(3)$ and we have that the reduced space $J^{-1}(\mu)/G_\mu$ is given by the coadjoint orbit $\mathcal{O}_\mu$ through $\mu \in \mathfrak{so}(3)^*$, which we again may identify with an invariant sphere in $\mathbb{R}^3$. This is a very special case of a reduction procedure due to Marsden and Weinstein [MW74], which also ensures that the reduced space is symplectic; in the case of the rigid body the symplectic form is given by (a multiple of) the area two-form.

So far we only have been concerned with the rotational degrees of freedom of the rigid body. In addition, translational degrees of freedom arise if we allow the body to move in euclidean space $\mathbb{R}^3$, such that the configuration space is taken to be $\mathbb{R}^3 \times \mathrm{SO}(3) =: P$ with corresponding cotangent bundle $\mathrm{T}^*P \simeq \mathrm{T}^*\mathbb{R}^3 \times \mathrm{T}^*\,\mathrm{SO}(3)$ which, by using body or space coordinates, is identified with $\mathrm{T}^*\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathfrak{so}(3)^*$. Since $\mathrm{SO}(3)$ acts trivially on the factor $\mathbb{R}^3$ the existence of a moment map for the cotangent lift of the action is immediate. It is given by

$$J(x, \xi, p_g)(\zeta) = \langle p_g, \zeta_P \rangle, \quad p_g \in \mathrm{T}^*\,\mathrm{SO}(3),$$

where $\zeta_P$ denotes the infinitesimal cotangent-action of $\zeta \in \mathfrak{so}(3)$ on $\mathrm{T}^*P$, and the corresponding reduced space $(\mathrm{T}^*P)_\mu = J^{-1}(\mu)/G_\mu$ is given by

$$\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\mu \simeq \mathrm{T}^*\mathbb{R}^d \times \mathrm{S}^2,$$

if we again identify $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$. Furthermore, according to the properties of the reduction procedure, $(\mathrm{T}^*P)_\mu$ is endowed with a symplectic structure given by

$$\omega = \omega_{\mathrm{T}^*\mathbb{R}^d} + \omega_{\mathcal{O}_\mu},$$

where $\omega_{\mathrm{T}^*\mathbb{R}^d}$ denotes the canonical symplectic structure of $\mathrm{T}^*\mathbb{R}^d$ and $\omega_{\mathcal{O}_\mu}$ the symplectic structure of the coadjoint orbit that is symplectomorphic to $\mathrm{S}^2 \subset \mathbb{R}^3$ equipped with the area two-form $\omega_{\mathrm{S}^2}$.

Let us now consider the behaviour of the dynamics under reduction: we start with a free motion and suppose that we are given a Riemannian metric on $\mathbb{R}^3$ as well as an invariant metric on $\mathrm{SO}(3)$, e.g. the one defined by the body's inertia tensor. If, in addition, we are given a connection on $P = \mathbb{R}^3 \times \mathrm{SO}(3)$ these metrics can be used to define an invariant metric[2] on P. We define the Hamiltonian function as the kinetic energy associated with the Riemannian metric on $P$, which therefore is $\mathrm{SO}(3)$ invariant and produces a reduced Hamiltonian on $(\mathrm{T}^*P)_\mu$. The corresponding Hamiltonian vector field on $(\mathrm{T}^*P)_\mu$ can be obtained by a restriction to $J^{-1}(\mu)$ and subsequent projection $J^{-1}(\mu) \to (\mathrm{T}^*P)_\mu$. Let us remark that this is not only true for the free Hamiltonian but also for any invariant Hamiltonian on $\mathrm{T}^*P$. If we introduce the Hamiltonian

$$H(x, \xi, u) = \frac{1}{2m}\|\xi - e\boldsymbol{A}(x)\|^2 + V(x) + \mu s u \cdot \boldsymbol{B}(x) \tag{1.2}$$

[2]In fact, there is a bijective correspondence between invariant Riemannian metrics on a principal fiber bundle $P \to M$ and triples consisting of a Riemannian metric on the base $M$, an (bi-)invariant metric on the structure group $G$ and a connection on the principal fiber bundle, see [Lan98a, Zel92, ST89, ST84, HPS83]. An explicit construction of the connection starting from an invariant metric is given by the so-called mechanical connection, see [Mar92, MMPR88, MRW84, Mar94].

for a magnetic field $\boldsymbol{B}$ on $\mathbb{R}^3$ with corresponding potential $\boldsymbol{A}$, then Hamilton's equations are the equations of motion for a spinning particle with charge $e$ and magnetic moment $\mu$. This Hamiltonian can also be thought of as the reduction of an invariant Hamiltonian on $\mathrm{T}^*P$ given by the pull-back of $H$ to $\mathrm{T}^*P$ under the projection $\mathrm{T}^*P \to (\mathrm{T}^*P)_\mu$. If we turn to the quantum mechanical description of the rigid body, according to general principles we expect the $\mathrm{SO}(3)$-invariance to be reflected also in the quantum mechanical description. Therefore, the quantum space corresponding to the classical phase space $\mathrm{T}^*\mathbb{R}^3 \times \mathcal{O}_\mu$ should carry a (projective) unitary representation of the symmetry group $\mathrm{SO}(3)$. In the case at hand, the quantum space $\mathscr{H}$ for the rigid body can be modeled as $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{H}_{\mathrm{S}^2}$, where $L^2(\mathbb{R}^d)$ represents the quantization of the translational degrees of freedom and $\mathscr{H}_{\mathrm{S}^2}$ the quantum space corresponding to the rotational degrees of freedom described on the coadjoint orbit $\mathcal{O}_\mu \simeq \mathrm{S}^2$ of $\mathrm{SO}(3)$. It is a famous result due to Kirillov [Kir62] that the quantum space associated to a coadjoint orbit is given by the unitary irreducible representations of the respective group. In particular, if the orbit is compact these spaces are finite dimensional. Concerning the Hilbert space $\mathscr{H}$ we therefore are concerned with square integrable functions on the translational configuration space, which take values in the representation space corresponding to the symmetry group.

Before generalizing these considerations to arbitrary principal bundles, we consider our example from a slightly different point of view: Let us write $\mathbb{R}^3 = M$, $G = \mathrm{SO}(3)$, $P = \mathbb{R}^3 \times G$ and construct the fiber product $P \times_M \mathrm{T}^*M$, which is is a fiber bundle over the base manifold $M$ defined by

$$P \times_M \mathrm{T}^*M = \{(p, \alpha);\, \pi_P(p) = \pi_M(\alpha)\},$$

such that the projection $\pi : P \times_M \mathrm{T}^*M \to M$ is given by $\pi(p, \alpha) = \pi_P(p) = \pi_M(\alpha)$ and the typical fiber is the product of the fibers of $P$ and $\mathrm{T}^*M$. Then we have a $G$ equivariant diffeomorphism (e.g. defined by spatial coordinates)

$$\mathrm{T}^*P \simeq (P \times_M \mathrm{T}^*M) \times \mathfrak{g}^* \tag{1.3}$$

and the moment map for the cotangent lift of the $G$ action is given by

$$J(p_m, \alpha_m, \eta) = \eta. \tag{1.4}$$

Furthermore the reduced space $J^{-1}(\mu)/G_\mu$ is isomorphic to $J^{-1}(\mathcal{O}_\mu)/G$, which in turn is diffeomorphic to the associated bundle $(P \times_M \mathrm{T}^*M) \times_G \mathcal{O}_\mu$. These are precisely the symplectic leaves of $(\mathrm{T}^*P)/G \simeq (P \times_M \mathrm{T}^*M) \times_G \mathfrak{g}^*$. Now the principal bundle $P \to M$ is (isomorphic to) the frame bundle $\mathrm{B}(M)$ of the base manifold $M$ and we can rephrase the above constructions as follows: The principal bundle $P \to M$ is given as the frame bundle $\mathrm{B}(M)$ of the manifold $M$, and the reduced cotangent bundle $(\mathrm{T}^*\mathrm{B}(M))/G \simeq (P \times_M \mathrm{T}^*M) \times_G \mathfrak{g}$ is the same as the associated bundle to the pull-back bundle $P \to \mathrm{T}^*M$ with respect to the canonical projection $\mathrm{T}^*M \to M$. This point of view immediately reveals the relevance of *spin structures* that in principle are double coverings of the frame bundle structure: as mentioned above, in the quantization process we have to construct projective representations of the structure group $G = \mathrm{SO}(3)$ that can be obtained by using representations of its double covering $\mathrm{SU}(2) = \mathrm{Spin}(3)$, see e.g. [BGV92, Roe88, LM89, Fri97, Gil95, VGB93, GVF01]. Given a representation of the spin group Spin(3) we can construct the associated spinor bundle, whose sections can be seen as vector valued and equivariant functions on the Spin(3) principal fiber bundle, i.e. the double covering of the frame bundle. Thus, the vector character encodes the spin degrees of freedom on the quantum level. A full quantum description can finally be obtained if one also quantizes the degrees of freedom described on $\mathrm{T}^*M$.

The above situation on a more general level is thus given by a principal fiber bundle $P \to M$ with structure group $G$, which corresponds to the internal degrees of freedom. In analogy to the previous discussion, the reduction process associates a symplectic manifold to both the translational degrees of freedom, described on the base manifold's cotangent bundle $\mathrm{T}^*M$, and the internal degrees of freedom in terms of coadjoint orbits. Also in this general setting we will recover the non-scalar structure of the quantum space. Let us consider the quantization of classical observables: since the canonical Poisson structure of $\mathrm{T}^*P$ is invariant with respect to the cotangent action of $G$ there is an induced Poisson structure on $(\mathrm{T}^*P)/G$, and the associated Poisson algebra[3] is the algebra of observables describing a particle moving on $P/G$. By definition, a representation of this Poisson algebra is a linear map $\rho$ from the algebra to the space of smooth and real valued functions on a symplectic manifold such that $\rho(f \circ g) = \rho(f)\rho(g)$ and $\rho(\{f, g\}) = \{\rho(f), \rho(g)\}$, where the latter Poisson bracket is induced by the symplectic structure. We can use the equivariant diffeomorphism (1.3)

$$\mathrm{T}^*P \simeq P \times_M \mathrm{T}^*M \times \mathfrak{g}^* \tag{1.3}$$

and the moment map (1.4)

$$J : \mathrm{T}^*P \to \mathfrak{g}^*$$

also in the case of general principal bundles and obtain that each symplectic leaf of the Poisson manifold $(\mathrm{T}^*P)/G$ is given by a reduced space $(\mathrm{T}^*P)_\mu = J^{-1}(\mu)/G_\mu \simeq J^{-1}(\mathcal{O}_\mu)/G = (\mathrm{T}^*P)_{\mathcal{O}_\mu}$. If, in addition, we are given a connection on $P$ this induces a diffeomorphism, see [Wei77a, Mon84],

$$(\mathrm{T}^*P)/G \simeq P \times_M \mathrm{T}^*M \times_G \mathfrak{g}^*,$$

and

$$(\mathrm{T}^*P)_{\mathcal{O}_\mu} \simeq P \times_M \mathrm{T}^*M \times_G \mathcal{O}_\mu$$

seen as fiber bundles over $\mathrm{T}^*M$ associated with the principal $G$ bundle $P \times_M \mathrm{T}^*M$ by the coadjoint action on $\mathfrak{g}^*$, see Appendix B for more details on these constructions. According to a general result, up to equivalence each irreducible representation of a Poisson algebra is given by the restriction to the symplectic leaves of the underlying Poisson manifold (see e.g. [Lan98a]). We therefore know that (up to equivalence) each irreducible representation of the Poisson algebra $C^\infty((\mathrm{T}^*P)/G)$ is realized on the symplectic manifolds of the type $(\mathrm{T}^*P)_{\mathcal{O}_\mu}$.

[3] i.e. the algebra of smooth functions on $\mathrm{T}^*P/H$ together with their composition and the Poisson bracket.

In order to see how the notion of gauge invariance is connected with the reduction procedure, we first consider the automorphism group $\mathrm{Aut}(P)$ of $P$. It consists of those diffeomorphisms of $P$ that commute with the right action of $G$ on $P$. Thus, under the bundle projection $\pi : P \to M$ an automorphism $\phi$ defines an diffeomorphism $\phi_M := \pi \circ \phi$ of $M$. The kernel of the map $\phi \mapsto \phi_M$ is given by the gauge group $\mathrm{Gau}(P)$. The Lie algebra $\mathfrak{gau}(P)$ of $\mathrm{Gau}(P)$ is isomorphic to the space of sections of $P \times_G \mathfrak{g}$, i.e. the bundle associated to $P$ by the adjoint action. Therefore for trivial bundles $P = M \times G$ the Lie algebra $\mathfrak{gau}(P)$ is given by $C^\infty(M, \mathfrak{g})$. A local section $s : M \to P$ defines a local trivialization $\psi_s : P \to M \times G$ by letting $\psi_s(s(m)) = (m, e)$ and extending by equivariance $\psi_s(s(m)g) = (m, g)$. Thus, a gauge transformation $g$ induces a local diffeomorphism $g^s : M \times G \to M \times G$ according to $g^s \circ \psi_s = \psi_s \circ g$, i.e.

$$g^s(m, h) = (m, g(s(m))h).$$

Furthermore, the gauge transformation defines a new section $s_g(m) := s(m)g(s(m))$. Two sections $s_\alpha$ and $s_\beta$ are related according to

$$s_\beta(m) = s_\alpha(m) g_{\alpha\beta}(m),$$

where $g_{\alpha\beta} : M \to G$ denotes the transition function. The local representations of a connection form $A$ on $P$ then transform according to

$$s_\beta^* A = \mathrm{Ad}_{g_{\alpha\beta}} s_\alpha^* A + g_{\beta\alpha} \, \mathrm{d} g_{\alpha\beta}.$$

Analogously to the action of $G$ on $P$ the action of the gauge group can be lifted to an action on $\mathrm{T}^*P$ that commutes with the $G$-action and thus defines an action of the gauge group on $(\mathrm{T}^*P)_{\mathcal{O}_\mu}$. In local coordinates, where $(\mathrm{T}^*P)_{\mathcal{O}_\mu} \simeq \mathrm{T}^*M \times \mathcal{O}_\mu$, one can write this action as

$$(z_m, \eta) \mapsto (z_m + \eta(\mathrm{d}g^{-1}(s(m))g(s(m))), \mathrm{Ad}^*_{g(s(m))} \eta),$$

where $z_m \in \mathrm{T}^*M$ and $\mathrm{d}g(s(m))g(s(m))$ is a one-form with values in $\mathfrak{g}$. A one-form $\gamma^A$, which also depends on the connection $A$, is said to be gauge covariant, if it is changed to $\gamma^{g^*A}$ under the above gauge transformation. For example, the covariant momentum (compare with (1.2))

$$\xi^A(z_m, \eta) = z - \eta(s^*A(m))$$

is gauge covariant. If we consider $(\mathrm{T}^*P)_{\mathcal{O}} \simeq P \times_M \mathrm{T}^*M \times_G \mathcal{O}$, then the (reduced) action of the gauge group simply reads

$$[x, \xi, \eta]_G \mapsto [x, \xi, \mathrm{Ad}^*_{g(x)} \eta]_G,$$

upon representing a point in $(\mathrm{T}^*P)_{\mathcal{O}_\mu}$ by a representative $(x, \xi, \eta)$ in $P \times_M \mathrm{T}^*M \times \mathcal{O}$. Thus, if $\mathrm{g}$ denotes a Riemannian metric on $M$ then the Hamiltonian $h([x, \xi, \eta]_G) := \frac{1}{2}\mathrm{g}^{-1}(\xi, \xi)$ is gauge-covariant and the equations of motion, which result if we use the symplectic structure on $(\mathrm{T}^*P)_{\mathcal{O}}$, are called *Wong's equations* [Won70]. These equations describe the motion of a charged particle in the external gauge field $A$.

If we want to obtain a quantum mechanical description of the above structures, we have to construct a representation of the Poisson algebra $C^\infty((\mathrm{T}^*P)/G, \mathbb{R})$ or equivalently of the space $C^\infty(\mathrm{T}^*P, \mathbb{R})^G$ of the $G$ invariant functions on $\mathrm{T}^*P$ on a Hilbert space. That means, we want to obtain a quantization of the classical observables. We could start by defining the Hilbert space

$$\mathscr{H} = L^2(P) = L^2(P, \mu),$$

where we suppose that we have an $G$ invariant measure[4] $\mu$ on $P$. The Hilbert space $\mathscr{H}$ carries a representation $\mathrm{U}_R$ of $G$ given by

$$\mathrm{U}_R(g)\psi(x) = \psi(xg).$$

We can define the C*-algebras $\mathrm{B}_0(\mathscr{H})^G$ and $\mathrm{B}(\mathscr{H})^G$ of compact and bounded operators on $L^2(P)$, respectively, that commute with each $\mathrm{U}_R(g)$, $g \in G$. In general we may think of $\mathrm{B}(L^2(P))$ as the quantization of the Poisson algebra $C^\infty(\mathrm{T}^*P, \mathbb{R})$. By averaging over the group $G$, which we assume to be compact, we can construct from a given quantization on $\mathrm{T}^*P$, such as Weyl quantization, a quantization $C^\infty((\mathrm{T}^*P)/G, \mathbb{R}) \to \mathrm{B}(L^2(P))^G$. Let us consider the structure of $\mathrm{B}(L^2(P))^G$ more closely. To this end notice that the invariant measure $\mu$ on $P$ gives a measure $\nu$ on $M$ satisfying

$$\int_P f(x)\,\mathrm{d}\mu(x) = \int_M \mathrm{d}\nu(m) \int_G f(s(m)g)\,\mathrm{d}g$$

for any $f \in L^1(P)$ and any measurable section $s :\ M \to P$. Furthermore, each such section gives a trivialization of $P$ and thus leads to a unitary transformation

$$U_s :\ L^2(P) \to L^2(M \times G) \simeq L^2(M) \otimes L^2(G),$$

defined by $U_s\psi(m, g) = \psi(s(m)g)$. Consider the space $\mathrm{B}_2(L^2(P))^G$ of $G$-invariant Hilbert-Schmidt operators on $L^2(P)$ whose elements can be characterized by a kernel $K \in L^2(P \times P)^G$ satisfying

$$K(xg, yg) = K(x, y).$$

We can then construct a map $L^2(P \times P)^G \to L^2(M \times M \times G)$ by sending $K(x, y)$ to $K(s(m)g, s(m'))$ and we identify $L^2(M \times M \times G)$ with $\mathrm{B}_2(L^2(M)) \otimes L^2(G)$, where $\mathrm{B}_2(L^2(Q))$ is a subspace of $\mathrm{B}(L^2(M))$. This can be used to show that each measurable section $s : M \to P$ determines an isomorphism $\mathrm{B}_0(L^2(P))^G \simeq \mathrm{B}_0(L^2(M)) \otimes \mathrm{C}^*(G)$. Since there is a correspondence between representations of $\mathrm{C}^*(G)$ and representations of $G$ we obtain that there is a bijective correspondence between the irreducible representations of the Poisson algebra $C^\infty(\mathrm{T}^*P/G)$ and irreducible representations of $G$. If we have a representation $(\pi, V)$ of $G$ the representations of this Poisson algebra can be realized on

$$\mathscr{H}_\pi = L^2(M) \otimes V.$$

[4] Such a measure can e.g. be obtained from an $G$ invariant Riemannian metric g on $P$.

Therefore, if we want to construct a quantization of the Poisson algebra $C^\infty((\mathrm{T}^*P)/G, \mathbb{R})$ we could start from an irreducible representation $(\pi, V)$ of the group $G$ and obtain a full quantization by also quantizing the cotangent bundle $\mathrm{T}^*M$, e.g. by Weyl quantization. Let us also remark that the above formalism is a generalization of the induced quantization method, dealing with homogeneous $G$-principal bundles $H \to H/G$, with $G$ a subgroup of $H$. According to Mackey's theory of induced representations [Mac68, Var68, Var70, Sim68] there are many inequivalent representations of $H$, and these are labeled by the irreducible representations of the subgroup $G$. As shown by Landsman [Lan90c, Lan90d, Lan90a, Lan90b] and Linden [LL91, LL92] this situation fits in the above formalism since the different sectors, i.e. the inequivalent representations, come equipped with a specific type of Yang-Mills field, see also [MT95] for an overview. What is characteristic also in this setting is that there occur vector valued functions rather than the usual scalar valued ones.

The semiclassical properties of the Laplacian on gauge fields have been extensively studied, see e.g. [GU90, ST84, ST89, HPS83, TU92, Wu98]. In our considerations we will adopt the more general point of view that we are given a quantum mechanical system whose Hilbert space is given as the product $\mathscr{H}_M \otimes \mathscr{H}_{\mathrm{int}}$, where the second factor is a finite-dimensional Hilbert space and $\mathscr{H}_M$ is the Hilbert space associated with a Riemannian manifold $M$, i.e. $\mathscr{H}_M \simeq L^2(M)$. All the situations mentioned above fall into this class. In a first step we only treat the degrees of freedom described on $\mathrm{T}^*M$ semiclassically while we leave the quantum character of the *internal degrees of freedom* described on $\mathscr{H}_{\mathrm{int}}$ unchanged. This leads us to consider hermitian vector bundles over $\mathrm{T}^*M$, whose characteristic fibers are given by $\mathscr{H}_{\mathrm{int}}$. Thus we have to deal with the quantization of functions taking values in the morphisms of these vector bundles.

In Chapter 2, after a short digression to general (required) properties of a quantization, we give an introduction to matrix-valued microlocal analysis. In particular, we concentrate on matrix-valued pseudodifferential calculus which is used to model (semiclassical) observables. The methods of semiclassical (or microlocal) analysis are used as a basis for the description of the semiclassical behaviour of the translational degrees of freedom, thought of as coordinate functions on a symplectic manifold. The internal degrees of freedom, described as components in a general (not further specified) hermitian space, are still being treated as quantum objects. Therefore, the scalar microlocal analysis is "tensored" with spaces of matrices, i.e. the semiclassical quantities are no longer scalar valued but take values in spaces of matrices. For these objects we primarily study how the time evolution in the quantum mechanical systems is reflected in the classical counterpart. In particular, we will consider the classical analogue of a time evolved quantum observable, which we will show to be well-defined only if one restricts to observables that possess a classical meaning. Since the quantum nature of observables is reminiscent in this analysis, not all quantities occuring have a classical interpretation. We will encounter subspaces in the quantum Hilbert space that are (semi-)classically decoupled and construct semiclassical projection operators that establish this splitting in "semiclassical sectors". Only observables that respect the splitting of the quantum space into these subspaces, i.e. observables that do not induce transitions from one sector to the other, have a classical meaning and therefore a semiclassically well-defined time evolution. Our main result connected with this situation is an Egorov theorem for matrix-valued pseudodifferential operators stated in Section 2.3.2. The second main result that we will prove in Chapter 2 is concerned with the mean semiclassical behaviour of eigenfunctions and gives another hint to the fact, that transition between different subspaces are non-semiclassical: if we consider general quantum observables, then, in a spectral mean, the observable can be replaced by their restriction to the different subspaces. Thus the mean expectation values of observables taken in eigenstates of the quantum Hamiltonian are given by the expectation values taken in the eigenstates that are projected to each sector. Furthermore, since the expectation values of the restricted observables can be calculated from classical quantities, also the expectation values of the full observables can be expressed by using classical objects. The precise statement is given by a limit formula of Szegö-type.

In Chapter 3 we set up the formalism we will use to provide a classical description of the internal degrees of freedom, represented by the matrix-valued operators on the finite dimensional Hilbert space $\mathscr{H}_{\mathrm{int}} \simeq \mathbb{C}^n$. Since this formalism is mainly based on the techniques of geometric quantization, we start by outlining its basic tools and results. In this context we consider a symplectic manifold and construct a hermitian line bundle with connection, the *pre-quantum line bundle*. The states are given by sections of these line bundles that are covariantly constant in the direction of a polarization, i.e. a subbundle of the (complexified) tangent bundle. Thus the quantum space, i.e. the space of covariantly constant sections, depends on the subbundle chosen. Different quantum spaces are related through a non-degenerate pairing given by the *Blattner-Kostant-Sternberg* kernel, on which we will focus in Section 3.4. The pairing between the different Hilbert spaces also allows for the construction of operators: If one lifts the action of symplectic transformations to morphisms of the pre-quantum line bundle over the symplectic manifold, and also considers the transformation of the polarization induced by the symplectic transformation, the infinitesimal version of the combined transformations of the pre-quantum line bundle and the polarization gives the quantum operator. This then is a differential operator acting on sections of the line bundle (more precisely, sections that take values in the half-form bundle, i.e. the line bundle associated with the bundle of metalinear frames for the polarization). A particular case arises when the underlying symplectic manifold is a coadjoint orbit of a compact Lie group. In this case the quantum spaces correspond to irreducible representation spaces of the Lie group, and the sections in the pre-quantum bundle are closely related to Perelomov's coherent states. We also discuss the lift of symplectic transformations which generates the quantum time evolution of coherent states. Furthermore, coherent states yield a tool for a quantum-classical correspondence between representation operators of the Lie group and classical objects, in this case functions on the respective coadjoint orbits, leading to Berezin's quantization. Using Berezin's quantization we can construct a Weyl-type quantization for the intrinsic degrees of freedom, which has additional useful properties.

In Chapter 4 another typical representative of symplectic manifolds, the cotangent bundle $\mathrm{T}^*\mathbb{R}^d$ of euclidean space, is considered. For this particular base manifold we perform the geometric quantization procedure and choose a class of polarizations that are associated with a subset of the complex symmetric matrices. Covariantly constant sections with respect to these polarizations turn out to be closely related to the coherent states for the Heisenberg group under the pairing between the quantum spaces corresponding to the polarizations described above and the vertical polarization corresponding to the Schrödinger representation. For these coherent states we discuss the relation between the classical and the quantum mechanical time evolution generated by quadratic Hamiltonian functions on the classical level, and the corresponding Weyl operators on the quantum level. Combining these results and the ones from the preceeding Chapter, we construct quantization data for the case of a symplectic manifold given as the product of a cotangent bundle and a coadjoint orbit: We use the product of the pre-quantum structures on this product manifold together with the product polarizations and show that the we can obtain covariantly constant sections as the product consisting of the coherent states for the Heisenberg group and the coherent states for the Lie group. This provides us with an explicit construction of covariantly constant sections of the pre-quantum line bundles. Since the product manifolds are the symplectic leaves of $(\mathrm{T}^*P)/G$, where $P = \mathbb{R}^d \times G$ is the trivial $G$-bundle over $\mathbb{R}^d$, this also yields an explicit construction for the geometric quantization procedure on principal bundles, which is also described in [Rob96b, Rob96a]. As a further consequence we obtain results on the semiclassical time evolution of these sections.

While these results deal with an exact classical propagation of quantum mechanical states for a very special class of Hamiltonians, Chapter 5 deals with an extension to more general dynamical behaviour: for general Hamiltonians (composed of a scalar term and a representation operator) we show how to approximate the quantum time evolution by the classical one with an accuracy of arbitrarily high order in the semiclassical parameter(s). This discussion is separated into two different semiclassical scenarios: In the first one only the translational degrees of freedom are treated in a semiclassical manner; we consider the case that the semiclassical parameter $\hbar$ associated with the translational degrees of freedom tends to zero, while the semiclassical parameter for the internal degrees of freedom, which is closely related to the dimension of the space on which the internal degrees of freedom are represented, is fixed. In the second scenario we assume that the latter parameter tends to infinity as $\hbar$ goes to zero; more precisely, we assume that the product of both parameters is constant. As a result, we find that in both scenarios the quantum mechanical time evolution of coherent states may be approximated by semiclassically localized states that are propagated under the classical time evolution. However, there is a restriction on these results: We can only show the approximation to be well-defined for times smaller than the so-called *Ehrenfest time*. This is a time scale $T(\hbar)$, such that $T(\hbar) \to \infty$ as $\hbar \to 0$, that depends on the stability of the respective trajectory of the classical flow.

In Chapter 6 we turn again to the more general situations already described in Chapter 2. We do no longer assume that the Lie group describing the internal degrees of freedom is explictly given nor that the quantum Hamiltonian is composed of a scalar part and a representation operator. We thus again deal with operators that take values in the morphisms of (finite-dimensional) hermitian vector bundles. However, the semiclassical projection operators provide a method of (partially) reducing this situation to the previous, simpler one: By projecting the quantum Hamiltonian to the semiclassically invariant subspaces of the full Hilbert space we obtain restricted Hamiltonians that contain a scalar leading order term and a contribution from the unitary group (or a subgroup hereof) corresponding to the hermitian structure of the invariant subspace. The degrees of freedom described in these subspaces can be given a classical meaning through the Moyal-Weyl formalism developed in Chapter 3, without assuming that these degrees of freedom become semiclassical. In terms of this description we can reformulate the Egorov theorem and the Szegö limit formula, obtained in Chapter 2, and employ this to a generalization to the matrix valued setting of a well-established result in the area of quantum chaos: The quantum ergodicity theorem, originally due to [Shn74, CdV85, Zel87, HMR87]. The original version of this theorem is concerned with distributions on the algebra of classical observables, defined by the quantum states $\psi_\hbar$ according to the map $B \mapsto \langle \psi_\hbar, \mathcal{B}\psi_\hbar \rangle$, where $\mathcal{B}$ denotes the quantization of the classical observable $B \in C^\infty(\mathrm{T}^*M)$. As a consequence of the (scalar) Egorov theorem [Ego69] these distributions weakly converge to probability measures on phase space that are invariant under the classical time evolution and are called *quantum limits* [Zel90]. According to a hypothesis due to Berry [Ber83] quantum limits should be given by classical probability measures on invariant sets in phase space. For classically ergodic systems, e.g., the quantum limits should be given as the Liouville measure on the energy shell. In fact, quantum ergodicity yields a mathematical proof of this assertion in the sense that it shows that the phase space lifts of "almost all" eigenfunctions of the quantum Hamiltonian converge to Liouville measure. This statement is generalized to the case of matrix-valued Hamiltonians: We show that "almost all" of the projected eigenfunctions tend towards an invariant measure which consists of the Liouville measure on an energy shell and the volume measure on the coadjoint orbit. Moreover, this measure is uniquely associated with the considered semiclassical subspace.

Finally, in Chapter 7, the results of Chapter 2 and Chapter 6 are applied to the Dirac equation. In particular, the physical significance of the semiclassical projection operators and the invariant algebra of observables with regard to *Zitterbewegung* is explained.

# Chapter 2

# Matrix-valued microlocal analysis

## 2.1 Prelude: Functorial properties of quantization

Semiclassical analysis is the fundamental tool in the study of the relationship between classical and quantum mechanics. This section is devoted to the description of the classical-quantum connection in a conceptional manner. In classical mechanics the underlying fundamental object is the phase space, i.e. a symplectic manifold $(X, \omega)$. On the quantum mechanical level this corresponds to a (pre-)Hilbert space $\mathscr{H}_X$. Furthermore, the Poisson algebra $C^\infty(X)$ of smooth functions on $X$ with the Poisson structure induced by the symplectic structure $\omega$ is associated with the symmetric operators on $\mathscr{H}_X$. This correspondence is given by a quantization map

$$C^\infty(X) \ni f \longmapsto \mathrm{op}[f],$$

which gives a linear map from the Poisson algebra $C^\infty(X)$ to a $*$-algebra $\mathfrak{A}$ of operators. Originally, the following properties, the so-called *Dirac axioms* [Dir58], were postulated:

**Definition 2.1.1.** A linear map $\mathrm{op} : C^\infty(X) \to \mathfrak{S}(\mathscr{H}_X)$ from the Poisson algebra to the symmetric operators on $\mathscr{H}_X$ is called a *quantization* provided that it satisfies

(i) $\mathrm{op}[1] = \mathrm{id}$.

(ii) $\frac{\hbar}{\mathrm{i}} \mathrm{op}[\{f, g\}] = [\mathrm{op}[f], \mathrm{op}[g]]$.

(iii) For some complete set $f_1, \ldots, f_n$ of functions[1] in involution the operators $\mathrm{op}[f_1], \ldots, \mathrm{op}[f_n]$ form a complete commuting set[2].

It turned out, however, that these requirements are too restrictive and that a quantization of this type does not exist in general[3].

[1] See Definition 3.2.1.
[2] See Definition 3.2.
[3] See the Groenwald-Van Howe Theorem [AM78]

The general approach to this problem is to enlarge the set of classical observables and to relax the criteria for the quantum-classical correspondence, which, of course, led to many different notions of quantization, many of which look related and have similar properties. For now we do not specify a particular quantization but are concerned with the general properties.

Let us replace the classical observables by the groups of which they are the infinitesimal generators, i.e. the classical observables generate Hamiltonian vector fields whose integral curves are symplectomorphisms. These should be represented by unitary operators on a quantum Hilbert space. Consider the class of symplectic manifolds. For two elements $(X, \omega)$ and $(X', \omega')$ we define the *product* as the symplectic manifold $(X \times X', \pi^*\omega + (\pi')^*\omega')$, where $\pi$ and $\pi'$ are the cartesian projections. The *symplectic dual* $\overline{X}$ of an element $(X, \omega)$ is $(X, -\omega)$. Now a diffeomorphism from a symplectic manifold $X$ to a symplectic manifold $X'$ is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $X \times \overline{X}'$, where $\overline{X}'$ denotes the symplectic dual of $X'$. In general, an immersed Lagrangian submanifold of $X \times \overline{X}'$ is called a *canonical relation*. We then define the morphisms $\mathrm{Mor}(X, X')$ to consist of all canonical relations in $X \times \overline{X}'$. Of course, immersed Lagrangian submanifolds of $X \times \overline{X}'$ coincide with those of the symplectic dual, and we therefore have an adjoint canonical relation of $L \in \mathrm{Mor}(X, X')$ as the element $L^* \in \mathrm{Mor}(X', X)$ defined by the same equivalence class of immersions into the product $X' \times \overline{X}$. If composition of morphisms were defined for all $L_1 \in \mathrm{Mor}(X, X')$ and $L_2 \in \mathrm{Mor}(X', Y)$, the class of symplectic manifolds together with the morphisms defined above would be a category[4]. However, consider $X \times \Delta_{X'} \times \overline{Y}$ in $X \times \overline{X}' \times X' \times \overline{Y}$, where $\Delta_{X'}$ denotes the diagonal in $\overline{X}' \times X'$, which is a reducible[5] coisotropic submanifold of $X \times \overline{X}' \times X' \times \overline{Y}$, and the immersed Lagrangian submanifold given as the product $L_1 \times L_2$. Then $L_1 \times L_2$ is called *clean* if it intersects the coisotropic submanifold $X \times \Delta_{X'} \times \overline{Y}$ cleanly. Then we obtain that $L_1 \circ L_2$ is an immersed Lagrangian submanifold of $X \times \overline{Y}$, that is $L_1 \circ L_2 \in \mathrm{Mor}(X, Y)$. Furthermore, associativity of composition of morphisms holds, as soon as they are defined. The elements of the symplectic category have a minimal element given by the zero-dimensional symplectic manifold $Z$ (consisting of a single point) equipped with the null symplectic structure. Morphisms from $Z$ to any other object $X$ naturally identify with immersed Lagrangian submanifolds of $X$. A canonical relation $L \in \mathrm{Mor}(X, Y)$ is called a *monomorphism* if the projection of $L$ onto $X$ is surjective and the projection of $L$ onto $Y$ injective, which implies that $L^* \circ L = \mathrm{id}_X$. Dually, we define $L$ to be an epimorphism if $L^*$ is a monomorphism.

A special feature exists for morphisms between the objects which are cotangent bundles. This is connected with

**Lemma 2.1.2.** *If $M$ and $N$ are smooth manifolds then the map*

$$S_{M,N} : \overline{\mathrm{T}^*M} \times \mathrm{T}^*N \longrightarrow \mathrm{T}^*(M \times N), \quad ((x, \xi), (y, \eta)) \longmapsto (x, y, \xi, -\eta),$$

[4] Neglecting the fact that it is not a proper category it nevertheless is sometimes called the *symplectic category*, see [BW97] and also [Wei77b, Wei71, GU90].

[5] See Appendix A for a definition.

*called the* Schwartz transform*, is a symplectomorphism. Furthermore, let $\theta_M$ and $\theta_N$ be the Liouville one-forms on $M$ and $N$, respectively, then*

$$S^*_{M,N}\theta_{M\times N} = \theta_M \oplus -\theta_N,$$

*and $S_{M,N}$ induces a diffeomorphism of the zero sections of the cotangent bundles*

$$\mathrm{Z}_M \times \mathrm{Z}_N \simeq \mathrm{Z}_{M\times N},$$

*and an isomorphism of the vertical bundles*

$$\mathrm{V}M \oplus \mathrm{V}N \simeq \mathrm{V(M \times N)}.$$

Of course, we can use the Schwartz transform to assign a Lagrangian embedding $\mathrm{i}_F : \mathrm{T}^*M \to \mathrm{T}^*(M \times N)$ corresponding to a symplectomorphism $F : \mathrm{T}^*M \to \mathrm{T}^*N$ as the composition of $S_{M,N}$ with the graph $\Gamma_F : \mathrm{T}^*M \to \overline{\mathrm{T}^*M} \times \mathrm{T}^*N$ of $F$. Also, the multiplication of cotangent vectors with $-1$ defines a symplectomorphism $\mathrm{T}^*M \to \overline{\mathrm{T}^*M}$ which combined with the Schwartz transform leads to the symplectomorphism $\mathrm{T}^*M \times \mathrm{T}^*N \simeq \mathrm{T}^*(M \times N)$. Therefore, the Schwartz transform allows us to identify canonical relations $\mathrm{Mor}(\mathrm{T}^*M, \mathrm{T}^*N)$ with immersed Lagrangian submanifolds of $\mathrm{T}^*(M \times N)$.

Now the quantization task can be described as the search for a functor from the symplectic category to the category of (hermitian) linear spaces, see also [Lan02, GU90]. This means that to each symplectic manifold $X$ we have to assign a Hilbert space $\mathscr{H}_X$ in such a way that $\mathscr{H}_{\overline{X}}$ is dual to $\mathscr{H}_X$ and $\mathscr{H}_{X\times Y}$ is canonically isomorphic to $\mathscr{H}_X \otimes \mathscr{H}_Y$. Furthermore, each morphism $L \in \mathrm{Mor}(X, Y)$ must then be assigned to a linear operator $\mathrm{op}[L] \in \mathrm{Hom}(\mathscr{H}_X, \mathscr{H}_Y) \simeq \mathscr{H}_X^* \otimes \mathscr{H}_Y$ in a way that commutes with compositions

$$\mathrm{op}[L' \circ L] = \mathrm{op}[L']\,\mathrm{op}[L].$$

In the sequel we will encounter several examples for quantizations. These will show that in general there is more information than just a Lagrangian submanifold required to determine an element in $\mathscr{H}_X$. Furthermore, the Hilbert space $\mathscr{H}_X$ may carry some filtration, such as powers of $\hbar$ or degrees of smoothness, and the quantization may be correct only to within a certain accuracy.

### 2.1.1 Semiclassical quantization in cotangent bundles

As we have already seen above, cotangent bundles have a very special structure and corresponding features. For this class of symplectic manifolds we can tackle the quantization problem by a technique that is based on the WKB approximation, yielding an explicit expression for the quantization map. To give a motivation for this procedure, consider the Schrödinger operator on $\mathbb{R}^d$,

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta + V(x).$$

An approximate solution of the eigenvalue equation $(\mathcal{H}-E)\psi=0$ is then given by

$$\psi(x)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x)},$$

where the phase $S$ has to fulfill the Hamilton-Jacobi equation

$$\frac{1}{2m}\|\nabla S\|^2+(V-E)=0.$$

Therefore, $S$ satisfies the Hamilton-Jacobi equation iff the image $L$ of the differential $\mathrm{d}S$, viewed as a mapping $\mathbb{R}^d\to\mathrm{T}^*\mathbb{R}^d$, lies in the level set $H^{-1}(E)$, where $H(x,\xi)=\frac{\xi^2}{2m}+V(x)$. In this case the phase function is called *admissible*, a property that can be characterized by the following conditions:

(i) $L$ is a $d$-dimensional submanifold of $H^{-1}(E)$.

(ii) The pull-back of the canonical one-form $\theta$ to $L$ is exact.

(iii) The restriction of the canonical projection $\pi:\mathrm{T}^*\mathbb{R}^d\to\mathbb{R}^d$ to $L$ induces a diffeomorphism $L\simeq\mathbb{R}^n$, i.e. $L$ is *projectable*.

See also [GS90]. If we want to go to higher order approximations of the time independent Schrödinger equation we have to make an ansatz of the form

$$\psi(x)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x)}(a_0(x)+\hbar a_1(x)+\cdots),$$

where we obtain that $a_j$ has to fulfill the transport equation

$$a_j\Delta S+2(\nabla a_j)(\nabla S)=(1-\delta_{j0})i\Delta a_{j-1},$$

whose homogeneous version, for $j=0$, can be rewritten as

$$\sum_{\nu=1}^{d}\partial_{x_\nu}\left(a^2\partial_{x_\nu}S\right)=0.$$

Now consider the restriction of the Hamiltonian vector field to $L=\mathrm{im}(\mathrm{d}S)$ which reads

$$\mathrm{X}_H\big|_L=\sum_{\nu=1}^{n}\left((\partial_{x_\nu}S)\partial_{x_\nu}-(\partial_{x_\nu}V)\partial_{\xi_\nu}\right),$$

and whose projection to $\mathbb{R}^d$ coincides with $\nabla S$. Using the correspondence between the divergence of a vector field and a density[6], the homogeneous transport equation says that $a^2\pi_*\mathrm{X}_H$ is divergence free, which can be written as

$$\mathscr{L}_{\pi_*X_H}(a^2|\,\mathrm{d}x|)=0,$$

[6]See [Fra97], where a density is called a pseudoform, or [MR94].

where $|\,\mathrm{d}x|$ denotes the canonical density on $\mathbb{R}^d$. Now as a consequence of the Hamilton-Jacobi Theorem, see [BW97], $\mathrm{X}_H$ is tangent to $L$ and the Lie derivative is invariant under diffeomorphisms, so the above equation is fulfilled iff the pull-back of $a^2|\,\mathrm{d}x|$ to $L$ via the projection $\pi$ is invariant under the flow $\mathrm{X}_H$. This suggests that a solution of the Hamilton-Jacobi equation should be a half-density on $L$ which is invariant by $\mathrm{X}_H$, or more generally that quantum states themselves should be represented by half-densities on the configuration space. Summarizing, we have

**Proposition 2.1.3.** *If $S$ is an admissible phase function and $a$ is a half-density on $L = \mathrm{im}(\,\mathrm{d}S)$ which is invariant under the flow of the classical Hamiltonian, then $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}(\,\mathrm{d}S)^*a$ is a second order approximate solution to the time-independent Schrödinger equation.*

Leaving the scenario of euclidean space we turn to the WKB quantization on general cotangent bundles and start from a Lagrangian immersion $i : L \longrightarrow \mathrm{T}^*M$ and set $\pi_L = \pi \circ i$ where $\pi : \mathrm{T}^*M \longrightarrow M$. The critical points and critical values of $\pi_L$ are called *singular points* and *caustic points* of $L$. We say that $(L, i)$ is projectable if $\pi_L$ is a diffeomorphism. Now a one-form $\alpha \in \Lambda^1(M)$ can be thought of as a map $\gamma_\alpha : M \longrightarrow \mathrm{T}^*M$ and the relation $\alpha \leftrightarrow (M, \gamma_\alpha)$ defines a bijective correspondence between the vector space of closed one-forms on $M$ and the set of projectable Lagrangian submanifolds of $\mathrm{T}^*M$, since

$$\gamma_\alpha^*\theta_M = \alpha.$$

This follows from $\pi\gamma_\alpha = \mathrm{id}_M$ and the definition of the Liouville one-form[7], and therefore

$$\mathrm{d}\alpha = \gamma_\alpha^*\omega_M.$$

In this setting we call $S : M \longrightarrow \mathbb{R}$ a phase function for a projectable Lagrangian embedding $(L, i) \subset \mathrm{T}^*M$ provided that $i(L) = \mathrm{d}S(M)$. This is fulfilled iff

$$\mathrm{d}(S \circ \pi_L) = i^*\theta_M.$$

Therefore $L$ is the image of an exact one-form on $M$ iff the restriction of the Liouville form to $L$ is itself exact, in which case we call the Lagrangian immersion exact.

In analogy to the WKB-approximation on $\mathbb{R}^d$ we try to quantize an exact Lagrangian immersion $(L, i) \subset \mathrm{T}^*M$ by choosing a primitive $\phi : L \longrightarrow \mathbb{R}$ such that $\mathrm{d}\phi = \theta_M$ and form the half-density on $M$ given by

$$\mathrm{I}_\hbar(L, i, \phi, a) := (\pi_L^{-1})^*\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi}\,a,$$

where $a$ is a half-density on $L$. Now suppose that the Lagrangian immersion is no longer exact but projectable, then since $i^*\theta_M$ on $L$ is closed it is locally exact by the Poincaré lemma and we can find a cover $\{L_j\}$ of $L$ and functions $\phi_j : L_j \longrightarrow \mathbb{R}$ such that $\mathrm{d}\phi_j = i^*\theta_M|_{L_j}$ and we can quantize $(L_j, i|_{L_j}, \phi_j, a_j)$ as above to obtain

$$I_j = \left(\pi_{L_j}^{-1}\right)^*\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi_j}\,a_j.$$

[7] $\gamma_\alpha^*\theta_M(p)(v) = \theta(\gamma_\alpha(p))((\gamma_\alpha)_*v) = \langle\gamma_\alpha(p), v\rangle$, where $v \in \mathrm{T}M$.

In order to quantize $(L, i, a)$ we have to piece together the $I_j$ to form a well-defined global half-density. This is possible only if the functions $\phi_j$ can be chosen such that the oscillatory coefficients $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi_j}$ agree on the overlaps, i.e. we must have

$$\phi_j - \phi_k \in 2\pi\hbar\mathbb{Z}$$

on each $L_j \cap L_k$, which by definition is the condition that the Liouville class $\lambda_{L,i}$ of the immersed Lagrangian submanifold is $\hbar$-integral. We will use the following terminology:

**Definition 2.1.4.** A projectable Lagrangian submanifold $(L, i) \subset \mathrm{T}^*M$ is quantizable if its Liouville class $\lambda_{L,i}$ is $\hbar$-integral.

However, this simple quantization scheme does not generalize immediately to non-projectable immersed Lagrangian submanifolds, since the projection cannot be used to push-forward half-densities from $L$ to $M$. Consider an arbitrary immersed Lagrangian submanifold $(L, i) \subset \mathrm{T}^*M$ and a half-density $a$ on $L$. If $p \in \pi_L(L)$ is non-caustic and $\pi_L$ is proper[8] there is a contractible neighbourhood $U \subset M$ of $p$ for which $\pi_L^{-1}(U)$ consists of finitely many disjoint open subset $L_j \subset L$ such that each $(L_j, i|_{L_j})$ is a projectable Lagrangian submanifold of $\mathrm{T}^*U$. Choosing a generalized phase function $\phi_j : L_j \to \mathbb{R}$, we expect that the quantization looks like

$$\sum_j (\pi_L^{-1})^* \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi_j} \, a_j$$

on $U$. We generalize the previous definition to

**Definition 2.1.4'.** An immersed Lagrangian submanifold $(L, i) \subset \mathrm{T}^*M$ is said to be prequantizable if its Liouville class $\lambda_{L,i}$ is $\hbar$-integral for some $\hbar \in \mathbb{R}_+$.

In order to give a geometrical description of the prequantizability condition and to show the close connection between geometrical and WKB quantization, we set up

**Definition 2.1.5.** For $\hbar \in \mathbb{R}_+$ the prequantum $\mathbb{T}_\hbar$ bundle associated to a cotangent bundle $(\mathrm{T}^*M, \omega_M)$ consists of the trivial principal bundle $Q_{M,\hbar} := \mathrm{T}^*M \times \mathbb{T}_\hbar$ together with the connection one-form $\varphi = -\pi^*\alpha_M + \mathrm{d}\sigma$.

Here $\sigma$ denotes the (multiple-valued) linear variable in $\mathbb{T}_\hbar = \mathbb{R}/(2\pi\hbar\mathbb{Z})$ and $\pi : Q_{M,\hbar} \to \mathrm{T}^*M$ is the bundle projection. If $i : L \to \mathrm{T}^*M$ is any Lagrangian immersion the curvature of the induced connection on $i^*Q_{M,\hbar}$ coincides with $i^*\omega_M$ and therefore vanishes. We can associate to $Q_{M,\hbar}$ the line bundle $E_{M,\hbar}$ through the representation $x \mapsto \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}x}$ of $\mathbb{T}_\hbar$ in $\mathrm{U}(1)$. Then $\phi$ induces a parallel section of $i^*E_{M,\hbar}$ by means of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi}$ and we have

**Theorem 2.1.6.** *An immersed Lagrangian submanifold $(L, i) \subset \mathrm{T}^*M$ is prequantizable if and only if for some $\hbar > 0$ there exists a nonzero parallel section over $L$ of the line bundle $i^*E_{M,\hbar}$.*

[8] i.e. the pre-image of a compact set is compact.

We now turn to non-projectable Lagrangian immersions. In order to parameterize these submanifolds we start with a function $\varphi : U \times \mathbb{R}^m \to \mathbb{R}$, where $U \subset M$, and a point $\tilde{p} = (p, v) \in (U, \mathbb{R}^m)$. In the projectable case, we used a function $f : \mathrm{T}^*M \supset U \to \mathbb{R}$ to obtain a mapping $\mathrm{d}f : U \to \mathrm{T}^*M$, i.e. for any smooth curve $\gamma$ in $M$ satisfying $\gamma(0) = p$ we have $\mathrm{d}f_p(\dot{\gamma}(0)) = (f \circ \gamma)'(0)$. If we now try to define $\mathrm{d}\varphi_{\tilde{p}} \in \mathrm{T}^*_p M$ analogously by $\mathrm{d}\varphi_{\tilde{p}}(\dot{\gamma}(0)) = (\varphi \circ \tilde{\gamma})'(0)$ for any lift $\tilde{\gamma}$ of $\gamma$ to the product $U \times \mathbb{R}^m$ such that $\tilde{\gamma}(0) = \tilde{p}$, we have to realize, that in general this fails, since the value of the directional derivative $(\varphi \circ \tilde{\gamma})'(0)$ depends on the lift $\tilde{\gamma}$. If, however, the fiber-derivative $\partial\varphi/\partial\vartheta$, where $\vartheta$ are coordinates of $\mathbb{R}^m$, vanishes at $\tilde{p}$, the expression for $\mathrm{d}\varphi_{\tilde{p}}$ is a well-defined element of $\mathrm{T}^*_p M$. We define the fiber critical set

$$\Sigma_\varphi := \left\{ \tilde{p} \in U \times \mathbb{R}^m;\ \frac{\partial\varphi}{\partial\vartheta}(\tilde{p}) = 0 \right\}.$$

The assumption that the map $\partial\varphi/\partial\vartheta : U \times \mathbb{R}^m \to \mathbb{R}^m$ is transverse to 0 implies that the fiber critical set is a smooth submanifold of $U \times \mathbb{R}^m$ and the assignment $\tilde{p} \mapsto \mathrm{d}\varphi_{\tilde{p}}$ defines a Lagrangian immersion of $\Sigma_\varphi$ into $\mathrm{T}^*M$. To see this, consider the general situation that we have smooth manifolds $M$ and $B$ together with a smooth submersion $p_M : B \to M$. Dualizing the inclusion $E := \ker(p_M)_* \overset{i}{\hookrightarrow} \mathrm{T}B$ gives a short exact sequence of vector bundles

$$0 \leftarrow E^* \overset{i^*}{\leftarrow} \mathrm{T}^*B \leftarrow E^\perp \leftarrow 0, \tag{2.1.1}$$

where $E^\perp \subset \mathrm{T}^*B$ denotes the annihilator of $E$ in $\mathrm{T}^*B$. The fiber derivative of a function $\phi : B \to \mathbb{R}$ is the composition $\mathrm{d}_\vartheta\phi := i^* \circ \mathrm{d}\phi$, and its fiber critical set is defined as

$$\Sigma_\phi = (\,\mathrm{d}_\vartheta\phi)^{-1}\mathrm{Z}_{E^*},$$

where $\mathrm{Z}_{E^*}$ denotes the zero section in $E^*$. The function $\phi$ is called *non-degenerate* if its fiber derivative is transverse to $\mathrm{Z}_{E^*}$, in which case $\Sigma_\phi$ is a smooth submanifold of $B$. Furthermore, on $\Sigma_\phi$ the fiber derivative $\mathrm{d}_\vartheta\phi$ has a well-defined intrinsic derivative $\nabla\,\mathrm{d}_\theta\phi$, see [GG73, GS90].

Let us make a short digression in order to give the notion of an intrinsic derivative. To this end, we consider a vector bundle map $\sigma : E \to F$ over a manifold $M$ and let $m \in M$. Choose local trivializations $E \simeq E_x \times U$, $F \simeq F_x \times U$ for a neighbourhood $U \ni x$. For each $e \in E_x$ choose a function $s : U \to E_x$ with $s(x) = e$. Then $\sigma$ determines a map $A : E_x \to F_x$ on $U$ and therefore $As$ is a function on $U$ with values in $F_x$ for which we may compute the differential at $x$. Now if $v \in \mathrm{T}_x M$ is any tangent vector we may define a map

$$\nabla_x\sigma : \ker\sigma(x) \otimes \mathrm{T}_x M \to \operatorname{coker}\sigma(x)$$

by

$$\nabla_x\sigma(e \otimes v) := \pi_{\operatorname{coker}\sigma(x)}\langle v,\, \mathrm{d}(As)\rangle,$$

which for $e \in \ker\sigma(x)$ is independent of the choice of the section $s$ and of the local trivializations and therefore well-defined, see [GS90].

In our case we have $\sigma = \mathrm{d}_\theta\phi$ whose intrinsic derivative is well-defined on the fiber critical set $\Sigma_\phi$ and a corresponding induced exact sequence of vector bundles

$$0 \longrightarrow \mathrm{T}\Sigma_\phi \longrightarrow \mathrm{T}_{\Sigma_\phi}B \xrightarrow{\nabla \mathrm{d}_\theta \phi} E^*|_{\Sigma_\phi} \longrightarrow 0. \tag{2.1.2}$$

Furthermore, we define the *fiber Hessian* $\operatorname{Hess}\phi$ of $\phi$ at $p \in \Sigma_\phi$ as the composition

$$\operatorname{Hess}\phi := \nabla \mathrm{d}_\theta\phi \circ i : E_p \longrightarrow E_p^*.$$

Now we may identify the annihilator $E^\perp$ of $E$ in $\mathrm{T}^*B$ with the pull back of $p^*\mathrm{T}^*M$ which gives a natural projection $\pi : E^\perp \to \mathrm{T}^*M$. On $\Sigma_\phi$ the differential $\mathrm{d}\phi$ defines a section of $E^\perp$ which we can project by $\pi$ to $\mathrm{T}^*M$. For

$$\lambda_\phi := \pi \circ \mathrm{d}\phi : \Sigma_\phi \longrightarrow \mathrm{T}^*M$$

we have the following

**Theorem 2.1.7.** *If $\phi$ is non-degenerate, then the map $\lambda_\phi : \Sigma_\phi \to \mathrm{T}^*M$ is an exact Lagrangian immersion.*

According to the definition,

$$\dim\ker\operatorname{Hess}\phi = \dim(\mathrm{T}_p\Sigma_\phi \cap E_p)$$

for all $p \in \Sigma_\phi$, which also equals

$$\dim\ker(\pi \circ \lambda_\phi)_*$$

on $\Sigma_\phi$. Therefore the dimension of the fibers of $B \to M$ has to be at least $\dim\ker\operatorname{Hess}\phi$; if it is equal $k$ we say that $\phi$ is *reduced*. This is the case when the fiber hessian vanishes.

We are now heading towards a concept which describes the local parameterization of Lagrangian submanifolds. To this end we will need some more definitions.

**Definition 2.1.8.** A triple $(B, p, \phi)$ is called a Morse family over a manifold $M$ if $p : B \to M$ is a smooth submersion and $\phi$ is a non-degenerate phase function on $B$ such that $\lambda_\phi$ is an embedding. A Morse family is said to be reduced at $b \in B$ if $\phi$ is a reduced phase function at $b$.

We then say that the Lagrangian submanifold $\lambda_\phi(\Sigma_\phi) = \Lambda_\phi$ is generated by the Morse family $(B, p, \phi)$. If $i : \Lambda \to \mathrm{T}^*M$ is a Lagrangian immersion and $b \in \Lambda$ then we denote by $\mathfrak{M}(\Lambda, i, b)$ the class of Morse families $(B, p, \phi)$ which generate $i(U)$ for some neighbourhood $U \subset \Lambda$ of $b$.

If we have a Morse family $(B, p, \phi) \in \mathfrak{M}(\Lambda, i, b)$ then we can produce further elements of the same class by the following operations:

1. Addition: For any $c \in \mathbb{R}$ also $(B, p, \phi + c)$ is in $\mathfrak{M}(\Lambda, i, b)$.
2. Composition: If $p' : B' \to M$ is a second submersion and $g : B' \to B$ a fiber-preserving diffeomorphism then $(B', p', \phi \circ g) \in \mathfrak{M}(\Lambda, i, b)$.

3. Suspension: The suspension of $(B, p, \phi)$ by a non-degenerate quadratic form $Q$ on $\mathbb{R}^n$ is defined as the Morse family given by the submersion $\tilde{p} : B \times \mathbb{R}^n \to M$, obtained by composing $p$ with the projection along $\mathbb{R}^n$ and the phase function $\tilde{\phi} = \phi + Q$. Then also $(B \times \mathbb{R}^n, \tilde{p}, \tilde{\phi}) \in \mathfrak{M}(\Lambda, i, b)$.

4. Restriction: If $B'$ is any open subset of $B$ containing $\lambda_\phi^{-1}(b)$ then the restrictions of $p$ and $\phi$ to $B'$ define a Morse family on $M$ which belongs to $\mathfrak{M}(\Lambda, i, b)$.

This shows that the phase function which generates a neighbourhood of a certain point of a Lagrangian manifold is far from unique. However, the above operations generate an equivalence relation among the Morse families which is called *stable equivalence*. We have the following fundamental result

**Theorem 2.1.9.** *Let $i : \Lambda \to \mathrm{T}^*M$ be a Lagrangian immersion and $b \in \Lambda$. Then*

*(i) $\mathfrak{M}(\Lambda, i, b)$ contains a reduced Morse family over $M$. In particular $\mathfrak{M}(\Lambda, i, b)$ is non-empty.*

*(ii) Any two members of $\mathfrak{M}(\Lambda, i, b)$ are stably equivalent.*

This means that an immersed Lagrangian submanifold can always be parameterized in the way described above; for a proof we refer to [BW97].

## 2.1.2 Maslov objects

So far we have only been concerned with the quantization of Lagrangian submanifolds away from caustic points. In the study of Lagrangian immersions $i : L \to \mathrm{T}^*M$ in the neighbourhood of caustic points the relationship of $\mathrm{T}L$ to the vertical bundle $i^*\mathrm{V}M$ of $i_*\mathrm{T}L$ is important. We enlighten this fact by the following

### Preliminaries

Consider $\mathrm{T}^*\mathbb{R} \simeq \mathbb{R}^2$ and an immersion $i : L \to \mathrm{T}^*\mathbb{R}$. We denote by $(x, \xi) \in \mathrm{T}^*\mathbb{R}$ the canonical symplectic coordinates and by $\pi_\xi$ the projection onto the $\xi$-axis, which takes over the role of the vertical bundle $\mathrm{V}(\mathrm{T}^*\mathbb{R})$. If $\pi_\xi$ is a diffeomorphism, then $(L, i)$ is said to be $\xi$-projectable, in which case there exists an alternative generating function $\tau : L \to \mathbb{R}$ satisfying $\mathrm{d}\tau = i^*(-x\,\mathrm{d}\xi)$, obtained by thinking of $\mathbb{R}^2$ as the cotangent bundle of $\xi$-space.

Consider the Lagrangian submanifold $i(\eta) = (q, \eta)$ which obviously is not projectable on the $x$-axis. Since the wavefunction corresponding to a constant half-density on $L$ should correspond to a probability distribution describing the position of a particle at $q$ with completely indetermined momentum, it should be a delta function supported at $q$. The basic idea due to Maslov is given by analyzing this situation by pretending that $\xi$ is position and $x$ momentum, then quantizing to obtain a function on $\xi$-space. Using the phase function $\tau(\eta) = -q\eta$ on $L$ we obtain

$$(\pi_\xi^{-1})^*\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tau}\, |\,\mathrm{d}\eta|^{1/2} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\xi}\, |\,\mathrm{d}\xi|^{1/2}.$$

In the simplest form, Maslov's technique is to suppose that $(L, i) \subset \mathrm{T}^*\mathbb{R}^d$ is $\xi$-projectable so that $\mathrm{d}\tau = i^*(-x\,\mathrm{d}\xi)$ for some phase function $\tau$ on $L$. If $a$ is a half-density on $L$, we define a function $B$ on $\xi$-space by the equation

$$B|\,\mathrm{d}\xi|^{1/2} = (\pi_\xi^{-1})^*\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tau}\,a.$$

Then the Maslov quantization of $(L, i, \tau, a)$ is defined by the half-density

$$\mathrm{J}_\hbar(L, i, \tau, a) := \mathcal{F}_\hbar^{-1}(B)|\,\mathrm{d}x|^{1/2},$$

where $\mathcal{F}_\hbar$ denotes the $\hbar$-Fourier transform. To relate this construction to the one given by pulling back half-densities, we have to compare the results in the case of a Lagrangian submanifold which is both $x$- and $\xi$-projectable. We look at a linear example, where the embedding $i : \mathbb{R} \to \mathrm{T}^*\mathbb{R}$ is given by $i(x) = (x, kx)$ with $k \neq 0$. Generalized phase functions on $(L, i)$ of the forms $\xi\,\mathrm{d}x$ and $-x\,\mathrm{d}\xi$ are given by $\phi(x) = \frac{k}{2}x^2$ and $\tau(x) = -\frac{k}{2}x^2$ respectively. If $a$ is a constant half-density on $L$, then according to the transformation properties of half-densities we have

$$(\pi_L^{-1})^*a = A|\,\mathrm{d}x|^{1/2}, \quad (\pi_\xi^{-1})^*a = |k|^{-1/2}A|\,\mathrm{d}\xi|^{1/2},$$

for a real constant $A \in \mathbb{R}$ determined by $a$. Quantization by pull-back therefore gives

$$\mathrm{I}_\hbar(L, i, \phi, a) = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}kx^2}\,A|\,\mathrm{d}x|^{1/2}.$$

On the other hand, we have $(\pi_\xi^{-1})^*\tau(\xi) = \frac{\xi^2}{2k}$ and a computation shows that

$$\mathcal{F}_\hbar^{-1}((\pi_\xi^{-1})^*\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tau})(x) = |k|^{1/2}\,\mathrm{e}^{-\mathrm{i}\,\pi\mathrm{sgn}(k)/4}\,\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}kx^2}.$$

Thus, by identifying $A = |k|^{1/2}$, Maslov's technique yields

$$\mathrm{J}_\hbar(L, i, \tau, a) = \mathrm{e}^{-\mathrm{i}\,\pi\mathrm{sgn}(k)/4}\,\mathrm{I}_\hbar(L, i, \phi, a),$$

i.e. both methods differ by a constant phase shift.

A similar correspondence for more general bi-projectable Lagrangian embeddings can be established by choosing

$$\phi = \tau + i^*(x\xi)$$

which means that the trivial additive constants in the phase functions are chosen in a suitable way. Next let $S(x)$ and $T(\xi)$ be the functions defined by pull-back:

$$S = (\pi_L^{-1})^*\phi, \quad T = (\pi_\xi^{-1})^*\tau.$$

Then $S$ and $T$ satisfy the Legendre relation, see [Arn78],

$$S(x) = -\xi(x)T'(\xi(x)) + T(\xi(x)),$$

where $\xi(x) = S'(x)$ and it follows that

$$T''(\xi(x)) = -(S''(x))^{-1}.$$

A half-density $a$ on $L$ determines functions $A(x)$ and $B(\xi)$ such that

$$(\pi_L^{-1})^* a = A|\,\mathrm{d}x|^{1/2}, \quad (\pi_\xi^{-1})^* a = B|\,\mathrm{d}\xi|^{1/2},$$

and

$$A(x) = |S''(x)|^{1/2} B(\xi(x)).$$

For each $x$ we have to compare the Maslov half-density

$$\mathrm{J}_\hbar(L, i, \tau, a) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x\xi + T(\xi))} B(\xi)\,\mathrm{d}\xi|\,\mathrm{d}x|^{1/2}$$

with that one obtained by pull-back

$$\mathrm{I}_\hbar(L, i, \phi, a) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} A|\,\mathrm{d}x|^{1/2}.$$

To this end set $k(x) = T''(\xi(x))$ and apply the method of stationary phase, which gives [BW97]

$$\mathrm{J}_\hbar(L, i, \tau, a) = \mathrm{e}^{-\mathrm{i}\,\pi \mathrm{sgn}(k)/4}\,\mathrm{I}_\hbar(L, i, \phi, a) + O(\hbar).$$

The essential difference between the naive quantization and Maslov's description lies in the relative phase constants of the summands of $\mathrm{I}_\hbar$ as illustrated by the above example. However, the main advantage of Maslov's description is that the objects occuring are smooth, even at caustic points.

According to the above example the relative phase depends on $T(\xi)$: This will have an inflection point at precisely those $\xi$ for which $(T'(\xi), \xi)$ is a singular point of $L$. Moreover, the sign of $T''$ at nearby points depends only on $L$ and not on the choice of $T$. Keeping this in mind, suppose that $(L, i)$ is a closed immersed curve in $\mathbb{R}^2$ that is non-degenerate in the sense that if $T$ is a $\xi$-dependent phase function of a subset of $L$, then $T'$ only has non-degenerate critical points. Under this assumption $\mathrm{sgn}(T'')$ changes by $\pm 2$ in the vicinity of a critical point of $T'$, and we can assign an index to $(L, i)$ by summing up these changes while transversing $L$ in a prescribed direction. The result is twice an integer, known as the *Maslov index* of $(L, i)$, see [Arn67].

### The Maslov class and its line bundle

In the general case, the $\xi$-plane is replaced by the vertical bundle $i^*\mathrm{V}M$. There is a degree-one cohomology class associated with $i^*\mathrm{V}M$ and $i_*\mathrm{T}L$, where $(L, i)$ denotes a Lagrangian immersion. We will now shortly describe, how the Maslov class is defined.

Let $V$ be a symplectic vector space and $\mathfrak{Lag}(V)$ the collection of all Lagrangian subspaces, i.e. the *Lagrangian Grassmannian.* We have, see [BW97] and Appendix A,

$$\pi_1(\mathfrak{Lag}(V)) \simeq \pi_1(\mathrm{S}^1) \simeq \mathbb{Z},$$

which gives a natural homomorphism

$$\mathrm{H}^1(\mathrm{S}^1, \mathbb{Z}) \to \mathrm{H}^1(\mathfrak{Lag}(V), \mathbb{Z}).$$

The image of the canonical generator of $\mathrm{H}^1(\mathrm{S}^1, \mathbb{Z})$ under this map is called the *universal Maslov class* $\mu_V$.

Now let $E$ be a symplectic vector bundle over $M$, then any pair $L$, $L'$ of Lagrangian subbundles of $E$ defines a cohomology class $\mu(L, L') \in \mathrm{H}^1(M, \mathbb{Z})$ as follows: Assume that $E$ admits a symplectic trivialization $f : E \to M \times V$ with some symplectic vector space $V$, and we denote by $f_L$, $f_L : M \to \mathfrak{Lag}(V)$ the maps induced by the Lagrangian subbundles $f(L)$ and $f(L')$ of $M \times V$. Then

$$\mu(L, L') = (f_L^* - f_{L'}^*)\mu_V.$$

Associated with any Morse family $(B, p_B, \phi)$ over a manifold $M$ is an index function $\mathrm{ind}_\phi : L_\phi \to \mathbb{Z}$ defined by

$$\mathrm{ind}_\phi(p) = \mathrm{ind}(\mathrm{Hess}\, \phi_{\lambda_\phi^{-1}(p)}),$$

where the index of a quadratic form is the dimension of the largest subspace on which it is negative-definite. Since the fiber Hessian is nondegenerate where $L_\phi$ is projectable, the index function $\mathrm{ind}_\phi$ is constant on any connected projectable subset of $L_\phi$. From the fact that any two Morse families are stably equivalent it follows that two index functions $\mathrm{ind}_\phi$ and $\mathrm{ind}_{\phi'}$ differ by an integer on each connected component of $L_\phi \cap L_{\phi'}$.

Since the Maslov class is a degree one cohomology class it determines an isomorphism class of flat hermitian line bundles, see [Ste51, Hus75]. Instead of constructing this isomorphism class, we provide a canonical representative. To this end consider the union

$$\mathfrak{M}(L, i) = \bigcup_{p \in L} \mathfrak{M}(L, i, p),$$

with the discrete topology (see e.g. [Mun00]). On the subset $L \times \mathfrak{M}(L, i) \times \mathbb{Z}$ consisting of all $(p, (B, p_B, \phi), n)$ such that $(B, p_B, \phi) \in \mathfrak{M}(L, i, p)$ we introduce an equivalence relation by setting

$$(p, (B, p_B, \phi), n) \sim (\tilde{p}, (\tilde{B}, p_{\tilde{B}}, \tilde{\phi}), \tilde{n}),$$

provided that $p = \tilde{p}$, and

$$n + \mathrm{ind}_\phi(p) = \tilde{n} + \mathrm{ind}_{\tilde{\phi}}(\tilde{p}).$$

The corresponding quotient space is a principal $\mathbb{Z}$-bundle $M_{L,i}$ over $L$ that is called the *Maslov principal bundle*. Associated to this bundle is the *Maslov line bundle*, where we represent $\mathbb{Z}$ on $\mathrm{U}(1)$ by $n \mapsto \mathrm{e}^{\mathrm{i}\pi n/2}$. This line bundle has a natural flat connection with holonomy $\mathbb{Z}_4$.

**Alternative description of the Maslov bundle** We now want to give an alternative description of the Maslov bundle, see also Section A.3.3. Given two Lagrangian subspaces $L$ and $L'$ of a symplectic vector space $V$, we denote by $\mathfrak{Lag}_{L,L'}$ the subset of the Lagrangian Grassmannian $\mathfrak{Lag}(V)$ comprised of those Lagrangian subspaces which are transverse to both $L$ and $L'$. For $W$, $W' \in \mathfrak{Lag}_{L,L'}$ the *cross index* of the quadruple $(L, L'; W, W')$ is defined as the integer

$$\sigma(L, L'; W, W') = \operatorname{ind}(L, L'; W) - \operatorname{ind}(L, L', W").$$

First we have to explain how $\operatorname{ind}(L, L'; W)$ is defined for $W$ transversal to both $L$ and $L'$: There is a natural linear symplectomorphism from $V$ to $L \oplus L^*$ which sends $W$ to $0 \oplus L^*$ and $L'$ to the graph of some self-adjoint linear map $T : L \to L^*$, see [GS90] and Appendix A. Denoting by $Q_T$ the quadratic form on $L$ induced by $T$ we define

$$\operatorname{ind}(L, L'; W) = \operatorname{ind} Q_T \quad \text{and} \quad \operatorname{sgn}(L, L'; W) = \operatorname{sgn} Q_T.$$

Because of the property

$$\operatorname{ind}(L, L'; W) = -\operatorname{ind}(L', L; W) = -\operatorname{ind}(L, W; L')$$

one immediately verifies the cocycle relation

$$\sigma(L, L'; W, W') + \sigma(L, L'; W', W'') + \sigma(L, L'; W'', W) = 0,$$

where $W$, $W'$, $W'' \in \mathfrak{Lag}_{L,L'}$. Denote by $\mathrm{F}_{L,L'}(V)$ the space of functions $f : \mathfrak{Lag}_{L,L'} \to \mathbb{Z}$ such that

$$f(W) - f(W') = \sigma(L, L'; W, W')$$

for all $W, W' \in \mathfrak{Lag}_{L,L'}$. Since any such function is determined up to an additive constant by its value at a single point of $\mathfrak{Lag}_{L,L'}$, it follows that $\mathbb{Z}$ acts simply and transitively by addition on $\mathrm{F}_{L,L'}(V)$.

We can assign a principal $\mathbb{Z}$-bundle to a triple $(E, \mathfrak{F}, \mathfrak{F}')$, where $E$ is a symplectic vector bundle and $\mathfrak{F}$ and $\mathfrak{F}'$ are Lagrangian subbundles. Associated to $E$ is the Lagrangian Grassmannian bundle $\mathfrak{Lag}(E)$ whose fiber at $x \in M$ is simply $\mathfrak{Lag}(E_x)$. The Lagrangian subbundles $\mathfrak{F}$ and $\mathfrak{F}'$ are then smooth sections of $\mathfrak{Lag}(E)$ and we denote by $\mathrm{M}_{\mathfrak{F},\mathfrak{F}'}(E)$ the principal $\mathbb{Z}$-bundle whose fiber over $x \in M$ equals $\mathrm{F}_{\mathfrak{F}_x,\mathfrak{F}'_x}(E_x)$.

A special case of this set-up occurs, when $L$ is an immersed Lagrangian submanifold of $\mathrm{T}^*M$ such that $E = i^*\mathrm{T}(\mathrm{T}^*M)$ is a symplectic vector bundle over $L$. Natural Lagrangian subbundles are then $\mathfrak{F} = i_*\mathrm{T}L$ and $= \mathfrak{F}' = i^*\mathrm{V}M$.

We quote the following result, and refer to [BW97] and [Arn67, Hör71, DH72, Dui96] for a proof.

**Theorem 2.1.10.** *In the above notation, the Maslov bundle* $\mathrm{M}_{L,i}$ *is canonically isomorphic to* $M_{\mathfrak{F},\mathfrak{F}'}(E)$.

### 2.1.3 WKB quantization

We now explicitly turn to WKB quantization and define the *phase bundle* associated with an immersed Lagrangian submanifold $i: L \to \mathrm{T}^*M$ as the tensor product

$$\Phi_{L,i,\hbar} := M_{L,\hbar} \otimes i^* E_{M,\hbar} \tag{2.1.3}$$

of the Maslov line bundle and the prequantum line bundle over $\mathrm{T}^*M$, see Definition 2.1.5 and the following discussion. The product of the flat connections defines a flat connection on the phase bundle. The phase bundle $\Phi_{L,i,\hbar}$ can be described explicitly as the collection of all quintuples $(p, t, (B, p_B, \phi), n, z)$ where $(p, t, n, z) \in L \times \mathbb{T}_\hbar \times \mathbb{Z} \times \mathbb{C}$ and $(B, p_B, \phi) \in \mathfrak{M}(L, i, p)$, modulo the equivalence relation given by

$$(p, t, (B, p_B, \phi), n, z) \sim (\tilde{p}, \tilde{t}, (\tilde{B}, p_{\tilde{B}}, \tilde{\phi}), \tilde{n}, \tilde{z})$$

whenever $p = \tilde{p}$ and

$$z \cdot \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} t}\, \mathrm{e}^{\mathrm{i}\,\pi(n + \mathrm{ind}_\phi(p))/2} = \tilde{z} \cdot \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\tilde{t}}\, \mathrm{e}^{\mathrm{i}\,\pi(\tilde{n} + \mathrm{ind}_{\tilde{\phi}}(\tilde{p}))/2}.$$

Now a Morse family $(B, p_B, \phi)$ which generates an open subset $L_\phi$ of $L$ defines a non-vanishing parallel section of the phase bundle by

$$s_{\phi,\hbar}(p) = \left[p, 0, (B, p_B, \phi), 0, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\phi(y)}\right], \tag{2.1.4}$$

where $\lambda_\phi(y) = p$. For each $\hbar \in \mathbb{R}_+$ we denote by $\Gamma_{\mathrm{par}}(\Phi_{L,i,\hbar})$ the space of parallel sections. The product

$$\Gamma_{L,i} := \prod_{\hbar > 0} \Gamma_{\mathrm{par}}(\Phi_{L,i,\hbar})$$

has the structure of a $\mathbb{C}$-module. An element $s \in \Gamma_{L,i}$ is a function which assigns to a $\hbar \in \mathbb{R}_+$ an element $s_\hbar \in \Gamma_{\mathrm{par}}(\Phi_{L,i,\hbar})$ so that the map $p \mapsto s_\hbar(p)$ defines a parallel section of $\Phi_{L,i,\hbar}$.

The *symbol space* of $(L, i)$ is defined as the complex vector space

$$\mathfrak{S}_{L,i} := |\Lambda|^{1/2} L \otimes \Gamma_{L,i},$$

and the *amplitude bundle* $A_\phi$ associated with a Morse family $(B, p_B, \phi)$ over a smooth manifold $M$ is defined as the complex line bundle

$$A_\phi := |\Lambda|^{1/2} B \otimes |\Lambda|^{1/2} E$$

over $B$, where again $E$ denotes the subbundle $\ker(p_{B*})$ of $\mathrm{T}B$. An amplitude is a section $\mathfrak{a}$ of $A_\phi$. We say that $\mathfrak{a}$ is properly supported provided that the restriction of $p_B : B \to M$ to $\mathrm{supp}(\mathfrak{a})$ is a proper map. Now because of the exact sequence of vector bundles (2.1.1) it follows that $|\Lambda|^{1/2} B$ is naturally isomorphic to $|\Lambda|^{1/2} p_B^* \mathrm{T}M \otimes |\Lambda|^{1/2} E$, see e.g. [BW97, Dui96], which yields

$$A_\phi \simeq |\Lambda|^{1/2} p_B^* \mathrm{T}M \otimes |\Lambda| E.$$

Under this isomorphism the image of an amplitude $\mathfrak{a}$ on $B$ can be written as $p_B^* |\,\mathrm{d}x|^{1/2} \otimes \sigma$, where $\sigma$ is a family of 1-densities on the fibers of $p_B$, i.e. $\sigma_x$ is a density on each non-empty $p_B^{-1}(x)$. By integration over the fibers we can pass to a half-density on $M$,

$$\mathrm{I}_\hbar(\phi, \mathfrak{a})(x) = (2\pi\hbar)^{-n/2}\,\mathrm{e}^{-\,\mathrm{i}\,n\pi/4}\left(\int_{p_B^{-1}(x)} \mathrm{e}^{\mathrm{i}\,\phi/\hbar}\,\sigma_x\right)|\,\mathrm{d}x|^{1/2},$$

where $n = \dim(p_B^{-1}(x))$. Geometrically, we pass from $a$ to a symbol on $L_\phi$ by using the exact sequence (2.1.2) and the natural isomorphism $|\Lambda|^{-1/2}E \simeq |\Lambda|^{1/2}E^*$. This shows that the restriction of $A_\phi$ to $\Sigma_\phi$ is isomorphic to $|\Lambda|^{1/2}\Sigma_\phi$. If the restriction of an amplitude $\mathfrak{a}$ corresponds to a half-density $a$ on $\Sigma_\phi$, we define the associated symbol on $L_\phi$ according to

$$\mathfrak{s}_a := ((\lambda_\phi^{-1} \circ i)^{-1})^* a \otimes s_\phi,$$

where $s_\phi$ is the section of the phase bundle defined in (2.1.4). For projectable $(L, i)$ the symbol $\mathfrak{s}_a$ and the half-density $\mathrm{I}_\hbar(\phi, a)$ are linked by

**Theorem 2.1.11.** *Suppose that two Morse families $(B, p_B, \phi)$ and $(\tilde{B}, p_{\tilde{B}}, \tilde{\phi})$ generate the same projectable Lagrangian embedding $(L, i)$, and let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be amplitudes on $B$ and $\tilde{B}$, respectively. Then $\mathfrak{s}_a = \mathfrak{s}_{\tilde{a}}$ on $L$ if and only if*

$$|\mathrm{I}_\hbar(\phi, a) - \mathrm{I}_\hbar(\tilde{\phi}, \tilde{a})| = O(\hbar).$$

Morse families allow for the quantization of symbols on immersed Lagrangian manifolds locally: Suppose that $(B, p_B, \phi)$ is a Morse family such that the phase function $\phi$ generates a subset $L_\phi \subset L$ and consider a symbol $s$ on $L$ supported in $L_\phi$. Then there exists a unique half-density $a$ supported in $L_\phi$ such that $\mathfrak{s} = a \otimes s_\phi$ and $(\lambda_\phi^{-1} \circ i)^* a$ can be identified with a section of the amplitude bundle of $B$ over $\Sigma_\phi$, which may be extended to an amplitude $\mathfrak{a}$ on $B$, compactly supported on the fibers. Let us then set for now

$$\mathrm{I}_\hbar(L, i, \mathfrak{s}) = \mathrm{I}_\hbar(\phi, \mathfrak{a})(X).$$

As one realized immediately from the preceding discussion, if $(\tilde{B}, p_{\tilde{B}}, \tilde{\phi})$ is another Morse family which generates $L_\phi$ and $\tilde{\mathfrak{a}}$ is a corresponding amplitude on $\tilde{B}$, then $\mathfrak{s}_{\mathfrak{a}} = \mathfrak{s}_{\tilde{\mathfrak{a}}} = \mathfrak{s}$ and according to Theorem 2.1.11

$$|\mathrm{I}_\hbar(\phi, \mathfrak{a}) - \mathrm{I}_\hbar(\tilde{\phi}, \tilde{\mathfrak{a}})| = O(\hbar),$$

and $\mathrm{I}_\hbar(L, i, \mathfrak{s})$ is well-defined up to terms of order $\hbar$. In order to quantize arbitrary symbols $\mathfrak{s}$ on $L$ we use a partition of unity subordinate to a locally finite covering $\{L_j\}$ of $L$ and glue the local constructions together. In order to obtain non-vanishing parallel sections of the phase bundle we employ the *Maslov quantization condition:* The phase class of $\Phi_{L,i}$ has to be $\hbar$-integral, i.e. the local constructions have to coincide on the intersection of their domains of definition.

**Example 2.1.12.** Let $N$ be a closed submanifold of a smooth manifold $M$, together with an embedding $i: N \hookrightarrow M$, and let $U$ be a tubular neighbourhood[9] of $N$. Then the image of the normal bundle $\mathrm{N}(N) \subset \mathrm{T}_N M$ under the embedding $\psi: \mathrm{N}(N) \to M$ satisfies $\psi = \pi$ on the zero section of $\mathrm{N}(N)$, where $\pi: \mathrm{N}(N) \to N$ is the natural projection in the normal bundle. Define $r := \pi \circ \psi^{-1}$, which is the retraction of $U$ on $N$, and the natural submersion $p_N: r^* N^\perp \to M$, where $N^\perp$ denotes the conormal bundle of $N$ in $\mathrm{T}^* M$. Furthermore, define

$$\phi: r^* N^\perp \to \mathbb{R}, \quad \phi(p) = \langle p, \psi^{-1}(p_N(p)) \rangle.$$

These objects together define a Morse family $(r^* N^\perp, p_N, \phi)$, the fiber critical set of $\phi$ is given by $\Sigma_\phi = p_N^{-1}(N) = N^\perp$ and the map $\lambda_\phi: N^\perp \to \mathrm{T}^* M$ is the inclusion. Thus the conormal bundle of $N$ is a Lagrangian submanifold of $\mathrm{T}^* M$ that admits a global generating function, and therefore both the Liouville and the Maslov classes of $N^\perp$ are trivial. In particular, this implies that the conormal bundle of any submanifold of $M$ satisfies the Maslov quantization condition.

So far the quantization of general symbols was concerned with regular values of $\pi_L$. However, the above considerations are no longer valid at caustics, since the principle of stationary phase (see [Hör90b, GS94]) no longer applies to

$$\int_{p_B^{-1}(x)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi} \, \sigma_x$$

when $x$ is a caustic point, at which the phase function $\phi$ has a degenerate critical point. However, for a compactly supported amplitude $\mathfrak{a}$ we can define a distributional half-density on $M$ according to

$$\langle \mathrm{I}_\hbar(\phi, \mathfrak{a}), u \rangle = (2\pi\hbar)^{-b/2} \, \mathrm{e}^{-\frac{\mathrm{i}}{4}\pi b} \int_B \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi} \, \mathfrak{a} \otimes p_B^* u,$$

where $b = \dim B$ and $u \in |\Lambda_0|^{1/2} M$. As above the local constructions can be glued together to give a well-defined quantization prescription. Then, the statement analogous to the one of Theorem 2.1.11 is given by

**Theorem 2.1.13.** *Suppose that two Morse families $(B, p_B, \phi)$ and $(\tilde{B}, p_{\tilde{B}}, \tilde{\phi})$ generate the same Lagrangian embedding $(L, i)$ and let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be amplitudes on $B$ and $\tilde{B}$, respectively. If $\psi: M \to \mathbb{R}$ is a smooth function whose differential intersects $i(L)$ at exactly one point $i(p)$ transversally, then $\mathfrak{s}_{\mathfrak{a}}(p) = \mathfrak{s}_{\tilde{\mathfrak{a}}}(p)$ if and only if*

$$\left| \langle \mathrm{I}_\hbar(\phi, \mathfrak{a}), \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi} \, u \rangle - \langle \mathrm{I}_\hbar(\tilde{\phi}, \tilde{\mathfrak{a}}), \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi} \, u \rangle \right| = O(\hbar).$$

The above considerations show in principle how to quantize immersed Lagrangian submanifolds of cotangent bundles. In order to verify that this quantization fulfills the fundamental properties of Section 2.1 we now verify its composition properties.

[9] The definition of a tubular neighbourhood is as follows: Consider the normal bundle $\mathrm{N}(N) = i^* \mathrm{T}M / i_* \mathrm{T}N$. Then $N$ is naturally identified with the zero section in $\mathrm{N}(N)$. In this case there is a neighbourhood $U \subset M$ of $i(N)$ in $M$ and a diffeomorphism $\psi: \mathrm{N}(N) \to U$ satisfying $\psi(m) = i(m)$ for all $m \in N$.

### 2.1.4 Composition of semiclassical states

Let $L_1$ and $L_2$ be immersed Lagrangian submanifolds of $\mathrm{T}^*M$ and $\mathrm{T}^*N$, respectively. Then their product $L_2 \times L_1$ gives a Lagrangian submanifold of $\mathrm{T}^*(M \times N)$ under the Schwartz-transform defined in Lemma 2.1.2, $\mathrm{S}_{M,N} : \mathrm{T}^*M \times \mathrm{T}^*N \to \mathrm{T}^*(M \times N)$. The phase bundle (2.1.3) $\Phi_{L_2 \times L_1,\hbar}$ is canonically isomorphic to the tensor product $\Phi_{L_2,\hbar} \otimes \Phi^{-1}_{L_1,\hbar}$. Furthermore, we have an isomorphism of the density bundle $|\Omega|^{1/2}(L_2 \times L_1) \to |\Omega|^{1/2} L_2 \otimes |\Omega|^{1/2} L_1$ and thus a natural isomorphism of the symbol spaces

$$\mathfrak{S}_{L_2} \otimes \mathfrak{S}_{L_1} \to \mathfrak{S}_{L_2 \times L_1}, \quad (\mathfrak{s}_2, \mathfrak{s}_1) \mapsto \mathfrak{s}_2 \otimes \mathfrak{s}_1^*, \tag{2.1.5}$$

and it makes sense to set up

**Definition 2.1.14.** The product of semiclassical states $(L_1, \mathfrak{s}_1)$ and $(L_2, \mathfrak{s}_2)$ in $\mathrm{T}^*M$ and $\mathrm{T}^*N$, respectively is the semi-classical state $(L_2 \times L_1, \mathfrak{s}_2 \otimes \mathfrak{s}_1^*)$ in $\mathrm{T}^*(M \times N)$.

Turning to compositions, we recall that the Schwartz-transform identifies a canonical relation in $\mathrm{Mor}(\mathrm{T}^*M, \mathrm{T}^*V)$ with an immersed Lagrangian submanifold in $\mathrm{T}^*(M \times V)$. Consider the diagram

$$\begin{tikzcd}
\mathrm{T}^*V \times \overline{\mathrm{T}^*N} \times \mathrm{T}^*N \times \overline{\mathrm{T}^*M} \arrow[r, "\mathrm{S}_{V\times N, N\times M} \circ (\mathrm{S}_{V,N} \times \mathrm{S}_{N,M})"] \arrow[d] & \mathrm{T}^*(M \times N \times N \times V) \arrow[d] \\
\mathrm{T}^*V \times \overline{\mathrm{T}^*M} \arrow[r, "\mathrm{S}_{V,M}"'] & \mathrm{T}^*(M \times V)
\end{tikzcd}$$

where the left vertical arrow denotes the reduction defined by the coisotropic submanifold $C = \mathrm{T}^*V \times \Delta_{\mathrm{T}^*N} \times \mathrm{T}^*N$ and the right vertical arrow the reduction defined by the image of $C$ under the Schwartz-transform. Now let $L$ and $L'$ be Lagrangian immersions in $\mathrm{T}^*(M \times N)$ and $\mathrm{T}^*(N \times V)$ respectively. Then $L' \circ L \in \mathrm{Mor}(\mathrm{T}^*M, \mathrm{T}^*V)$ is the reduction of $L_2 \times L_1$ by $C$. And the Schwartz transform of $L_2 \times L_1$ reduced by the image $C_{V,N,M}$ of $C$ gives the image of $L_2 \circ L_1$ under the Schwartz transform.

**Definition 2.1.15.** A reducible pair $(C, L)$ in a symplectic manifold $P$ is called properly reducible, if the quotient of $I = L \times_P C$ by its characteristic foliation is a smooth Hausdorff manifold and the map $I \to L_C = (L \cap C)/(L \cap C^\perp)$ is proper.

We refer to Appendix A for a definition and an outline of the reduction procedure used here. Since our aim in this section is of a mainly conceptional nature, the reader who is not familiar with these notions may proceed with Definition 2.1.19 without asking for the technical conditions which allow semiclassical states to be composed.

**Lemma 2.1.16.** *If $(C, L)$ is a properly reducible pair in a symplectic manifold $P$, there exists a natural linear map*

$$|\Omega|^{1/2} L \otimes |\Omega|^{1/2} C \to |\Omega|^{1/2} L_C.$$

Now let $M$ be a smooth manifold and consider a submanifold $N \subset M$ equipped with a foliation $\mathcal{F}$ such that the leaf space $N_{\mathcal{F}}$ is a smooth Hausdorff manifold. Then

$$C_N = \{(x, p) \in \mathrm{T}^*M;\ x \in N,\ \mathcal{F}_x \subset \ker p\}$$

is a coisotropic submanifold of $\mathrm{T}^*M$ whose reduced space is the cotangent bundle $\mathrm{T}^*N_{\mathcal{F}}$ of the leaf space. If $L$ is an immersed Lagrangian submanifold of $\mathrm{T}^*M$ such that $(C_N, L)$ form a reducible pair, we denote by $I$ the fiber product $L \times_{\mathrm{T}^*M} C_N$ and consider the diagram

$$\begin{tikzcd}
L \arrow[r] & \mathrm{T}^*M \\
I \arrow[u, "r_L"] \arrow[r, "r_{C_N}"] \arrow[d, "\pi"'] & C_N \arrow[u, "i"'] \arrow[d, "p_C"] \\
L_C \arrow[r, "j"'] & \mathrm{T}^*N
\end{tikzcd}.$$

Then we have

**Lemma 2.1.17.** *In the notation of the diagram above, there is a natural isomorphism*

$$r_L^*\Phi_{L,\hbar} \to \pi^*\Phi_{L_C,\hbar}.$$

Together with the map for the density bundles we therefore obtain a natural map for the symbol spaces,

$$\mathfrak{S}_L \to \mathfrak{S}_{L_C}.$$

In the following we will employ

**Definition 2.1.18.** Semiclassical states $(L_1, \mathfrak{s}_1)$ in $\mathrm{T}^*(M \times N)$ and $(L_2, \mathfrak{s}_2)$ in $\mathrm{T}^*(N \times V)$ are called *composable* if the conormal submanifold $C_{V,N,M}$ and the immersed Lagrangian submanifold $L_2 \times L_1$ form a properly reducible pair.

We obtain for any properly reducible pair a natural linear map of symbol spaces

$$\mathfrak{S}_{L_2} \otimes \mathfrak{S}_{L_1} \to \mathfrak{S}_{L_2 \circ L_1} \tag{2.1.6}$$

given by the composition of the product map (2.1.5) and the reduction map (2.1.6). We denote the image of $\mathfrak{s}_2 \otimes \mathfrak{s}_1$ under this map by $\mathfrak{s}_2 \circ \mathfrak{s}_1$.

**Definition 2.1.19.** If a semi-classical state $(L_1, \mathfrak{s}_1)$ in $\mathrm{T}^*(M \times N)$ is composable with another semiclassical state $(L_2, \mathfrak{s}_2)$ in $\mathrm{T}^*(N \times V)$, their composition is defined as the semi-classical state $(L_2 \circ L_1, \mathfrak{s}_2 \circ \mathfrak{s}_1)$ in $\mathrm{T}^*(M \times V)$.

We have used the Schwartz transform to give a meaning to the composed symbols on the image of $L_2 \circ L_1$. On the quantum level, the analogous correspondence is given by the Schwartz kernel theorem: Let $M$ be a smooth manifold and consider distributional half-densities on $M$ as continuous $\mathbb{C}$-linear functionals on the compactly supported half-densities $|\Lambda|_0^{1/2}M$, which is given the topology of $C^\infty$-convergence. The space of distributional half-densities is denoted by $|\Lambda|_{-\infty}^{1/2}M$, which is supposed to be equipped with the weak $*$ topology[10]. If $N$ is any other manifold, a *kernel* is any element of $|\Lambda|_{-\infty}^{1/2}(M \times N)$. This then defines a linear map $K : |\Lambda|_0^{1/2}M \to |\Lambda|_{-\infty}^{1/2}N$ by duality:

$$\langle K(u), v\rangle := \langle K, u \otimes v\rangle. \tag{2.1.7}$$

We have (see e.g. [Köt69, Trè67])

**Theorem 2.1.20 (Schwartz kernel theorem).** *Every $K \in |\Omega|_{-\infty}^{1/2}(M \times N)$ defines a linear map $K : |\Omega|_0^{1/2}M \to |\Omega|_{-\infty}^{1/2}N$ by (2.1.7), which is continuous in the sense that $K(\phi_j) \to 0$ in $|\Omega|_{-\infty}^{1/2}N$ if $\phi_j \to 0$ in $|\Omega|_0^{1/2}M$. Conversely, to any such linear map, there is exactly one distribution $K$ such that (2.1.7) holds.*

Therefore, in the same way as the Schwartz transform provides an identification between $\mathrm{Mor}(\mathrm{T}^*M, \mathrm{T}^*N)$ with Lagrangian submanifolds in $\mathrm{T}^*(M \times N)$ the Schwartz kernel theorem gives a correspondence

$$|\Omega|_{-\infty}^{1/2}(M \times N) \longleftrightarrow \mathrm{Mor}(\mathscr{H}_M, \mathscr{H}_N).$$

We summarize the basic quantum-classical correspondence in

| **Object** | **Classical version** | **Quantum version** |
|---|---|---|
| basic spaces | symplectic manifold $(X, \omega)$ | Hermitian vector space $\mathscr{H}_X$ |
| | product $X \times Y$ | tensor product $\mathscr{H}_X \otimes \mathscr{H}_Y$ |
| | dual $\overline{X}$ | dual $\mathscr{H}_X^*$ |
| | morphisms $X \times \overline{Y}$ | morphisms $\mathrm{Mor}(\mathscr{H}_X, \mathscr{H}_Y)$ |
| | point | $\mathbb{C}$ |
| states | Lagrangian submanifolds $L \subset X$ | elements in $\mathscr{H}_X$ |
| space of observables | Poisson algebra $C^\infty(X)$ | symmetric operators on $\mathscr{H}_X$ |

### 2.1.5 Pseudodifferential and Fourier integral operators

A special case of the Lagrangian quantization procedure described in the previous section is given by pseudodifferential and Fourier integral operators. In this section we will concentrate on these objects and give a schematic presentation, while we postpone a more technical treatment of pseudodifferential operators to Section 2.2.

[10] That is the weakest topology such that all the maps $|\Lambda|_{-\infty}^{1/2}M \to \mathbb{C}$, $\lambda \mapsto \lambda(x)$, $x \in |\Lambda|_0^{1/2}M$ are continuous, see [RS72, Rud91, Wer97].

As a motivation consider $\mathrm{T}^*\mathbb{R}^d = \{(x,\xi);\ x, \xi \in \mathbb{R}^d\}$ together with the canonical quantization, where

$$\xi_j \mapsto \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_j} =: \mathcal{D}_j$$

and $x_j$ quantizes to the corresponding multiplication operator. Then an *$\hbar$-differential operator* is defined to be

$$\mathcal{B}_\hbar = \sum_{m=0}^{N} \mathcal{B}_m \hbar^{k+m}, \quad k \in \mathbb{Z}, \tag{2.1.8}$$

where each $\mathcal{B}_m$ is a polynomial in $\mathcal{D}_j$. An application to a compactly supported oscillatory test function $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi}\, u$ gives

$$\left(\mathcal{B}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi}\, u\right)(x) = (2\pi\hbar)^{-d} \iint_{\mathbb{R}^{2d}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}((x-y)\xi - \psi(y))}\, B_\hbar(x,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi,$$

where $B_\hbar = \mathrm{symb}[\mathcal{B}_\hbar]$ is the *symbol* of the differential operator, which is obtained from (2.1.8) by formally replacing $\mathcal{D}_j$ with $\xi_j$. For fixed $x$ the function $(y,\xi) \mapsto (x-y)\xi - \psi(y)$ has a non-degenerate critical point when $y = x$ and $\xi = \mathrm{d}\psi(x)$. Therefore, the principle of stationary phase gives

$$(\mathcal{B}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi}\, u)(x) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi(x)}\, u(x) B_0(x,\, \mathrm{d}\psi(x))\hbar^k + O(\hbar^{k+1}).$$

To make the connection with the Lagrangian quantization, outlined in the previous Section, we note that the phase function $\phi(x,y,\xi) = (x-y)\xi$ on $B = \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d)^*$ together with $V = \mathbb{R}^d \times \mathbb{R}^d$ and the projection $p:\ B \to V$ defines a Morse family $(B,p,\phi)$ which generates the conormal bundle $L_\Delta$ to the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$. The symbol $B_0$ can be written, by using the canonical global densities, as

$$\mathfrak{a} = B_0(x,\xi)|\,\mathrm{d}x|^{1/2}|\,\mathrm{d}y|^{1/2}|\,\mathrm{d}\xi|.$$

This gives an amplitude on $B$ whose restriction to the fiber critical set $\Sigma_\phi = \{(x,x,\xi);\ x\cdot\xi = 0\}$ induces a well-defined symbol $\mathfrak{s}_B = B_0(x,x,\xi)$ on $L_\Delta$. Now we have

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi}\, u = \mathrm{I}_\hbar(L,\mathfrak{s}),$$

where $L$ is the projectable Lagrangian submanifold of $\mathrm{T}^*\mathbb{R}^d$ defined as the image of $\mathrm{d}\psi$, and $\mathfrak{s}$ is obtained by pulling back $u$ to $L$. Then $(L,\mathfrak{s}) = (L_\Delta \circ L, \mathfrak{s}_B \circ \mathfrak{s})$, which means that under a composition with $(L_\Delta, \mathfrak{s}_B)$ the symbol $\mathfrak{s}$ is multiplied by the values of $B_0$ on $L \simeq \Sigma_\phi$, and we obtain

$$\left(\mathcal{B}_\hbar\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\psi}\, u\right)(x) = \mathrm{I}_\hbar(L_\Delta \circ L, \mathfrak{s}_B \circ \mathfrak{s})\hbar^k + O(\hbar^{k+1}).$$

The Schwartz kernel for the operator $\mathcal{B}_\hbar$ is given by $\mathrm{I}_\hbar(L_\Delta, \mathfrak{s}_B)$, such that the application of a $\hbar$-differential operator to a semiclassical state gives a semiclassical state whose underlying

Lagrangian manifold is the composition of the original one with $L_\Delta$, and the amplitude is given as a product of the amplitudes of the state and of the operator.

In particular, this shows that if the symbol $B_0$ vanishes on $L$, then

$$\mathcal{B}_\hbar \mathrm{I}_\hbar(L, \mathfrak{s}) = O(\hbar^{k+1}),$$

i.e. $I_\hbar(L, \mathfrak{s})$ is a first-order approximate solution to the equation $\mathcal{B}u = 0$. In general, the zero set of the *principal symbol* $B_0$ is called the *characteristic variety* of $\mathcal{B}_\hbar$. If 0 is a regular value of the principal symbol, $B_0^{-1}(0)$ is a coisotropic submanifold of $\mathrm{T}^*M$ and semiclassical states contained in $B_0^{-1}(0)$ represent asymptotic solutions to the equation $\mathcal{B}u = 0$. In this sense, the coisotropic submanifold $B_0^{-1}(0)$[11] corresponds to the kernel of $\mathcal{B}_\hbar$ in $\mathscr{H}_M$ and the reduction projection quantizes to the orthogonal projection onto this subspace.

A generalization of the above constructions is given by the notion of *pseudodifferential operators*, whose local version reads

$$(\mathcal{B}u)(x) = (2\pi\hbar)^{-d} \iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi}\, B_\hbar(x,\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi, \tag{2.1.9}$$

where $B_\hbar(x, \xi)$ is no longer restricted to be a polynomial in $\xi$. As in the case of a differential operator, the invariance of the principal symbol of a pseudodifferential operator under coordinate transformations (see e.g. [Rob87, DS99, Shu01, Tay81]) can be used to extend the above definition to a well-defined notion on manifolds, see [Hör90b, Hör85a, GS94]. In this case, the principal symbol of a pseudodifferential operator is a well-defined function on the cotangent bundle of the manifold.

A further generalization of pseudodifferential operators is given by the quantization of Lagrangian submanifolds that are no longer conormal bundles to the diagonal, i.e. the phase function has to be replaced by a generating function $\phi$ of the Lagrangian submanifold. Therefore, we replace the function $(x, y, \xi) \mapsto (x-y)\cdot\xi$ by a phase function $\phi$ that generates $L$,

$$(\mathcal{B}u)(x) = (2\pi\hbar)^{-d} \iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi(x,y,\xi)}\, a(x, y, \xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi.$$

Since we won't need the general framework of Fourier integral operators, we will be concerned with only a few special cases and refer to [Dui96, GS90, Hör71, DH72, Trè80a, Trè80b] for a detailed description.

The simplest generalizations of a pseudodifferential operator are semiclassical states whose underlying Lagrangian submanifold is a conormal bundle[12]. Let $M$ be a smooth manifold with smooth submanifold $N$. The conormal bundle $L_N \subset \mathrm{T}^*M$ is generated by a single Morse family $(B, V, p_V, \phi)$, where $V$ is a tubular neighbourhood of $N$ in $M$ and $B$ is a vector bundle over $V$ with fiber-dimension equal to the codimension of $N$ in $M$, see Example 2.1.12. As in the case of a pseudodifferential operator, an amplitude $\mathfrak{a}$ on $B$ gives rise to a well-defined family of distributional half-densities $\mathrm{I}_\hbar(L, \mathfrak{s})$ on $M$. Now suppose that

[11] more precisely, the reduced manifold
[12] but not necessarily of the diagonal!

the amplitude $\mathfrak{a}$ has an asymptotic expansion in terms that are positively homogeneous with respect to the natural $\mathbb{R}_+$-action on the fibers of $p_V : B \to V$. Then the fiber critical set is invariant under the $\mathbb{R}_+$-action and the identification $\lambda_\phi : \Sigma_\phi \xrightarrow{\sim} L_N$ is equivariant with respect to the natural $\mathbb{R}_+$-action on $L_N$. Therefore, positively homogeneous symbols on $L_N$ induce positively homogeneous amplitudes on $B$ of the same order and we can define $\mathrm{I}_\hbar(L, \mathfrak{s})$ as before by requiring that $\mathfrak{s}$ be homogeneous.

Let $M = X \times Y$ be a product manifold. Then the distributions $\mathrm{I}_\hbar(L, \mathfrak{s})$ represent Schwartz kernels for continuous linear operators $|\Lambda|_0^{1/2} X \to |\Lambda|_\infty^{1/2} Y$. Under certain additional assumptions, which are in particular fulfilled if $N$ is the graph of a canonical transformation, these operators map $|\Lambda|_0^{1/2} X$ to $|\Lambda|_0^{1/2} Y$ and therefore extend continuously to $|\Lambda|_{-\infty}^{1/2} X \to |\Lambda|_{-\infty}^{1/2} Y$. From the semiclassical point of view the above considerations yield the following correspondence: The quantization of a (classical) observable results in a pseudodifferential operator, while canonical transformations find their quantum counterparts in Fourier integral operators.

## 2.2 Semi-quantum-classical analysis

After the fairly general and schematic considerations above we now turn to the question of how to incorporate internal degrees of freedom. In this Section we will focus on pseudo-differential calculus which is considered to model operators on the product Hilbert space $\mathscr{H}_M \otimes \mathscr{H}_{\text{int}}$, where $\mathscr{H}_M$ is the Hilbert space associated to the cotangent bundle of a manifold $M$ and $\mathscr{H}_{\text{int}}$ is a finite-dimensional Hilbert space corresponding to the internal degrees of freedom. For the moment, however, we do not specify the character of the internal degrees of freedom further and therefore have to consider Hilbert bundles, which in the finite-dimensional case are hermitian vector bundles, over the symplectic manifolds on which the translational degrees of freedom are modeled. A suitable modification of the Lagrangian quantization procedure is given by allowing the amplitudes to take values in the homomorphisms of these vector bundles. This is equivalent to only describing the degrees of freedom corresponding to the cotangent bundle on a semiclassical level and retaining the quantum nature of the internal degrees of freedom. The states are given by distributional sections of the corresponding vector bundles, see [Hör85a, Hör85b]. These objects are very rich in structure and have appeared in various fields of physics and mathematics during the last years, see [RVW96]. In particular, symbols with values in the homomorphisms of a vector bundle over a symplectic manifold provide an accessible example for a Lie algebroid, see also [Con94, Wei01, CW99, Wei91a, Wei91b] and also [Mac87, Ren80]. Therefore we are led to consider a new (pseudo-)category whose objects are hermitian vector bundles over symplectic manifolds, and the morphisms are the canonical relations taking values in the endomorphisms of the vector bundles. Here the minimal object is the zero-dimensional symplectic manifold together with a rank-one projection. The connection between our type of problem and some areas of noncommutative geometry becomes even more apparent, if we interpret the internal degrees of freedom as manifestation of a gauge bundle structure, see Section B.4, which then naturally leads to the notion of a gauge groupoid [Lan93, Lan92, CW99, Ati57].

We start by considering a hermitian vector bundle with typical fiber $\mathbb{C}^n$ over $\mathrm{T}^*\mathbb{R}^d \simeq \mathbb{R}^d \times \mathbb{R}^d$, which of course corresponds to the local construction on manifolds if we ignore global topological effects and their influence on the bundle structure. In this case, the base is contractible and the bundle is trivial[13]. So we have to consider quantization of conormal bundles to the diagonal where the amplitudes take values in the matrices.

### 2.2.1 Matrix-valued pseudodifferential calculus

In this section, we give an outline of the matrix-valued pseudodifferential calculus which will be used in the sequel. The corresponding results for scalar objects are well-known, cf. [Rob87, DS99, Hör]. Most of the results carry over to the matrix-valued case, if one takes into account some mild modifications; this program has been performed in [Gla00], see also [BG00, Cor95, BK85, Teu03, MF81]. Here we will closely follow [BG04a].

The quantities we are primarily concerned with are linear and continuous operators $\mathcal{B} : \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d) \otimes \mathbb{C}^n$ with Schwartz kernels $K_\mathcal{B}$ taking values in the $n \times n$ matrices $\mathrm{M}_n(\mathbb{C})$. Instead of using a kernel $K_\mathcal{B} \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ an operator $\mathcal{B}$ can alternatively be represented by its (Weyl) symbol $B \in \mathscr{S}'(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ that is related to the Schwartz kernel through[14]

$$K_\mathcal{B}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi}\, B\Big(\frac{x+y}{2},\xi\Big)\ \mathrm{d}\xi. \tag{2.2.1}$$

Here $\hbar \in (0,\hbar_0]$, with $\hbar_0 > 0$, serves as a semiclassical parameter and $\mathrm{T}^*\mathbb{R}^d := \mathbb{R}^d \times \mathbb{R}^d$ denotes the cotangent bundle of the configuration space $\mathbb{R}^d$, i.e., $\mathrm{T}^*\mathbb{R}^d$ is the phase space of the translational degrees of freedom. Below (see Chapters 4, 5 and 6) $\mathrm{T}^*\mathbb{R}^d$ will provide one component of a certain combined phase space, which also represents the degrees of freedom described by the matrix character of the symbol in terms of points on a suitable symplectic manifold.

According to the Schwartz kernel theorem every continuous linear map $\mathcal{B} : \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d) \otimes \mathbb{C}^n$ can be viewed as an operator with kernel of the above form. However, operators with kernels in $\mathscr{S}'(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ are too general for many purposes; e.g., they can in general not be composed with each other. One therefore has to restrict to smaller sets of kernels and hence to smaller classes of symbols. To achieve this we make use of order functions $m : \mathrm{T}^*\mathbb{R}^d \to [1,\infty)$, which have to fulfill a certain growth property in the sense that there are positive constants $C$, $N$ such that

$$m(x,\xi) \le C\left(1+(x-y)^2+(\xi-\eta)^2\right)^{N/2} m(y,\eta) \tag{2.2.2}$$

for all $(x,\xi),(y,\eta) \in \mathrm{T}^*\mathbb{R}^d$. A typical example for such an order function is given by

$$m(x,\xi) = \left(1+x^2+\xi^2\right)^M, \quad M \ge 0.$$

[13] Of course, in the case of a manifold that is not euclidean one has to deal with the problem of gluing together the (local) constructions on euclidean space.

[14] Note that this prescription differs (in an unessential way) from the one given in equation (2.1.9), where pseudodifferential operators according to [KN65] are used.

This notion allows us to define the symbol classes which we will employ in the subsequent discussions (see [DS99]).

**Definition 2.2.1.** Let $m : \mathrm{T}^*\mathbb{R}^d \to [1,\infty)$ be an order function. Then define the symbol class $\mathrm{S}(m) \subset C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ to be the set of $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ such that for every $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$ and all $\alpha,\beta \in \mathbb{N}_0^d$ there exist constants $C_{\alpha,\beta} > 0$ with

$$\|\partial_\xi^\alpha \partial_x^\beta B(x,\xi)\|_{n\times n} \le C_{\alpha,\beta} m(x,\xi). \tag{2.2.3}$$

Here $\|\cdot\|_{n\times n}$ denotes an arbitrary (matrix) norm on $\mathrm{M}_n(\mathbb{C})$. If in addition the symbol $B(x,\xi;\hbar)$ depends on the parameter $\hbar \in (0,\hbar_0]$, we say that $B \in \mathrm{S}(m)$ if $B(\cdot,\cdot;\hbar)$ is uniformly bounded in $\mathrm{S}(m)$ when $\hbar$ varies in $(0,\hbar_0]$. In particular, for $q \in \mathbb{R}$ let $\mathrm{S}^q(m)$ consist of $B : \mathrm{T}^*\mathbb{R}^d \times (0,\hbar_0] \to \mathrm{M}_n(\mathbb{C})$ belonging to $\hbar^{-q}\,\mathrm{S}(m)$, i.e.,

$$\|\partial_\xi^\alpha \partial_x^\beta B(x,\xi;\hbar)\|_{n\times n} \le C_{\alpha,\beta} \hbar^{-q} m(x,\xi)$$

for all $\alpha,\beta \in \mathbb{N}_0^d$, $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$, and $\hbar \in (0,\hbar_0]$.

An asymptotic expansion of $B \in \mathrm{S}^{q_0}(m)$ is defined by a sequence $\{B_j \in \mathrm{S}^{q_j}(m)\}_{j\in\mathbb{N}_0}$ of symbols, where $q_j$ decreases monotonically to $-\infty$ and

$$B - \sum_{j=0}^{N} B_j \in \mathrm{S}^{q_{N+1}}(m)$$

for all $N \in \mathbb{N}_0$. In this case we write

$$B \sim \sum_{j=0}^{\infty} B_j.$$

In the following we will often use the class $\mathrm{S}^q_{\mathrm{cl}}(m)$ of classical symbols, whose elements have asymptotic expansions in integer powers of $\hbar$,

$$B \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B_j,$$

where $B_j \in \mathrm{S}(m)$ is independent of $\hbar$. We also use the notation

$$\mathrm{S}^\infty(m) := \bigcup_{q\in\mathbb{R}} \mathrm{S}^q(m) \quad \text{and} \quad \mathrm{S}^{-\infty}(m) := \bigcap_{q\in\mathbb{R}} \mathrm{S}^q(m).$$

An operator with a kernel of the form (2.2.1) and symbol $B \in \mathrm{S}(m)$ clearly maps both $\mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ and $\mathscr{S}'(\mathbb{R}^d) \otimes \mathbb{C}^n$ into themselves, whereby according to (2.2.1) it acts on $\mathbb{C}^n$-valued functions $\psi \in \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ as

$$(\mathcal{B}\psi)(x) = \big(\mathrm{op}^W[B]\psi\big)(x) = \frac{1}{(2\pi\hbar)^d} \iint_{\mathrm{T}^*\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi}\, B\Big(\frac{x+y}{2},\xi\Big)\psi(y)\ \mathrm{d}y\ \mathrm{d}\xi.$$

Operators $\mathcal{B} = \operatorname{op}^W[B]$ of this type are called *Weyl operators*, and $\operatorname{symb}^W[\mathcal{B}] = B$ denotes the *Weyl symbol* of $\mathcal{B}$. Furthermore, we will sometimes use $\mathrm{OPS}(m)$ to denote Weyl operators with symbols in $\mathrm{S}(m)$. If the Weyl symbol of an operator is a classical symbol with asymptotic expansion $B \sim \sum_{j\in\mathbb{N}_0} \hbar^{-q+j} B_j$, we also call $\operatorname{op}^W[B]$ a semiclassical pseudodifferential operator. The leading order term $\operatorname{symb}^W_P[\mathcal{B}] = B_0$ is then referred to as the principal symbol of $\mathcal{B}$, and the subsequent term $B_1$ as the subprincipal symbol.

The set of Weyl operators with symbols from the classes $\mathrm{S}(m)$ is stable under operator multiplication, in the sense that the operator product is again a Weyl operator with symbol in a certain class:

**Lemma 2.2.2.** *Let $m_1, m_2$ be order functions. Then for $B_j \in \mathrm{S}(m_j)$, $j = 1, 2$, the product of the corresponding operators $\mathcal{B}_j = \operatorname{op}^W[B_j]$ is again a Weyl operator that can be expressed in terms of the symbols $B_1, B_2$ as*

$$\mathcal{B}_1\mathcal{B}_2 = \operatorname{op}^W[B_1]\operatorname{op}^W[B_2] = \operatorname{op}^W[B_1\#B_2],$$

*where the symbol product $(B_1, B_2) \mapsto B_1\#B_2$ is continuous from $\mathrm{S}(m_1)\times\mathrm{S}(m_2)$ to $\mathrm{S}(m_1m_2)$ in the topology generated by the seminorms associated with the estimate (2.2.3). In explicit terms the symbol product reads*

$$(B_1\#B_2)(x,\xi) = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\omega(\partial_x,\partial_\xi;\partial_y,\partial_\eta)}\, B_1(x,\xi)B_2(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}},$$

*where $\omega(v_x, v_\xi; w_x, w_\xi) = v_x \cdot w_\xi - v_\xi \cdot w_x$ denotes the symplectic two-form on $\mathrm{T}^*\mathbb{R}^d$. Furthermore, $B_j \in \mathrm{S}^0_{\mathrm{cl}}(m_j)$ are mapped to $B_1\#B_2 \in \mathrm{S}^0_{\mathrm{cl}}(m_1m_2)$ with (classical) asymptotic expansion*

$$(B_1\#B_2)(x,\xi) \sim \sum_{k,j_1,j_2\in\mathbb{N}_0} \frac{\hbar^{k+j_1+j_2}}{k!}\left(\frac{\mathrm{i}}{2}\omega(\partial_x,\partial_\xi;\partial_y,\partial_\eta)\right)^k B_{1,j_1}(x,\xi)B_{2,j_2}(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}}.$$

The following result, which in its original version is due to Beals [Bea77], is useful in situations where one wishes to identify a given operator as a pseudodifferential operator.

**Lemma 2.2.3.** *Let $\mathcal{B}(\hbar) : \mathscr{S}(\mathbb{R}^d)\otimes\mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d)\otimes\mathbb{C}^n$ be a linear and continuous operator depending on the semiclassical parameter $\hbar \in (0, \hbar_0]$. The following statements are then equivalent:*

*(i)* *$\mathcal{B}(\hbar) = \operatorname{op}^W[B]$ is a Weyl operator with symbol $B \in \mathrm{S}^0(1)$.*

*(ii)* *For every sequence $l_1(x,\xi), \dots, l_N(x,\xi)$, $N \in \mathbb{N}$, of linear forms on $\mathrm{T}^*\mathbb{R}^d$ the operator given by the multiple commutator $[\operatorname{op}^W[l_N], [\operatorname{op}^W[l_{N-1}], \cdots, [\operatorname{op}^W[l_1], \mathcal{B}]\cdots]$ is bounded as an operator on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ and its norm is of the order $\hbar^N$.*

The direction (i)⇒(ii) is a simple consequence of the symbolic calculus outlined above. For the reverse direction see [Gla00, HS88, DS99].

In the discussions below we will basically encounter two types of (Weyl) operators: quantum Hamiltonians $\mathcal{H} = \mathrm{op}^W[H]$ with symbols $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ generating the quantum mechanical time evolution, and observables $\mathcal{B} = \mathrm{op}^W[B]$. In typical cases a Hamiltonian $\mathcal{H}$ is given and one is interested in a suitable algebra of observables that allows us to study dynamical properties of the quantum system. For this purpose it is often convenient to consider bounded operators. In the scalar case it is sufficient to know the boundedness of the symbols in order to obtain a bounded Weyl operator. This result, originally going back to Calderón and Vaillancourt [CV71], generalises to the context of pseudodifferential operators with matrix valued symbols without changes.

**Proposition 2.2.4.** *Let $B(\hbar) \in \mathrm{S}(1)$, then the Weyl quantisation $\mathrm{op}^W[B(\hbar)]$ of this symbol is continuous on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. Furthermore, for $\hbar \in (0, \hbar_0]$ there exists an upper bound for the operator norm of $\mathrm{op}^W[B(\hbar)]$.*

For a proof of this result in the context of semiclassical pseudodifferential operators (depending on a parameter $\hbar$) see [Rob87, HS88, DS99].

A quantum Hamiltonian is required to be (essentially) selfadjoint. Thus, in the case of a Weyl operator $\mathcal{H} = \mathrm{op}^W[H]$ one requires the symbol $H$ to take values in the hermitian $n \times n$ matrices. In order to trace back spectral properties of $\mathcal{H}$ to properties of the principal symbol $H_0$ we will have to construct (asymptotic) inverses of $\mathcal{H} - z$ and relate them to $(H_0 - z)^{-1}$. In this context an operator $\mathcal{B} = \mathrm{op}^W[B]$ is called *elliptic*, if its symbol $B \in \mathrm{S}(m)$ is invertible, i.e., if the matrix inverse $B^{-1}$ exists in $\mathrm{S}(m^{-1})$. In such a case one can construct a parametrix $Q \in \mathrm{S}(m^{-1})$ which is an asymptotic inverse of $B$ in the sense of symbol products,

$$B \# Q \sim Q \# B \sim 1.$$

To see that such an inverse exists for elliptic operators, consider

$$\mathrm{op}^W[B]\,\mathrm{op}^W[B^{-1}] = 1 - \hbar\,\mathrm{op}^W[R],$$

with $R \in \mathrm{S}(m)$. For sufficiently small $\hbar$ the operator $1 - \hbar\,\mathrm{op}^W[R]$ possesses a bounded inverse and one can define a (left and right) inverse $\mathrm{op}^W[B^{-1}](1 - \hbar\,\mathrm{op}^W[R])^{-1}$ for $\mathrm{op}^W[B]$. Furthermore, the Beals characterisation of pseudodifferential operators (Lemma 2.2.3) implies that this inverse is again a bounded pseudodifferential operator, see also [DS99]. To obtain an asymptotic expansion for the parametrix $Q$ one next defines the operator $\mathcal{Q}_N := \mathrm{op}^W[B^{-1}](1 + \hbar\mathcal{R} + \cdots + \hbar^N \mathcal{R}^N)$, with $\mathcal{R} = \mathrm{op}^W[R]$, which is equivalent to $\mathcal{Q} = \mathrm{op}^W[Q]$ modulo terms of order $\hbar^{N+1}$. One can hence write

$$Q \sim B^{-1} + \hbar(B^{-1} \# R) + \hbar^2(B^{-1} \# R \# R) + \cdots, \tag{2.2.4}$$

and finally observes:

**Lemma 2.2.5.** *Let $B \in \mathrm{S}(m)$ be elliptic in the sense that $B^{-1}(x,\xi)$ exists for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$ and is in the class $\mathrm{S}(m^{-1})$. Then there exists a parametrix $Q \in \mathrm{S}(m^{-1})$ with an asymptotic expansion of the form (2.2.4) such that*

$$B \# Q \sim Q \# B \sim 1.$$

From (2.2.4) one moreover observes that an elliptic operator with classical symbol has a parametrix that is a classical symbol.

Frequently it is very convenient to have a functional calculus of pseudodifferential operators available. In some places, e.g., we would like to apply the Helffer-Sjöstrand formula, which shows that a smooth and compactly supported function $f \in C_0^\infty(\mathbb{R})$ of an essentially selfadjoint operator $\mathcal{B}$ with symbol in $B \in \mathrm{S}(m)$ yields a pseudodifferential operator

$$f(\mathcal{B}) = -\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{f}(z)(\mathcal{B}-z)^{-1}\ \mathrm{d}z,$$

whose symbol is in $\mathrm{S}(m^{-N})$ for every $N \in \mathbb{N}$. Here $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an almost-analytic extension of $f$, i.e., $\tilde{f}(z) = f(z)$ for $z \in \mathbb{R}$ and $|\bar{\partial}\tilde{f}(z)| \leq C_N|\mathrm{Im}\, z|^N$ for all $N \in \mathbb{N}_0$. In the scalar case these results were shown in [HS89] (see also [DS99]) and have been extended to the matrix valued situation in [Dim93, Dim98]. A criterion that guarantees the essential selfadjointness of $\mathcal{B}$ is that first its symbol $B \in \mathrm{S}(m)$ is hermitian and, second, that $B + \mathrm{i} \in \mathrm{S}(m)$ is elliptic in the sense described above. If $B \in \mathrm{S}^0_{\mathrm{cl}}(m)$ one can even write down a classical asymptotic expansion for the symbol of the operator $f(\mathcal{B})$ whose principal symbol reads $f(B_0)$, where $B_0$ denotes the principal symbol of $\mathcal{B}$, see [Rob87, DS99].

## 2.3 Time evolution and semiclassical restriction to eigenspaces

### 2.3.1 Semiclassical projections

We motivate the following construction of semiclassical projection operators by considering the time evolution generated by a quantum Hamiltonian $\mathcal{H}$, i.e., the Cauchy problem

$$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\psi(t) = \mathcal{H}\psi(t) \tag{2.3.1}$$

for an essentially selfadjoint operator $\mathcal{H}$ defined on a dense domain $D(\mathcal{H})$ in the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. If one introduces the strongly continuous one-parameter group of unitary operators $\mathcal{U}(t) := \exp(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t)$, $t \in \mathbb{R}$, a solution of (2.3.1) can be obtained by defining $\psi(t) := \mathcal{U}(t)\psi_0$ for $\psi_0 \in D(\mathcal{H})$. Therefore the time evolution $\mathcal{B}(t) := \mathcal{U}(t)^*\mathcal{B}\mathcal{U}(t)$ of a bounded operator $\mathcal{B} \in B(L^2(\mathbb{R}^d) \otimes \mathbb{C}^n)$ in the Heisenberg picture has to fulfill the (Heisenberg) equation of motion

$$\frac{\partial}{\partial t}\mathcal{B}(t) = \frac{\mathrm{i}}{\hbar}[\mathcal{H}, \mathcal{B}(t)]. \tag{2.3.2}$$

If one assumes $\mathcal{B}$ and $\mathcal{H}$ to be semiclassical pseudodifferential operators with symbols in the classes $\mathrm{S}^q_{\mathrm{cl}}(1)$ and $\mathrm{S}^0_{\mathrm{cl}}(m)$, respectively, equation (2.3.2) yields in leading semiclassical order an equation for the principal symbols:

$$\frac{\partial}{\partial t}B_0(t) = \frac{\mathrm{i}}{\hbar}[H_0, B_0(t)] + O(\hbar^0), \quad \hbar \to 0.$$

If one now requires the time evolution to respect the filtration of the algebra $\mathrm{S}^{\infty}_{\mathrm{cl}}(1) := \bigcup_{q\in\mathbb{Z}} \mathrm{S}^{q}_{\mathrm{cl}}(1)$ then, in particular, the principal symbol $B_0(t)$ should stay in its class which derives from the associated grading $\mathrm{S}^{q}_{\mathrm{cl}}(1)/\mathrm{S}^{q-1}_{\mathrm{cl}}(1)$, $q \in \mathbb{Z}$. One thus has to restrict to operators whose principal parts $B_0$ commute with the principal symbol $H_0$ of the operator $\mathcal{H}$. This condition is equivalent to a block-diagonal form of $B_0$,

$$B_0(x,\xi) = \sum_{\mu=1}^{l} P_{\mu,0}(x,\xi) B_0(x,\xi) P_{\mu,0}(x,\xi), \tag{2.3.3}$$

with respect to the projection matrices $P_{\mu,0} : \mathrm{T}^*\mathbb{R}^d \to \mathrm{M}_n(\mathbb{C})$, $\mu = 1,\dots,l$, that project onto the eigenspaces corresponding to the eigenvalue functions $\lambda_\mu : \mathrm{T}^*\mathbb{R}^d \to \mathbb{R}$ of the hermitian principal symbol matrix $H_0 : \mathrm{T}^*\mathbb{R}^d \to \mathrm{M}_n(\mathbb{C})$. Since (2.3.3) is the semiclassical limit of the symbol of the operator $\hbar^q \sum_{\mu=1}^{l} \mathrm{op}^W[P_{\mu,0}]\mathcal{B}\,\mathrm{op}^W[P_{\mu,0}]$, when $\mathcal{B}$ is a semiclassical Weyl operator with symbol $B \in \mathrm{S}^{q}_{\mathrm{cl}}(1)$, one can ask how the symbols $P_{\mu,0}$, which are projectors onto the eigenspaces of $H_0$ in $\mathbb{C}^n$, are related to projection operators onto (almost) invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ with respect to $\mathcal{H} = \mathrm{op}^W[H]$. We are hence looking for quantisations $\tilde{\mathcal{P}}_\mu$ of symbols $P_\mu \in \mathrm{S}^{0}_{\mathrm{cl}}(1)$, with principal symbols $P_{\mu,0}$, which are (almost) orthogonal projections, i.e.,

$$\tilde{\mathcal{P}}_\mu \tilde{\mathcal{P}}_\mu = \tilde{\mathcal{P}}_\mu = \tilde{\mathcal{P}}_\mu^* \mod O(\hbar^\infty) \tag{2.3.4}$$

in the operator norm. Moreover, in order that these operators map to almost invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ with respect to the time evolution $\mathcal{U}(t) = \exp(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t)$ generated by $\mathcal{H}$, we require them to fulfill

$$\|[\mathcal{H}, \tilde{\mathcal{P}}_\mu]\|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n} = 0 \mod O(\hbar^\infty). \tag{2.3.5}$$

As it will turn out, it is even possible to modify the operators $\tilde{\mathcal{P}}_\mu$ in such a way that they satisfy the relation (2.3.4) exactly, i.e., not only mod $O(\hbar^\infty)$.

The above requirements lead us to consider (formal) asymptotic expansions for the symbols $P_\mu$,

$$P_\mu(x,\xi) \sim \sum_{j=0}^{\infty} \hbar^j P_{\mu,j}(x,\xi),$$

which satisfy (2.3.4) and (2.3.5) on a (formal) symbol level:

$$P_\mu \# P_\mu \sim P_\mu \sim P_\mu^*, \tag{2.3.6}$$

and

$$[P_\mu, H]_\# := P_\mu \# H - H \# P_\mu \sim 0. \tag{2.3.7}$$

The solutions of the above equations will be called semiclassical projections and can be constructed by two different methods. The first one is based on solving the recursive problem arising from (2.3.6) and (2.3.7) by employing asymptotic expansions of $P_\mu$ and

$H$ in $\mathrm{S}^0_{\mathrm{cl}}(1)$ and $\mathrm{S}^0_{\mathrm{cl}}(m)$, respectively, using the symbolic calculus outlined in section 2.2.1 and finally equating equal powers of the semiclassical parameter $\hbar$. For this procedure cf. [EW96, BN99]. The second method employs the Riesz projection formula in the context of pseudodifferential calculus [HS88, NS01, Nen93, Nen02]. In the following we will pursue the latter method.

To this end we consider the matrix valued hermitian principal symbol $H_0 \in \mathrm{S}(m)$ of the operator $\mathcal{H}$, and in the following we assume:

(H0) The (real) eigenvalues $\lambda_\mu$, $\mu = 1, \dots, l$, of $H_0$ have constant multiplicities $k_1, \dots, k_l$ and fulfill the hyperbolicity condition

$$|\lambda_\nu(x,\xi) - \lambda_\mu(x,\xi)| \geq Cm(x,\xi) \quad \text{for} \quad \nu \neq \mu \quad \text{and} \quad |x| + |\xi| \geq c.$$

This requirement is analogous to a condition imposed in [Cor82] on the eigenvalues of the symbol of an operator in a strictly hyperbolic system, i.e., where the eigenvalues are non-degenerate. In particular, the problem of mode conversion that arises from points where multiplicities of eigenvalues change is avoided. Since the eigenvalues are solutions of the algebraic equation

$$\det\big(H_0(x,\xi) - \lambda\big) = \sum_{\nu=0}^{n} \eta_\nu(x,\xi)\lambda^\nu = 0, \tag{2.3.8}$$

they are smooth functions on $\mathrm{T}^*\mathbb{R}^d$. Moreover, since $H_0$ is supposed to be hermitian, the eigenvalues are bounded by the matrix norm of $H_0$. Using the smoothness of the eigenvalues and the hyperbolicity condition (H0), one obtains:

**Proposition 2.3.1.** *Let $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ be hermitian and let the hyperbolicity condition (H0) be fulfilled. Then there exist symbols $P_\mu \in \mathrm{S}^0_{\mathrm{cl}}(1)$ with asymptotic expansions*

$$P_\mu \sim \sum_{j=0}^{\infty} \hbar^j P_{\mu,j}, \quad \mu = 1, \dots, l, \tag{2.3.9}$$

*that fulfill the conditions (2.3.6) and (2.3.7). In particular, the coefficients $P_{\mu,j}$, $j \in \mathbb{N}_0$, are unique, i.e., the symbols $P_\mu$ are uniquely determined modulo $\mathrm{S}^{-\infty}(1)$.*

*Furthermore, the corresponding almost projection operators $\tilde{\mathcal{P}}_\mu = \mathrm{op}^W[P_\mu]$ provide a semiclassical resolution of the identity on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^d$,*

$$\tilde{\mathcal{P}}_1 + \dots + \tilde{\mathcal{P}}_l = \mathrm{id}_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n} \mod O(\hbar^\infty).$$

*Proof.* We use the technique of [HS88, NS01] and consider the Riesz projections

$$P_\mu(x,\xi) := \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_\mu(x,\xi)} Q(x,\xi,z)\ \mathrm{d}z, \tag{2.3.10}$$

where $\Gamma_\mu(x,\xi)$ is a simply closed and positively oriented regular curve in the complex plane enclosing the, and only the, eigenvalue $\lambda_\mu(x,\xi) \in \mathbb{R}$ of $H_0(x,\xi)$. Moreover, $Q(x,\xi,z)$

denotes a parametrix for $H-z$, i.e., $(H-z)\#Q \sim Q\#(H-z) \sim 1$ that will be constructed below. For technical considerations one may choose the contour as $\Gamma_\mu(x,\xi) = \{\lambda_\mu(x,\xi) + \rho_\mu(x,\xi)\,\mathrm{e}^{\mathrm{i}\varphi},\ 0 \le \varphi \le 2\pi\}$ with $0 < c \le \rho_\mu < \frac{1}{2}\min_{\nu\ne\mu}\{|\lambda_\mu - \lambda_\nu|\}$. Since $H_0$ is hermitian with eigenvalues $\lambda_\nu$, $\nu = 1,\dots,l$, one can estimate the matrix norm of $(H_0 - z)^{-1}$ for $z \in \Gamma_\mu(x,\xi)$ as

$$\|(H_0(x,\xi) - z)^{-1}\|_{n\times n} \le \frac{C}{\rho_\mu(x,\xi)}.$$

The condition (H0) then allows us to choose $\rho_\mu(x,\xi) \ge cm(x,\xi)$, so that $H_0 - z$ is elliptic for $z \in \Gamma_\mu$. If then $\hbar$ is sufficiently small, also $H - z = H_0 - z + O(\hbar)$ is elliptic and the relation

$$(H-z)\#(H_0-z)^{-1} = 1 - \hbar R$$

enables one to construct a parametrix $Q(x,\xi,z) \in \mathrm{S}^0_{\mathrm{cl}}(m^{-1})$ for $H - z$ with asymptotic expansion

$$Q(x,\xi,z) \sim \sum_{j=0}^{\infty} \hbar^j Q_j(x,\xi,z)$$

in the same manner as in (2.2.4), see also [Rob87, DS99]. Plugging this expansion into (2.3.10) one obtains

$$\begin{aligned} P_\mu(x,\xi) &= \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_\mu(x,\xi)} Q(x,\xi,z)\ \mathrm{d}z \\ &\sim \sum_{j=0}^{\infty} \hbar^j \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_\mu(x,\xi)} Q_j(x,\xi,z)\ \mathrm{d}z =: \sum_{j=0}^{\infty} \hbar^j P_{\mu,j}(x,\xi) \end{aligned} \tag{2.3.11}$$

by using the Borel construction, see e.g. [Fol89], to sum asymptotic series of symbols. According to the properties of the Riesz integral, the symbols $P_\mu$ therefore fulfill (2.3.6) and (2.3.7). Since these equations have unique solutions modulo $O(\hbar^\infty)$ [EW96], the coefficients $P_{\mu,j}$ are unique.

We now consider more general $z \in \mathbb{C}$, and by inspecting the above construction notice that the parametrix $Q(z)$ is well-defined as long as $z$ has a sufficiently large distance from the eigenvalues of $H_0$. According to equation (2.2.4) its asymptotic expansion then reads

$$Q(z) \sim (H_0 - z)^{-1} + \hbar(H_0 - z)^{-1}\#R(z)\#(\mathrm{id}_{\mathbb{C}^n} + \hbar R(z) + \hbar^2 R(z)\#R(z) + \cdots).$$

Since

$$R(z) = \frac{1}{\hbar}\left(1 - (H-z)\#(H_0-z)^{-1}\right),$$

it follows according to the composition formula of Lemma 2.2.2 that $R(z)$ contains a factor $(H_0 - z)^{-1}$, and therefore the only singularities of $Q(z)$ are caused by the eigenvalues of $H_0$. Thus, according to the Cauchy formula the expression

$$P_1 + \cdots + P_l = \frac{1}{2\pi\,\mathrm{i}} \int_{\bigcup_{\mu=1}^{l}\Gamma_\mu} Q(z)\ \mathrm{d}z$$

can be replaced by

$$\frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma(r)} Q(z)\ \mathrm{d}z,$$

where $\Gamma(r)$ is a contour with minimal distance $r$ from the origin in $\mathbb{C}$ that encloses all eigenvalues of $H_0$ while keeping a sufficient distance from them. The value of the above integral does not depend on the particular choice of $\Gamma(r)$ so that one can take the limit $r \to \infty$ and hence obtains

$$\lim_{r\to\infty}\frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma(r)} Q(z)\ \mathrm{d}z = \lim_{r\to\infty}\frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma(r)} (H_0 - z)^{-1}\ \mathrm{d}z = \mathrm{id}_{\mathbb{C}^n} \mod O(\hbar^\infty).$$

□

The so constructed symbols $P_\mu$ yield semiclassical almost projection operators

$$\tilde{\mathcal{P}}_\mu := \mathrm{op}^W[P_\mu]$$

which according to Proposition 2.2.4 are bounded and obviously satisfy the relations (2.3.4) and (2.3.5). Following [Nen02] one can even construct pseudodifferential operators $\mathcal{P}_\mu$ that are semiclassically equivalent to $\tilde{\mathcal{P}}_\mu$ in the sense that $\|\tilde{\mathcal{P}}_\mu - \mathcal{P}_\mu\| = O(\hbar^\infty)$, and which fulfill (2.3.4) exactly, see also [WO93, MT02]. To see this, consider the operator

$$\mathcal{P}_\mu := \frac{1}{2\pi\,\mathrm{i}}\int_{|z-1|=\frac{1}{2}} (\tilde{\mathcal{P}}_\mu - z)^{-1}\ \mathrm{d}z, \tag{2.3.12}$$

which is well-defined since the spectrum of $\tilde{\mathcal{P}}_\mu$ is concentrated near 0 and 1. Thus $\mathcal{P}_\mu$ is an orthogonal projector acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$, with $\|[\mathcal{P}_\mu,\mathcal{H}]\| \le c\|[\tilde{\mathcal{P}}_\mu,\mathcal{H}]\| = O(\hbar^\infty)$[15] Since $\mathcal{P}_\mu$ is close to $\tilde{\mathcal{P}}_\mu$ in operator norm, Beals' characterisation of pseudodifferential operators (see Lemma 2.2.3) yields that $\mathcal{P}_\mu$ is again a pseudodifferential operator with symbol in the class $\mathrm{S}^0(1)$. This has already been noticed in [NS01] and follows from the fact that $(\tilde{\mathcal{P}}_\mu - z)^{-1}$ for $|z-1| = 1/2$ is a pseudodifferential operator according to the parametrix construction of Lemma 2.2.5. Having projectors available, one can also construct (pseudo-differential) unitary transformations of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ which convert $\mathcal{H}$ by conjugation in an almost block-diagonal form, see [Cor83b, LF91, BR99, NS01, PST03]. Such unitary transformations are obviously not unique, and since for most purposes it suffices to work with the projectors we hence refrain from using the unitary operators here.

In view of the fact that $\mathcal{P}_\mu$ is an orthogonal projector on the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$, one can ask if it is possible to satisfy also the relation (2.3.5) exactly. In other words, to what extent can $\mathcal{P}_\mu$ be related to a spectral projection of $\mathcal{H}$? (See [HS88, Cor00, Cor01] for examples.) We want to illustrate this question in the case where the principal symbol $H_0$ of $\mathcal{H}$ possesses two well-separated eigenvalues $\lambda_\nu < \lambda_{\nu+1}$ with constant multiplicities $k_\nu$ and $k_{\nu+1}$, respectively, among the eigenvalues $\lambda_1,\dots,\lambda_l$. For $l = 2$ this is exactly the

[15] That the commutator is bounded in the operator norm follows from the observation that $\dot{\mathcal{P}}_\mu(t) = [\mathcal{P}_\mu(t),\mathcal{H}]$ and the Egorov Theorem 2.3.4.

situation that occurs in the case of a Dirac-Hamiltonian that we discuss in some detail in [BG04c, BG03] and in Chapter 7. We also assume that there exists $\lambda \in \mathbb{R}$ separated from the spectrum $\operatorname{spec}(\mathcal{H})$ of $\mathcal{H}$ such that

$$\lambda - \lambda_\nu(x,\xi) > Cm(x,\xi) \quad \text{and} \quad \lambda_{\nu+1}(x,\xi) - \lambda > C'm(x,\xi) \tag{2.3.13}$$

for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$. It follows from these assumptions that one can replace the contour $\Gamma_< := \bigcup_{\mu=1}^{\nu} \Gamma_\mu$ in

$$P_<(x,\xi) := \sum_{\mu=1}^{\nu} P_\mu(x,\xi) = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_<} Q(x,\xi,z)\ \mathrm{d}z, \tag{2.3.14}$$

see (2.3.10), by a straight line $\Gamma_+ := \{z \in \mathbb{C};\ z = \lambda + \mathrm{i}\,t,\ t \in \mathbb{R}\}$ that avoids the eigenvalues of the principal symbol $H_0$ as well as the spectrum of $\mathcal{H}$. Correspondingly, $\Gamma_> := \bigcup_{\mu=\nu+1}^{l} \Gamma_\mu$ is deformed into $\Gamma_-$ given by $\Gamma_+$ with reversed orientation. Thus,

$$P_{\lessgtr}(x,\xi) = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_\pm} Q(x,\xi,z)\ \mathrm{d}z.$$

and $\tilde{\mathcal{P}}_< = \mathrm{op}^W[P_<]$ is semiclassically equivalent to the spectral projection of $\mathcal{H}$ onto the interval $(-\infty,\lambda)$ given by

$$\mathbb{1}_{(-\infty,\lambda)}(\mathcal{H}) = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_+} (\mathcal{H} - z)^{-1}\ \mathrm{d}z,$$

whereas $\tilde{\mathcal{P}}_>$ corresponds to $\mathbb{1}_{(\lambda,\infty)}(\mathcal{H})$. Therefore we have

**Proposition 2.3.2.** *If the eigenvalues $\lambda_1,\dots,\lambda_l$ of the principal symbol $H_0$ are separated according to (H0) and the condition (2.3.13) is fulfilled, the almost projection operators $\tilde{\mathcal{P}}_{\lessgtr} := \mathrm{op}^W[P_{\lessgtr}]$, whose symbols are defined in (2.3.14), can be semiclassically identified with the spectral projections $\mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})$ and $\mathbb{1}_{(\lambda,\infty)}(\mathcal{H})$ of the operator $\mathcal{H}$ to the intervals $(-\infty,\lambda)$ and $(\lambda,\infty)$, respectively. This means*

$$\|\tilde{\mathcal{P}}_< - \mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})\| = O(\hbar^\infty) \quad \textit{and} \quad \|\tilde{\mathcal{P}}_> - \mathbb{1}_{(\lambda,\infty)}(\mathcal{H})\| = O(\hbar^\infty).$$

*A corresponding statement holds for the related orthogonal projectors $\mathcal{P}_{\lessgtr}$,*

$$\|\mathcal{P}_< - \mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})\| = O(\hbar^\infty) \quad \textit{and} \quad \|\mathcal{P}_> - \mathbb{1}_{(\lambda,\infty)}(\mathcal{H})\| = O(\hbar^\infty).$$

### 2.3.2 Time evolution and Egorov Theorem

In this section our aim is to identify a suitable class of operators that is left invariant by the time evolution. Recalling the reasoning from the beginning of Section 2.3.1, we are interested in a subalgebra of $\mathrm{S}^\infty_{\mathrm{cl}}(1)$ whose filtration is respected by the time evolution generated by the one-parameter group $\mathcal{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t\right)$, where $\mathcal{H}$ is an essentially selfadjoint pseudodifferential operator with symbol $H$ in the class $\mathrm{S}^0_{\mathrm{cl}}(m)$. The following assumptions on the symbol $H$ guarantee the essential selfadjointness of $\mathcal{H}$ on $\mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ (see [DS99]):

(H1) $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ is hermitian,

(H2) $H_0 + \mathrm{i}$ is elliptic in the sense that $\|(H_0(x,\xi)+\mathrm{i})^{-1}\|_{n\times n} \leq cm(x,\xi)^{-1}$.

Under the assumptions (H1) and (H2), $\mathcal{U}(t)$ therefore defines a strongly-continuous unitary one-parameter group.

We now consider the time evolution of an operator $\mathcal{B} \in B(L^2(\mathbb{R}^d)\otimes\mathbb{C}^n)$ given by

$$\mathcal{B}(t) := \mathcal{U}(t)^*\mathcal{B}\mathcal{U}(t), \tag{2.3.15}$$

which is, of course, a bounded operator on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. According to Proposition 2.2.4 the boundedness of $\mathcal{B}$ is guaranteed by choosing $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$. Moreover, a conjugation of (2.3.15) with $\sum_{\mu=1}^l \mathcal{P}_\mu = \mathrm{id}_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n} + O(\hbar^\infty)$ results in a bounded operator so that

$$\mathcal{B}(t) = \sum_{\nu,\mu=1}^{l} \mathcal{P}_\mu\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t}\, \mathcal{B}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\, \mathcal{P}_\nu = \sum_{\nu,\mu=1}^{l} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t}\, \mathcal{P}_\mu\mathcal{B}\mathcal{P}_\nu\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t} \mod O(\hbar^\infty) \tag{2.3.16}$$

in the operator norm. Here we have used the property $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}t}\,\mathcal{P}_\nu = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\,\mathcal{P}_\nu$ modulo $O(\hbar^\infty)$ that follows from the Duhamel principle. Now, the principal symbol[16] of $\mathcal{H}\mathcal{P}_\mu$ is a scalar multiple of the identity in the eigenspace $P_{\mu,0}\mathbb{C}^n$ of $H_0$ corresponding to $\lambda_\mu$, i.e., $H_0 P_{\mu,0} = \lambda_\mu P_{\mu,0}$. Thus, for $\mu = \nu$ the operator $\exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t\right)\mathcal{B}\exp\left(-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t\right)$ is a pseudodifferential operator with symbol in the class $\mathrm{S}^0(1)$, see [Ivr98, BG00]. But when $\mu \neq \nu$ the corresponding expressions are semiclassical Fourier integral operators. In that case the semiclassical limit at time $t \neq 0$ is different in nature from that at time zero. For a Dirac-Hamiltonian this phenomenon is related to the so-called "Zitterbewegung" which we discuss in more detail in Chapter 7. Therefore, we are here interested in operators $\mathcal{B}$ with symbols in $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ for which $\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$ is again a semiclassical pseudodifferential operator with symbol $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$. We hence introduce the following notion:

**Definition 2.3.3.** A symbol $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ is in the invariant subalgebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ of the algebra $\mathrm{S}^\infty_{\mathrm{cl}}(1)$, if and only if for all finite $t$ the (bounded) operator $\mathcal{B}(t) = \mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$, $\mathcal{B} = \mathrm{op}^W[B]$, is a semiclassical pseudodifferential operator with symbol $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$, i.e.,

$$\mathrm{S}^\infty_{\mathrm{inv}}(1) := \left\{B \in \mathrm{S}^q_{\mathrm{cl}}(1)\,;\, \mathrm{symb}^W[\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)] \in \mathrm{S}^q_{\mathrm{cl}}(1) \quad \text{for} \quad t \in [0,T],\ q \in \mathbb{Z}\right\}.$$

This means that the invariant algebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ has a filtration, induced by the filtration of $\mathrm{S}^\infty_{\mathrm{cl}}(1)$, which is invariant under conjugation of the corresponding operators with $\mathcal{U}(t)$. Due to the results of [BG00] we expect that operators which are block-diagonal with respect to the projections $\mathcal{P}_\mu$ are in the associated invariant operator algebra. This statement is made precise in Theorem 2.3.4 which is a variant of the Egorov theorem [Ego69] for general hyperbolic systems.

[16] We remark that before transferring operators to symbol level one can replace $\mathcal{P}_\mu$ by $\tilde{\mathcal{P}}_\mu$ and employ the classical asymptotic expansion of the symbol $P_\mu$. This will only amount to an error of order $\hbar^\infty$.

Let us first consider an operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ that is block-diagonal with respect to the semiclassical projections, i.e.,

$$B \sim \sum_{\mu=1}^{l} P_\mu \# B \# P_\mu \quad \text{in} \quad \mathrm{S}^q_{\mathrm{cl}}(1).$$

According to the Heisenberg equation of motion (2.3.2) its time evolution $B(t)$ is governed by

$$\frac{\partial}{\partial t} B(t) \sim \frac{\mathrm{i}}{\hbar}[H, B(t)]_{\#}. \tag{2.3.17}$$

Suppose now that $B(t)$ has a (formal) asymptotic expansion

$$B(t) \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_j$$

and use the composition formula of Lemma 2.2.2 together with the fact that the block-diagonal form of an operator $\mathcal{B}$ is preserved under the time evolution, see (2.3.16). On the symbol level the diagonal blocks $\mathcal{P}_\nu \mathcal{B}(t) \mathcal{P}_\nu$ then obey the following equation:

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\nu,j} \sim \\
&\quad \sum_{l,j=0}^{\infty} \sum_{|\alpha|+|\beta| \geq 0} \gamma(\alpha,\beta) \hbar^{-q+l+j+|\alpha|+|\beta|-1} \left( B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)}\, H_{\nu,j}{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|} H_{\nu,j}{}^{(\alpha)}_{(\beta)} B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} \right).
\end{aligned}$$

Here we introduced the notation $F^{(\alpha)}_{(\beta)} := \partial^\alpha_\xi \partial^\beta_x F$ for $F \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$, as well as

$$\begin{aligned}
\gamma(\alpha,\beta) &:= \frac{\mathrm{i}^{|\alpha|-|\beta|-1}}{2^{|\alpha|+|\beta|}|\alpha|!|\beta|!}, \\
H_\nu &:= P_\nu \# H \# P_\nu \sim H \# P_\nu \sim \sum_{j=0}^{\infty} \hbar^j H_{\nu,j}, \\
B(t)_{\nu\nu} &:= P_\nu \# B(t) \# P_\nu \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\nu,j}.
\end{aligned}$$

One hence has to solve, by taking $[H_{\nu,0}, B(t)_{\nu\nu,0}] = 0$ into account,

$$\begin{aligned}
[H_{\nu,0}, &B(t)_{\nu\nu,n+1}] = \\
&- \frac{\partial}{\partial t} B(t)_{\nu\nu,n} - \frac{1}{2}\Big( \{B(t)_{\nu\nu,n}, H_{\nu,0}\} - \{H_{\nu,0}, B(t)_{\nu\nu,n}\} \Big) - \mathrm{i}[B(t)_{\nu\nu,n}, H_{\nu,1}] \\
&+ \sum_{\substack{0 \leq l \leq n-1 \\ j+|\alpha|+|\beta| = n-l+1}} \gamma(\alpha,\beta) \left( B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} H_{\nu,j}{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|} H_{\nu,j}{}^{(\alpha)}_{(\beta)} B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} \right).
\end{aligned} \tag{2.3.18}$$

Upon multiplying this commutator equation with the projection matrices $P_{\mu,0}$ from both sides one first realises that it is only solvable, if the diagonal blocks of the right-hand side, that we denote by $R_{n,\nu}(t)$, vanish. The off-diagonal blocks on both sides of the relation (2.3.18) then yield the general structure of the solution, which reads

$$B(t)_{\nu\nu,n+1} = \sum_{\mu=1}^{l} P_{\mu,0} B(t)_{\nu\nu,n+1} P_{\mu,0} + \sum_{\mu\neq\eta} \frac{P_{\mu,0} R_{n,\nu}(t) P_{\eta,0}}{\lambda_\mu - \lambda_\eta}, \tag{2.3.19}$$

see also [Cor95]. This demonstrates that one obtains the off-diagonal parts of $B(t)_{\nu\nu,n+1}$ with respect to the projection matrices $P_{\mu,0}$ from the preceding coefficients of the asymptotic expansion of $B(t)_{\nu\nu}$. The diagonal parts then have to be determined by the condition that the commutator equation (2.3.18) must possess a (non-trivial) solution with initial value $B(t)_{\nu\nu,n+1}|_{t=0} = B_{\nu\nu,n+1}$. Starting with $n = 0$, where the sum in (2.3.18) is empty, one has to solve

$$P_{\mu,0}\left(\frac{\partial}{\partial t} B(t)_{\nu\nu,0} + \frac{1}{2}\Big(\{B(t)_{\nu\nu,0}, H_{\nu,0}\} - \{H_{\nu,0}, B(t)_{\nu\nu,0}\}\Big) + \mathrm{i}[B(t)_{\nu\nu,0}, H_{\nu,1}]\right) P_{\mu,0} = 0.$$

Expressions of this type have already been considered in [Spo00], where it was shown that the above equation is equivalent to

$$\frac{\partial}{\partial t}\left(P_{\mu,0} B(t)_{\nu\nu,0} P_{\mu,0}\right) - \delta_{\nu\mu}\{\lambda_\nu, P_{\mu,0} B(t)_{\nu\nu,0} P_{\mu,0}\} - \mathrm{i}[\tilde{H}_{\nu\mu,1}, P_{\mu,0} B(t)_{\nu\nu,0} P_{\mu,0}] = 0, \tag{2.3.20}$$

see also appendix D. Here we have defined the hermitian $n \times n$ matrix

$$\tilde{H}_{\nu\mu,1} := \mathrm{i}(-1)^{\delta_{\nu\mu}} \frac{\lambda_\nu}{2} P_{\mu,0}\{P_{\nu,0}, P_{\nu,0}\} P_{\mu,0} - \mathrm{i}\,\delta_{\nu\mu}[P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}] + P_{\mu,0} H_{\nu,1} P_{\mu,0} \tag{2.3.21}$$

according to (D.4) and (D.5) of appendix D. Now, equation (2.3.20) is trivially fulfilled for $\nu \neq \mu$, and the case $\nu = \mu$ has already been considered in [Ivr98, BN99, BG00], where it was shown that the solution reads

$$B(t)_{\nu\nu,0}(\xi, x) = d_{\nu\nu}^{-1}(x,\xi,t) B_{\nu\nu,0}\big(\Phi_\nu^t(x,\xi)\big) d_{\nu\nu}(x,\xi,t).$$

In this expression $\Phi_\nu^t : \mathrm{T}^*\mathbb{R}^d \to \mathrm{T}^*\mathbb{R}^d$ denotes the Hamiltonian flow generated by the eigenvalue $\lambda_\nu$ of $H_0$, and the transport matrix $d_{\nu\nu}$ is determined by the equation

$$\dot{d}_{\nu\nu}(x,\xi,t) + \mathrm{i}\,\tilde{H}_{\nu\nu,1}\big(\Phi_\nu^t(x,\xi)\big) d_{\nu\nu}(x,\xi,t) = 0, \quad d_{\nu\nu}(x,\xi,0) = \mathrm{id}_{\mathbb{C}^n}. \tag{2.3.22}$$

One has thus fixed the coefficients $B(t)_{\nu\nu,0} = P_{\nu,0} B(t)_0 P_{\nu,0}$, i.e., the principal symbol of $\mathcal{B}(t)$, since the off-diagonal terms $B(t)_{\nu\mu,0} = P_{\nu,0} B(t)_0 P_{\mu,0}$ vanish and therefore trivially fulfill (2.3.18). According to (2.3.19) we hence have also determined the off-diagonal parts of the sub-principal term $B(t)_{\nu\nu,1}$, which vanish as well. The diagonal contributions $P_{\mu,0} B(t)_{\nu\nu,1} P_{\mu,0}$ with respect to the projection matrices obey $[P_{\eta,0}, P_{\mu,0} B(t)_{\nu\nu,1} P_{\mu,0}] = 0$ and thus can be determined from the relation (2.3.18). As in [Ivr98, BG00], we hence obtain a recursive Cauchy problem for the coefficients $B(t)_{\nu\nu,n}$ and are now in a position to state:

**Theorem 2.3.4.** *Let $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ be hermitian with the property*

$$\|H_j{}^{(\alpha)}_{(\beta)}(x,\xi)\|_{n\times n} \le C_{\alpha,\beta} \quad \text{for all} \quad (x,\xi) \in \mathrm{T}^*\mathbb{R}^d \text{ and } |\alpha|+|\beta|+j \ge 2-\delta_{j0}, \tag{2.3.23}$$

*and such that the conditions (H0) and (H2) are fulfilled. Furthermore, suppose that $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ is block-diagonal with respect to the semiclassical projections defined in (2.3.9),*

$$B \sim \sum_{\mu=1}^{l} P_\mu \# B \# P_\mu.$$

*Then $B$ is in the invariant algebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ introduced in Definition 2.3.3, i.e., $\mathcal{B}(t)$ is again a semiclassical pseudodifferential operator with symbol $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$. Furthermore, the principal symbol of $\mathcal{B}(t)$ is given by*

$$B(t)_0(x,\xi) = \sum_{\nu=1}^{l} d^*_{\nu\nu}(x,\xi,t) B_{\nu\nu,0}\big(\Phi^t_\nu(x,\xi)\big) d_{\nu\nu}(x,\xi,t), \tag{2.3.24}$$

*where $\Phi^t_\nu$ is the Hamiltonian flow generated by the eigenvalue $\lambda_\nu$ of $H_0$, and $d_{\nu\nu}$ is a unitary $n\times n$ matrix which is determined by the transport equation (2.3.22).*

*Proof.* As in [Ivr98, BG00] we start by rewriting (2.3.18) for the diagonal block of $B(t)_{\nu\nu,n}$ with respect to $P_{\mu,0}$ in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[d^{-1}_{\nu\mu}(x,\xi,-t)(P_{\mu,0}B(t)_{\nu\nu,n}P_{\mu,0})\circ\Phi^{-t\delta_{\nu\mu}}_\nu(x,\xi) d_{\nu\mu}(x,\xi,-t)\right] \\ &\quad = \sum_{\substack{0\le l\le n-1 \\ j+|\alpha|+|\beta|=n-l+1}} \gamma(\alpha,\beta)P_{\mu,0}\left(B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)}H_{\nu,j}{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|}H_{\nu,j}{}^{(\alpha)}_{(\beta)}B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)}\right)P_{\mu,0}, \end{aligned} \tag{2.3.25}$$

where $d_{\nu\mu}$ is determined by the transport equation

$$\dot d_{\nu\mu}(x,\xi,t) + \mathrm{i}\,\tilde H_{\nu\mu,1}(\Phi^{t\delta_{\nu\mu}}_\nu(x,\xi))d_{\nu\mu}(x,\xi,t), \quad d_{\nu\mu}(x,\xi,0) = \mathrm{id}_{\mathbb{C}^n},$$

that generalises (2.3.22) also to the off-diagonal transport. And since $\tilde H_{\nu\mu,1}$ is hermitian, the solution $d_{\nu\mu}$ is a unitary $n\times n$ matrix, which in the case $\nu \ne \mu$ is obviously given by

$$d_{\nu\mu}(x,\xi,t) = \mathrm{e}^{-\mathrm{i}\,\tilde H_{\nu\mu,1}(x,\xi)t}.$$

In order to obtain estimates on the derivatives of the symbols $P_{\mu,0}B(t)_{\nu\nu,n}(t)P_{\mu,0}$ one has to control the behaviour of the flow $\Phi^t_\nu$ generated by the eigenvalue $\lambda_\nu$ of $H_0$. To this end we first notice that $H_0 \in \mathrm{S}(m)$ implies the bound $|\lambda_\nu(x,\xi)| \le cm(x,\xi)$ on its eigenvalues. Furthermore, due to the hyperbolicity condition (H0) the projections $P_{\nu,0}$ onto the eigenspaces of $H_0$ are in $\mathrm{S}(1)$. We then consider the first order derivatives ($|\alpha|+|\beta|=1$) of the relation

$$H_0(x,\xi)P_{\nu,0}(x,\xi) = \lambda_\nu(x,\xi)P_{\nu,0}(x,\xi),$$

which exist since the eigenvalues $\lambda_\nu$ are smooth functions on the phase space $\mathrm{T}^*\mathbb{R}^d$, see equation (2.3.8). One thus obtains

$$\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = \big(H_0(x,\xi)P_{\nu,0}(x,\xi)\big)^{(\alpha)}_{(\beta)} - \lambda_\nu(x,\xi)P_{\nu,0}{}^{(\alpha)}_{(\beta)}(x,\xi).$$

Now, since $P_{\nu,0}(x,\xi)P_{\nu,0}{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = 0$, a multiplication of the above equation with $P_{\nu,0}(x,\xi)$ from both sides yields

$$\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = P_{\nu,0}(x,\xi)H_0(x,\xi)^{(\alpha)}_{(\beta)}P_{\nu,0}(x,\xi), \tag{2.3.26}$$

and hence

$$\big|\lambda_\nu{}^{(\alpha)}_{(\beta)}\big| = c\big\|\lambda_\nu{}^{(\alpha)}_{(\beta)}P_{\nu,0}\big\|_{n\times n} = c\big\|P_{\nu,0}H_0{}^{(\alpha)}_{(\beta)}P_{\nu,0}\big\|_{n\times n} \le \tilde{c}\big\|H_0{}^{(\alpha)}_{(\beta)}\big\|_{n\times n}.$$

$H_0 \in \mathrm{S}(m)$ therefore implies that the first order derivatives of $\lambda_\nu$ are bounded by the order function $m$. One can continue this argument by successively differentiating equation (2.3.26), concluding that $\lambda_\nu\,\mathrm{id}_{\mathbb{C}^n} \in \mathrm{S}(m)$ for all $\nu = 1,\dots,l$. In particular, the property

$$\big\|H_0{}^{(\alpha)}_{(\beta)}(x,\xi)\big\|_{n\times n} \le C_{\alpha,\beta} \quad \text{for} \quad |\alpha|+|\beta| \ge 1,$$

which follows from (2.3.23), transfers to a corresponding growth property of the eigenvalues of $H_0$:

$$\big|\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)\big| \le C_{\alpha,\beta} \quad \text{for} \quad |\alpha|+|\beta| \ge 1.$$

Therefore, the Hamiltonian flows $\Phi^t_\nu$ exist globally on $\mathrm{T}^*\mathbb{R}^d$ such that $|\Phi^t_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}$ for all $\alpha$, $\beta \in \mathbb{N}^d_0$ and for all finite times $t \in [0,T]$, see [Rob87]. This property guarantees that $B \circ \Phi^t_\nu \in \mathrm{S}(1)$ for all $B \in \mathrm{S}(1)$. Concerning the unitary matrices $d_{\nu\mu}$ the following is true:

**Lemma 2.3.5.** *If the subprincipal symbol $H_1$ of $\mathcal{H}$ satisfies $\|H_1{}^{(\alpha)}_{(\beta)}\|_{n\times n} \le C_{\alpha,\beta}$ for all $|\alpha|+|\beta| \ge 1$, then $\|d_{\nu\mu}{}^{(\alpha)}_{(\beta)}(x,\xi,t)\|_{n\times n} \le C'_{\alpha,\beta}$ for all $t \in [0,T]$, $|\alpha|+|\beta| \ge 1$ and $\nu$, $\mu = 1,\dots,l$.*

For the proof of this Lemma see [Ivr98] and [Gla00]. With these properties at hand one can integrate equation (2.3.25) and solve for $P_{\mu,0}B(t)_{\nu\nu,n}P_{\mu,0}$ by conjugating with $d_{\nu\mu}(x,\xi,-t)$ and shifting the arguments by $\Phi^{\delta_{\nu\mu}t}_\nu$ (which only amounts to an actual shift in the case $\nu=\mu$). For the principal symbol of $\mathcal{B}(t)$ one thus obtains

$$B(t)_{\nu\nu,0}(x,\xi) = d_{\nu\nu}(\Phi^t_\nu(x,\xi),-t)B_{\nu\nu,0}(\Phi^t_\nu(x,\xi))d^{-1}_{\nu\nu}(\Phi^t_\nu(x,\xi),-t),$$

which is the only block of $B(t)_{\nu\nu,0}$ with respect to $P_{\mu,0}$, $\mu = 1,\dots,l$, that is different from zero. Using

$$d_{\nu\mu}(\Phi^{t\delta_{\nu\mu}}_\nu(x,\xi),-t) = d^{-1}_{\nu\mu}(x,\xi,t) = d^*_{\nu\mu}(x,\xi,t), \tag{2.3.27}$$

see [BN99], one finally obtains (2.3.24). For the higher coefficients $B(t)_{\nu\nu,n}$, $n \ge 1$, one employs the Duhamel principle and uses that fact that the sum in (2.3.25) is taken over indices with $|\alpha| + |\beta| + j \geq 2$, and thus involves terms in $\mathrm{S}(1)$, in order to conclude that $B(t)_{\nu\nu,n} \in \mathrm{S}(1)$. This shows that one has found an asymptotic expansion in $\mathrm{S}^q_{\mathrm{cl}}(1)$ for the symbol of $\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$ that can be summed with the Borel method to yield a complete symbol. $\square$

This theorem shows that, for finite times $t$, one can associate to a (semiclassically) block-diagonal symbol $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ a symbol $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$ whose quantisation $\mathrm{op}^W[B(t)]$ is semiclassically close to $\mathcal{B}(t) = \mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$, i.e.,

$$\|\mathcal{B}(t) - \mathrm{op}^W[B(t)]\| = O(\hbar^\infty) \quad \text{for all} \quad t \in [0, T].$$

This is a semiclassical version of the Egorov theorem [Ego69], which was originally formulated for the case of scalar symbols. A weaker version that is also sometimes referred to as an Egorov theorem (see, e.g., [PST03]) would only assert that one can evolve the principal symbol $B_0$ of $\mathcal{B}$ into a symbol $B(t)_0$, as given in (2.3.24), such that its quantisation $\mathrm{op}^W[B(t)_0]$ is $\hbar$-close to the time-evolved operator $\mathcal{B}(t)$, i.e.,

$$\|\mathcal{B}(t) - \mathrm{op}^W[B(t)_0]\| = O(\hbar).$$

This statement is clearly covered by Theorem 2.3.4, since the quantisation of the difference $B(t) - B(t)_0 \in \mathrm{S}^{q-1}_{\mathrm{cl}}(1)$ yields a bounded operator with norm of order $\hbar$, see Proposition 2.2.4.

We will now show (generalising results of Cordes [Cor83a, Cor00, Cor01]) that the semiclassical block-diagonal operators exhaust all operators with symbols in the invariant algebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$.

**Proposition 2.3.6.** *The invariant subalgebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ of $\mathrm{S}^\infty_{\mathrm{cl}}(1)$ consists of precisely those $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ which are semiclassically block-diagonal with respect to the projections $P_\mu$, $\mu = 1, \dots, l$, defined in (2.3.11) of Proposition 2.3.1, i.e.,*

$$B \in \mathrm{S}^\infty_{\mathrm{inv}} \subset \mathrm{S}^\infty_{\mathrm{cl}}(1) \quad \Leftrightarrow \quad B \sim \sum_{\mu=1}^{l} P_\mu \# B \# P_\mu.$$

*Proof.* Consider an operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^\infty_{\mathrm{cl}}(1)$, whose equation of motion is given by (2.3.17). For the symbol of the time-evolved operator we now assume an asymptotic expansion

$$B(t) \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_j$$

in $\mathrm{S}^q_{\mathrm{cl}}(1)$. Furthermore, one can use (2.3.7) to separate (2.3.17) into blocks with respect to $P_\mu$, $\mu = 1, \dots, l$. For the off-diagonal blocks ($\nu \neq \mu$) one therefore obtains

$$\frac{\partial}{\partial t} B(t)_{\nu\mu} \sim \frac{\mathrm{i}}{\hbar}[H, B(t)_{\nu\mu}]_\#, \tag{2.3.28}$$

where $B(t)_{\nu\mu} := P_\nu \# B(t) \# P_\mu \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\mu,j}$. In leading semiclassical order the factor $\hbar^{-1}$ on the right-hand side of equation (2.3.28) enforces the condition

$$[H_0, B(t)_{\nu\mu,0}] = (\lambda_\nu - \lambda_\mu) B(t)_{\nu\mu,0} = 0.$$

Since $\lambda_\mu \neq \lambda_\nu$ for $\mu \neq \nu$, this immediately yields $B(t)_{\nu\mu,0} = 0$. Furthermore,

$$\frac{\partial}{\partial t} \sum_{j=1}^{\infty} \hbar^{-q+j} B(t)_{\nu\mu,j} \sim \mathrm{i} \Big[ H, \sum_{j=1}^{\infty} \hbar^{-q+j-1} B(t)_{\nu\mu,j} \Big]_{\#}.$$

Again the leading order on the right-hand side has to vanish, i.e.,

$$[H_0, B(t)_{\nu\mu,1}] = 0.$$

This means that the symbol $B(t)_{\nu\mu,1}$ must be block-diagonal with respect to the projection matrices $P_{\mu,0} \in \mathrm{S}(1)$. But

$$P_{\lambda,0} B(t)_{\nu\mu,1} P_{\lambda,0} = \mathrm{symb}_P^W \left[ \hbar^{-1} \left( P_\lambda \# (B(t)_{\nu\mu} - B(t)_{\nu\mu,0}) \# P_\lambda \right) \right] = 0,$$

since $B(t)_{\nu\mu,0} = 0$ for $\nu \neq \mu$. Iterating the above procedure we see that if $B \in \mathrm{S}_{\mathrm{inv}}(1)$, then it has to be block-diagonal with respect to $P_\mu$, $\mu = 1, \dots, l$. This proves one direction asserted in the proposition. The other direction, that the block-diagonal operators form a subset of the invariant algebra, is contained in the Egorov theorem 2.3.4. $\square$

At this point it is instructive to include a comment on the transport equation (2.3.22) that not only occurs in connection with an Egorov theorem, but also in a WKB-type framework. In this context Littlejohn and Flynn [LF91] introduced a splitting of the analogue to $\tilde{H}_{\nu\nu,1}$ (defined in equation (2.3.21)) into two contributions, one of which is related to a Berry connection [Ber84]. Subsequently Emmrich and Weinstein [EW96] generalised the approach of [LF91] and gave a geometrical interpretation for the second contribution, which they related to a Poisson curvature. We now want to identify the two contributions in the present situation, i.e., in $\tilde{H}_{\nu\nu,1}$. To this end we calculate $H_{\nu,1} = P_{\nu,1} H_0 + P_{\nu,0} H_1 + \frac{\mathrm{i}}{2}\{P_{\nu,0}, H_0\}$ using

$$-P_{\nu,0} P_{\nu,1} P_{\nu,0} + (1 - P_{\nu,0}) P_{\nu,1} (1 - P_{\nu,0}) = \frac{\mathrm{i}}{2} \{P_{\nu,0}, P_{\nu,0}\},$$

which follows from the condition $P_\nu \# P_\nu \sim P_\nu$ and the composition formula in Lemma 2.2.2. Thus

$$P_{\nu,0} H_{\nu,1} P_{\nu,0} = P_{\nu,0} H_1 P_{\nu,0} + \mathrm{i} \frac{\lambda_\nu}{2} P_{\nu,0} \{P_{\nu,0}, P_{\nu,0}\} P_{\nu,0} + \frac{\mathrm{i}}{2} \sum_{\eta=1}^{l} \lambda_\eta P_{\nu,0} \{P_{\nu,0}, P_{\eta,0}\} P_{\nu,0}.$$

The relation $P_{\nu,0}\{P_{\nu,0}, P_{\eta,0}\} P_{\nu,0} = -P_{\nu,0}\{P_{\eta,0}, P_{\eta,0}\} P_{\nu,0}$ and the spectral representation $H_0 = \sum_\mu \lambda_\mu P_{\mu,0}$ now allow us to rewrite the expression (2.3.21) for $\tilde{H}_{\nu\nu,1}$ as

$$\tilde{H}_{\nu\nu,1} = H_{\nu,\mathrm{Berry}} + H_{\nu,\mathrm{Poisson}} + P_{\nu,0} H_1 P_{\nu,0}$$

with

$$\begin{aligned} H_{\nu,\mathrm{Berry}} &:= -\,\mathrm{i}[P_{\nu,0},\{\lambda_\nu,P_{\nu,0}\}],\\ H_{\nu,\mathrm{curvature}} &:= \frac{\mathrm{i}}{2}\Big(\lambda_\nu P_{\nu,0}\{P_{\nu,0},P_{\nu,0}\}P_{\nu,0}+P_{\nu,0}\{P_{\nu,0},H_0-\lambda_\nu P_{\nu,0}\}P_{\nu,0}\Big). \end{aligned} \tag{2.3.29}$$

This corresponds exactly to the splitting discussed in [EW96], see also [Spo00], whose geometrical significance will be discussed in the next Section.

### 2.3.3 Dynamics in the eigenspaces

According to the Egorov theorem 2.3.4, the semiclassical calculus outlined above results not only in a transport of the principal symbols of observables by the Hamiltonian flows $\Phi_\nu^t$, but also in a conjugation by the (unitary $n\times n$) transport matrices $d_{\nu\nu}$. The latter define the dynamics of those degrees of freedom that on the quantum mechanical level are described by the factor $\mathbb{C}^n$ of the total Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. Our intention in this section now is to develop combined classical dynamics of both types of degrees of freedom, i.e., those described by the Hamiltonian flows and those that are represented by the conjugations. In this context the fact that the conjugations enter along integral curves of the Hamiltonian flows introduces a hierarchy among the two types of degrees of freedom.

In a first step we confirm that the dynamics represented by the transport matrices $d_{\nu\nu}$ take place in the eigenspaces of the principal symbol $H_0$ in $\mathbb{C}^n$. To this end we notice that since at every point $(x,\xi)\in \mathrm{T}^*\mathbb{R}^d$ the projection matrices $P_{\nu,0}(x,\xi)$ yield an orthogonal splitting of $\mathbb{C}^n$ and have constant rank $k_\nu$, they define $k_\nu$-dimensional subbundles $\pi_\nu : E^\nu \to \mathrm{T}^*\mathbb{R}^d$ of the trivial vector bundle $\mathrm{T}^*\mathbb{R}^d\times\mathbb{C}^n$ over phase space. The fibre $E^\nu_{(x,\xi)}=\pi_\nu^{-1}(x,\xi)$ over $(x,\xi)\in\mathrm{T}^*\mathbb{R}^d$ is given by the range of the projection, i.e., $E^\nu_{(x,\xi)}=P_{\nu,0}(x,\xi)\mathbb{C}^n$. Furthermore, the canonical hermitian structure of $\mathbb{C}^n$ induces a hermitian structure on the fibres. We now intend to interpret the conjugation by $d_{\nu\nu}$ as a dynamics in the eigenvector bundle $E^\nu$, and for this purpose notice:

**Lemma 2.3.7.** *The restricted transport matrices $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ provide unitary maps between the fibres $E^\nu_{(x,\xi)}$ and $E^\nu_{\Phi^t_\nu(x,\xi)}$.*

*Proof.* In order to see that $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ maps $E^\nu_{(x,\xi)}$ into $E^\nu_{\Phi^t_\nu(x,\xi)}$ we show

$$P_{\nu,0}(\Phi^t_\nu(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)=d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi). \tag{2.3.30}$$

This relation is certainly true for $t=0$ where both sides yield $P_{\nu,0}$. Moreover, the derivative with respect to $t$ of the left-hand side reads

$$\{\lambda_\nu,P_{\nu,0}\}(\Phi^t_\nu(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)-\mathrm{i}\,P_{\nu,0}(\Phi^t_\nu(x,\xi))\tilde H_{\nu\nu,1}(\Phi^t_\nu(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi),$$

which equals

$$-\,\mathrm{i}\,\tilde H_{\nu\nu,1}(\Phi^t_\nu(x,\xi))P_{\nu,0}(\Phi^t_\nu(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi),$$

since the commutator $[P_{\nu,0}, \tilde{H}_{\nu\nu,1}]$ can be calculated as (see equation (2.3.21))

$$-\,\mathrm{i}[P_{\nu,0}, [P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}]] = -\,\mathrm{i}\,(P_{\nu,0}\{\lambda_\nu, P_{\nu,0}\} + \{\lambda_\nu, P_{\nu,0}\}P_{\nu,0}) = -\,\mathrm{i}\{\lambda_\nu, P_{\nu,0}\};$$

here we have used (D.2) and $P_{\nu,0}^2 = P_{\nu,0}$. Thus, $P_{\nu,0}(\Phi_\nu^t(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ fulfills the same differential equation with respect to $t$ as $d_{\nu\nu}(x,\xi,t)P_{\nu,0}$, and this finally implies the validity of equation (2.3.30).

In order to see the unitarity, one has to show that $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ is an isometry whose range is the complete fibre $E^\nu_{\Phi_\nu^t(x,\xi)}$. The first point is clear since $d_{\nu\nu}$ is unitary on $\mathbb{C}^n$ and the fibres inherit their hermitian structures from $\mathbb{C}^n$. The second point follows from the observation that the transport provided by $d_{\nu\nu}$ can be reversed: Given $v(\Phi_\nu^t(x,\xi)) \in E_{\Phi_\nu^t(x,\xi)}$, the vector $P_{\nu,0}(x,\xi)d_{\nu\nu}(\Phi_\nu^t(x,\xi),-t)v(\Phi_\nu^t(x,\xi))$ lies in $E^\nu_{(x,\xi)}$ and is mapped to $v(\Phi_\nu^t(x,\xi))$ by $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$, see (2.3.27). □

It is possible to recover the splitting of $\tilde{H}_{\nu\nu,1}$ given in (2.3.29) in geometrical terms closely related to parallel transport in the eigenbundles. Let us introduce a connection in the eigenbundles in terms of the covariant derivative $\nabla_\nu := P_{\nu,0}\,\mathrm{d}$. This is the covariant derivative on $E^\nu$ that is naturally associated with the trivial connection on the trivial $\mathbb{C}^n$-bundle over phase space, having $\mathrm{d}$ as its covariant derivative, and the projection $P_{\nu,0}$ from the trivial bundle to the eigenbundle. It was observed by Simon [Sim83a] that such projected connections are those whose holonomy in certain physical situations is called Berry's phase [Ber84]. Therefore, the corresponding terms in [LF91] are called Berry terms. In order to see how $H_{\nu,\mathrm{Berry}}$ is connected with the covariant derivative we consider the parallel displacement induced by $\nabla_\nu$. We claim that parallel transport with respect to $\nabla_\nu$ in $E^\nu$ along an integral curve $\Phi_\nu^t$ is given by $E^\nu_{(x,\xi)} \ni \varphi_\nu \mapsto \varphi_\nu(t)(x,\xi) = \tilde{d}_\nu(x,\xi,t)\varphi(\Phi_\nu^t(x,\xi))$, where we define

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{d}_\nu(t) - [P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}]\tilde{d}_\nu = 0.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\nu(t) = [P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}]\varphi_\nu(t) + \{\lambda_\nu, \varphi_\nu(t)\}$$

which according to $P_{\nu,0}\{\lambda_\nu, P_{\nu,0}\}P_{\nu,0} = 0$ and $\{\lambda_\nu, \varphi_\nu\} = \{\lambda_\nu, P_{\nu,0}\varphi_\nu\} = P_{\nu,0}\{\lambda_\nu, \varphi_\nu\} + \{\lambda_\nu, P_{\nu,0}\}\varphi_\nu$ equals

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\nu(t)\right|_{t=0} = P_{\nu,0}\{\lambda_\nu, \varphi_\nu\},$$

such that $H_{\nu,\mathrm{Berry}}$ indeed generates the parallel transport corresponding to $\nabla_\nu$.

In order to obtain a geometrical interpretation for $H_{\nu,\mathrm{curvature}}$ we consider the curvature of $\nabla_\nu$ which is given by a two-form $F_\nu$ with values in the endomorphisms of $E^\nu$ according to

$$\nabla_\nu^2\varphi = F_\nu\varphi$$

for any section $\varphi$ of $E^\nu$. As in [EW96] we find that $F_\nu = P_{\nu,0}\,\mathrm{d}P_{\nu,0}\wedge(\,\mathrm{d}P_{\nu,0})P_{\nu,0}$. Therefore, the first term of $H_{\nu,\mathrm{curvature}}$ is just $\lambda_\nu F_\nu(\Pi)$, where $\Pi$ denotes the cosymplectic structure[17].

[17] or Poisson tensor, see [MR94, AMR88]

To describe the second term, we define the one-form with values in the vector bundle morphisms from $E^\nu$ to the kernel of $P_{\nu,0}$, i.e. to the image of $\mathbb{1}-P_{\nu,0}$, by $S\varphi = (\mathbb{1}-P_{\nu,0})\,\mathrm{d}\varphi$, which equals

$$S = (\mathbb{1} - P_{\nu,0})\,\mathrm{d}P_{\nu,0}.$$

This quantity measures to what extent the trivial parallel transport corresponding to $\mathrm{d}$ moves a vector out of the $E^\nu$ eigenbundle into its complement. Similarly, we can define a one-form from the kernel of $P_{\nu,0}$ to $E^\nu$ by

$$S^*\varphi = -P_{\nu,0}\,\mathrm{d}\varphi = -P_{\nu,0}\,\mathrm{d}(\mathbb{1} - P_{\nu,0})\varphi$$

for any section of the complement of $E^\nu$. Using $S$ and $S^*$ define

$$S^* \wedge ((H_0 - \lambda_\nu P_{\nu,0})S),$$

where $H_0 - \lambda_\nu P_{\nu,0}$ is considered as an endomorphism of $\ker P_{\nu,0}$, on which it equals $H_0$. As shown in [EW96] the contraction of this two-form with the cosymplectic structure gives the second term of $H_{\nu,\mathrm{curvature}}$.

According to the above, the action of $d_{\nu\nu}(x,\xi,t)$ on a section in $E^\nu$ can be viewed as a parallel transport along the integral curves of the flow $\Phi^t_\nu$. If one now introduces sections of $E^\nu$ that yield orthonormal bases $\{e_1(x,\xi),\dots,e_{k_\nu}(x,\xi)\}$ of the fibres $E^\nu_{(x,\xi)}$, the representations of $d_{\nu\nu}(x,\xi,t)$ in these bases are unitary $k_\nu \times k_\nu$ matrices $D_\nu(x,\xi,t)$. Since the principal symbol $H_0$ of the operator $\mathcal{H}$ is hermitian (on $\mathbb{C}^n$), a preferred choice for the sections $\{e_1,\dots,e_{k_\nu}\}$ would consist of orthonormal eigenvectors of $H_0$. However, this choice is obviously not unique because it amounts to fixing an isometry $V_\nu(x,\xi) : \mathbb{C}^{k_\nu} \to E^\nu_{(x,\xi)}$, such that $V_\nu(x,\xi)V_\nu^*(x,\xi) = P_{\nu,0}(x,\xi)$ and $V_\nu^*(x,\xi)V_\nu(x,\xi) = \mathrm{id}_{\mathbb{C}^{k_\nu}}$. Here one still has a freedom to change the isometry by an arbitrary unitary automorphism of $\mathbb{C}^{k_\nu}$. Having chosen an isometry $V_\nu(x,\xi)$ for every fibre $E^\nu_{(x,\xi)}$ in a smooth way, the $n\times n$ transport matrices $d_{\nu\nu}(x,\xi,t)$ are mapped to unitary $k_\nu \times k_\nu$ matrices

$$D_\nu(x,\xi,t) := V_\nu^*\big(\Phi^t_\nu(x,\xi)\big)\,d_{\nu\nu}(x,\xi,t)V_\nu(x,\xi). \tag{2.3.31}$$

Their dynamics follows from the transport equation (2.3.22) as

$$\dot{D}_\nu(x,\xi,t) + \mathrm{i}\,\tilde{H}_\nu\big(\Phi^t_\nu(x,\xi)\big)D_\nu(x,\xi,t) = 0 \quad \text{with} \quad D_\nu(x,\xi,0) = \mathrm{id}_{\mathbb{C}^{k_\nu}}, \tag{2.3.32}$$

where the hermitian $k_\nu \times k_\nu$ matrix $\tilde{H}_\nu$ is derived from (2.3.21) for $\mu = \nu$,

$$\tilde{H}_\nu = -\,\mathrm{i}\,\frac{\lambda_\nu}{2}V_\nu^*\{P_{\nu,0},P_{\nu,0}\}V_\nu + \mathrm{i}\{\lambda_\nu, V_\nu^*\}V_\nu + V_\nu^* H_{\nu,1}V_\nu.$$

What is of more importance for later purposes than the non-uniqueness of this representation, however, is the fact that the above construction allows us to introduce a skew-product flow over the Hamiltonian flow $\Phi^t_\nu$, thus reflecting the hierarchy of the two types of degrees of freedom. See [CFS82] for a definition of skew-product flows and cf.

[BK99b] where these occur in the context of a semiclassical trace formula for matrix valued operators. At this stage provisionally consider

$$\hat{Y}^t_\nu : \mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu) \to \mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu),$$

defined by $\hat{Y}^t_\nu(x,\xi,g) := (\Phi^t_\nu(x,\xi), D_\nu(x,\xi,t)g)$, which yields a flow on the product space $\mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu)$ due to the cocycle relation $D_\nu(x,\xi,t+t') = D_\nu(\Phi^t_\nu(x,\xi),t')D_\nu(x,\xi,t)$. Later we are interested in ergodic properties of such skew-product flows, and these are independent of the particular choice of the sections $\{e_1,\dots,e_{k_\nu}\}$. Here we remark that in some cases the point of view advertised above might turn out too general. It can indeed happen that the fibre part of the skew-product flow does not require the complete group $\mathrm{U}(k_\nu)$. E.g., in [BGK01] a situation was considered where $k_\nu = 2j+1$, $j \in \frac{1}{2}\mathbb{N}$, and the transport matrices $D_\nu$ were operators in a $2j+1$-dimensional unitary irreducible representation of $\mathrm{SU}(2)$. This fact could be identified by the observation that when $(x,\xi)$ ranges over $\mathrm{T}^*\mathbb{R}^d$, the skew-hermitian matrices $\mathrm{i}\,\tilde{H}_\nu(x,\xi)$ generate a Lie subalgebra of $\mathfrak{u}(2j+1)$ which is isomorphic to $\mathfrak{su}(2)$.

In the general case one therefore should not necessarily expect that the transport matrices $D_\nu$ generate all of $\mathrm{U}(k_\nu)$, but only a certain Lie subgroup. In order to identify this group we consider the Lie subalgebra

$$\left\langle \mathrm{i}\,\tilde{H}_\nu(x,\xi);\ (x,\xi) \in \mathrm{T}^*\mathbb{R}^d \right\rangle \subset \mathfrak{u}(k_\nu) \tag{2.3.33}$$

generated by the skew-hermitian matrices $\mathrm{i}\,\tilde{H}_\nu(x,\xi)$. Via exponentiation of this subalgebra one hence obtains a Lie subgroup $G \subset \mathrm{U}(k_\nu)$ that is compact and connected. To be more precise, the result of the exponentiation is a $k_\nu$-dimensional unitary representation $\rho$ of $G$. Its Lie algebra $\mathfrak{g}$ then is embedded in (2.3.33) via the derived representation $\mathrm{d}\rho$. In this setting the transport matrices $D_\nu$ are operators in the representation $\rho$, i.e., $D_\nu(x,\xi,t) = \rho(g_\nu(x,\xi,t))$. Hence we are now in a position to define the skew-product flows

$$\tilde{Y}^t_\nu : \mathrm{T}^*\mathbb{R}^d \times G \to \mathrm{T}^*\mathbb{R}^d \times G \tag{2.3.34}$$

through

$$\tilde{Y}^t_\nu(x,\xi,g) = \big(\Phi^t_\nu(x,\xi), g_\nu(x,\xi,t)g\big). \tag{2.3.35}$$

These flows leave the product measure $\mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}g$ on $\mathrm{T}^*\mathbb{R}^d \times G$ invariant, which consists of Lebesgue measure $\mathrm{d}x\ \mathrm{d}\xi$ on $\mathrm{T}^*\mathbb{R}^d$ and the normalised Haar measure $\mathrm{d}g$ on $G$. Moreover, if one restricts the Hamiltonian flows $\Phi^t_\nu$ to compact level surfaces of the eigenvalue functions $\lambda_\nu$ at non-critical values $E$,

$$\Omega_{\nu,E} := \lambda_\nu^{-1}(E) = \big\{(x,\xi) \in \mathrm{T}^*\mathbb{R}^d;\ \lambda_\nu(x,\xi) = E\big\},$$

the restrictions of the skew-product flows $\tilde{Y}^t_\nu$ to $\Omega_{\nu,E} \times G$ leave the measures $\mathrm{d}\ell(x,\xi)\ \mathrm{d}g$ invariant, where $\mathrm{d}\ell(x,\xi)$ denotes the normalised Liouville measure on $\Omega_{\nu,E}$.

In Chapter 6 we are interested in the question under which conditions imposed on suitable classical dynamics quantum ergodicity holds. In analogy to [BG00] one approach to this problem would be to consider the restriction of the skew-product flow $\hat{Y}^t_\nu$ to

$\Omega_{\nu,E} \times \mathrm{U}(k_\nu)$: Its ergodicity with respect to the product measure that consists of Liouville measure on $\Omega_{\nu,E}$ and Haar measure on $\mathrm{U}(k_\nu)$ implies quantum ergodicity. Since, however, the dynamics in the eigenspaces is completely fixed by a restriction to the group $G$, the dynamical behaviour of the flow $\hat{Y}^t_\nu$ is determined by that of $\tilde{Y}^t_\nu$. One hence concludes that in order to prove quantum ergodicity one requires the following condition (see Remark 6.2.3):

(Irr$_\nu$) The representation $\rho : G \to \mathrm{U}(k_\nu)$ is irreducible.

In the sequel we always assume this to be the case.

## 2.4 Semiclassical asymptotics of eigenfunctions

### 2.4.1 A Szegö-type limit formula

A fundamental ingredient in the asymptotics of eigenvectors that will be discussed in Chapter 6 is a semiclassical limit formula for the expectation values of bounded operators $\mathcal{B}$ on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. Below we will obtain a Szegö-type formula which connects semiclassically averaged expectation values with objects that can be calculated from the principal symbol $B_0$ of the operator $\mathcal{B}$ and therefore allow for a classical interpretation. On the so defined classical side we fix a value $E$ for all eigenvalue functions $\lambda_\nu$, $\nu = 1, \dots, l$, of the principal symbol $H_0$ with the following properties:

(H3$_\nu$) There exists some $\varepsilon > 0$ such that all $\lambda_\nu^{-1}([E-\varepsilon, E+\varepsilon]) \subset \mathrm{T}^*\mathbb{R}^d$ are compact.

(H4$_\nu$) The functions $\lambda_\nu$ shall possess no critical values in $[E-\varepsilon, E+\varepsilon]$.

(H5$_\nu$) Among the level surfaces $\Omega_{\nu,E} = \lambda_\nu^{-1}(E)$, $\nu = 1, \dots, l$, at least one is non-empty.

In addition to (H1) and (H2), which imply the essential selfadjointness of the operator $\mathcal{H}$, these conditions guarantee as in the scalar case [DS99] that for sufficiently small $\hbar$ the spectrum of $\mathcal{H}$ is discrete in any open interval contained in $[E-\varepsilon, E+\varepsilon]$. This setting now allows us to generalise the constructions made in [BG00] to Hamiltonians with non-scalar principal symbols: The expectation values of an operator $\mathcal{B}$ will be considered in normalised eigenvectors $\psi_j$ of $\mathcal{H}$ with corresponding eigenvalues $E_j$ in an interval $I(E,\hbar) = [E-\hbar\omega, E+\hbar\omega]$, $\omega > 0$, such that $I(E,\hbar) \subset [E-\varepsilon, E+\varepsilon]$ if $\hbar$ is small enough. Let us denote by $N_I := \mathrm{card}\{E_j \in I(E,\hbar)\}$ the number of eigenvalues in $I(E,\hbar)$. On the classical side the Hamiltonian flows $\Phi^t_\nu$ generated by the eigenvalue functions $\lambda_\nu$ will enter on the level surfaces $\Omega_{\nu,E}$. Regarding these we assume:

(H6$_\nu$) The periodic points of $\Phi^t_\nu$ with non-trivial periods form a set of Liouville measure zero in $\Omega_{\nu,E}$.

The quantities appearing on the classical side of the limit formula turn out to be averages of smooth matrix valued functions $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ over $\Omega_{\nu,E}$ with respect to Liouville measure, for which we introduce the notation

$$\ell_{\nu,E}(B) := \int_{\Omega_{\nu,E}} B(x,\xi)\, \mathrm{d}\ell(x,\xi).$$

The main result of this section is now summarised in the following Szegö-type limit formula:

**Proposition 2.4.1.** *Let $\mathcal{H}$ be a semiclassical pseudodifferential operator with symbol $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$, such that the principal symbol $H_0$ satisfies the assumptions (H0)–(H2) and (H3$_\nu$)–(H6$_\nu$) for all $\nu = 1, \dots, l$. Furthermore, let $\mathcal{B}$ be an operator with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ and principal symbol $B_0$. Then the limit formula*

$$\lim_{\hbar\to 0} \frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \psi_j, \mathcal{B}\psi_j \rangle = \frac{\sum_{\nu=1}^l \mathrm{vol}\,\Omega_{\nu,E}\,\mathrm{tr}\,\ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0})}{\sum_{\nu=1}^l k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}} \tag{2.4.1}$$

*holds.*

*Proof.* Adapted to the spectral localisation mentioned above we choose a smooth and compactly supported function $g \in C_0^\infty(\mathbb{R})$ such that $g(\lambda) = \lambda$ on a neighbourhood of $[E-\varepsilon, E+\varepsilon]$. Furthermore, we apply the semiclassical splitting of the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ given by the projection operators $\mathcal{P}_\nu$,

$$L^2(\mathbb{R}^d) \otimes \mathbb{C}^n = \mathrm{ran}\,\mathcal{P}_1 \oplus \cdots \oplus \mathrm{ran}\,\mathcal{P}_l \mod \hbar^\infty, \tag{2.4.2}$$

and the corresponding decomposition $\mathcal{H} = \sum_{\nu=1}^l \mathcal{H}\mathcal{P}_\nu$ (mod $O(\hbar^\infty)$) of the Hamiltonian. By employing the generalisation of the Helffer-Sjöstrand formula to matrix valued operators developed in [Dim93, Dim98], we represent $g(\mathcal{H}) = \sum_{\nu=1}^l g(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu$ (mod $O(\hbar^\infty)$) with

$$g(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu = -\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{g}(z)(\mathcal{H}-z)^{-1}\mathcal{P}_\nu \,\mathrm{d}z,$$

where $\tilde{g}$ is an almost-analytic extension of $g$. Since the principal symbol $H_0P_{\nu,0}$ of $\mathcal{H}\mathcal{P}_\nu$ is scalar, $H_0P_{\nu,0} = \lambda_\nu P_{\nu,0}$, when considered to act on sections in the eigenvector bundle $E^\nu$, one can use the methods of [DS99] to show that on $\lambda_\nu^{-1}([E-\varepsilon, E+\varepsilon])$ the asymptotic expansions of $\mathrm{symb}^W[g(\mathcal{H}\mathcal{P}_\nu)]$ and of $\mathrm{symb}^W[\mathcal{H}\mathcal{P}_\nu]$ coincide. Below we will always employ the spectral localisation to the interval $I(E,\hbar)$, and since $\mathrm{symb}^W[g(\mathcal{H}\mathcal{P}_\nu)] \in \mathrm{S}^0_{\mathrm{cl}}(1)$, one can therefore now assume that $H \in \mathrm{S}^0_{\mathrm{cl}}(1)$. Furthermore, the decomposition (2.4.2) allows us to employ the techniques of [DS99] in the same manner as in [BG00]. Hence, if $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $I(E,\hbar)$ and $\mathrm{supp}\,\chi \subset [E-\varepsilon, E+\varepsilon]$, the operator

$$\mathcal{U}_\chi(t) := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}t}\chi(\mathcal{H})\sum_{\nu=1}^l \mathcal{P}_\nu = \sum_{\nu=1}^l \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\chi(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu \mod O(\hbar^\infty),$$

has a pure point spectrum. Moreover, each of the operators $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\chi(\mathcal{H}\mathcal{P}_\nu)$ can be approximated in trace norm up to an error of $O(\hbar^\infty)$ by a semiclassical Fourier integral operator with a kernel of the form

$$K_\nu(x,y,t) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} a_\nu(x,y,t,\xi)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\nu(x,\xi,t)-\xi y)}\,\mathrm{d}\xi. \tag{2.4.3}$$

Here, as in [BK99a], the phases $S_\nu$ have to fulfill the Hamilton-Jacobi equations

$$\lambda_\nu\big(x, \partial_x S_\nu(x,\xi,t)\big) + \partial_t S_\nu(x,\xi,t) = 0, \quad S_\nu(x,\xi,0) = x\xi.$$

The amplitudes $a_\nu \in \mathrm{S}^0_{\mathrm{cl}}(1)$ with asymptotic expansions $a_\nu \sim \sum_{j=0}^\infty \hbar^j a_{\nu,j}$ are determined as solutions of certain transport equations [BK99a] with initial conditions $a_\nu|_{t=0} = \chi(\lambda_\nu)P_{\nu,0} + O(\hbar)$. Following [BG00] further, we choose test functions $\rho \in C^\infty(\mathbb{R})$ with compactly supported Fourier transforms $\hat\rho \in C_0^\infty(\mathbb{R})$ such that

$$\operatorname{Tr} \frac{1}{2\pi} \int_{\mathbb{R}} \hat\rho(t)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Et}\, \mathcal{B}\mathcal{U}_\chi(t)\ \mathrm{d}t = \sum_j \chi(E_j)\langle\psi_j, \mathcal{B}\psi_j\rangle \rho\left(\frac{E_j - E}{\hbar}\right),$$

where Tr denotes the operator trace on the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. Using the semiclassical approximation (2.4.3) one now has to calculate

$$\frac{1}{2\pi(2\pi\hbar)^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat\rho(t) \sum_{\nu=1}^{l} \operatorname{tr}\big(B_0(x, \partial_x S_\nu) a_{\nu,0}(x,x,t,\xi)\big)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\nu(x,\xi,t) - x\xi + Et)}\ \mathrm{d}\xi\ \mathrm{d}x\ \mathrm{d}t \tag{2.4.4}$$

in leading semiclassical order. This can be done with the method of stationary phase, where the stationary points $(x_{\nu,\mathrm{st}}, \xi_{\nu,\mathrm{st}}, t_{\nu,\mathrm{st}})$ of the phase $S_\nu(x,\xi,t) - x\xi + Et$ determine periodic points $(x_{\nu,\mathrm{st}}, \xi_{\nu,\mathrm{st}}) \in \Omega_{\nu,E}$ of the Hamiltonian flow $\Phi^t_\nu$ with periods $t_{\nu,\mathrm{st}}$. Since the eigenvalue function $\lambda_\nu$ is supposed to be non-critical at $E$, the periods $t_{\nu,\mathrm{st}}$ of the flow $\Phi^t_\nu$ cannot accumulate at zero, see [Rob87]. One can hence split $\hat\rho = \hat\rho_1 + \hat\rho_2$ in such a way that $\hat\rho_1$ is supported only in a small neighbourhood of zero and $\hat\rho_2 = 0$ in the vicinity of zero, so that the only period in $\operatorname{supp}\hat\rho_1$ is the trivial one, $t_{\nu,\mathrm{st}} = 0$. The contribution coming from $\hat\rho_1$ to (2.4.4) is therefore determined by the periodic points with $t_{\nu,\mathrm{st}} = 0$. These build up the entire level surface $\Omega_{\nu,E}$ which, according to assumption (H3$_\nu$), is compact. The result then reads (see [DS99, BG00])

$$\sum_j \chi(E_j)\langle\psi_j, \mathcal{B}\psi_j\rangle \rho_1\left(\frac{E_j - E}{\hbar}\right) = \chi(E)\frac{\hat\rho_1(0)}{2\pi} \sum_{\nu=1}^{l} \frac{\operatorname{vol}\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}} \big(\operatorname{tr}\ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0}) + O(\hbar)\big). \tag{2.4.5}$$

Coming to the contribution of the term with $\hat\rho_2$ to the expression (2.4.4), we recall that $\hat\rho_2$ has been chosen to vanish in a neighbourhood of zero. The relevant stationary points are hence related to periodic orbits of the flow $\Phi^t_\nu$ with non-vanishing periods. The condition (H6$_\nu$) now allows us to employ the methods of [DS99], leading to the estimate

$$\sum_j \chi(E_j)\langle\psi_j, \mathcal{B}\psi_j\rangle \rho_2\left(\frac{E_j - E}{\hbar}\right) = o(\hbar^{1-d}). \tag{2.4.6}$$

The relations (2.4.5) and (2.4.6) together therefore imply that for every test function $\rho \in C^\infty(\mathbb{R})$ with Fourier transform $\hat\rho \in C_0^\infty(\mathbb{R})$ the estimate (2.4.5) holds with $\rho_1$ replaced by $\rho$. Hence, the Tauberian argument developed in [BPU95] can be applied to yield

$$\sum_{E_j \in I(E,\hbar)} \langle \psi_j, \mathcal{B}\psi_j \rangle = \frac{\omega}{\pi} \sum_{\nu=1}^{l} \frac{\operatorname{vol}\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}} \operatorname{tr} \ell_{\nu,E}(P_{\nu,0} B_0 P_{\nu,0}) + o(\hbar^{1-d}). \tag{2.4.7}$$

In this relation one can set the operator $\mathcal{B}$ equal to the identity and thus obtains a semiclassical expression for the number $N_I$ of eigenvalues of $\mathcal{H}$ in $I(E,\hbar)$,

$$N_I := \#\{E_j \in I(E,\hbar)\} = \frac{\omega}{\pi} \sum_{\nu=1}^{l} k_\nu \frac{\operatorname{vol}\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}} + o(\hbar^{1-d}), \tag{2.4.8}$$

where $k_\nu = \operatorname{tr} P_{\nu,0}$ denotes the dimension of the fibre $\operatorname{ran} P_{\nu,0} = E^\nu$ corresponding to the eigenvalue $\lambda_\nu$ of $H_0$. The proof is now finished by combining the expressions (2.4.7) and (2.4.8). □

Let us remark that the operators $\mathcal{B}$ considered in the limit formula (2.4.1) have not been restricted to those with symbols in the invariant subalgebra $\mathrm{S}^0_{\mathrm{inv}}(1) \subset \mathrm{S}^0_{\mathrm{cl}}(1)$. Nevertheless, only the diagonal blocks of their principal symbols $B_0$ with respect to the projection matrices $P_{\nu,0}$ enter on the right-hand side of (2.4.1). In particular, this implies that for an operator $\mathcal{B}$ with a purely off-diagonal principal symbol, i.e., $P_{\mu,0} B_0 P_{\mu,0} = 0$ for all $\mu = 1, \dots, l$, the semiclassical average vanishes. Thus one can replace an operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ by its diagonal part $\sum_\mu \tilde{\mathcal{P}}_\mu \mathcal{B} \tilde{\mathcal{P}}_\mu$, whose symbol is in the invariant algebra $\mathrm{S}^0_{\mathrm{inv}}(1)$, without changing the value of the limit on the right-hand side of (2.4.1).

So far we have considered expectation values in normalised eigenvectors of $\mathcal{H}$. Our intention now is to discuss the projections $\mathcal{P}_\nu \psi_j$ of the eigenvectors of $\mathcal{H}$ to a fixed almost invariant subspace of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. One thus expresses averaged expectation values in the projected eigenvectors in terms of classical quantities related to the single Hamiltonian flow $\Phi^t_\nu$. In order to achieve this one applies Proposition 2.4.1 to operators $\mathcal{P}_\nu \mathcal{B} \mathcal{P}_\nu$ and exploits the selfadjointness of $\mathcal{P}_\nu$. This results in

**Corollary 2.4.2.** *Under the assumptions stated in Proposition 2.4.1, for each $1 \le \nu \le l$ the restricted limit formula*

$$\lim_{\hbar\to 0} \frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \mathcal{P}_\nu \psi_j, \mathcal{B}\mathcal{P}_\nu \psi_j \rangle = \frac{\operatorname{vol}\Omega_{\nu,E} \operatorname{tr} \ell_{\nu,E}(P_{\nu,0} B_0 P_{\nu,0})}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}}. \tag{2.4.9}$$

*holds.*

Thus the semiclassical average of the projected eigenvectors $\mathcal{P}_\nu \psi_j$, with $E_j \in I(E,\hbar)$, localises on the corresponding level surface $\Omega_{\nu,E} \subset \mathrm{T}^*\mathbb{R}^d$. If one considers (2.4.9) for different $\nu$, the relative weights of the corresponding projections are determined by the relative volumes of the associated level surfaces and the dimensions of the eigenspaces $E^\nu$, which equal the volumes of the coadjoint orbits $\mathcal{O}_\lambda$.

In general, however, the projected eigenvectors $\mathcal{P}_\nu\psi_j$ are neither normalised, nor are they genuine eigenvectors of $\mathcal{H}$. We therefore now introduce the normalised vectors

$$\phi_{j,\nu} := \frac{\mathcal{P}_\nu\psi_j}{\|\mathcal{P}_\nu\psi_j\|}. \tag{2.4.10}$$

Since the projectors $\mathcal{P}_\nu$ only commute with $\mathcal{H}$ up to a term of $O(\hbar^\infty)$, the pairs $(E_j, \phi_{j,\nu})$ are quasimodes [Arn72] with discrepancies $r_{j,\nu}$, i.e.,

$$(\mathcal{H} - E_j)\phi_{j,\nu} = \frac{[\mathcal{H}, \mathcal{P}_\nu]\psi_j}{\|\mathcal{P}_\nu\psi_j\|} \quad \text{and} \quad r_{j,\nu} = \frac{\|[\mathcal{H}, \mathcal{P}_\nu]\psi_j\|}{\|\mathcal{P}_\nu\psi_j\|}.$$

This observation only ensures the existence of an eigenvalue of $\mathcal{H}$ in the interval $[E_j - r_{j,\nu}, E_j + r_{j,\nu}]$, which is a trivial statement; it does not imply that $\phi_{j,\nu}$ is close to an eigenvector of $\mathcal{H}$, see [Laz93]. It therefore is of somewhat more interest to consider the operator $\mathcal{H}\mathcal{P}_\nu$, whose spectrum inside the interval $[E-\varepsilon, E+\varepsilon] \supset I(E,\hbar)$ is also purely discrete. Following the above reasoning, one then concludes that $(E_j, \phi_{j,\nu})$ is a quasimode with discrepancy $r_{j,\nu}$ also for this operator. Thus, if $\|\mathcal{P}_\nu\psi_j\| \geq c\hbar^N$ for some $N \geq 0$ and hence $r_{j,\nu} = O(\hbar^\infty)$, the operator $\mathcal{H}\mathcal{P}_\nu$ has an eigenvalue with distance $O(\hbar^\infty)$ away from $E_j$. Since there are $N_I$ eigenvalues $E_j \in I(E,\hbar)$ one finds as many quasimodes for $\mathcal{H}\mathcal{P}_\nu$. But this operator has only

$$N_I^\nu = \frac{k_\nu\omega}{\pi}\frac{\operatorname{vol}\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}} + o(\hbar^{1-d})$$

eigenvalues in $I(E,\hbar)$, compare (2.4.8). This observation might suggest that only approximately $N_I^\nu$ of the $N_I$ projected eigenvectors $\mathcal{P}_\nu\psi_j$ are of considerable size, such that the discrepancies of the associated quasimodes are smaller than the distance of $E_j$ to neighbouring eigenvalues of $\mathcal{H}$. This expectation can be strengthened by an application of the limit formula (2.4.9) with the choice $\mathcal{B} = \mathrm{id}$,

$$\lim_{\hbar\to 0}\frac{1}{N_I}\sum_{E_j\in I(E,\hbar)}\|\mathcal{P}_\nu\psi_j\|^2 = \frac{k_\nu \operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}}, \tag{2.4.11}$$

which implies that

$$N_I^\nu = \sum_{E_j\in I(E,\hbar)}\|\mathcal{P}_\nu\psi_j\|^2 + o(1), \quad \hbar \to 0. \tag{2.4.12}$$

One could thus expect that roughly $N_I^\nu$ of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ are close to $\psi_j$, and the rest is such that $\|\mathcal{P}_\nu\psi_j\|$ is semiclassically small. However, (2.4.12) does not rule out the other extreme situation, provided by projected eigenvectors $\mathcal{P}_\nu\psi_j$, $\nu = 1, \dots, l$, equidistributing in the sense that their squared norms are asymptotic to $N_I^\nu/N_I$ as $\hbar \to 0$. In that case the discrepancies of the associated quasimodes for the operators $\mathcal{H}\mathcal{P}_\nu$ can be estimated as $r_{j,\nu} = O(\hbar^\infty)$. In order that these quasimodes do not produce more than $N_I^\nu$ eigenvalues of $\mathcal{H}\mathcal{P}_\nu$ in $I(E,\hbar)$ a finite fraction of the eigenvalues $E_j$ of $\mathcal{H}$ must possess spacings to their nearest neighbours of the order $\hbar^\infty$. Since in general there exist no sufficient lower bounds on eigenvalue spacings none of the two extreme situations discussed above can be excluded so far.

What is possible, however, is to derive from (2.4.11) an upper bound for the fraction of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ that are close in norm to $\psi_j$,

$$\lim_{\hbar\to 0}\frac{1}{N_I}\#\big\{E_j\in I(E,\hbar);\ \|\mathcal{P}_\nu\psi_j-\psi_j\|=o(1)\big\}\leq\frac{k_\nu\operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\operatorname{vol}\Omega_{\mu,E}},$$

see also [Sch01]. To obtain lower bounds is more difficult. The limit formula (2.4.11) only allows for an estimate of the fraction of projected eigenvectors with norms that tend to a finite limit as $\hbar\to 0$. One conveniently measures this fraction in units of the value that is expected for equidistributed projections. Therefore, with $\delta:=\tilde{\delta}\frac{k_\nu\operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\operatorname{vol}\Omega_{\mu,E}}$, we consider

$$N_{\nu,I}^{\delta}:=\#\big\{E_j\in I(E,\hbar);\ \|\mathcal{P}_\nu\psi_j\|^2\geq\delta\big\}.$$

Since

$$\begin{aligned}\frac{1}{N_I}\sum_{E_j\in I(E,\hbar)}\|\mathcal{P}_\nu\psi_j\|^2&\leq\frac{1}{N_I}\sum_{\substack{E_j\in I(E,\hbar)\\ \|\mathcal{P}_\nu\psi_j\|^2\geq\delta}}1+\frac{1}{N_I}\sum_{\substack{E_j\in I(E,\hbar)\\ \|\mathcal{P}_\nu\psi_j\|^2<\delta}}\|\mathcal{P}_\nu\psi_j\|^2\\ &\leq\frac{N_{\nu,I}^{\delta}}{N_I}+\frac{\delta}{N_I}(N_I-N_{\nu,I}^{\delta}),\end{aligned}$$

the relative fraction of projected eigenvectors with finite semiclassical limit can be estimated from below as

$$\lim_{\hbar\to 0}\frac{N_{\nu,I}^{\delta}}{N_I}\geq\frac{(1-\tilde{\delta})k_\nu\operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\operatorname{vol}\Omega_{\mu,E}}.\tag{2.4.13}$$

A refinement of the above considerations is given by the study of individual eigenfunctions $\psi_j$. As we already have seen above, a quasimode (or a projected eigenfunction) can only be close to a genuine eigenfunction if it localizes on one and only one energy shell in the semiclassical limit; this behaviour then certainly transfers to the eigenfunction which the quasimode approximates. In order to ask the question if a certain quasimode is close to an eigenfunction, we can employ the following statement (see [Sch01]):

**Proposition 2.4.3.** *Let $(E,\phi)$ be a quasimode of $\mathcal{H}$ with discrepancy $r$ and assume that the spectrum of $\mathcal{H}$ is discrete in a neighbourhood of $[E-r,E+r]$. Denote the distance of $[E-r,E+r]$ to the part of the spectrum outside of $[E-r,E+r]$ by $\epsilon$ and let $\pi$ be the spectral projection corresponding to the complement of $[E-r,E+r]$ in* $\operatorname{spec}(\mathcal{H})$. *Then*

$$\|\pi\phi\|\leq\frac{r}{\epsilon}.$$

Of course, this Proposition only becomes useful if the discrepancy $r$ is smaller than the distance of $[E-r, E+r]$ from $\mathrm{spec}(\mathcal{H})$.

Thus, if the eigenvalues of $\mathcal{H}$ were separated by a minimal distance greater than the discrepancy, the above statement would imply that the quasimode is close to an eigenfunction. In general, however, no such minimal distance is known; there are a lot of examples where eigenvalues are quasidegenerate[18]which allows us to construct quasimodes as superposition of eigenvectors whose eigenvalues are quasidegenerate, see e.g. the Schrödinger operator in a double well potential [HS84, Sim84b, Sim84a, Sim83b, Pet97, Dav82, Dav84]. In these cases, the quasidegeneracies are due the symmetry of the underlying system. However, this effect immediately breaks down if the symmetry is destroyed, see e.g. [Sim85]. This suggests that even if a quasimode is not close to an eigenfunction it might be close if one perturbes the system. Therefore, let $\mathcal{H} + \hbar^p \mathcal{A}$ be a perturbation with $\mathcal{A} \in \mathrm{OPS}(1)$ and $A \in \mathrm{S}(1)$ hermitian, such that the system's classical limit remains unchanged. First of all, we have to ensure that under this perturbation a quasimode $(E, \phi)$ is taken to a quasimode with (almost) the same discrepancy. That this, we assume that $(E, \phi)$ can be extended to a family of quasimodes $\mathcal{A} \mapsto (E(\mathcal{A}), \phi(\mathcal{A}))$ whose discrepancies $r = r(\mathcal{A})$ are all of order $\hbar^N$ with $N$ independent of $\mathcal{A}$. Furthermore, we assume that the quasimode doesn't change too much under small variations of the perturbation, i.e.

$$\|\phi(\mathcal{A}) - \phi(\mathcal{A}')\| \leq C\hbar^p \|\mathcal{A} - \mathcal{A}'\|. \tag{2.4.14}$$

If these conditions are fulfilled, we call $(E, \phi)$ a *stable quasimode*. From [Sch01] we quote

**Conjecture 2.4.4.** *Let the Hamiltonian $\mathcal{H}$ fulfill the conditions stated in Proposition 2.4.1 and let $(E, \phi)$ be a stable quasimode with discrepancy of order $\hbar^{d+\delta}$ for some $\delta > 0$. Then the set of perturbations for which $\phi(\mathcal{A})$ converges to an eigenfunction is of second Baire category in* $\mathrm{OPS}(1)$.

We remark that a set in a complete metrizable space is said to be of second Baire category if it can be represented as a countable intersection of open dense subsets, see [Mun00]. We say that a property is *generically* true, if it is true on a set of second Baire category. Therefore, the above conjecture means that in a generic situation quasimodes with sufficiently small discrepancies should be close to genuine eigenfunctions.

There are a few indications towards the correctness of this Conjecture. In [Uhl72, Uhl76, Alb78] it is shown for example that eigenvalues are generically simple, or in [MO04] that the eigenfunctions for special systems localize on classically invariant sets.

[18] with a distance $\mathrm{e}^{-c/\hbar}$, $c > 0$

# Chapter 3

# Geometric quantization

The purpose of this chapter is to provide the mathematical tools needed to give a classical interpretation to the intrinsic degrees of freedom, still remanescent in its quantum nature in the analysis performed in Chapter 2 . Before we give an outline of the underlying mathematical principles, following [Śni80, SW76, Sou70], we want to state an observation that is very striking when one enters the realm of geometric quantization:

> *We have found that when the notion of what physicists mean by quantizing a function is suitably generalized and made rigorous, one may develop a theory which goes a long way towards constructing all the irreducible unitary representations of a connected Lie group. In the compact case it encompasses the Borel-Weil theorem. [Kos70]*

## 3.1 Line bundles and connections

Quantization in general assigns a Hilbert space to a symplectic manifold $X$. In Chapter 2 we were concerned with symplectic manifolds that were given as cotangent bundles of manifolds. In the present context we will not restrict to this class of symplectic manifolds. This fact will be important in the quantization procedure for the internal degrees of freedom since, as we will see, the phase space for these will in general not be given as cotangent bundles.

Now let $\mathbb{C}^\times$ be the multiplicative group of complex numbers, whose Lie algebra is identified with $\mathbb{C}$ by associating to each $c \in \mathbb{C}$ the one-parameter group $\mathrm{e}^{2\pi\,\mathrm{i}\,ct} \subset \mathbb{C}^\times$. Let $L \to X$ be a complex line bundle over $X$, and $L^\times$ the line bundle obtained from $L$ by removing the zero section. In fact $L^\times$ is the $\mathbb{C}^\times$-principal fibre bundle associated with $L$. As usual, the space of sections $\Gamma(X, L)$ is isomorphic to the space of complex valued functions $\tilde{\lambda} : L^\times \to \mathbb{C}$ which are equivariant with respect to the $\mathbb{C}^\times$ action, i.e.

$$\tilde{\lambda}(cz) = c^{-1}\tilde{\lambda}(z) \tag{3.1.1}$$

for each $c \in \mathbb{C}^\times$ and $z \in L^\times$, see e.g. [BGV92, Lan98a, Kob87]. Explicitly, the isomorphism

$\tilde{\lambda} \stackrel{\sim}{\mapsto} \lambda$ is given by

$$\lambda(\pi(z)) = \tilde{\lambda}(z)z,$$

where $\pi : L \to X$ is the bundle projection. Associated with each $c \in \mathbb{C}$ is the fundamental vector field[1] $\eta_c$ on $L^\times$ defined by

$$(\eta_c f)(z) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{2\pi\,\mathrm{i}\,ct}\, z)\Big|_{t=0}.$$

For equivariant functions (3.1.1) we get

$$\eta_c \tilde{\lambda} = -2\pi\,\mathrm{i}\, c\tilde{\lambda}.$$

For each function $f$ on $X$ we can define a vector field $\eta_f$ on $L^\times$ by

$$\eta_f(z) = \eta_{f(\pi(z))}(z)$$

such that

$$\eta_f \tilde{\lambda} = -2\pi\,\mathrm{i}(f \circ \pi)\tilde{\lambda}. \tag{3.1.2}$$

Let $\alpha$ be a *connection form* on $L^\times$, i.e. a $\mathbb{C}^\times$ valued one-form on $L^\times$ such that

$$\alpha(\eta_c) = c$$

for each $c \in \mathbb{C}$. A connection, in this case defined by the connection one-form $\alpha$, is equivalent to a splitting of the exact sequence

$$0 \to \operatorname{ver} L^\times \to \mathrm{T}L^\times \to \pi^*\mathrm{T}X \to 0,$$

where $\operatorname{ver} L^\times \subset \mathrm{T}L^\times$ is the subbundle generated by the vector fields $\eta_c$. This is accomplished by defining

$$\operatorname{hor} L^\times := \{u \in \mathrm{T}L^\times; \alpha(u) = 0\},$$

and we have

$$\mathrm{T}L^\times = \operatorname{hor} L^\times \oplus \operatorname{ver} L^\times.$$

So for any vector field $\zeta$ on $L^\times$ we obtain the horizontal and vertical components, $\operatorname{hor}\zeta$ and $\operatorname{ver}\zeta$, such that

$$\zeta(z) = \operatorname{hor}\zeta(z) + \operatorname{ver}\zeta(z)$$

for each $z \in L^\times$. Furthermore, there is a unique *horizontal lift* $\tilde{\xi}$ of a vector field $\xi$ on $X$, i.e. a vector field $\tilde{\xi}$ on $L^\times$ which projects down to $\xi$ and fulfills $\alpha(\tilde{\xi}) = 0$. This also allows for[2] the definition of a covariant derivative $\nabla$ acting on sections of $L$ according to

$$\nabla_\xi \lambda(\pi(z)) = (\tilde{\xi}\tilde{\lambda})(z)z, \tag{3.1.3}$$

[1] This is the general construction for principal $G$-bundles: An element of the corresponding Lie algebra $\mathfrak{g}$ generates the so-called fundamental vector field on the principal bundle, see [KN63, KN69].

[2] And in fact is equivalent to

which can easily be shown to be a derivation. In many cases it is useful that the covariant derivative can be directly related to the connection one-form by the following construction: Let $\lambda$ be a non-vanishing section of $L$, i.e. a map $X \to L^\times$. Then $\lambda_*\xi(x)$ is a vector in $\mathrm{T}_{\lambda(x)}L^\times$ and can be decomposed in its vertical and horizontal components,

$$\lambda_*(\xi(x)) = \mathrm{hor}\,\lambda_*\xi(x) + \mathrm{ver}\,\lambda_*\xi(x) = \tilde{\xi}(x) + \eta_{\alpha(\lambda_*\xi(x))}\lambda(x),$$

since $\pi\lambda = \mathrm{id}_X$. Therefore,

$$(\tilde{\xi}\tilde{\lambda})(\lambda(x)) = \mathrm{d}\tilde{\lambda}(\tilde{\xi}(x)) = \mathrm{d}\tilde{\lambda}(\lambda_*\xi(x)) - \mathrm{d}\tilde{\lambda}(\eta_{\alpha(\lambda_*\xi(x))}(\lambda(x))).$$

But $\mathrm{d}\tilde{\lambda}(\lambda_*\xi(x)) = (\xi(\tilde{\lambda}\circ\lambda))(x)$ vanishes since $\lambda(x) = \tilde{\lambda}(\lambda(x))\lambda(x)$, i.e. $\tilde{\lambda}\circ\lambda = \mathrm{id}$. In addition,

$$\mathrm{d}\tilde{\lambda}(\eta_{\alpha(\lambda_*\xi(x))}(\lambda(x))) = -2\pi\,\mathrm{i}\,\alpha(\lambda_*\xi(x))\tilde{\lambda}(\lambda(x)) = -2\pi\,\mathrm{i}\,\lambda^*\alpha(\xi(x)).$$

Summarizing we have

$$(\tilde{\xi}\tilde{\lambda})(\lambda(x)) = 2\pi\,\mathrm{i}(\lambda^*\alpha)(\xi(x)).$$

and thus

$$\nabla_\xi\lambda = 2\pi\,\mathrm{i}(\lambda^*\alpha)(\xi)\lambda.$$

The curvature form of the connection $\alpha$ is given by $\mathrm{d}\alpha$ and we have

$$(\nabla_\xi\nabla_{\xi'} - \nabla_{\xi'}\nabla_\xi - \nabla_{[\xi,\xi']})\lambda = 2\pi\,\mathrm{i}(\lambda^*\,\mathrm{d}\alpha)(\xi,\xi')\lambda. \tag{3.1.4}$$

### 3.1.1 The pre-quantum line bundle

A pre-quantum line bundle of a symplectic manifold $(X,\omega)$ is a complex line bundle $L \to X$ with a connection $\nabla$ such that the connection form satisfies the *pre-quantum condition*

$$\mathrm{d}\alpha = -\pi^*\omega. \tag{3.1.5}$$

Such a line bundle exists iff $\omega$ defines an integral de Rham cohomology class, see Appendix E and [Kos70, Woo97, EMRV98]. If this condition is fulfilled the set of equivalence classes of such line bundles with connection can be parameterized by the group of all unitary characters of the fundamental group of $X$. Furthermore, we assume that there exists a hermitian structure $\langle\cdot,\cdot\rangle$ on $L$ that is invariant with respect to $\nabla$, i.e.

$$\xi\langle\lambda,\lambda'\rangle = \langle\nabla_\xi\lambda,\lambda'\rangle + \langle\lambda,\nabla_\xi\lambda'\rangle$$

for each pair $\lambda$ and $\lambda'$ of sections and each vector field $\xi$ on $X$. Such an invariant hermitian structure exists if and only if the one-form

$$2\pi\,\mathrm{i}(\alpha - \overline{\alpha})$$

is exact. It is then determined by $\alpha$ up to a multiplicative positive constant.

Now let $\zeta$ be a real vector field on $L^\times$ preserving $\alpha$,

$$\mathscr{L}_\zeta \alpha = \imath_\zeta \, \mathrm{d}\alpha + \mathrm{d}\imath_\zeta \alpha = 0. \tag{3.1.6}$$

By evaluating this on $\eta_c$ and taking into account that $\imath_{\eta_c} \mathrm{d}\alpha = 0$[3] we get

$$\eta_c(\alpha(\zeta)) = 0,$$

which means that $\alpha(\zeta)$ is constant along the fibers of $L^\times \to X$. Consequently there exists a function $f$ on $X$ such that

$$\alpha(\zeta) = -f \circ \pi.$$

So $f$ determines the vertical part of $\zeta$ by

$$\operatorname{ver} \zeta = -\eta_f$$

and according to (3.1.6) we have

$$\imath_{\operatorname{hor} \zeta} \pi^* \omega = -\,\mathrm{d}(f \circ \pi).$$

Therefore, $f$ is real valued and $\operatorname{hor} \zeta$ is the horizontal lift of the Hamiltonian vector field $\mathrm{X}_f$ of $f$

$$\operatorname{hor} \zeta = \tilde{\mathrm{X}}_f.$$

We denote by $\zeta_f$ the vector field

$$\zeta_f = \tilde{\mathrm{X}}_f - \eta_f. \tag{3.1.7}$$

Indeed, this is the first step in the quantization procedure: The association $f \mapsto \zeta_f$ is a linear isomorphism of the Poisson algebra $(X, \omega)$ onto the Lie algebra of real connection-preserving vector fields on $L^\times$. Each vector field $\zeta_f$ is $\mathbb{C}^\times$-invariant and we can define its action on the space of functions satisfying (3.1.1), and consequently on the sections of $L$. According to equations (3.1.2) and (3.1.3) we get

$$\zeta_f \tilde{\lambda} = \left(\nabla_{\xi_f} \lambda + \mathrm{i}\, f \lambda\right)^{\sim}. \tag{3.1.8}$$

### 3.1.2 The pre-quantization map

Let $f$ be a function on $X$ such that its Hamiltonian vector field $\mathrm{X}_f$ is complete. Then $f$ generates a one-parameter group $\Phi_f^t$ of canonical transformations of $(X, \omega)$ and induces a one-parameter group $\tilde{\Phi}_f^t$ of connection preserving diffeomorphisms of $L^\times$ such that for all $t \in \mathbb{R}$

$$\pi \circ \tilde{\Phi}_f^t = \Phi_f^t \circ \pi.$$

The group $\tilde{\Phi}_f^t$ is the flow corresponding to the connection preserving vector field (3.1.7). Since each $\tilde{\Phi}_f^t$ preserves the connection form $\alpha$ it commutes with the action of $\mathbb{C}^\times$ on $L^\times$.

[3] Since $\imath_{\eta_c} \alpha = c$.

Hence for each $t \in \mathbb{R}$ and each section $\lambda$ of $L$ the function $\tilde{\lambda} \circ \tilde{\Phi}_f^{-t}$ defines a section of $L$ which we denote by $\Phi_f^t \lambda$:

$$(\Phi_f^t \lambda)^\sim = \tilde{\lambda} \circ \tilde{\Phi}_f^{-t}.$$

The mapping $\lambda \mapsto \Phi_f^t \lambda$ defines a one-parameter group of linear transformations on the space of sections of $L$ and enables us to define the *pre-quantized operator* $\mathrm{op}[f]_{\mathrm{pre}}$ corresponding to $f$ according to

$$\mathrm{op}[f]_{\mathrm{pre}} \lambda := \mathrm{i} \, \frac{\mathrm{d}}{\mathrm{d}t} (\Phi_f^t \lambda) \bigg|_{t=0} .$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} (\Phi_f^t \lambda)^\sim = -\zeta_f (\Phi_f^t \lambda)^\sim ,$$

we explicitly have

$$(\mathrm{op}[f]_{\mathrm{pre}} \lambda)^\sim = -\mathrm{i} \, \zeta_f \tilde{\lambda}. \tag{3.1.9}$$

Furthermore, according to (3.1.8)

$$\mathrm{op}[f]_{\mathrm{pre}} \lambda = (-\mathrm{i} \, \nabla_{\xi_f} + f) \lambda. \tag{3.1.10}$$

Following this prescription, constant functions are mapped to a multiplication by that constant function. Therefore the map $f \mapsto \mathrm{op}[f]_{\mathrm{pre}}$ is a linear monomorphism of the Poisson algebra of $(X, \omega)$ into the algebra of differential operators on the space of sections of $L$. Using the commutation relations for the covariant derivative in equation (3.1.4) and the properties of the Poisson bracket we get that the pre-quantization map indeed is a morphism

$$[\mathrm{op}[f]_{\mathrm{pre}}, \mathrm{op}[g]_{\mathrm{pre}}] = \mathrm{op}[\{f, g\}]_{\mathrm{pre}}.$$

However, this morphism still lacks the typical quantum mechanical factor $\hbar$ missing, which we also expect to show up in the last equation. To incorporate this factor in the above description, we can replace the pre-quantum condition (3.1.5) by

$$\mathrm{d}\alpha = -\frac{1}{\hbar} \pi^* \omega,$$

and the pre-quantizing assignment (3.1.9) by

$$(\mathrm{op}[f]_{\mathrm{pre}} \lambda)^\sim = -\mathrm{i} \, \hbar \zeta_f \tilde{\lambda},$$

which results in

$$\mathrm{op}[f]_{\mathrm{pre}} \lambda = (-\mathrm{i} \, \hbar \nabla_{\xi_f} + f) \lambda.$$

## 3.2 The representation space

In general the operators obtained by the pre-quantization map do not provide an irreducible representation of classical observables. We characterize the reducibility more explicitly in terms of

**Definition 3.2.1.** Let $(X, \omega)$ be a symplectic manifold. A set of smooth functions $\{f_j\}$ is said to be a complete set of classical observables iff every other smooth function $g$ that fulfills

$$\{f_j, g\} = 0 \quad \text{for all } f_j$$

is constant.

This is the classical notion of a complete set of observables as opposed to its quantum version

**Definition 3.2.2.** Let $\mathscr{H}$ be a Hilbert space. A set of selfadjoint operators $\{\mathcal{A}_j\}$ on $\mathscr{H}$ is said to be a complete set of operators iff every other selfadjoint operator $\mathcal{B}$ commuting with all of the $\mathcal{A}_j$ is a multiple of the identity.

By Schur's lemma, the quantum notion of a complete set of operators is equivalent to the fact that $\mathscr{H}$ is irreducible under the action of the $\mathcal{A}_j$, see also [EMRV98, Woo97].

The general property a quantization has to fulfill is formulated by

**Irreducibility Postulate.** If $\{f_j\}$ is a complete set of classical observables of a physical system then the associated quantum operators should form a complete set of operators (which implies that the Hilbert space is irreducible under the action of $\{\mathrm{op}[f_j]\}$).

It is precisely this postulate which prevents the pre-quantization being a quantization in general. This defect can be removed by the construction that we now introduce.

**Definition 3.2.3.** A *polarization* $\mathfrak{F}$ of a symplectic manifold $(X, \omega)$ is a complex involutive Lagrangian distribution on $X$ such that $\dim(\mathfrak{F}_x \cap \overline{\mathfrak{F}}_x)$ is constant.

The complex distributions $\mathfrak{F} \cap \overline{\mathfrak{F}}$ and $\mathfrak{F} + \overline{\mathfrak{F}}$ are complexifications of certain real distributions

$$\mathfrak{D}^{\mathbb{C}} = \mathfrak{F} \cap \overline{\mathfrak{F}}, \quad \mathfrak{E}^{\mathbb{C}} = \mathfrak{F} + \overline{\mathfrak{F}}. \tag{3.2.1}$$

For each $x \in X$ the we have the relation

$$\mathfrak{E}_x = \mathfrak{D}^{\perp},$$

where $\mathfrak{D}^{\perp}$ denotes the characteristic distribution corresponding to $\mathfrak{D}$. Since $\mathfrak{F}$ is involutive also $\mathfrak{D}$ is involutive, so that $\mathfrak{D}$ defines a foliation of $X$. We denote by $X/\mathfrak{D}$ the space of all integral manifolds and by $\pi_{\mathfrak{D}} : X \to X/\mathfrak{D}$ the canonical projection.

**Definition 3.2.4.** A polarization $\mathfrak{F}$ is called *strongly admissible* if $\mathfrak{E}$ is an involutive distribution, the spaces $X/\mathfrak{D}$ and $X/\mathfrak{E}$ of the integral manifolds of $\mathfrak{D}$ and $\mathfrak{E}$ are quotient manifolds of $X$ and the canonical projection $\pi_{\mathfrak{E}\mathfrak{D}} : X/\mathfrak{D} \to X/\mathfrak{E}$ is a submersion.

Strongly admissible polarizations are particular interesting, since

**Theorem 3.2.5.** *Let $\mathfrak{F}$ be a strongly admissible polarization. Then for each integral manifold $D$ of $\mathfrak{D}$ the tangent bundle $\mathrm{T}D$ is globally spanned by commuting vector fields. This defines a global parallelism in $D$ in which the parallel vector fields are the restrictions of the Hamiltonian vector field in $\mathfrak{D}$ to $D$.*

*Furthermore, each fiber $M$ of $\pi_{\mathfrak{E}\mathfrak{D}}$ has a Kähler structure such that $\mathfrak{F}|_{\pi_{\mathfrak{D}}^{-1}(M)}$ projects onto the distribution of anti-holomorphic vectors on $M$.*

*Proof.* Let $f$ be a function on $X$ such that $\mathrm{X}f = 0$ for any $\mathrm{X} \in \mathfrak{E}$. Then $\imath_{\mathrm{X}_f}\omega = 0$ and therefore $\mathrm{X}_f \in \mathfrak{D}_x$. This means that the Hamiltonian vector fields of functions constant along $\mathfrak{E}$ are in $\mathfrak{D}$. Furthermore, these vector fields commute since $\{f, g\} = -\mathrm{X}_f g$ for $\mathrm{X}_f \in \mathfrak{D}_x \subset \mathfrak{E}_x$. Conversely, it is clear that if $\mathrm{X}_f \in \mathfrak{D}_x$ then $f$ is constant along $\mathfrak{E}$.

Let $D$ be an integral manifold of $\mathfrak{D}$ and let $x$ be any point on $D$ and $(V, q_1, \ldots, q_d)$, with $d = \dim \mathfrak{D} = \operatorname{codim} \mathfrak{E}$, a chart on $X/\mathfrak{E}$ at $\pi_{\mathfrak{E}}(x)$. Then the Hamiltonian vector fields $\mathrm{X}_{\pi_{\mathfrak{E}}^* q_1}, \ldots, \mathrm{X}_{\pi_{\mathfrak{E}}^* q_d}$ commute and span $\mathfrak{D}|_{\pi_{\mathfrak{E}}^{-1}(V)}$. The tangent bundle is globally spanned by $\mathrm{X}_{\pi_{\mathfrak{E}}^* q_j}$.

Now let $M$ be a fiber of $\pi_{\mathfrak{E}\mathfrak{D}}$. The projection onto $M$ of $\mathfrak{F}|_{\pi_{\mathfrak{D}}^{-1}(M)}$ is an involutive complex distribution $\mathfrak{F}_M$ on $M$ such that $\mathfrak{F}_M + \overline{\mathfrak{F}}_M = \mathrm{T}^{\mathbb{C}}M$ and each vector $w \in \mathrm{T}M$ can be expressed as $\pi_{\mathfrak{D}*}(w + \overline{w})$ for some $w \in \mathfrak{F}$. Define

$$\mathrm{J}_M : \mathrm{T}M \to \mathrm{T}M, \quad \mathrm{J}_M(\pi_{\mathfrak{D}*}(w + \overline{w})) = -\pi_{\mathfrak{D}*}(\mathrm{i}\, w + \overline{\mathrm{i}\, w}), \tag{3.2.2}$$

and

$$h_M : \mathrm{T}M \times \mathrm{T}M \to \mathbb{C}, \quad h_M(\pi_{\mathfrak{D}*}(w + \overline{w}), \pi_{\mathfrak{D}*}(w' + \overline{w}')) = 2\,\mathrm{i}\,\omega(w, w').$$

Then $\mathrm{J}_M$ is an integrable almost complex structure such that $\mathfrak{F}_M$ is the distribution of anti-holomorphic vectors and $h_M$ is a hermitian form on $M$ with associated Kähler two-form given by $\omega$. □

### 3.2.1 The half-form bundle

Given a polarization $\mathfrak{F}$ on $(X, \omega)$ the first attempt in constructing an appropriate Hilbert space would be to take those sections of the pre-quantum line bundle $L \to X$ that are covariantly constant along $\mathfrak{F}$. If $\lambda_1$ and $\lambda_2$ are two such sections their product

$$\langle \lambda_1, \lambda_2 \rangle$$

is constant along $\mathfrak{D}$ and therefore the integral

$$\int_X \langle \lambda_1, \lambda_2 \rangle \omega^n$$

diverges unless the leaves of $\mathfrak{D}$ are compact. However, since $\langle \lambda_1, \lambda_2 \rangle$ defines a function on $X/\mathfrak{D}$ we could define a product on $X/\mathfrak{D}$ if there was a suitable measure. Unfortunately, there is no canonically defined measure on $X/\mathfrak{D}$. If $\langle\lambda_1,\lambda_2\rangle$ was defined as a density rather than a function on $X/\mathfrak{D}$, we could integrate this density over $X/\mathfrak{D}$. This can be achieved by using covariantly constant functions of the tensor product of $L$ with the half-form bundle $\sqrt{\Lambda^d}\mathfrak{F}$, which we now define.

Let $\mathfrak{F}$ be a polarization of $(X,\omega)$. Then a *linear frame* $\boldsymbol{w}=(w_1,\ldots,w_n)$ of $\mathfrak{F}$ at $x$ is an ordered basis of $\mathfrak{F}_x$. The collection of all linear frames of $\mathfrak{F}$ forms a right $\mathrm{GL}(n,\mathbb{C})$ bundle $\mathrm{B}(\mathfrak{F})$ over $X$, called the *bundle of linear frames* of $\mathfrak{F}$. Associated to this principal bundle is the complex line bundle $\Lambda^n\mathfrak{F}$ over $X$, the $n$-th exterior product of $\mathfrak{F}^*$. The space of sections of $\Lambda^n\mathfrak{F}$ is isomorphic to the space of complex valued functions $\tilde{\mu}$ on $\mathrm{B}(\mathfrak{F})$ such that

$$\tilde{\mu}(\boldsymbol{w}C)=\det(C^{-1})\tilde{\mu}(\boldsymbol{w})$$

for each $\boldsymbol{w}\in\mathrm{B}_x(\mathfrak{F})$ and $C\in\mathrm{GL}(n,\mathbb{C})$. Explicitly, this isomorphism is given by

$$\mu(x)=\tilde{\mu}(w_1,\ldots,w_d)w_1\wedge\cdots\wedge w_d.$$

Let $\mathrm{ML}(n,\mathbb{C})$ be the double covering group of $\mathrm{GL}(n,\mathbb{C})$, see Section 3.3, and let the covering homomorphism be $\rho:\ \mathrm{ML}(n,\mathbb{C})\to\mathrm{GL}(n,\mathbb{C})$. Then the *bundle of metalinear frames* of $\mathfrak{F}$ is a right principal $\mathrm{ML}(n,\mathbb{C})$ bundle $\mathrm{MB}(\mathfrak{F})$ over $X$ together with a map $\tau:\ \mathrm{MB}(\mathfrak{F})\to\mathrm{B}(\mathfrak{F})$ such that

$$\begin{array}{ccc}
\mathrm{MB}(\mathfrak{F})\times\mathrm{ML}(n,\mathbb{C}) & \longrightarrow & \mathrm{MB}(\mathfrak{F})\\
{\scriptstyle\tau\times\rho}\big\downarrow & & \big\downarrow{\scriptstyle\tau}\\
\mathrm{B}(\mathfrak{F})\times\mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{B}(\mathfrak{F})
\end{array}\tag{3.2.3}$$

commutes. We will examine the existence and structure of metalinear and also metaplectic bundles more closely in the next section. For the moment we only exploit that there is a well-defined character

$$\chi:\ \mathrm{ML}(n,\mathbb{C})\to\mathbb{C},$$

whose square equals $\det\circ\rho:\ \mathrm{ML}(n,\mathbb{C})\to\mathbb{C}$. The bundle $\sqrt{\Lambda^n}\mathfrak{F}$ is the line bundle over $X$ associated to $\mathrm{MB}(\mathfrak{F})$ with standard fiber $\mathbb{C}$ on which $\mathrm{ML}(n,\mathbb{C})$ acts by multiplication with $\chi(\tilde{C})$, $\tilde{C}\in\mathrm{ML}(n,\mathbb{C})$. The space of sections of $\sqrt{\Lambda^n}\mathfrak{F}$ is isomorphic to the space of complex valued functions $\tilde{\nu}$ on $\mathrm{MB}(\mathfrak{F})$ which are covariant with respect to the $\mathrm{ML}(n,\mathbb{C})$ action, i.e.

$$\tilde{\nu}(\tilde{\boldsymbol{w}}\tilde{C})=\chi(\tilde{C}^{-1})\tilde{\nu}(\tilde{\boldsymbol{w}}),$$

for each $\tilde{\boldsymbol{w}}\in\mathrm{MB}(\mathfrak{F})$ and $\tilde{C}\in\mathrm{ML}(n,\mathbb{C})$.

Now a strongly admissible polarization $\mathfrak{F}$ can be spanned by complex Hamiltonian vector fields $\xi_1,\ldots,\xi_n$ locally on $W\subseteq X$. Let $\boldsymbol{\xi}:\ W\to\mathrm{B}(\mathfrak{F})$ be the frame field defined by $\boldsymbol{\xi}(x)=(\xi_1(x),\ldots,\xi_n(x))$ and suppose that $W$ is contractible, then there exists a lift (see e.g. [GH81]) of $\boldsymbol{\xi}$ to a metalinear frame field $\tilde{\boldsymbol{\xi}}$. Let $\nu_{\tilde{\boldsymbol{\xi}}}$ be the unique section of $\sqrt{\Lambda^n}\mathfrak{F}$ over $W$ such that

$$\tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\boldsymbol{\xi}})=1,\tag{3.2.4}$$

then any section $\nu$ of $\sqrt{\Lambda^n\mathfrak{F}}$ can be represented on $W$ as

$$\nu|_W = (\tilde{\nu} \circ \tilde{\boldsymbol{\xi}})\nu_{\tilde{\boldsymbol{\xi}}}. \tag{3.2.5}$$

We can define a covariant derivative in the direction of $\mathfrak{F}$ acting on sections of $\sqrt{\Lambda^n\mathfrak{F}}$ as follows: A local section $\nu$ of $\sqrt{\Lambda^n\mathfrak{F}}$ is said to covariantly constant along $\mathfrak{F}$ if $\tilde{\nu} \circ \tilde{\boldsymbol{\xi}}$ is constant along $\mathfrak{F}$, where $\tilde{\boldsymbol{\xi}}$ is a any metalinear frame field that projects down to complex Hamiltonian vector fields spanning $\mathfrak{F}|_W$. Since there exist nonvanishing sections of $\sqrt{\Lambda^n\mathfrak{F}}$ that are covariantly constant, see (3.2.4), and since every section can be represented as in (3.2.5) we define $\nabla_u\nu$ as

$$(\nabla_u\nu)|_W = u(\tilde{\nu} \circ \tilde{\boldsymbol{\xi}})\nu_{\tilde{\xi}}$$

for each $u \in \mathfrak{F}|_W$. It can be shown that this definition is independent of the choice of $\tilde{\boldsymbol{\xi}}$ projecting to $\tilde{\boldsymbol{\xi}}$. It satisfies the properties of a covariant derivative if one restricts to vectors in $\mathfrak{F}$, see [Śni80].

### 3.2.2 Square integrable sections

The connection on the pre-quantum bundle $L$ and the covariant derivative of sections of the half-form bundle together induce a partial covariant derivative of sections of $L \otimes \sqrt{\Lambda^n\mathfrak{F}}$. By definition the quantum states are given as sections $\sigma : X \to L \otimes \sqrt{\Lambda^n\mathfrak{F}}$ that are covariantly constant along $\mathfrak{F}$. Now for each complex valued function $\psi$ on $X/\mathfrak{D}$, such that the restrictions of $\psi$ to the fibers of $\pi_{\mathfrak{E}\mathfrak{D}} : X/\mathfrak{D} \to X/\mathfrak{E}$ are holomorphic with respect to the complex structure defined on the fibers, see (3.2.2). The section $(\pi_{\mathfrak{D}}^*\psi)\sigma$ of $L \otimes \sqrt{\Lambda^n\mathfrak{F}}$ is also covariantly constant along $\mathfrak{F}$. Therefore, quantum states are given by sections $\sigma$ of $L \otimes \sqrt{\Lambda^n\mathfrak{F}}$ which are covariantly constant along $\mathfrak{F}$ and holomorphic along the fibers of $\pi_{\mathfrak{E}\mathfrak{D}}$.

Now assume that $\mathfrak{F}$ is a positive polarization, i.e.

$$\mathrm{i}\,\omega(\xi, \overline{\xi}) \geq 0$$

for any $\xi \in \mathfrak{F}$. To each pair $\sigma_1$ and $\sigma_2$ of $L \otimes \sqrt{\Lambda^n\mathfrak{F}}$ we now associate a (complex) density $\langle\sigma_1, \sigma_2\rangle_{X/\mathfrak{D}}$ on $X/\mathfrak{D}$: To any point $x \in X$ we have a neighbourhood $V \ni x$ in which the sections can be represented as

$$\sigma_i|_V = \lambda_i \otimes \nu_i, \quad i = 1,\, 2,$$

where $\lambda_i$ are covariantly constant sections of $L|_V$ and $\nu_i$ are covariantly constant sections of $\sqrt{\Lambda^n\mathfrak{F}}|_V$. Given a point $x \in V$, consider a basis

$$(v_1, \dots, v_m, u_1, \dots, u_{n-m}, \overline{u}_1, \dots, \overline{u}_{n-m}, w_1, \dots, w_m)$$

of $\mathrm{T}^{\mathbb{C}}X$ such that $(v_1, \dots, v_m)$ is basis of $\mathfrak{D}_x$ and $\boldsymbol{b} = (v_1, \dots, v_m, u_1, \dots, u_{n-m})$ is a basis of $\mathfrak{F}_x$. Moreover, we require that

$$\omega(v_i, w_j) = \delta_{ij}, \quad \mathrm{i}\,\omega(u_k, \overline{u}_r) = \delta_{kr}, \quad \omega(u_k, w_j) = \omega(w_i, w_j) = 0.$$

This basis projects down to a basis

$$(\pi_{\mathfrak{D}*}u_1,\ldots\pi_{\mathfrak{D}*}u_{n-m},\pi_{\mathfrak{D}*}\overline{u}_1,\ldots,\pi_{\mathfrak{D}*}\overline{u}_{n-m},\pi_{\mathfrak{D}*}w_1,\ldots,\pi_{\mathfrak{D}*}w_m)$$

of $\mathrm{T}^{\mathbb{C}}X/\mathfrak{D}$. The value of the density $\langle\sigma_1,\sigma_2\rangle_{X/\mathfrak{D}}$ on the above basis is defined to be

$$\langle\lambda_1(x),\lambda_2(x)\rangle\tilde{\nu}_1(\tilde{\boldsymbol{b}})\overline{\tilde{\nu}_2(\tilde{\boldsymbol{b}})},$$

where $\tilde{\boldsymbol{b}}$ is a metalinear frame of $\mathfrak{F}$ at $x$ projecting down to $\boldsymbol{b}\in\mathrm{B}(\mathfrak{F})$. This density only depends on the sections $\sigma_1$ and $\sigma_2$ and the basis of $\mathrm{T}^{\mathbb{C}}X/\mathfrak{D}$. It furthermore defines a hermitian inner product

$$\langle\sigma_1,\sigma_2\rangle := \int_{X/\mathfrak{D}}\langle\sigma_1,\sigma_2\rangle_{X/\mathfrak{D}}$$

on the space of covariantly constant sections. Let us denote by $\mathscr{H}_0$ the completion of the pre-Hilbert space of those covariantly constant sections for which $\langle\sigma,\sigma\rangle$ is finite. In the case of a real polarization, $\mathscr{H}_0$ can be represented as the space of square integrable functions on $X/\mathfrak{D}$, see [Śni80], which for $X=\mathrm{T}^*\mathbb{R}^d$ leads to $L^2(\mathbb{R}^d)$.

### 3.2.3 Distributional sections and Bohr-Sommerfeld conditions

The complement of $\mathscr{H}_0$ in the representation Hilbert space $\mathscr{H}$ is spanned by distributional sections of $L\otimes\sqrt{\Lambda^n}\mathfrak{F}$ that are covariantly constant along $\mathfrak{F}$. However, the (singular) supports of these sections are restricted by the Bohr-Sommerfeld conditions. We won't be much concerned with distributional sections and the strongly involved and technical considerations connected with these, since we will apply the technique of geometric quantization only in the case where $X$ is compact and $\mathfrak{F}\cap\overline{\mathfrak{F}}=\{0\}$, which is called a *totally complex* or *Kähler polarization*. But for the sake of completeness we shortly discuss distributional sections.

Let $D$ be an integral manifold of $\mathfrak{D}$. The covariant derivative induces a flat connection on the restriction to $L\otimes\sqrt{\Lambda^n}\mathfrak{F}$. The holonomy group $G_D$ of this connection is a subgroup of $\mathbb{C}^\times$, whose elements can be obtained by multiplication of the elements of the holonomy groups of $L|_D$ and $\sqrt{\Lambda^n}\mathfrak{F}|_D$ corresponding to the same loop in $D$. Let $\sigma$ be a covariantly constant section of $L\otimes\sqrt{\Lambda^n}\mathfrak{F}$ that contains $D$ in its domain of definition, then parallel transport along a loop in $D$ results in the multiplication of $\sigma$ by elements of $G_D$. However, since $\sigma$ is covariantly constant, it does not change under parallel transport and therefore either $\sigma|_D$ is the zero section or the holonomy group of $G_D$ is trivial. The union of all integral manifolds of $\mathfrak{D}$ such that the corresponding holonomy group is trivial is called the *Bohr-Sommerfeld variety* $S$. For each $x\in X$ let $D_x$ be the integral manifold of $\mathfrak{D}$ passing through $x$. Then we have

$$S=\{x\in X;\ G_{D_x}=\{1\}\},$$

and the covariantly constant sections of $L\otimes\sqrt{\Lambda^n}\mathfrak{F}$ vanish in the complement of $S$.

Now consider a contractible open set $U \subset X$ such that $L|_U$ admits a trivializing section $\lambda_0$. Then

$$\nabla\lambda_0 = -\frac{\mathrm{i}}{\hbar}\theta \otimes \lambda_0,$$

where $\theta$ is a local symplectic potential, i.e. $\mathrm{d}\theta = \omega$ on $U$. Then, for each loop $\gamma$ in $U$ the element of the holonomy group is given by

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_\gamma \theta}.$$

If $\gamma$ is contained in the integral manifold $D$ we denote by $\mathrm{e}^{-2\pi\,\mathrm{i}\,d_\gamma}$ the element of the holonomy group of the flat connection in $\sqrt{\Lambda^n \mathfrak{F}}|_D$ corresponding to $\gamma$. Then the condition that $G_D$ is trivial is equivalent to

$$\int_\gamma \theta = (n_\gamma + d_\gamma)\hbar \tag{3.2.6}$$

for any loop $\gamma$ in $D$. Here $n_\gamma$ is an integer, and (3.2.6) is called *Bohr-Sommerfeld quantization condition.*

## 3.3 Metalinear manifolds and half-forms

In this section we give a systematic definition of half-forms and the corresponding structure on manifolds. As a motivation consider an $s$-density on an $n$-dimensional manifold $X$ that to each $v \in \mathrm{T}_x X$ assigns a number $\sigma_x(v)$ and transforms as

$$\sigma(vA) = |\det A|^s \sigma(v)$$

for any $A \in \mathrm{GL}(n, \mathbb{R})$. We would now like to define a half-form to be an object which transforms according to

$$\sigma(vA) = \sqrt{\det A}\,\sigma(v). \tag{3.3.1}$$

The trouble with this definition is that the square root is not well-defined on $\mathrm{GL}(n, \mathbb{R})$. In order to remedy this we will have to pass to the double covering of $\mathrm{GL}(n, \mathbb{R})$ as well as of the tangent bundle $\mathrm{T}X$. Note that any $A \in \mathrm{GL}(n, \mathbb{C})$ can be decomposed as $A = PU$, where $U = \mathrm{e}^{\mathrm{i}\,\theta} U_1$, $U_1 \in \mathrm{SU}(n)$ is unitary and $P = P^*$ is positive definite[4]. Since $\mathrm{SU}(n)$ is simply connected the fundamental group is given by

$$\pi_1 \mathrm{GL}(n, \mathbb{R}) = \pi_1(S^1) = \mathbb{Z}.$$

Therefore a double covering of $\mathrm{GL}(n, \mathbb{C})$ exists[5]. We let $\mathbb{Z}$ act on $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$ by

$$(k, (u, A)) \longmapsto \left(u + \frac{2\pi\,\mathrm{i}\,k}{n}, \mathrm{e}^{-2\pi\,\mathrm{i}\,k/n} A\right).$$

[4]This is known as the polar decomposition. $AA^*$ is positive definite and therefore has a positive definite square root $P$. Write $A = PP^{-1}A$ and observe that $(P^{-1}A)(P^{-1}A)^* = P^{-1}P^2P^{-1} = \mathbb{1}$, so $U = P^{-1}A$ is unitary. See also [RS72]

[5] See e.g. [Hu59, GH81].

Then the map

$$\mathbb{C} \times \mathrm{SL}(n,\mathbb{C}) \xrightarrow{\pi} \mathrm{GL}(n,\mathbb{C}), \quad (u,A) \longmapsto \mathrm{e}^u A,$$

is invariant under the action of $\mathbb{Z}$ and provides an isomorphism

$$(\mathbb{C} \times \mathrm{SL}(n,\mathbb{C}))/\mathbb{Z} \simeq \mathrm{GL}(n,\mathbb{C}).$$

This isomorphism can be used to pull back det defined on $\mathrm{GL}(n,\mathbb{C})$ to give $\pi^* \det = \det \circ \pi$ which has a well-defined holomorphic square root:

$$\pi^* \det(u,A) = \mathrm{e}^{nu}, \quad \sqrt{\pi^* \det}(u,A) = \mathrm{e}^{nu/2}.$$

The holomorphic function

$$u \longmapsto \mathrm{e}^{nu/2}$$

is already defined on

$$(\mathbb{C} \times \mathrm{SL}(n,\mathbb{C}))/2\mathbb{Z} =: \mathrm{ML}(n,\mathbb{C}),$$

which we call the *metalinear group*. This is a double cover of $\mathrm{GL}(n,\mathbb{C})$. Denote by $\chi$ the holomorphic square root on $\mathrm{ML}(n,\mathbb{C})$, i.e.

$$\chi^2(C) = \det(\rho(C)),$$

where $\rho : \mathrm{ML}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ is the covering map. We can consider $\mathrm{GL}(n,\mathbb{R})$ as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ and define

$$\mathrm{ML}(n,\mathbb{R}) := \rho^{-1}\mathrm{GL}(n,\mathbb{R}), \tag{3.3.2}$$

which gives a subgroup of $\mathrm{ML}(n,\mathbb{C})$ and a double covering of $\mathrm{GL}(n,\mathbb{R})$. Since $\chi|_{\mathrm{ML}(n,\mathbb{R})}$ can take values in the four half lines $\mathbb{R}_\pm$, $\mathrm{i}\,\mathbb{R}_\pm$, $\mathrm{ML}(n,\mathbb{R})$ has four components.

Now let $V$ be an $n$-dimensional vector space, and denote by $\mathrm{B}(V)$ the set of bases of $V$. A *metalinear structure* MB(V) is by definition a covering $\mathrm{MB}(V) \to \mathrm{B}(V)$ together with an action $\mathrm{MB}(V) \times \mathrm{ML}(n,\mathbb{R}) \to \mathrm{MB}(V)$ consistent with the covering $\mathrm{ML}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$, i.e. such that the diagram

$$\begin{tikzcd}
\mathrm{MB}(V) \times \mathrm{ML}(n,\mathbb{R}) \arrow[r] \arrow[d, "\tilde\rho\times\rho"'] & \mathrm{MB}(V) \arrow[d, "\tilde\rho"] \\
\mathrm{B}(V) \times \mathrm{GL}(n,\mathbb{R}) \arrow[r] & \mathrm{B}(V)
\end{tikzcd}$$

commutes, compare diagram (3.2.3). Given a metalinear structure we define a *half-form* to be a map $\sigma : \mathrm{MB}(V) \to \mathbb{C}$ such that

$$\sigma(fB) = \chi(B)\sigma(f). \tag{3.3.3}$$

For two half-forms it clearly follows

$$\sigma_1(fB)\sigma_2(fB) = \det(\rho(B))\sigma_1(f)\sigma_2(f),$$

and $\sigma_1\sigma_2$ gives an $n$-form on $\mathrm{B}(V)$. If we denote by $\Lambda^{1/2}V$ the space of half-forms we have a bilinear pairing

$$\Lambda^{1/2}V \times \Lambda^{1/2}V \longrightarrow \Lambda^n V.$$

The following will be important when we consider cotangent bundles:

**Proposition 3.3.1.** *A metalinear structure on $V$ induces a metalinear structure on its dual space $V^*$*

$$\Lambda^{1/2}V \simeq \Lambda^{1/2}V^*.$$

Now let $E \to X$ be a real vector bundle over some manifold $X$. We let $\mathrm{B}(E)$ denote the bundle of bases of $E$, thus $\mathrm{B}(E)$ is a $\mathrm{GL}(n,\mathbb{R})$-principal bundle over $X$ with typical fiber isomorphic to $\mathbb{R}^n$. We define a metalinear structure on $E$ to be a lifting of the $\mathrm{GL}(n)$ bundle to an $\mathrm{ML}(n)$ bundle $\mathrm{MB}(E)$ with a projection onto $B(E)$ that is consistent with the bundle structure

$$\begin{array}{ccc} \mathrm{MB}(E) \times \mathrm{ML}(n) & \longrightarrow & \mathrm{MB}(E) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{B}(E) \times \mathrm{GL}(n) & \longrightarrow & \mathrm{B}(E) \end{array} .$$

Not every vector bundle admits a metalinear structure[6]; we give conditions for the existence in Appendix E. Furthermore, to every metalinear bundle one can associate a bundle of half-forms. In particular, if $X$ is a differentiable manifold and $\mathrm{T}X$ carries a metalinear structure, we say that $X$ is a metalinear manifold. Let $\Lambda^{1/2}X$ be the space of smooth sections of the half-form bundle $\Lambda^{1/2}\mathrm{T}X$. Then, if $\sigma_1$ and $\sigma_2$ are half-forms, $\sigma_1\overline{\sigma}_2$ [7] is a density on $X$. Thus if $\Lambda_0^{1/2}X$ denotes the space of of smooth half-forms of compact support we can make $\Lambda_0^{1/2}X$ into a pre-Hilbert space under the scalar product

$$(\sigma_1, \sigma_2) = \int_X \rho_1\overline{\rho}_2.$$

Let $X$ and $X'$ be two metalinear manifolds. Given a bundle morphism $\Lambda^{1/2}X \to \Lambda^{1/2}X'$, this can be used to define the pull back map on half-forms[8].

### 3.3.1 Metaplectic manifolds

During the procedure of geometric quantization we will have to choose a metalinear structure on each Lagrangian subspace of $\mathrm{T}X$, where $X$ is a symplectic manifold. We will have to lift the bundle of symplectic frames to a double covering as it was done for the case of a metalinear structure. To this end we use the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ which possesses a

[6] However, it clearly does if it is orientable.

[7] We define the conjugate half-forms by using the character $\overline{\chi}$ in equation (3.3.3)

[8] For the definition of bundle morphisms see e.g. [GVF01, Ste51, Hus75].

double covering $\mathrm{Mp}(n,\mathbb{R})$ called the *metaplectic group*. Now we have a natural embedding $\mathrm{GL}(n,\mathbb{R}) \hookrightarrow \mathrm{Sp}(n,\mathbb{R})$ given by

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}.$$

Thus $\mathrm{GL}(n,\mathbb{R})$ can be viewed as subgroup in $\mathrm{Sp}(n,\mathbb{R})$. Denote by $G$ the inverse image of this subgroup in $\mathrm{Mp}(n,\mathbb{R})$, the double covering of $\mathrm{Sp}(n,\mathbb{R})$. Then

**Proposition 3.3.2.** *The group $G$ defined above is isomorphic to* $\mathrm{ML}(n,\mathbb{R})$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{ML}(n,\mathbb{R}) & \longrightarrow & \mathrm{Mp}(n,\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{GL}(n,\mathbb{R}) & \longrightarrow & \mathrm{Sp}(n,\mathbb{R}) \end{array}$$

*commutes.*

Let $X$ be a symplectic manifold and let $\mathrm{Bp}(X)$ denote the bundle of symplectic frames of $\mathrm{T}X$, thus $\mathrm{Bp}(X)$ is a right principal $\mathrm{Sp}(n)$ bundle. A lifting of $\mathrm{Bp}(X)$ to a right principal $\mathrm{Mp}(n)$ bundle is called a metaplectic structure on $X$ and the corresponding bundle is called the bundle of metaplectic frames $\mathrm{Mp}(X)$.

Now a metaplectic structure on $X$ gives rise to a metalinear structure on each Lagrangian subspace $P_x \subset \mathrm{T}_xX$. More precisely let $\mathfrak{Lag}(X)$ be the bundle of all Lagrangian subspaces of $\mathrm{T}X$ and let $E$ a section Let $\mathrm{B}(E)$ be the bundle of bases of $E$. It is a (right) $\mathrm{GL}(n,\mathbb{R})$ bundle. Each basis of $E_x$ extends to a symplectic basis of $\mathrm{T}_xX$. Conversely, any symplectic basis $\{e_1,\dots,e_n,f_1,\dots,f_n\}$ determines a Lagrangian subspace, namely the one spanned by $\{e_1,\dots,e_n\}$. Thus the map

$$\lambda:\ \mathrm{Bp}(X) \longrightarrow \mathrm{B}(E), \quad (e_1,\dots,e_n,f_1,\dots,f_n) \longmapsto (e_1,\dots,e_n)$$

is a surjective smooth bundle map. Furthermore, any two elements in $\lambda^{-1}(e_1,\dots,e_n)$ differ by the action of a matrix of the form

$$\begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix}$$

where $S$ is a symmetric matrix. Let $N$ be the group of all these matrices, then $\mathrm{Bp}(X) \xrightarrow{\lambda} \mathrm{B}(E)$ is a principal $N$ bundle. In addition $\mathrm{GL}(n,\mathbb{R}) \subset \mathrm{Sp}(n,\mathbb{R})$ normalizes $N$ and $\lambda$ is equivariant with respect to the action of $N$. Since $N$ is simply connected it lifts isomorphically to a subgroup of $\mathrm{Mp}(n)$. We have

$$\begin{array}{ccc} \mathrm{Mp}(X) & \xrightarrow{\tilde\lambda} & \mathrm{Mp}(X)/N = \mathrm{MB}(E) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r} \\ \mathrm{Bp}(X) & \xrightarrow[\lambda]{} & \mathrm{B}(E) \end{array}.$$

Since $\mathrm{Mp}(X)$ is a double covering of $\mathrm{Bp}(X)$ and $\mathrm{ML}(n,\mathbb{R})$ normalizes $N$ in $\mathrm{Mp}(n,\mathbb{R})$[9], we can conclude that $r$ provides a double covering of $\mathrm{B}(E)$, giving $E$ a metalinear structure.

Now let $Y \hookrightarrow X$ be a submanifold and $E_1$ a bundle of Lagrangian subspaces over $Y$ then $E_1$ carries a metalinear structure. If $E_1$ and $E_2$ are two transverse Lagrangian bundles then each carries a metalinear structure. Now the symplectic structure on $X$ allows us to identify the linear structure on $E_1$ with the dual of the linear structure on $E_2$[10].

**Proposition 3.3.3.** *In the situation described above, the metaplectic structure on $X$ determines an isomorphism between the metalinear structure on $E_1$ and the complex conjugate dual metalinear structure on $E_2$. In particular, there is a natural sesquilinear pairing between $\Lambda^{1/2}E_1$ and $\Lambda^{1/2}E_2$ induced by the metalinear structure on $X$.*

For a proof of this statement we refer to [GS90]. We will meet more general situations and constructions in the subsequent discussion.

### 3.3.2 Positive Lagrangian frames

In Section 3.3.1 the metaplectic structure on a symplectic manifold was used to assign a metalinear structure to each Lagrangian submanifold and to introduce a pairing between the half-forms defined on transverse Lagrangian submanifolds. In this section we will give an analogous construction for positive polarizations, closely following [Śni80, Woo97, SW76, Tuy87a]. In particular, we will show we can eliminate the ambiguity in choosing the metalinear structure by using the metaplectic structure.

We consider the bundle of positive Lagrangian frames $\mathfrak{Lag}^+X$, which is a fiber bundle over $X$ with typical fiber

$$P = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} \in \mathbb{C}^{2n\times n};\ U^{\mathrm{T}}V = V^{\mathrm{T}}U,\ \operatorname{rank}\begin{pmatrix} U \\ V \end{pmatrix} = n,\ \mathrm{i}(V^*U - U^*V) \text{ pos. semidefinite} \right\}, \tag{3.3.4}$$

see [Śni80]. The structure group $\mathrm{Sp}(n,\mathbb{R})$ acts on $P$ according to

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} T_1U + T_2V \\ T_3U + T_4V \end{pmatrix}, \tag{3.3.5}$$

where $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$. This bundle is associated to the bundle of symplectic frames $\mathrm{Bp}(X)$. The elements of $\mathfrak{Lag}^+X$ are complex linear $n$-frames. Consequently, there is a right $\mathrm{GL}(n,\mathbb{C})$ action, which induces an action on $P$ given by

$$\left(\begin{pmatrix} U \\ V \end{pmatrix}, C\right) \longmapsto \begin{pmatrix} UC \\ VC \end{pmatrix}$$

[9] We continue to denote the subgroup in $\mathrm{Mp}(n)$ covering $N$ by $N$.

[10] The fibers $E_{1,x}$ and $E_{2,x}$ are non-singularly paired under the symplectic form $\omega_x$.

for any $C \in \mathrm{GL}(n,\mathbb{C})$. Now the conditions (3.3.4) and (3.3.5) imply that

$$C := U - \mathrm{i}\,V$$

is non-singular and

$$W := (U + \mathrm{i}\,V)(U - \mathrm{i}\,V)^{-1}$$

is symmetric and satisfies $\|W\| \le 1$. If we denote

$$B := \{W \in \mathrm{M}_n(\mathbb{C});\ W^{\mathrm{T}} = W,\ \|W\| \le 1\},$$

the map $(U,V) \mapsto (W,C)$ defined above gives a bijection between $P$ and $B \times \mathrm{GL}(n,\mathbb{C})$, where the inverse is given by

$$U = \frac{1}{2}(\mathbb{1} + W)C, \quad V = \frac{\mathrm{i}}{2}(\mathbb{1} - W)C.$$

Hence the $\mathrm{GL}(n,\mathbb{C})$-action on $P$ induces an action on $B \times \mathrm{GL}(n,\mathbb{C})$ according to

$$(W,C)C' = (W, CC'),$$

such that $P \simeq B \times \mathrm{GL}(n,\mathbb{C})$ becomes a $\mathrm{GL}(n,\mathbb{C})$ principal bundle over $B$. Furthermore, the left $\mathrm{Sp}(n)$ action transfers to $B \times \mathrm{GL}(n,\mathbb{C})$,

$$g(W,C) = (gW, \alpha(g,W)C),$$

where

$$\begin{aligned} gW = {} & ((T_1 + \mathrm{i}\,T_3)(\mathbb{1} + W) - (T_4 - \mathrm{i}\,T_2)(\mathbb{1} - W)) \\ & \times ((T_1 - \mathrm{i}\,T_3)(\mathbb{1} + W) \quad + (T_4 + \mathrm{i}\,T_2)(\mathbb{1} - W))^{-1} \end{aligned}$$

and

$$\alpha(g,W) = \frac{1}{2}((T_1 - \mathrm{i}\,T_3)(\mathbb{1} + W) + (T_4 + \mathrm{i}\,T_2)(\mathbb{1} - W)) \in \mathrm{GL}(n,\mathbb{C}),$$

for any $g = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$. We note that $\alpha : \mathrm{Sp}(n,\mathbb{R}) \times B \to \mathrm{GL}(n,\mathbb{C})$ satisfies

$$\alpha(g_1 g_2, W) = \alpha(g_1, g_2 W)\alpha(g_2, W).$$

It is possible to imbed $\mathrm{U}(n) \hookrightarrow \mathrm{Sp}(n,\mathbb{R})$ as follows,

$$\mathrm{U}(n) \ni S + \mathrm{i}\,T \longrightarrow \begin{pmatrix} S & T \\ -T & S \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}),$$

where $S$, $T \in \mathrm{M}_n(\mathbb{R})$ such that $S^{\mathrm{T}}S + T^{\mathrm{T}}T = \mathbb{1}$ and $S^{\mathrm{T}}T = T^{\mathrm{T}}S$. Thus if $g \in \mathrm{Sp}(n)$ is the image of $S + \mathrm{i}\,T$ under this embedding, we obtain

$$\alpha(g,W) = S + \mathrm{i}\,T.$$

This gives rise to a unique lift $\tilde{\alpha}: \mathrm{Mp}(n,\mathbb{R}) \times B \to \mathrm{ML}(n,\mathbb{C})$ of $\alpha$ such that

$$\begin{tikzcd} \mathrm{Mp}(n,\mathbb{R}) \times B \arrow[r, "\tilde{\alpha}"] \arrow[d, "\rho\times\mathrm{id}"'] & \mathrm{ML}(n,\mathbb{C}) \arrow[d] \\ \mathrm{Sp}(n,\mathbb{R}) \times B \arrow[r, "\alpha"'] & \mathrm{GL}(n,\mathbb{C}) \end{tikzcd}$$

commutes. For each $W \in B$ and for $\tilde{e} \in \mathrm{Mp}(n,\mathbb{R})$ the identity in $\mathrm{Mp}(n,\mathbb{R})$, the quantity $\tilde{\alpha}(\tilde{e}, W)$ is the identity in $\mathrm{ML}(n,\mathbb{C})$. Let $\tilde{P} := B \times \mathrm{ML}(n,\mathbb{C})$ be the trivial $\mathrm{ML}(n,\mathbb{C})$ bundle over $B$. There is a left action of $\mathrm{Mp}(n,\mathbb{R})$ on $\tilde{P}$ defined by $\tilde{\alpha}$,

$$\tilde{g}(W,\tilde{C}) = (\rho(\tilde{g})W, \tilde{\alpha}(\tilde{g},W)\tilde{C}), \tag{3.3.6}$$

for each $\tilde{g} \in \mathrm{Mp}(n,\mathbb{R})$ and each $(W,\tilde{C}) \in \tilde{P}$. $\tilde{P}$ is a double covering space of $P$ with covering map $\tau: \tilde{P} \to P$, given by

$$\tau(W,\tilde{C}) = \begin{pmatrix} U \\ V \end{pmatrix},$$

where $U = \frac{1}{2}(\mathbb{1} + W)\rho(\tilde{C})$ and $V = \frac{\mathrm{i}}{2}(\mathbb{1} - W)\rho(\tilde{C})$.

**Definition 3.3.4.** The bundle of metalinear positive Lagrangian frames $\widetilde{\mathfrak{Lag}}^{+}(X)$ is the fiber bundle over $X$ with typical fiber $\tilde{P}$ on which $\mathrm{Mp}(n,\mathbb{R})$ acts by (3.3.6). It is associated to the metaplectic frame bundle $\mathrm{Mp}(X)$ and yields a double covering $\tau : \tilde{\mathfrak{Lag}}^{+}(X) \to \mathfrak{Lag}^{+}(X)$ of the bundle of positive Lagrangian frames.

The usual description of associated bundles, see e.g. [KN63, Hus75, Ste51, BGV92], implies that a metalinear positive Lagrangian frame $\tilde{\boldsymbol{w}} \in \widetilde{\mathfrak{Lag}}^{+}(X)_x$ can be identified with a function $\tilde{w}^{\#} : \mathrm{Mp}(X) \to \tilde{P}$ such that for each metaplectic frame $(\tilde{u},\tilde{v}) \in \mathrm{Mp}_x(X)$ and each $g \in \mathrm{Mp}(n,\mathbb{R})$

$$\tilde{w}^{\#}((\tilde{u},\tilde{v})g) = \tilde{g}^{-1}\tilde{w}^{\#}(\tilde{u},\tilde{v}).$$

The double covering $\tau : \tilde{\mathfrak{Lag}}^{+}(X) \to \mathfrak{Lag}^{+}(X)$ is constructed as follows: for each $\tilde{\boldsymbol{w}} \in \tilde{\mathfrak{Lag}}^{+}(X)$, $w = \tau(\tilde{w})$ is the unique element of $\mathfrak{Lag}^{+}(X)$ such that

$$\begin{tikzcd} \mathrm{Mp}(X) \arrow[r, "\tilde{w}^{\#}"] \arrow[d, "\tau"'] & \tilde{P} \arrow[d, "\tau"] \\ \mathrm{Bp}(X) \arrow[r, "w^{\#}"'] & P \end{tikzcd}$$

commutes. The right action of $\mathrm{GL}(n,\mathbb{C})$ and the left action of $\mathrm{Sp}(n,\mathbb{R})$ on $P$ commute with each other, which implies that also the right action of $\mathrm{ML}(n,\mathbb{C})$ on $\tilde{P}$ commutes with the left action of $\mathrm{Mp}(n,\mathbb{R})$, and $\tilde{\mathfrak{Lag}}^+(X)$ inherits a right action of $\mathrm{ML}(n,\mathbb{C})$ such that

$$\begin{tikzcd} \widetilde{\mathfrak{Lag}}^+(X) \times \mathrm{ML}(n,\mathbb{C}) \arrow[r] \arrow[d, "\tau\times\rho"'] & \tilde{\mathfrak{Lag}}^+(X) \arrow[d, "\tau"] \\ \mathfrak{Lag}^+(X) \times \mathrm{GL}(n,\mathbb{C}) \arrow[r] & \mathfrak{Lag}^+(X) \end{tikzcd} ,$$

where the horizontal arrows are given by the group actions, commutes.

Now let $\mathfrak{F}$ be a positive polarization of the symplectic manifold $(X,\omega)$. The bundle $\mathrm{B}(\mathfrak{F})$ of linear frames of $F$ is a subbundle of $\mathfrak{Lag}^+(X)$, invariant under the action of $\mathrm{GL}(n,\mathbb{C})$. The inverse image of $\mathrm{B}(\mathfrak{F})$ under the double covering $\tau : \widetilde{\mathfrak{Lag}}^+(X) \to \mathfrak{Lag}^+(X)$ is a subbundle of $\mathrm{MB}(\mathfrak{F})$ of $\tilde{\mathfrak{Lag}}^+(X)$ invariant under the action of $\mathrm{ML}(n,\mathbb{C})$, and $\tau$ restricted to $\mathrm{MB}(\mathfrak{F})$ defines a double covering of $\mathrm{B}(\mathfrak{F})$. Therefore, $\mathrm{MB}(\mathfrak{F})$ is a $\mathrm{ML}(n,\mathbb{C})$-principal bundle and the diagram

$$\begin{tikzcd} \mathrm{MB}(\mathfrak{F}) \times \mathrm{ML}(n,\mathbb{C}) \arrow[r] \arrow[d, "\tau\times\rho"'] & \mathrm{MB}(\mathfrak{F}) \arrow[d] \\ \mathrm{B}(\mathfrak{F}) \times \mathrm{GL}(n,\mathbb{C}) \arrow[r] & \mathrm{B}(\mathfrak{F}) \end{tikzcd}$$

commutes, where again the horizontal arrows denote the group actions.

Let us assume that we have chosen a metaplectic structure on $(X,\omega)$, and, for each positive polarization $\mathfrak{F}$ we consider only the induced metalinear frame bundle. Let $\boldsymbol{w}_1 = \{e_1,\dots,e_n\}$ and $\boldsymbol{w}_2 = \{f_1,\dots,f_n\}$ be arbitrary positive Lagrangian frames on which we impose the condition

$$\mathrm{i}\,\omega(e_j,\overline{f}_k) = \delta_{jk} \tag{3.3.7}$$

for all $j$, $k \in \{1,\dots,n\}$. Let $(u,v) \in \mathrm{Bp}(X)$ and $\begin{pmatrix} U_i \\ V_i \end{pmatrix} \in P$ such that

$$w_i = uU_i + vV_i$$

for $i = 1$, 2. Then (3.3.7) reads

$$V_2^* U_1 - U_2 V_1 = -\mathrm{i}\,\mathbb{1}.$$

We can use the isomorphism $P \simeq B \times \mathrm{GL}(n,\mathbb{C})$ to transfer this condition and obtain

$$\mathbb{1} - W_2^* W_1 = 2(C_2^*)^{-1} C_1^{-1}. \tag{3.3.8}$$

Now let $B_0$ be given as the set of those $S \in \mathrm{M}_n(\mathbb{C})$ such that $\|S\| \leq 1$ and 1 is not an eigenvalue of $S$. Then consider the map

$$\gamma : B_0 \longrightarrow \mathrm{GL}(n, \mathbb{C}), \quad S \longmapsto \mathbb{1} - S.$$

Since $B_0$ is contractible, see [Mun00, GH81, Hu59, Bre93], there exists a unique map $\tilde{\gamma} : B_0 \to \mathrm{GL}(n, \mathbb{C})$ such that

$$\rho \circ \tilde{\gamma} = \gamma$$

and $\tilde{\gamma}(0) = \tilde{\mathrm{id}}$. We want to lift condition (3.3.7) to the bundle of metalinear Lagrangian frames. Let $\tilde{\boldsymbol{w}}_1$ and $\tilde{\boldsymbol{w}}_2$ be two positive metalinear Lagrangian frames, $(\tilde{u}, \tilde{v})$ is a metaplectic frame and

$$(W_i, \tilde{C}_i) = \tilde{w}_i^{\#}(\tilde{u}, \tilde{v}).$$

Then we can lift (3.3.8) to obtain[11]

$$\tilde{\gamma}(W_2^* W_1) = 2(\tilde{C}_2^*)^{-1} \tilde{C}_1^{-1}. \tag{3.3.9}$$

This equation is invariant under the action of the kernel of $\rho : \mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$. Furthermore, projecting the last equation to $\mathrm{GL}(n, \mathbb{C})$ gives (3.3.8), which is invariant under the action of $\mathrm{Sp}(n, \mathbb{R})$. Therefore, (3.3.9) is invariant under the action of $\mathrm{Mp}(n, \mathbb{R})$ so that it defines a relation in $\tilde{\mathfrak{Lag}}^+(X)$, independent of the metaplectic frame chosen. Therefore we have indeed lifted condition (3.3.7) to the bundle of positive metalinear frames

### 3.3.3 Induced metaplectic structures

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be a pair of strongly admissible polarizations. Set

$$\mathfrak{D}_{12} = \mathfrak{D}_1 \cap \mathfrak{D}_2, \quad \mathfrak{E}_{12} = \mathfrak{E}_1 + \mathfrak{E}_2.$$

Then we have

**Lemma 3.3.5.** *For strongly admissible positive polarizations we have*

$$\mathfrak{F}_1 \cap \overline{\mathfrak{F}}_2 = \mathfrak{D}_{12}^{\mathbb{C}}, \quad \mathfrak{F}_1 + \overline{\mathfrak{F}}_2 = \mathfrak{E}_{12}^{\mathbb{C}}.$$

*Proof.* If $w \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ then by positivity

$$\mathrm{i}\,\omega(w, \overline{w}) \geq 0 \quad \text{and} \quad \mathrm{i}\,\omega(\overline{w}, w) \geq 0,$$

so that $\omega(w, \overline{w}) = 0$ and $w \in \mathfrak{D}_i^{\mathbb{C}}$. Therefore $w \in \mathfrak{D}_{12}^{\mathbb{C}}$. The inclusion $\mathfrak{D}_{12}^{\mathbb{C}} \subseteq \mathfrak{F}_1 \cap \mathfrak{F}_2$ is obvious. □

[11] Note that conjugation and scalar multiplication lift from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{ML}(n, \mathbb{C})$.

**Definition 3.3.6.** The pair $(\mathfrak{F}_1, \mathfrak{F}_2)$ of polarizations is said to be strongly admissible if $\mathfrak{E}_{12}$ is an involutive distribution on $X$ and the spaces $X/\mathfrak{D}_{12}$ and $X/\mathfrak{E}_{12}$, of integral manifolds of $\mathfrak{D}_{12}$ and $\mathfrak{E}_{12}$ respectively, are quotient manifolds of $X$. We denote the corresponding projections by

$$\pi_{\mathfrak{D}_{12}} : X \longrightarrow X/\mathfrak{D}_{12}, \quad \pi_{\mathfrak{E}_{12}} : X \longrightarrow X/\mathfrak{E}_{12}.$$

Now for each integral manifold $E$ of $\mathfrak{E}_{12}$ its projection $N := \pi_{\mathfrak{D}_{12}}(M)$ to $X/\mathfrak{D}_{12}$ is a submanifold of $X/\mathfrak{D}_{12}$ endowed with a symplectic form $\omega_N$ such that the pull-back of $\omega_N$ to $E$ equals the restriction of $\omega$ to $E$. Then the metaplectic structure on $X$ induces a metaplectic structure on $(N, \omega_N)$. We will briefly sketch how this works. At $x \in X$ let $\mathrm{Bp}_x^{12}(X)$ the collection of symplectic frames in $\mathrm{Bp}_x(X)$ of the form

$$(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{t}, \boldsymbol{v}) = (s_1, \ldots, s_m, u_1, \ldots, u_{n-m}; t_1, \ldots, t_m, v_1, \ldots, v_{n-m})$$

such that

$$s_1, \ldots, s_m \in \mathfrak{D}_{12}$$

and

$$s_1, \ldots, s_m, u_1, \ldots, u_{n-m}, v_1, \ldots, v_{n-m} \in \mathfrak{E}_{12}.$$

The collection of all symplectic frames of this form is a subbundle

$$\mathrm{Bp}^{12}(X) = \bigcup_{x \in X} \mathrm{Bp}_x^{12}(X)$$

of the symplectic frame bundle $\mathrm{Bp}(X)$. It is a principal fiber bundle with structure group $G$ that is a subgroup of $\mathrm{Sp}(n, \mathbb{R})$. Furthermore $G$ contains a normal subgroup $G_0$ isomorphic to $\mathrm{Sp}(n-m, \mathbb{R})$, and $\rho^{-1}(G_0)$, where $\rho : \mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$ denotes the covering map, has two connected components; we denote by $\tilde{G}_0$ the one containing the identity element. This is a normal subgroup in $\tilde{G} = \rho^{-1} G$ and the quotient $\tilde{G}/\tilde{G}_0$ is isomorphic to $\mathrm{Mp}(n-m, \mathbb{R})$.

Now let X be a Hamiltonian vector field in $\mathfrak{D}_{12}$ with corresponding flow $\Phi^t$ inducing a mapping $\mathrm{Bp}(X) \to \mathrm{Bp}(X)$ by its differential. Since $X \in \mathfrak{D}_{12}$ it maps $\mathrm{Bp}^{12}(X)$ to itself, a property which transfers to the lift $\tilde{\Phi}^t : \mathrm{Mp}^{12}(X) \to \mathrm{Mp}^{12}(X)$. By means of the group $G_0$ we define an equivalence relation in $\mathrm{Bp}^{12}(X)$ through

$$\boldsymbol{b} \sim \boldsymbol{b}' \quad \Leftrightarrow \quad \boldsymbol{b} = \Phi_*^t(\boldsymbol{b}')g, \tag{3.3.10}$$

for some $g \in G_0$. The quotient space $\mathrm{Bp}^{12}(X)/\sim$ is a right principal $\mathrm{Sp}(n-m, \mathbb{R})$ bundle over $X/\mathfrak{D}_{12}$. And the relation (3.3.10) can be lifted to $\mathrm{Mp}^{12}(X)$ according to

$$\tilde{\boldsymbol{b}} \sim \tilde{\boldsymbol{b}}' \quad \Leftrightarrow \quad \tilde{\boldsymbol{b}} = \tilde{\Phi}_*^t(\tilde{\boldsymbol{b}}')\tilde{g}$$

for some $\tilde{g} \in \tilde{G}_0$. The quotient $\mathrm{Mp}^{12}(X)/\sim$ is a right principal $\mathrm{Mp}(n-m, \mathbb{R})$ bundle covering $\mathrm{Bp}^{12}(X)/\sim$. For each integral manifold $M$ of $\mathfrak{E}_{12}$ the restriction of $\mathrm{Bp}^{12}(X)/\sim$ to $N = \pi_{\mathfrak{D}_{12}}(M)$ is isomorphic to $\mathrm{Bp}(N)$. The restriction of $\mathrm{Mp}^{12}(X)/\sim$ to $N$ defines a bundle $\mathrm{Mp}(N)$ of metaplectic frames for $(N, \omega_N)$. Let $\widetilde{\mathfrak{Lag}}^+(N)$ be a bundle of metalinear positive Lagrangian frames of $(N, \omega_N)$ covering $\widetilde{\mathfrak{Lag}}^+(N)$; these bundles can be expressed in terms of quotients of subbundles of $\tilde{\mathfrak{Lag}}^+(X)$ and $\mathfrak{Lag}^+(X)$: Define $\mathfrak{Lag}^+_{12}(X)$ by

$$\mathfrak{Lag}^+_{12} := \{(\boldsymbol{v}, \boldsymbol{u}) \in \mathfrak{Lag}^+(X);\ \boldsymbol{v} = (v_1, \ldots, v_m),\ v_i \in \mathfrak{D}_{12}\}$$

and $\widetilde{\mathfrak{Lag}}^+_{12}(X)$ by $\tau^{-1}(\mathfrak{Lag}^+_{12}(X))$. Then we have a mapping $\mathfrak{Lag}^+_{12}(X) \to \mathfrak{Lag}^+(N)$ defined by

$$(\boldsymbol{v}, \boldsymbol{u}) \mapsto (\pi_{\mathfrak{D}_{12}})_* \boldsymbol{u},$$

which lifts to $\widetilde{\mathfrak{Lag}}^+_{12}(X) \to \widetilde{\mathfrak{Lag}}^+(N)$, and the metaplectic structure on $N$ is induced by the one of $X$.

## 3.4 Blattner-Kostant-Sternberg kernels

The Fourier transform and the Bargmann transform [Bar61] are the most prominent examples of the pairing between Hilbert spaces corresponding to different polarizations. The Fourier transform gives a pairing between sections (i.e. functions on $\mathbb{R}^d \times \mathbb{R}^d = \{(x, \xi)\}$) that are constant in the $x$- and $\xi$-direction, respectively, see e.g. [GS90]. The Bargmann transform provides a pairing between holomorphic functions on $\mathbb{C}^n$ and functions on $\mathbb{R}^n$.

In general two polarizations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ come together with two different representation spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ associated with them. For strongly admissible pairs of polarizations $(\mathfrak{F}_1, \mathfrak{F}_2)$ there is a canonically defined sesquilinear map

$$\mathcal{K}_{12} : \mathscr{H}_1 \times \mathscr{H}_2 \longrightarrow \mathbb{C},$$

called the *Blattner-Kostant-Sternberg kernel*. This kernel induces a linear map

$$\mathcal{U}_{12} : \mathscr{H}_2 \longrightarrow \mathscr{H}_1$$

such that

$$\langle \sigma_1, \mathcal{U}_{12} \sigma_2 \rangle = \mathcal{K}_{12}(\sigma_1, \sigma_2).$$

for each $\sigma_i \in \mathscr{H}_i$. If $\mathcal{U}_{12}$ is unitary, the representation spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are said to be unitarily related. By using this pairing, half-forms and metaplectic structures one can achieve the same results as the ones based on half-densities and the Maslov index.

Let us first consider the case we already encountered in Section 3.3.

### 3.4.1 Transverse polarizations

We consider a pair of complete strongly admissible positive polarizations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that

$$\mathfrak{F}_1 + \overline{\mathfrak{F}}_2 = \mathrm{T}^{\mathbb{C}} X. \tag{3.4.1}$$

Then each pair $(\sigma_1, \sigma_2)$ of sections of $L \otimes \sqrt{\Lambda^n}\mathfrak{F}_1$ and $L \otimes \sqrt{\Lambda^n}\mathfrak{F}_2$ gives rise to a function (density) $|\sigma_1, \sigma_2|$ on $X$. We will now show how to define this function. Because of the transversality condition (3.4.1) we can find $\boldsymbol{w}_i \in \mathrm{B}_x\mathfrak{F}_i$ such that

$$\mathrm{i}\,\omega(\boldsymbol{w}_1, \boldsymbol{w}_2) = \mathbb{1}. \tag{3.4.2}$$

Let us factorize the sections $\sigma_i$ in some neighbourhood of $x$ in such a way that

$$\sigma_i = \lambda_i \otimes \nu_i.$$

Let $\mathrm{MB}(\mathfrak{F}_i) \ni \tilde{\boldsymbol{w}}_i,\ i = 1,\, 2,$ be metalinear frames projecting to $w_i$. Then the expression

$$\langle\lambda_1(x), \lambda_2(x)\rangle\tilde{\nu}_1(\tilde{\boldsymbol{w}}_1)\overline{\tilde{\nu}_2(\tilde{\boldsymbol{w}}_2)}$$

is defined by $\sigma_1(x)$ and $\sigma_2(x)$ up to a factor of $\pm 1$. If we could restrict the arbitrariness in the choice of the metalinear frames $\tilde{w}_i$ such that their projections satisfy (3.4.2), one could define a function $|\sigma_1, \sigma_2| : X \to \mathbb{C}$ by setting

$$|\sigma_1, \sigma_2|(s) := \langle\lambda_1(s), \lambda_2(s)\rangle\nu_1^{\#}(\tilde{w}_1)\nu_2^{\#}(\tilde{w}_2).$$

This is indeed achieved by means of the metaplectic structure, that allows us to lift the condition (3.4.2) to the bundle of metalinear frames, see Section 3.3.2. Thus $|\sigma_1, \sigma_2|$ is well-defined by the above expression, where $\tilde{w}_i \in \mathrm{MB}_x\mathfrak{F}_i$ satisfy the condition (3.3.9).

Now let $\boldsymbol{\zeta}_i$ be local linear frame fields for $\mathfrak{F}_i$, which consist of (complex) Hamiltonian vector fields, and let $\tilde{\boldsymbol{\zeta}}_i$ be a lift of $\boldsymbol{\zeta}_i$ to a metalinear frame field for $\mathfrak{F}_i$. Assume that the factorizing sections $\nu_i$ are chosen such that

$$\tilde{\nu}_i \circ \tilde{\boldsymbol{\zeta}}_i = 1,\ i = 1,\, 2.$$

For $j,\, k \in \{1, \ldots, n\}$ set

$$d_{jk} := \mathrm{i}\,\omega(\zeta_{1,j}, \overline{\zeta}_{2,k}).$$

Then the frame fields $\zeta_1$ and

$$\zeta_2' := \left(\sum_k \zeta_{2,k}\overline{d}_{k1}, \ldots \sum_k \zeta_{2,k}\overline{d}_{kn}\right)$$

satisfy (3.4.2) at each point of their common domain. Let $\tilde{\zeta}_2'$ be a lift of $\zeta_2'$ to a metalinear frame field. Then we have

$$(\tilde{\nu}_1 \circ \tilde{\boldsymbol{\zeta}}_1)(\tilde{\nu}_2^{\circ}\tilde{\boldsymbol{\zeta}}_2') = \det(d_{jk})^{-1/2},$$

where the ambiguity of the sign is absorbed into the choice of the branch of the square root. Therefore, we have

$$|\sigma_1, \sigma_2| = \sqrt{\det(\omega(\boldsymbol{\zeta}_1, \overline{\boldsymbol{\zeta}}_2))}\langle\lambda_1, \lambda_2\rangle.$$

For $\sigma_i \in \mathscr{H}_i$ the kernel $\mathcal{K}_{12}$ is now defined by integrating the density

$$\mathcal{K}(\sigma_1, \sigma_2) = \int_X |\sigma_1, \sigma_2||\omega^n|.$$

In general, in order to ensure the convergence of this integral one needs additional assumptions on the polarizations. We will not deal with this problem here since we will always be concerned with special sections of $\mathscr{H}_i$ that are smooth and sufficiently regular, such that the arising integrals exist. Let us remark that it is also possible to define the above pairing for distributional sections, see [Śni80].

### 3.4.2 Strongly admissible pairs of polarizations

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be strongly admissible positive polarizations such that also the pair $(\mathfrak{F}_1, \mathfrak{F}_2)$ is strongly admissible, see Section 3.3.3.

Now let $\mathscr{H}_i$ be the representation spaces associated with the polarizations $\mathfrak{F}_i$. We consider a pair of sections $\sigma_i \in \mathscr{H}_i$ and construct a density $|\sigma_1, \sigma_2|$ on $X/\mathfrak{D}_{12}$. Consider a basis

$$(\boldsymbol{v}, \boldsymbol{u}_1, \boldsymbol{u}_2, \overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2, \boldsymbol{t})$$

of $\mathrm{T}^{\mathbb{C}}_x X$, where

- $\boldsymbol{v}$ is a basis of $\mathfrak{D}_{12}$,
- $\boldsymbol{w}_i = (\boldsymbol{v}, \boldsymbol{u}_i) \in \mathrm{B}_x\mathfrak{F}_i$,
- $\mathrm{i}\,\omega(u_{1,j}, u_{2,k}) = \delta_{jk}$,
- $\omega(v_r, t_s) = \delta_{fs}$,
- $\omega(u_{1,j}, t_s) = \omega(\overline{u}_{1,j}, t_s) = \omega(u_{2,j}, t_s) = \omega(\overline{u}_{2,j}, t_s) = 0$,

for $j,\ k \in \{1, \dots, n-m\},\ r,\ s \in \{1, \dots, m\}$ and $m = \dim \mathfrak{D}_{12}$. This gives a basis

$$b = \pi_{\mathfrak{D}_{12}*}(\boldsymbol{u}_1, \boldsymbol{u}_2, \overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2, \boldsymbol{t})$$

of $\mathrm{T}^{\mathbb{C}}_z(X/\mathfrak{D}_{12})$, where $z = \pi_{\mathfrak{D}_{12}}(x)$. Now let us choose a factorization of the sections

$$\sigma_i = \lambda_i \otimes \nu_i,$$

and metalinear frames $\tilde{\boldsymbol{w}}_i$ for $\mathfrak{F}_i$ that project down to $\boldsymbol{w}_i$. Then we define the density according to

**Proposition 3.4.1.** *The quantity*

$$|\sigma_1, \sigma_2| := \langle \lambda_1(x), \lambda_2(x) \rangle \tilde{\nu}_1(\tilde{\boldsymbol{w}}_1) \overline{\tilde{\nu}_2(\tilde{\boldsymbol{w}}_2)},$$

*where $\tilde{\boldsymbol{w}}_i$ are metalinear frames for $\mathfrak{F}_i$, defines a density on the quotient manifold $X/\mathfrak{D}_{12}$.*

For a complete proof of this statement we refer to [Śni80] since we won't need the BKS pairing in the general case of strongly admissible pairs of polarizations. Let us only remark that this definition in the case $m = 0$ coincides with the one given in Section 3.4.1.

## 3.5 Lifting the action of canonical transformations

Having available all the ingredients for the geometric quantization scheme, the metaplectic frame bundle $\mathrm{Mp}(X)$, an associated bundle $\tilde{\mathfrak{Lag}}^{+}(X)$ of positive metalinear Lagrangian frames, a complete strongly admissible polarization $\mathfrak{F}$, and a pre-quantization line bundle $L$ with connection and invariant Hermitian form, we have to define the lift of canonical transformations in order to obtain an explicit expression for the quantized operators. The representation space consists of sections of $L \otimes \sqrt{\Lambda^n}\mathfrak{F}$ covariantly constant along $\mathfrak{F}$ where $\sqrt{\Lambda^n}\mathfrak{F}$ is associated to $\mathrm{MB}(\mathfrak{F})$.

Now consider a function $f$ on $X$ such that its Hamiltonian vector field $\mathrm{X}_f$ is complete, and let $\Phi^t$ be the corresponding one-parameter group of canonical transformations generated by $\mathrm{X}_f$. For each $t \in \mathbb{R}$ the image of $\mathfrak{F}$ under $\Phi^t_*$ is a complete strongly admissible positive polarization $\Phi^t_*\mathfrak{F}$. The one-parameter group $\Phi^t$ lifts to a one-parameter group $(\Phi^t)^{\#}$ of diffeomorphisms of $\mathfrak{Lag}^+(X)$ which preserves the structure of the bundle of positive Lagrangian frames: For $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathfrak{Lag}^+(X)$ we define

$$(\Phi^t)^{\#}(\boldsymbol{w}) = (\Phi^t_* w_1, \ldots, \Phi^t_* w_n).$$

There exists a unique lift $\tilde{\Phi}^t$ of $(\Phi^t)^{\#}$ to a one-parameter group of bundle morphisms of $\widetilde{\mathfrak{Lag}}^{+}(X, \omega)$. For each $t \in \mathbb{R}$ the bundle $\tilde{\mathfrak{Lag}}^{+}(X)$ induces a metalinear frame bundle of $\Phi^t_*\mathfrak{F}$ given by

$$\mathrm{MB}(\Phi^t_*\mathfrak{F}) = \tau^{-1}(\mathrm{B}(\Phi^t_*\mathfrak{F})),$$

where $\tau$ is the covering of the bundle of positive Lagrangian frames introduced in Section 3.3.2. Furthermore, if $\tilde{\boldsymbol{w}} \in \mathrm{MB}(\mathfrak{F})$ then $\tilde{\Phi}^t(\tilde{\boldsymbol{w}}) \in \mathrm{MB}(\Phi^t_*(\mathfrak{F}))$. Thus $\tilde{\Phi}^t$ restricted to $\mathrm{MB}(\mathfrak{F})$ yields an isomorphism of the $\mathrm{ML}(n, \mathbb{C})$ principal fibre bundles $\mathrm{MB}(\mathfrak{F})$ and $\mathrm{MB}(\Phi^t_*\mathfrak{F})$. Now let $\sqrt{\Lambda^n}\Phi^t_*\mathfrak{F}$ be the line bundle associated to $\mathrm{MB}(\Phi^t_*\mathfrak{F})$ where $\mathrm{ML}(n, \mathbb{C})$ acts on $\mathbb{C}$ by multiplication with $\chi(C)$, $C \in \mathrm{ML}(n, \mathbb{C})$. We denote by $\mathscr{H}_t$ the representation space consisting of those sections of $L \otimes \sqrt{\Lambda^n}\Phi^t_*\mathfrak{F}$ that are covariantly constant along $\Phi^t_*\mathfrak{F}$. If $\nu$ is a local section of $\sqrt{\Lambda^n}\mathfrak{F}$, for each $t \in \mathbb{R}$ we have a section $\Phi^t\nu$ of $\sqrt{\Lambda^n}\mathfrak{F}$ defined by

$$\widetilde{(\Phi^t\nu)}(\tilde{\boldsymbol{w}}) = \tilde{\nu}(\tilde{\Phi}^{-t}\tilde{\boldsymbol{w}})$$

for each $\tilde{\boldsymbol{w}} \in \mathrm{MB}(\Phi^t_*\mathfrak{F})$. If $\nu$ is covariantly constant along $\mathfrak{F}$ then $\Phi^t\nu$ is covariantly constant along $\Phi^t_*\mathfrak{F}$.

For each $\sigma \in \mathscr{H}$, which can be locally factorized as

$$\sigma = \lambda \otimes \nu,$$

we define $\Phi^t\sigma$ by

$$\Phi^t\sigma = \Phi^t\lambda \otimes \Phi^t\nu,$$

where

$$(\Phi^t\lambda)^{\sim} = \tilde{\lambda} \circ (\tilde{\Phi}^{-t}),$$

with $\tilde{\Phi}^t$ the pre-quantization lift defined in Section 3.1.2. This definition extends linearly to all sections in $\mathscr{H}$ and the map $\Phi^t : \mathscr{H} \to \mathscr{H}_t$ is a vector space isomorphism with inverse defined in terms of $\Phi^{-t}$. In particular, this map is unitary.

### 3.5.1 Polarization preserving functions

In this section we assume that

$$\Phi^t_* \mathfrak{F} = \mathfrak{F}$$

for all $t \in \mathbb{R}$. This means that

$$[\mathrm{X}_f, \xi] \in \mathfrak{F}$$

for any $\xi \in \mathfrak{F}$. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a local frame field for $\mathfrak{F}$, then

$$[\mathrm{X}_f, \xi_i] = \sum_j a_{ij}(x) \xi_j(x).$$

The function $f$ generates a one-parameter group $\Phi^t : \mathscr{H} \to \mathscr{H}$ of unitary transformations. Then the quantum operator $\hat{f}$ on $\mathscr{H}$ associated to $f$ is then defined by

$$\hat{f}(\sigma) := \mathrm{i}\,\hbar \frac{\mathrm{d}}{\mathrm{d}t} (\Phi^t \sigma)|_{t=0},$$

for every $\sigma \in \mathscr{H}$. It is a selfadjoint first order differential operator.

Let us give a local description of $\hat{f}$. We use a metalinear frame field $\tilde{\boldsymbol{\xi}}$ corresponding to $\boldsymbol{\xi} \in \mathrm{B}(\mathfrak{F})$ and a local section $\nu_{\tilde{\boldsymbol{\xi}}}$ defined by

$$\tilde{\nu}_{\tilde{\boldsymbol{\xi}}} \circ \tilde{\boldsymbol{\xi}} = 1. \tag{3.5.1}$$

With the local factorization

$$\sigma = \lambda \otimes \nu_{\tilde{\boldsymbol{\xi}}},$$

for some covariantly constant section $\lambda \in \Gamma(X, L)$, we obtain

$$\hat{f}(\lambda \otimes \nu_{\tilde{\boldsymbol{\xi}}}) = \left( \mathrm{i}\,\hbar \frac{\mathrm{d}}{\mathrm{d}t} \Phi^t \lambda \right)_{t=0} \otimes \nu_{\tilde{\xi}} + \lambda \otimes \left( \mathrm{i}\,\hbar \frac{\mathrm{d}}{\mathrm{d}t} \Phi^t \nu_{\tilde{\boldsymbol{\xi}}} \right)_{t=0}.$$

The first term is given by the pre-quantization map

$$(-\mathrm{i}\,\hbar \nabla_{\xi_f} + f)\lambda,$$

and it remains to calculate the second term: Let $\tilde{\Phi}^t \tilde{\boldsymbol{\xi}}$ denote the local metalinear frame field for $\mathfrak{F}$ obtained by

$$\tilde{\Phi}^t \tilde{\boldsymbol{\xi}}(x) = \tilde{\Phi}^t_* (\tilde{\boldsymbol{\xi}}(\Phi^{-t}(x))).$$

The frames $\tilde{\boldsymbol{\xi}}$ and $\tilde{\Phi}^t_* \tilde{\boldsymbol{\xi}}$ are related by an element $\tilde{C}_t(x) \in \mathrm{ML}(n, \mathbb{C})$

$$\Phi^t \tilde{\boldsymbol{\xi}}(x) = \tilde{\boldsymbol{\xi}}(x) \tilde{C}_t(x),$$

and we have

$$\begin{aligned}
(\widetilde{\Phi}^t \nu_{\tilde{\boldsymbol{\xi}}})(\tilde{\boldsymbol{\xi}}(x)) &= \tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\Phi}^{-t} \tilde{\boldsymbol{\xi}}(x)) \\
&= \tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\Phi}^{-t} \tilde{\boldsymbol{\xi}}(\Phi^t(x))) \\
&= \tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\boldsymbol{\xi}}(\Phi^{-t}(x)) \tilde{C}_{-t}(\Phi^{-t}(x))) \\
&= \chi((\tilde{C}_{-t}(\Phi^{-t}(x)))^{-1}) \tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\boldsymbol{\xi}}(\Phi^{-t}(x))) \\
&= \chi((\tilde{C}_{-t}(\Phi^{-t}(x)))^{-1}) \tilde{\nu}_{\tilde{\boldsymbol{\xi}}}(\tilde{\boldsymbol{\xi}}(x)),
\end{aligned}$$

where the last equality follows from (3.2.4). Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t\nu_{\tilde{\xi}}(x))_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\chi(\tilde{C}_{-t}(\Phi^{-t}(x))^{-1})_{t=0}\nu_{\tilde{\xi}}(x).$$

Since $\tilde{C}_0(\Phi^t(x)) = \tilde{\mathbb{1}}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(\tilde{C}_{-t}(\Phi^{-t}(x))^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t}\chi(\tilde{C}_{-t}(x)^{-1})_{t=0}.$$

Let $C_t$ be the projection of $\tilde{C}_t$ to $\mathrm{GL}(n,\mathbb{R})$,

$$C_t(x) = \rho(\tilde{C}_t(x)),$$

then

$$\Phi^t_*\boldsymbol{\xi}(x) = \boldsymbol{\xi}(x)C_t(x), \tag{3.5.2}$$

and

$$\chi(\tilde{C}_{-t}(x)^{-1}) = (\det C_{-t}(x))^{1/2}.$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\chi\left(\tilde{C}_{-t}(x)^{-1}\right)\Big|_{t=0} = \frac{1}{2}\operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}C_t(x)\right)_{t=0}.$$

So we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t\nu_{\tilde{\xi}}(x))|_{t=0} = \frac{1}{2}\operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}C_t(x)\right)_{t=0}\nu_{\tilde{\xi}}(x).$$

In order to evaluate the trace of the derivative of $C_t$, we differentiate (3.5.2) with respect to $t$ at $t=0$,

$$\begin{aligned}\boldsymbol{\xi}(x)\frac{\mathrm{d}}{\mathrm{d}t}C_t(x)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t\xi_1(x),\dots,\Phi^t\xi_n(x)) \\ &= (-[\xi_f,\xi_1],\dots,-[\xi_f,\xi_n]) \\ &= -\left(\sum_{j=1}^n a_{1j}(x)\xi_j(x),\dots,\sum_{j=1}^n a_{nj}(x)\xi_j(x)\right)\end{aligned}$$

and therefore

$$\operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}C_t(x)\right)_{t=0} = -\sum_{j=1}^n a_{jj}(x).$$

So we have

$$\hat{f}(\lambda\otimes\nu_{\tilde{\xi}}) = \left(-\,\mathrm{i}\,\hbar\nabla_{\xi_f} + f - \frac{1}{2}\,\mathrm{i}\,\hbar\sum_{j=1}^n a_{jj}\right)\lambda\otimes\nu_{\tilde{\xi}}. \tag{3.5.3}$$

Suppose that $f$ is constant along $\mathfrak{F}$ so that the Hamiltonian vector field $\xi_f$ is contained in $\mathfrak{F}$. Since the Hamiltonian vector fields contained in a polarization commute, the matrix $(a_{jk})_{1\le j,k\le n}$ vanishes identically. It is also possible to obtain an explicit expression for the lift of a symplectomorphism in the case of a Kähler polarization: If the canonical map is generated by a real valued Hamiltonian function on $X$, then the transported polarization and the original one are transversal such we can use the BKS-pairing as in the situation described above, see [Tuy87a], and as a result we obtain the quantum operator also in the form (3.5.3). The above statements will be particularly useful, when we discuss the time evolution of coherent states for the Heisenberg group in Section 4.1. There we will be concerned with a Kähler polarization that is determined by a complex symmetric matrix. The explicit form of the quantum operator will be used as the infinitesimal version of the quantum dynamics of the coherent states and thus allows for an explicit calculation of a time evolved coherent state.

## 3.6 Coherent states, coadjoint orbits and representations

The most frequently encountered symplectic manifolds are cotangent bundles, Kähler manifolds and coadjoint orbits. As we will see in the sequel, some coadjoint orbits also have a natural Kähler structure.

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, whose complexification we denote by $\mathfrak{g}^{\mathbb{C}}$. The complexified Lie algebra can be identified with the space of complex tangent vectors at the identity of $G$. Furthermore, by translation each $\xi \in \mathfrak{g}^{\mathbb{C}}$ determines a left and a right invariant vector field $\mathrm{l}_\xi$ and $\mathrm{r}_\xi$ on $G$. Now let $\eta \in \mathfrak{g}^*$, the dual of $\mathfrak{g}$, and $(M, \omega)$ the reduction of $(G, \omega_\eta)$, where we define $\omega_\eta(\xi, \zeta) = \eta([\xi, \zeta])$, see also Appendix B. Then we have $M \simeq G/(G_\eta)_0 \simeq \mathcal{O}_\eta$, the coadjoint orbit through $\eta \in \mathfrak{g}^*$, where $(G_\eta)_0$ means the identity component of the isotropy subgroup of $\eta$ under the coadjoint action. Now suppose that $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\eta([\xi, \zeta]) = 0, \quad \text{for all } \xi,\, \zeta \in \mathfrak{b}, \tag{3.6.1}$$

and

$$2 \dim_{\mathbb{C}} \mathfrak{b} = \dim G + \dim G_\eta. \tag{3.6.2}$$

Let $\tilde{\mathfrak{F}}$ be the complex distribution on $G$ spanned by the vector fields $\mathrm{r}_\xi$ for $\xi \in \overline{\mathfrak{b}}$ and let $\mathfrak{C}$ be the characteristic distribution corresponding to $\omega_\eta$, which is spanned by $\mathrm{r}_\zeta$ for $\zeta \in \underline{G_\eta}$. The first of the above two conditions then implies that $\omega_\eta(\mathrm{r}_\xi, \mathrm{r}_{\underset{\sim}{\zeta}}) = 0$ for all $\xi, \zeta \in \overline{\mathfrak{b}}$, and therefore $\tilde{\mathfrak{F}}$ is isotropic. The second condition shows that $\tilde{\mathfrak{F}}$ has maximal possible dimension. It follows therefore that $\tilde{\mathfrak{F}}$ contains the complexification of $\mathfrak{g}_\eta$[12].

Now the projection of $\tilde{\mathfrak{F}}$ to $M$ is isotropic, of dimension $\frac{1}{2}\dim M$, and is integrable because $\mathfrak{b}$ was supposed to be a subalgebra. Since a polarization determines an almost complex structure, which in this case is integrable, we have shown that a coadjoint orbit has a natural Kähler structure.

If $G$ is compact and connected and $\eta \in \mathfrak{g}^*$ it contains a maximal torus $T$, which is unique up to conjugation, see [BtD85, Sim96, BR80]. Since the elements $T$ of commute, we can

[12] otherwise $\tilde{\mathfrak{F}} + \mathfrak{C}^{\mathbb{C}}$ would be isotropic of higher dimension

decompose $\mathfrak{g}^{\mathbb{C}}$ into simultaneous eigenspaces of the generators of $T$. The corresponding eigenvalues are imaginary and determine a collection of linear forms on $\mathfrak{t}$, the Lie algebra of $T$ also called the *Cartan algebra*, see Appendix C.

**Definition 3.6.1.** The *roots* of $\mathfrak{g}$ are the non-zero linear forms $\alpha \in (\mathfrak{t}^{\mathbb{C}})^*$ such that

$$\mathfrak{g}_\alpha := \{\xi \in \mathfrak{g}^{\mathbb{C}};\ \mathrm{ad}_\tau \xi = \alpha(\tau)\xi \quad \forall \tau \in \mathfrak{t}\}$$

is non-trivial. We denote the set of roots by $\Delta$.

We collect some facts about roots, mainly from [BtD85, Sim96, Sam90, FH91, Tay86], see also [Wal73, War72, HR79] and Appendix C. These results are obtained by elementary manipulations and consist mainly of vocabulary:

> *The word "root" (and also "fundamental" and "Weyl"!) is terribly overworked in the theory of semisimple Lie algebras. There are roots, root vectors, root spaces, root elements. Neither roots nor root vectors lie in root spaces, although root elements do. [Sim96]*

If $\alpha \in \Delta$ then also $-\alpha = \overline{\alpha} \in \Delta$ and $\mathfrak{g}_{-\alpha} = \overline{\mathfrak{g}}_\alpha$. The eigenspaces $\mathfrak{g}_\alpha$ corresponding to the roots are one-dimensional and $\mathfrak{g}^{\mathbb{C}}$ decomposes into the direct sum

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \cdots .$$

If $\xi_\alpha \in \mathfrak{g}_\alpha$ and $\xi_\beta \in \mathfrak{g}_\beta$ then

$$[\xi_\alpha, \xi_\beta] \in \mathfrak{g}_{\alpha+\beta}$$

whenever $\alpha + \beta \in \Delta$, otherwise $[\xi_\alpha, \xi_\beta] = 0$; moreover,

$$[\xi_\alpha, \xi_\beta] \in \mathfrak{t}^{\mathbb{C}} \tag{3.6.3}$$

if $\alpha = -\beta$. Let $\xi_\alpha \in \mathfrak{g}_\alpha$ be non-zero then $\zeta_\alpha := \frac{\mathrm{i}}{2}[\overline{\xi}_\alpha, \xi_\alpha]$ is also nonzero and lies in $\mathfrak{t}$, and by a suitable rescaling we can arrange that $\alpha(\zeta_\alpha) = \mathrm{i}$. Then put $\xi_\alpha = x_\alpha + \mathrm{i}\, y_\alpha$ for real $x_\alpha$ and $y_\alpha$ so that

$$[x_\alpha, y_\alpha] = -\zeta_\alpha, \quad [y_\alpha, \zeta_\alpha] = -x_\alpha, \quad [\zeta_\alpha, x_\alpha] = -y_\alpha.$$

According to these relations $x_\alpha$, $y_\alpha$ and $\zeta_\alpha$ span a subalgebra in $\mathfrak{g}$ which is isomorphic to $\mathfrak{su}(2)$, the Lie algebra of SU(2). Since $\mathfrak{t} \subset \mathfrak{g}_\eta$ [13] we clearly have $\eta([\zeta, \cdot]) = 0$ for any $\zeta \in \mathfrak{t}$, and therefore

$$0 = \eta([\zeta, \xi]) = \alpha(\zeta)\eta(\xi)$$

for any $\xi \in \mathfrak{g}_\alpha$ and for all $\zeta \in \mathfrak{t}$, which implies $\eta(\xi) = 0$. This means that $\eta$ vanishes on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \cdots$. If $\eta(\zeta_\alpha) \neq 0$ for any root $\alpha$, then $\mathfrak{g}_\eta = \mathfrak{t}$. In this case $\eta$ is said to be *regular*. In general, $\mathfrak{g}_\eta$ is the sum of $\mathfrak{t}$ and the span$\{x_\alpha,\ y_\alpha,\ \zeta_\alpha\}$ for which $\eta(\zeta_\alpha) = 0$.

[13] All maximal tori are conjugate. We can use this freedom to ensure that $T \subset G_\eta$, see also [BtD85, Sim80].

In both cases, whether $\eta$ is regular or not, we can define an invariant polarization on $M = G/(G_\eta)_0$, i.e. an polarization that is invariant under the $G$ action on $M$, which can be identified with the coadjoint action on $\mathcal{O}_\eta$: Define the positive and negative roots as

$$\Delta_\eta^\pm := \{\alpha \in \Delta;\ \pm\eta(\zeta_\alpha) > 0\}$$

and put

$$\mathfrak{b} := \mathfrak{t}^\mathbb{C} \oplus \bigoplus_{\alpha \notin \Delta_\eta^+} \mathfrak{g}_\alpha.$$

Now since $\alpha$ and $-\alpha$ both are in $\Delta_\eta^+$ only if $\eta(\zeta_\alpha) = 0$, this satisfies condition (3.6.1) because of equation (3.6.3), and it also satisfies (3.6.2), see e.g. [Lan98b]. In the regular case $\mathfrak{g}_\alpha \subset \overline{\mathfrak{b}}$ only if $\alpha \in \Delta_\eta^+$.

Now consider the projection down to $G/(G_\eta)_0$. The tangent space at any point is $\mathfrak{g}/\mathfrak{g}_\eta$ and can be identified with the subspace of $\mathfrak{g}$ spanned by the $x_\alpha$, $y_\alpha$ corresponding to positive roots $\alpha \in \Delta_\eta^+$. The polarization at $m \in M$ is the sum over $\mathfrak{g}_\alpha$; $\alpha \in \Delta_\eta^+$, and if $\alpha \in \Delta_\eta^+$ then

$$\mathrm{i}\,\eta([x_\alpha - \mathrm{i}\,y_\alpha, x_\alpha + \mathrm{i}\,y_\alpha]) = 2\eta(\zeta_\alpha) > 0,$$

so that the polarization is always positive.

The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ generates a subgroup $B \subset G^\mathbb{C}$ that is called a *Borel subgroup*, if $\eta$ is regular, and a *parabolic subgroup* otherwise.

We have thus constructed an invariant positive polarization on the symplectic manifold $M$. In order to perform the geometric quantization procedure, we in addition have to construct a pre-quantizing line bundle $L \to M$ with a connection whose curvature equals $\omega$. As a matter of fact, such a line bundle will not exist in general. We have to impose additional assumptions on $\eta$: We say that $\eta$ is *integral* if $\eta|_{\mathfrak{g}_\eta}$ exponentiates to a character of $G_\eta$, i.e. if there is a character $\chi_\eta$ of $G_\eta$ such that its differential equals $\eta$. The set of coadjoint orbits corresponding to integral characters is in one-to-one correspondence with the unitary dual of $G$, i.e. the set of equivalence classes of unitary irreducible representations of $G$, see e.g. [Kir62, Kir76, Kir99] and also [Vog97, GV98, Vog00].

In the case of an integral coadjoint orbit $\mathcal{O}_\eta$ we can explicitly perform the construction of a pre-quantum line bundle. Let us exploit the fact that $G \to G/G_\eta$ is naturally a $G_\eta$ principal bundle over $\mathcal{O}_\eta \simeq G/G_\eta$. Using the character $\chi_\eta : G_\eta \to \mathrm{S}^1$, we can define the action of $G_\eta$ on $G \times \mathbb{C}$ by

$$h(g, z) = (gh^{-1}, \chi_\eta(h)z)$$

and put

$$L := (G \times \mathbb{C})/G_\eta.$$

Then

**Theorem 3.6.2.** *$L$ is a line bundle with connection over $\mathcal{O}_\eta \simeq G/G_\eta$ possessing a natural hermitian structure, such that*

$$\mathrm{i}\,\mathrm{curv}(L) = \omega.$$

For a proof see e.g. [Kos70].

Furthermore, as we have pointed out in above the discussion, the coadjoint orbits have a natural Kähler structure and therefore a totally complex (invariant) polarization. Thus the representation space is given as the space of holomorphic sections of the line bundle $L$. The question of the existence of holomorphic sections of the above line bundle, in a first step, is (asymptotically) answered by the following observation: Consider $k$-fold tensor powers $L^{\otimes k}$ of $L$, which can be defined by raising the transition functions for $L$ to the $k$-th power, such that

$$\mathrm{i}\,\mathrm{curv}(L^{\otimes k}) = k\omega.$$

This follows from the functorial properties of the Chern class [Hus75, KN69]. Therefore, also $L^{\otimes k}$ is a pre-quantum line bundle over $\mathcal{O}_\eta$ and the representation space (in the case that $\mathcal{O}_\eta$ is compact) and is taken to be

$$\mathscr{H}_k := L^2_{\mathrm{hol}}(\mathcal{O}_\eta, L^{\otimes k}),$$

on which we have a natural inner product given by

$$\langle \lambda_1, \lambda_2 \rangle = \int_{\mathcal{O}_\eta} h^{(k)}(\lambda_1, \lambda_2) \frac{\omega^n}{n!},$$

where $h^{(k)}$ is the hermitian structure on $L^{\otimes k}$ induced by the hermitian structure $h$ on $L$. The geometric quantization procedure gives a morphism from a (suitable) space of functions on $\mathcal{O}_\eta$ to the space of operators on $\mathscr{H}_k$. As a consequence of the Riemann-Roch-Hirzebruch formula [Hir78, GH87] the dimension of $\mathscr{H}_k$ is a polynomial in $k$ which can be explicitly computed in terms of the Kähler structure on $\mathcal{O}_\eta$. We only state the leading order term here, which reads

$$\dim \mathscr{H}_k \sim \frac{k^n}{n!} \operatorname{vol}(\mathcal{O}_\eta),$$

where $n = \frac{1}{2} \dim \mathcal{O}_\eta$.

Since all the constructions employed here are $G$-equivariant $G$ also acts on $\mathscr{H}_k$. In the case of $\mathrm{SU}(2)$ the coadjoint orbits are given as $\mathbb{CP}^1$ and $\mathscr{H}_k$ consists of the homogeneous polynomials of degree $k$ in two variables, see [Bor00]. This observation has a generalization in a deep and famous result in representation theory, known as Borel-Weil theorem, which states that the representations of $G$ on $\mathscr{H}_k$ are unitary and irreducible, and, in particular, all representations arise in this way, see e.g. [Bot57, Tay86, War72, Ser54, BE89, Hel00, Hel01]. Apart from the method of geometric quantization one could use a technique known as *Berezin-Toeplitz quantization*, which is defined as follows: Consider the orthogonal projection, the so-called *generalized Szegö projector*, see [BdMG81, Sch96, Sch98],

$$\Pi_k : L^2(X, L^{\otimes k}) \to \mathscr{H}_k = L^2_{\mathrm{hol}}(X, L^{\otimes k}),$$

which is defined on any quantizable Kähler manifold analogously to the approach described above for coadjoint orbits. Then for $f \in C^\infty(X)$ and $\phi \in \mathscr{H}_k$ we can define the associated Berezin-Toeplitz operator as

$$\mathrm{op}^{\mathrm{BT}}[f]\phi = \Pi_k(f\phi).$$

The relation between the Berezin-Toeplitz operators and the geometric quantization operators is not obvious, but they are closely related, see [Tuy87b, Sch96]. In particular, the semiclassical properties[14] of the Szegö projector and Toeplitz operators have been extensively studied, see e.g. [BU03, BU00, BPU98, Bor98, Cha03b, Cha03a, Cha00] and also [Zel03, SZ02, SZ03, Zel98, Zel97]. The importance for us is the close connection with coherent states, that we are now going to examine. To this end we first have to point out that there is an alternative picture of looking at the quantizing line bundle: The associated principal bundle. Let $Z \subset L^*$ be the unit circle bundle in the dual of $L$, which of course inherits a hermitian structure from that on $L$. Thus, a point in $Z$ is a pair $(x, \lambda)$ where $\lambda$ is a complex linear functional on $L_x$ which maps the unit circle in $L_x$[15] to the unit circle in $\mathbb{C}$. The circle action is $\lambda\, \mathrm{e}^{\mathrm{i}\,\theta}(w) = \mathrm{e}^{\mathrm{i}\,\theta}\,\lambda(w)$. Then a section of the associated bundle $Z \times_k \mathbb{C}$[16] can be represented by an equivariant function $f : Z \to \mathbb{C}$, i.e. a function satisfying

$$f(x, \lambda\, \mathrm{e}^{\mathrm{i}\,\theta}) = \mathrm{e}^{\mathrm{i}\,k\theta}\, f(x, \lambda).$$

Let $C^\infty(Z)_k$ be the set of these equivariant functions, then we have the natural identification

$$C^\infty(Z)_k \simeq C^\infty(X, L^{\otimes k}),$$

given by

$$f(x, \lambda) = \lambda^k(s_x),$$

where $\lambda^k$ denotes the element in the dual of $L^{\otimes k}$ given by $\lambda \otimes \cdots \otimes \lambda$ and $s_x$ is a section of $L^{\otimes k}$. On $Z$ w the fundamental vector field $\xi_\theta$, given as the generator of the $\mathrm{S}^1$-action, generates the vertical bundle $\operatorname{ver} \mathrm{T}Z$. Suppose that we are given a connection $\alpha$ on $Z$, which determines the horizontal distribution $\operatorname{ver} \mathrm{T}Z$, then this determines a covariant derivative on $L^{\otimes k}$. Since $\alpha(\xi_\theta) = 1$ and $\omega$ is non-degenerate, we see immediately that

$$\frac{\alpha}{2\pi} \wedge \omega^n$$

is a non-vanishing volume form on $Z$. Such a form is called *contact form* and can be viewed as a substitute for the symplectic form in the case of odd-dimensional manifolds. This form gives a natural Hilbert space isomorphism

$$L^2(X, L^{\otimes k}) \simeq L^2(Z)_k,$$

and we can view each $\mathscr{H}_k$ as a subspace of $L^2(Z)$, see also [Zel97, Cha00]. This suggests the following construction, see [BU00, BPU98, Raw77, RCG90].

**Definition 3.6.3.** Let $\Pi_k(x, y)$ be the Schwartz kernel of the Szegö projector

$$\Pi_k : L^2(Z) \to \mathscr{H}_k.$$

[14] In the case of compact Kähler quantization "semiclassical" means the limit $k \to \infty$!

[15] determined by the hermitian structure

[16] Which is defined as $L^{\otimes k} \times \mathbb{C}$ divided by the equivalence relation $(p\, \mathrm{e}^{\mathrm{i}\,\theta}, z) \sim_k (p, \mathrm{e}^{\mathrm{i}\,k\theta}\, z)$

Then a coherent state in $\mathscr{H}_k$ associated to $p \in Z$ is given by

$$\Pi_k(\cdot, p) \in \mathscr{H}_k,$$

and the *coherent state map* is

$$\Psi_k : Z \longrightarrow \mathscr{H}_k, \quad p \mapsto \Pi_k(\cdot, p). \tag{3.6.4}$$

The relation of this definition of coherent states to Perelomov's definition [Per72, Per86] is not obvious. However, in the subsequent discussion we will show an equivalence between the two.

**Remark 3.6.4.** Let us briefly note some properties of coherent states:

1. By their definition coherent states possess the *reproducing property*

   $$\forall f \in \mathscr{H}_k,\ p \in Z : \quad f(p) = \langle \Psi_k(p), f \rangle,$$

   since the coherent state map can be viewed as an application of the Szegö projector to the delta function, see [Bor00].

2. The coherent state map induces a map

   $$\Psi_k^0 : X \longrightarrow \mathbb{P}\mathscr{H}_k, \quad x \mapsto [\Psi_k(p_x)],$$

   where $p_x \in Z$ is any point in $Z$ projecting to $x \in X$, and $[\Psi_k(x)]$ denotes the complex line through $\Psi_k(p)$ in $\mathscr{H}_k$.

3. The coherent state map $Z \longrightarrow \mathscr{H}_k$ can be dualized to obtain a map $Z \longrightarrow \mathbb{P}\mathscr{H}_k^*$, which turns out to be a holomorphic embedding for sufficiently large $k$. This statement is equivalent to

   **Theorem 3.6.5 (Kodaira embedding).** *Integral Kähler manifolds are projectively algebraic.*

   See, e.g. [GH87] for a proof. The embedding $F$ described above is not symplectic, but approximately symplectic, see [Tia90, Zel98]:

   **Theorem 3.6.6 (Tian).** *For large $k$*

   $$\frac{1}{k} F^* \Omega_{\mathrm{FS}} = \omega + O(k^{-1}),$$

   *where $\Omega_{\mathrm{FS}}$ is the Fubini-Study symplectic form on $\mathbb{P}\mathscr{H}_k$ and $\omega$ the symplectic form on $X$.*

The Berezin-Toeplitz quantization described briefly above, is similar to a quantization technique due to Berezin [Ber75b, Ber74, Ber75a]. To describe this procedure we give an alternative way of looking at the coherent state map. Let $s \in \mathscr{H}_k$ be a holomorphic section, then the evaluation map

$$s \mapsto s(\pi(p)) = \hat{p}(s) \cdot p$$

is a linear form

$$\hat{p} : \mathscr{H}_k \to \mathbb{C}.$$

Because of the inner product and the Riesz theorem [RS72, Rud91, BB93] there exists a unique holomorphic section of $e_p$ such that

$$\langle e_p, s \rangle = \hat{p}(s)$$

for all $s \in \mathscr{H}_k$, which is precisely the reproducing property for the definition of the coherent states given above. Thus $e_p$ can be identified with $\Psi_k(p)$.

If we consider the manifold $X$ as the phase space of a physical system, we have an anti-holomorphic embedding $X \hookrightarrow \mathbb{P}\mathscr{H}_k$. Also dynamical properties can be described in this language, see [BBdM93, Ber97b, Ber97a, Ber99, BG92, Odz92, Odz88].

Let us consider the projection $\mathrm{P}_{\pi(p)}$ onto the subspaces spanned by $e_p$, which is well-defined, see e.g. [Sch98, BS00]. We call these operators *coherent projectors*, see Appendix C. Let us also define, following [Raw77],

$$\epsilon(\pi(p)) = |p|^2 \langle e_p, e_p \rangle$$

with $|p|^2 := h(\pi(p))(p,p)$. Let $s_1$, $s_2 \in \mathscr{H}_k$ be two holomorphic sections and put $x = \pi(p)$, such that $s_1(x) = \hat{p}(s_1)p$ and $s_2(x) = \hat{p}(s_2)p$. Therefore,

$$h(s_1, s_2)(x) = \langle s_1, e_p \rangle \langle e_p, s_2 \rangle |q|^2 = \langle s_1, \mathrm{P}_x s_2 \rangle \epsilon(x).$$

By integration we obtain

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, \mathrm{P}_x s_2 \rangle \epsilon(x) \omega^n(x),$$

a property which is called *over-completeness* of coherent states.

**Definition 3.6.7.** Let $\mathcal{B}$ be a linear operator on $\mathscr{H}_k$. The *lower or covariant (Berezin) symbol* $\sigma_{\mathrm{l}}(\mathcal{B})$ is defined by the map

$$X \to \mathbb{C};\ x \mapsto \sigma_{\mathrm{l}}(\mathcal{B})(x) := \mathrm{Tr}(\mathcal{B}\mathrm{P}_x) = \frac{\langle e_p, \mathcal{B} e_p \rangle}{\langle e_p, e_p \rangle}.$$

The *upper or contravariant (Berezin) symbol* $\sigma_{\mathrm{u}}(\mathcal{B})$ is defined through

$$\mathcal{B} = \int_X \sigma_{\mathrm{u}}(x) \mathrm{P}_x \epsilon(x) \omega^n(x). \tag{3.6.5}$$

Having these notions at hand, we can state some connections between the Berezin-Toeplitz and Berezin quantization:

**Proposition 3.6.8.** *The following relations hold:*

*(a) The upper symbol of a Toeplitz operator* $\mathrm{op}^{\mathrm{BT}}[f]$ *is given by the function itself*

$$\sigma_{\mathrm{u}}(\mathrm{op}^{\mathrm{BT}}[f]) = f.$$

*(b) The Toeplitz quantization map* $f \mapsto \mathrm{op}^{\mathrm{BT}}[f]$ *and the covariant symbol map* $\mathcal{B} \mapsto \sigma_{\mathrm{l}}(\mathcal{B})$ *are conjugate to each other, i.e.*

$$\mathrm{Tr}(\mathcal{B}^* \,\mathrm{op}^{\mathrm{BT}}[f]) = \int \sigma_{\mathrm{l}}(\mathcal{B}) f \epsilon \omega^n. \tag{3.6.6}$$

For a proof of these statements see [Sch98, Sch96, Cha00] and also Appendix C, where some analytic properties of coherent projections are discussed.

The symbolic properties of Toeplitz and Berezin-Toeplitz operators have been extensively discussed in literature, mainly in the context of deformation quantization, see e.g. [Fed96, BMS94, RCG90, RCG93]. We will use the algebraic properties of the upper and lower symbols later on, where we will briefly come back to the notion of Berezin symbols.

For the moment, however, we are more interested in a point of view that is in a sense opposite to the one alluded to above: Suppose that we are given a compact Lie group together with a unitary irreducible representation $\pi : G \to \mathrm{End}(V)$ on a finite dimensional Hilbert space $V$. This means we are given the (representation) operators and are interested in their classical counterparts.

To start with consider the projectivized space $\mathbb{P}V$, which is a natural Kähler (and therefore symplectic) manifold: Let $n+1 = \dim V$ and denote by $[z_0, \dots, z_n]$ the subspace of $V$[17] generated by $(z_0, \dots, z_n) \in V \setminus \{0\}$. Then we have charts $U_j := \{z_j \neq 0\}$ and a diffeomorphism

$$\phi_j : (z_0, \dots, \not{z}_j, \dots, z_n) \mapsto [z_0, \dots, z_{j-1}, 1, z_{j+1}, \dots, z_n]$$

yielding an atlas of $\mathbb{PC}^n$. On the overlaps $U_j \cap U_k$ the change of variables is given in terms of multiplication by $\frac{z_j}{z_k}$ and thus is holomorphic. Obviously we also have a natural line bundle $L' \to \mathbb{PC}^{n+1}$, called the *canonical bundle* defined by $L' = \mathbb{C}^{n+1} \setminus \{0\}$ and the projection $(z_0, \dots, z_n) \mapsto [z_0, \dots, z_n]$. In each chart there is a natural non-vanishing section $s_j : (z_0, \dots, \not{z}_j, \dots, z_n) \mapsto (z_0, \dots, z_{j-1}, 1, z_{j+1}, \dots, z_n)$. For these trivializations the transition functions are given by the corresponding change of variables $\frac{z_j}{z_k}$, and therefore the canonical line bundle is holomorphic. Also, $L' \to \mathbb{CP}^n$ inherits a hermitian structure from the standard hermitian structure on $\mathbb{C}^{n+1}$ and the associated curvature is related to the Fubini-Study form[18]. We already know from the above considerations that a coadjoint orbit

[17] We can always assume that $V \simeq \mathbb{C}^{n+1}$.

[18] This is not quite true! The hermitian structure induced from that one on $\mathbb{C}^{n+1}$ has opposite sign. This flaw can be repaired by dualizing the bundle, see [Bor00]

can be holomorphically embedded in the projectivized representation space $\mathbb{P}V$. However, in general this embedding is not symplectic. We are therefore now going to investigate the question under what circumstances a coadjoint orbit can be symplectically embedded. We shall see that the only symplectic orbits are contained in those corresponding to projectivized weight vectors. However, not all orbits induced by weight vectors are symplectic; there is an additional restriction, [GS84, KS82].

**Theorem 3.6.9.** *Let $G$ be a compact semisimple Lie group with a unitary, irreducible representation on $V$. Under the action of $G$ on $\mathbb{P}V$ the orbit through a point $[\psi] \in PV$ is symplectic relative to the restriction of the Fubini-Study form $\omega_{\mathrm{FS}}$ on $\mathbb{P}V$ if and only if $\psi \in V$ is a weight vector of some maximal torus of $G$ that satisfies*

$$\lambda(\mathfrak{g}_\alpha) = 0 \quad \textit{implies that } \mathrm{d}\pi(\mathfrak{g}_\alpha)\psi = 0. \tag{3.6.7}$$

Let us remark that condition (3.6.7) is equivalent to the condition that the stabilizer group $G_{[\psi]}$ of $[\psi]$ coincides with the stabilizer $G_\lambda$ of $\lambda$, see [KS82]. This in particular is fulfilled if $\lambda$ is a regular weight or a maximal weight.

There is only one orbit that is Kähler, and that is the orbit through $[\psi]$ where $\psi$ is a maximal weight vector.

This result shows the important consequences of the conditions stated in the following definition of coherent states for compact Lie groups. [Per72, Per86, Gil72, Gil74].

**Definition 3.6.10 (Gilmore-Perelomov).** Let $G$ be a compact Lie group and $(\pi, V)$ a unitary irreducible representation on a finite dimensional Hilbert space $V$. Furthermore, let $\psi_\lambda$ be the (unique) normalized weight vector corresponding to the maximal weight $\lambda$. A set of coherent states associated to $\psi_\lambda$ is given by the image of

$$\{\pi(g)\psi_\lambda; g \in G\}$$

under the projection $V \to \mathbb{P}V$.

It is obvious that such coherent states can be labeled by points in $G/G_\lambda$, since $\pi(h)\psi_\lambda = \mathrm{e}^{\mathrm{i}\,\lambda(h)}\,\psi_\lambda$ for $h \in T$. The following diagram commutes,

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d] & \mathrm{Aut}(V) \arrow[d, "\Pi"] \\ G/T \arrow[r, "\rho"'] & \mathrm{Aut}\,\mathbb{P}V \end{tikzcd},$$

and in these terms we have a coherent state map given by

$$G/G_\lambda \to \mathbb{P}V, \quad gG_\lambda \mapsto [\pi(g)\psi_\lambda].$$

According to the above theorem, if $\lambda$ is a maximal weight the orbit $[\pi(G)\psi_\lambda]$ is a Kähler submanifold of the Kähler manifold $\mathbb{P}V$. Explicitly, partly reviewing the discussion on page 100 we identify $V \simeq \mathbb{C}^{n+1}$ for some $n \in \mathbb{N}$, and therefore $\mathbb{P}V$ with $\mathbb{CP}^n$. This gives an equivalence of $V \setminus \{0\} \to \mathbb{P}V$ with the canonical line bundle $L' \to \mathbb{CP}^n$, described above. This correspondence can be explicitly implemented by associating to a vector $\psi \in V$ the projection onto the one-dimensional subspace that it spans,

$$V \ni \psi \mapsto \frac{\langle \cdot, \psi \rangle}{\langle \psi, \psi \rangle} \psi =: \mathrm{P}_{[\psi]}.$$

Then an analytic atlas of $\mathbb{P}V$ is given by

$$\{(U_{[\phi]}, u_{[\phi]}),\ [\phi] \in \mathbb{P}V\},$$

where $U_{[\phi]} = \{[\psi] \in \mathbb{P}V;\ \langle \phi, \psi \rangle \neq 0\}$ is an open (dense) set in $\mathbb{P}V$. Now

$$u_{[\phi]}([\psi]) := \frac{\mathrm{id} - \mathrm{P}_{[\phi]}}{\langle \phi, \psi \rangle} \psi$$

with a fixed vector $\phi \in \operatorname{ran} \mathcal{P}_{[\phi]}$, is a diffeomorphism $U_{[\phi]} \to \ker \mathcal{P}_{[\phi]} = \operatorname{ran}(\mathrm{id} - \mathcal{P}_{[\phi]}) \simeq \mathbb{C}^n$, see [AAGM95, KS82]. Again, the canonical bundle over $\mathbb{P}V$ is a $\mathrm{GL}(1, \mathbb{C})$ principal bundle. Elements of the associated (holomorphic) line bundle $L_{\text{can}}$ can be written as $([\psi], \psi)$ where $\psi \in \operatorname{ran} \mathcal{P}_{[\psi]}$. A local trivialization of this line bundle over $U_{[\phi]}$ is given by the unit section

$$s([\psi]) = \left( [\psi], \frac{\psi}{\langle \phi, \psi \rangle} \right), \quad \phi \in \operatorname{ran} \mathcal{P}_{[\phi]},\ \|\phi\| = 1.$$

The non-vanishing sections of the holomorphic line bundle can be related to elements of $V$ by the *identification map*

$$I:\ L_{\text{can}} \to V, \quad I([\psi], \psi) = \psi. \tag{3.6.8}$$

For any $\psi \in V$ let $l_\psi$ be its dual element in $V^*$. The restriction of $l_\psi$ to the fiber of $[\psi']$ in $L_{\text{can}}$ then yields a section $s^*_{[\psi']}$ of the (anti-holomorphic) dual line bundle $L^*_{\text{can}}$, and the map $[\psi'] \mapsto s^*_{[\psi']}$ is anti-linear between $\mathscr{H}$ and the sections of the dual bundle. Hence, we may realize $V$ as the space of anti-holomorphic sections. Now the complex structure of $\operatorname{ran} u_{[\psi]}$ endows the tangent space $\mathrm{T}_{[\psi]}\mathbb{P}V$ with an integrable complex structure $\mathrm{J}_{[\psi]}$ [19], making $\mathbb{P}V$ into a Kähler manifold, whose corresponding canonical two-form, the *Fubini-Study* two-form, reads pointwise

$$\omega_{\mathrm{FS}}(X_{[\psi]}, Y_{[\psi]}) = \frac{1}{2\,\mathrm{i}} \left( \langle \xi, \zeta \rangle - \langle \zeta, \xi \rangle \right)$$

where $\xi$, $\zeta \in \operatorname{ran} u_{[\phi]}$ correspond to $X_{[\psi]}$, $Y_{[\psi]}$ under the identification $\mathrm{T}_{[\psi]}\mathbb{P}V \simeq \operatorname{ran} u_{[\psi]}$[20]. The associated Riemannian metric is given by

$$\mathrm{g}_{\mathrm{FS}}(X_{[\psi]}, Y_{[\psi]}) = \omega_{\mathrm{FS}}(X_{[\psi]}, \mathrm{J}_{[\psi]} Y_{[\psi]}) = \frac{1}{2} \left( \langle \xi, \zeta \rangle + \langle \zeta, \xi \rangle \right),$$

[19] For example, because of the Newlander-Nirenberg theorem, see [Hör90a].
[20] This identification can e.g. be obtained by differentiating curves through $[\psi]$ in $\mathbb{P}V$.

and a hermitian structure $\mathrm{H}_{\mathrm{FS}}$ through

$$\mathrm{H}_{\mathrm{FS}}(([\psi],\psi),([\psi'],\psi')) = \langle\psi,\psi'\rangle.$$

Furthermore, define

$$\alpha(\psi) = \frac{\langle\, \mathrm{d}\psi,\psi\rangle}{\|\psi\|^2},$$

which is a one-form on $V$, and consider its pull-back under the identification map (3.6.8),

$$\alpha_{\mathrm{FS}} = \mathrm{I}^*\alpha.$$

This defines a $\mathbb{C}^\times$-invariant one-form on $L_{\mathrm{can}}$ whose horizontal space at $([\psi],\psi)$ is given by $\operatorname{ran} u_{[\psi]}$. For an arbitrary section $s : U_{[\phi]} \to L^\times_{\mathrm{can}}$ the pull-back

$$\theta_{\mathrm{FS}} = s^*\alpha_{\mathrm{FS}}$$

defines a local one-form on $\mathbb{P}V$ yielding a covariant derivative $\nabla_{\mathrm{FS}}$ such that

$$\omega_{\mathrm{FS}} = \mathrm{i}\,\mathrm{curv}(\nabla_{\mathrm{FS}}).$$

We thus have

**Proposition 3.6.11.** *The data $(L_{\mathrm{can}},\mathrm{H}_{\mathrm{FS}},\nabla_{\mathrm{FS}})$ define a pre-quantization of $(\mathbb{P}V,\omega_{\mathrm{FS}})$.*

For the connection to coadjoint orbits we use the equivariant moment map, see e.g. Appendix B,

$$J : \mathbb{P}V \to \mathfrak{g}^*, \quad J([\psi])(X) = \langle\psi,\, \mathrm{d}\pi(X)\psi\rangle \tag{3.6.9}$$

where $\psi \in V$ is an arbitrary lift of $[\psi] \in \mathbb{P}V$. We use a result from [Lan98b]:

**Proposition 3.6.12.** *The coadjoint orbit $\mathcal{O}_\lambda$ corresponding to a maximal weight $\lambda \in \mathfrak{t}^*$ contains the image of $[\psi_\lambda]$ under the moment map (3.6.9), i.e.*

$$J([\psi_\lambda]) \in \mathcal{O}_\lambda \simeq G/G_\lambda.$$

*In particular, the restriction of $J$ to the orbit $\pi(G)[\psi_\lambda]$ is a symplectomorphism when $\mathcal{O}_\lambda$ is endowed with the negative Lie-Kirillov symplectic form*

$$\omega_\eta(X_\xi, X_\zeta) = -\eta([\xi,\zeta]),$$

*where $X_\xi$, $X_\zeta$ are the fundamental vector field corresponding to $\xi$, $\zeta \in \mathfrak{g}$.*

*Proof.* According to the properties of weight spaces, we have that

$$J([\psi_\lambda])(X) = \begin{cases} \lambda(X) & X \in \mathfrak{t} \\ 0 & X \in \mathfrak{t}^\perp \end{cases},$$

hence $J([\psi_\lambda])$ defines a trivial extension of $\lambda$ to all of $\mathfrak{g}$. This proves the first claim.

The stability group $G_{J([\psi_\lambda])}$ of $J([\psi_\lambda])$ consists of those $g \in G$ for which

$$\langle \pi_\lambda(g)\psi_\lambda,\, \mathrm{d}\pi_\lambda(Y)\pi_\lambda(g)\psi_\lambda\rangle = \langle \psi_\lambda,\, \mathrm{d}\pi_\lambda(Y)\psi_\lambda\rangle$$

for all $Y \in \mathfrak{g}$. Since $\pi_\lambda$ is irreducible this implies that $\psi_\lambda$ and $\pi_\lambda(g)\psi_\lambda$ define the same element of $\mathbb{P}V$, proving that $G_{J([\psi_\lambda])} \subseteq G_{[\psi_\lambda]}$, and the inverse inclusion follows from the covariance of the moment map. □

This shows that the orbit $[\pi(G)\psi_\lambda] \subset \mathbb{P}V$ of the maximal weight vector may be identified (symplectically) with the coadjoint orbit $G/G_\lambda \simeq \mathcal{O}_\lambda$, i.e.

$$J^*\omega_{\mathcal{O}_\lambda} = \omega_{\mathrm{FS}}.$$

As already seen above, the coherent state map explicitly implements an holomorphic embedding of the coadjoint orbit $\mathcal{O}_\lambda$ into the projective space $\mathbb{P}V$ together with a holomorphic line bundle over $\mathcal{O}_\lambda$. This fact is usually expressed by saying that the quantization of coadjoint orbits is projectively induced. We enlighten this situation by the following diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \pi(G)V \arrow[r, hook] \arrow[d, "\Pi"] & V \arrow[d, "\Pi"] \\ G/T \arrow[r] & {[\pi(G)\psi_\lambda]} \arrow[r, hook] & \mathbb{P}V \end{tikzcd}.$$

We also have an equivalence of the pre-quantum line bundles over $\mathcal{O}_\lambda$ and $[\pi(G)\psi_\lambda]$ in the sense that the connection forms $\alpha_{\mathcal{O}_\lambda}$ on the coadjoint orbit and the restriction of $\theta_{\mathrm{FS}}$ to $[\pi(G)\psi_\lambda]$ coincide. Since the above date define a pre-quantum structure, this also implies that $\omega_{\mathcal{O}_\lambda}$ is the same as the restriction of the Fubini-Study form $\omega_{\mathrm{FS}}$ on $\mathbb{P}V$ to the orbit $[\pi(G)\psi_\lambda]$.

**Remark 3.6.13.** Summarizing the above discussion, we have that coadjoint orbits corresponding to maximal weights can be symplectically embedded as Kähler submanifolds into $\mathbb{P}V$. Furthermore the coherent state map in (3.6.4) can be implemented by using the Gilmore-Perelomov coherent states of Definition 3.6.10. Compared with Theorem 3.6.6, this reveals a very particular case. As shown by Rawnsley [Raw77], the obstruction for the two pre-quantum data of $\mathbb{P}V$ and the Kähler manifold $X$ to coincide is given by a one-form $\mathrm{d}\epsilon$ of type $(1,0)$, where $\epsilon$ is a real-valued function on $X$. In the homogeneous case, i.e. when $X$ is given as a homogeneous space, $\epsilon$ is invariant under holomorphic symplectic diffeomorphisms and therefore has to be constant. Thus if we deal with coadjoint orbits corresponding to maximal weights we are faced with the particular simple situation where the obstruction $\mathrm{d}\epsilon$ vanishes, and the pre-quantum structures coincide.

### 3.6.1 Time evolution of coherent states for compact Lie groups

We now consider the time evolution of coherent states generated by Hamiltonian functions $f : \mathcal{O}_\lambda \to \mathbb{R}$ that leave the polarization invariant. The type of functions that we will consider are precisely the lower symbols $\sigma_1[\,\mathrm{d}\pi_\lambda(\mathfrak{c})] = f$ of (derived) representation operators that arise from an irreducible representation $\pi_\lambda : G \to \operatorname{End} V_\lambda$. However, the lower symbol of the representation operator $\sigma_1[\,\mathrm{d}\pi_\lambda(\mathfrak{c})](\eta) = \eta(\mathfrak{c})$ coincides with the (evaluation of the) moment map of the coadjoint action, which is given simply by the embedding $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, evaluated on $\mathfrak{c}$. The flow generated by $f(\eta)$, viewed as Hamiltonian function on $\mathcal{O}_\lambda$, can be identified with the coadjoint action of $G$ on $\mathcal{O}_\lambda$; the Hamiltonian vector field generated by $f$ at $\eta \in \mathcal{O}_\lambda$ can be identified with the fundamental vector field $\mathrm{ad}^*_{\mathfrak{c}}$ at $\eta$. Therefore the integral curve $\eta(t)$ starting at $\eta$ is the same as $\mathrm{Ad}_{g(t)}\,\eta$, where $g(t)$ is a one-parameter group in $G$ such that

$$\dot{g}(t) = \mathrm{i}\,\mathfrak{c}g(t). \tag{3.6.10}$$

Since the polarization on $\mathcal{O}_\lambda$ is invariant with respect to the coadjoint action, see Section 3.6, the lift of the flow $\Phi^t_f : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ to a connection preserving diffeomorphism is in this case given by

$$\phi_\eta \mapsto \phi_{\eta(t)} \exp\left(-\mathrm{i}\int_0^t \theta_{\mathcal{O}_\lambda}(\mathrm{X}_f)(\eta(t')) - f(\eta(t'))\,\mathrm{d}t'\right).$$

This is precisely the integral version of equation (3.5.3). Now we know that the prequantum structures of $\mathcal{O}_\lambda$ and $[\pi(G)\psi_\lambda]$ coincide. Therefore, the above lift of $\Phi^t_f$, which on $[\pi(G)\psi_\lambda]$ is given by $[\pi(g(t))\phi_\eta]$, is equivalent to $\pi(g(t))\phi_\eta$.

We now exploit the one-to-one correspondence between irreducible representations and dominant weights, i.e. weights that lie in a (fixed) Weyl chamber, see Appendix C. In particular, the correspondence is such that the representation associated to a dominant weight $\lambda$ has $\lambda$ as maximal weight. Therefore we label the representation $(\pi_\lambda, V_\lambda)$ by the (unique) maximal weight, to account for the correspondence between maximal weights and representations. Since the weights form an integral lattice and the Weyl chambers are convex cones, the set $\{k\lambda\}_{k\in\mathbb{N}}$ is a sublattice of the lattice of weights and is contained in the Weyl chamber of $\lambda$. In particular, each $k\lambda$ is a maximal weight associated to an irreducible unitary representation $(\pi_{k\lambda}, V_{k\lambda})$ whose dimension may be determined by the Weyl dimension formula,

$$\dim V_{k\lambda} = \Pi_{\alpha\in\Delta^+}\frac{\langle\alpha, k\lambda+\rho\rangle}{\langle\alpha,\rho\rangle} = \Pi_{\alpha\in\Delta^+}\frac{k\langle\alpha,\lambda\rangle + \langle\alpha,\rho\rangle}{\langle\alpha,\rho\rangle}. \tag{3.6.11}$$

Here $\Delta^+$ denotes the set of positive roots and $\rho$ is the half sum of the positive roots

$$\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.$$

This shows that multiplication of the maximal weight $\lambda$ results in a higher dimensional representation of the underlying compact Lie group $G$. Repeating the above discussion for the coadjoint orbit $\mathcal{O}_{k\lambda}$ corresponding to the maximal weight $k\lambda$, we see that we must consider the coadjoint orbit $\mathcal{O}_{k\lambda}$ through the maximal weight $k\lambda$ associated to $(\pi_{k\lambda}, V_{k\lambda})$. Then the pre-quantum line bundle $L_{\mathcal{O}_{k\lambda}} \to \mathcal{O}_{k\lambda}$ inherits a connection whose curvature equals $\mathrm{i}\, k\omega_{\mathcal{O}_\lambda}$, where $\omega_{\mathcal{O}_\lambda}$ is the Lie-Kirillov symplectic structure on the coadjoint orbit $\mathcal{O}_\lambda$.

**Lemma 3.6.14.** *We can identify $\mathcal{O}_{k\lambda}$ with $G/G_\lambda$ which is symplectically embedded in $\mathbb{P}V_{k\lambda}$ as a Kählerian submanifold, where the pre-quantizing structure of $\mathcal{O}_{k\lambda}$ can be identified with the Fubini-Study structure of $\mathbb{P}V_{k\lambda}$. This implies an identification*

$$\mathcal{O}_\lambda \simeq G/T \simeq \mathcal{O}_{k\lambda},$$

*that uses the linear structure of $\mathfrak{g}^*$ and can be made explicit by the multiplication $\mathcal{O}_\lambda \ni \eta \overset{\sim}{\mapsto} k\eta \in \mathcal{O}_{k\lambda}$ with $k \in \mathbb{N}$ fixed. Let $I : \mathcal{O}_\lambda \overset{\sim}{\to} \mathcal{O}_{k\lambda}$ denote the above identification, then under this identification the symplectic structures are related through*

$$\omega_{\mathcal{O}_{k\lambda}} = k\omega_{\mathcal{O}_\lambda}.$$

*Furthermore, the pre-quantizing structures of $\mathcal{O}_\lambda$ and $\mathcal{O}_{k\lambda}$ correspond to each other. More precisely, $I^*L_{\mathcal{O}_{k\lambda}}$ is equivalent to $L_{\mathcal{O}_\lambda}^{\otimes k}$ and $I^*\theta_{\mathcal{O}_{k\lambda}} = k\theta_\lambda$. Thus we have an equivalence of the hermitian line bundles $L_{\mathcal{O}_\lambda}^{\otimes k}$ and $I^*L_{\mathcal{O}_{k\lambda}}$ over the coadjoint orbit $\mathcal{O}_\lambda$.*

Since the symplectic structure corresponding to $L_{\mathcal{O}_\lambda}^{\otimes k}$ is given by $k\omega_{\mathcal{O}_\lambda}$, where $\omega_{\mathcal{O}_\lambda}$ denotes the Lie-Kirillov symplectic structure, and the moment map for the coadjoint action of $G$ on $\mathcal{O}_{k\lambda}$ is given by $k$ times the moment map of the $G$-action on $\mathcal{O}_\lambda$, also the (coadjoint) dynamics on $\mathcal{O}_\lambda$ and $\mathcal{O}_{k\lambda}$ can be identified[21]. Summarizing, we observe that the transition $\lambda \mapsto k\lambda$ is the same as taking higher tensor powers of the pre-quantizing line bundle $L_{\mathcal{O}_\lambda} \to \mathcal{O}_\lambda$, in the way described above. Therefore, the pre-quantization lift of a coadjoint dynamics on $\mathcal{O}_\lambda$ is given by (the pull-back under $I : \mathcal{O}_\lambda \overset{\sim}{\to} \mathcal{O}_{k\lambda}$ of)

$$\pi_{k\lambda}(g(t))\phi_{k\eta} = \phi_{k\eta(t)} \exp\left(-\mathrm{i}\, k \int_0^t \theta_{\mathcal{O}_\lambda}(\mathrm{X}_f)(\eta(t')) - \eta(t')(\mathfrak{c})\, \mathrm{d}t'\right).$$

### 3.6.2 Localization properties

In this section we briefly consider semiclassical properties of the coherent states $\phi_{k\eta}$ corresponding to a unitary irreducible representation $(\pi_{k\lambda}, V_{k\lambda})$. To begin with, let us fix $k \in \mathbb{N}$. We remark that, in the case that the translational degrees of freedom are absent, the semiclassical limit is performed in terms of $k \to \infty$. Let us consider the function $f(\eta) := \langle \phi_\eta, \phi_\mu \rangle$ defined on $\mathcal{O}_\lambda$. Obviously we have that $|f(\eta)| \leq 1$ and because of Definition 3.6.10 we can write

$$f(\eta) = \langle \psi_\lambda, \pi_\lambda(g_\eta^{-1} g_\mu)\psi_\lambda \rangle,$$

which shows that $|f(\eta)|$ equals 1 if and only if $g_\eta^{-1} g_\mu \in G_\lambda$, the isotropy group of $\lambda$. However, since $\mathcal{O}_\lambda \simeq G/G_\lambda$, this implies that $|f(\eta)|$ attains its unique maximal value 1

[21] Of course this has to be the case, since the coadjoint action also behaves linearly.

precisely for $\eta = \mu$. The behaviour of $f(\eta)$ in the semiclassical limit $k \to \infty$ can be obtained from a result of Gilmore [Gil79], see also [Duf90], which uses that

$$\langle \psi_{k\lambda}, \pi_{k\lambda}(g)\psi_{k\lambda}\rangle = \langle \psi_\lambda, \pi_\lambda(g)\psi_\lambda\rangle^k.$$

This might suggest that in the semiclassical limit $f(\eta)$ becomes increasingly concentrated at $\eta$. That this is indeed the case is shown in the following result [Duf90], see also [Lie73]:

**Proposition 3.6.15.** *Let*

$$\nu_k(g) := \dim V_{k\lambda} \left|\langle \psi_{k\lambda}, \pi_{k\lambda}(g)\psi_{k\lambda}\rangle\right|^2 = \dim V_{k\lambda} \left|\langle \psi_\lambda, \pi_\lambda(g)\psi_\lambda\rangle\right|^{2k}.$$

*Then $\nu_k$ defines a $G_\lambda$-invariant probability measure on $G$, which fulfills*

$$\lim_{k\to\infty} \nu_k(f) = \int_{G_\lambda} f(h)\,\mathrm{d}h$$

*for all $f \in L^1(G)$, where $\mathrm{d}h$ denotes the normalized Haar measure on $G_\lambda$. In particular, if $f$ is $G_\lambda$ invariant we have*

$$\lim_{k\to\infty} \nu_k(f) = f(e).$$

The importance of this proposition lies in the fact that the probability measure defined above is closely related to the Husimi transform $H[\phi_\mu](\eta)$ of the coherent state $\phi_\mu$, which is defined as

$$H[\phi_\mu](\eta) := \left|\langle \phi_\mu, \phi_\eta\rangle\right|^2.$$

It is immediately clear that $H[\phi_\mu](\eta)$ coincides with $(\dim V_{k\lambda})^{-1}\nu_k(g)$ for $g = g_\mu^{-1}g_\eta$, and therefore also carries the localization properties of $\nu_k(g)$.

## 3.7 A Moyal-Weyl quantizer for the intrinsic degrees of freedom

The aim of this section is to construct a so-called *Moyal-Weyl quantizer* for the intrinsic degrees of freedom described by a compact semi-simple Lie group $G$.

**Definition 3.7.1 ([GVF01]).** Let $X$ be a (finite-dimensional) symplectic manifold, $\mu$ a multiple of the Liouville measure on $X$ and $\mathscr{H}$ a Hilbert space (somehow) associated with $X$. A *Moyal quantizer* for $(X, \mu, \mathscr{H})$ is a mapping $\Delta$ from $X$ to the space of bounded selfadjoint operators on $\mathscr{H}$ such that $\Delta(X)$ is weakly dense in $\mathrm{B}(\mathscr{H})$ and fulfills

$$\mathrm{Tr}\,\Delta(u) = 1, \quad \mathrm{Tr}(\Delta(u)\Delta(v)) = \delta(u - v)$$

in the distributional sense, where $\delta(u-v)$ denotes the reproducing kernel for the measure $\mu$.

If one has a Moyal quantizer at hand one can solve the quantization problem as follows: For a (sufficiently regular) function $a$ on $X$ one defines the (Stratonovich-Weyl) operator

$$\mathrm{op}^{\mathrm{SW}}[a] := \int_X a(x)\Delta(x)\,\mathrm{d}\mu(x),$$

and has the corresponding phase space representation of an operator $\mathcal{A} \in B(\mathscr{H})$ by

$$\mathrm{symb}^{\mathrm{SW}}[A] := \mathrm{Tr}(A\Delta(\cdot)).$$

It is obvious that

$$\mathrm{symb}^{\mathrm{SW}}[\mathrm{id}] = 1,$$

and that $\mathrm{symb}^{\mathrm{SW}}$ inverts $\mathrm{op}^{\mathrm{SW}}$ according to

$$\mathrm{symb}^{\mathrm{SW}}[\mathrm{op}^{\mathrm{SW}}[a]](x) = \mathrm{Tr}\left(\int_X a(y)\Delta(y)\,\mathrm{d}\mu(y)\Delta(x)\right) = a(x).$$

In particular, $\mathrm{symb}^{\mathrm{SW}}[\mathrm{op}^{\mathrm{SW}}[1]] = 1$ implies that $1 \mapsto \mathrm{id}_{\mathscr{H}}$ by the weak density of $\Delta(X)$, which amounts to the reproducing property

$$\int_X \Delta(x)\,\mathrm{d}\mu(x) = \mathrm{id}_{\mathscr{H}}\,.$$

Finally we obtain the so-called *tracial property*

$$\mathrm{Tr}[\mathrm{op}^{\mathrm{SW}}[a]\,\mathrm{op}^{\mathrm{SW}}[b]] = \int_X a(x)b(x)\,\mathrm{d}\mu(x). \tag{3.7.1}$$

The concept of a Moyal quantizer was introduced in [GBV88b, VGB89], where the Moyal quantizer for spin was worked out. In [VGB89] however, this quantizer was called *Stratonovich-Weyl* quantizer, inspired by [Str57]. This construction is particularly interesting in the equivariant setting: If $X$ is a symplectic homogeneous $G$-space, $\mu$ is an $G$-invariant measure on $X$ and $G$ acts by a projective unitary irreducible representation $\pi$ on the Hilbert space $\mathscr{H}$, then a Moyal quantizer is a combination $(X, \mu, \mathscr{H}, G, \pi)$ together with a map $\Delta$ which takes $X$ to the bounded selfadjoint operators on $\mathscr{H}$ satisfying satisfies the properties of Definition 3.7.1, and in addition the covariance property

$$\pi(g)\Delta(x)\pi(g)^{-1} = \Delta(gx) \tag{3.7.2}$$

for all $g \in G$ and $x \in X$. The quantizer then allows Fourier analysis to be performed essentially as in the abelian case, see also [GBV88a, GBV88b, Tay84]: the distribution

$$E(g, x) := \mathrm{Tr}(\pi(g)\Delta(x))$$

takes over the role of the exponential kernel of the Fourier transform. It will in general be a distribution on the space of smooth sections of a non-trivial line bundle over $G \times X$.

The situation we are faced with is precisely the symplectic homogeneous setting, and we will give an explicit construction of the Moyal quantizer in this case, following [FGV90, Gra92]. To this end we will use the analytical properties of the lower and upper symbols described in Section 3.6. An alternative way to obtain a Moyal quantizer is given by the use of the notion of *groupoids* and associated *algebroids* in the context of geometric quantization, as in [GBV95, dMRT02][22]

Let us consider a fixed irreducible unitary representation $\pi_\lambda : G \to V_\lambda$ of $G$ and fix a set of positive roots $\Delta^+$. Let $\mathcal{O}_\lambda$ be the coadjoint orbit through $\lambda \in \mathfrak{g}^*$[23]. Let us consider the lower symbol map (3.6.6) $\sigma_1[A](\eta)$ corresponding to an operator $A \in \mathcal{L}(\mathscr{H}_\lambda)$, which is a continuous function on $\mathcal{O}_\lambda$. Define the (finite-dimensional) space of functions

$$\mathcal{S}_\lambda := \{\sigma_1[A];\ A \in \mathcal{L}(\mathscr{H}_\lambda)\}.$$

The essential point in this construction is that no information about the operator $A$ is lost by passing to the lower symbol, see also Appendix C.

**Proposition 3.7.2.** *The symbol map*

$$\sigma_1 :\ \mathcal{L}(\mathscr{H}_\lambda) \to \mathcal{S}_\lambda$$

*is linear and one-to-one.*

*Proof.* We give a short sketch of the proof that can be found in [Sim80] and [Kla64], see also [Wil86]. It consists of showing that the kernel of the lower symbol map is trivial. Let $\sigma_1[A] = 0$, this is equivalent to

$$\langle \psi_\lambda, \pi(g) A \pi(g)^{-1} \psi_\lambda \rangle$$

for all $g \in G$, where $\psi_\lambda$ denotes the maximal weight vector corresponding to the representation $(\pi_\lambda, V_\lambda)$. Taking derivatives $n$ times at $g = e$ gives

$$\langle \psi_\lambda, \mathrm{ad}_{\xi_1} \mathrm{ad}_{\xi_2} \cdots \mathrm{ad}_{\xi_n} A \psi_\lambda \rangle = 0.$$

[22] Groupoids arose as a far reaching extension of groups. See e.g. [Wei01, CW99, Wei91a, Wei91b, Con94, CDW87, Ren80, Mac87, Ram98]. Groupoids are also useful in the description of the dynamical behaviour of particles in external gauge fields, see [Lan98a, Lan92, Lan93, Lan03, Lan99]. It seems that the groupoids (in particular Lie groupoids) and the corresponding algebroids could give interesting geometrical insights and extensions of the objects we are studying. There are recent developments to define microlocal objects such as pseudodifferential operators on groupoids, see e.g. [NWX99, LMN00, LN01, WX91, Wei91a, KSM02]. However, considerations in this direction are far beyond the scope of this thesis; therefore we leave it at just quoting Weinstein [Wei01]: " *Mathematicians tend to think of the notion of symmetry as being virtually synonymous with the theory of groups and their actions, perhaps largely because of the well-known Erlanger programme [...], which virtually defined the geometric structures by their groups of automorphisms. In fact, though groups are indeed sufficient to characterize homogeneous structures, there are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the symmetry, and hence much of the structure, of such objects can be characterized algebraically if we use groupoids and not just groups.*"

[23] Initially, $\lambda$ is defined as linear form on the Cartan algebra $\mathfrak{t}$. However, since $G$ is supposed to be semi-simple we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by using the (non-degenerate) Killing-form. This gives $\lambda \in \mathfrak{g}^*$, which is equivalent to a (trivial) extension of $\lambda \in \mathfrak{t}^*$ to all of $\mathfrak{g}$.

This is valid for all $\xi_j \in \mathfrak{g}$ and by linearity for all $\xi_j \in \mathfrak{g}^{\mathbb{C}}$. If we choose the $\xi_j$ to be the root elements $\xi_\alpha$, which can be chosen such that $\mathrm{d}\pi(\xi_\alpha)^* = \pm\,\mathrm{d}\pi(\xi_{-\alpha})$ and use the fact that the representation space is spanned by $\mathrm{d}\pi(X_{-\alpha})\psi_\lambda$ for $\alpha \in \Delta^+$, we can immediately conclude that

$$\langle\, \mathrm{d}\pi(\xi_{-\alpha_n}) \cdots \mathrm{d}\pi(\xi_{-\alpha_1})\psi_\lambda, A\psi_\lambda\rangle = 0$$

and therefore $A\psi_\lambda = 0$. Since the same reasoning is valid for $A$ replaced by $\pi(g)A\pi(g)^{-1}$ for any $g \in G$, we can conclude that $A = 0$. $\square$

The flaw of the lower symbols however is that they lack the tracial property (3.7.1), which the existence of a Moyal quantizer automatically implies. From Section 3.6 we know that the upper and lower symbols fulfill

$$\mathrm{Tr}(A\,\mathrm{op}^{BT}[f]) = \int_{\mathcal{O}_\lambda} f(\eta)\sigma_{\mathrm{l}}[A](\eta)\,\mathrm{d}\eta$$

and that the upper symbol corresponding to a Berezin-Toeplitz operator is given by the respective function on $\mathcal{O}_\lambda$. The Berezin-Toeplitz quantization, and thus the upper symbol map, defines a linear bijection between $B(V_\lambda)$ and $\mathcal{S}_\lambda$. Obviously both the upper and the lower symbol are covariant

$$\sigma_{\mathrm{l}}[\pi(g)A\pi(g)^{-1}](\eta) = \sigma_{\mathrm{l}}[A](\mathrm{Ad}^*_g\,\eta), \quad \sigma_{\mathrm{u}}[\pi(g)A\pi(g)^{-1}](\eta) = \sigma_{\mathrm{u}}[A](\mathrm{Ad}^*_g\,\eta).$$

This implies the important property

**Proposition 3.7.3.** *The mapping $K : \sigma_{\mathrm{u}}[A] \mapsto \sigma_{\mathrm{l}}[A]$ is a positive invertible operator on $\mathcal{S}_\lambda$. Furthermore, it commutes with the regular representation $(\tau(g)f)(\eta) = f(g^{-1}\eta)$ restricted to $f \in \mathcal{S}_\lambda$.*

*Proof.* The bijectivity of $K$ follows from the fact that $\sigma_{\mathrm{l}}[A] \mapsto A$ and $A \mapsto \sigma_{\mathrm{u}}[A]$ are bijections. The positivity follows from

$$\int_{\mathcal{O}_\lambda} K\overline{\sigma_{\mathrm{l}}[A]}\sigma_{\mathrm{l}}[A]\,\mathrm{d}\eta = \mathrm{Tr}(A^*A) \geq 0.$$

The fact that $K$ commutes with the regular representation is just a restatement of the covariance of the upper and lower symbols. $\square$

**Definition 3.7.4.** For $A \in \mathcal{L}(V_\lambda)$ we define the Stratonovich-Weyl symbol $\mathrm{symb}^{\mathrm{SW}}[A] \in \mathcal{S}_\lambda$ by

$$\mathrm{symb}^{\mathrm{SW}}[A] := K^{1/2}\sigma_{\mathrm{l}}[A] = K^{-1/2}\sigma_{\mathrm{u}}[A].$$

The Stratonovich-Weyl symbol has the following properties

**Theorem 3.7.5.** *The symbol map $\mathcal{L}(V_\lambda) \ni A \mapsto \mathrm{symb}^{\mathrm{SW}}[A] \in \mathcal{S}_\lambda$ fulfills*

*(i) It is a linear one-to-one map from $\mathcal{L}(V_\lambda)$ onto $\mathcal{S}_\lambda$.*

*(ii)* $\operatorname{symb}^{\mathrm{SW}}[A^*] = \overline{\operatorname{symb}}^{\mathrm{SW}}[A]$.

*(iii)* $\operatorname{symb}^{\mathrm{SW}}[\mathrm{id}_{V_\lambda}] = 1$.

*(iv)* $\operatorname{symb}^{\mathrm{SW}}[\pi(g)A\pi(g)^{-1}](\eta) = \operatorname{symb}^{\mathrm{SW}}[A](\mathrm{Ad}_g^*\eta)$.

*(v)* $\displaystyle\int_{\mathcal{O}_\lambda} \operatorname{symb}^{\mathrm{SW}}[A](\eta)\operatorname{symb}^{\mathrm{SW}}[B](\eta)\,\mathrm{d}\eta = \operatorname{Tr}(AB)$.

*Proof.* The bijectivity follows from the bijectivity of the lower symbol map and the bijectivity of $K$. That $\sigma_\mathrm{l}[A^*] = \overline{\sigma}_\mathrm{l}[A]$ holds an immediate consequence of its definition and $\sigma_\mathrm{u}[A^*] = \overline{\sigma}_\mathrm{u}[A]$ follows from the fact that the coherent projection (Szegö-projector) is orthogonal. Therefore $K$ commutes with complex conjugation and $K^{1/2}$ inherits this property. Obviously $\sigma_\mathrm{l}[\mathrm{id}_{V_\lambda}] = 1$ and $\sigma_\mathrm{u}[\mathrm{id}_{V_\lambda}] = 1$[24], thus $\operatorname{symb}^{\mathrm{SW}}[\mathrm{id}_{V_\lambda}] = 1$.

The tracial property follows from the fact that $K$ commutes with complex conjugation and

$$\int_{\mathcal{O}_\lambda} \operatorname{symb}^{\mathrm{SW}}[A](\eta)\operatorname{symb}^{\mathrm{SW}}[B](\eta)\,\mathrm{d}\eta = \int_{\mathcal{O}_\lambda} K\sigma_\mathrm{u}[A](\eta)\sigma_\mathrm{l}[B](\eta)\,\mathrm{d}\eta = \operatorname{Tr}(AB).$$

□

Now the inverse $\mathcal{S}_\lambda \ni \operatorname{symb}^{\mathrm{SW}}[A] \mapsto A \in \mathcal{L}(V_\lambda)$ can be expressed in terms of a Moyal quantizer $\Delta : \mathcal{O}_\lambda \to \mathcal{L}(V_\lambda)$, i.e. an operator valued function on $\mathcal{O}_\lambda$ such that

$$A = \int_{\mathcal{O}_\lambda} \operatorname{symb}^{\mathrm{SW}}[A](\eta)\Delta(\eta)\,\mathrm{d}\eta.$$

Consider the lower symbol

$$\sigma_\mathrm{l}[A](\eta) = \langle\phi_\eta, A\phi_\eta\rangle = \langle\phi_\eta, \int_{\mathcal{O}_\lambda} \sigma_\mathrm{u}[A](\eta')\mathrm{P}_{\eta'}\,\mathrm{d}\eta'\phi_\eta\rangle = \int_{\mathcal{O}_\lambda} |\langle\phi_\eta, \phi_{\eta'}\rangle|^2\sigma_\mathrm{u}[A](\eta')\,\mathrm{d}\eta'.$$

so that $|\langle\phi_\eta, \phi_{\eta'}\rangle|^2$ is the kernel of $K^{-1}$. The reproducing kernel of $\mathcal{S}_\lambda$, expanded in on an orthonormal basis of $\mathcal{S}_\lambda$, can be used to diagonalize $K$. Thus yielding an explicit expression for the quantizer $\Delta$, see e.g. [FGV90].

**Remark 3.7.6.** The localization properties of coherent states described in Section 3.6.2 imply that upper and lower symbols coincide in the semiclassical limit $k \to \infty$, and, as an effect, also are the same as the Stratonovich-Weyl symbol. See [Sim80, Duf90].

[24]This requires a normalization of the Liouville measure $\mathrm{d}\eta$ on $\mathcal{O}_\lambda$ such that $\int_{\mathcal{O}_\lambda}\mathrm{d}\eta = \dim V_\lambda$.

# Chapter 4

# Coherent states for particles with intrinsic degrees of freedom

In this Chapter, we will use the notions developed in Chapter 3 to give a suitable definition of coherent states for particles which besides the translational degrees of freedom also possess internal degrees of freedom that are described by (irreducible representations of) a compact Lie group. The most popular representative for this class is a non-relativistic particle with spin, where the Lie group is given by SU(2):

> SU(2) *is the simplest of the simple Lie groups. It is the unique rank 1, compact, connected, simply connected, semisimple Lie group (Wow! Five adjectives to define it!) [Sim96]*

But let us first inspect coherent states for the translational degrees of freedom:

## 4.1 Coherent States for the Heisenberg group

We consider the symplectic space $\mathrm{T}^*\mathbb{R}^d \simeq \mathbb{R}^d \times \mathbb{R}^d$, a central role in the study of which is played by the *Heisenberg group* $H(\mathbb{R}^d)$. Its Lie algebra can be realized by taking the coordinate functions $q_j$, $p_j$ as well as the unit function on $\mathrm{T}^*\mathbb{R}^d$ as basis elements and equating the Lie bracket with (minus) the Poisson bracket on $\mathrm{T}^*\mathbb{R}^d$. If we denote this basis by $\{Q_j, P_j, Z\}$ then we have the relations

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, Z] = [Q_i, Z] = 0, \quad [P_i, Q_j] = -\delta_{ij} Z. \tag{4.1.1}$$

**Definition 4.1.1.** The Heisenberg group $H(\mathbb{R}^d)$ is the unique connected and simply connected Lie group of dimension $2d + 1$ with Lie algebra $\mathfrak{h}_d$ defined by the relations (4.1.1).

Since the Lie algebra $\mathfrak{h}_d$ is nilpotent, the exponential map $\mathrm{Exp} : \mathfrak{h}_d \to H(\mathbb{R}^d)$ is a diffeomorphism and we parameterize $H(\mathbb{R}^d)$ by $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ and $s \in \mathbb{R}$ such that

$$(u, v, s) := \mathrm{Exp}(-uQ + vP + sZ).$$

The algebraic relations (4.1.1) then define the group multiplication in $H(\mathbb{R}^d)$:

$$(w, s) \cdot (w', s') = (w + w', \frac{1}{2}\omega(w, w')),$$

where $w = (u, v)$ and $\omega$ denotes the standard symplectic form on $\mathrm{T}^*\mathbb{R}^d$. Furthermore, the orbits of the coadjoint action of $H(\mathbb{R}^d)$ on the dual $\mathfrak{h}_d^*$ of its Lie algebra are given by

$$\mathrm{Ad}^*_{(u,v,s)}(q, p, c) = (q + cu, p + cv, c).$$

In particular, for $c = 1$ we obtain the "usual" $\mathrm{T}^*\mathbb{R}^d$, which of course is a symplectic manifold.

In the following we are going to describe the quantization of this symplectic manifold. First however, we remark that $\mathrm{T}^*\mathbb{R}^d$ is contractible, therefore all the additional structures needed in the geometric quantization scheme exist and are unique up to isomorphisms. The pre-quantum line bundle $L = \mathrm{T}^*\mathbb{R}^d \times \mathbb{C}$ is trivial with a trivializing section given by $\lambda_0(x, \xi) = (x, \xi, 1)$ for each $(x, \xi) \in \mathbb{R}^d$. We define the connection in $L$ by

$$\nabla\lambda_0 = -\frac{\mathrm{i}}{\hbar}\theta \otimes \lambda_0,$$

where $\theta = \sum_{j=1}^d \xi_j \,\mathrm{d}x_j$ is the canonical Liouville one-form on $\mathrm{T}^*\mathbb{R}^d$ and thus fulfills $\mathrm{d}\theta = \omega$. For any polarization $\mathfrak{F}$, which we will specify further in the following, also the bundle $\mathrm{MB}(\mathfrak{F})$ of metalinear frames is trivial.

### 4.1.1 Holomorphic sections for a complex polarization

The aim of this section is to calculate covariantly constant sections with respect to a polarization, which is determined by the Lagrangian subspace

$$(\mathrm{T}^*\mathbb{R}^d)^{\mathbb{C}} \supset \mathfrak{F}_B = \{(z, Bz);\; z \in \mathbb{C}^d\} \tag{4.1.2}$$

defined by the symmetric complex matrix $B \in \Sigma_d$ from the Siegel upper half space, see Section A.5. If we use the canonical symplectic frame $\{e_1, \dots, e_n; f_1, \dots, f_n\}$[1] of $\mathrm{T}^*\mathbb{R}^d$ then $\mathfrak{F}$ is spanned by $\{\xi_j := e_j + Bf_j\}_{1\le j\le d}$. The complex Hamiltonian functions associated to these basis vectors are given by

$$\overline{z}_j = \left(\xi_j - \sum_{k=1}^n B_{jk}x_k\right).$$

So we get

$$x_j = -\frac{1}{2}\,\mathrm{i}\sum_{k=1}^d (\Im B)^{-1}_{jk}(z_k - \overline{z}_k), \quad \xi_j = \frac{1}{2}\left((z_j + \overline{z}_j) - \mathrm{i}\sum_{k=1}^d (\Re B(\Im B)^{-1})_{jk}(z_k - \overline{z}_k)\right),$$

[1]Where we may identify $e_j = \partial_{x_j}$ and $f_j = \partial_{\xi_j}$, see [RR89].

and a calculation gives

$$\frac{\partial}{\partial z_j} = -\mathrm{i}\sum_{k=1}^{d} (\Im B)^{-1}_{jk}\left(\frac{\partial}{\partial x_k} + \sum_{l=1}^{d} B_{kl}\frac{\partial}{\partial \xi_l}\right) \in L,$$

such that the $\{z_j\}_{1\leq j\leq d}$ are the holomorphic and the $\{\overline{z}_j\}_{1\leq j\leq d}$ the anti-holomorphic coordinates, in which the symplectic form reads

$$\omega = -\frac{1}{2}\mathrm{i}\sum_{j,k=1}^{d} (\Im B)^{-1}_{jk}\,\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

and

$$\theta_1 = \frac{1}{2}\mathrm{i}\sum_{j,k=1}^{d} (\Im B)^{-1}_{jk}\overline{z}_k\,\mathrm{d}z_j$$

is a symplectic potential adapted to $\mathfrak{F}$.[2].

Now let $\mathfrak{F}_B$ be the Kähler polarization defined by $B$. Since $\mathrm{T}^*\mathbb{R}^d$ is contractible the bundle $\mathrm{MB}(\mathfrak{F}_B)$ of metalinear frames for $\mathfrak{F}_B$ is trivial. Let $\tilde{\boldsymbol{\xi}}_z$ be a metalinear frame field covering $\boldsymbol{\xi}_z$, which consists of the Hamiltonian vector fields corresponding to $z = \{z_1, \dots z_d\}$ and $\nu_{\tilde{\boldsymbol{\xi}}_z}$ a section of the half-form bundle $\sqrt{\Lambda^d}\mathfrak{F}_B$ defined by

$$\tilde{\nu}_{\tilde{\boldsymbol{\xi}}_z} \circ \tilde{\boldsymbol{\xi}}_z = 1,$$

see Section 3.5. Let $L \to \mathrm{T}^*\mathbb{R}^d$ be the pre-quantum line bundle, which in this case also is trivial $L = \mathrm{T}^*\mathbb{R}^d \times \mathbb{C}$, with a trivializing section

$$\lambda_0 : X \longrightarrow L, \quad x \longmapsto (x, 1).$$

Thus, according to Section 3.1,

$$\nabla\lambda_0 = -\frac{\mathrm{i}}{\hbar}\left(\sum_k \xi_k\,\mathrm{d}x_k\right) \otimes \lambda_0$$

and $\langle\lambda_0, \lambda_0\rangle = 1$. Let us introduce a new trivializing section given by

$$\begin{aligned}\lambda_1 = \exp\Bigg\{-\frac{1}{4}\sum_{j,k=1}^{d} \big((\Im B)^{-1}_{jk}\xi_j\xi_k - 2((\Im B)^{-1}\Re B)_{jk}(x_k\xi_j) \\ -2\,\mathrm{i}(\xi_k x_j) + (\Re B(\Im B)^{-1}\Re B + \Im B)_{jk}x_j x_k\big)\Bigg\}\lambda_0.\end{aligned} \tag{4.1.3}$$

For this we have

$$\nabla\lambda_1 = -\frac{\mathrm{i}}{\hbar}\theta_1 \otimes \lambda_1.$$

[2]This means that $\imath_{\overline{X}}\theta_1 = 0$ for any $X \in \mathfrak{F}_B$.

If we introduce the positive definite real symmetric form

$$\mathrm{g}_B := \begin{pmatrix} \Im B + \Re B(\Im B)^{-1}\Re B & -\Re B(\Im B)^{-1} \\ -(\Im B)^{-1}\Re B & (\Im B)^{-1} \end{pmatrix},$$

associated with the complex structure corresponding to $\mathfrak{F}_B$ and the symplectic form (see [Hör85a, KN69, Jos00, BW97] and also Appendix A), we can write

$$\lambda_1 = \exp\left(-\frac{1}{4}((x,\xi),(\mathrm{g}_B - \mathrm{i}\,\mathrm{k})(x,\xi))\right)\lambda_0,$$

where

$$\mathrm{k} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Any other section which is covariantly constant with respect to $\mathfrak{F}_B$ can then be written as

$$\lambda = \psi(x,\xi)\lambda_1,$$

where $\psi(x,\xi)$ is covariantly constant with respect to $\mathfrak{F}_B$. In order that $\lambda_1$ is normalized with respect to the canonical volume form $\omega^d$ on $\mathrm{T}^*\mathbb{R}^d$ we redefine[3]

$$\lambda_1 = (\det \Im B)^{-1/4} 2^{-d}(\pi\hbar^{-3d/4}) \exp\left(-\frac{1}{4}((x,\xi),(\mathrm{g}_B - \mathrm{i}\,\mathrm{k})(x,\xi))\right)\lambda_0. \tag{4.1.4}$$

### 4.1.2 The Schrödinger representation for the covariant sections

The polarization $\mathfrak{F}_B$ defined in equation (4.1.2) is transversal to the vertical polarization $\mathfrak{F}_{\mathrm{ver}}$, defined as the kernel of the differential of the canonical projection $\mathrm{T}^*\mathbb{R}^d \to \mathbb{R}^d$, since $\Im B$[4] is non-singular. The vertical polarization is spanned by the vectors $\{f_j\}_{1\le j\le d}$ of the symplectic basis. Therefore, there is a well-defined pairing between sections covariantly constant along $\mathfrak{F}_B$ and those covariantly constant along $\mathfrak{F}_{\mathrm{ver}}$. This pairing can be used to translate the holomorphic section in equation (4.1.4) into the position representation: According to Section 3.4 we know that for any section

$$\sigma_1 = \psi\lambda_1 \otimes \nu_{\tilde{\boldsymbol{\xi}}_z} \in \mathscr{H}_B$$

covariantly constant along $\mathfrak{F}_B$ and any section

$$\sigma_2 = \varphi\lambda_0 \otimes \nu_{\tilde{\boldsymbol{\xi}}_{\mathrm{ver}}} \in \mathscr{H}_{\mathrm{ver}}$$

covariantly constant along $\mathfrak{F}_{\mathrm{ver}}$ the BKS-pairing is given by

$$\langle \sigma_1, \sigma_2 \rangle = \int_{\mathrm{T}^*\mathbb{R}^d} \mathrm{e}^{-\frac{1}{4\hbar}\langle (x,\xi),(\mathrm{g}_B - \mathrm{i}\,\mathrm{k})(x,\xi)\rangle}\,\psi(x,\xi)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\xi.$$

[3] That this indeed defines a normalized section can be seen by using Lemma 4.1.2 below.
[4] And hence also $B$!

This pairing induces a map $\mathcal{U}: \mathscr{H}_B \to \mathscr{H}_{\mathrm{ver}}$ given by

$$\mathcal{U}(\psi\lambda_1 \otimes \nu_{\tilde{\boldsymbol{\xi}}_z}) = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{4\hbar}\langle (x,\xi),(\mathrm{g}_B - \mathrm{i}\,\mathrm{k})(x,\xi)\rangle} \psi(x,\xi)\,\mathrm{d}\xi\, \lambda_0 \otimes \nu_{\tilde{\boldsymbol{\xi}}_z},$$

which induces a linear operator U from the space of sections that are covariantly constant along $\mathfrak{F}_B$ and square integrable with respect to the weight

$$\mathrm{e}^{-\frac{1}{4\hbar}\langle (x,\xi),\mathrm{g}_B(x,\xi)\rangle}$$

to the space of square integrable functions on $\mathbb{R}^d$. This operator is defined by

$$(\mathrm{U}\psi)(x) = \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{1}{4\hbar}\langle (x,\xi),(\mathrm{g}_B - \mathrm{i}\,\mathrm{k})(x,\xi)\rangle} \psi(x,\xi)\,\mathrm{d}\xi,$$

such that

$$\mathcal{U}(\psi\lambda_1 \otimes \nu_{\tilde{\boldsymbol{\xi}}_z}) = (\mathrm{U}\psi)\lambda_0 \otimes \nu_{\tilde{\boldsymbol{\xi}}}.$$

We remark that the integral kernel of $\mathcal{U}$ arises from the kernel corresponding to the (standard) Bargmann transform under a symplectic change of coordinates[5]. This immediately implies that U is unitary and simultaneously justifies the name *generalized Bargmann-transform* for the map U.

In order to calculate this generalized Bargmann transform explicitly we recall the following result on Gaussian integrals (see e.g. [Hör90b]):

**Lemma 4.1.2.** *Let $C \in \mathrm{M}_d(\mathbb{C})$ be non-singular and symmetric with $\Im C \geq 0$ then*

$$\int_{\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{2}\langle \xi, C\xi\rangle}\, \mathrm{e}^{\mathrm{i}\langle y,\xi\rangle}\,\mathrm{d}\xi = \left(\det\left(\frac{C}{2\pi\,\mathrm{i}}\right)\right)^{-1/2} \mathrm{e}^{-\frac{\mathrm{i}}{2}\langle y, C^{-1}y\rangle}.$$

*The branch of the square root is defined by demanding continuity and $(\det(C/2\pi\,\mathrm{i}))^{1/2} > 0$ if $C/\mathrm{i}$ is real. Note that $-C^{-1}$ has non-negative imaginary part.*

We want to apply this result to the trivializing section (4.1.3) by setting

$$C = \frac{\mathrm{i}}{2\hbar}(\Im B)^{-1} \quad \text{and} \quad y = -\frac{\mathrm{i}}{2\hbar}((\Im B)^{-1}\Re B + \mathrm{i}\,\mathbb{1})x.$$

A direct calculation for $\psi = 1$ gives

$$(\mathcal{U}\psi)(x) = (\pi\hbar)^{-d/4}(\det \Im B)^{1/4} \exp\left(\frac{\mathrm{i}}{2\hbar}\langle x, Bx\rangle\right),$$

which is a translational coherent state centered at $(0,0) \in \mathrm{T}^*\mathbb{R}^d$. It could have also been constructed by using the Schrödinger representation of the Heisenberg group in order to obtain a coherent state in accordance with the scheme of Section 3.6, see [Fol89, BG04b].

[5] and this is precisely the one, which maps $\mathfrak{F}_B$ to $\mathfrak{F}_{\mathrm{i}\,\mathbb{1}}$

In order to obtain a coherent state centered at a different point $(q,p) \neq (0,0)$ in phase space, we have to employ phase space translations, e.g. provided by the Schrödinger representation of the Heisenberg group. We consider the linear function $f(x,\xi) = px - q\xi = \omega((x,\xi),(q,p))$ and the time one map $\Phi_f$ corresponding to the flow $\Phi_f^t$ generated by $f$. Since $f$ is linear it preserves the polarization and we get that the quantization (coinciding with the pre-quantization in this case) of $\Phi_f$ is given by

$$\varphi^B_{(q,p)} := (\pi\hbar)^{-d/4} (\det \Im B)^{1/4} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(x-q)} \, \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} \langle x-q, B(x-q) \rangle} \,. \tag{4.1.5}$$

### 4.1.3 Time evolution of holomorphic sections

Let us now turn to the time evolution of the holomorphic section (4.1.4). The image of a polarization $\mathfrak{F}_B$ under a canonical transformation $\Phi^t$ is given by $\Phi^t_* \mathfrak{F}_B$ and the one-parameter group of canonical transformations $\Phi^t$ lifts to a one-parameter group $\tilde{\Phi}^t$ of diffeomorphisms of $\mathfrak{Lag}^+(X)$ preserving the structure of the bundle of positive Lagrangian frames, see Section 3.5. For each $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathfrak{Lag}^+(X)$, $\tilde{\Phi}^t(\boldsymbol{w}) = (\Phi^t_* w_1, \ldots \Phi^t_* w_n)$ is a positive Lagrangian frame, see Section 3.5. Furthermore, there exists a unique lift, also denoted by $\tilde{\Phi}^t$, of $\tilde{\Phi}^t$ to a one-parameter group of diffeomorphisms of $\widetilde{\mathfrak{Lag}}^+(X)$. For each $t \in \mathbb{R}$ the bundle $\widetilde{\mathfrak{Lag}}^+(X)$ induces a metalinear frame bundle of $\Phi^t_* \mathfrak{F}_B$ given by $\mathrm{MB}(\Phi^t_* \mathfrak{F}_B) = \tau^{-1}(\mathrm{B}(\Phi^t_* \mathfrak{F}_B))$, where $\tau$ denotes the covering of the bundle of positive Lagrangian frames defined in Section 3.3.2. Furthermore, if $\tilde{\boldsymbol{w}} \in \mathrm{MB}(\mathfrak{F}_B)$ then $\tilde{\Phi}^t(\boldsymbol{w}) \in \mathrm{MB}(\Phi^t_* \mathfrak{F}_B)$. Thus, $\tilde{\Phi}^t$ restricted to $\mathrm{MB}(\mathfrak{F}_B)$ yields an isomorphism of the $\mathrm{ML}(n,\mathbb{C})$-principal bundles $\mathrm{MB}(\mathfrak{F}_B)$ and $\mathrm{MB}(\Phi^t_* \mathfrak{F}_B)$. Let $\sqrt{\Lambda^n} \Phi^t_* \mathfrak{F}_B$ be the half-form bundle associated to $\mathrm{MB}(\Phi^t_* \mathfrak{F}_B)$ and denote by $\mathscr{H}_{B,t}$ the representations space consisting of those sections of $L \otimes \sqrt{\Lambda^n} \Phi^t_* \mathfrak{F}$ that are covariantly constant along $\Phi^t_* \mathfrak{F}_B$. If $\nu$ is a local section of $\sqrt{\Lambda^n} \mathfrak{F}$, then for each $t \in \mathbb{R}$ we have a section $\Phi^t \nu$ of $\sqrt{\Lambda^n} \Phi^t_* \mathfrak{F}_B$ defined by

$$(\tilde{\Phi}^t \tilde{\nu})(\tilde{\boldsymbol{w}}) = \tilde{\nu}(\tilde{\Phi}^t(\tilde{\boldsymbol{w}})),$$

for each $\tilde{\boldsymbol{w}} \in \mathrm{MB}(\Phi^t_* \mathfrak{F})$. If $\tilde{\nu}$ is constant along $\mathfrak{F}_B$ then $\tilde{\Phi}^t \nu$ is covariantly constant along $\Phi^t_* \mathfrak{F}_B$. Therefore, we can define a map $\mathscr{H}_B \to \mathscr{H}_{B,t}$, also denoted by $\Phi^t$, according to

$$\sigma = \lambda \otimes \nu \longmapsto \Phi^t \sigma = \Phi^t \lambda \otimes \Phi^t \nu,$$

where, according to Section 3.5, the pre-quantization of the canonical transformation is given by, see equation (3.5.3),

$$(\Phi^t \lambda)(x) = \exp\left( \frac{\mathrm{i}}{\hbar} \int_0^t (\theta_0(\mathrm{X}_H) - H)(\Phi^s(x)) \, \mathrm{d}s \right) \lambda(\Phi^t(x)).$$

Here $H$ denotes the Hamiltonian function $H : \mathrm{T}^*\mathbb{R}^d \to \mathbb{R}$ that generates the flow $\Phi^t$ as an integral curve of the Hamiltonian vector field $\mathrm{X}_H$.

We remark that our polarization $\mathfrak{F}_B$ is totally complex (i.e. Kählerian) and hence has the property

$$\mathfrak{F}_B \cap \overline{\mathfrak{F}}_B = \{0\}, \quad \mathfrak{F}_B + \overline{\mathfrak{F}}_B = \mathrm{T}^{\mathbb{C}}(\mathrm{T}^*\mathbb{R}^d).$$

The assumption that the Hamiltonian function is real valued implies that $\mathscr{L}_{\mathrm{X}_H}\mathfrak{D} \subset \mathfrak{D} = \{0\}$ and therefore that

$$\mathfrak{F}_B \cap \overline{(\Phi^t_*\mathfrak{F}_B)} = \{0\}, \quad \mathfrak{F}_B + \overline{(\Phi^t\mathfrak{F}_B)} = \mathrm{T}^{\mathbb{C}}(\mathrm{T}^*\mathbb{R}^d),$$

which means that the polarization and its image under the canonical transformation are transversal, see [Tuy87a, Tuy87b, Tuy89]. Therefore, we again have a well-defined BKS-pairing between sections in $\mathscr{H}_B$ and $\mathscr{H}_{B,t}$ given by

$$\langle\sigma, \sigma(t)\rangle = \sqrt{\det\omega(\boldsymbol{\zeta}, \overline{\Phi^t_*\boldsymbol{\zeta}})}\langle\lambda, \lambda(t)\rangle$$

for $\sigma \in \mathscr{H}_B, \sigma(t) \in \mathscr{H}_{B,t}$.

Let us now consider the case of a Hamiltonian of the form

$$H_q(x,\xi;t) = \frac{1}{2}\langle x, H_{xx}(z(t))x\rangle + \frac{1}{2}\langle \xi, H_{\xi\xi}(z(t))\xi\rangle + \langle x, H_{x\xi}(z(t))\xi\rangle = \langle (x,\xi), H''(z(t))(x,\xi)\rangle \tag{4.1.6}$$

where $z(t) = (q(t), p(t)) = \Phi^t(q,p)$ is an integral curve of $\mathrm{X}_H$. Thus $H_q$ precisely is the quadratic term in the Taylor expansion of $H$ about the trajectory $z(t)$. The Hamiltonian vector field $\mathrm{X}_{H_q}$ thus reads

$$\mathrm{X}_{H_q} = (H_{x\xi}(z(t))x + H_{\xi\xi}(z(t))\xi)\,\frac{\partial}{\partial x} - (H_{x\xi}(z(t))\xi + H_{xx}(z(t))x)\,\frac{\partial}{\partial \xi}.$$

Let $\Phi^t_q$ be the flow generated by $H_q$, and consider the transport of the holomorphic section (4.1.3) under the lift of $\Phi^t_q$ to a connection preserving diffeomorphism of the pre-quantum line bundle and of the half-form bundle. In order to calculate the action

$$\int_0^t \xi_q(t')\dot{x}_q(t') - H_q(x_q(t'), \xi_q(t'), z(t'))\,\mathrm{d}t'$$

along an integral curve $(x_q(t'), \xi_q(t'))$ of $\mathrm{X}_{H_q}$ we use that

$$\begin{aligned}
H_q(x_q(t), \xi_q(t); t) &= \langle (x_q(t), \xi_q(t)), H''(z(t))(x_q(t), \xi_q(t))\rangle \\
&= \langle \mathrm{J}(x_q(t), \xi_q(t)), \mathrm{J}H''(z(t))(x_q(t), \xi_q(t))\rangle \\
&= \langle \mathrm{J}(x_q(t), \xi_q(t)), (\dot{x}_q(t), \dot{\xi}_q(t))\rangle.
\end{aligned}$$

Plugging this into the expression for the action we obtain

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \frac{1}{2}\left(\dot{\xi}_q(t')x_q(t') + \xi_q(t')\dot{x}_q(t')\right)\,\mathrm{d}t'\right) = \exp\left(-\frac{\mathrm{i}}{2\hbar}\left(x_q(t)\xi_q(t) - x\xi\right)\right).$$

Furthermore, we use that fact that the integral curve $(x_q(t), \xi_q(t))$ is given by

$$\begin{pmatrix} x_q(t) \\ \xi_q(t)\end{pmatrix} = S(t)\begin{pmatrix} x \\ \xi\end{pmatrix},$$

where $S(t)$ denotes the linearized flow along the trajectory $z(t)$, see Appendix A, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \mathrm{J}H_q''(z(t))S(t) = \mathrm{J}H''(z(t))S(t).$$

Thus we obtain that the section $\lambda_t = \Phi^t\lambda$, for $\lambda$ defined in (4.1.4), is given by

$$\lambda_t(x,\xi) = (\det \Im B)^{-1/4} 2^{-d}(\pi\hbar)^{-3d/4}\alpha(t)\,\mathrm{e}^{-\frac{1}{4\hbar}\langle S(-t)(x,\xi),\mathrm{g}_B S(-t)(x,\xi)\rangle}\,\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}x\xi}\,\lambda_0(x,\xi).$$

It is holomorphic with respect to the polarization $\mathfrak{F}_{B,t}$, where we denote by $\alpha(t)$ the half-form contribution

$$\alpha(t) = \sqrt{\det\omega(\boldsymbol{\zeta},\overline{(\Phi_q^t)_*\boldsymbol{\zeta}})},$$

see also [Śni80]. We remark that because of the quadratic structure of $H_q$ the differentials of $\Phi^t$ and $\Phi_q^t$ along $z(t)$ coincide: They are given by $S(t)$. We do not calculate $\alpha(t)$ for the moment, which in principle is given by the differential equation

$$\dot\alpha(t) = \sum_{j=1}^{d}\alpha_{jj}(t)\alpha(t),$$

where the $\alpha_{jk}$ are determined by

$$(\mathscr{L}_{\mathrm{X}_{H_q}}\zeta_j)\circ\Phi_q^t = \sum_{k=1}^{d}\alpha_{jk}(t)(\Phi_q^t)_*\zeta_k + \beta_{jk}(t)(\Phi_q^t)_*\bar\zeta_k,$$

see [Tuy87a]. This fact also reveals the importance of using the description of coherent states as covariantly constant sections along the polarization $\mathfrak{F}_B$, since in general, we do not have such a simple description for the half-form pairing. Furthermore, using the property that $S(-t)^{\mathrm{T}}\mathrm{g}_B S(-t) = \mathrm{g}_{B(t)}$, see Section A.5, we obtain

$$\lambda_t(x,\xi) = (\det \Im B)^{-1/4} 2^{-d}(\pi\hbar)^{-3d/4}\alpha(t)\,\mathrm{e}^{-\frac{1}{4\hbar}\langle(x,\xi),g_{B(t)}(x,\xi)\rangle}\,\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}x\xi}\,. \tag{4.1.7}$$

Now the polarization $(\Phi_q^t)_*\mathfrak{F}_B = \Phi_*^t\mathfrak{F}_B = \mathfrak{F}_{B(t)}$ is again transversal to the vertical polarization that corresponds to the Schrödinger representation. In analogy to the previous section we can therefore calculate the image of $\lambda_t$ under the generalized Bargmann transform to obtain

$$\varphi(t) := (\pi\hbar)^{-d/4}(\det \Im B)^{1/4}\alpha(t)\exp\left(\frac{\mathrm{i}}{2\hbar}\langle x, B(t)x\rangle\right). \tag{4.1.8}$$

We claim that $\alpha(t)$ is given by

$$\alpha(t) = \mathrm{m}(S(t),B),$$

where $\mathrm{m}(S(t),B)$ is the multiplier defined in (A.5.6). This is an immediate consequence of a direct calculation of the propagation of coherent states:

**Proposition 4.1.3.** *Let $H_Q(x,\xi,t)$ be the Taylor expansion of $H$ about the trajectory $z(t)$*
$H_Q(x,\xi,t) = H(z(t)) + H_x(z(t))(x-q(t)) + H_\xi(z(t))(\xi - p(t)) + H_q(x-q(t), \xi - p(t), t)$.
*Then the solution of the Cauchy problem*

$$\mathrm{i}\,\hbar \frac{\partial}{\partial t}\psi(t,x) = \mathcal{H}_Q(t)\psi(t,x)$$

*with initial condition*

$$\psi(0,x) = (\pi\hbar)^{-d/4}(\det \Im B)^{1/4}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(p(x-q)+\frac{1}{2}\langle x-q, B(x-q)\rangle\right)}$$

*is given by the translation of $\varphi(t)$ defined in equation (4.1.8), i.e.*

$$\psi(t,x) = (\pi\hbar)^{-d/4}(\det \Im B(t))^{d/4}\,\mathrm{e}^{\mathrm{i}\left(\frac{R(t)}{\hbar}+\frac{\pi}{2}\sigma(t)\right)}\,\varphi^{B(t)}_{(q(t),p(t))},$$

*where*

$$\varphi^{B(t)}_{(q(t),p(t))} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(t)(x-q(t))}\,\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\langle x-q(t), B(t)(x-q(t))\rangle}\,.$$

*and the principal function is*

$$R(t) = \int_0^t p(t')q(t') - H(q(t'), p(t'))\,\mathrm{d}t'. \tag{4.1.9}$$

*Furthermore, the Maslov phase $\mathrm{e}^{\mathrm{i}\,\frac{\pi}{2}\sigma(t)}$ is determined by*

$$\mathrm{m}(S(t), B)(\det \Im B)^{1/4} = (\det \Im B(t))^{1/4}\,\mathrm{e}^{\mathrm{i}\,\frac{\pi}{2}\sigma(t)},$$

*and $B(t) = S(t)[B]$ is given according to the action of the symplectic group on the Siegel upper half-plane.*

*Proof.* See the techniques used in [Sch01] and [BG04b] and use an ansatz of the form

$$\psi(t) = (\pi\hbar)^{-d/4}\gamma(t)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\vartheta(t)}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(p(t)(x-q(t))+\frac{1}{2}\langle x-q(t), B(t)(x-q(t))\rangle\right)}$$

which yields a set of equations

$$\begin{aligned} \dot{\vartheta} &= \dot{q}p - H \\ -\dot{p} + B\dot{q} &= H_x + BH_\xi \\ -\dot{B} &= H_{xx} + H_{x\xi}B + BH_{\xi x} + BH_{\xi\xi}B \\ \frac{\dot{\gamma}}{\gamma} &= -\frac{1}{2}\,\mathrm{tr}\,(H_{\xi x} + H_{\xi\xi}B) \end{aligned}$$

so that the identification $\vartheta = R$ immediately yields equation (4.1.9). The next two equations are related with the action of the linearized flow on the Siegel upper half space and indeed are fulfilled by $B(t) = S(t)[B]$, see Section A.5. The last equation requires the multiplier $\mathrm{m}(S(t), B)$, which is also considered in A.5. □

If we compare the time evolutions of the coherent states corresponding to $H_q$ and $H_Q$, we see that the quadratic term in $H_Q$, which is given by $H_q$, determines the shape of the coherent state's Gaussian part. This is due to the fact, that the linearized flow is determined by $H_q$. So, apart from the translation about $z(t)$ and corresponding action, the function (4.1.8) is the same state as $\psi(t,x)$ given in the above Proposition.

## 4.2 Coherent states for particles with internal degrees of freedom and their time evolution

So far we have been concerned exclusively with coherent states that arose as holomorphic sections of a pre-quantum line bundle over the translational phase space $\mathrm{T}^*\mathbb{R}^d$. If in addition to the translational degrees of freedom a particle has an intrinsic structure described by a compact semi-simple Lie group $G$, the phase space for these degrees of freedom is given by a coadjoint orbit $\mathcal{O}$ corresponding to the action of $G$ on the dual of its Lie algebra $\mathfrak{g}$. We consider the phase space

$$X := \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}$$

and make it into a symplectic manifold by equipping it with the symplectic structure

$$\omega := \pi_1^*\omega_{\mathrm{T}^*\mathbb{R}^d} + \pi_2^*\omega_{\mathcal{O}},$$

where $\pi_1$, $\pi_2$ denote the cartesian projections of $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}$ onto the first and second factor, respectively. Furthermore, $\omega_{\mathrm{T}^*\mathbb{R}^d}$ is the canonical symplectic structure on $\mathrm{T}^*\mathbb{R}^d$ and $\omega_{\mathcal{O}}$ the Lie-Kirillov symplectic structure on the coadjoint orbit, see Appendix B. We remark that $(M, \omega)$ as defined above is precisely the phase space of particle in an external Yang-Mills field, see [GS82, GS78, Mon84, Wei77a] and also Appendix B. According to the functorial properties that we require for a quantization, see Chapter 2, we are looking for a quantization of $(X, \omega)$ such that the resulting Hilbert space $\mathscr{H}_X$ is given as the tensor product of the representations spaces $\mathscr{H}_{\mathrm{T}^*\mathbb{R}^d}$ and $\mathscr{H}_{\mathcal{O}}$. From the theory of representations of compact Lie groups, we know that the representations of $G$ are in one-to-one correspondence with integral coadjoint orbits. Therefore suppose that $\mathcal{O}_\lambda$ quantizes to a unitary irreducible representation $\pi_\lambda : G \to V$[6], where $\lambda$ denotes the corresponding maximal weight. Furthermore, from Section 3 we know that the coherent state map $G \ni g \mapsto \pi(g)\psi_\lambda$ defines a holomorphic section of the pre-quantum line bundle $L_{\mathcal{O}_\lambda} \to \mathcal{O}_\lambda$ with respect to the invariant polarization defined by the distribution generated by the algebra $\mathfrak{b}$ defined in Section 3.6. Since the polarization is invariant under translations on $\mathcal{O}_\lambda$, which are generated by the coadjoint action, and the moment map of the coadjoint action is simply given by the embedding $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, the polarization is invariant under the flow generated by the lower symbol $\sigma_{\mathrm{l}}[\,\mathrm{d}\pi_\lambda(\mathfrak{c})](\eta)$ for any representation operator $\mathrm{d}\pi_\lambda(\mathfrak{c})$ of $x \in \mathfrak{g}$. Therefore, the representation theory of $G$ together with the notion of coherent states provides us with suitable quantization data $(L_{\mathcal{O}_\lambda}, \mathrm{H}_{\mathcal{O}_\lambda}, \nabla_{\mathcal{O}_\lambda})$ together with a polarization $\mathfrak{F}_{\mathcal{O}_\lambda}$. In order to obtain a quantization of $M$ we want to use this data, together with the quantization scheme described in Section 4.1.

To this end we define a pre-quantizing line bundle $L \to X$ as follows: let $L_{\mathrm{T}^*\mathbb{R}^d}$ and $L_{\mathcal{O}_\lambda}$ be the pre-quantization line bundles over $\mathrm{T}^*\mathbb{R}^d$ and $\mathcal{O}_\lambda$, respectively. Under the cartesian projections $\pi_1$ and $\pi_2$ we can define the pull-back bundles

$$L_1 := \pi_1^* L_{\mathrm{T}^*\mathbb{R}^d} := \{(x, \xi, z; x', \xi', \eta) \in L_{\mathrm{T}^*\mathbb{R}^d} \times \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda;\ (x, \xi) = (x', \xi')\}$$

[6] Actually, in our applications we start from a given representation.

and

$$L_2 := \pi_2^* L_{\mathcal{O}_\lambda} := \{(\eta, z; x, \xi, \eta') \in L_{\mathcal{O}_\lambda} \times \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda;\ \eta = \eta'\}$$

which are then line bundles over $M = \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$.

**Proposition 4.2.1.** *The product bundle $L := L_1 \otimes L_2 \to (\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda)$ is a pre-quantizing line bundle, whose connection is defined by the product connection*

$$\nabla_L := \nabla_{L_1} \otimes \mathrm{id}_{L_2} + \mathrm{id}_{L_1} \otimes \nabla_{L_2}, \tag{4.2.1}$$

*where $\nabla_{L_j}$, $j = 1, 2$, is induced by the connections on $L_{\mathrm{T}^*\mathbb{R}^d}$ and $L_{\mathcal{O}_\lambda}$, respectively, under the pull-back with respect to the cartesian projections. In particular, the curvature of $\nabla$ is given by*

$$\mathrm{curv}(\nabla) = \pi_1^* \,\mathrm{curv}(\nabla_{\mathrm{T}^*\mathbb{R}^d}) + \pi_2^* \,\mathrm{curv}(\nabla_{\mathcal{O}_\lambda}) = \mathrm{i}\, \pi_1^* \omega_{\mathrm{T}^*\mathbb{R}^d} + \mathrm{i}\, \pi_2^* \omega_{\mathcal{O}_\eta} = \mathrm{i}\, \omega.$$

*Proof.* It is obvious that the connections on the two line bundles induce connections on their corresponding pull-back bundles. Because of the naturality axiom of Chern classes (see e.g. [KN69]) it follows that the curvatures of the pull-back connections are given by the pull-back of the respective curvatures. That the curvature behaves additively under taking the tensor product is an immediate consequence of the definition of the connection or, again, the functorial properties of characteristic classes [Hus75]. Of course, the bundle $L$ inherits a hermitian structure from the hermitian structures of its two factors, and this hermitian structure is compatible with $\nabla_L$. □

As already mentioned above we want to construct sections of $L$ that are covariantly constant with respect to a certain polarization, which we now define: since the tangent bundle

$$\mathrm{T}M = \mathrm{T}(\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda) = \mathrm{T}(\mathrm{T}^*\mathbb{R}^d) \times \mathrm{T}\mathcal{O}_\lambda$$

is a product, the polarization is also a product. We use the product of the polarizations $\mathfrak{F}_B \subset \mathrm{T}(\mathrm{T}^*\mathbb{R}^d)^{\mathbb{C}}$ used in Section 4.1 and the invariant polarization $\mathfrak{F}_{\mathfrak{b}} \subset \mathrm{T}\mathcal{O}_\lambda^{\mathbb{C}}$ which is generated by the Borel subalgebra $\mathfrak{b} = \mathfrak{b}(\lambda)$ corresponding to the maximal weight $\lambda$, see Section 3.6:

$$\mathfrak{F} := \mathfrak{F}_B \times \mathfrak{F}_{\mathfrak{b}}.$$

Now if $\sigma_1$ is a section of $L_{\mathrm{T}^*\mathbb{R}^d} \to \mathrm{T}^*\mathbb{R}^d$ and $\sigma_2$ is a section of $L_{\mathcal{O}_\lambda} \to \mathcal{O}_\lambda$ both can be pulled back to sections of the pull-back bundles, then

$$\sigma := \pi_1^* \sigma_1 \otimes \pi_2^* \sigma_2$$

is a section of $L$. Furthermore, if $\sigma_1$ is covariantly constant along $\mathfrak{F}_B$ and $\sigma_2$ is covariantly constant along $\mathfrak{F}_{\mathfrak{b}}$, then $\sigma$ is covariantly constant along $\mathfrak{F}$. Therefore we have

**Proposition 4.2.2.** *Let $\sigma_1$ be the section of $L_{\mathrm{T}^*\mathbb{R}^d}$ defined in equation (4.1.4) and let $\phi_\eta$ be a section of $L_{\mathcal{O}_\lambda}$ defined by the coherent state map as $\phi_\eta := \pi(g_\eta)\psi_\lambda$, where $g \in G$ is such that $\mathrm{Ad}^*_{g_\eta} \lambda = \eta$. Then*

$$s := \sigma_1 \otimes \sigma_2 = \sigma_1 \otimes \phi_\eta$$

*is a section of $L \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. In particular, it is covariantly constant along $\mathfrak{F}$ with respect to the connection defined in (4.2.1).*

The Hilbert space $\mathscr{H}$ for particles with internal degrees of freedom is given as the tensor product

$$\mathscr{H} := \mathscr{H}_B \otimes \mathscr{H}_{\mathcal{O}_\lambda},$$

and by the reasoning of Section 4.1 we have a pairing between $\mathscr{H}_B \otimes \mathscr{H}_{\mathcal{O}_\lambda}$ and $\mathscr{H}_{\mathrm{ver}} \otimes \mathscr{H}_{\mathcal{O}_\lambda}$, where the pairing for the second factor is simply given by the inner product in $\mathscr{H}_{\mathcal{O}_\lambda}$. Under this paring the Schrödinger representation of the section $s$ of Proposition 4.2.2 is given by

$$\varphi^B_{(0,0)} \otimes \phi_\eta$$

where $\varphi^B_{(q,p)}$ has been defined in equation (4.1.5). Of course, under shifts generated by linear Hamiltonians on $\mathrm{T}^*\mathbb{R}^d$ the above representation maps to

$$\varphi^B_{(q,p)} \otimes \phi_\eta.$$

### 4.2.1 Time evolution

We now turn to the time evolution of the covariantly constant section of the product line bundle described above. Let $H : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathbb{R}$ be a Hamiltonian function on the phase space whose flow we denote by $\Phi^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. In the following, we will always assume that this Hamiltonian takes the special form

$$H(x,\xi,\eta) = H_0(x,\xi) + \hbar k \eta(\mathfrak{c}(x,\xi)), \tag{4.2.2}$$

where $\eta(\mathfrak{c}(x,\xi)) = \frac{1}{k}\,\mathrm{symb}_{\mathrm{l}}[\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}(x,\xi))]$ is given as the lower symbol of the representation of a $\mathfrak{g}$-valued function $\mathfrak{c} : \mathrm{T}^*\mathbb{R}^d \to \mathfrak{g}$. We remark that we are again working with tensor powers $L^{\otimes k}_{\mathcal{O}_\lambda}$ of the pre-quantum line bundle over the coadjoint orbit $\mathcal{O}_\lambda$. As a consequence, the maximal weight $\lambda$ is multiplied by $k$ such that $k\lambda$ is the maximal weight corresponding to the representation $\pi_{k\lambda}$. In the next Chapter one of the two scenarios considered will have the property that the product $\hbar k := K$ is fixed. Therefore, the above Hamiltonian function may equivalently be written as

$$H(x,\xi,\eta) = H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi)). \tag{4.2.3}$$

Furthermore, concerning the classical dynamics, we remark that taking tensor powers of the pre-quantum line bundle $L^{\otimes k}_{\mathcal{O}_\lambda}$ also has the effect that the symplectic form on $\mathcal{O}_{k\lambda}$ is multiplied by a factor $k$. Now the lower symbol of $\mathrm{d}\pi_{k\lambda}(\mathfrak{c}(x,\xi))(k\eta)$ coincides with the moment map corresponding to the coadjoint action,

$$\mathrm{ad}^*_{\mathfrak{c}(x,\xi)}(k\eta) = k\,\mathrm{ad}^*_{\mathfrak{c}(x,\xi)}\,\eta.$$

The occuring homogeneity in $k$ can be used to view the dynamics to be given still on $\mathcal{O}_\lambda$ instead of $\mathcal{O}_{k\lambda}$. Now let

$$\Phi^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$$

be the Hamiltonian flow generated by $H$ and let us denote $\Phi^t(q,p,\eta) = (z(t),\eta(t))$, where $\pi_1\Phi^t(q,p,\eta) = z(t) = (q(t),p(t))$ denotes the projection of $\Phi^t$ to the first factor; this of course has to fulfill

$$\dot{q}(t) = \frac{\partial H}{\partial \xi}(q(t),p(t),\eta(t)), \quad \dot{p}(t) = \frac{\partial H}{\partial x}(q(t),p(t),\eta(t)).$$

The projection of the flow onto the second factor is determined by

$$\dot{\eta}(t) = \mathrm{ad}^*_{\mathfrak{c}(q(t),p(t))}\,\eta(t).$$

As in the case of the time evolution of coherent states for the Heisenberg group, in a first step we consider the time evolution generated by a quadratic Hamiltonian whose coefficient corresponds to the quadratic term in the Taylor expansion of the Hamiltonian $H$ about the integral curve $(z(t),\eta(t))$. However, since the second term in the Hamiltonian (4.2.3) is linear in the coordinates we do not have a corresponding quadratic term for the internal degrees of freedom. Let us therefore define

$$H_q(x,\xi,t) := \frac{1}{2}\langle (x,\xi), H_0''(z(t))(x,\xi)\rangle + \frac{1}{2}K\langle (x,\xi), \eta(t)(\mathfrak{c}(z(t)))(x,\xi)\rangle \tag{4.2.4}$$

which can be seen as a time-dependent Hamiltonian function on the translational phase space $\mathrm{T}^*\mathbb{R}^d$. It therefore describes a trivial dynamics for the intrinsic degrees of freedom. As in Section 4.1 the proper dynamics for the internal degrees of freedom will be generated by a shift, i.e. the time evolution of the sections under a linear Hamiltonian.

In order to deal with the time evolution generated by $H_q$ we proceed as in Section 4.1. Since the flow is defined on $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ such that it acts trivially on $\mathcal{O}_\lambda$ we obtain

**Lemma 4.2.3.** *The lift of the flow $\Phi_q^t$ generated by $H_q$ to a connection preserving diffeomorphism of the quantizing line bundle $L \otimes \sqrt{\Lambda^n}\mathfrak{F} \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$*[7] *maps the section $s = \sigma_1 \otimes \phi_\eta$ defined in Proposition 4.2.2 to the section*

$$s_q(t) := \lambda_t(x,\xi) \otimes \phi_\eta,$$

*Here $\lambda_t$ is as in (4.1.7), but now the Hamiltonian (4.1.6) is replaced by (4.2.4) that is used to define $B(t)$ and $\alpha(t)$.*

As an immediate consequence we obtain

**Corollary 4.2.4.** *The Schrödinger representation of the section $s_q(t)$ described in the above Lemma is given by*

$$\varphi^{B(t)}_{(0,0)} \otimes \phi_\eta,$$

*where $B(t)$ arises from $B$ under the action of the differential of $\Phi_q^t$ on the Siegel upper half space $\Sigma_d$.*

[7] where $n = \dim \mathfrak{F} = d + \frac{1}{2}\dim \mathcal{O}_\lambda$.

Now we want to obtain the time evolution under the Hamiltonian $H_Q$ that arises as the Taylor expansion of $H$ about the trajectory $(z(t), \eta(t))$ up to second order in $(x, \xi)$ and up to first order in the internal degrees of freedom, i.e.

$$\begin{aligned} H_Q(x,\xi,\eta,t) &= \sum_{|\nu|=0}^{2} \frac{1}{\nu!} H_0^{(\nu)}(z(t))(w-z(t))^\nu \\ &\quad + \sum_{|\nu|=0}^{2} \frac{1}{\nu!} \eta(t)(\mathfrak{c}^{(\nu)}(z(t)))(w-z(t))^\nu + (\eta-\eta(t))(\mathfrak{c}(z(t))) \\ &= \sum_{|\nu|=0}^{2} \frac{1}{\nu!} H_0^{(\nu)}(z(t))(w-z(t))^\nu \\ &\quad + \sum_{|\nu|=1}^{2} \frac{1}{\nu!} \eta(t)(\mathfrak{c}^{(\nu)}(z(t)))(w-z(t))^\nu + \eta(\mathfrak{c}(z(t))). \end{aligned}$$

Again using the fact that the lower symbol $\eta(\mathfrak{c}(x,\xi)) = \sigma_{\mathrm{l}}[\mathrm{d}\pi_\lambda(\mathfrak{c}(x,\xi))](\eta)$ is the same as the moment map for the coadjoint action of $\mathfrak{c}(x,\xi)$ on the coadjoint orbit $\mathcal{O}_\lambda$ we see that $\eta(\mathfrak{c})$ quantizes to $\mathrm{d}\pi_\lambda(\mathfrak{c})$, seen as operator acting on the pre-quantum line bundle $L_{\mathcal{O}_\lambda} \to \mathcal{O}_\lambda$ defined in Section 3.6 under the geometrical quantization procedure. Therefore the coherent states follow exactly the time evolution generated by $\eta(\mathfrak{c}(z(t)))$ seen as a Hamiltonian function on $\mathcal{O}_\lambda$ if a suitable action is taken into account. This amounts to the lift of the time evolution on $\mathcal{O}_\lambda$ to a connection preserving diffeomorphism. According to Section 3.6.1 we know that

$$\pi(g(t))\phi_\eta = \phi_{\eta(t)}\, \mathrm{e}^{-\mathrm{i}\,k\int_0^t \theta_{\mathcal{O}_\lambda}(\mathrm{X}_{\eta(t')(\mathfrak{c}(z(t')))}) - \eta(t')(\mathfrak{c}(z(t')))\,\mathrm{d}t'}$$

where $\eta(t)$ is the integral curve on $\mathcal{O}_\lambda$ corresponding to the vector field generated by the Hamiltonian $\eta(\mathfrak{c}(z(t)))$. Here $g(t)$ denotes a lift of $\eta(t)$, seen as an orbit on $G/G_\lambda \simeq \mathcal{O}_\lambda$ to a curve in $G$ such that

$$\dot{g}(t) = \mathrm{i}\,\mathfrak{c}(z(t))g(t), \quad g(0) = 1.$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\eta(t)} = \mathrm{d}\pi_{k\lambda}(\mathfrak{c}(z(t)))\phi_{\eta(t)} + \mathrm{i}\left(\theta_{\mathcal{O}_\lambda}(\mathrm{X}_{\eta(t)(\mathfrak{c}(z(t)))}) - \eta(t)\right)\phi_{\eta(t)}, \tag{4.2.5}$$

and it follows

**Proposition 4.2.5.** *The solution of the Cauchy problem*

$$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\psi_Q(t) = \mathcal{H}_Q(t)\psi_Q(t), \quad \psi_Q(0) = \varphi^B_{(q,p)} \otimes \phi_\eta,$$

*with*

$$\mathcal{H}_Q(t) = \mathrm{op}^W[H_Q - \eta(\mathfrak{c}(z(t)))] + \hbar\,\mathrm{d}\pi_\lambda(\mathfrak{c}(z(t))) \tag{4.2.6}$$

*is given by*

$$\psi_Q(t) = \mathrm{e}^{\mathrm{i}\left(\frac{R}{\hbar} + \frac{\pi}{2}\sigma(t)\right)} \varphi^{B(t)}_{(q(t),p(t))} \otimes \phi_{\eta(t)}.$$

*Here* $(q(t), p(t), \eta(t))$ *is the solution curve of Hamilton's equations on* $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ *generated by*

$$H(x,\xi,\eta) = H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi)),$$

*and the phase is determined by*

$$R(t) = \int_0^t p(t')\dot{q}(t') - H_0(q(t'),p(t')) + \imath_{\mathrm{X}_{\eta(\mathfrak{c}(q(t'),p(t'))}}\theta_{\mathcal{O}_\lambda} - \eta(t')(\mathfrak{c}(q(t'),p(t')))\,\mathrm{d}t'.$$

*Proof.* We can closely follow the proof of Proposition 4.1.3 and employ the Ansatz for $\psi(t)$ defined there. By using relation (4.2.5) we end up with the equations

$$\begin{aligned} \dot{\vartheta} &= \dot{q} - H_0 + K\dot{\varrho} \\ -\dot{p} + B\dot{q} &= H'_x + BH'_\xi \\ -\dot{B} &= H''_{xx} + H''_{\xi x}B + BH''_{\xi x} + BH''_{\xi\xi}B \\ \frac{\dot{\gamma}}{\gamma} &= -\frac{1}{2}\operatorname{tr}\left(H''_{\xi x} + H_{\xi\xi}B\right), \end{aligned}$$

where $H$ and $H_0$ were defined in (4.2.2) and we have abbreviated

$$\rho(t) := \theta_{\mathcal{O}_\lambda}(\mathrm{X}_{\eta(t)(\mathfrak{c}(z(t)))}) - \eta(t)(\mathfrak{c}(z(t))).$$

The only difference between the present situation and the one occuring in Proposition 4.1.3 is that there occurs an additional action term $\varrho(t)$, which already has been taken into account in equation 4.2.5, and the fact that now the modified Hamiltonian $H$ generates dynamics. Nevertheless, the above equations can be solved analogously to the proof of Proposition 4.1.3, since the projection of the linearized flow corresponding to $H$ about the trajectory $z(t)$ can be viewed as generated by the time-dependent Hamiltonian $\tilde{H}(x,\xi;t) = H_0(x,\xi) + \eta(t)(\mathfrak{c}(z(t)))$. □

# Chapter 5

# Quantum dynamics of coherent states and its semiclassical approximation

In this Chapter we want to examine the time evolution for particles with intrinsic degrees of freedom more closely. We will also provide a semiclassical approximation for the quantum time evolution of coherent states generated by more general than quadratic Hamiltonians. This scheme will be performed in two semiclassical scenarios, depending on the relationship between the two semiclassical parameters which we have to identify first. For the translational degrees of freedom a semiclassical limit is given by $\hbar \to 0$. For the internal degrees of freedom, described by an irreducible representation $(\pi_{k\lambda}, V_{k\lambda}),\ k \in \mathbb{N}$, the semiclassical limit is performed in terms of $k \to \infty$. Then the two different scenarios can be described as follows:

1. Only the translational degrees of freedom are considered to be semiclassical while the intrinsic degrees of freedom are still being treated on a quantum level. That means that we are concerned with the limit $\hbar \to 0$ and leave $k$ fixed. In this scenario we will suppress the index $k$ and refer to the irreducible representation describing the internal degrees of freedom as $(\pi_\lambda, V_\lambda)$. Yet, we will map the internal degrees of freedom to a classical model, whose underlying phase space is given by a coadjoint orbit.

2. Both types of degrees of freedom are supposed to become semiclassical in the following sense: as $\hbar \to 0$ the product $\hbar k =: K$ is kept fixed, i.e. the semiclassical limit for the internal degrees of freedom is described by $k = \frac{K}{\hbar} \to \infty$. Thus, the dimension of the representation space corresponding to the intrinsic degrees of freedom tends to infinity.

Let us turn to the first scenario:

## 5.1 Semiclassics in $\hbar$

In the present situation $\hbar$ is assumed to be the only semiclassical parameter and we consider Hamiltonians acting on sections of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ according to

$$(\mathcal{H}\psi)(x) = \left(\mathrm{op}^{\mathrm{W}}[H]\psi\right)(x) = \frac{1}{(2\pi\hbar)^d} \iint_{\mathrm{T}^*\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi}\, H\left(\frac{x+y}{2}, \xi\right) \psi(y)\, \mathrm{d}y\, \mathrm{d}\xi$$

with symbol

$$H(x,\xi) = H_0(x,\xi) + \hbar\, \mathrm{d}\pi_\lambda(\mathfrak{c}(x,\xi)). \tag{5.1.1}$$

Here $H_0(x,\xi)$ denotes a scalar valued function on the translational phase space $\mathrm{T}^*\mathbb{R}^d$ and $\mathfrak{c} : \mathrm{T}^*\mathbb{R}^d \to \mathfrak{g}$ takes values in the Lie algebra $\mathfrak{g}$ corresponding to the Lie group $G$ describing the internal degrees of freedom. A comparison of the lower symbol of (5.1.1) and the one given in equation (4.2.3) shows that, due to the fact that $k$ is fixed, the leading order part is given by $H_0$. Let us assume that $H \in \mathrm{S}(m)$, see Definition 2.2.1. The time evolution generated by the Hamiltonian $\mathcal{H}$ corresponding to (5.1.1) is given by the (strongly-continuous) one-parameter group $\mathcal{U}(t) = \exp(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t)$ which is unitary provided that $\mathcal{H}$ is essentially selfadjoint. This is guaranteed by the condition that $H + \mathrm{i}$ is elliptic in the sense

$$\|(H(x,\xi) + \mathrm{i})^{-1}\| \leq c m(x,\xi) \tag{5.1.2}$$

for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$, see hypothesis (H2) on page 48. In the following we assume this to be the case and do not distinguish between $\mathcal{H}$ and its self-adjoint extension.

In a first step we want to construct an approximate Hamiltonian that propagates a coherent state exactly. Regarding the translational part we use the fact that the time evolution generated by a quadratic Hamiltonian fulfills this requirement, see Proposition 4.1.3. Furthermore, for the internal degrees of freedom we have seen that a coherent state $\phi_\eta$ is propagated under a Hamiltonian corresponding to a representation operator by the pre-quantization lift of the one-parameter subgroup defined by

$$\dot{g}(t) = \mathrm{i}\, \mathfrak{c} g(t), \;\; g(0) = 1,$$

see equation (3.6.10). This equation is a version of equation (2.3.32) for the present setting, where the Hamiltonian's principal symbol is scalar. This leads us to consider a Taylor expansion of the symbol (5.1.1) about some smooth curve $z(t) = (x(t), \xi(t))$ in translational phase space $\mathrm{T}^*\mathbb{R}^d$

$$H_Q(t,w) = \sum_{|\nu|=0}^{2} H_0^{(\nu)}(z(t))(w - z(t))^\nu + \hbar\, \mathrm{d}\pi_\lambda(\mathfrak{c}(z(t))),$$

up to different orders in the principal and in the subprincipal symbol, where we have used the convenient abbreviation $w = (x,\xi) \in \mathrm{T}^*\mathbb{R}^d$. Thus we have an operator which is quadratic in position and momentum and linear in the representation operator. A glance at Proposition 4.1.3 and the results of Section 3.6.1 immediately suggest that the following is true:

**Proposition 5.1.1.** *The solution of the Schrödinger equation*

$$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\psi_Q(t) = \mathcal{H}_Q(t)\psi_Q(t) \quad \textit{with} \quad \psi_Q(0) = \varphi^B_{(q,p)} \otimes \phi_\eta$$

*is a time-dependent coherent state*

$$\psi_Q(t) = \mathrm{e}^{\mathrm{i}\left(\frac{R(t)}{\hbar}+\rho(t)+\frac{\pi}{2}\sigma(t)\right)}\,\varphi^{B(t)}_{(q(t),p(t))} \otimes \phi_{\eta(t)}, \tag{5.1.3}$$

*where $(q(t'),p(t'))$ is the integral curve to the Hamiltonian vector field generated by $H_0$. Moreover,*

$$R(t) = \int_0^t p(t')\dot{q}(t') - H_0(q(t'),p(t'))\,\mathrm{d}t',$$

*is the action corresponding to the pre-quantization lift of the flow $(q(t'),p(t'))$ to the line bundle $L_{\mathrm{T}^*\mathbb{R}^d} \to \mathrm{T}^*\mathbb{R}^d$, and*

$$\rho(t) := \int_0^t \theta_{\mathcal{O}_\lambda}(X_f)(\eta(t')) - \eta(t')(\mathfrak{c}(z(t')))\,\mathrm{d}t'$$

*is the pre-quantization lift of the flow generated by the time-dependent Hamiltonian $f(\eta;t) = \eta(\mathfrak{c}(t))$ on the coadjoint orbit $\mathcal{O}_\lambda$. Furthermore, $B(t)$ is related to $B$ according to the action of the linearization of the flow $\Phi^t(q,p) = (q(t),p(t))$ corresponding to $H_0$, see Appendix A.5.*

Let us compare the latter result with the one stated in Proposition 4.2.5: There the flow was generated by the Hamiltonian (4.2.3) on $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ and a genuine coupling between the translational motion and the dynamics of the intrinsic degrees of freedom occured. In the present situation, we have an hierarchy of dynamics: The translational degrees of freedom are propagated with respect to the flow $\Phi^t$ generated by $H_0$. The internal degrees of freedom are subject to $\eta \mapsto \mathrm{Ad}^*_{g(t,x,\xi)}\,\eta = \eta(t)$ where

$$\dot{g}(t,x,\xi) = \mathrm{i}\,\mathfrak{c}(\Phi^t(x,\xi))g(t,x,\xi).$$

Thus the translational dynamics drives the one of the internal degrees of freedom and we obtain a skew-product dynamics

$$(x,\xi,\eta) \mapsto (\Phi^t(x,\xi), \mathrm{Ad}^*_{g(t,x,\xi)}\,\eta),$$

analogously to equation (2.3.35).

Our aim now is to compare the time evolution generated by the original quantum Hamiltonian $\mathcal{H}$ with the one generated by the approximate Hamiltonian $\mathcal{H}_Q(t)$. For this we will follow the method devised in [CR97] for the case without internal degrees of freedom, whose presence requires some modifications that, however, are modest when the quantum number $k$ is fixed. But for the clarity of the presentation, and to prepare for the more involved situation to be dealt with in the second semiclassical scenario, we will now present the argument in some detail.

As stated above the Hamiltonian $\mathcal{H}$ generates a unitary and strongly continuous one-parameter group $\mathcal{U}(t,t_0)$, if its symbol satisfies the ellipticity condition (5.1.2). When considering the limit $\hbar \to 0$ and keeping $k$ fixed this requirement only need to be imposed on the principal symbol, i.e. we demand

$$|H_0(x,\xi) + \mathrm{i}\,| \geq cm(x,\xi). \tag{5.1.4}$$

Let now $\mathcal{U}_Q(t,t_0)$ be the corresponding unitary group generated by $\mathcal{H}_Q(t)$. Using Duhamel's principle, see e.g. [Tay96], we may then express the difference between these unitary operators as

$$\mathcal{U}(t,t_0) - \mathcal{U}_Q(t,t_0) = \frac{1}{\mathrm{i}\,\hbar}\int_{t_0}^{t} \mathcal{U}(t,t')\big(\mathcal{H} - \mathcal{H}_Q(t')\big)\,\mathcal{U}_Q(t',t_0)\ \mathrm{d}t'\ . \tag{5.1.5}$$

Since we are interested in the difference

$$\mathcal{U}(t,t_0)(\varphi^B_{(q,p)} \otimes \phi_\eta) - \psi_Q(t),$$

we have to consider the action of (5.1.5) on the initial state $\varphi^B_{(q,p)} \otimes \phi_\eta$ with $t_0 = 0$. This requires an estimate of

$$\|(\mathcal{H} - \mathcal{H}_Q(t'))\psi_Q(t')\|\ , \tag{5.1.6}$$

where $\psi_Q(t')$ is the time dependent coherent state (5.1.3). One can achieve this with the help of the following lemma, which is an immediate extension of a result given in [CR97].

**Lemma 5.1.2.** *Let $f,g \in \mathrm{S}(1)$ and $F : \mathrm{T}^*\mathbb{R}^d \to \mathrm{T}^*\mathbb{R}^d$ be a linear map with Hilbert-Schmidt norm $\|F\|_{\mathrm{HS}}$. Fix $\alpha,\beta \in \mathbb{N}^{2d}$ with $k := |\alpha| = |\beta| + 2 > 2$ and introduce the symbol*

$$A(w) := (Fw)^\alpha f(Fw) + \hbar\,(Fw)^\beta g(Fw)\ .$$

*Then for any real number $\kappa > 0$ there exist $C > 0$ and $N \in \mathbb{N}$ such that*

$$\|\operatorname{op}^W[A]\psi_\hbar\| \leq C\hbar^{k/2}\left(\|F\|^k_{\mathrm{HS}} \sup_{|\gamma|\leq k+N} |\partial^\gamma_w f(w)| + \|F\|^{k-2}_{\mathrm{HS}} \sup_{|\gamma|\leq k-2+N} |\partial^\gamma_w g(w)|\right) \tag{5.1.7}$$

*holds for any function $\psi_\hbar(x) = \hbar^{-d/4}\psi\big(x/\sqrt{\hbar}\big)$ with $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $0 < \hbar + \sqrt{\hbar}\|F\|_{\mathrm{HS}} < \kappa$.*

We intend to apply this lemma to the difference (5.1.6), with $f$ chosen to be the Taylor remainder of $H_0$ of order three and $g$ the Taylor remainder of $\mathrm{d}\pi_\lambda(\mathfrak{c})$ of order one. But first we replace (5.1.6) by

$$\|\mathcal{U}_Q(t',0)^*\big(\mathcal{H} - \mathcal{H}_Q(t')\big)\mathcal{U}_Q(t',0)\psi_Q(0)\| \tag{5.1.8}$$

and invoke an appropriate Egorov theorem. Since the Hamiltonian generating $\mathcal{U}_Q(t,0)$ has a symbol that is composed of a scalar and quadratic principal part as well as a matrix valued subprincipal part, one can combine the techniques used in [BG00, Sch01] and Section 2.3. This shows that

$$\mathcal{W}(t) := \mathcal{U}_Q(0,t)\big(\mathcal{H} - \mathcal{H}_Q(t)\big)\mathcal{U}_Q(t,0) \tag{5.1.9}$$

is a Weyl operator with symbol

$$W(t,w) = \pi_\lambda(g(t))^* \left(H - H_Q(t)\right)\left(z - S^{-1}(t)(w - z(t))\right) \pi_\lambda(g(t)) \ . \tag{5.1.10}$$

Since the principal part of the symbol $H - H_Q(t)$ is scalar it is not affected by the conjugation with $\pi(g(t,x,\xi))$. In the subprincipal term this conjugation results in the transformation $\mathfrak{c} \mapsto \mathrm{Ad}_{g(t)}\, \mathfrak{c}$ and therefore the part concerning the transport of the internal degrees of freedom in the Egorov relation (5.1.10) does not contribute to an estimate of (5.1.8) in an essential way.

If one now localizes the symbol (5.1.10) in $w$ with some smooth function that is compactly supported around $z(t)$, leading to an error of size $O(\hbar^\infty)$ when one applies $\mathcal{W}$ to a coherent state located at $z(t)$, one can proceed to use Lemma 5.1.2 as in [CR97]. This shows that there exists a constant $K > 0$ such that

$$\|(\mathcal{H} - \mathcal{H}_Q(t))\psi_Q(t)\| \le K\hbar^{3/2}\vartheta(t)^3\,\delta(t)^L \ , \tag{5.1.11}$$

where

$$\vartheta(t) := \max\Big\{1, \sup_{t'\in[0,t]} \|S_{0,z}(t')\|_{\mathrm{HS}}\Big\} \quad \text{and} \quad \delta(t) := \sup_{t'\in[0,t]} \big(1 + |z(t')|\big) \tag{5.1.12}$$

depend on the classical trajectory $z(t) = (q(t), p(t))$. The constant $L \ge N/2$ is related to $N$ appearing in the definition of an order function (2.2.2). We then obtain:

**Theorem 5.1.3.** *Let the Hamiltonian $\mathcal{H} \in \mathrm{OPS}(m)$ and the ellipticity condition (5.1.4) be fulfilled. Then the coherent state $\psi_Q(t)$ defined in (5.1.3) semiclassically approximates $\psi(t) = \mathcal{U}(t,0)\big(\varphi^B_{(q,p)} \otimes \phi_\eta\big)$ in the following sense,*

$$\|\psi(t) - \psi_Q(t)\| \le K\sqrt{\hbar}\, t\, \vartheta(t)^3 \ . \tag{5.1.13}$$

*The right-hand side vanishes in the combined limits $\hbar \to 0$ and $t \to \infty$ as long as $t \ll T_z(\hbar)$. The time scale $T_z(\hbar)$ depends on the linear stability of the trajectory $z(t)$. If the latter has a positive and finite maximal Lyapunov exponents $\lambda_{\max}(z)$, then $T_z(\hbar) = \frac{1}{6\lambda_{\max}(z)}|\log\hbar|$. In the case of a trajectory on a (non-degenerate) KAM-torus this time scale is $T_z(\hbar) = C\,\hbar^{-1/8}$.*

*Proof.* Conservation of energy, $H_0\big(z(t)\big) = E$, together with the ellipticity condition (5.1.4) implies that $\delta(t)$ is bounded from above by some constant depending on $E$. Thus the estimate (5.1.11) immediately yields (5.1.13) when used in (5.1.5).

If $z(t)$ is a trajectory with positive, but finite, maximal Lyapunov exponent the dominant behaviour as $t \to \infty$ comes from the term $\vartheta(t)^3$. This is due to the relation

$$\lambda_{\max}(z) = \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\|_{\mathrm{HS}},$$

see Section 5.4, which readily implies that $T_z(\hbar) = \frac{1}{6\lambda_{\max}}|\log\hbar|$. In Section 5.4 we also discuss a sufficient condition under which finite maximal Lyapunov exponents occur.

If $z(t)$ is a trajectory on a KAM-torus one can introduce local action-angle variable $(I,\alpha)$ in a neighbourhood of that torus such that in these coordinates the flow reads $I(t)=I$ and $\alpha(t)=\alpha+\omega(I)t$, see Section 5.5. One therefore finds

$$\|S(t)\|_{\mathrm{HS}}^2 = 2d + f(I)t^2$$

such that $\vartheta(t)\sim Ct$ as $t\to\infty$ which finally yields $T_z(\hbar)=C\hbar^{-1/8}$. In the maximally degenerate case, we have $f(I)=0$ and this changes to $T_z(\hbar)=C\hbar^{-1/2}$. □

In a next step we want to improve the semiclassical error in (5.1.13) to an arbitrary (half-integer) power of $\hbar$. This requires higher order approximations that may be achieved as in [CR97] by iterating Duhamel's principle (5.1.5), resulting in the Dyson expansion

$$\begin{aligned}\mathcal{U}(t,0)-\mathcal{U}_Q(t,0) = \sum_{j=1}^{N-1}(\mathrm{i}\,\hbar)^{-j}\int_0^t\dots\int_{t_{j-1}}^t \mathcal{U}_Q(t,0)\,\mathcal{W}(t_j)\dots\mathcal{W}(t_1)\;\mathrm{d}t_j\dots\mathrm{d}t_1\\ +\mathcal{R}_N(t;\hbar)\end{aligned} \tag{5.1.14}$$

with remainder term

$$\mathcal{R}_N(t;\hbar) = (\mathrm{i}\,\hbar)^{-N}\int_0^t\dots\int_{t_{N-1}}^t \mathcal{U}(t,t_N)\mathcal{U}_Q(t_N,0)\,\mathcal{W}(t_N)\dots\mathcal{W}(t_1)\;\mathrm{d}t_N\dots\mathrm{d}t_1\ . \tag{5.1.15}$$

In order to estimate the contribution of the remainder when (5.1.14) is applied to the initial coherent state $\psi(0)=\varphi^B_{(q,p)}\otimes\phi_\eta$ we use the argument leading to (5.1.11) repeatedly. This yields the estimate

$$\|\mathcal{R}_N(t;\hbar)\psi(0)\| \le K_N\,\hbar^{N/2}\,t^N\,\vartheta(t)^{3N}\,\delta(t)^{mN}\ . \tag{5.1.16}$$

We then replace the symbols of each difference $\mathcal{H}-\mathcal{H}_Q(t_k)$ appearing in the sum in (5.1.14) by their Taylor expansions,

$$\sum_{|\nu|=3}^{n_k}\frac{1}{\nu!}H_0^{(\nu)}\big(z(t_k)\big)\big(w-z(t_k)\big)^\nu+\hbar\sum_{|\nu|=1}^{n_k-2}\frac{1}{\nu!}\big(w-z(t_k)\big)^\nu\,\mathrm{d}\pi_\lambda(\mathfrak{c}^{(\nu)}(z(t_k)))+r_k(t_k,w)\ . \tag{5.1.17}$$

The integers $n_k$ are chosen sufficiently large such that, after quantization, the contribution of the remainders $r_k$ to an application of (5.1.14) to $\psi(0)$ yields terms that can be absorbed in the estimate (5.1.16). Similar to the case without internal degrees of freedom treated in [CR97] the quantization of the main terms in (5.1.17) produces matrix valued differential operators $\hat p_{kj}(t)=\mathrm{op}^W[p_{kj}(t)]$ with time dependent coefficients acting on the coherent state $\varphi^B_{(q,p)}\otimes\phi_\eta$. The symbols $p_{kj}(t)(x,\xi)$ are polynomials in $(x,\xi)$ of degree $\le k$. Lemma 5.1.2 finally leads to the following result:

**Theorem 5.1.4.** *Suppose that the quantum Hamiltonian $\mathcal{H}$ with symbol (5.1.1) satisfies the conditions specified in Theorem 5.1.3 and the ellipticity condition (5.1.4). Then for $t>0$*

*and any $N \in \mathbb{N}$ there exists a state $\psi_N(t) \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\dim V_\lambda}$, localized at $\big(q(t), p(t), \eta(t)\big)$, that approximates the full time evolution $\psi(t) = \mathcal{U}(t,0)\big(\varphi^B_{(q,p)} \otimes \phi_\eta\big)$ of a coherent state up to an error of order $\hbar^{N/2}$. More precisely,*

$$\|\psi(t) - \psi_N(t)\| \le C_N \sum_{j=1}^{N-1} \left(\frac{t}{\hbar}\right)^j (\sqrt{\hbar}\vartheta(t))^{2j+N} \ . \tag{5.1.18}$$

*The right-hand side vanishes in the combined limits $\hbar \to 0$ and $t \to \infty$ as long as $t \ll T_z(\hbar)$, where $T_z(\hbar)$ denotes the same time scale as in Theorem 5.1.3.*

*Furthermore, $\psi_N(t)$ arises from $\varphi^B_{(q,p)} \otimes \phi_\eta$ through the application of certain (time dependent) differential operators $\hat{p}_{kj}(t) = \mathrm{op}^W[p_{kj}(t)]$ of order $\le k$, followed by the time evolution generated by $\mathcal{H}_Q(t)$, according to*

$$\psi_N(t) = \psi_Q(t) + \sum_{(k,j)\in\Delta} \mathcal{U}_Q(t,0)\, \hat{p}_{kj}(t)\, \psi(0) \ .$$

*Here we have defined $\Delta := \{(k,j) \in \mathbb{N} \times \mathbb{N};\ 1 \le k - 2j \le N-1,\ k \ge 3j, 1 \le j \le N-1\}$.*

We remark that the (matrix-valued) differential operators $\hat{p}_{kj}(t)$ do not increase the frequency set of a semiclassical distribution such as the initial state $\varphi^B_{(q,p)} \otimes \phi_\eta$. This follows for the translational part from the respective statement without internal degrees of freedom [Rob87]. For the part concerning the internal degrees of freedom one concludes this from the fact that it is only acted upon by a matrix. Moreover, $\hat{U}_Q(t,0)$ propagates the frequency set by the flow $\Phi^t_0$ so that both $\psi_Q(t)$ and $\psi_N(t)$ are semiclassically localized at $\big(q(t), p(t), \eta(t)\big)$.

Note that we are using the definition of the frequency set according to [GS90, Cha80, DG75] and [Hel97, Mar02, Rob87, CdV] in the component-wise sense and not the refinement hereof that can be found in [Den82, Den92, Den93] and [Kra00, SV01, FV02].

## 5.2 Semiclassics in $\hbar$ and $k$

We now consider the second semiclassical scenario in which both semiclassical parameters, $\hbar$ and $k$, are used. For this purpose we still represent the Hamiltonian $\mathcal{H}$ as a matrix valued Weyl operator. This way $\hbar$ appears as before, whereas the second parameter $k \in \mathbb{N}$ controls the dimension of the representation space according to the Weyl dimension formula (3.6.11). As we will see the parameter $k$ enters relevant estimates through the expression $\hbar\, \mathrm{d}\pi_{k\lambda}(\mathfrak{c})$. To leading order this will produce factors $\hbar k$ and our desire to perform systematic semiclassical expansions therefore enforces us to keep the combination

$$K := \hbar k$$

fixed in the semiclassical limit. This means that from now on we consider $\hbar \to 0$ and $k \to \infty$ with $\hbar k = K$.

We already know that in this semiclassical limit the relevant classical dynamics is given by the flow generated by $H(x,\xi,\eta) = H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi))$, see Proposition 4.2.5. The main difference with the first scenario, considered in the previous section, is that there occurs a mutual influence of the dynamics of the two different types of degrees of freedom. While in the first scenario the dynamics for the internal degrees of freedom was driven by the translational dynamics by means of a skew product dynamics in the present, second scenario both degrees of freedom are treated on an equal level. This also implies that the action corresponding to the internal degrees of freedom is of the same order as the one of the translational degrees of freedom.

Furthermore, we know how to choose an approximate Hamiltonian which propagates an initial coherent state in the present scenario exactly, see Proposition 4.2.5. It remains, however, to estimate the difference between the full time evolution and the one generated by the approximate Hamiltonian $\mathcal{H}_Q$ of Proposition 4.2.5. This is the main task for this section. Since in this situation the term $\hbar\,\mathrm{d}\pi_{k\lambda}$ in the Weyl symbol (5.1.1) is of the same order as $H_0$ we will have to impose the ellipticity condition on the full symbol, i.e.

$$cm(x,\xi)^{-1} \geq \|H(x,\xi) + \mathrm{i}\,\| \geq \frac{\|(H(x,\xi)+\mathrm{i})\psi\|}{\|\psi\|}, \tag{5.2.1}$$

where in the middle $\|\cdot\|$ denotes the operator norm on $V_{k\lambda}$, that we identify with $\mathbb{C}^N$, $N = \dim V_{k\lambda}$, and on the right-hand side $\psi$ is any non-zero vector in $V_{k\lambda}$. We want to translate this condition to one to be imposed on the scalar symbol

$$H(x,\xi,\eta) = H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi)).$$

To this end we use

**Lemma 5.2.1.** *For any $\mathfrak{c} \in \mathfrak{g}$ and $\eta \in \mathcal{O}_\lambda$ and $N \in \mathbb{N}$ there exist differential operators $D_\eta^{(j)}$ of degree $2j$ on $C^\infty(\mathcal{O}_\lambda) \otimes \mathbb{C}^{\dim V_{k\lambda}}$ and constants $C_N > 0$ such that*

$$\left\| \mathrm{d}\pi_{k\lambda}(\mathfrak{c})\phi_\eta - \left(k + \frac{2\langle\lambda,\delta\rangle}{\langle\lambda,\lambda\rangle}\right) \Pi_{\alpha\in\Delta^+}\left(\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\rho\rangle} + \frac{1}{k}\right) \sum_{j=0}^{N} \frac{1}{k^j} D_\eta^{(j)}(\eta(\mathfrak{c})\phi_\eta) \right\| \leq \frac{C_{N+1}}{k^{N+1}}. \tag{5.2.2}$$

*The leading order in this asymptotic expansion is determined by the constant $D_\eta^{(0)} = 1$,*

$$\mathrm{d}\pi_{k\lambda}(\mathfrak{c})\phi_\eta = k\eta(\mathfrak{c})\phi_\eta\left(1 + O(k^{-1})\right).$$

*Proof.* We start by expressing a linear map $L$ on the representation space $V_{k\lambda}$ in terms of its upper symbol

$$L = \dim V_{k\lambda} \int_{\mathcal{O}_\lambda} \sigma_{\mathrm{u}}[L](\eta)\mathrm{P}_\eta\,\mathrm{d}\eta, \tag{5.2.3}$$

where $\mathrm{d}\eta$ denotes the normalized (invariant) volume form on $\mathcal{O}_\lambda$ and $\mathrm{P}_\eta$ stands for the projector onto the one-dimensional subspace in $V_{k\lambda}$ spanned by the coherent state vector $\phi_\eta$. In the present context the relevant linear maps are representation operators of Lie algebra elements $\mathfrak{c} \in \mathfrak{g}$ and their upper symbols are connected with the lower symbols according to

$$\begin{aligned}\sigma_{\mathrm{u}}[\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c})](\eta) &= \big(1 + 2\langle k\lambda, k\lambda\rangle^{-1}\langle k\lambda, \delta\rangle\big)\,\sigma_{\mathrm{l}}[\,\mathrm{d}\pi_{k\lambda}](\eta) = \left(1 + \frac{2\langle\lambda,\delta\rangle}{k\langle\lambda,\lambda\rangle}\right) k\eta(\mathfrak{c}) \\ &= (k+c)\eta(\mathfrak{c})\end{aligned} \tag{5.2.4}$$

where $\delta$ is the magic weight, i.e. the sum of the fundamental weights, see [Sim80] and Appendix C. Therefore an application of a representation operator to a coherent state reads

$$\mathrm{d}\pi_{k\lambda}(\mathfrak{c})\phi_\eta = \dim V_{k\lambda}(k+c)\int_{\mathcal{O}_\lambda} \eta'(\mathfrak{c})\langle\phi_{\eta'},\phi_\eta\rangle\phi_{\eta'}\,\mathrm{d}\eta'. \tag{5.2.5}$$

An asymptotic expansion of the above integral as $k \to \infty$ can be achieved with the method of steepest descent, which is a variant of the stationary phase method with a complex phase function and is described in detail in [Hör90b]. The first step consists of identifying the relevant phase factor, which in the present case is given by

$$\langle\phi_{\eta'},\phi_\eta\rangle = \mathrm{e}^{\mathrm{i}\,kS^\eta_\lambda(\eta')},$$

where

$$\Im S^\eta_\lambda(\eta') = -\log|\langle\phi_\eta,\phi_{\eta'}\rangle|\,,$$

which takes values in $[0,\infty)$ and assumes its unique absolute minimum 0 at $\eta' = \eta$. The real part of $S^\eta_\lambda(\eta')$ can be interpreted as the action of a curve joining $\eta$ and $\eta'$. Since we will not need an explicit expression for $S^\eta_\lambda(\eta')$ we do not further specify it for now. We only note that $\eta' = \eta$ is a unique, non-degenerate stationary point of the phase. Up to an error of size $O(\mathrm{e}^{-k})$ one can hence cut out a neighbourhood of $\eta$ in the integral (5.2.5), and allow for an error of that size but achieving that the argument is well-defined and finite such that we can apply the method of stationary phase, which now implies the existence of differential operators $D^{(j)}_\eta$ of order $2j$ on $C^\infty(\mathcal{O}_\lambda)\otimes\mathbb{C}^{\dim V_{k\lambda}}$ and constants $C_N > 0$ such that for any $N \in \mathbb{N}$ the expansion (5.2.2) holds. These differential operators carry an over all factor $k^{-\frac{1}{2}\dim\mathcal{O}_\lambda}$ which has to be compared with

$$\dim V_{k\lambda} = k^{\operatorname{card}\Delta^+_\alpha}\Pi_{\alpha\in\Delta^+}\left(\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\rho\rangle} + \frac{1}{k}\right),$$

where $\Delta^+_\alpha := \{\alpha \in \Delta^+;\ \langle\alpha,\lambda\rangle > 0\}$, which has the same leading order behaviour as $k \to \infty$, provided that $\frac{1}{2}\dim\mathcal{O}_\lambda = \operatorname{card}\Delta^+_\alpha$. This indeed is the case since

$$\dim\mathcal{O}_\lambda = \dim\mathfrak{g} - \dim\mathfrak{t} - 2\operatorname{card}\{\alpha\in\Delta^+;\ \langle\alpha,\lambda\rangle = 0\} = 2\operatorname{card}\Delta^+_\alpha.$$

In addition, the constant $D^{(0)}_\eta$ can be determined using

$$\mathrm{d}\pi_{k\lambda}(\mathfrak{c})\phi_\eta = k\eta(\mathfrak{c})\phi_\eta,$$

for $\mathfrak{c} \in g_\eta$, the isotropy algebra, which implies that $D^{(0)}_\eta = 1$ and concludes the proof. $\square$

This allows us to translate the ellipticity condition (5.2.1) to

$$|H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi)) + \mathrm{i}\,| \geq cm(x,\xi) \tag{5.2.6}$$

by using $\psi = (H(x,\xi)+\mathrm{i})^2\phi_\eta$ in (5.2.1).

Now consider the time evolution generated by the approximate Hamiltonian (4.2.6) which transports a coherent state to a state of the same form. This should be compared with the time evolution of the full Hamiltonian with Weyl symbol

$$H(x,\xi) = H_0(x,\xi) + \hbar\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}(x,\xi)). \tag{5.2.7}$$

This is done by methods similar to those of the previous section, however in the present case also explicit estimates for the internal degrees of freedom are needed. Again we base our investigation of the difference of the two types of quantum dynamics on the Duhamel principle, whose input is given by the difference $\mathcal{H} - \mathcal{H}_Q$, which is the Weyl quantization of

$$\begin{aligned}\Delta H(w,t) := \sum_{|\nu|=1}^{2} \frac{1}{\nu!}\left(\hbar\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}^{(\nu)}(z(t))) - K\eta(t)(\mathfrak{c}^{(\nu)}(z(t)))\right)(w-z(t))^\nu \\ + H_0^{(3)}(t,w) + \hbar\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}^{[3]}(t,w)),\end{aligned} \tag{5.2.8}$$

where $H_0^{[3]}$ and $\mathfrak{c}^{[3]}$ denote the Taylor remainders of order three. Introducing an operator $\mathcal{W}(t)$ as in (5.1.9) the same type of Egorov theorem applies, leading to the symbol

$$W(t,w) = \pi(g(t))^*\Delta H(z - S^{-1}(t)(w-z(t)),t)\pi(g(t)) \tag{5.2.9}$$

of W(t), where $S(t)$ here is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \mathrm{J}\tilde{H}''(z(t),t)S(t), \tag{5.2.10}$$

and $\tilde{H}(w,t) = H_0(w) + K\eta(t)(\mathfrak{c}(w))$. Furthermore, as in the previous section $g(t)$ is a one-parameter subgroup in $G$ determined by

$$\dot{g}(t) = \mathrm{i}\,\mathfrak{c}(z(t))g(t).$$

We remark that $z(t)$ being the projection of the flow $\Phi^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ generated by $H(x,\xi,\eta) = H_0(x,\xi) + K\eta(\mathfrak{c}(x,\xi))$ to $\mathrm{T}^*\mathbb{R}^d$ here requires the differential $S(t)$ of $\Phi^t$ with respect to $z$. The conjugation with $\pi(g(t))$ has no effect on the scalar terms in (5.2.8), whereas it rotates the representation operators $\mathrm{d}\pi_{k\lambda}(\mathfrak{c})$ to $\mathrm{d}\pi_{k\lambda}(\mathrm{Ad}_{g(t)}\,\mathfrak{c})$. Hence for an application of $\mathcal{W}(t)$ to $\phi_\eta$ we can use Lemma 5.2.1. By also converting estimates with respect to $k$ to ones with respect to $\hbar$ this yields in leading order

$$(\,\mathrm{d}\pi_{k\lambda}(\mathrm{Ad}_{g(t)^{-1}}\,\mathfrak{c}) - K\eta(t)(\mathfrak{c}))\phi_\eta = K(\eta(\mathrm{Ad}_{g(t)^{-1}}\,\mathfrak{c}) - \eta(t)(\mathfrak{c}))\phi_\eta + O(k^{-1}) = O(\hbar). \tag{5.2.11}$$

Moreover, the complete asymptotic series in powers of $k^{-1}$ provided by Lemma 5.2.1 results in a full asymptotic expansion of (5.2.11). This observation now enables us to apply Lemma 5.1.2 in a completely analogous way to that used previously, yielding

$$\|\mathcal{H}-\mathcal{H}_Q(t)\psi_Q(t)\| \le C\hbar^{3/2}\vartheta(t)^3\delta(t)^3,$$

where the quantities $\vartheta(t)$ and $\delta(t)$ are defined as in (5.1.12), however now with $S(t)$ and $z(t)$ defined in the way described above.

The stability of the trajectory $z(t)$ is encoded in the quantity

$$\tilde{\lambda}_{\max}(z) = \limsup_{t\to\infty}\frac{1}{t}\log\|S(t)\|_{\mathrm{HS}} \ . \tag{5.2.12}$$

Since $z(t)$ is not the integral curve of a flow, rather than calling $\tilde{\lambda}_{\max}(z)$ a Lyapunov exponent we refer to it as a stability exponent. This can, however, be bounded by the maximal Lyapunov exponent of the flow-line $(z(t),\eta(t))$ in $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$, see Section 5.4. Thus, in close analogy to Theorem 5.1.3 we finally obtain:

**Theorem 5.2.2.** *Let the conditions imposed on the Hamiltonian in Section 5.1 and the ellipticity condition (5.2.6) hold. Then the coherent state $\psi_Q(t)$ defined in Proposition 4.2.5 semiclassically approximates $\psi(t)=\mathcal{U}(t,0)\big(\varphi^B_{(q,p)}\otimes\phi_\eta\big)$ in the following sense,*

$$\|\psi(t)-\psi_Q(t)\| \le K\sqrt{\hbar}\,t\,\vartheta(t)^3 \ ,$$

*when $\hbar k$ is kept fixed. The right-hand side vanishes in the combined limits $\hbar\to 0$, $k\to\infty$ and $t\to\infty$ as long as $t\ll T_z(\hbar)$. The time scale $T_z(\hbar)$ depends on the linear stability of the trajectory $z(t)$. If the latter possesses a positive and finite stability exponent $\tilde{\lambda}_{\max}(z)$, one has $T_z(\hbar)=\frac{1}{6\tilde{\lambda}_{\max}(z)}|\log\hbar|$. In case $z(t)$ is a projection to $\mathrm{T}^*\mathbb{R}^d$ of a trajectory $(z(t),\eta(t))$ on a (non-degenerate) KAM-torus in $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$ this time scale is $T_z(\hbar)=C\,\hbar^{-1/8}$.*

As in the previous case an improvement of the semiclassical error can be achieved with the Dyson expansion (5.1.14). The present case, however, requires an additional estimate of the contribution corresponding to the internal degrees of freedom. Concerning the error term $\mathcal{R}_N(t;\hbar)\psi(0)$, the translational part is dealt with by a repeated application of the argument leading to Theorem 5.2.2. For the part corresponding to the internal degrees of freedom an inspection of the relations (5.2.8) and (5.2.9) reveals the necessity to estimate the successive application of the operators

$$\Lambda(t_k) := \big(\hbar\,\mathrm{d}\pi_\lambda(\mathrm{Ad}_{g(t_k)^{-1}}\,\mathfrak{c}(z(t_k))) - K\eta(t_k)(\mathfrak{c}(z(t_k)))\big)$$

to the coherent state $\phi_\eta$. Representing these operators in the form (5.2.3), the result of their $l$-fold ($l\le j$) application reads

$$\begin{aligned}\Lambda(t_l)\ldots\Lambda(t_1)\phi_\eta = (\dim V_{k\lambda})^l\int\limits_{\mathcal{O}_\lambda}\ldots\int\limits_{\mathcal{O}_\lambda}&\sigma_{\mathrm{u}}[\Lambda(t_l)](\eta_l)\ldots\sigma_{\mathrm{u}}[\Lambda(t_1)](\eta_1)\times\\ &\times\Pi(\eta_l)\ldots\Pi(\eta_1)\phi_\eta\ \mathrm{d}\eta_l\ldots\mathrm{d}\eta_1\ ,\end{aligned} \tag{5.2.13}$$

with the upper symbols

$$\sigma_{\mathrm{u}}(\eta_k) = K\eta_k(\mathrm{Ad}_{g(t_k)^{-1}}\,\mathfrak{c}(z(t_k)) - \eta(t_k)(\mathfrak{c}(z(t_k))) + c\hbar\eta_k(\mathrm{Ad}_{g(t_k)^{-1}}\,\mathfrak{c}(z(t_k))) \tag{5.2.14}$$

with $c = \frac{2\langle\lambda,\delta\rangle}{\langle\lambda,\lambda\rangle}$, see equation (5.2.4). Starting with $\eta_l$, the integral (5.2.13) can be successively evaluated with the method of steepest descent similar to the proof of Lemma 5.2.1. The relation

$$\Pi(\eta_l)\ldots\Pi(\eta_1)\phi_\eta = \langle\phi_{\eta_l},\phi_{\eta_{l-1}}\rangle\cdots\langle\phi_{\eta_1},\phi_\eta\rangle\phi_{\eta_l}$$

then shows that the critical points of the phase are given by $\eta_l = \eta_{l-1} = \cdots = \eta_1 = \eta$. At these points, however, the upper symbols $\sigma_{\mathrm{u}}[\Lambda(t_k)](\eta_k)$ are of order $\hbar$, compare (5.2.14). The application of the method of steepest descent therefore yields in leading order a contribution $O(\hbar^l) = O(k^{-l})$. Derivatives of total order $n$ contribute terms of the order $O(k^{-n}\hbar^{l-n}) = O(s^{-l})$, if $n \le l$, and of the order $O(k^{-n})$ otherwise. Altogether there hence exist differential operators $\mathcal{D}^{(\kappa)}$ of order $\le 2\kappa$ on $C^\infty((\mathcal{O}_\lambda)^l)\otimes\mathbb{C}^{\dim V_{k\lambda}}$ such that

$$\Lambda(t_l)\ldots\Lambda(t_1)\phi_\eta - \sum_{\kappa=l}^{K}\frac{1}{s^\kappa}\mathcal{D}^{(\kappa)}\big(\sigma_{\mathrm{u}}[\Lambda(t_l)](\eta_l)\ldots\sigma_{\mathrm{u}}[\Lambda(t_1)](\eta_1)\phi_\eta\big)_{\eta_l=\cdots=\eta_1=\eta} \tag{5.2.15}$$

is of the order $k^{-(M+1)}$ for any $M \ge l$. The left-hand side of (5.2.13) hence is of the order $O(k^{-l}) = O(\hbar^l)$, meaning that every factor $\Lambda(t_k)$ contributes a factor of $\hbar$. We therefore finally obtain an estimate of the remainder term to the Dyson series given by

$$\|\mathcal{R}_N(t;\hbar)\psi(0)\| \le K_N\hbar^{N/2}t^N\vartheta(t)^{3N}\delta(t)^{mN}\ .$$

The main terms in the Dyson expansion are treated by replacing each factor of (5.2.8), occurring at $t = t_k$, with the Taylor expansions

$$\sum_{|\nu|=1}^{2}\frac{1}{\nu!}(\hbar\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}^{(\nu)}(z(t_k)) - K\eta(t_k)(\mathfrak{c}(z(t_k))))(w-z(t_k))^\nu$$
$$+\sum_{\nu=3}^{n_k}\frac{1}{\nu!}\Big(H_0^{(\nu)}(z(t_k)) + \hbar\,\mathrm{d}\pi_{k\lambda}(\mathfrak{c}^{(\nu)}(z(t_k)))\Big)(w-z(t_k))^\nu + r_k(t_k,w)\ ,$$

where again the integers $n_k$ are chosen sufficiently large. The contribution of the translational degrees of freedom can be dealt with as in the previous semiclassical scenario, and the internal degrees of freedom contribution follows from the expansion (5.2.15). Finally grouping together terms of corresponding orders in $\hbar$, we arrive at a statement analogous to Theorem 5.1.4.

**Theorem 5.2.3.** *Suppose that the quantum Hamiltonian $\mathcal{H}$ with symbol (5.2.7) satisfies the conditions specified above. Then for $t > 0$ and any $N \in \mathbb{N}$ there exists a state $\psi_N(t) \in L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\dim V_{k\lambda}}$, localized at $\big(q(t),p(t),\eta(t)\big)$, that approximates the full time evolution*

*$\psi(t) = \mathcal{U}(t,0)\big(\varphi^B_{(q,p)} \otimes \phi_\eta\big)$ of a coherent state up to an error of order $\hbar^{N/2}$ when $\hbar k$ is fixed. More precisely,*

$$\|\psi(t) - \psi_N(t)\| \leq C_N \sum_{j=1}^{N-1} \left(\frac{t}{\hbar}\right)^j (\sqrt{\hbar}\vartheta(t))^{2j+N} \ .$$

*The right-hand side vanishes in the combined limits $\hbar \to 0$, $k \to \infty$ and $t \to \infty$ as long as $t \ll T_z(\hbar)$, where $T_z(\hbar)$ denotes the same time scale as in Theorem 5.2.2.*

*Furthermore, $\psi_N(t)$ arises from $\varphi^B_{(q,p)} \otimes \phi_\eta$ through the application of certain (time dependent) differential operators $\hat{q}_{k\kappa j}(t) = \mathrm{op}^W[p_{kj}(t)] \otimes r_\kappa$,*

$$\psi_N(t) = \psi_Q(t) + \sum_{(k,\kappa,j)\in\Delta_N} \mathcal{U}_Q(t,0)\, \hat{q}_{k\kappa j}(t)\, \psi(0) \ ,$$

*where $p_{kj}(t)$ is a polynomial in $(x,\xi)$ of degree $\leq k$ and $r_\kappa$ is a differential operator of order $\leq 2\kappa$ on $C^\infty(\mathcal{O}_\lambda) \otimes \mathbb{C}^{\dim V_{k\lambda}}$. Here we have also defined*

$$\Delta_N := \{(k,\kappa,j) \in \mathbb{N}^3;\ 1 \leq k + 2\kappa - 2j \leq N-1,\ k + 2\kappa \geq 3j,\ 1 \leq j \leq N-1\} \ .$$

The semiclassical localization of $\psi_N(t)$ here is different from the situation covered by Theorem 5.1.4 in that the operators $r_\kappa$ act on $\phi_\eta$. But these are differential operators and hence do not increase the frequency set. This means that $\psi_N(t)$ is semiclassically localized at $\Phi^t(q,p,\eta)$ and in in this respect is not different from the classically propagated coherent state $\psi_Q(t)$.

## 5.3 Discussion

In the previous section we analyzed the semiclassical behaviour of coherent states in two different limits. In various places we saw that the difference between the two cases is expressed in the way the classical translational and internal motion are coupled. Otherwise the final results agree to a large extent. This includes the mechanisms of semiclassical localization in the product phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$.

The problem of how the localization of an initial coherent state develops with time can be made more explicit by using semiclassical phase-space lifts of the coherent states. At $t = 0$ the state $\psi(0) = \varphi^B_{(q,p)} \otimes \phi_\eta$ is concentrated in a neighbourhood of the point $(q,p,\eta) \in \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. This concentration can be measured in terms of expectation values $\langle\psi(0), \mathcal{A}\psi(0)\rangle$ of operators $\mathcal{A} = \mathrm{op}^W[A]$ that are quantizations of well localized symbols $A \in C_0^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_{\dim V_{k\lambda}}(\mathbb{C})$. For simplicity we also assume that $A$ is independent of $\hbar$. At later times $\psi(t)$ can in both semiclassical scenarios be approximated by an appropriate coherent state $\psi_Q(t)$, such that

$$\langle\psi(t), \mathcal{A}\psi(t)\rangle = \langle\psi_Q(t), \mathcal{A}\psi_Q(t)\rangle + o(1) \ , \quad t \ll T_z(\hbar) \ . \tag{5.3.1}$$

The expectation value on the right-hand side has a phase-space representation

$$\langle\psi_Q(t), \mathcal{A}\psi_Q(t)\rangle = \frac{1}{(2\pi\hbar)^d} \iint_{\mathrm{T}^*\mathbb{R}^d} W[\varphi^{B(t)}_{z(t)}](w)\, \langle\phi_{\eta(t)}, A(w)\phi_{\eta(t)}\rangle_{V_{k\lambda}}\ \mathrm{d}w \ , \tag{5.3.2}$$

where $W[\varphi^B_{(q,p)}]$ denotes the Wigner transform of a translational coherent state which is given by

$$\begin{aligned} W[\varphi^B_{(q,p)}](x,\xi) &= \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\xi y}\,\overline{\varphi}^B_{(q,p)}(x-\tfrac{1}{2}y)\varphi^B_{(q,p)}(x+\tfrac{1}{2}y)\,\mathrm{d}y \\ &= 2^d\,\mathrm{e}^{-\frac{1}{\hbar}\langle (x-q,\xi-p),\mathrm{g}_B(x-q,\xi-p)\rangle}, \end{aligned} \tag{5.3.3}$$

with the metric $\mathrm{g}_B$ defined in Section 4.1.1. Thus $\psi_Q(t)$ is concentrated at the point $(q(t),p(t),\eta(t))$ in the semiclassical limit as long as the quadratic form $\mathrm{g}_{B(t)}/\hbar$ is strictly positive definite. Either of the time evolutions of $B$ specified in Propositions 4.2.5 and 5.1.1 now imply, see Appendix A,

$$\mathrm{g}_{B(t)} = (S_z(t)^{-1})^*\,\mathrm{g}_B\,S_z(t)^{-1}\ ,$$

where $S(t)$ is a solution of (5.2.10) or (A.5.5), respectively. Thus the spreading of $\psi_Q(t)$ in $\mathrm{T}^*\mathbb{R}^d$, which can be measured in terms of the combined uncertainty in position and momentum

$$\frac{1}{(2\pi\hbar)^d}\iint_{\mathrm{T}^*\mathbb{R}^d}((x-q)^2+(\xi-p)^2)W[\varphi^B_{(q,p)}](x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \frac{\hbar}{2}\,\mathrm{tr}\,\mathrm{g}_B,$$

is bounded according to

$$\hbar\,\mathrm{tr}\,\mathrm{g}_{B(t)} \le \hbar\|\mathrm{g}_{B(t)}\|_{\mathrm{HS}} \le \hbar\|S(t)\|^2_{\mathrm{HS}}\|\mathrm{g}_B\|_{\mathrm{HS}}.$$

If $z(t)$ is a trajectory with maximal Lyapunov (or stability) exponent $\lambda_{\max}(z) > 0$, the requirement for the state $\psi_Q(t)$ to remain localized therefore is $t \ll \frac{1}{2\lambda_{\max}(z)}|\log\hbar|$. This time scale is three times larger than $T_z(\hbar)$, which is the estimated time in (5.3.1) for the coherent state $\psi_Q(t)$ to still well approximate the full time evolution $\psi(t)$.

Let us remark that the limitations in (5.3.1), to approximate the expectation value in terms of a coherent state, derive from estimating the difference $\psi(t)-\psi_Q(t)$ in $L^2$-norm. But the error term on the right-hand side of (5.3.1) measures this difference in a considerably weaker form so that one might expect it to vanish as $\hbar\to 0$ and $t\to\infty$ also for times $T_z(\hbar)\le t \ll 3T_z(\hbar)$. In the case without internal degrees of freedom Bouzouina and Robert [BR02] proved that this indeed holds, suggesting that the same is true in the present setting.

Expectation values in coherent states such as (5.3.1) can also be used to obtain the leading semiclassical description of the propagation of observables. To see this let $\hat{A}$, as above, be a bounded Weyl operator and denote its quantum time evolution by $\hat{A}(t) = \hat{U}(t,0)^*\hat{A}\,\hat{U}(t,0)$. Here, however, we do not necessarily require the symbol to be compactly supported. The relations (5.3.1) and (5.3.2) then remain valid so that

$$\langle\psi(0),\hat{A}(t)\psi(0)\rangle = \frac{1}{(2\pi\hbar)^d}\iint_{\mathrm{T}^*\mathbb{R}^d} W[\varphi^{B(t)}_{z(t)}](w)\,\langle\phi_{\eta(t)},A(w)\phi_{\eta(t)}\rangle\ \mathrm{d}w + o(1)\ .$$

Since $\mathcal{A}(t)$ is bounded it may also be expressed as a Weyl operator, with symbol $A(t)$ such that for $t \ll T_z(\hbar)$ equation (5.3.1) can be rewritten as

$$\frac{1}{(2\pi\hbar)^d} \iint_{\mathrm{T}^*\mathbb{R}^d} W[\varphi_{z(0)}^{B(0)}](w) \, \langle \phi_{\eta(0)}, A(t)(w)\phi_{\eta(0)} \rangle \; \mathrm{d}w \\ - \frac{1}{(2\pi\hbar)^d} \iint_{\mathrm{T}^*\mathbb{R}^d} W[\varphi_{z(t)}^{B(t)}](w) \, \langle \phi_{\eta(t)}, A(w)\phi_{\eta(t)} \rangle \; \mathrm{d}w = o(1) \; .$$

The semiclassical localisation properties of the coherent states discussed above therefore imply that in leading order the symbol of the time evolved observable $\hat{A}(t)$ can be expressed in terms of the symbol of $\hat{A}$ transported along the classical flow $\big(q(t), p(t), \eta(t)\big)$,

$$\langle \phi_\eta, A(t)(q,p)\phi_\eta \rangle - \langle \phi_{\eta(t)}, A\big(q(t), p(t)\big)\phi_{\eta(t)} \rangle = o(1) \; .$$

The $\mathbb{C}^{\dim V_{k\lambda}}$-expectation values in $G$-coherent states are lower symbols, see Section 3.6, of the matrix valued functions $A(t)$ and $A$, respectively. In terms of this mixed phase space representation of operators, employing Weyl calculus for the translational part and lower symbols for the intrinsic degrees of freedom part, this means that the quantum time evolution of observables follows the classical dynamics in leading semiclassical order. This statement represents a limited version of an Egorov theorem and again is valid for both semiclassical scenarios discussed in the preceding section, up to the time scale $t \ll T_z(\hbar)$.

## 5.4 Linear stability of Hamiltonian flows

In this section we consider the Hamiltonian flows $\Phi_0^t : \mathrm{T}^*\mathbb{R}^d \to \mathrm{T}^*\mathbb{R}^d$ and $\Phi^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ generated by $H_0$ and $H = H_0 + K\eta(\mathfrak{c})$, respectively. For these flows we want to recall the notion of Lyapunov exponents and give sufficient criteria of their existence in terms of properties of the Hamiltonian function. To this end we consider a general flow $\tilde{\Phi}^t : M \to M$ on a symplectic manifold $M$ with dimension $2n$.

The linear stability of a flow $\tilde{\Phi}^t$ is determined by properties of the differential $\tilde{\Phi}^t_*$ which is a linear map from the tangent space $\mathrm{T}_\alpha M$ to $\mathrm{T}_{\tilde{\Phi}^t(\alpha)}M$. It, moreover, is a multiplicative cocycle over the flow $\tilde{\Phi}^t$, i.e. $\tilde{\Phi}^{t+t'}_*(\alpha) = \tilde{\Phi}^{t'}_*(\tilde{\Phi}^t(\alpha))\, \tilde{\Phi}^t_*(\alpha)$. If one introduces a euclidean scalar product in the tangent spaces this gives rise to the adjoint $(\tilde{\Phi}^t_*)^{\mathrm{T}}$. Then $(\Phi^t_*)^{\mathrm{T}}\Phi^t_*$ is a non-negative symmetric linear map on $\mathrm{T}_\alpha M$ whose eigenvalues we denote by

$$\mu_t^{(1)}(\alpha) \geq \cdots \geq \mu_t^{(2n)}(\alpha) \geq 0 \; .$$

The $2n$ Lyapunov exponents of the flow $\tilde{\Phi}^t$ at $\alpha \in M$ are now given by the expressions

$$\lambda_k(\alpha) := \limsup_{t\to\infty} \frac{1}{2t} \log \mu_t^{(k)}(\alpha) \; ,$$

if these are finite. The largest Lyapunov exponent $\lambda_{\max}(\alpha)$ provides a quantitative measure for the linear stability of $\tilde{\Phi}^t$ since it measures the leading rate of local phase space expansion; it can be obtained from the relation

$$\lambda_{\max}(\alpha) = \limsup_{t\to\infty} \frac{1}{2t} \log \mathrm{tr}\big((\Phi^t_*(\alpha))^{\mathrm{T}}\, \Phi^t_*(\alpha)\big) \; . \tag{5.4.1}$$

Hamiltonian flows leave the energy shells

$$\Omega_E := \{\alpha \in M;\ H(\alpha) = E\}$$

invariant. If $E$ is a regular value of the Hamiltonian function $H$, the energy shell $\Omega_E$ is a smooth submanifold of $M$ of dimension $2n-1$. In such a case two Lyapunov exponents are always zero. They correspond to the direction of the flow and the direction transversal to the energy shell. Of the remaining $2n-2$ Lyapunov exponents half are non-negative (if they exist) and the rest of the Lyapunov spectrum is given by minus the first half.

In general it is not known whether the Lyapunov exponents are finite. If, however, an energy shell $\Omega_E$ is compact, one can introduce the normalised Liouville measure as a flow invariant probability measure on $\Omega_E$. In this case one can apply Oseledec' multiplicative ergodic theorem to the restriction of $\tilde{\Phi}^t$ to this energy shell [Ose68, Arn98]; it guarantees that the Lyapunov exponents are finite for almost all points on $\Omega_E$ with respect to Liouville measure. Moreover, if the flow is ergodic with respect to Liouville measure $\lambda_k(\alpha)$ is constant on a set of full measure. Since we want to consider also non-compact energy shells we now give alternative sufficient criteria for the finiteness of Lyapunov spectra.

**Proposition 5.4.1.** *Let $H \in C^\infty(M)$ be a Hamiltonian function such that the Hilbert-Schmidt norm of $\mathrm{D}^2 H$ is bounded on the energy shell $\Omega_{E,\alpha}$ that contains the point $\alpha \in M$. Then the Lyapunov exponents $\lambda_1(\alpha), \dots, \lambda_{2n}(\alpha)$ are finite.*

*Proof.* Fix $\alpha \in M$ and introduce canonical coordinates $(q,p) \in U \subset \mathbb{R}^n \times \mathbb{R}^n$ in a neighbourhood of $\alpha$. Then in this neighbourhood $\mathrm{D}^2 H$ is represented by the matrix $H''(q,p)$ of second derivatives with respect to $(q,p)$. In these coordinates we denote the flow by $\hat{\Phi}^t(q,p)$; its differential satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Phi}^t_*(q,p) = \mathrm{J}\, H''\big(\hat{\Phi}^t(q,p)\big)\,\hat{\Phi}^t_*(q,p)\ , \qquad \hat{\Phi}^t_*(q,p)|_{t=0} = \mathbb{1}_{2n}\ , \tag{5.4.2}$$

where $J = \left(\begin{smallmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{smallmatrix}\right)$. By integrating (5.4.2) and taking the Hilbert-Schmidt norm one obtains

$$\|\hat{\Phi}^t_*(q,p)\|_{\mathrm{HS}} \leq 2n + \int_0^t \|J\, H''\big(\hat{\Phi}^s(q,p)\big)\|_{\mathrm{HS}}\, \|\hat{\Phi}^s_*(q,p)\|_{\mathrm{HS}}\ \mathrm{d}s\ .$$

For simplicity we here assume that for the points $\Phi^s(\alpha)$, $s \in [0,t]$, one can use the same system of canonical coordinates. Gronwall's inequality [Tay96, AM78] then yields the estimate $(t>0)$

$$\|\hat{\Phi}^t_*(q,p)\|_{\mathrm{HS}} \leq 2n \exp\Big\{t \sup_{s\in[0,t]} \|\mathrm{J}\, H''\big(\hat{\Phi}^s(q,p)\big)\|_{\mathrm{HS}}\Big\} \leq 2n\ \mathrm{e}^{Ct}\ ,$$

with some constant $C>0$. The last line follows from the boundedness of $\mathrm{D}^2 H$ on $\Omega_{E,\alpha}$. Since on the other hand

$$\|\hat{\Phi}^t_*(q,p)\|_{\mathrm{HS}} = \sqrt{\mu_t^{(1)}(\alpha) + \cdots + \mu_t^{(2n)}(\alpha)}\ ,$$

the bound

$$\frac{1}{2t} \log \mu_t^{\max}(\alpha) \leq K$$

for the maximal eigenvalue $\mu_t^{\max}(\alpha)$ follows. This finally implies the assertion. □

An application of this Proposition to the two flows $\Phi_0^t$ (defined on $M = \mathrm{T}^*\mathbb{R}^d$) and $\Phi^t$ (defined on $M = \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$) immediately yields

**Corollary 5.4.2.** *If the norm of $H_0''$ is bounded on $\Omega_{E,(x,\xi)} \subset \mathrm{T}^*\mathbb{R}^d$, the $2d$ Lyapunov exponents $\lambda_{0,k}(x,\xi)$ of the flow $\Phi_0^t$ are finite. If, in addition, the derivatives $\mathfrak{c}(x',\xi')$ of order $|\nu| \leq 2$ are bounded for all $(x',\xi',\eta') \in \Omega_{E,(x,\xi,\eta)} \subset \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$, the $2d + \dim \mathcal{O}_\lambda$ Lyapunov exponents $\lambda_k(x,\xi,\eta)$ of the flow $\Phi^t$ are also finite.*

In the second semiclassical scenario, however, rather than the Lyapunov exponent $\lambda_k(q,p,\eta)$ of a point $(q,p,\eta) \in \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ the stability exponent (5.2.12) of the projection to $\mathrm{T}^*\mathbb{R}^d$ entered Theorem 5.2.2. Revisiting the proof of Proposition 5.4.1 shows that in view of (5.2.10) such a stability exponent is finite under the same conditions as stated in Corollary 5.4.2 for $\lambda_k$. Moreover, a simple estimate yields the bound

$$\tilde{\lambda}_{\max} \leq \lambda_{\max} \; .$$

## 5.5 Regular motion on KAM tori and Lyapunov exponents

A KAM torus in a symplectic dynamical system is a particular case of an invariant set with complete dynamics. It is diffeomorphic to an $n$-torus, where $2n$ is the dimension of phase space. It bears a *quasiperiodic motion*; this is defined on the torus $\mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$, with frequencies $\omega \in \mathbb{R}^k$, as the flow $g_\omega^t : \mathbb{T}^k \to \mathbb{T}^k$ whose lift is given by $\tilde{g}_\omega^t : \mathbb{R}^k \to \mathbb{R}^k$, $\tilde{g}_\omega^t(x) = x + \omega t$.

Now consider a symplectic dynamical system $(X,\omega)$, $\dim X = 2n$. An *invariant torus with quasiperiodic motion* is a subset $T \subset X$ satisfying the conditions

(IT1) $T$ is an invariant set with complete dynamics. We denote by $f^t$ the flow generated on $T$ by the system.

(IT2) $T$ is an isotropic submanifold.

(IT3) There exists a transitive quasiperiodic motion $g_\omega^t : \mathbb{T}^k \to \mathbb{T}^k$, with $k \leq n$, and an embedding $\chi : \mathbb{T}^k \to X$ such that $T = \chi(\mathbb{T}^k)$ and $\chi$ intertwines $f^t$ and $g_\omega^t$, i.e. the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "f^t"] & T \subset X \\ \mathbb{T}^k \arrow[u, "\chi"] \arrow[r, "g_\omega^t"'] & \mathbb{T}^k \arrow[u, "\chi"'] \end{tikzcd} \; .$$

We want to consider the dynamical behaviour of the system near $T$, restricting ourselves to the case $k = n$, which implies that $T$ is isotropic with maximal dimension, i.e. Lagrangian. There exists a neighbourhood of $T$ in $X$ which is symplectomorphic to a neighbourhood of the zero section in $\mathrm{T}^*T$, the symplectomorphism being the identity on $T$, see [BW97]. The neighbourhood can be chosen to be $\mathbb{T}^k \times B$, where $B \subset \mathbb{R}^n$ is an open ball with center at zero. The preimage of the dynamical system under this diffeomorphism gives a Hamiltonian system with Hamiltonian $H(\varphi, I)$. The corresponding flow is given by

$$I(t) = I, \quad \varphi(t) = \varphi + \omega(I)t.$$

The KAM-torus is called *non-degenerate* if $\left(\frac{\partial^2 H}{\partial I_j \partial I_k}\right)$ is non-singular. We call the torus maximally degenerate if the above matrix has only eigenvalues equal to zero. In these coordinates we can calculate

$$\|S(t)\|_{\mathrm{HS}}^2 = 2d + f(I)t^2,$$

where in the maximally degenerate case $f(I) = 0$.

# Chapter 6

# Quantum ergodicity for particles with internal degrees of freedom

## 6.1 Classical description of the intrinsic degrees of freedom

The purpose of this section is to give a classical interpretation to the intrinsic degrees of freedom and their dynamics described in Section 2.3.3. To this end we employ the Stratonovich-Weyl quantizer presented in Section 3.7. In Chapter 2 we were concerned with general Hamiltonians that generate a time evolution which leaves certain subspaces of the Hilbert space (almost) invariant. We used semiclassical projection operators $\mathcal{P}_\nu$ to these subspaces in order to study the time evolution in these: For each subspace it was generated by the projected Hamiltonian which has a scalar principal part. Since the semiclassical limit is given in terms of $\hbar \to 0$ the leading order terms $P_{\nu,0}$ of $\mathcal{P}_\nu$ define a subbundle $E^\nu$ of the trivial bundle $\mathrm{T}^*\mathbb{R}^d \times \mathbb{C}^n$, see Section 2.3.3. Therefore, in the setting described there the group $G$ under consideration is a matrix Lie group, i.e. a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ and its Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathrm{M}_n(\mathbb{C})$, whose Lie algebra structure is defined by the matrix commutator. Consider now the unitary irreducible representation $(\rho, V)$ described in Section 2.3.3 and fix a (real) highest weight $\lambda \in \mathfrak{t}^*$ together with an (essentially) unique highest weight vector $\psi_\lambda$ which allows us to define upper and lower symbols as described in Section 3.6 and 3.7.

With this formalism at hand one can now transfer the dynamics of a (hermitian) $B \in \mathcal{L}(V)$ given by a conjugation with $D(t) = \rho(g(t))$, $B \mapsto B(t) = D^{-1}(t)BD(t)$, to the coadjoint action of $g(t)$ on the symplectic manifold $\mathcal{O}_\lambda$ via the relation $\mathrm{symb}^{SW}[B(t)](\eta) = \mathrm{symb}^{SW}[B](\mathrm{Ad}^*_{g(t)}\,\eta)$. The symplectic structure on $\mathcal{O}_\lambda$ allows us to identify the dynamics $\eta \mapsto \mathrm{Ad}^*_{g(t)}\,\eta$ as being Hamiltonian.

As an ultimate outcome of the above formalism we are now in a position to introduce a skew-product flow on the symplectic phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ that completely determines the time evolution of the $\nu$-th diagonal block of an observable on the level of its principal symbol. Explicitly, this flow is given by

$$Y_\nu^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \tag{6.1.1}$$

with

$$Y_\nu^t(x,\xi,\eta) := \big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta\big); \tag{6.1.2}$$

it leaves the product measure $\mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}\eta$ invariant.

Consider now a semiclassical pseudodifferential operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^\infty_{\mathrm{cl}}(1)$. Mod $O(\hbar^\infty)$ the quantum dynamics preserves the diagonal structure of its blocks $\mathcal{P}_\nu\mathcal{B}\mathcal{P}_\nu$. According to the Egorov theorem 2.3.4, together with the definition (2.3.31), the principal symbol of $\mathcal{P}_\nu\mathcal{B}(t)\mathcal{P}_\nu$ hence reads

$$V_\nu(x,\xi)D_\nu^*(x,\xi,t)\big(V_\nu^*B_0V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)D_\nu(x,\xi,t)V_\nu^*(x,\xi). \tag{6.1.3}$$

We now exploit the possibility, explicitly provided by the Moyal quantizer for compact groups defined in 3.7, to uniquely represent the value of $V_\nu^*B_0V_\nu : \mathrm{T}^*\mathbb{R}^d \to \mathcal{L}(\mathbb{C}^{k_\nu})$ in terms of a Stratonovich-Weyl symbol,

$$b_{0,\nu}(x,\xi,\eta) := \mathrm{symb}^{SW}\big[(V_\nu^*B_0V_\nu)(x,\xi)\big](\eta). \tag{6.1.4}$$

The dynamics of the principal symbol in this representation is now summarised in the following variant of the Egorov theorem:

**Proposition 6.1.1.** *The Stratonovich-Weyl symbol $b(t)_{0,\nu}$ associated with the principal symbol of the operator $\mathcal{P}_\nu\mathcal{B}(t)\mathcal{P}_\nu$ is the time evolution of $b_{0,\nu}$ under the skew-product flow $Y_\nu^t$ defined in equations (6.1.1)–(6.1.2), i.e.,*

$$b(t)_{0,\nu}(x,\xi,\eta) = b_{0,\nu}\big(Y_\nu^t(x,\xi,\eta)\big).$$

*Proof.* According to (6.1.3) and (6.1.4), $b(t)_{0,\nu}$ is given by

$$b(t)_{0,\nu}(x,\xi,\eta) = \mathrm{symb}^{SW}\big[\rho(g_\nu^{-1}(x,\xi,t))\big(V_\nu^*B_0V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)\rho(g_\nu(x,\xi,t))\big](\eta),$$

which due to the covariance property $(iv)$ of Theorem 3.7.5 reads

$$\begin{aligned} b(t)_{0,\nu}(x,\xi,\eta) &= \mathrm{symb}^{SW}\big[\big(V_\nu^*B_0V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)\big](\mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta) \\ &= b_{0,\nu}\big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta\big). \end{aligned}$$

□

## 6.2 Quantum Ergodicity

Our intention in this section is to consider quantum ergodicity for the normalized eigenvectors $\psi_j$, $E_j \in I(E,\hbar)$, of the quantum Hamiltonian $\mathcal{H}$. In the case of scalar pseudodifferential operators one denotes by quantum ergodicity a weak convergence of the phase space lifts of almost all eigenfunctions to Liouville measure on the level surface $\Omega_E = H_0^{-1}(E)$, and proves this to hold if the flow generated by the principal symbol $H_0$ of the quantum Hamiltonian is ergodic on $\Omega_E$. In the present situation of operators with matrix valued symbols, however, each eigenvalue $\lambda_\nu$ of $H_0$ defines its own classical dynamics. One hence can only expect quantum ergodicity to be concerned with statements about the projections $\mathcal{P}_\nu\psi_j$ of the eigenvectors to the different almost invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ in relation to the behaviour of the associated classical systems. In Section 2.4.1 we discussed the question of identifying those projected eigenvectors whose norms are not semiclassically small. Since presently this problem cannot be resolved directly, quantum ergodicity can only be formulated by restricting to those eigenvectors whose squared norms exceed a value of $\delta$ in the semiclassical limit, without specifying them further.

Conventionally the convergence of quantum states determined by the eigenvectors $\psi_j$ of $\mathcal{H}$ is discussed in terms of expectation values of observables in these states. Explicit lifts of the eigenfunctions to phase space are then, e.g., provided by their Wigner transforms. The choice of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ leads us to consider expectation values of diagonal blocks $\mathcal{P}_\nu\mathcal{B}\mathcal{P}_\nu$ of operators $\mathcal{B}$ with symbols $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$. On the symbol level the time evolution of these blocks is covered by the Egorov theorem 2.3.4. Representing the blocks of the principal symbols by Stratonovich-Weyl symbols we are, according to Proposition 6.1.1 faced with the skew-product flows $Y^t_\nu$ on the product phase spaces $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_{\lambda,\nu}$. Since the Stratonovich-Weyl symbols $b_{0,\nu}$ defined in equation (6.1.4) that are associated with symbols $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ are clearly integrable with respect to the measures $\mathrm{d}\ell\ \mathrm{d}\eta$ on the (compact) manifolds $\Omega_{\nu,E}\times\mathcal{O}_\lambda$, the (assumed) ergodicity of the flow $Y^t_\nu$ implies that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(b_{0,\nu}\circ Y^t_\nu\big)(x,\xi,\eta)\ \mathrm{d}t &= \frac{1}{\operatorname{vol}\mathcal{O}_\lambda}\int_{\Omega_{\nu,E}}\int_{\mathcal{O}_{\lambda,\nu}} b_{0,\nu}(x',\xi',\eta')\ \mathrm{d}\eta'\,\mathrm{d}\ell(x',\xi') \\ &= \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\end{aligned} \tag{6.2.1}$$

holds for almost all initial conditions $(x,\xi,\eta)\in\Omega_{\nu,E}\times\mathcal{O}_\lambda$. In particular, one immediately realizes that the supposed ergodicity of $Y^t_\nu$ implies ergodicity for the flow $\Phi^t_\nu$ on $\Omega_{\nu,E}$ with respect to Liouville measure $\mathrm{d}\ell$. As a consequence the condition (H6$_\nu$) of Section 2.4.1 is automatically fulfilled.

For the subsequent formulation and proof of quantum ergodicity we choose to follow in principle the approach of [Zel96, ZZ96]. This means that we investigate the variance of expectation values about their mean in the semiclassical limit. In order to avoid the problem of explicitly estimating the norms of projected eigenvectors we here consider the normalised vectors $\phi_{j,\nu}$, defined in (2.4.10), which have been identified as quasimodes for both the operators $\mathcal{H}$ and $\mathcal{H}\mathcal{P}_\nu$. Moreover, we concentrate on vectors corresponding to projected eigenvectors with norms that do not vanish semiclassically, i.e., with $\|\mathcal{P}_\nu\psi_j\|^2 \geq \delta$ for some fixed $\delta\in(0,1)$. This approach is similar to the one introduced by Schubert [Sch01] in the context of local quantum ergodicity, where an equidistribution was shown for quasimodes associated with ergodic components of phase space. In Section 2.4.1 we estimated the relative number $N^\delta_{\nu,I}/N_I$ of the associated eigenvectors among all eigenvectors of $\mathcal{H}$ in the semiclassical limit from below, see (2.4.13). A non-trivial bound could only be obtained for $\tilde{\delta} < 1$ corresponding to

$$\delta < \delta_\nu := \frac{k_\nu \operatorname{vol} \Omega_{\nu,E}}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol} \Omega_{\mu,E}}.$$

Therefore, from now on we confine $\delta$ to the interval $\delta \in (0, \delta_\nu)$, and are thus in a position to state one of our main results.

**Theorem 6.2.1.** *Let $\mathcal{H}$ be a pseudodifferential operator with hermitian symbol $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ whose principal part $H_0$ fulfills the conditions (H1) and (H2) of section 2.3.2. The eigenvalues $\lambda_1, \dots, \lambda_l$ of $H_0$ are required to have constant multiplicities and shall obey the conditions (H3$_\nu$)–(H5$_\nu$) of section 2.4.1 for all $\nu \in \{1, \dots, l\}$. Moreover, they shall be separated according to the hyperbolicity condition (H0),*

$$|\lambda_\nu(x,\xi) - \lambda_\mu(x,\xi)| \geq Cm(x,\xi) \quad \textit{for} \quad \nu \neq \mu \quad \textit{and} \quad |x| + |\xi| \geq c.$$

*Assume now that the symbol $H \sim \sum_{j=0}^{\infty} \hbar^j H_j$ satisfies the growth condition*

$$\|H_j{}^{(\alpha)}_{(\beta)}(x,\xi)\|_{n\times n} \leq C_{\alpha,\beta} \quad \textit{for all} \quad (x,\xi) \in \mathrm{T}^*\mathbb{R}^d \textit{ and } |\alpha| + |\beta| + j \geq 2 - \delta_{j0}, \tag{2.3.23}$$

*and that the condition (Irr$_\nu$) of section 2.3.3 holds. If then the flow $Y^t_\nu$ defined in (6.1.2) is ergodic on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ with respect to the invariant measure $\mathrm{d}\ell\, \mathrm{d}\eta$, in every sequence of normalised projected eigenvectors $\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$, with $\|\mathcal{P}_\nu \psi_j\|^2 \geq \delta$, $\delta \in (0, \delta_\nu)$ fixed, one finds a subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ of density one, i.e.,*

$$\lim_{\hbar\to 0} \frac{\#\{\alpha;\ \|\mathcal{P}_\nu \psi_{j_\alpha}\|^2 \geq \delta\}}{\#\{j;\ \|\mathcal{P}_\nu \psi_j\|^2 \geq \delta\}} = 1,$$

*such that for every operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ and principal symbol $B_0$*

$$\lim_{\hbar\to 0} \langle \phi_{j_\alpha,\nu}, \mathcal{B}\phi_{j_\alpha,\nu}\rangle = \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}), \tag{6.2.2}$$

*where $b_{0,\nu}$ denotes the Stratonovich-Weyl symbol associated with $P_{\nu,0} B_0 P_{\nu,0}$. Furthermore, the density-one subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ can be chosen to be independent of the operator $\mathcal{B}$.*

*Proof.* We start with considering expectation values of the operator $\mathcal{B}$ taken in the quasimodes $\{\phi_{j,\nu}\}$ and denote their variance about the mean $\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})$ of the corresponding Stratonovich-Weyl symbol $b_{0,\nu}$ defined in (6.1.4) as

$$S^\delta_{2,\nu}(E,\hbar) := \frac{1}{N^\delta_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar) \\ \|\mathcal{P}_\nu \psi_j\|^2 \geq \delta}} \left| \langle \phi_{j,\nu}, \mathcal{B}\phi_{j,\nu}\rangle - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}) \right|^2.$$

Due to the definition (2.4.10) of the normalised vectors $\phi_{j,\nu}$, this variance can also be written as

$$\begin{aligned} S^{\delta}_{2,\nu}(E,\hbar) &= \frac{1}{N^{\delta}_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar)\\ \|\mathcal{P}_\nu \psi_j\|^2 \geq \delta}} \big|\langle \phi_{j,\nu}, \big(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big)\phi_{j,\nu}\rangle\big|^2 \\ &= \frac{1}{N^{\delta}_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar)\\ \|\mathcal{P}_\nu \psi_j\|^2 \geq \delta}} \|\mathcal{P}_\nu \psi_j\|^{-2} \big|\langle \psi_j, \mathcal{P}_\nu\big(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big)\mathcal{P}_\nu \psi_j\rangle\big|^2 . \end{aligned}$$

Allowing for an error of $O(\hbar^\infty)$, in this expression the expectation values can be replaced by those of the operator $\tilde{\mathcal{P}}_\nu(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}))\tilde{\mathcal{P}}_\nu$ whose symbol is in the invariant subalgebra $\mathrm{S}^0_{\mathrm{inv}}(1) \subset \mathrm{S}^0_{\mathrm{cl}}(1)$. Therefore, since all further requirements are also met, the Egorov theorem 2.3.4 applies and yields that for finite times $t \in [0,T]$ the evolution $\mathcal{U}^*(t)\tilde{\mathcal{P}}_\nu(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}))\tilde{\mathcal{P}}_\nu\mathcal{U}(t)$ of this operator is again a pseudodifferential operator with symbol in the class $\mathrm{S}^0_{\mathrm{cl}}(1)$. Taking into account that the $\psi_j$s are eigenvectors of $\mathcal{H}$ with eigenvalues $E_j$, the above expression can be rewritten as

$$S^{\delta}_{2,\nu}(E,\hbar) = \frac{1}{N^{\delta}_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar)\\ \|\mathcal{P}_\nu \psi_j\|^2 \geq \delta}} \big|\langle \psi_j, \mathcal{B}_{\nu,T}\psi_j\rangle\big|^2 \|\mathcal{P}_\nu \psi_j\|^{-2},$$

where we have defined the auxiliary operator

$$\mathcal{B}_{\nu,T} := \frac{1}{T}\int_0^T \mathcal{U}^*(t)\tilde{\mathcal{P}}_\nu\big(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big)\tilde{\mathcal{P}}_\nu\mathcal{U}(t)\ \mathrm{d}t. \tag{6.2.3}$$

Furthermore, by using using the Cauchy-Schwarz inequality and the lower bound on the norms $\|\mathcal{P}_\nu\psi_j\|^2 \geq \delta > 0$ we obtain as an upper bound

$$S^{\delta}_{2,\nu}(E,\hbar) \leq \frac{1}{\delta}\frac{N_I}{N^{\delta}_{\nu,I}}\frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \psi_j, \mathcal{B}^2_{\nu,T}\psi_j\rangle.$$

According to equation (2.4.13) the factor $N_I/N^{\delta}_{\nu,I}$ can be estimated from above in the semiclassical limit. We hence consider the semiclassical limit of the expression

$$\frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \psi_j, \mathcal{B}^2_{\nu,T}\psi_j\rangle,$$

to which Proposition 2.4.1 can be applied. To this end one requires the principal symbol $B_{\nu,T,0}$ of the auxiliary operator $\mathcal{B}_{\nu,T}$, which follows from Theorem 2.3.4 as

$$B_{\nu,T,0} = \frac{1}{T}\int_0^T d^*_{\nu\nu}\big((P_{\nu,0}B_0P_{\nu,0})\circ\Phi^t_\nu\big)d_{\nu\nu}\ \mathrm{d}t - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})P_{\nu,0}.$$

Given this, the limit formula (2.4.1) and the estimate (2.4.13) yield

$$\begin{aligned}\lim_{\hbar\to 0} S^{\delta}_{2,\nu}(E,\hbar) &\leq \frac{1}{\delta}\,\frac{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}}{(1-\tilde{\delta})k_\nu \operatorname{vol}\Omega_{\nu,E}}\;\frac{\operatorname{vol}\Omega_{\nu,E}\operatorname{tr}\ell_{\nu,E}(B^2_{\nu,T,0})}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}} \\ &= \frac{1}{\delta}\,\frac{1}{1-\tilde{\delta}}\,\mathrm{M}_{E,\nu,\lambda}\big((\operatorname{symb}^{SW}[B_{\nu,T,0}])^2\big),\end{aligned} \tag{6.2.4}$$

when employing the tracial property $(v)$ of Theorem 3.7.5.

According to Proposition 6.1.1 the Stratonovich-Weyl symbol of $B_{\nu,T,0}$ can now be easily calculated as

$$\operatorname{symb}^{SW}[B_{\nu,T,0}(x,\xi)](\eta) = \frac{1}{T}\int_0^T \big(b_{0,\nu}\circ Y^t_\nu\big)(x,\xi,\eta)\;\mathrm{d}t - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}).$$

Since we assume the skew-product flow $Y^t_\nu$ to be ergodic with respect to $\mathrm{d}\ell\,\mathrm{d}\eta$, the relation (6.2.1) implies that $\operatorname{symb}^{SW}[B_{\nu,T,0}(x,\xi)](\eta)$ vanishes in the limit $T\to\infty$ for almost all points $(x,\xi,\eta)\in\Omega_{\nu,E}\times\mathcal{O}_\lambda$. Now, on the right-hand side of (6.2.4) the square of $\operatorname{symb}^{SW}[B_{\nu,T,0}]$ enters integrated over $\Omega_{\nu,E}\times\mathcal{O}_\lambda$, so that this expression vanishes as $T\to\infty$. We hence conclude that

$$\lim_{\hbar\to 0} S^{\delta}_{2,\nu}(E,\hbar) = 0.$$

This, in turn, is equivalent to the existence of a subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}\subset\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$ of density one, such that equation (6.2.2) holds, see [Wal82]. Finally, by a diagonal construction as in [Zel87, CdV85] one can extract a subsequence of $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ that is still of density one in $\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$, such that (6.2.2) holds independently of the operator $\mathcal{B}$. □

The version of quantum ergodicity asserted in Theorem 6.2.1 means that in the semiclassical limit the lifts of almost all quasimodes $\phi_{j,\nu}$ to the phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$ equidistribute in the sense that suitable Wigner functions (weakly) converge to an invariant measure on $\Omega_{\nu,E}\times\mathcal{O}_\lambda$ that is proportional to $\mathrm{d}\ell\,\mathrm{d}\eta$. In order to identify the proper Wigner transform consider

$$\begin{aligned}&\langle\phi_{j_\alpha,\nu},\mathcal{B}\phi_{j_\alpha,\nu}\rangle = \\ &\qquad\frac{1}{(2\pi\hbar)^d}\iint_{\mathrm{T}^*\mathbb{R}^d}\operatorname{tr}\Big(W[\phi_{j_\alpha,\nu}](x,\xi)P_\nu(x,\xi)\#B(x,\xi)\#P_\nu(x,\xi)\Big)\;\mathrm{d}x\;\mathrm{d}\xi + O(\hbar^\infty),\end{aligned}$$

with the matrix valued Wigner transform

$$W[\psi](x,\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\xi y}\,\overline{\psi}(x-\tfrac{y}{2})\otimes\psi(x+\tfrac{y}{2})\;\mathrm{d}y \tag{6.2.5}$$

defined for $\psi\in L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. We now exploit the Stratonovich-Weyl calculus to conclude that on the level of principal symbols

$$\begin{aligned}\operatorname{tr}\Big(W[\phi_{j_\alpha,\nu}]P_{\nu,0}B_0P_{\nu,0}\Big) &= \operatorname{tr}\Big(\big(V^*_\nu W[\phi_{j_\alpha,\nu}]V_\nu\big)\big(V^*_\nu B_0 V_\nu\big)\Big) \\ &= \int_{\mathcal{O}_\lambda}\operatorname{symb}^{\mathrm{SW}}[V^*_\nu W[\phi_{j_\alpha,\nu}]V_\nu](\eta)\operatorname{symb}^{SW}[V^*_\nu B_0 V_\nu](\eta)\;\mathrm{d}\eta.\end{aligned}$$

The second factor in the integral has been defined as $b_{0,\nu}$ in (6.1.4). In analogy to this we therefore introduce for $\psi \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ the scalar Wigner transform (see also [BGK01])

$$w_\nu[\psi](x,\xi,\eta) := \mathrm{symb}^{SW}[V_\nu^*(x,\xi)W[\psi](x,\xi)V_\nu(x,\xi)](\eta),$$

that indeed provides a lift of $\psi$ to the phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. The statement of Theorem 6.2.1 can thus be rephrased in that under the given conditions one obtains (in the sense of a weak convergence),

$$\lim_{\hbar\to 0} \frac{1}{(2\pi\hbar)^d}\, w_\nu[\phi_{j_\alpha,\nu}](x,\xi,\eta)\ \mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}\eta = \frac{1}{\mathrm{vol}\,\mathcal{O}_\lambda}\ \mathrm{d}\ell(x,\xi)\ \mathrm{d}\eta$$

along the subsequence of density one. However, since in $\phi_{j,\nu}$ the normalisation of $\mathcal{P}_\nu\psi_j$ is hidden, an equivalent equidistribution for the lifts of the projected eigenvectors is only shown up to a constant. In analogy to the discussion in [Sch01] this means that in the sequence $\{\psi_j;\ E_j \in I(E,\hbar)\}$ there exists a subsequence $\{\psi_{j_\alpha}\}$ of density one such that as $\hbar \to 0$,

$$\langle \psi_{j_\alpha}, \mathcal{P}_\nu \mathcal{B} \mathcal{P}_\nu \psi_{j_\alpha}\rangle = \|\mathcal{P}_\nu\psi_{j_\alpha}\|^2 \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}) + o(1),$$

with a corresponding statement for the scalar Wigner transforms $w_\nu[\mathcal{P}_\nu\psi_{j_\alpha}]$. Notice that the factor $\|\mathcal{P}_\nu\psi_{j_\alpha}\|^2$ is independent of the operator $\mathcal{B}$ so that the subsequence can again be chosen independently of $\mathcal{B}$. Therefore, a non-vanishing semiclassical limit only exists for those subsequences along which the norms $\|\mathcal{P}_\nu\psi_{j_\alpha}\|$ do not tend to zero as $\hbar \to 0$. These subsequences are excluded in the formulation of Theorem 6.2.1 since $\delta$ is fixed and positive.

The difficulties with estimating norms of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ arise from the presence of several level surfaces $\Omega_{\nu,E}$ on which the lifts of eigenfunctions potentially condense in the semiclassical limit. The situation simplifies considerably, if at the energy $E$ all of the $l$ level surfaces except one are empty.

**Corollary 6.2.2.** *If under the conditions stated in Theorem 6.2.1 only the level surface $\Omega_{\nu,E} \subset \mathrm{T}^*\mathbb{R}^d$ is non-empty, there exists a subsequence $\{\psi_{j_\alpha}\}$ of density one in $\{\psi_j;\ E_j \in I(E,\hbar)\}$, independent of the operator $\mathcal{B}$, such that*

$$\lim_{\hbar\to 0}\langle \psi_{j_\alpha}, \mathcal{P}_\mu \mathcal{B} \mathcal{P}_\mu \psi_{j_\alpha}\rangle = \delta_{\mu\nu}\, \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}).$$

In this situation the norms $\|\mathcal{P}_\mu\psi_{j_\alpha}\|$ converge to one for $\mu = \nu$ and to zero otherwise as $\hbar \to 0$ along the subsequence. The lifts of the eigenvectors therefore condense on the only available level surface in $\mathrm{T}^*\mathbb{R}^d$, as one clearly would have expected.

**Remark 6.2.3.** As a condition for quantum ergodicity to hold we have assumed the skew-product flow $Y_\nu^t$ on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ to be ergodic. The reason for introducing this flow was to formulate a genuinely classical criterion in terms of a dynamics on the symplectic phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. The formulation will be somewhat simpler, if one refrains from insisting on a completely classical description and employs the skew-product flow $\tilde{Y}_\nu^t$ defined on $\mathrm{T}^*\mathbb{R}^d \times G$, see (2.3.35), instead. Then the use of the Stratonovich-Weyl calculus can be avoided. Such a formulation is based on a hybrid of the classical Hamiltonian flow $\Phi_\nu^t$ on $\mathrm{T}^*\mathbb{R}^d$ and the dynamics represented by the conjugation with the unitary matrices $D_\nu$, which appears to be quantum mechanical in nature. Both formulations, however, are equivalent in the sense that, first, the Stratonovich-Weyl calculus relates the quantum dynamics in the eigenspace to a classical dynamics on the coadjoint orbit in a one-to-one manner. Second, in Section 6.3 we show that the skew-product $Y_\nu^t$ on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ is ergodic, if and only if the skew-product $\tilde{Y}_\nu^t$ is ergodic on $\Omega_{\nu,E} \times G$. One can therefore formulate Theorem 6.2.1 without recourse to the Stratonovich-Weyl calculus once the limit $\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})$ is expressed as

$$\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}) = \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(P_{\nu,0} B_0 P_{\nu,0}),$$

see (6.2.1). Up to equation (6.2.4) the proof of Theorem 6.2.1 proceeds in the same manner as shown. From this point on one can then basically follow the method of [BG00], and to this end represents the principal symbol $B_{\nu,T,0}$ of the auxiliary operator (6.2.3) in terms of the isometries $V_\nu$,

$$V_\nu^* B_{\nu,T,0} V_\nu = \frac{1}{T} \int_0^T D_\nu^* \big( (V_\nu^* B_0 V_\nu) \circ \Phi_\nu^t \big) D_\nu \ \mathrm{d}t - \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu).$$

We now suppose that the flow $\tilde{Y}_\nu^t$ is ergodic on $\Omega_{\nu,E} \times G$ and choose the function $F(x,\xi,g) := \rho(g)^* (V_\nu^* B_0 V_\nu)(x,\xi) \rho(g) \in L^1(\Omega_{\nu,E} \times G) \otimes \mathrm{M}_{k_\nu}(\mathbb{C})$ to exploit the ergodicity. This yields for almost all initial values $(x,\xi,g) \in \Omega_{\nu,E} \times G$,

$$\begin{aligned} &\lim_{T\to\infty} \rho(g)^* V_\nu^*(x,\xi) B_{\nu,T,0}(x,\xi) V_\nu(x,\xi) \rho(g) \\ &\qquad = \int_{\Omega_{\nu,E}} \int_G \rho(h)^* (V_\nu^* B_0 V_\nu)(y,\zeta) \rho(h) \ \mathrm{d}h \, \mathrm{d}\ell(y,\zeta) - \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu). \end{aligned}$$

Furthermore, since the representation $(\rho, \mathbb{C}^{k_\nu})$ is assumed to be irreducible and the integral in the above expression is invariant under conjugation with arbitrary elements of $U(k_\nu)$, Schur's lemma implies that this integral is a multiple of the identity in $\mathbb{C}^{k_\nu}$, leading to

$$\int_{\Omega_{\nu,E}} \int_G \rho(h)^* (V_\nu^* B_0 V_\nu)(y,\zeta) \rho(h) \ \mathrm{d}h \, \mathrm{d}\ell(y,\zeta) = \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu).$$

Due to the way the principal symbol $B_{\nu,T,0}$ enters on the right-hand side of (6.2.4), the conjugation with $V_\nu(x,\xi)\rho(g)$ as well as the restriction to almost all $(x,\xi,g)$ is inessential, so that again one concludes a vanishing of $S_{2,\nu}^\delta(E,\hbar)$ as $\hbar \to 0$.

**Remark 6.2.4.** Let us briefly comment on the possible consequences in the case that Conjecture 2.4.4 holds true. The quasimodes $\mathcal{P}_\nu \psi_j$ clearly fulfill condition (2.4.14) since by construction $\|\mathcal{P}_\nu(\mathcal{A}) - \mathcal{P}_\nu(\mathcal{A}')\| \leq \hbar^p \|\mathcal{A} - \mathcal{A}'\|$ and therefore

$$\|\mathcal{P}_\nu(\mathcal{A})\psi_j - \mathcal{P}_\nu(\mathcal{A}')\psi_j\| \leq \hbar^p \|\mathcal{A} - \mathcal{A}'\|.$$

Generically the quasimodes (and thus the eigenfunctions) therefore would be associated with one energy shell in the sense that the projection to all others is semiclassically small. Thus in a generic situation Theorem 6.2.1 would result in the fact that a subsequence of projected eigenfunctions of density $\frac{k_\nu \operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}}$ is semiclassically concentrated on $\Omega_{\nu,E}$ and in particular converges to Liouville measure on that energy shell.

Before we now turn to a concrete example, we summarize the results of the preceding Chapters. One of the main topics is the comparison between quantum mechanical and classical time evolution, where we have started with a discussion in the Heisenberg picture, i.e. we were concerned with the time evolution of observables for quantum systems with internal degrees of freedom. It turns out that the quantum evolution in leading semiclassical order is given by a classical dynamics given in terms of a skew-product flow. The base flow is generated by a Hamiltonian on the translational phase space and the skewing function is given by the transport matrices corresponding to the internal degrees of freedom. The appearance of a skew-product structure reflects the different levels of description for the translational and the internal degrees of freedom in the situation that only one of the available semiclassical parameters is used. Thus, the same skew-product turns up when one considers the semiclassical time evolution of combined coherent states in this limit. Coherent states allow for a refined analysis concerning the approximation of quantum time evolution. Thus, this procedure was also performed in the second semiclassical scenario where both types of degrees of freedom were considered to be semiclassical. In this case then the relevant classical dynamics is given by a Hamiltonian flow on the combined phase space for both types of degrees of freedom. Both semiclassical scenarios have in common, that the quantum dynamics can be approximated in classical terms. Another indication that classical dynamics is relevant for the semiclassical limit of the quantum time evolution is that classical ergodicity implies quantum ergodicity.

## 6.3 The relation between the ergodicity of two skew-product flows

In the previous Section and Section 2.3 we considered two types of skew-product dynamics built over the Hamiltonian flows $\Phi_\nu^t$ on $\mathrm{T}^*\mathbb{R}^d$. Both derive from the dynamics in the eigenvector bundles $E^\nu \to \mathrm{T}^*\mathbb{R}^d$ given by conjugating the diagonal blocks of principal symbols with the transport matrices $d_{\nu\nu}$ along integral curves of the Hamiltonian flows. After having fixed local orthonormal bases in the fibres, or isometries $V_\nu(x,\xi) : \mathbb{C}^{k_\nu} \to E^\nu(x,\xi)$, respectively, the transport matrices $d_{\nu\nu}$ are represented by unitary $k_\nu \times k_\nu$ matrices $D_\nu$, leading to the skew-product flows $\hat{Y}_\nu^t$ on $\mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu)$. We then noticed that the dynamics in the fibres might not exhaust the whole group $\mathrm{U}(k_\nu)$, but only some subgroup $G$, which is then represented in $\mathrm{U}(k_\nu)$. This led us to consider the skew-product flows $\tilde{Y}_\nu^t$ on $\mathrm{T}^*\mathbb{R}^d \times G$, given as $\tilde{Y}_\nu^t(x,\xi,g) = (\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)g)$, see (2.3.34) and (2.3.35). Assuming that the representation $\rho$ of $G$ in $\mathrm{U}(k_\nu)$ is irreducible, we constructed a representation of the fibre dynamics on the coadjoint orbit $\mathcal{O}_\lambda$ of $G$ determined by $\rho$. We thus arrived at the skew-product flows $Y_\nu^t$ on the symplectic phase spaces $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$, with $Y_\nu^t(x,\xi,\eta) = (\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta)$, see (6.1.1) and (6.1.2). In section 6.2 we required either the flows $\tilde{Y}_\nu^t$ or $Y_\nu^t$, restricted to the level surfaces $\Omega_{\nu,E} \subset \mathrm{T}^*\mathbb{R}^d$ in the base manifold, to be ergodic relative to the respective invariant measures $\mathrm{d}\ell\ \mathrm{d}g$ or $\mathrm{d}\ell\ \mathrm{d}\eta$. We now show:

**Proposition 6.3.1.** *The flow $\tilde{Y}_\nu^t : \Omega_{\nu,E} \times G \to \Omega_{\nu,E} \times G$ is ergodic with respect to $\mathrm{d}\ell\ \mathrm{d}g$, if and only if the associated flow $Y_\nu^t : \Omega_{\nu,E} \times \mathcal{O}_\nu \to \Omega_{\nu,E} \times \mathcal{O}_\nu$ is ergodic with respect to $\mathrm{d}\ell\ \mathrm{d}\eta$.*

*Proof.* A convenient characterisation for the ergodicity of a flow $\Phi^t$ on a probability space $(\Sigma, \mathrm{dm})$ with invariant measure $\mathrm{dm}$ employs the flow-invariant subsets of $\Sigma$: The flow is ergodic with respect to $\mathrm{dm}$, if and only if every measurable flow-invariant set has either measure zero or full measure. We now first consider the 'if' direction asserted in the proposition and to this end assume that $Y_\nu^t$ on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ is ergodic with respect to $\mathrm{d}\ell\ \mathrm{d}\eta$. Hence every measurable $Y_\nu^t$-invariant set $B \subset \Omega_{\nu,E} \times \mathcal{O}_\lambda$ has either measure zero or full measure. In order to relate these sets with subsets of $\Omega_{\nu,E} \times G$ we recall the composed map $G \xrightarrow{\pi} G/G_\lambda \xrightarrow{\kappa} \mathcal{O}_\lambda$ from section 2.3.3, where $\pi$ denotes the canonical projection of $G$ onto $G/G_\lambda$ and $\kappa$ is the diffeomorphism that identifies $G/G_\lambda$ with $\mathcal{O}_\lambda$. One then realises that the following diagram commutes:

$$\begin{tikzcd}
(x,\xi,g) \arrow[r, mapsto, "\tilde{Y}_\nu^t"] \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi"'] & \big(\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)g\big) \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi"] \\
(x,\xi,gG_\lambda) \arrow[r, mapsto, "\overline{Y}_\nu^t"] \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa"'] & (\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)gG_\lambda) \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa"] \\
(x,\xi,\eta) \arrow[r, mapsto, "Y_\nu^t"] & \big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}(\eta)\big)
\end{tikzcd}\ , \tag{6.3.1}$$

where $\overline{Y}_\nu^t$ is induced by $Y_\nu^t$ under $\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi$. According to this diagram a $\tilde{Y}_\nu^t$-invariant set $A \subset \Omega_{\nu,E} \times G$ projects to a $Y_\nu^t$-invariant subset $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ of $\Omega_{\nu,E} \times \mathcal{O}_\lambda$. The assumed ergodicity of $Y_\nu^t$ then implies that the measure of $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ is zero or one. Now the normalised Haar measure $\mathrm{d}g$ on $G$ projects under $\kappa\circ\pi$ to the volume measure $\mathrm{d}\eta$ on the coadjoint orbit $\mathcal{O}_\lambda$. This can be obtained from the Fubini theorem (cf. [BtD85]) which states for every $f \in L^1(\mathcal{O}_\lambda)$ that

$$\begin{aligned}
\int_G (\pi^*\kappa^* f)(g)\ \mathrm{d}g &= \int_{G/G_\lambda}\left(\int_{G_\lambda}(\kappa^* f)\circ\pi(gh)\ \mathrm{d}h\right)\ \mathrm{d}(gG_\lambda) \\
&= \int_{G/G_\lambda}(\kappa^* f)(gG_\lambda)\ \mathrm{d}(gG_\lambda).
\end{aligned} \tag{6.3.2}$$

Here $\mathrm{d}h$ denotes the normalised Haar measure on $G_\lambda$ and $\mathrm{d}(gG_\lambda)$ is the normalised left invariant volume form on $G/G_\lambda$ arising from the volume form on the coadjoint orbit under the pullback $\kappa^*$. Hence, the sets $A$ and $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ have identical measures and thus the measure of $A$ is either zero or one. Therefore, the assumed ergodicity of $Y_\nu^t$ implies ergodicity of $\tilde{Y}_\nu^t$.

In order to prove the opposite direction one simply reverses the above argument: Starting with $Y_\nu^t$-invariant subsets of $\Omega_{\nu,E}\times\mathcal{O}_\lambda$, one lifts these to $\Omega_{\nu,E}\times G$. Due to the commuting diagram (6.3.1) these lifts are $\tilde{Y}_\nu^t$-invariant and therefore, according to the assumed ergodicity of $\tilde{Y}_\nu^t$, have measure zero or one. Again the Fubini theorem (6.3.2) implies equal measures of the sets and their lifts. Hence $Y_\nu^t$ is ergodic. $\square$

# Chapter 7

# A remark on Zitterbewegung and semiclassical observables for the Dirac equation

In this final Chapter we outline the application of the techniques developed in the preceding Chapters to the Dirac equation, which can be seen as a prototype of a system that carries internal degrees of freedom that are modeled by matrix-valued Weyl operators. In particular, we use the fact that we are in the situation of Section 2.3 and Chapter 6. The Dirac equation is a system of partial differential equations given as a Cauchy problem for a matrix valued Weyl operator. In particular, the physical significance of the semiclassical projection operators (2.3.12) reveal their importance when one studies the Dirac equation. On $\mathrm{T}^*\mathbb{R}^d$ the Dirac equation reads

$$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\psi(t,x) = \mathcal{H}_{\mathrm{D}}\psi(t,x). \tag{7.i}$$

It describes a particle with mass $m$ and charge $e$ coupled to external (time-independent) electromagnetic field $E(x) = -\operatorname{grad}\phi(x)$ and $B(x) = \operatorname{rot} A(x)$. The Hamiltonian reads

$$\mathcal{H}_{\mathrm{D}} = c\alpha\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{e}{c}A(x)\right) + \beta mc^2 + e\phi(x), \tag{7.ii}$$

where matrices

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

with the Pauli matrices $\sigma_j$, define a realization of the Dirac algebra, i.e.

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \quad \alpha_k\beta + \beta\alpha_k = 0.$$

If we restrict to smooth potentials $\phi$ and $A$, this operator is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d) \otimes \mathbb{C}^4$, see [Tha92, EL99]. Since the Dirac operator is a matrix valued differential operator, it can be written as Weyl operator, see Section 2.2.1, whose symbol is given by

$$H_{\mathrm{D}}(x,\xi) = c\alpha\left(\xi - \frac{e}{c}A(x)\right) + \beta mc^2 + e\phi(x). \tag{7.iii}$$

In a first step we deal with the case without interactions, i.e. $A = 0$ and $\phi = 0$.

## 7.1 The free Dirac equation

Dirac had introduced the relativistic quantum theory of spin $1/2$ particles, which reproduced the spectrum of the hydrogen atom with overwhelming accuracy. However, it turned out, that the theory also had to deal with a number of inconsistencies as, e.g. Klein's paradox [Kle29]. In addition, Schrödinger observed [Sch30] that the (free) time evolution of the naive position operator, that he had taken over from the non-relativistic quantum theory, contained a contribution without any classical interpretation. Since this term was rapidly oscillating he introduced the notion of *Zitterbewegung (trembling motion)*. Schrödinger's intention was to explain this phenomenon as being caused by spin. However, later on it turned out that all occuring paradoxa had their origin in the coexistence of particles and anti-particles. Only a quantum field theoretic description, which incorporates particle creation and annihilation, can therefore completely solve these problems.

Within the context of relativistic quantum mechanics, however, it is possible to remove Zitterbewegung of free particles by introducing modified position operators that are associated with either particles or anti-particles only. These operators are free from particle/anti-particle interferences that cause Zitterbewegung, see [Tha92] for details. In the case of interactions (with external potentials) Zitterbewegung cannot be exactly eliminated. It is, however, possible to devise an asymptotic construction, for example one based on the Foldy-Wouthuysen transformation. This involves a non-relativistic expansion, and when applied to the Dirac equation reproduces in leading order the Pauli equation. The genuinely relativistic coexistence of particles and anti-particles is therefore removed in a natural way.

In a semiclassical context similar constructions are possible while the relativistic level of description is maintained. This can be achieved through a semiclassical decoupling of particles and anti-particles, i.e. asymptotically order by order in powers of $\hbar$. In this direction several approaches have been developed recently, aiming at semiclassical expansions for scattering phases [BN99, BR99] or at recovering Thomas precession of spin [Spo00].

Let us consider the origin of Zitterbewegung for the free Dirac equation more closely: in the case of vanishing potentials the spectrum of the Dirac operator (7.ii) is absolutely continuous and consists of $(-\infty, -mc^2) \cup (mc^2, \infty)$, see e.g. [Tha92]. The time evolution

$$\mathcal{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\mathcal{H}_{\mathrm{D}} t\right)$$

generated by the free Dirac Hamiltonian is unitary, since $\mathcal{H}_{\mathrm{D}}$ is essentially selfadjoint. Schrödinger considered the time evolution of the standard position operator $\hat{x}$, whose components are just the multiplication with the respective coordinate functions acting on a suitable domain in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. His intention was to resolve the apparent paradox that the spectrum of the velocity operator resulting from the Heisenberg equation of motion of the standard position operator, $\dot{\hat{x}}(t) = c\alpha(t)$, was $\pm c$, whereas the classical velocity of a free relativistic particle reads $c^2\xi/\sqrt{\xi^2c^2+m^2c^4}$ and thus is smaller than $c$ in magnitude. Integrating the equations of motion Schrödinger found

$$\begin{aligned}\hat{x}(t) &= \mathcal{U}(t)^*\hat{x}\,\mathcal{U}(t)\\ &= \hat{x}(0) + c^2\hat{p}\mathcal{H}^{-1}t + \frac{\hbar}{2\,\mathrm{i}}\mathcal{H}^{-1}\left(\mathrm{e}^{\frac{2\,\mathrm{i}}{\hbar}\mathcal{H}t} - \mathrm{id}\right)\mathcal{F}.\end{aligned} \tag{7.1.1}$$

The first two terms of this result exactly correspond to the respective classical dynamics of a free relativistic particle. The third term however contains the operator

$$\mathcal{F} := c\alpha - c^2\hat{p}\mathcal{H}^{-1},$$

which is well-defined since zero is not contained in the spectrum of the free Dirac operator. This quantity expresses the difference between the standard velocity operator and the quantization of the classical velocity. It actually introduces a rapidly oscillating time dependence and hence was named Zitterbewegung (trembling motion) by Schrödinger. In order to interpret $\mathcal{F}$ by means of interference terms between particle and anti-particle, we introduce the projection operators

$$\mathcal{P}_\pm := \frac{1}{2}\left(\mathrm{id} \pm |\mathcal{H}|^{-1}\mathcal{H}\right) \tag{7.1.2}$$

that fulfill $\mathcal{P}_\mp\mathcal{P}_\pm = 0$ and $\mathcal{P}_+ + \mathcal{P}_- = \mathrm{id}$. Then the time evolution of the projected position operators

$$\hat{x}_\pm = \mathcal{P}_\pm\hat{x}\mathcal{P}_\pm$$

is given by, see [Tha92],

$$\hat{x}_\pm(t) = \hat{x}_\pm(0) + c^2\hat{p}\mathcal{H}^{-1}t\mathcal{P}_\pm$$

and hence exactly corresponds to the respective classical expressions. So particles and anti-particles show characteristically different time evolutions and the Zitterbewegung in equation (7.1.1) represents the interference term

$$\mathcal{P}_+\hat{x}\mathcal{P}_- + \mathcal{P}_-\hat{x}\mathcal{P}_+ = \frac{\mathrm{i}\,\hbar}{2}\mathcal{H}^{-1}\mathcal{F}$$

which is absent in a classical description.

Turning to the general case of non-vanishing external fields, we want to give a recipe that allows us to separate particles from anti-particles in a semiclassical fashion: the Hilbert space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ splits into mutually orthogonal subspaces and observables are to be projected to these subspaces. To perform this procedure we have already developed the necessary techniques in Section 2.3. We will now specify them to the present problem.

## 7.2 Semiclassical decoupling of particles and anti-particles

In the case of non-vanishing, smooth external potentials the Dirac Hamiltonian is essentially selfadjoint and therefore again generates a unitary time evolution. The Weyl symbol of the Hamiltonian (7.iii) is a function on the phase space $\mathrm{T}^*\mathbb{R}^3$ which takes values in the hermitian $4 \times 4$ matrices. In particular, the two doubly degenerate eigenvalue functions

$$h_\pm(x,\xi) = e\phi(x) \pm \sqrt{(c\xi - eA(x))^2 + m^2c^4}$$

can be identified as the classical Hamiltonians of particles and anti-particles, respectively, without spin. Associated with these eigenvalues are the eigenprojections

$$P_0^\pm(x,\xi) := \frac{1}{2}\left(\mathrm{id} \pm \frac{\alpha \cdot (c\xi - eA(x)) + \beta mc^2}{\sqrt{(c\xi - eA(x))^2 + m^2c^4}}\right) \tag{7.2.1}$$

from $\mathrm{T}^*\mathbb{R}^3 \times \mathbb{C}^4$ to the respective eigenbundles $E^\pm(x,\xi) \to \mathrm{T}^*\mathbb{R}^3$, compare Section 2.3.3. In the absence of potentials the projections (7.2.1) are independent of $x$, which implies that their square in the sense of the symbol product of Lemma 2.2.2 reduces to the pointwise product and therefore they quantize to give the projections (7.1.2). In the general case we therefore have to find analogous objects, whose quantizations give projections that decouple particle and anti-particle. Following Section 2.3.1, we have to construct the corresponding semiclassical projection operators. Let us remark that under the assumption that $\phi(x)$ is bounded in the $C^\infty$-topology and that $A(x)$ grows at most like some power of $x$, the quantity

$$\varepsilon(x,\xi) := \sqrt{(c\xi - eA(x))^2 + m^2c^4} \tag{7.2.2}$$

serves as an order function in the sense of (2.2.2) for the symbol (7.iii). In particular, the assumptions of Proposition 2.3.1 are fulfilled, and we have the existence of semiclassical projections operators $\mathcal{P}_\pm = \mathrm{op}^{\mathrm{W}}[P_\pm]$ with $P_\pm \in \mathrm{S}^0_{\mathrm{cl}}(1)$ whose principal symbols coincide with (7.2.1).

These semiclassical projections provide a semiclassical splitting of the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ in particle and anti-particle subspaces $\mathscr{H}_\pm := \mathcal{P}_\pm \mathscr{H}$. Moreover, due to the fact that the projections almost commute with the Hamiltonian, see (2.3.5), these subspaces are almost invariant with respect to the time evolution generated by $\mathcal{H}_{\mathrm{D}}$, i.e.

$$\mathcal{U}(t)\mathcal{P}_\pm\psi - \mathcal{P}_\pm\mathcal{U}(t)\psi = O(t\hbar^\infty)$$

for all $\psi \in \mathscr{H}$. Hence, up to a small error every spinor in one of the subspaces $\mathscr{H}_\pm$ remains under the time evolution within this subspace; and this is true for semiclassically long times $t << \hbar^{-N}$, for $N$ arbitrarily large. This result is in agreement with the (heuristic) physical picture that particles interact with anti-particles via tunneling; the latter is a genuine quantum process with pair production and annihilation rates that are exponentially small in terms of $\hbar$. Related to this observation is the fact that eigenspinors of the Hamiltonian can only almost be associated with particles or anti-particles: if $\psi_n \in \mathscr{H}$ is an eigenspinor $\mathcal{H}_{\mathrm{D}}\psi_n = E_n\psi_n$ its projections $\mathcal{P}_\pm\psi_n$ are in general only almost eigenspinors, i.e. quasimodes, see Section 2.4. Thus, the discrete spectrum of the Dirac Hamiltonian cannot truly be divided into a particle and anti-particle part.

We also want to specify the conditions which guarantee that the semiclassical projections $\mathcal{P}_\pm$ are semiclassically associated to spectral projections of $\mathcal{H}_{\mathrm{D}}$, see Proposition

2.3.2 for the general case. In the free case, $\mathcal{P}_\pm$ are obviously the spectral projections to $(-\infty, -mc^2)$ and $(mc^2, \infty)$, respectively. In the presence of potentials, depending on their behaviour at infinity, see Section 2.4, there exist constants $E_+ > E_-$ such that the spectrum inside $(E_-, E_+)$ is discrete and absolutely continuous outside; e.g. if the potentials and their derivatives vanish at infinity one finds $E_\pm = \pm mc^2$, see [Tha92]. For $E \in (E_-, E_+)$ not in the spectrum of $\mathcal{H}_{\mathrm{D}}$ let us denote by $\mathcal{P}^E_\pm$ the spectral projections to $(E, \infty)$ and $(-\infty, E)$ respectively. Let us assume that

$$h_-(x,\xi) < E < h_+(x,\xi)$$

for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^3$. On the submanifold $\xi = \frac{e}{c}A(x)$ this condition requires the variation of the potential $\phi(x)$ to be strictly restricted by $2mc^2$. In particular, this guarantees that the assumptions of Proposition 2.3.2 are fulfilled such that the semiclassical projectors $\mathcal{P}_\pm$ are close to the spectral projections,

$$\|\mathcal{P}_\pm - \mathcal{P}^E_\pm\| = O(\hbar^\infty).$$

Of course, the notion of invariant observables defined in Section 2.3.2 and the semiclassical dynamics in the eigenspaces also carry over to the case of the Dirac equation. We will give some details in the next section.

## 7.3 Invariant observables and the classical limit of dynamics

The example of the standard position operator for the free particle and its Zitterbewegung demonstrates that not all quantum observables possess a direct (semi-)classical interpretation. Only the diagonal blocks with respect to the projections $\mathcal{P}_\pm$ allow for establishing a quantum-classical correspondence. This observation applies in particular to dynamical questions because only after projection there exists an unambiguous classical Hamiltonian given by the respective eigenvalue of the quantum Hamiltonians principal symbol, see Theorem 2.3.4. According to the discussion in Section 2.3.2 we know that if we model observables as Weyl operators with symbols in $\mathrm{S}^\infty_{\mathrm{cl}}(1)$, and define the subalgebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ of invariant operators therein, this invariant subalgebra is modulo $O(\hbar^\infty)$ given by the operators that are block-diagonal with respect to $\mathcal{P}_\pm$, see Proposition 2.3.6. Therefore, an observable remains semiclassical under the quantum dynamics if it does not contain off-diagonal blocks representing an interference of particle and anti-particle dynamics, since this has no classical equivalent. Terms exceptional to this must be smaller than any power of $\hbar$, indicating that they arise from pure quantum effects. Furthermore, up to an error of order $\hbar^\infty$ the time evolution of $\mathcal{P}_\pm \mathcal{B} \mathcal{P}_\pm$, where $\mathcal{B} \in \mathrm{OPS}(1)$, is governed by the projected Hamiltonians $\mathcal{H}_{\mathrm{D}}\mathcal{P}_\pm$ on the quantum level. By using the description of Sections 2.3.3 and 6.1 this can be given a classical description: given an observable $\mathcal{O} \in \mathrm{S}^\infty_{\mathrm{inv}}(1)$ the diagonal blocks are (approximately) propagated with the projected Hamiltonians $\mathcal{H}_{\mathrm{D}}\mathcal{P}_\pm$ whose symbols in leading order read

$$H_{\mathrm{D}}(x,\xi)P_{0,\pm}(x,\xi) = h_\pm(x,\xi)P_{0,\pm}(x,\xi).$$

And the equations of motion are given by

$$\dot{x}_\pm(t) = \{h_\pm, x_\pm(t)\}, \quad \dot{\xi}_\pm(t) = \{h_\pm, \xi_\pm(t)\}, \tag{7.3.1}$$

whose solutions $(x_\pm(t),\xi_\pm(t)) = \Phi^t_\pm(x,\xi)$ with $(x_\pm(0),\xi_\pm(0)) = (x,\xi)$ define two Hamiltonian flows on $\mathrm{T}^*\mathbb{R}^3$. These flows represent the classical dynamics of relativistic particles (and anti-particles), whereas spin is absent from these expressions, reflecting the fact that a priori spin is a quantum mechanical concept and its classical counterpart has to be recovered in systematic semiclassical approximations to the quantum system.

In the present situation the kinematics and dynamics of spin are encoded in the matrix part of the symbols. For an observable $\mathcal{B} \in \mathrm{S}^\infty_{\mathrm{inv}}(1)$ the leading semiclassical order of its symbol is composed of the functions

$$(P_{0,\pm}B_0P_{0,\pm})(x,\xi), \tag{7.3.2}$$

taking values in the hermitian $4\times4$ matrices. These matrices act on the two dimensional subspaces $P_{0,\pm}(x,\xi)\mathbb{C}^4 = E^\pm(x,\xi)$ of $\mathbb{C}^4$, which can be viewed as the Hilbert spaces of a spin $1/2$ attached to a classical particle or anti-particle, respectively, at the point $(x,\xi)$ in phase space. Upon expanding vectors from $E^\pm$, e.g. as in Section 2.3.3 in a basis of eigenvectors, one introduces isometries $V_\pm(x,\xi) : \mathbb{C}^2 \to P_{0,\pm}(x,\xi)\mathbb{C}^4$, see e.g. [Rom92], such that $V_\pm(x,\xi)V^*_\pm(x,\xi) = P_{0,\pm}(x,\xi)$ and $V^*_\pm(x,\xi)V_\pm(x,\xi) = \mathbb{1}_2$. A possible choice for these isometries is

$$\begin{aligned} V_+(x,\xi) &= \frac{1}{\sqrt{2\varepsilon(x,\xi)(\varepsilon(x,\xi)+mc^2)}}\begin{pmatrix} \big(\varepsilon(x,\xi)+mc^2\big)\mathbb{1}_2 \\ \big(c\xi - eA(x)\big)\cdot\sigma \end{pmatrix}, \\ V_-(x,\xi) &= \frac{1}{\sqrt{2\varepsilon(x,\xi)(\varepsilon(x,\xi)+mc^2)}}\begin{pmatrix} \big(c\xi - eA(x)\big)\cdot\sigma \\ -\big(\varepsilon(x,\xi)+mc^2\big)\mathbb{1}_2 \end{pmatrix}. \end{aligned}$$

That way the diagonal blocks (7.3.2) can be represented in terms of the $2\times2$ matrices

$$\big(V^*_\pm\, B_0\, V_\pm\big)(x,\xi)\ , \tag{7.3.3}$$

which are hermitian if $B_0(x,\xi)$ is hermitian on $\mathbb{C}^4$.

According to the constructions of Section 3.7 and the fact that the coadjoint orbits of SU(2) are (isomorphic to) $\mathrm{S}^2$, the matrix valued functions (7.3.3) can be mapped in a one-to-one manner to real valued functions on the sphere $\mathrm{S}^2$ via

$$b^\pm_0(x,\xi,n) := \mathrm{tr}\Big(\Delta_{1/2}(n)\,\big(V^*_\pm\, B_0\, V_\pm\big)(x,\xi)\Big)\ , \tag{7.3.4}$$

where $n\in\mathbb{R}^3$ with $|n|=1$ is viewed as a point on $\mathrm{S}^2$ and

$$\Delta_{1/2}(n) := \frac{1}{2}\big(\mathbb{1}_2 + \sqrt{3}\, n\cdot\sigma\big)$$

takes values in the hermitian $2 \times 2$ matrices, see [VGB89]. The quantizers $\Delta_{1/2}(n)$ provide a quantum-classical correspondence in the sense of Definition 3.7.1 in the context of homogeneous spaces. Thus, the semiclassically leading term of a (block-diagonal) semiclassical observable $\mathcal{B}$ can therefore be represented by two real valued functions on the space $\mathrm{T}^*\mathbb{R}^3 \times \mathrm{S}^2$. Through the relation

$$\big(V_\pm^* \, B_0 \, V_\pm\big)(x,\xi) = \int_{\mathrm{S}^2} b_0^\pm(x,\xi,n)\, \Delta_{1/2}(n)\ \mathrm{d}n \ ,$$

where $\mathrm{d}n$ is the normalised area form on $\mathrm{S}^2$, the matrix valued expressions (7.3.3) are unambiguously recovered from the functions (7.3.4), see also [BGK01].

The point $n \in \mathrm{S}^2$ that arises as an additional variable in a general classical observable (7.3.4) can be viewed as a classical equivalent of spin. This interpretation is suggested by the fact that the classical observable (7.3.4) associated with quantum spin,

$$\mathcal{B} = \frac{\hbar}{2}\,\Sigma_k \qquad \text{with} \qquad \Sigma_k := \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \ ,$$

can be calculated as

$$\begin{aligned} b_0^\pm(x,\xi,n) &= \frac{\hbar}{2}\,\mathrm{tr}\Big(\Delta_{1/2}(n)\, V_\pm^*(x,\xi)\, \Sigma_k\, V_\pm(x,\xi)\Big) \\ &= \sqrt{\tfrac{1}{2}\big(\tfrac{1}{2}+1\big)}\,\hbar\, n_k \ . \end{aligned}$$

Up to its normalization, which depends on $\hbar$ and the spin quantum number $s = 1/2$, the point $n \in \mathrm{S}^2$ can thus be considered as a classical spin. The latter is therefore represented on its natural phase space $\mathrm{S}^2$, and the space $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{S}^2$ on which the classical observables (7.3.4) are defined can be viewed as the combined phase space of the translational and the spin degrees of freedoms.

We now want to identify the classical dynamics of both the translational and the spin degrees of freedom that emerge in the semiclassical limit of the quantum dynamics generated by the Hamiltonian $\hat{H}$. To this end we recall that $\mathrm{S}_{\mathrm{inv}}$ was defined to contain those semiclassical observables $\mathcal{B} \in \mathrm{S}_{\mathrm{cl}}$ whose time evolutions $\mathcal{B}(t) = \hat{U}(t)^* \mathcal{B} \hat{U}(t)$ are again semiclassical observables. Hence $\mathcal{B}(t)$ is a Weyl operator with symbol $B(t)(x,\xi;\hbar) \in \mathrm{S}^0_{\mathrm{cl}}(1)$. The leading term $B(t)_0(x,\xi)$ then yields the classical time evolution of both types of degrees of freedom and is given by

$$B(t)_0(x,\xi) = \sum_{\nu \in \{+,-\}} d_\nu^*(x,\xi,t)\, \big(\Pi_0^\nu B_0 \Pi_0^\nu\big)\big(\Phi_\nu^t(x,\xi)\big)\, d_\nu(x,\xi,t) \ . \tag{7.3.5}$$

Here, as in Theorem 2.3.4, the unitary $4 \times 4$ matrices $d_\pm(x,\xi,t)$ are determined by the transport equations, cf. equation (2.3.22),

$$\dot{d}_\pm(x,\xi,t) + \mathrm{i}\, H_\pm\big(\Phi_\pm^t(x,\xi)\big)\, d_\pm(\xi,x,t) = 0 \ , \qquad d_\pm(\xi,x,0) = \mathbb{1}_4 \ , \tag{7.3.6}$$

and the effective spin-Hamiltonians $H_\pm$, in analogy to (2.3.21), are defined as

$$H_\pm := -\mathrm{i}[\Pi_0^\pm, \{h_\pm, \Pi_0^\pm\}] + \frac{\mathrm{i}}{2}\big(h_\pm \Pi_0^\pm \{\Pi_0^\pm, \Pi_0^\pm\}\Pi_0^\pm + \Pi_0^\pm\{\Pi_0^\pm, H - h_\pm \Pi_0^\pm\}\Pi_0^\pm\big) \ . \tag{7.3.7}$$

We refrain from giving explicit expressions for the $4 \times 4$ effective spin-Hamiltonians (7.3.7) here since below we will only work in a $2 \times 2$ representation; instead we refer to [Spo00].

As a consequence of Lemma 2.3.7, the $4\times4$ matrices $d_\pm(x, \xi, t)$ map the two dimensional subspaces $\Pi_0^\pm(x, \xi)\mathbb{C}^4$ of $\mathbb{C}^4$ unitarily to the propagated subspaces $\Pi_0^\pm(\Phi_\pm^t(x, \xi))\mathbb{C}^4$ [BG04a]. They hence describe the transport of the (anti-) particle spin along the classical trajectories $(x_\pm(t), \xi_\pm(t))$. If one prefers to work in the orthonormal eigenbases $\{e_1^\pm(x, \xi), e_2^\pm(x, \xi)\}$ of the spaces $\Pi_0^\pm(x, \xi)\mathbb{C}^4$ one can introduce the unitary $2 \times 2$ matrices

$$D_\pm(x, \xi, t) := V_\pm^*\big(\Phi_\pm^t(x, \xi)\big)\, d_\pm(x, \xi, t)\, V_\pm(x, \xi) \ ,$$

see (2.3.31), which are determined by the equations

$$\dot{D}_\pm(x, \xi, t) + \frac{\mathrm{i}}{2} C_\pm\big(\Phi_\pm^t(x, \xi)\big) \cdot \sigma\, D_\pm(\xi, x, t) = 0 \ , \qquad D_\pm(\xi, x, 0) = \mathbb{1}_2 \ , \tag{7.3.8}$$

following from (7.3.6). The transformed effective spin-Hamiltonians $C_\pm \cdot \sigma/2$ derive from (7.3.7) and can be expressed in terms of the electromagnetic fields through

$$C_\pm(x, \xi) = \mp \frac{ec}{\varepsilon(x, \xi)} \left( B(x) \pm \frac{1}{\varepsilon(x, \xi) + mc^2} \Big( cE(x) \times \big(c\xi - eA(x)\big)\Big)\right) \ ,$$

where $\varepsilon(x, \xi)$ is defined in (7.2.2). Since therefore the effective spin-Hamiltonians are hermitian and traceless $2 \times 2$ matrices, the solutions $D_\pm$ of (7.3.8) are in $\mathrm{SU}(2)$. Related results have previously been obtained in a WKB-type situation [RK63, BK98, BK99a] and in the context of a semiclassical propagation of Wigner functions [Spo00].

The transport matrices $D_\pm(x, \xi, t)$ carry two-component spinors along the trajectories $\Phi_\pm^t(x, \xi)$ and induce a classical spin dynamics along (anti-) particle trajectories. These combined classical dynamics can be recovered upon representing the leading symbol (7.3.5) in terms of the functions (7.3.4) on the combined phase space,

$$\begin{aligned} b(t)_0^\pm(x, \xi, n) &= \mathrm{tr}\Big(\Delta_{1/2}(n)\, \big(V_\pm^*\, B(t)_0\, V_\pm\big)(x, \xi)\Big) \\ &= \mathrm{tr}\Big(D_\pm(x, \xi, t)\, \Delta_{1/2}(n)\, D_\pm^*(x, \xi, t)\, \big(V_\pm^*\, B_0\, V_\pm\big)\big(\Phi_\pm^t(x, \xi)\big)\Big) \\ &= b_0^\pm\big(\Phi_\pm^t(x, \xi), \mathrm{Ad}^*_{D_\pm(x,\xi,t)}\, n\big) \ , \end{aligned}$$

see Proposition 6.1.1. Hence, in leading semiclassical order the dynamics of an observable $\mathcal{B} \in \mathrm{S}_{\mathrm{inv}}$ can be expressed in terms of the classical time evolutions

$$(x, \xi) \mapsto \Phi_\pm^t(x, \xi) \qquad \text{and} \qquad n \mapsto n_\pm(t) = \mathrm{Ad}^*_{D_\pm(x,\xi,t)}\, n = R(D_\pm(x, \xi, t))n \ . \tag{7.3.9}$$

Here $R:\ \mathrm{SU}(2)\to\mathrm{SO}(3)$ denotes the covering of SO(3) by SU(2) according to $g(n\cdot\sigma)g^{-1}=(R(g)n)\cdot\sigma$, and the spin motion on the sphere thus emerging obeys the equation

$$\dot{n}_\pm(t)=C_\pm\big(\Phi^t_\pm(x,\xi)\big)\times n_\pm(t)\ ,\qquad n_\pm(0)=n\ , \tag{7.3.10}$$

that is implied by (7.3.8). These spin dynamics exactly coincide with the Thomas precession that was derived in a purely classical context [Tho27].

Both types of time evolutions in (7.3.9) can be combined to yield the skew-product dynamics

$$(x,\xi,n)\mapsto Y^t_\pm(x,\xi,n):=\big(\Phi^t_\pm(x,\xi),R(D_\pm(x,\xi,t))n\big) \tag{7.3.11}$$

on the phase space $\mathrm{T}^*\mathbb{R}^3\times\mathrm{S}^2$, cf. Section 6.1. These flows are composed of the Hamiltonian flows $\Phi^t_\pm$ defined in (7.3.1) on the ordinary phase space $\mathbb{R}^3\times\mathbb{R}^3$ and the spin dynamics (7.3.10) on $\mathrm{S}^2$ that are driven by the Hamiltonian flows. They leave the normalised measures

$$\mathrm{d}\ell^\pm_E(x,\xi)\ \mathrm{d}n=\frac{1}{\mathrm{vol}\,\Omega^\pm_E}\,\delta\big(h_\pm(x,\xi)-E\big)\ \ \mathrm{d}x\ \mathrm{d}p\ \mathrm{d}n$$

on the combined phase space invariant that are products of Liouville measure $\mathrm{d}\ell^\pm_E(x,\xi)$ on $\Omega^\pm_E$ and the normalised area measure $\mathrm{d}n$ on $\mathrm{S}^2$. We hence now conclude that the classical limit of the quantum dynamics generated by the Dirac-Hamiltonian (7.iii) is given by the two skew-product flows (7.3.11) combining the Hamiltonian relativistic motion of (anti-) particles with the spin precession along the (anti-) particle trajectories.

## 7.4 Semiclassical behaviour of eigenspinors

In this section we give an explicit application of the results obtained in Section 6.2 to the Dirac equation and study the semiclassical behaviour of eigenspinors in the semiclassical limit, this will involve partially repeating some of the considerations of Chapter 6. In Section 6.2 we saw that in general eigenspinors $\psi_n$ of the Dirac Hamiltonian (7.ii) cannot uniquely be associated with either particles or anti-particles. For the purpose of semiclassical studies we therefore prefer to work with the normalised projected eigenspinors

$$\phi^\pm_n:=\frac{\mathcal{P}^\pm\psi_n}{\|\mathcal{P}^\pm\psi_n\|}\ . \tag{7.4.1}$$

Due to the fact that the projectors almost commute with the Hamiltonian, the quantities (7.4.1) are almost eigenspinors (quasimodes) of both $\hat{H}$ and $\hat{H}\mathcal{P}^\pm$,

$$\big\|\big(\hat{H}-E_n\big)\phi^\pm_n\big\|=r^\pm_n\qquad\text{and}\qquad\big\|\big(\hat{H}\mathcal{P}^\pm-E_n\big)\phi^\pm_n\big\|=s^\pm_n\ ,$$

with error terms $r^\pm_n$ and $s^\pm_n$ given by

$$r^\pm_n=\frac{\|[\hat{H},\mathcal{P}^\pm]\psi_n\|}{\|\mathcal{P}^\pm\psi_n\|}\qquad\text{and}\qquad s^\pm_n=\frac{\|[\hat{H}\mathcal{P}^\pm,\mathcal{P}^\pm]\psi_n\|}{\|\mathcal{P}^\pm\psi_n\|}\ .$$

As in Section 6.2 we can give quantitative statements on the numbers $N_{E,\omega}$ and $N^{\pm}_{E,\omega}$ of eigenvalues that the operators $\mathcal{H}$ and $\mathcal{H}\mathcal{P}_{\pm}$, respectively, have in $[E-\hbar\omega, E+\hbar\omega]$. Namely, as $\hbar \to 0$,

$$\begin{aligned} N_{E,\omega} &\sim \frac{2\omega}{\pi}\,\frac{\mathrm{vol}\,\Omega^+_E + \mathrm{vol}\,\Omega^-_E}{(2\pi\hbar)^2}\ , \\ N^{\pm}_{E,\omega} &\sim \frac{2\omega}{\pi}\,\frac{\mathrm{vol}\,\Omega^{\pm}_E}{(2\pi\hbar)^2}\ . \end{aligned} \tag{7.4.2}$$

Thus, if at energy $E$ both energy shells $\Omega^{\pm}_E$ in phase space have positive volumes, the ratios $N^{\pm}_{E,\omega}/N_{E,\omega}$ governing the previous discussion are semiclassically determined by the relative fraction of the volumes of the associated energy shells.

We now explore the consequences an ergodic behaviour of the classical skew-product dynamics $Y^t_{\pm}$ exerts on the projected eigenspinors. In an analogous situation for Schrödinger-Hamiltonians the ergodicity of the associated Hamiltonian flow implies that the Wigner transforms of almost all eigenfunctions equidistribute on the corresponding energy shell. This result, known as quantum ergodicity, goes back to Shnirelman [Shn74] and has been fully proven in [Zel87, CdV85, HMR87]. In the case under study we now suppose that the energy $E$ lies in an interval $(E_-, E_+)$, in which the spectrum of the Dirac-Hamiltonian is discrete. Moreover, at least one of the classical energy shells $\Omega^+_E$ and $\Omega^-_E$ shall be non-empty and the periodic orbits of the Hamiltonian flows $\Phi^t_{\pm}$ shall be of measure zero on $\Omega^{\pm}_E$. If then $Y^t_{\pm}$ is ergodic on $\Omega^{\pm}_E \times \mathrm{S}^2$, the time average of a classical observable of the type (7.3.4) equals its phase-space average,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T b^{\pm}_0\big(Y^t_{\pm}(x,\xi,n)\big)\ \mathrm{d}t = \int_{\Omega^{\pm}_E}\int_{\mathrm{S}^2} b^{\pm}_0(x',\xi',n')\ \mathrm{d}\ell^{\pm}_E(x',\xi')\ \mathrm{d}n' =: M^{\pm}_E\big(b^{\pm}_0\big)\ ,$$

for almost all initial conditions $(x,\xi,n)$. In this setting, as a consequence of Theorem 6.2.1, in any sequence of projected and normalized projected eigenspinors eigenspinors with associated eigenvalues $E_n \in [E-\hbar\omega, E+\hbar\omega]$ and $\|\mathcal{P}^{\pm}\psi_n\|^2 \geq \delta$ there exists a subsequence $\{\phi^{\pm}_{n_\alpha}\}_{\alpha\in\mathbb{N}}$ of density one such that for every semiclassical observable $\mathcal{B} \in \mathrm{S}_{\mathrm{inv}}$

$$\lim_{\hbar\to 0}\langle \phi^{\pm}_{n_\alpha}, \mathcal{B}\phi^{\pm}_{n_\alpha}\rangle = M^{\pm}_E\big(b^{\pm}_0\big)\ . \tag{7.4.3}$$

And this density-one subsequence can be chosen independent of the observable. Again following Section 6.2, we introduce the matrix-valued Wigner functions (6.2.5) and corresponding scalar representation

$$w_{\pm}[\psi](x,\xi,n) := \frac{1}{2}\,\mathrm{tr}\Big(\Delta_{1/2}(n)\ \big(V^*_{\pm}\,W[\psi]\,V_{\pm}\big)(x,\xi)\Big)\ .$$

Thus, due to the projections inherent in the expectation value on left-hand side of (7.4.3) only one diagonal block of the observable contributes, so that this expectation value can be reformulated as

$$\begin{aligned}\langle\phi^{\pm}_{n_\alpha},\mathcal{B}\phi^{\pm}_{n_\alpha}\rangle &= \frac{1}{(2\pi\hbar)^3}\iint \mathrm{tr}\big(W[\phi^{\pm}_{n_\alpha}]\,P^{\pm}\#B\#P^{\pm}\big)(x,\xi)\ \mathrm{d}x\ \mathrm{d}\xi\\ &= \frac{1}{(2\pi\hbar)^3}\iint \mathrm{tr}\Big(\big(V^*_{\pm}\,W[\phi^{\pm}_{n_\alpha}]\,V_{\pm}\big)\big(V^*_{\pm}\,B_0\,V_{\pm}\big)(x,\xi)(1+O(\hbar))\Big)\ \mathrm{d}x\ \mathrm{d}\xi\\ &= \frac{1}{(2\pi\hbar)^3}\iiint \big(w_{\pm}[\phi^{\pm}_{n_\alpha}](x,\xi,n)\,b^{\pm}_0(x,\xi,n)(1+O(\hbar))\big)\ \mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}n\ .\end{aligned}$$

The principal result (7.4.3) of quantum ergodicity can hence be read as saying that

$$\lim_{\hbar\to 0}\frac{w_{\pm}[\phi^{\pm}_{n_\alpha}](x,\xi,n)}{(2\pi\hbar)^3} = \frac{1}{\mathrm{vol}\,\Omega^{\pm}_E}\,\delta\big(h_{\pm}(x,\xi)-E\big)\ .$$

This relation has to be understood in a weak sense, i.e. after integration with a symbol $b^{\pm}_0(x,\xi,n)$. Hence, the scalar Wigner transforms of projected eigenspinors become semiclassically equidistributed on the associated phase space $\Omega^{\pm}_E\times \mathrm{S}^2$ once the classical time evolution $Y^t_{\pm}$ is ergodic on this space.

As the discussion at the beginning of this Section and Section 2.4 shows, the difficulties with statements about genuine eigenspinors derive from the coexistence of the particle and anti-particle subspaces; and because interactions between these subspaces on a scale $\hbar^\infty$ cannot be controlled within the present setting. However, if one of the energy shells $\Omega^{\pm}_E$ were empty there is only one classical manifold onto which phase-space lifts of eigenspinors could condense, namely $\Omega^{\mp}_E\times \mathrm{S}^2$. In such a case, say when $\Omega^-_E$ is empty, the statement of Theorem 6.2.1 applies to a density-one subsequence $\{\psi_{n_\alpha}\}$ of eigenspinors themselves, see also Corollary 6.2.2, such that

$$\lim_{\hbar\to 0}\langle\psi_{n_\alpha},\mathcal{P}^+\mathcal{B}\mathcal{P}^+\psi_{n_\alpha}\rangle = M^{\pm}_E\big(b^{\pm}_0\big)\qquad \text{and}\qquad \lim_{\hbar\to 0}\langle\psi_{n_\alpha},\mathcal{P}^-\mathcal{B}\mathcal{P}^-\psi_{n_\alpha}\rangle = 0\ . \tag{7.4.4}$$

A corresponding statement then holds for the associated scalar Wigner transforms $w_{\pm}[\psi_{n_\alpha}]$. It has been mentioned previously that the restriction to only one non-empty energy shell at energy $E$ requires potentials that do not vary too strongly; e.g., $\phi(x)$ must not vary as much as $2mc^2$. This condition is fulfilled in many physically relevant situations, in which (7.4.4) therefore applies if only the classical particle dynamics $Y^t_+$ are ergodic.

# Appendix A

# Elements of symplectic geometry

*I like to think of symplectic geometry as playing the role in mathematics of a language which can facilitate communication between geometry and analysis. On the one hand, since the cotangent bundle of any manifold is a symplectic manifold, many phenomena and constructions of differential topology and geometry have symplectic "interpretations", some of which lead to the consideration of symplectic manifolds other than cotangent bundles. On the other hand, the category of symplectic manifolds has formal similarities to the categories of linear spaces used in analysis.* [Wei77b]

## A.1 Symplectic vector spaces

In this appendix we will be concerned with some basic geometric properties of symplectic vector spaces.

Suppose that $V$ is a $m$-dimensional real vector space. A bilinear form $\omega : V \times V \to \mathbb{R}$ induces a map

$$\tilde{\omega} : V \to V^*,$$

defined by

$$\tilde{\omega}(x)(y) = \omega(x, y).$$

Let $W \subset V$ be a subspace then the orthogonal with respect to $\omega$ is defined as

$$W^{\perp} = \{x \in V;\ W \subset \ker \tilde{\omega}(x)\}.$$

If $V^{\perp} = \{0\}$ then $\tilde{\omega}$ is an isomorphism. In this case $\omega$ is called *non-degenerate*.

A non-degenerate skew-symmetric bilinear form is called a linear symplectic structure on $V$. The group of linear endomorphisms that preserve $\omega$ is called the group of linear symplectic transformation $\mathrm{Sp}(V)$ or *symplectic group*. Furthermore, the conformal symplectic group $\mathrm{CSp}(V)$ consists of those endomorphisms $B : V \to V$

$$\omega(Bv, Bw) = \mu_B \omega(v, w)$$

with some $\mu_B \in \mathbb{R}$ for all $v, w \in V$. We denote by $\mathfrak{sp}(V)$ and $\mathfrak{csp}(V)$ the Lie algebras corresponding to $\mathrm{Sp}(V)$ and $\mathrm{CSp}(V)$ respectively.

Since the determinant of skew-symmetric matrices is zero when the dimension is odd we conclude that the dimension of a symplectic vector space is necessarily even. Therefore, $V$ also admits a complex structure, i.e. a linear endomorphism $\mathrm{J} : V \to V$ such that $\mathrm{J}^2 = -\,\mathrm{id}_V$. We say that J is compatible with $\omega$ if

$$\omega(\mathrm{J}x, \mathrm{J}y) = \omega(x, y), \quad \omega(x, \mathrm{J}x) \geq 0$$

for all $x, y \in V$. Then

$$g_{\mathrm{J}}(\cdot, \cdot) = \omega(\cdot, \mathrm{J}\cdot)$$

defines a symmetric, positive-definite bilinear form on $V$. We have

**Theorem A.1.1.** *Every symplectic vector space admits a compatible complex structure. Furthermore, the collection of $\omega$-compatible complex structures is contractible.*

For an $\omega$-compatible J we may also define a hermitian structure by

$$\langle \cdot, \cdot \rangle = g_{\mathrm{J}}(\cdot, \cdot) + \mathrm{i}\,\omega(\cdot, \cdot),$$

for which $\langle Jx, Jy \rangle = \langle x, y \rangle$ for any $x, y \in V$. A linear transformation from $\mathrm{GL}(V)$ which leaves any two of the structures $\omega$, $g_{\mathrm{J}}$, J on $V$ invariant preserves the third and therefore preserves the hermitian structure. In terms of the automorphism groups $\mathrm{Sp}(V)$, $\mathrm{O}(V)$, $\mathrm{GL}(V, J)$ of $\omega$, $g_{\mathrm{J}}$, J this means that the intersection of any two equals $\mathrm{U}(V)$.

## A.2 Lagrangian subspaces

There are several distinguished subspaces $W \subset V$ in a symplectic vector space $(V, \omega)$ which are characterized by their relation to their symplectic orthogonal $W^{\perp}$. Before we introduce some of them, we state the following rules:

**Lemma A.2.1.** *Let $A$, $B \subset V$ be any subspaces of the symplectic vector space $(V, \omega)$. Then*

$$A^{\perp\perp} = A, \quad \dim A^{\perp} = \dim V - \dim A,$$

*and*

$$(A + B)^{\perp} = A^{\perp} \cap B^{\perp}, \quad (A \cap B)^{\perp} = A^{\perp} + B^{\perp}.$$

**Definition A.2.2.** We call a subspace $W \subset V$ of a symplectic vector space

1. *isotropic* if it is contained in its orthogonal: $W \subset W^{\perp}$,

2. *coisotropic* or *involutive* if $W^{\perp} \subset W$,

3. *Lagrangian* or *maximally isotropic* if $W^{\perp} = W$, in which case $\dim W = \frac{1}{2} \dim V$,

4. *symplectic* if $W + W^\perp = V$ or equivalently if $W \cap W^\perp = \{0\}$.

**Example A.2.3.** Let $E$ be a real vector space with dual $E^*$. Then $E \oplus E^*$ is symplectic when endowed with the symplectic structure

$$\omega((x,\xi),(y,\eta)) = \eta(x) - \xi(y).$$

Both $E$ and $E^*$ are Lagrangian subspaces since the restriction of $\omega$ to either space vanishes and $\dim E = \dim E^* = \frac{1}{2}\dim(E \oplus E^*)$. In terms of a basis $\{x_j\}$ of E and dual basis $\{\xi_j\}$ of $E^*$ the form $\omega$ can be represented as the matrix

$$\omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

This shows that if a linear operator $T$ on $V$ is given by the matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then $T$ is symplectic if $A^{\mathrm{T}}C$ and $B^{\mathrm{T}}D$ are symmetric and $A^{\mathrm{T}}D - C^{\mathrm{T}}B = \mathbb{1}$. These conditions in particular are fulfilled if $A \in \mathrm{GL}(E),\ D = (A^{\mathrm{T}})^{-1}$ and $B = C = 0$, so $\mathrm{GL}(E)$ is isomorphic to a subgroup of $\mathrm{Sp}(V)$. More generally, if $K : E \to F$ is an isomorphism then the association

$$(x,\xi) \mapsto (Kx, (K^{-1})^*\xi)$$

defines a linear symplectomorphism between $E \oplus E^*$ and $F \oplus F^*$.

Furthermore, the graph of a linear map $B : E \to E^*$ is a Lagrangian subspace of $E \oplus E^*$ iff $B$ is selfadjoint.

Now consider the symplectic vector space $V \oplus \overline{V}$, denoting $V \oplus V$ endowed with the symplectic structure $\omega \oplus (-\omega)$. Let $T : V \to V$ be a linear symplectic map, then the graph of $T$ is a Lagrangian submanifold of $V \oplus \overline{V}$.

It is a fundamental result that Lagrangian subspaces exist at all:

**Lemma A.2.4.** *Any finite-dimensional symplectic vector space contains a Lagrangian subspace.*

*Proof.* Suppose we are given a $2n$-dimensional symplectic vector space and $W \subset V$ any isotropic subspace (with $\dim W < n$). Since $2n = \dim W + \dim W^\perp$, there exists a nonzero vector $w \in W^\perp/W$. Then the subspace $W'$ of $V$ spanned by $W \cup \{w\}$ is isotropic with $\dim W' = \dim W + 1$. Therefore it follows, that starting from any isotropic subspace which is not Lagrangian we can construct isotropic subspaces properly containing the original space. Starting from any one-dimensional subspace we can iterate this procedure which has to end up with a Lagrangian subspace because $\dim V$ was supposed to finite. $\square$

A pair $L, L' \subset V$ of transverse Lagrangian subspaces define a *Lagrangian splitting* of $V$. In this case $\tilde{\omega} : V \to V^*$ defines an isomorphism $L' \simeq L^*$ which gives a linear symplectomorphism $V \simeq L \oplus L^*$ equipped with the canonical symplectic structure defined in Example A.2.3. If J is an $\omega$-compatible complex structure on $V$ then $L \oplus \mathrm{J}L$ defines a Lagrangian splitting. According to Lemma A.2.4 any symplectic vector space contains a Lagrangian subspace and this is isomorphic to $\mathbb{R}^n$. So we have

**Theorem A.2.5.** *Every $2n$-dimensional symplectic vector space is linearly symplectomorphic to $(\mathbb{R}^{2n}, \omega)$.*

**Lemma A.2.6.** *Let $V$ be a symplectic vector space with an $\omega$-compatible complex structure J and define $T_\epsilon : V \to V$ by $T_\epsilon(x) := x + \epsilon \mathrm{J}x$.*

1. *If $L, L'$ are any Lagrangian subspaces of $V$, then $L_\epsilon := T_\epsilon(L)$ is a Lagrangian subspace transversal to $L'$ for small $\epsilon > 0$.*

2. *For any two Lagrangian subspaces $L, L'$ of $V$ there is a Lagrangian subspace $L''$ transverse both to $L'$ and $L$.*

*Proof.* We clearly have $T_\epsilon \in \mathfrak{csp}(V)$ with $\omega(T_\epsilon(x), T_\epsilon(y)) = (1+\epsilon^2)\omega(x,y)$. Therefore $L_\epsilon$ is a Lagrangian subspace for all $\epsilon > 0$. Use orthonormal bases[1] $\{v_i\}$ and $\{w_i\}$ of $L'$ and $L$ respectively so that $v_i = w_i$, $i = 1, \dots, k$ span the intersection $L \cap L'$. Then $\{w_i + \epsilon J w_i\}$ form a basis of $L_\epsilon$ and $L'$ and $L_\epsilon$ are transversal precisely when the matrix $M := \{\omega(v_i, w_j + \epsilon \mathrm{J} w_j)\} = \{\omega(v_i, w_j) + \epsilon g_{\mathrm{J}}(v_i, w_j)\}$ is non-singular. According to our choice we have

$$M = \begin{pmatrix} \epsilon \mathbb{1} & 0 \\ 0 & A + \epsilon B \end{pmatrix}$$

where $A = \{\omega(v_i, w_j)\}_{k+1 \le i,j \le n}$ and $B$ is some $(n-k) \times (n-k)$ matrix. Now let us define $I := \mathrm{span}\{v_i\}_{k+1 \le i \le n}$ and $W := \mathrm{span}\{w_i\}_{k+1 \le i \le n}$. Then $I$ is isotropic since $\{v_i\}$ is a basis of the Lagrangian subspace $L'$. Furthermore $I \cap L = \{0\}$ and $W \subset L$ is complementary to $I^\perp \cap L$ so $I + W \subset I^\perp + W$. Therefore $W^\perp \cap I \subset I^\perp \cap L$. Thus $W^\perp \cap I \subset I \cap L = \{0\}$. Also $I^\perp \cap W = \{0\}$ and therefore $I + W$ is a symplectic subspace of $V$ and therefore the restriction of $\omega$ to $I + W$ is non-singular. This shows that $A$ is non-singular, thus proving the first part of our Lemma.

By the same method we prove that $L_\epsilon$ is transverse to $L$ for small $\epsilon$ □

# A.3 Symplectic vector bundles

We now consider symplectic manifolds $P$, that is manifolds endowed with a *symplectic structure*, which by definition is a closed, non-degenerate two-form $\omega$ on $P$. Generalizing the notions from symplectic vector spaces we say that a submanifold $C \subset P$ is *(co-)isotropic* if each tangent space is (co-)isotropic. In particular, *Lagrangian submanifolds* are (co-)isotropic manifolds whose dimension equals $\frac{1}{2} \dim P$.

[1] with respect to the inner product $g_{\mathrm{J}}$ induces by J

Since the symplectic form on a $2n$-dimensional manifold $P$ defines a smooth family of linear symplectic forms on the fibers of $\mathrm{T}P$, the frame bundle of $P$ can be reduced to a principal $\mathrm{Sp}(n)$ bundle over $P$, [KN63, Fri97]. We call any vector bundle with this structure a symplectic vector bundle.

**Theorem A.3.1.** *Every symplectic vector bundle admits a compatible complex vector bundle structure.*

For a proof see [MS98, KN69].

Now a *Lagrangian subbundle* $L$ of a symplectic vector bundle $E$ is a subbundle $L \subset E$ such that each fiber $L_x$ is a Lagrangian subspace in the fiber $E_x$. We have

**Theorem A.3.2.** *Let $E \to M$ be a symplectic vector bundle and suppose that $L$ and $L'$ are Lagrangian subbundles such that $L_x$ is transverse to $L'_x$ for each $x \in M$. Then there exists a compatible complex structure $J$ on $E$ such that*

$$JL = L'.$$

*Proof.* Let $J_0$ be any complex structure on $E$. Then both $L'$ and $JL$ are transverse to $L$. So we can construct a symplectomorphism $T : L \oplus L' \to L \oplus J_0 L$ which preserves $L$ and maps $L'$ to $J_0 L$. Then $J = T^{-1} J_0 T$ is a compatible complex structure for which $JL = L'$ □

## A.3.1 The Lagrangian Grassmannian

Let us denote by $\mathfrak{Lag}(V)$ the collection of all Lagrangian subspaces of the symplectic vector space $(V, \omega)$ and define a natural action of $\mathrm{Sp}(V)$ on $\mathfrak{Lag}(V)$ by

$$j : \mathrm{Sp}(V) \times \mathfrak{Lag}(V) \to \mathfrak{Lag}(V), \quad j(T, L) := j_L(T) := T(L).$$

This action is characterized by [dG97]):

**Lemma A.3.3.** *The unitary group $\mathrm{U}(V)$ associated with an $\omega$-compatible complex structure $J$ acts transitively on $\mathfrak{Lag}(V)$.*

*Proof.* Let $L_1$ and $L_2$ be two Lagrangian subspaces of $V$. Then $L_1 \oplus JL_1$ and $L_2 \oplus JL_2$ define Lagrangian splittings of $V$. Now let $K : V \to V$ be an orthogonal transformation such that $K(L_1) = L_2$. Then, since $g_J(L_i, JL_i) = 0$, the transformation may be chosen such that $K(JL_1) = JL_2$. Thus by a calculation we see that $K \in \mathrm{Sp}(V)$, therefore

$$K \in \mathrm{Sp}(V) \cap \mathrm{O}(V) = \mathrm{U}(V).$$

□

It will be important to have

**Proposition A.3.4.** *The fundamental group of* $\mathrm{U}(n)$ *is isomorphic to* $\mathbb{Z}$ *and the determinant map* $\det : \mathrm{U}(n) \to \mathrm{S}^1$ *induces an isomorphism of fundamental groups.*

*Proof.* The determinant map $\det : \mathrm{U}(n) \to \mathrm{S}^1$ is a fibration [BT82, Bre93]. Therefore we have an associated homotopy sequence

$$\cdots \to \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(\mathrm{S}^1) \to \pi_0(\mathrm{SU}(n)) \to \cdots ,$$

see [BT82, Ste51], where $\pi_0$ denotes the set of all path components. This shows that

$$\pi_1(\mathrm{U}(n)) \simeq \pi_1(\mathrm{S}^1) \simeq \mathbb{Z}.$$

□

According to the proof of Lemma A.3.3 the stabilizer of a fixed element $L \in \mathfrak{Lag}(V)$ is obviously isomorphic to $\mathrm{O}(n)$ and we have a (non-canonical) identification of the Lagrangian Grassmannian $\mathfrak{Lag}(V)$ with $\mathrm{U}(n)/\mathrm{O}(n)$: we consider the mapping $\det^2 : \mathrm{U}(n) \to \mathrm{S}^1$, which maps $\mathrm{O}(n) \to \{1\}$ and therefore gives a well defined map

$$\det^2 : \mathrm{U}(n)/\mathrm{O}(n) \to \mathrm{S}^1.$$

Let $\mathrm{S}(\mathrm{U}(n)/\mathrm{O}(n)) := (\det^2)^{-1}(1)$ on which $\mathrm{SU}(n)$ acts transitively with isotropy group $\mathrm{SO}(n)$. Thus we have a fibration of $\mathrm{U}(n)/\mathrm{O}(n)$ where the fiber over each point is diffeomorphic to $\mathrm{SU}(n)/\mathrm{SO}(n)$. Since $\mathrm{SU}(n)$ is simply connected and $\mathrm{SO}(n)$ is connected also the quotient $\mathrm{SU}(n)/\mathrm{SO}(n)$ is simply connected. Therefore the exact sequence in homotopy associated to the fibering shows that

$$\pi_1(\mathfrak{Lag}(V)) \simeq \mathbb{Z} \simeq \pi_1(\mathrm{S}^1).$$

We claim that the above isomorphism does not depend on the choice of the complex structure or on the choice of the Lagrangian plane $L$. By using the fact that the set of $\omega$-compatible complex structures is contractible and the connectedness of the unitary group which acts transitively on $\mathfrak{Lag}(V)$, we can construct a null-homotopy, which leaves $\pi_1(\mathfrak{Lag}(V))$ invariant [Mun00, GH81], thus showing the claimed independence.

Now we pass to homology by using the Hurewicz isomorphism [Bre93]. We have that

$$\pi_1(\mathfrak{Lag}(V)) \simeq \mathrm{H}_1(\mathfrak{Lag}(V), \mathbb{Z})$$

and by the universal coefficient theorem

$$\mathrm{H}_1(\mathfrak{Lag}(V), \mathbb{Z}) \simeq \mathrm{H}^1(\mathfrak{Lag}(V), \mathbb{Z}).$$

Now consider the canonical generator

$$\frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}$$

of $\mathrm{H}^1(\mathrm{S}^1, \mathbb{Z})$, which we pull back

$$(\det^2)^* \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}$$

to obtain a generator of $\mathrm{H}^1(\mathfrak{Lag}(V), \mathbb{Z})$, which is called the *universal Maslov class.* We will enlighten the geometrical meaning of this class in the subsequent discussion.

### A.3.2 The universal covering space of $\mathfrak{Lag}(V)$

We again choose a complex structure on $V$ and thus can identify $V \simeq \mathbb{C}^n$. Any Lagrangian subspace is then given as $X = A\mathbb{R}^n$ for some $A \in U(n)$ and

$$A\mathbb{R}^n = B\mathbb{R}^n \quad \Leftrightarrow \quad A\overline{A}^{-1} = B\overline{B}^{-1}$$

since we also have $\overline{A}\mathbb{R}^n = \overline{B}\mathbb{R}^n$. Thus we have a well-defined map

$$v : \mathfrak{Lag}(V) \to \mathrm{U}(n), \quad v(X) = A\overline{A}^{-1},$$

if $X = A\mathbb{R}^n$, which obviously has the covariance property

$$v(BX) = Bv(X)\overline{B}^{-1}.$$

If $\mathbb{C}^n \ni z = Ar$ then $r \in \mathbb{R}^n$ iff $z = A\overline{A}^{-1}\overline{z}$. Therefore we have that

$$z \in X \quad \text{iff} \quad z = v(X)\overline{z}.$$

If $X \cap Y \neq \{0\}$ then the equations

$$z = v(X)\overline{z} \quad \text{and} \quad z = v(Y)\overline{z}$$

have a non-trivial solution, such that $v(X) - v(Y)$ is not invertible. Conversely, let us assume that $X = \mathbb{R}^n$ and $Y = A\mathbb{R}^n$ for some $A \in \mathrm{U}(n)$. Now let $\mathbb{1} - A\overline{A}^{-1}$ be non-invertible such that there is $u \in \mathbb{C}^n$ with that $u = A\overline{A}^{-1}\overline{u}$ and $\overline{u} = A\overline{A}^{-1}u$ such that we can find a real solution. We have proved

**Lemma A.3.5.** *$X$ and $Y$ are transversal iff $v(X) - v(Y)$ is invertible.*

Now define $\tilde{\mathrm{U}}(n)$ to be the set of all pairs

$$\{(A, \varphi);\ A \in \mathrm{U}(n),\ \varphi \in \mathbb{R} \text{ such that } \det A = \mathrm{e}^{\mathrm{i}\varphi}\},$$

which is made into a group by declaring the multiplication as

$$(A, \varphi) \cdot (A', \varphi') = (AA', \varphi + \varphi').$$

Furthermore, $\tilde{\mathrm{U}}(n)$ is a covering group of $\mathrm{U}(n)$ with projection $(A, \varphi) \mapsto A$. We have an isomorphism

$$\mathrm{SU}(n) \times \mathbb{R} \to \tilde{\mathrm{U}}(n), \quad (B, \psi) \mapsto (B\,\mathrm{e}^{\mathrm{i}\psi}, n\psi),$$

and since $\mathrm{SU}(n)$ is simply connected it follows that $\tilde{\mathrm{U}}(n)$ is the universal covering group of $\mathrm{U}(n)$ and it is immediate that $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$. This also implies that the fundamental group of $\mathrm{Sp}(V)$ is $\mathbb{Z}$, see [GS90, dG97].

Denote by $\widetilde{\mathfrak{Lag}}(V)$ the set of pairs

$$(X, \theta), \quad X \in \mathfrak{Lag}(V),\ \theta \in \mathbb{R} \text{ such that } \det v(X) = \mathrm{e}^{\mathrm{i}\theta}.$$

Now $\mathbb{Z}$ acts on $\mathfrak{Lag}(V)$ by

$$k(X,\theta) = (X, \theta + 2k\pi),\ k \in \mathbb{Z},$$

and $\mathfrak{Lag}(V) = \widetilde{\mathfrak{Lag}}(V)/\mathbb{Z}$, making $\widetilde{\mathfrak{Lag}}(V)$ into a covering space of $\mathfrak{Lag}(V)$. In addition, $\tilde{\mathrm{U}}(n)$ acts on this covering space by

$$(A,\varphi)(X,\theta) = (AX, \theta + 2\varphi),$$

which is easily seen to be transitive and well-defined because of the covariance property of $v(X)$. Now the isotropy subgroup of $(\mathbb{R}^n, 0) \in \widetilde{\mathfrak{Lag}}(V)$ consists of all $(A, 0)$ with

$$\overline{A} = A, \quad \det A = 1,$$

that is $A \in \mathrm{SO}(n)$ and it follows that

$$\widetilde{\mathfrak{Lag}}(V) = \tilde{\mathrm{U}}(n)/\,\mathrm{SO}(n),$$

which is simply connected since $\tilde{\mathrm{U}}(n)$ is simply connected and $\mathrm{SO}(n)$ is connected. Thus we have

**Lemma A.3.6.** $\widetilde{\mathfrak{Lag}}(V)$ *is the universal covering space of* $\mathfrak{Lag}(V)$.

Now let $\tilde{\gamma}$ be a lift to $\widetilde{\mathfrak{Lag}}(V)$ of a curve $\gamma$ in $\mathfrak{Lag}(V)$, such that $\tilde{\gamma}(0) = (X,\theta)$ and $\tilde{\gamma}(1) = (X, \theta + 2\pi)$ such that

$$\int_\gamma (\det{}^2)^* \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z} = 1.$$

This is the simplest example for the Maslov index, i.e. the Maslov class evaluated on a curve on $\widetilde{\mathfrak{Lag}}(V)$

## A.3.3 The Maslov index

Let $u = (X,\theta)$ and $u' = (X',\theta')$ be two elements in $\widetilde{\mathfrak{Lag}}(V)$. We say that $u$ and $u'$ are transversal if this is true for $X$ and $X'$ in $\mathfrak{Lag}(V)$. Now the *Maslov index* $m(u,u')$ is associated with a pair $(u,u')$ of transverse elements in $\widetilde{\mathfrak{Lag}}(V)$. Recall that by the spectral theorem the logarithm of some $A \in \mathrm{GL}(n,\mathbb{C})$ is well-defined unless $A$ has an eigenvalue on the negative real axis. We have

$$\exp(\log(A)) = A, \quad \exp^{\mathrm{tr}(\log(A))} = \det A, \quad \log A^{-1} = -\log A,$$

if $\log A$ is defined. Now define

$$m(u,u') := \frac{1}{2\pi}\left(\theta - \theta' + \mathrm{i}\,\mathrm{tr}\log(-v(X)v(X')^{-1})\right), \tag{A.3.1}$$

which is well defined since for transverse Lagrangian planes $X$ and $X'$ we know that $v(X) - v(X') = (v(X)v(X')^{-1} - 1)v(X')$ is invertible and therefore $-v(X)v(X')^{-1}$ does not have the eigenvalue $-1$ and consequently no eigenvalue on the negative real axis, since it is unitary by definition. Then

$$e^{2\pi\,i\,m(u,u')} = (-1)^n = e^{i\,n\pi}$$

which implies

$$m(u,u') \in \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}+\frac{1}{2} & n \text{ odd} \end{cases}.$$

Furthermore, we have

$$m(ku, k'u') = k - k' + m(u,u')$$

for $k$, $k' \in \mathbb{Z}$. Since $\mathrm{Sp}(V)$ is connected[2], the action of $\mathrm{Sp}(V)$ on $\widetilde{\mathfrak{Lag}(V)}$ may be lifted in a unique way to an action of the universal covering $\tilde{\mathrm{Sp}}(V)$ of $\mathrm{Sp}(V)$ on $\widetilde{\mathfrak{Lag}}(V)$ by the lifting lemma [Mun00, GH81]. If $u$ and $u'$ are transverse then so are $\tilde{a}u$ and $\tilde{a}u'$ for all $\tilde{a} \in \tilde{\mathrm{Sp}}(V)$. Therefore the map

$$\tilde{a} \mapsto m(\tilde{a}u, \tilde{a}u')$$

is well-defined, continuous and takes values in a discrete set. It therefore has to be constant, i.e.

$$m(u,u') = m(\tilde{a}u, \tilde{a}u')$$

for all $\tilde{a} \in \tilde{\mathrm{Sp}}(V)$. By definition we have

$$m(u,u') + m(u',u) = 0.$$

Now let $X$, $X'$ and $X''$ be three transversal Lagrangian subspaces. These then define a quadratic form on $X$ as follows: there is a symplectomorphism $V \to X \oplus X^*$ that sends $X'' \mapsto 0 \oplus X^*$ and $X'$ to the graph of a selfadjoint map $T : L \to L^*$, whose associated quadratic form we denote by $Q_T$. Then define

$$\mathrm{ind}(X, X', X'') := \mathrm{ind}\, Q_T$$

which is a symplectic invariant, i.e.

$$\mathrm{ind}(aX, aX', aX'') = \mathrm{ind}(X, X', X'')$$

for any $a \in \mathrm{Sp}(V)$.

Such an invariant does not exist for pairs of Lagrangian planes since the symplectic group acts transitively on pairs of transversal Lagrangian subspaces. Now let $u = (X,\theta)$, $u' = (X',\theta')$ and $u'' = (X'',\theta'')$ be the points in $\widetilde{\mathfrak{Lag}}(V)$ over $X$, $X'$ and $X''$, respectively. Then we have the Leray formula [Ler74]

$$m(u,u') + m(u',u'') + m(u'',u) = \mathrm{ind}(X, X', X'').$$

[2] Which follows from the fact that $\mathrm{U}(n)$, being a maximal compact subgroup of $\mathrm{Sp}(V)$, is a deformation retract of $\mathrm{Sp}(V)$.

Now we can explain the relationship between the Maslov class and the Maslov index: let $Y$ be a fixed Lagrangian subspace and $X(t)$, $0 \le t \le 1$, a curve of Lagrangian planes such that $X(0)$ and $X(1)$ both are transversal to $Y$. Let $\tilde{Y} \in \widetilde{\mathfrak{Lag}}(V)$ be above $Y$ and $u(t)$ a covering of $X(t)$ in $\widetilde{\mathfrak{Lag}}(V)$. Then

$$l = m(\tilde{Y}, u(1)) - m(\tilde{Y}, u(0))$$

is independent of the lifts chosen. Now the function $m(\tilde{Y}, u(\cdot))$ is well-defined as long as $u(\cdot)$ is transversal to $\tilde{Y}$. Suppose we have an isolated point $s \in [0,1]$ such that $u(s)$ is not transversal to $\tilde{Y}$. Then let us choose a sufficiently small neighbourhood $(s-\varepsilon, s+\varepsilon)$ of $s$ and a Lagrangian subspace $Z$ which is transverse both to $Y$ and $X(t)$ for all $t \in (s-\varepsilon, s+\varepsilon)$, which exists according to Lemma A.2.6. Thus we have

$$\begin{aligned}\operatorname{ind}(Z, Y, X(t'')) - \operatorname{ind}(Z, Y, X(t')) &= \\ &m(\tilde{Y}, u(t'')) - m(\tilde{Y}, u(t')) - \Big(m(\tilde{Z}, u(t'')) - m(\tilde{Z}, u(t'))\Big)\end{aligned}$$

for $s - \varepsilon < t' < s < t'' < s + \varepsilon$. However, since $m(\tilde{Z}, u(\cdot))$ is continuous, we must have $m(\tilde{Z}, u(t')) = m(\tilde{Z}, u(t''))$ and therefore

$$\operatorname{ind}(Z, Y, X(t'')) - \operatorname{ind}(Z, Y, X(t')) = m(\tilde{Y}, u(t'')) - m(\tilde{Y}, u(t')).$$

Now suppose that $X(1) = X(0)$. Then

$$u(1) = lu(0)$$

and consequently

$$l = \int (\det{}^2)^* \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}.$$

Let $\Lambda$ be a Lagrangian submanifold of $V$, such that we may identify $\mathrm{T}_\lambda\Lambda$ with a Lagrangian subspace of $V$ and $\mathrm{T}_\lambda V$ with $V$. Then any curve $\gamma$ on $\Lambda$ gives rise to a curve of Lagrangian subspaces $\mathrm{T}_{\gamma(t)}\Lambda \subset V$. If $\gamma$ is closed then by the above technique we may associate an integer to $\gamma$. Therefore, we have defined an element of $\mathrm{H}^1(\Lambda, \mathbb{Z})$.

Now assume that $M$ is a differentiable manifold whose cotangent bundle $\mathrm{T}^*M$ is a symplectic manifold, which has a canonical Lagrangian subbundle VM. By choosing a Riemannian metric on $M$ we may identify $\mathfrak{Lag}(\mathrm{T}^*M)$, the bundle of Lagrangian subspace of $\mathrm{T}(\mathrm{T}^*M)$, with $\mathrm{U}(n)/\mathrm{O}(n)$ fiberwise. If $\Lambda$ is a Lagrangian submanifold of $\mathrm{T}^*M$, then again any (closed) curve $\gamma$ on $\Lambda$ determines a curve on $\mathfrak{Lag}(\mathrm{T}^*M)$ and we can define an element in $\mathrm{H}^1(\Lambda, \mathbb{Z})$ as above, which assigns an integer to $\gamma$. In particular, since all possible choices of Riemannian metrics may be continuously deformed into each other, the class in $\mathrm{H}^1(\Lambda, \mathbb{Z})$ is independent of the particular choice.

### A.3.4 The cross index

In this section we want to give a definition of the Maslov class following Hörmander. To this end we associate an integer $\sigma(A,B,C,D)$ to a quadruplet $A$, $B$, $C$, $D$, of Lagrangian subspaces with

$$C \cap A = \{0\} = C \cap B$$

and

$$D \cap A = \{0\} = D \cap B.$$

Define the isotropic subspace,

$$R = A \cap B,$$

whose symplectic orthogonal is coisotropic and therefore reducible, i.e. we have a well defined reduction map $\rho:\ R^\perp \to R^\perp/R$. Then the images $\rho(A)$ and $\rho(B)$ in $R^\perp/R$ are transversal Lagrangian subspaces. Now since $C$ and $D$ are both transversal to $A$ and $B$, $\rho(C)$ and $\rho(D)$ determine non-singular quadratic forms $Q_C$ and $Q_D$ on $\rho(B)$. Let

$$\sigma(A,B,C,D) := \mathrm{ind}(A,B,C) - \mathrm{ind}(A,B,D),$$

from which we immediately have that

$$\sigma(A,B,C,D) = -\sigma(A,B,D,C)$$

and

$$\sigma(A,B,C,D) + \sigma(A,B,D,E) + \sigma(A,B,E,C) = 0. \tag{A.3.2}$$

We shall now prove that $\sigma(A,B,C,D)$ is locally constant. To this end we given an alternative description. We have that both $C$ and $D$ are contained in $\mathfrak{Lag}_A$[3]. Since $\mathfrak{Lag}_A$ is diffeomorphic to the set of quadratic forms (see also [Dui96]) we conclude that we may choose a curve $\gamma_{CD} \subset \mathfrak{Lag}_A$ joining $C$ and $D$, and any two such curves are homotopic. Similarly, up to homotopy there is a unique curve $\gamma_{DC} \subset \mathfrak{Lag}_B$ joining $D$ to $C$ in $\mathfrak{Lag}_B$. This defines a closed curve $\gamma_{CDC}$ in $\mathfrak{Lag}(V)$ which is unique up to some element of $\pi_1(\mathfrak{Lag}(V))$.

**Lemma A.3.7.** *We have*

$$(A,B,C,D) = \int_{\gamma_{CDC}} (\det{}^2)^* \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}. \tag{A.3.3}$$

*Furthermore, the function $A,B,C,D \mapsto \sigma(A,B,C,D)$ is locally constant.*

**Proposition A.3.8.** *The symbol $(A,B,C,D)$ satisfies*

$$(A,B,C,D) = -(C,D,A,B). \tag{A.3.4}$$

[3]Denoting the set of Lagrangian subspaces transversal to $A$.

Now let $E \to N$ be a symplectic vector bundle. Then we have a fiber bundle $\mathfrak{Lag}(E) \to N$, where $\mathfrak{Lag}(E)_n$ consists of all Lagrangian subspaces of $E_n$. Suppose we are given two sections $A$ and $B$ of $\mathfrak{Lag}(E)$. Locally we can always find sections $C$ and $D$ (defined on $U_C$ and $U_D$ respectively) transversal both to $A$ and $B$ and we define a Čech one-cocycle

$$\mathfrak{L}(U_C, U_D) = (A, B, C, D).$$

It clearly is continuous and integer valued and therefore defines a Čech cochain, and it is a cocycle because of equation (A.3.2). We denote the corresponding cohomology class by $\alpha(E; A, B)$.

We compute the cohomology class for the following situation: Let $V$ be a symplectic vector space and $N = \mathfrak{Lag}(V)$ the Lagrangian Grassmannian. Define a symplectic vector bundle $E \to N$ as the tautologous vector bundle assigning to any point $V$ a copy of $V$. Then $\mathfrak{Lag}(E)$ has a canonical section $B$ defined as $B(n) = n$. Now let $A$ be a constant section of $E$[4]. We the obtain an element $\alpha(A, B)$ for which the following result holds

**Proposition A.3.9 (Hörmander).** *The class $\alpha(A, B)$ coincides with the fundamental generating class of* $\mathrm{H}^1(\mathfrak{Lag}(V))$.

As above let $E \to N$ be a symplectic vector bundle and $A$ and $B$ sections of $\mathfrak{Lag}(E)$. Let $\{U_i\}$ be a contractible open cover of $N$. Let us recall shortly again how $\alpha(A, B)$ was defined: we choose section

$$C_{U_i} : U_i \to \mathfrak{Lag}(E)$$

transversal to $A$ and $B$. The Čech cocycle defining $\alpha(A, B)$ has the value

$$\mathfrak{L}(U_i, U_j) = (A, B, C_{U_i}, C_{U_j})$$

on the pair of open sets $(U_i, U_j)$. Let us set

$$\tau(U_i, U_j) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathfrak{L}(U_i, U_j)},$$

which we can regard as transition functions for a line bundle on $N$, which is called the *Maslov line bundle* associated to $(E, A, B)$.

## A.4 Lagrangian submanifolds and coisotropic reduction

Let us first introduce the objects we will be concerned with: we consider the class of all smooth, finite-dimensional symplectic manifolds. Given two objects $(X, \omega_X)$ and $(Y, \omega_Y)$ we define their product as the symplectic manifold $(X \times Y, \pi_X^*\omega_X + \pi_Y^*\omega_Y)$, where $\pi_X$, $\pi_Y$ denote the cartesian projections of $X \times Y$ onto the respective factors. The symplectic dual of an object $(X, \omega_X)$ then is $(X, -\omega_X)$.

We have the following easy statement:

[4] A constant section may in the present setting be considered as a pull back of a section $\mathfrak{Lag}(V) \to$ point.

**Lemma A.4.1.** *A diffeomorphism $f : X \to Y$ between two symplectic manifolds is a symplectomorphism, i.e. $f^*\omega_Y = \omega_X$, iff its graph is a Lagrangian submanifold of the product $Y \times \overline{X}$, where we denote by $\overline{X}$ the symplectic dual of $X$.*

We now want to describe several operations on immersed Lagrangian submanifolds. However, we first have to discuss some geometric constructions. To this end we start with a closed two-form $\omega'$ on a smooth manifold $M$ which has constant rank and is not necessarily non-degenerate. In this case $(X, \omega')$ is called a pre-symplectic manifold and we have a subbundle $(\mathrm{T}M)^\perp$ whose fiber at $x \in X$ is given by $(\mathrm{T}_x M)^\perp$ which is called the characteristic subbundle.

**Proposition A.4.2.** *The characteristic subbundle of a presymplectic manifold $(X, \omega')$ is integrable.*

*Proof.* Clearly, $(\mathrm{T}M)^\perp$ is a subbundle, since $\omega'$ is supposed to have constant rank. Now, since $\imath_{[X,Y]} = [\mathscr{L}_X, \imath_Y]$, see e.g. [KN63], we have

$$\imath_{[X,Y]}\omega' = \mathscr{L}_X(\imath_Y\omega') - \imath_Y\mathscr{L}_X\omega.$$

So if $Y \in (\mathrm{T}M)^\perp$ then the first term on the right hand side vanishes and if $X \in (\mathrm{T}M)^\perp$ then by using Cartan's formula [KN63] we see that $[X, Y] \in (\mathrm{T}M)^\perp$ and the vector fields of $(\mathrm{T}M)^\perp$ form a Lie algebra. By Frobenius' theorem [Ste64] we have that $(\mathrm{T}M)^\perp$ is an integrable distribution. □

The foliation $M^\perp$ defined by $(\mathrm{T}M)^\perp$ is called the characteristic foliation of $M$. If the quotient $M/M^\perp$ is a smooth manifold, we call $M$ reducible. Then $\omega'$ induces a symplectic structure on $M/M^\perp$ which is a reduced (symplectic) manifold.

As a special case consider a coisotropic submanifold $C \subset M$ of a symplectic manifold, i.e. for $m \in C$

$$\{v \in \mathrm{T}_m M;\ \omega(v, w) = 0 \text{ for all } w \in \mathrm{T}_m C\} = (\mathrm{T}_m C)^\perp \subset \mathrm{T}_m C.$$

Then $\omega' := i^*\omega$, where $i : C \hookrightarrow M$ is the natural inclusion, is a presymplectic structure and $\ker \omega' = (\mathrm{T}C)^\perp$.

**Example A.4.3.** Let $H : M \to \mathbb{R}$ be a smooth function having $C$ as a regular level set, i.e. $C$ is a submanifold of codimension one in $M$ (a hypersurface) and therefore a coisotropic submanifold. Then the Hamiltonian vector field $X_H$ generated by $H$ is tangential to the characteristic distribution since $X_H H = 0$ and since $\dim(\mathrm{T}C)^\perp = \dim M - \dim C = 1$, we see that the Hamiltonian vector field generates the characteristic distribution.

Reducible coisotropic submanifolds define certain operations on immersed Lagrangian submanifolds, as we shall see shortly. Let us first set up

**Definition A.4.4.** Let two maps $i : N \to V$ and $j : M \to V$ be given. Then the *fiber product* of these is defined to be the subset

$$N \times_V M = (i \times j)^{-1}\Delta_V$$

of $N \times M$, where $\Delta_V$ denotes the diagonal in $V \times V$. This is also described by the following commutative diagram

$$\begin{tikzcd} N \times_V M \arrow[r, "r_M"] \arrow[d, "r_N"'] & M \arrow[d, "j"] \\ N \arrow[r, "i"'] & V \end{tikzcd}$$

where $r_N$, $r_M$ denote the restrictions of the cartesian product to the respective factors.

**Definition A.4.5.** Let $N$, $M$, $V$ be smooth manifolds, then two maps $i : N \to V$ and $j : M \to V$ are said to *intersect cleanly* if their fiber product $N \times_V M$ is a submanifold of $N \times M$ and

$$i_*\mathrm{T}N \cap j_*\mathrm{T}M = (j \circ r_M)_*\mathrm{T}(N \times_V M).$$

Furthermore, we will say that $i : N \to V$ intersects a submanifold $W \subset V$ cleanly if $i$ and the inclusion $W \hookrightarrow V$ intersect cleanly.

To enlighten these definitions we give some examples:

**Example A.4.6.** Let the product $i \times j$ of the two maps be transversal to the diagonal $\Delta_V$ in $V \times V$ then clearly $N \times_V M$ is a submanifold since

$$\mathrm{T}_{(i\times j)(n,m)}\Delta_V + ((i \times j)_*(N \times M))_{(i\times j)(n,m)} = \mathrm{T}_{(i\times j)(n,m)}(V \times V)$$

for all $(n, m) \in N \times M$. In particular, the above situation occurs if $j$ is a submersion and $i$ is any smooth map.

**Example A.4.7.** Let $i : N \to V$ and $j : M \to V$ be any smooth maps whose fiber product is a smooth submanifold of $N \times M$. Then for any tangent vector $(v, w)$ of $N \times_V M$ the vector $(i \times j)_*(v, w)$ is tangent to $\Delta_V$ with $i_*v = j_*w$. Since $r_{M*}(v, w) = w$ we have that $r_{M*}(v, w) = 0$ iff $w = 0$, in which case $j_*w = i_*v = 0$. Thus, if $i$ is an immersion then also $r_M$ is an immersion.

In particular, if $i$ and $j$ are immersions whose intersection is clean then $r_M$ and $r_N$ are immersions.

We give a description of clean intersections in local coordinates due to [Hör85a]. First, recall that two smooth submanifolds $M$ and $N$ of a manifold $P$ are said to intersect transversally at $m_0 \in M \cap N$ if $\mathrm{T}_mM + \mathrm{T}_mN = \mathrm{T}_mP$ when $m = m_0$. This is of course equivalent to the statement that the corresponding conormal bundles intersect trivially, i.e. $(\mathrm{C}_M)_m \cap (\mathrm{C}_N)_m = \{0\}$. Now let $M$ be locally defined by $f_1 = \ldots = f_k = 0$ where the differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_k$ are linearly independent at $m_0$ and similarly let $N$ be defined by $g_1 = \cdots = g_l = 0$. Then $\mathrm{d}f_1, \dots, \mathrm{d}f_k, \mathrm{d}g_1, \dots, \mathrm{d}g_l$ are all linearly independent at $m_0$. Thus the intersection $M \cap N$ defined by $f_1 = \cdots = f_k = g_1 = \cdots = g_l = 0$ is a smooth submanifold in a neighbourhood of $m_0$, in particular

$$\operatorname{codim}(M \cap N) = \operatorname{codim}(M) + \operatorname{codim}(N)$$

and

$$\mathrm{T}_m(M \cap N) = \mathrm{T}_m M \cap \mathrm{T}_m N, \ m \in M \cap N. \tag{A.4.1}$$

The intersection can be a submanifold even if it is not transversal. So in general we have

$$\operatorname{codim} M + \operatorname{codim} N = \operatorname{codim}(M \cap N) + \operatorname{codim}(M + N), \tag{A.4.2}$$

where $e := \operatorname{codim}(M + N)$ is called the *excess* of the intersection.

**Proposition A.4.8.** *Let $M$, $N$ and $M \cap N$ be smooth submanifolds of a smooth manifold $P$ and assume that (A.4.1) holds. For any $m_0 \in M \cap N$ one can then choose local coordinates such that $M$ and $N$ are defined by linear equations in their local coordinates. Therefore, (A.4.2) is valid with some integer $e \geq 0$, which vanishes precisely when the intersection is transversal.*

For a proof we refer to [Hör85a].

If $C$ is a coisotropic submanifold and $L$ an immersed Lagrangian submanifold, then we say that $(C, L)$ forms a *reducible pair* if $C$ is reducible and $C$ and $L$ intersect cleanly.

Let $P$ and $Q$ be symplectic manifolds, and $L$ a Lagrangian submanifold of $Q$, then $P \times L$ is a coisotropic submanifold of $P \times Q$ whose characteristic foliation consists of leaves of the form $\{p\} \times L$ for $p \in P$, since the symplectic form on $P$ is non-degenerate by assumption. Therefore, $P \times L$ is reducible and the reduction coincides with the restriction $P \times L \to P$. As a special case, we have that $R \times \Delta_Q \times \overline{P}$ is a reducible coisotropic submanifold of $R \times \overline{Q} \times Q \times \overline{P}$, where $\Delta_Q$ denotes the diagonal in $\overline{Q} \times Q$. Let $L_1$ be a canonical relation from $P$ to $Q$ and $L_2$ a canonical relation from $Q$ to $R$. The product $L_2 \times L_1$ then is an immersed Lagrangian submanifold of $R \times \overline{Q} \times Q \times \overline{P}$ and we call $L_2 \times L_1$ clean, if $(R \times \Delta_Q \times \overline{P}, L_2 \times L_1)$ is a reducible pair, i.e. if $R \times \Delta_Q \times \overline{P}$ and $L_2 \times L_1$ as submanifolds of $R \times \overline{Q} \times Q \times \overline{P}$ intersect cleanly.

**Proposition A.4.9.** *If $L_2 \times L_1$ is clean, then $L_2 \circ L_1$ is an immersed Lagrangian submanifold of $R \times \overline{P}$, i.e. $L_2 \circ L_1 \in \operatorname{Mor}(P, R)$ is a canonical relation.*

The proof of the above proposition is based on the following facts:

**Lemma A.4.10.** *Let $V$ be a symplectic vector space and $L, C \subset V$ a Lagrangian and a coisotropic subspace, respectively. Then*

$$L^C = L \cap C + C^\perp$$

*is a Lagrangian subspace of $V$ contained in $C$, and*

$$L_C := (L \cap C)/(L \cap C^\perp)$$

*is a Lagrangian subspace of $C/C^\perp$.*

*Proof.* Since $(L^C)^\perp = (L + C^\perp) \cap C$ and $C^\perp \subset C$ the first assertion follows immediately. Furthermore, since $L_C = L^C/C^\perp$ we have

$$L_C^\perp = (L^C)^\perp/C^\perp = L^C/C^\perp.$$

$\square$

A pointwise application of this result gives

**Theorem A.4.11.** *Let $(C, L)$ be a reducible pair in a symplectic manifold $P$. Then*

$$L^C := L \cap C + C^\perp$$

*is an immersed Lagrangian submanifold of $C/C^\perp$ and*

$$L_C := (L \cap C)/(L \cap C^\perp).$$

*is an immersed Lagrangian submanifold of $P$.*

Now to the

*Proof of Proposition A.4.9.* We want to apply Theorem A.4.11 to $C = R \times \Delta_Q \times \overline{P}$ and $L = L_2 \times L_1$ as well defined coisotropic resp. immersed Lagrangian submanifolds of $R \times \overline{Q} \times Q \times \overline{P}$. By assumption, these form a reducible pair such that $L_C$ and $L^C$ are well-defined Lagrangians. Now the composition is defined as

$$L_2 \circ L_1 = \{(r, p) \in R \times P; (r, q) \in L_2 \text{ and } (q, p) \in L_2 \text{ for some } q \in Q\},$$

which is the projection of $L \cap C$ onto $R \times P$. On the other hand $C/C^\perp = R \times \overline{P}$ and therefore

$$L_2 \circ L_1 = (L \cap C)/C^\perp = L_C,$$

which according to Theorem A.4.11 is an immersed Lagrangian submanifold of $C/C^\perp = R \times \overline{P}$. $\square$

## A.5 Complex symplectic vector spaces

So war we have only been concerned with real vector spaces carrying a symplectic structure. We will also need the analogous structures for complex vector spaces and their Lagrangian subspaces. We briefly provide some notions taken from [Sch01, Hör85a, Fol89], see also [CdS01, Mei94, HZ94, WC84].

Let $V$ be a complex vector space together with an anti-symmetric non-degenerate bilinear form $\omega$, i.e. a symplectic structure. The notions of Definition A.2.2 immediately carry over to complex subspaces of a symplectic vector space. Since we will be particularly interested in Lagrangian subspaces we give

**Definition A.5.1.** The set of all Lagrangian planes in the complexification $V^{\mathbb{C}}$ of a symplectic vector space $V$ is denoted by $\mathfrak{Lag}(V^{\mathbb{C}})$. A Lagrangian plane $L \in \mathfrak{Lag}(V^{\mathbb{C}})$ is called *positive* if

$$\mathrm{i}\,\omega(\bar{l}, l) \geq 0$$

for all $l \in L$, and *totally real* if

$$\mathrm{i}\,\omega(\bar{l}, l) = 0$$

for all $l \in L$. We will denote the set of all positive Lagrangian planes in $V^{\mathbb{C}}$ by $\mathfrak{Lag}^+(V^{\mathbb{C}})$.

If $L_0$ is a totally real Lagrangian plane in $V^{\mathbb{C}}$ we denote by $\mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}})$ the set of all positive Lagrangian planes transversal to $L_0$, i.e.

$$\mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}}) = \{L \in \mathfrak{Lag}^+(V^{\mathbb{C}});\ L \cap L_0 = \{0\}\}.$$

By choosing symplectic coordinates $(x, \xi)$ in $V^{\mathbb{C}}$ such that $L_0$ is represented by $x = 0$ any $L \in \mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}})$ can be written as the graph of a linear map $B : \mathbb{C}^n \to \mathbb{C}^n$,

$$\mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}}) \ni L = \{(x, Bx);\ x \in \mathbb{C}^d\}. \tag{A.5.1}$$

Then $L$ being Lagrangian means that

$$0 = \omega((x, Bx), (y, By)) = \langle x, By\rangle - \langle Bx, y\rangle$$

for all $x$, $y \in \mathbb{C}^n$. Thus $B$ is symmetric and the positivity of $L$ implies

$$\mathrm{i}\,\omega((\overline{x}, \overline{B}\overline{x}), (x, Bx)) = \mathrm{i}\left(\langle \overline{Bx}, x\rangle - \langle \overline{x}, Bx\rangle\right) = 2\langle \overline{x}, \Im Bx\rangle \geq 0,$$

which means that $\Im B$ is positive. Thus, the space $\mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}})$ is isomorphic to the space of all symmetric matrices in $\mathrm{M}_n(\mathbb{C})$ whose imaginary part is positive.

**Definition A.5.2.** The set of symmetric matrices in $\mathrm{M}_n(\mathbb{C})$ with $\Im B > 0$ is called the *Siegel upper half-space* $\Sigma_n$.

The symplectic group $\mathrm{Sp}(V)$ acts on $\mathfrak{Lag}(V)$. By linearity this action can be extended to an action of $\mathrm{Sp}(V)$ on $\mathfrak{Lag}(V^{\mathbb{C}})$; in particular, this induces an action on $\mathfrak{Lag}^+(V^{\mathbb{C}})$. We want to transfer this action to the Siegel upper half-plane; to this end assume that $L_B \in \mathfrak{Lag}^+_{L_0}(V^{\mathbb{C}})$ is defined by $B \in \Sigma_n$ according to (A.5.1). If $S \in \mathrm{Sp}(V)$ we are looking for $S[B] \in \Sigma_n$ such that

$$SL_B = L_{S[B]},$$

which can be calculated as

$$S[B] = (S_{22}B + S_{21})(S_{12}B + S_{11})^{-1}, \tag{A.5.2}$$

if we write $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. This indeed is well-defined and gives a group action of $\mathrm{Sp}(V)$ in $\Sigma_n$. We have [Fol89]

**Theorem A.5.3.** *(i) If $B \in \Sigma_n$ and $S \in \mathrm{Sp}(V)$ then $S[B] \in \Sigma_n$.*

*(ii) For any $B_1$, $B_2 \in \Sigma_n$ there is $S \in \mathrm{Sp}(V)$ such that $S[B_1] = B_2$, i.e. $\mathrm{Sp}(V)$ acts transitively on $\Sigma_n$.*

*(iii) The stabilizer of $\mathrm{i}\,\mathbb{1}_n \in \Sigma_n$ in $\mathrm{Sp}(V)$ is given by $\mathrm{Sp}(V) \cap O(V)$*

Thus we have that $\Sigma_n \simeq \mathrm{Sp}(V)/(\mathrm{Sp}(V) \cap O(V))$. Now a (totally) complex Lagrangian plane determines a complex structure on $V$ compatible with $\omega$, see e.g. [Woo97].

**Proposition A.5.4.** *Denote by $\mathrm{J}_L$ the complex structure on $V$ determined by $L$, by $\mathrm{g}_L$ the induced metric and by $\mathrm{h}_L$ the corresponding hermitian form. We have for any $S \in \mathrm{Sp}(V)$ that*

$$\mathrm{J}_{SL} = S\mathrm{J}_L S^{-1}, \quad \mathrm{h}_{SL} = (S^{-1})^* \mathrm{h}_L S^{-1}.$$

*Since $\mathrm{g}_L$ is the real part of $\mathrm{h}_L$ this immediately gives*

$$\mathrm{g}_{SL} = (S^{-1})^* \mathrm{g}_L S^{-1}.$$

*If we have chosen symplectic coordinates in $V^{\mathbb{C}}$ such that $L$ is represented by (A.5.1), then*

$$\mathrm{J}_L = \begin{pmatrix} -(\Im B)^{-1}\Re B & (\Im B)^{-1} \\ -(\Im B + \Re B(\Im B)^{-1}\Re B) & \Re B(\Im B)^{-1} \end{pmatrix}$$

*and*

$$\mathrm{g}_L = \begin{pmatrix} \Im B + \Re B(\Im B)^{-1}\Re B & -\Re B(\Im B)^{-1} \\ -(\Im B)^{-1}\Re B & (\Im B)^{-1} \end{pmatrix}. \tag{A.5.3}$$

*Furthermore, $\mathrm{g}_L \in \mathrm{Sp}(V)$ and $\det \mathrm{g}_L = 1$.*

Now let $H \in C^\infty(\mathrm{T}^*\mathbb{R}^d, \mathbb{R})$ be a Hamiltonian function and consider the linearized flow

$$S_z(t) : \mathrm{T}_{z(0)}(\mathrm{T}^*\mathbb{R}^d) \to \mathrm{T}_{z(t)}(\mathrm{T}^*\mathbb{R}^d) \tag{A.5.4}$$

about a trajectory $z(t)$ corresponding to the flow generated by $H$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} S_z(t) = \mathrm{J}H''(z(t))S_z(t), \tag{A.5.5}$$

where $\mathrm{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$. Since $S(t)$ is a symplectic map, we have an action $S(t)[B] = B(t)$ on the Siegel upper half plane. A direct calculation, see e.g. [Sch01], gives

$$\dot{B}(t) = -H''_{x,\xi}B - H''_{x,x} - BH''_{\xi,\xi}B - BH''_{\xi,x}.$$

If we define the multiplier

$$\mathrm{m}(S(t), B) := \sqrt{\det(S(t)_{12}B + S(t)_{22})}, \tag{A.5.6}$$

then the above equations can be used to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \mathrm{m}(S(t), B) = -\frac{1}{2}\,\mathrm{tr}(H''_{\xi,x} + H''_{\xi,\xi}B) = -\frac{1}{2}\,\mathrm{tr}((\dot{S}(t)_{12}B + \dot{S}(t)_{11})(S(t)_{12}B + S(t)_{22})).$$

# Appendix B

# The moment map and Lie algebra cohomology

## B.1 Lie algebra cohomology

In this Section we will briefly specify the Lie algebras that we will be concerned with in the following, on a algebraic level. To this end let $\mathfrak{g}$ be a Lie algebra and $V$ a $\mathfrak{g}$-module. Let $\Lambda^k(\mathfrak{g}, V)$ be the antisymmetric $k$-linear maps $\mathfrak{g} \to V$. We define the operator

$$\delta : \Lambda^k(\mathfrak{g}, V) \to \Lambda^{k+1}(\mathfrak{g}, V)$$

by

$$\begin{aligned}\delta f(\xi_0, \ldots, \xi_k) = &\sum_{i=0}^{k} (-1)^i \xi_i f(\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k) \\ &+ \sum_{\substack{i,j=1 \\ i<j}}^{k} (-1)^{i+j} f([\xi_i, \xi_j], \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_k).\end{aligned} \tag{B.1.1}$$

For $k = 0$, $f$ is an element of $V$ and the last term does not occur so

$$\delta f(\xi) = \xi f.$$

For $k = 1$, $f$ is a linear map from $\mathfrak{g}$ to $V$ and

$$\delta f(\xi_0, \xi_1) = \xi_0 f(\xi_1) - \xi_1 f(\xi_0) - f([\xi_0, \xi_1]),$$

whereas for $k = 2$ we have

$$\begin{aligned}\delta f(\xi_0, \xi_1, \xi_2) = &\xi_0 f(\xi_1, \xi_2) - \xi_1 f(\xi_0, \xi_2) + \xi_2 f(\xi_0, \xi_1) \\ &- f([\xi_0, \xi_1], \xi_2) + f([\xi_0, \xi_2], \xi_1) - f([\xi_1, \xi_2], \xi_0).\end{aligned}$$

Note, that if $V$ is a trivial $\mathfrak{g}$-module, then the first terms in (B.1.1) disappears. As a direct calculation shows that $\delta^2 = 0$. Let us then define

$$\ker \delta \cap \Lambda^k(\mathfrak{g}, V) =: \mathrm{Z}^k(\mathfrak{g}, V), \quad \text{and} \quad \mathrm{B}^k(\mathfrak{g}, V) := \delta(\Lambda^{k-1}(\mathfrak{g}, V)).$$

The cohomology groups (see, e.g. [GH87, Hir78, Dol72]) of the above complex are

$$\mathrm{H}^k(\mathfrak{g}, V) := \mathrm{Z}^k(\mathfrak{g}, V)/\mathrm{B}^k(\mathfrak{g}, V).$$

In the following we will always suppose that the Lie algebras under consideration are finite dimensional. The significance of the first cohomology group is expressed in

**Proposition B.1.1.** *A Lie algebra $\mathfrak{g}$ has the property that $\mathrm{H}^1(\mathfrak{g}, V) = \{0\}$ for all $V$ iff every representation of $\mathfrak{g}$ is completely reducible.*

In fact the assumptions of the above proposition have important consequences; in particular:

**Proposition B.1.2.** *Any Lie algebra satisfying the hypotheses of Proposition B.1.1 has $\mathrm{H}^2(\mathfrak{g}, V) = \{0\}$.*

We will be concerned with a special type of Lie algebras:

**Definition B.1.3.** A Lie algebra $\mathfrak{g}$ is called *semisimple* if it contains no commutative ideals.

Recall that a Lie group is called semisimple if it has no abelian connected normal subgroup other than $\{e\}$.

The following is a criterion for semisimplicity

**Proposition B.1.4 (Cartan).** *If $\mathfrak{g}$ is a semisimple Lie algebra over a field with characteristic zero and $(\rho, V)$ is a one-to-one finite-dimensional representation of $\mathfrak{g}$ then the trace form*

$$b_\rho(\xi, \eta) := \operatorname{tr} \rho(\xi)\rho(\eta)$$

*is nondegenerate. If the trace form of the adjoint representation of a Lie algebra $\mathfrak{g}$, called the Killing form, is nondegenerate, then $\mathfrak{g}$ is semisimple.*

Semisimple Lie algebras are particularly interesting since

**Theorem B.1.5.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the cohomology groups with values in a trivial $\mathfrak{g}$ module are trivial, i.e. $\mathrm{H}^1(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}) = \{0\}$.*

## B.2 Symplectic homogeneous spaces

In this Section we are interested in homogeneous[1] $G$ spaces, where $G$ is a Lie group with semisimple Lie algebra $\mathfrak{g}$. Our discussion will use results given in [GS84] and also [MR94, AMR88, AM78, Lan98a].

Let the Lie group $G$ act on a manifold $M$. Since we may identify the left-invariant vector fields on $G$ with the Lie algebra $\mathfrak{g}$ we also have an identification of left-invariant $q$-forms on $G$ with the exterior algebra $\Lambda^q(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$. Since the exterior derivative $\mathrm{d}$ commutes with pull-backs [Fra97, KN63, Ste64] we have an induced linear map

$$\delta : \Lambda^q(\mathfrak{g}^*) \longrightarrow \Lambda^{q+1}(\mathfrak{g}^*).$$

Let us calculate $\delta\omega$ for $\omega \in \Lambda^1(\mathfrak{g}^*)$: If $\xi$ is an invariant vector field then $\imath_\xi\omega$ is constant, thus

$$\mathscr{L}_\xi\omega = \imath_\xi \,\mathrm{d}\omega + \mathrm{d}\imath_\xi\omega = \imath_\xi \,\mathrm{d}\omega,$$

and for any invariant vector field $\eta$ clearly $\imath_\eta\omega$ is constant, which implies

$$0 = \mathscr{L}_\xi\omega(\eta) = (\imath_\xi \,\mathrm{d}\omega)(\eta) + \omega([\xi,\eta]) = \mathrm{d}\omega(\xi\wedge\eta) + \omega([\xi,\eta]).$$

So we obtain

$$\delta\omega(\xi\wedge\eta) = -\omega([\xi,\eta])$$

for $\omega \in \Lambda^1(\mathfrak{g}^*)$. If $\omega \in \Lambda^2(\mathfrak{g}^*)$ and $\xi$, $\eta$ and $\zeta$ are in $\mathfrak{g}$, we can apply the same argument to $0 = \mathscr{L}_\xi\omega(\eta\wedge\zeta)$, and use the above formula to conclude that

$$\delta\omega(\xi\wedge\eta\wedge\zeta) = -\omega([\xi,\eta]\wedge\zeta) - \omega([\eta,\zeta]\wedge\xi) - \omega([\zeta,\xi]\wedge\eta).$$

This allows for an inductive calculation of $\delta$ and explicitly shows that $\delta^2 = 0$. As usual we define

$$\mathrm{B}^k(\mathfrak{g}) := \delta(\Lambda^{k-1}(\mathfrak{g}^*)) \quad \text{and} \quad \mathrm{Z}^k(\mathfrak{g}) := \ker\delta \subset \Lambda^k(\mathfrak{g}^*).$$

We also have the corresponding cohomology groups

$$\mathrm{H}^k(\mathfrak{g}) := \mathrm{Z}^k(\mathfrak{g})/\mathrm{B}^k(\mathfrak{g}).$$

Since for compact $G$ the averaging over the groups allows us to replace differential forms by left-invariant ones, in this case $\mathrm{H}^k(\mathfrak{g})$ is the $k$th de Rham cohomology group of $G$ over the reals. This need not to be true for non-compact $G$.

Why we are particularly interested in $\mathrm{H}^1(\mathfrak{g})$ and $\mathrm{H}^2(\mathfrak{g})$ will become clear if we consider the following situation: Let $(M,\omega)$ be a symplectic manifold and denote by $C^\infty(M)$ the space of smooth functions on $M$ and by $\Xi_{\mathrm{H}}(M)$ the space of all Hamiltonian vector fields on $M$. Of course we have a linear map from $C^\infty(M)$ to $\Xi_{\mathrm{H}}(M)$, $C^\infty(M) \ni H \mapsto \mathrm{X}_H \in \Xi_{\mathrm{H}}(M)$ defined by

$$\mathrm{d}H = \imath_{\mathrm{X}_H}\omega.$$

[1] i.e. transitive

If $M$ is connected, the kernel of this map consists of the constant functions on $M$ resulting in the exact sequence

$$0 \to \mathbb{R} \to C^\infty(M) \to \Xi_{\mathrm{H}}(M) \to 0$$

of Lie algebras. Since $\mathrm{d}\{H_1, H_2\} = \imath_{\mathrm{X}_{\{H_1,H_2\}}}\omega$ the above maps are also homomorphisms of Lie algebras, where $\mathbb{R}$ is thought of as the trivial Lie algebra, and $\mathbb{R} \to C^\infty(M)$ is the injection of $\mathbb{R}$ into $C^\infty(M)$ as the constant functions. Furthermore, $\mathbb{R}$ is the center of the Lie algebra $C^\infty(M)$ and therefore $C^\infty(M)$ is a central extension of $\Xi_{\mathrm{H}}(M)$.

More generally, we consider the exact sequence of Lie algebras

$$0 \to \mathbb{R} \to \mathfrak{f} \xrightarrow{\rho} \mathfrak{h} \to 0, \quad [\mathbb{R}, f] = 0, \tag{B.2.1}$$

i.e. the central extension $\mathfrak{f}$ of $\mathfrak{h}$ by $\mathbb{R}$. Suppose, we are given a homomorphism $\kappa : \mathfrak{g} \to \mathfrak{h}$. Then we ask if there is a Lie-algebra homomorphism $\lambda : \mathfrak{g} \to \mathfrak{f}$ such that $\kappa = \rho \circ \lambda$. To this end consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathfrak{f} \arrow[r, "\rho"] & \mathfrak{h} \arrow[r] & 0 \\
 & & & \mathfrak{g} \arrow[ul, "\lambda"'] \arrow[u, "\kappa"'] &
\end{tikzcd}.$$

Since $\mathfrak{g}$ is a finite-dimensional vector space and $\mathfrak{f}$ is a vector space, we can find a linear map $\mu : \mathfrak{g} \to \mathfrak{f}$ such that $\rho \circ \mu = \kappa$ simply by defining $\mu$ for basis elements of $\mathfrak{g}$. Since $\kappa$ is a Lie algebra homomorphism there is $c \in \Lambda^2(\mathfrak{g}^*)$ defined through

$$c(\xi, \eta) := \mu([\xi, \eta]) - [\mu(\xi), \mu(\eta)],$$

and a direct calculation shows that $\delta c = 0$. Furthermore, $c$ defines an cohomology class $[c]$ since it is independent of the particular choice of $\mu$. This leads to

**Proposition B.2.1.** *Given a central extension (B.2.1) and a homomorphism $\kappa : \mathfrak{g} \to \mathfrak{h}$, there is a well-defined cohomology class $[c] \in \mathrm{H}^2(\mathfrak{g})$ that measures the obstruction to the possibility of finding a Lie-algebra homomorphism $\lambda : \mathfrak{g} \to \mathfrak{f}$ covering $\kappa$.*

*Such a $\lambda$ exists iff $[c] = 0$, and in that case the possible choices for $\lambda$ are parameterized by $\mathrm{H}^1(\mathfrak{g})$. In particular, if $\mathrm{H}^2(\mathfrak{g}) = \{0\}$ then a homomorphism $\lambda$ always exists, and if $\mathrm{H}^1(\mathfrak{g}) = \{0\}$ it is unique.*

Semisimple Lie algebras always have $\mathrm{H}^2(\mathfrak{g}) = \mathrm{H}^1(\mathfrak{g}) = \{0\}$, see Theorem B.1.5. Since left multiplication commutes with right multiplication, right multiplication defines a representation of $G$ on $\Lambda^q(\mathfrak{g}^*)$, which we denote by $\mathrm{Ad}^\#$. If $G$ acts on $M$ for each $g \in G$ and $m \in M$ we obtain maps $\psi_m : G \to M$ and $\phi_g : M \to M$ defined by

$$\psi_m(g) := gm \quad \text{and} \quad \phi_g(m) = gm.$$

Now let $\Omega$ be an invariant $q$-form, i.e. $\phi_g^*\Omega = \Omega$ for all $g \in G$. Then we have

**Proposition B.2.2.** *An invariant q-form $\Omega$ on $M$ defines a map $\Psi : M \to \Lambda^q(\mathfrak{g}^*)$ given by*

$$\Psi(m) = \psi_m^* \Omega.$$

*This map is a G-morphism, i.e.* $\Psi(am) = \mathrm{Ad}_a^\# \Psi(m)$.

In particular, if $(M, \omega)$ is a symplectic manifold then $\Psi(m) = \psi_m^* \omega$ is closed and we have

$$\Psi : M \to \mathrm{Z}^2(\mathfrak{g}).$$

**Theorem B.2.3.** *Any symplectic action of a Lie group $G$ an a symplectic manifold $(M, \omega)$ defines a $G$-morphism, $\Psi : M \to \mathrm{Z}^2(\mathfrak{g})$, and $\Psi(M)$ is a union of $G$-orbits. In particular, if the action of $G$ on $M$ is transitive, the image of $M$ under $\Psi$ consists of a single $G$-orbit in $\mathrm{Z}^2(\mathfrak{g})$.*

In the following we will use a variant of symplectic reduction, see Section A.4. Let $M$ be a manifold and $\omega$ a closed two-form of constant rank, i.e. with the dimension of $(\mathrm{T}_m M)^\perp$ the same for all $m \in M$[2]. The characteristic subbundle $(\mathrm{T}M)^\perp$ of a presymplectic manifold $(M, \omega)$ is integrable, see Proposition A.4.2 The foliation $\mathcal{M}^\perp$ defined by the characteristic subbundle is called characteristic foliation of $M$. If the quotient space $M/\mathcal{M}^\perp$ is a smooth manifold, we have an induced symplectic structure on $M/\mathcal{M}^\perp$, which therefore is a symplectic manifold, called the reduced manifold of $M$. In particular, this is the case if the characteristic foliation is fibrating[3].

Let us consider a symplectic manifold $(Y, \Omega)$ and a smooth map $i : X \to Y$ with $\omega = i^*\Omega$. Now let the rank of $\omega$ be constant, then we obtain the quotient map $\rho : X \to X/X^\perp =: M$ and hence the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle\rho}\big\downarrow & & \\ M & & \end{array}$$

where $i^*\Omega = \rho^*\overline{\omega}$ and $\overline{\omega}$ is the symplectic form on $M$. Suppose that $f_1$ and $f_2$ are two functions on $Y$ such that $i^*f_1$ and $i^*f_2$ are constant along the fibers of the projection $\rho$, therefore $i^*f_\nu = \rho^*F_\nu$ for some functions $F_\nu$ on $M$.

There is a nice connection between the Poisson brackets for these classes of functions and coisotropic immersion.

**Theorem B.2.4.** *If $i : X \to Y$ is a coisotropic immersion whose characteristic foliation is fibrating with projection map $\rho : X \to M$, and if $f_\nu$, $\nu = 1, 2$, are functions on $Y$ satisfying $i^*f_\nu = \rho^*F_\nu$ then*

$$i^*\{f_1, f_2\}_Y = \rho^*\{F_1, F_2\}.$$

[2] In this case $\omega$ is called a pre-symplectic structure on $M$ with characteristic subbundle $(\mathrm{T}M)^\perp$, see Section A.4.

[3] i.e. if there exists a manifold $X$ and a submersion $\rho : M \to X$ such that the leaves of the characteristic foliation are all of the form $\rho^{-1}(x)$ for $x \in X$.

Now let $\omega \in \mathrm{Z}^2(\mathfrak{g})$ and define

$$\mathfrak{h}_\omega = \{\xi \in \mathfrak{g};\ \imath_\xi \omega = 0\}.$$

Then $\mathfrak{h}_\omega \subset \mathfrak{g}_\omega$, the isotropy algebra of $\omega$, since $\mathfrak{g}_\omega = \{\xi \in \mathfrak{g}; \mathscr{L}_\xi \omega = 0\}$ and $\mathscr{L}_\xi \omega = \mathrm{d}\imath_\xi \omega$ since $\omega$ is closed. If $\xi \in \mathfrak{h}_\omega$ and $\eta \in \mathfrak{h}_\omega$ then

$$\imath_{[\eta,\xi]}\omega = \mathscr{L}_\eta \imath_\xi \omega = 0,$$

therefore $\mathfrak{h}_\omega$ is an ideal in $\mathfrak{g}_\omega$ and, in particular, $\mathfrak{h}_\omega$ is a subalgebra of $\mathfrak{g}$. Let $H_\omega$ be the subgroup generated by $\mathfrak{h}_\omega$. If $H_\omega$ is closed we can form the quotient space $M = G/H_\omega$ and obtain the projection $\rho:\ G \to M$. Then a check of the definitions shows that

$$\Psi(m) = \omega,$$

where $m$ is the identity coset in $G/H_\omega$. Therefore, the orbit in $\mathrm{Z}^2(\mathfrak{g})$ associated with $G/H_\omega$ is precisely the $G$-orbit through $\omega$.

Now suppose we start with a symplectic manifold of the form $M = G/H$ with symplectic form $\Omega$. Since $\mathrm{T}M = \mathfrak{g}/\mathfrak{h}$ and since $G$ acts transitively on $M$, the fundamental vector fields $\xi_M$ fill out the tangent space at each point. Therefore $\mathfrak{h} = \mathfrak{h}_\omega$ and hence $H_\omega$ is the connected component of $H$, or $M_\omega$ is a covering space of $M$. Thus, the set of transitive symplectic manifolds of $G$ is parameterized by the set of $G$-orbits through elements $\omega$ in $\mathrm{Z}^2(\mathfrak{g})$, whose corresponding subgroup $H_\omega$ is closed. Let $\tau \in \mathrm{Z}^2(\mathfrak{g})$, then a criterion which guarantees that the associated group $H_\tau$ is closed is given by

**Proposition B.2.5 (Kostant-Souriau).** *If $\tau = -\mathrm{d}\beta$ the subgroup $H_\tau$ is closed and is the connected component of the group*

$$G_\beta = \{c \in G;\ \mathrm{Ad}_c^\#\, \beta = \beta\}.$$

*Thus $G/H_\tau$ is a covering space of the orbit $G\beta$. In particular, each orbit $G\beta$ is a symplectic manifold with symplectic form $\omega$ given at the point $\beta$ by*

$$\omega(\xi_\beta, \eta_\beta) = \beta([\xi, \eta]). \tag{B.2.2}$$

*Proof.* Since $\xi$ and $\eta$ are left-invariant, we have

$$\mathscr{L}_\xi \beta = \imath_\xi\, \mathrm{d}\beta.$$

Moreover $\mathscr{L}_\xi(\beta(\eta)) = 0$ implies $(\mathscr{L}_\xi \beta)(\eta) = \beta([\xi,\eta])$, since $\beta(\eta)$ is constant. Therefore $H_\tau$ is the connected component of $G_\beta$. Now the vector $\xi_\beta$ corresponding to $\xi$ at $\beta$ is just the image of $\xi$ under the projection $\rho:\ G \to G/H$. Therefore the left-hand side of (B.2.2) is the value of $\tau$ evaluated on $(\xi, \eta)$. But since

$$0 = \mathscr{L}_\xi(\tau(\eta)) = (\imath_\xi\, \mathrm{d}\tau)(\eta) + \tau([\xi, \eta])$$

the proposition follows. □

We also have

**Theorem B.2.6 (Kostant-Souriau).** *If $\mathrm{H}^1(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}) = \{0\}$ then, up to coverings, every homogeneous symplectic manifold for $G$ is an orbit of $G$ acting on $\mathfrak{g}^*$, i.e. a coadjoint orbit.*

## B.3 The moment map

Suppose we are given a symplectic action of a Lie group $G$ on a symplectic manifold $(M, \omega)$, with the associated map $\Psi : M \to \mathrm{Z}^2(\mathfrak{g})$. If $\mathrm{H}^1(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}) = \{0\}$, then for each $m \in M$ there is a unique element $\Phi(m)$ of $\mathfrak{g}^*$ with $\delta\Phi(m) = \Psi(m)$. In other words, there is a unique map $\Phi : M \to \mathfrak{g}^*$ such that

$$\delta\Phi = \Psi.$$

Now $\mathrm{H}^1(\mathfrak{g}) = \{0\}$ means that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Let $\xi,\ \eta \in \mathfrak{g}$ with corresponding vector fields $\xi_M$ and $\eta_M$ on $M$, then

$$\mathscr{L}_{\xi_M} \imath_{\eta_M} \omega = \imath_{[\eta_M, \xi_M]} \omega$$

since $\mathscr{L}_{\xi_M} \omega = 0$ and also

$$\mathscr{L}_{\xi_M} \imath_{\eta_M} \omega = \mathrm{d} \imath_{\xi_M} \imath_{\eta_M}$$

according to Cartan's equation and $\mathrm{d}\imath_{\eta_M}\omega = 0$. Therefore, if $\zeta = [\eta, \xi]$ then $\imath_{\zeta_M}\omega = \mathrm{d}f$, where $f = \imath_{\xi_M} \imath_{\eta_M} \omega$ and $\zeta_M$ is the Hamiltonian vector field corresponding to the function $f$. However, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, which means[4] that any element of $\mathfrak{g}$ is a linear combination of elements of the form $[\xi, \eta]$ we have a homomorphism from the Lie algebra $\mathfrak{g}$ into the Lie algebra of Hamiltonian vector fields $\Xi_{\mathrm{H}}(M)$ on $M$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "\rho"] & \Xi_{\mathrm{H}}(M) \arrow[r] & 0 \\
 & & & \mathfrak{g} \arrow[ul, "\lambda"] \arrow[u] &
\end{tikzcd}$$

Therefore, according to Proposition B.2.1, there is a unique homomorphism

$$\lambda : \mathfrak{g} \longrightarrow C^\infty(M),$$

sending $\xi$ to $f_\xi$ for each $\xi \in \mathfrak{g}$, where

$$\mathrm{d}f_\xi = \imath_{\xi_M}\omega.$$

Furthermore, since $f_\xi$ depends linearly on $\xi$, for each $m \in M$ we can define $\Phi(m) \in \mathfrak{g}^*$ according to

$$\langle \Phi(m), \xi \rangle = f_\xi(m).$$

The last equation can also be expressed as

$$\langle \mathrm{d}\Phi, \xi \rangle = \imath_{\xi_M}\omega,$$

where $\mathrm{d}\Phi$ gives a linear map $\mathrm{T}_m M \to \mathfrak{g}^*$. Therefore, under the identification of $\mathrm{T}_m^* M$ with $\mathrm{T}_m M$ given by the symplectic form $\omega$, we can say that $\mathrm{d}\Phi_m$ is the transpose of the evaluation map from $\mathfrak{g}$ to $\mathrm{T}_m M$. It follows that

$$\ker \mathrm{d}\Phi_m = \mathfrak{g}_M(m)^\perp, \tag{B.3.1}$$

[4]and follows from the semisimplicity of $\mathfrak{g}$

where $\mathfrak{g}_M(m)$ denotes the subspace of $\mathrm{T}_m M$ generated by all vectors of the form $\xi_M(m)$. If $G$ acts transitively on $M$ then $\ker \mathrm{d}\Phi = 0$ and $\Phi$ is an immersion. If we denote by $(\mathrm{im}\ \mathrm{d}\Phi_m)^0 \subset \mathfrak{g}$ the space of vectors in $\mathfrak{g}$ that vanishes when evaluated on any element of $\mathrm{im}\ \mathrm{d}\Phi_m$, then

$$(\mathrm{im}\ \mathrm{d}\Phi_m)^0 = \{\xi \in \mathfrak{g};\ \xi_M(m) = 0\}.$$

One thus realizes that $\mathrm{d}\Phi_m$ is surjective iff the stabilizer group of $m$ is discrete.

Now $G$ acts on $\mathfrak{g}$ via the adjoint representation and the mapping $\xi \mapsto \xi_M$ is a $G$-morphism. It follows that the map $\Phi$ is also a $G$-morphism, i.e.

$$\Phi(am) = \mathrm{Ad}_a^* \Phi(m).$$

Summarizing, we obtain

**Theorem B.3.1.** *Let $G$ be a connected Lie group with $\mathrm{H}^1(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}) = \{0\}$. Suppose we have a symplectic group action of $G$ on a symplectic manifold $(M, \omega)$. Denote the corresponding homomorphism of $\mathfrak{g}$ to vector fields on $M$ by writing $\xi_M$ for the vector field corresponding to $\xi \in \mathfrak{g}$. Then there is a unique homomorphism*

$$\lambda : \mathfrak{g} \longrightarrow C^\infty(M)$$

*that satisfies*

$$\imath_{\xi_M}\omega = \mathrm{d}f_\xi,$$

*where $f_\xi = \lambda(\xi)$. The homomorphism $\lambda$ is a $G$-morphism and allows us to define the map $\Phi : M \to \mathfrak{g}^*$, called the* moment map, *by*

$$\langle \Phi(m), \xi \rangle = f_\xi(m).$$

In order to discuss further properties of the moment map we will have to introduce some notation, for which, besides Appendix A, we refer to [Wei77b, BW97, GS90, Hör85a].

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a $G$-orbit and suppose that $\Phi : M \to \mathfrak{g}^*$ intersects $\mathcal{O}$ cleanly, see Definition A.4.5. Let $m \in \Phi^{-1}(\mathcal{O})$ and let $W$ be the tangent space to $\Phi^{-1}(\mathcal{O})$. Then by the clean intersection hypothesis we have

$$\Phi_* W = \mathrm{T}\mathcal{O}.$$

Now $G$ acts transitively on $\mathcal{O}$, therefore the tangent space $\mathrm{T}_{\Phi(m)}\mathcal{O}$ is spanned by the vectors $\xi_{\mathfrak{g}^*}(\Phi(m))$. According to the equivariance of $\Phi$ we have

$$\mathrm{d}\Phi_m(\xi_M(m)) = \xi_{\mathfrak{g}^*}(\Phi(m))$$

and therefore

$$W = \mathfrak{g}_M(m) + \ker \mathrm{d}\Phi_m = \mathfrak{g}_M(m) + \mathfrak{g}_M(m)^\perp$$

according to (B.3.1). Therefore

$$W^\perp = \mathfrak{g}_M(m) \cap \mathfrak{g}_M^\perp(m) \subset W,$$

and $W$ is coisotropic.

Now if $\xi \in \mathfrak{g}$ such that $\xi_M(m) \subset \mathfrak{g}_M(m)^\perp$, then $\xi_{\mathfrak{g}^*}(\Phi(m)) = \mathrm{d}\Phi_m(\xi_M(m)) = 0$ and $\xi \in \mathfrak{g}_{\Phi(m)}$, the isotropy algebra of $\Phi(m)$. We therefore have

**Theorem B.3.2 (Kazhdan, Kostant, Sternberg).** *If $\Phi : M \to \mathfrak{g}^*$ intersects an orbit $\mathcal{O}$ cleanly, then $\Phi^{-1}(\mathcal{O})$ is coisotropic and the null foliation through a point $m \in \Phi^{-1}(\mathcal{O})$ is the orbit of $M$ under $G^0_{\Phi(m)}$, the connected component of the isotropy subgroup of $\Phi(m)$.*

More generally [GS84]:

**Theorem B.3.3.** *Let $C \subset M$ be an invariant submanifold of $\mathfrak{g}^*$. Given a Hamiltonian action of $G$ on a symplectic manifold $M$ whose moment map $\Phi$ intersects $C$ cleanly, then $\Phi^{-1}(C)$ is a coisotropic submanifold of $M$. The leaf of the null foliation through a point $m \in \Phi^{-1}(C)$ is the orbit of $m$ under the $G^0_{\Phi(m)}$.*

## B.3.1 Marsden-Weinstein reduction

Another important way to produce coisotropic submanifolds out of the moment map is given by the Marsden-Weinstein reduction [MW74]. See also [Mar92].

Let $M$ and $N$ be two symplectic manifolds with Hamiltonian $G$ actions whose moment maps are $\Phi_M$ and $\Phi_N$, respectively. Then $M \times N$ is a Hamiltonian $G$-space with moment map

$$\Phi_{M\times N}(m,n) = \Phi_M(m) + \Phi_N(n).$$

If we reverse the sign of the symplectic form on $N$ and denote the resulting symplectic manifold by $\overline{N}$, then the corresponding moment map reads

$$\Phi_{M\times \overline{N}}(m,n) = \Phi_M(m) - \Phi_N(n).$$

Now, as a particular case, let $N = \mathcal{O}$ be a coadjoint orbit, where the moment map is just the injection $\mathcal{O} \hookrightarrow \mathfrak{g}^*$, see e.g. [MR94]. Thus

$$\Phi_{M\times \overline{\mathcal{O}}}(m,\eta) = \Phi_M(m) - \eta.$$

Let us apply Theorem B.3.2 to $M \times \overline{\mathcal{O}}$ with the orbit taken to be the trivial orbit $\{0\}$. Then $\Phi_{M\times\overline{\mathcal{O}}}$ intersects this orbit cleanly iff

$$\Phi^{-1}_{M\times\overline{\mathcal{O}}}(0) = \{(m,\eta) \in M \times \mathcal{O}; \Phi_M(m) = \eta\}$$

is a submanifold of $M \times \mathcal{O}$ and $\Phi_M$ intersects $\mathcal{O}$ cleanly. Under the clean intersection hypothesis $\Phi^{-1}_{M\times\overline{\mathcal{O}}}(0)$ is a coisotropic submanifold of $M \times \overline{\mathcal{O}}$, and the null foliation through any point is the orbit of this point under the action of the connected component of the isotropy group of 0 which, however, is all of $G$. So if we assume $G$ to be connected then the corresponding quotient space $\Phi^{-1}_{M\times\mathcal{O}^-}(0)/G$ carries a natural symplectic structure. This symplectic space is called the Marsden-Weinstein reduced space $M_{\mathcal{O}}$ of $M$ at $\mathcal{O}$.

Since $G$ acts transitively on $\mathcal{O}$ we can identify in a set-theoretical sense

$$\Phi^{-1}_{M\times\overline{\mathcal{O}}}(0)/G \simeq \Phi^{-1}_M(\eta)/G_\eta.$$

Now Theorem B.3.2 associates a symplectic manifold to an orbit by taking the quotient of $\Phi^{-1}_M(\mathcal{O})$ by its null foliation, provided that this foliation is fibrating. Let us denote this space by $Z_{\mathcal{O}}$ and consider its relation to the Marsden-Weinstein reduced space $M_{\mathcal{O}}$. To this end assume that $\Phi_M$ intersects $\mathcal{O}$ cleanly and that the null foliation of $\Phi_M^{-1}(\mathcal{O})$ is fibrating over the symplectic manifold $Z_{\mathcal{O}}$. Since $\Phi^{-1}(\mathcal{O})$ is invariant under $G$ each of the vector field $\xi_M$ is tangent to $\Phi_M^{-1}(\mathcal{O})$ and hence the functions $f_\xi$, defined in Theorem B.3.1, are constant along the leaves of the null foliation; therefore then define functions $F_\xi$ on $Z_{\mathcal{O}}$. The group $G$ preserves the null foliation on $\Phi_M^{-1}(\mathcal{O})$ and hence defines a symplectic action on $Z_{\mathcal{O}}$. More precisely, this action is Hamiltonian with $F_\xi$ being the function on $Z_{\mathcal{O}}$ corresponding to $\xi \in \mathfrak{g}$. The moment map $\Phi_{Z_{\mathcal{O}}}$ is constant on the fibers of $\Phi_M^{-1}(\mathcal{O})$ over $Z_{\mathcal{O}}$ and we have the commuting diagram

$$\begin{tikzcd} \Phi_M^{-1}(\mathcal{O}) \arrow[d, "\rho"'] \arrow[dr, "\Phi_M"] & \\ Z_{\mathcal{O}} \arrow[r, "\Phi_{Z_{\mathcal{O}}}"] & \mathcal{O} \end{tikzcd} .$$

Now let $z \in Z_{\mathcal{O}}$ and $\eta = \Phi_{Z_{\mathcal{O}}}(z) \in \mathcal{O}$ and consider a point $m \in M$ over $z$, i.e. $\rho(m) = z$, so that $\Phi_M(m) = \eta$. If $a \in G_\eta^0$, then Theorem B.3.2 implies $\rho(am) = \rho(m) = z$. However, since $\rho(am) = a\rho(m)$ we have

$$G_\eta^0 \subset G_z,$$

and also

$$G_z \subset G_\eta,$$

since $\Phi_{Z_{\mathcal{O}}}$ is a $G$ morphism. Thus, if $G_\eta$ is connected, then $G_z = G_\eta$ and hence the map $\Phi_{Z_{\mathcal{O}}}$ gives a diffeomorphism of the orbit $G \cdot z$ through $z$ and the orbit $\mathcal{O}$. On the other hand, let $F_\eta = \Phi_{Z_{\mathcal{O}}}^{-1}(\eta) = \Phi_M^{-1}(\eta)/G_\eta$, that can be identified with $M_{\mathcal{O}}$. Then we have the map

$$\gamma : \mathcal{O} \times F_\eta \to Z_{\mathcal{O}}, \quad \gamma(a\eta, z) = az,$$

which is well defined since $G_z = G_\eta$ for all $z \in F_\eta$. We may also define

$$m \longmapsto (m, \Phi(m)), \quad \Phi_M^{-1}(\mathcal{O}) \longrightarrow \Phi_{M\times\mathcal{O}}^{-1}(0),$$

which induces

$$\pi : Z_{\mathcal{O}}/G \longrightarrow \Phi_{M\times\mathcal{O}}^{-1}(\mathcal{O})/G = M_{\mathcal{O}}.$$

We hence have

$$(\Phi_{Z_{\mathcal{O}}} \times \pi) : Z_{\mathcal{O}} \longrightarrow \mathcal{O} \times M_{\mathcal{O}},$$

whose inverse is $\gamma$, proving that $\gamma$ is a diffeomorphism. This can easily be verified to be symplectic.

**Theorem B.3.4.** *Under the hypotheses of Theorem B.3.2, assume that the null foliation is fibrating over a symplectic manifold $Z_{\mathcal{O}}$. Then $Z_{\mathcal{O}}$ is a Hamiltonian $G$ space with moment map $\Phi_{Z_{\mathcal{O}}}$. We can identify $Z_{\mathcal{O}}/G$ with the Marsden-Weinstein reduced space $M_{\mathcal{O}} = \Phi_{M\times\mathcal{O}}^{-1}(0)/G = \Phi_M^{-1}(\eta)/G_\eta$ for any $\eta \in \mathcal{O}$. If $G_\eta$ is connected for some, and hence all $\eta \in \mathcal{O}$, then we have a symplectic diffeomorphism of $Z_{\mathcal{O}}$ with $\mathcal{O} \times M_{\mathcal{O}}$, and this is a $G$ morphism identification when we regard $M_{\mathcal{O}}$ as a trivial $G$ space.*

## B.4 Motion of a particle in Yang-Mills fields

Let $P \to M$ be a $G$ principal fiber bundle. $G$ acts on $P$ from the right and $M = P/G$. This action induces an action of $G$ on $\mathrm{T}^*P$ as follows: In general any diffeomorphism $\phi$ of a manifold $P$ induces a transformation of $\mathrm{T}^*P$ into itself. This can be seen as follows: Let $(p, \gamma) \in \mathrm{T}^*P$. Then

$$\phi^* : \mathrm{T}^*_{\phi(p)}P \to \mathrm{T}^*_p P,$$

and we have an induced transformation on $\mathrm{T}^*P$ given by

$$\tilde{\phi}(p, \gamma) = (\phi(p), (\phi^*)^{-1}\gamma).$$

Let $\alpha \in \Lambda^1(\mathrm{T}^*P)$ be the fundamental one-form, then

$$(\tilde{\phi}^*\alpha) = \alpha$$

according to the definition of $\alpha$. Therefore $\tilde{\phi}$ preserves the symplectic form $\omega = \mathrm{d}\alpha$ on $\mathrm{T}^*P$. Now let $\phi$ be a flow on $P$ generated by a vector field $\xi$. According to the above considerations this flow induces a flow on $\mathrm{T}^*P$ with corresponding vector field $\tilde{\xi}$. Obviously

$$\pi_*\tilde{\xi} = \xi$$

where $\pi : \mathrm{T}^*P \to P$ is the canonical projection and

$$\alpha_{(p,\gamma)}(\tilde{\xi}_{(p,\gamma)}) = \langle \gamma, \xi_p \rangle.$$

Since the flow generated by $\tilde{\xi}$ preserves $\alpha$ we have

$$\mathscr{L}_{\tilde{\xi}}\alpha = 0 = \imath_{\tilde{\xi}}\,\mathrm{d}\alpha + \mathrm{d}\imath_{\tilde{\xi}}\alpha$$

and therefore

$$\imath_{\tilde{\xi}}\omega = \mathrm{d}f_{\xi}, \quad \text{with} \quad f_{\xi} = \langle \alpha, \tilde{\xi} \rangle. \tag{B.4.1}$$

This shows that the induced $G$ action on $\mathrm{T}^*P$ is Hamiltonian with moment map $\Phi' : \mathrm{T}^*P \to \mathfrak{k}^*$ given by the dual of the map $\mathfrak{g} \to \mathrm{T}P, \quad \xi \mapsto \xi_P$, see also [Wei77a, GS78, GS82, Ste77, Mon84]. Let $\mathcal{O}'$ be a $G$-orbit in $\mathfrak{k}^*$, then we can consider the Marsden-Weinstein reduced phase space

$$\mathrm{T}^*P_{\mathcal{O}'} = \Phi^{-1}_{\mathrm{T}^*P \times \mathcal{O}'}(0)/G$$

where

$$\Phi_{\mathrm{T}^*P \times \mathcal{O}'}(z, \beta) = \Phi'(z) = \beta.$$

Since $P$ is a principal bundle every point $m \in M$ has a coordinate neighbourhood $U$ such that $\pi^{-1}(U)$ is isomorphic as a $G$ space to $U \times G$, where $\pi$ denotes the projection $\pi : P \to M$, see [KN63, KN69, Kob87, Nak90, Fra97, Ste51, Hus75]. This isomorphism is equivalent to the choice of a section of $P$ over $U$. In this local trivialization we may also identify

$$\mathrm{T}^*(\pi^{-1}U) \simeq \mathrm{T}^*U \times \mathrm{T}^*G \simeq \mathrm{T}^*U \times G \times \mathfrak{k}^*,$$

where we have used the identification of $\mathfrak{k}$ with invariant vector fields on $G$, see, e.g. [BtD85]. In terms of this trivialization the moment map reads

$$\Phi'(z, c, \alpha) = -\alpha, \quad z \in \mathrm{T}^*U,\, c \in G,\, \alpha \in \mathfrak{k}^*,$$

and it is immediate that $\Phi'$ is a submersion. Let us consider the Marsden-Weinstein reduced space $(\mathrm{T}^*P)_{\mathcal{O}'} = \Phi^{-1}_{\mathrm{T}^*P\times\overline{\mathcal{O}}'}(0)/G$. In the local trivialization a point $(z, c, \alpha, \beta) \in \mathrm{T}^*P \times \overline{\mathcal{O}}'$ belongs to $\Phi^{-1}_{\mathrm{T}^*P\times\overline{\mathcal{O}}'}(0)$ iff $\beta \in \mathcal{O}'$ and $\beta = -\alpha$. Therefore,

$$\dim(\mathrm{T}^*P)_{\mathcal{O}'} = 2\dim M + \dim \mathcal{O}'.$$

Now let there be given a connection on $P$. According to [Wei77a] this allows us to consider $(\mathrm{T}^*P)_{\mathcal{O}}$ as being fibered over $\mathrm{T}^*X$. The choice of a connection is equivalent to the choice of a horizontal subspace of the tangent space at each point of $P$, see e.g. [BGV92]. For a fiber bundle $P$ over $X$ we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{Ver}\, P \longrightarrow \mathrm{T}P \longrightarrow \pi^*\mathrm{T}X \longrightarrow 0$$

such that we can identify $\pi^*\mathrm{T}X$ with the quotient of $\mathrm{T}P$ by its vertical subbundle $\mathrm{Ver}\, P$. There is no canonical choice of a splitting of the above exact sequence. A connection however gives such a splitting in the sense that the kernel of the connection one-form gives the horizontal subbundle $\mathrm{Hor}\, P$ such that

$$\mathrm{Hor}\, P \oplus \mathrm{Ver}\, P = \mathrm{T}P. \tag{B.4.2}$$

Then we also have an identification

$$\mathrm{T}^*_pP \simeq (\mathrm{Ver}_p\, P)^* \oplus (\mathrm{Hor}_p\, P)^* \simeq (\mathrm{Ver}_p\, P)^* \oplus \mathrm{T}^*_xX, \tag{B.4.3}$$

where $x = \pi(p)$. In particular, there is a projection

$$\kappa_p : \mathrm{T}^*_pP \longrightarrow \mathrm{T}^*_xX.$$

Now let $z \in \mathrm{T}_pP$ and $n \in \mathrm{T}^*_pP$ such that

$$z = v + w \quad \text{and} \quad n = l + m$$

with respect to the direct sum decompositions given in (B.4.2) and (B.4.3), and $\langle n, z\rangle = \langle l, v\rangle + \langle m, w\rangle$. Since $v = \xi_P$ for some $\xi \in \mathfrak{k}$ we have

$$\langle l, v\rangle = \langle l, \xi_P\rangle = \langle \Phi(p, n), \xi\rangle$$

according to the definition of the moment map on the cotangent bundle, see (B.4.1). Furthermore,

$$\langle m, w\rangle = \langle \kappa(n), \pi_*(z)\rangle.$$

Let $\eta \in \mathrm{T}_{(p,n)}\mathrm{T}^*P$ and $z$ its image in $\mathrm{T}_pP$. The fundamental one-form $\alpha_P$ on $\mathrm{T}^*P$ is given by

$$\alpha_P(\eta) = \langle n, z\rangle = \langle \Phi(p,n), \xi\rangle + \langle \kappa(n), \pi_*(z)\rangle,$$

where the last term equals $\kappa^*\alpha_X(\eta)$ such that we have

$$\alpha_P = \langle \Phi, \theta\rangle + \kappa^*\alpha_X,$$

where $\theta$ is the connection one-form on $P$.

Now let $f : Y \to X$ be a smooth map and consider the pull-back bundle

$$f^\# P = \{(y,p);\ f(y) = \pi(p)\},$$

with projection $\pi^\# : f^\# P \to Y$ onto the first factor. This implies the commuting diagram

$$\begin{tikzcd} f^\# P \arrow[r, "\tilde f"] \arrow[d, "\pi^\#"'] & P \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where $\tilde f$ is the projection onto the second factor of the pull-back bundle. Obviously, $f^\# P$ is a principal $G$-bundle over $Y$ and $f$ is a morphism of $G$ bundles.

If $\theta$ is a connection form on $P$ then $\tilde f^*\theta$ is a connection form on $f^\# P$. If we consider the special case $f : \mathrm{T}^*X \to X$ and denote $P^\# := f^\# P$ we have

$$\begin{tikzcd} P^\# \arrow[r] \arrow[d] & P \arrow[d] \\ \mathrm{T}^*X \arrow[r] & X \end{tikzcd}.$$

Now the map $\kappa : \mathrm{T}^*P \to \mathrm{T}^*X$ can be used to define

$$\sigma : \mathrm{T}^*P \to P^\#, \quad \sigma(p,n) = (\kappa(p,n), p)$$

Thus obtain

$$\begin{tikzcd} \mathrm{T}^*P \arrow[r, "\sigma"] \arrow[dr, "\kappa"] & P^\# \arrow[r] \arrow[d, "\pi^\#"] & P \arrow[d, "\pi"] \\ & \mathrm{T}^*X \arrow[r] & X \end{tikzcd},$$

where, of course, the maps $\sigma$ and $\kappa$ depend on the connection. Now let $Q$ be a Hamiltonian $G$-space with moment map $\Phi_Q$ and form the Hamiltonian $G$ space $\overline{\mathrm{T}^*P} \times Q$ whose symplectic form is $\mathrm{d}\alpha_P + \omega_Q$. Its moment map is given by $\overline{\Phi} = \Phi_{\overline{\mathrm{T}^*P}\times Q} = \Phi_Q - \Phi$. Consider the map $\sigma \times \mathrm{id} : \mathrm{T}^*P \times Q \to P^\# \times Q$ which for fixed $(p,q) \in P \times Q$ maps $\mathrm{T}^*_pP \times \{q\} \to \mathrm{T}^*_xX \times \{(p,q)\}$ and $n \in \mathrm{T}^*_pP$ to $\kappa_p(n)$. Now the horizontal component of $n$ is determined by its image under this map while the vertical component is determined by $\Phi(p,n) = \Phi_Q(q)$ when restricted to $\overline{\Phi}^{-1}(0)$. We therefore have a diffeomorphism, denoted by $\chi$, between $\overline{\Phi}^{-1}(0)$ and $P^\# \times Q$. Therefore

**Proposition B.4.1.** *Let $\chi$ denote the restriction of $\sigma \times \mathrm{id} : \mathrm{T}^*P \times Q \to P^\# \times Q$ to $\Phi^{-1}_{\overline{\mathrm{T}^*P}\times Q}(0)$. Then $\chi$ is a $G$-equivariant diffeomorphism. Furthermore, if we define the closed two-form $\nu$ on $P^\# \times Q$ by*

$$\nu = \mathrm{d}\left(\langle \Phi_Q, \theta^\# \rangle + \pi^{\#*}\alpha_X\right) + \omega_Q,$$

*then $\chi^*\nu$ is the restriction of the symplectic form of $\overline{\mathrm{T}^*P} \times Q$ to the coisotropic submanifold $\Phi^{-1}_{\overline{\mathrm{T}^*P}\times Q}(0)$.*

According the general reduction scheme, we know that the symplectic manifold associated with $\Phi^{-1}_{\overline{\mathrm{T}^*P}\times Q}(0)$ is just the quotient under $G$. But $(P^\# \times Q)/G$ is the associated bundle $Q(P^\#)$. Thus we have

**Proposition B.4.2.** *The leaves of the null foliation of the form $\nu$ on $P^\# \times Q$ are precisely the orbits of the $G$ action. In particular, there exists a unique symplectic form $\Omega$ on the associated bundle $Q(P^\#)$ such that*

$$\nu = \rho^*\Omega, \tag{B.4.4}$$

*where $\rho : P^\# \times Q \to Q(P^\#)$ is the natural projection. The diffeomorphism $\chi$ induced a symplectic diffeomorphism $\overline{\chi}$ of $\overline{\mathrm{T}^*P_Q}$ with $Q(P^\#)$.*

Now the choice of a connection on $P$ induces a unique covariant derivative on the associated bundle $Q(P^\#)$, see e.g. [BGV92] and thereby a curvature on $Q(P^\#)$. Let $Q$ be symplectic with symplectic form $\omega_Q$. This allows us to construct a two-form $\tilde{\omega}_Q$ on $Q(P^\#)$, see [GS84].

**Proposition B.4.3.** *Let $P \to X$ be a principal bundle with structure group $G$, and let $P^\# \to \mathrm{T}^*M$ be the pull-back of $P$ to the cotangent bundle $\mathrm{T}^*M$ of $M$. Let $\theta$ be a connection one form on $P$ and $\theta^\#$ the induced connection on $P^\#$. Let $Q$ be a Hamiltonian $G$ space and $Q(P^\#)$ the associated bundle. The moment map $\Phi : Q \to \mathfrak{k}^*$ induces a function $\tilde{\Phi} : Q(P^\#) \to \mathfrak{k}^*(P^\#)$. The symplectic form $\omega_Q$ on $Q$ together with the connection $\theta$ induces a two-form $\tilde{\omega}_Q$ on $Q(P^\#)$ and the curvature $\theta^\#$ gives rise to a $\mathfrak{k}(P^\#)$-valued two-form $F^\#$ on $Q(P^\#)$. Then*

$$\Omega = \mathrm{d}\pi^{\#*}\alpha_X + \langle \tilde{\Phi}, F^\# \rangle + \tilde{\omega}_Q \tag{B.4.5}$$

*is a symplectic form on $Q(P^\#)$, which coincides with the symplectic form in (B.4.4). In particular, there is a symplectic diffeomorphism of $\overline{\mathrm{T}^*P_Q}$ with $(Q(P^\#), \Omega)$.*

Since a connection on $P$ induces a projection $\mathrm{T}^*P_Q$ onto $\mathrm{T}^*X$, any Hamiltonian on $\mathrm{T}^*X$ pulls back to $\mathrm{T}^*P_Q$. Thus together with the symplectic form a connection induces a Hamiltonian vector field $\mathrm{X}_H$ on $\mathrm{T}^*P_Q$. This construction is called the *principle of minimal coupling* [GS84, Ste77, GS78, GS82]. For example, suppose that $G = \mathrm{U}(1)$ and $Q$ is taken to be a point in $\mathfrak{g}^*$ which in the case of an Abelian group is a coadjoint orbit. The term $\tilde{\omega}_Q$ in the above Proposition does not occur, and for $M = \mathbb{R}^3$ we get a modification of the canonical symplectic structure on $\mathrm{T}^*\mathbb{R}^3$ associated to a magnetic field. If we take $M = \mathbb{R}^4$ and $H(x,\xi) = \frac{1}{2}\|\xi\|^2$ the resulting equations of motion contain the Lorentz force. Thus the equations of the principle of minimal coupling generalize Lorentz forces to an arbitrary gauge field.

Let us examine what these equations look like in general: the symplectic form $\Omega$ in Proposition B.4.3 is the negative of the symplectic form on $\mathrm{T}^*P_Q$ and the Hamiltonian vector field $\mathrm{X}_H$ is determined by $\imath_{\mathrm{X}_H}\Omega = \mathrm{d}H$. Let us write

$$\mathrm{X}_H = \lambda(a\partial_x) + \lambda(b\partial_\xi) + w,$$

in terms of local coordinates on $\mathrm{T}^*M$, where $\lambda(v)$ is the horizontal vector corresponding to the vector $v \in \mathrm{T}(\mathrm{T}^*M)$, and $w$ is vertical. Then

$$\imath_w\Omega = \imath_w\tilde{\omega}_Q,$$

since the first two terms in (B.4.5) vanish when evaluated on a vertical vector. Since $\mathrm{d}H$ has no vertical component, we conclude that $w = 0$. The coefficient of $\mathrm{d}\xi$ in $\imath_{\mathrm{X}_H}\Omega$ is $-a$, so we see that

$$a = \partial_\xi H.$$

The $\mathrm{d}x$ term in $\imath_{\mathrm{X}_H}\Omega$ is

$$b\,\mathrm{d}x + \langle\tilde{\Phi}, \imath_{a\partial_x}F\rangle$$

which must equal $-\partial_x H\,\mathrm{d}x$. Now $\langle\tilde{\Phi}, \imath_{a\partial_x}F\rangle$ is a covector that we can consider as a point of $\mathrm{T}^*M$. Thus we obtain the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial\xi}, \qquad \frac{\mathrm{d}\xi}{\mathrm{d}t} + \langle\tilde{\Phi}, \imath_{\dot{x}}F\rangle = -\frac{\partial H}{\partial x}.$$

These equations, together with $w = 0$, are Hamilton's equations corresponding to a Hamiltonian function $H$ on $\mathrm{T}^*M$. Notice that $w = 0$ says that the solution curves are all horizontal, i.e. obtained by parallel transport from their projections onto $\mathrm{T}^*M$.

In the case $Q = \mathcal{O}$ a coadjoint orbit in $\mathfrak{k}^*$ these equations have an intrinsic interpretation: the connection $\theta$ induces a projection from $\mathrm{T}^*P$ to $\mathrm{T}^*M$ and the Hamiltonian $H$ pulls back to a function on $\mathrm{T}^*P$, which is $G$ invariant. The corresponding flow leaves $\Phi_P^{-1}(\eta)$ invariant and therefore determines a flow on the reduced space $\Phi^{-1}(\eta)/G_\eta$, which equals $\mathrm{T}^*P_{\mathcal{O}}$, where $\mathcal{O}$ is the orbit through $\eta$. Now two connections $\theta$ and $\theta_0$ differ by a $\mathfrak{g}$-valued horizontal one-form $\tilde{A}$ on $P$,

$$\theta - \theta_0 = \tilde{A}.$$

In particular $\tilde{A}$ is equivariant with respect to the action of $G$ on $P$. Let $A^{\#}$ be the pull-back of $\tilde{A}$ to $\mathrm{T}^*P$. Then $\langle \Phi, A^{\#} \rangle$ is an invariant horizontal one-form on $\mathrm{T}^*P$. If $\kappa$ and $\kappa_0$ are the projections $\mathrm{T}^*P \to \mathrm{T}^*M$ induced by $\theta$ and $\theta_0$, respectively, then

$$\kappa = \kappa_0 + \langle \Phi, A^{\#} \rangle.$$

Let us apply this to the case where $P$ is (locally) given as $P|_U = U \times G$ and where we take $\theta_0$ to be the flat connection corresponding to this product decomposition. Thus $\kappa_0 : \mathrm{T}^*P \simeq \mathrm{T}^*U \times \mathrm{T}^*G \to \mathrm{T}^*U$ is the projection onto the first factor. Then the form $\tilde{A}$ will be given at $(x, c) \in U \times G$ as

$$\tilde{A}_{(x,c)} = \mathrm{Ad}_{c^{-1}} \tilde{A}_{(x,e)}.$$

We denote by $A$ the $\mathfrak{g}$-valued one-form obtained as a representative under the trivializing section $s(x) = (x, e)$ according to

$$A = s^* \tilde{A} = s^* \theta.$$

Hence we can write

$$A_{(x,c)} = \mathrm{Ad}_{c^{-1}} A_x.$$

Finally, by the left-invariant identification $\mathrm{T}^*G \simeq G \times \mathfrak{g}^*$ the moment map $\Phi$ is given by $\Phi(c, \beta) = -\beta$ and we have

$$\kappa(x, \xi, c, \beta) = (x, \xi - \langle \beta, \mathrm{Ad}_{c^{-1}} A_x \rangle) = (x, \xi - \langle \mathrm{Ad}^*_c \beta, A_x \rangle),$$

where $x \in M$, $\xi \in \mathrm{T}^*_x M$, $c \in G$ and $\beta \in \mathfrak{g}^*$. So

$$\kappa^* H = H(x, \xi - \langle \mathrm{Ad}^*_c \beta, A_x \rangle),$$

and in canonical coordinates on $\mathrm{T}^*U$ Hamilton's equations read

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial \xi_i}, \quad \frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \frac{\partial H}{\partial x_i} + \frac{\partial H}{\partial \xi_i} \frac{\partial \langle \mathrm{Ad}^*_c \beta, A_x \rangle}{\partial x_i}.$$

In this setting they are also known as Wong's equations [Won70, Mon84].

# Appendix C

# Fundamentals of the structure and representation theory of Lie groups

In this Appendix we give a concise review of Weyl's theory of representations of compact Lie groups, see [Sam90, Ada69, Sim96, FH91, Ste97]. Let $G$ be a semi-simple compact Lie group and $\mathfrak{g}$ its Lie algebra. Since $G$ is compact, the adjoint representation of $G$ on $\mathfrak{g}$ can be made unitary; a possible choice of an inner product is the negative Killing form

$$(X, Y) := -\operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y),$$

where $X$, $Y \in \mathfrak{g}$ and $\operatorname{ad}_X(Z) = [X, Z]$ for $Z \in \mathfrak{g}$. This inner product allows us to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ in concrete situations.

The first part of Weyl's theory of representations of $G$ is to diagonalize as much as possible: We pick a maximal connected abelian subgroup $T$ of $G$, i.e. a *maximal torus*, whose Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ is called the *Cartan subalgebra*; its dimension is the *rank* of the Lie group $G$. Up to conjugation there exists only one maximal torus, i.e. all maximal tori are conjugate, see [BtD85].

Let there be given a representation $(\pi, V)$ of $G$ on a (finite-dimensional) complex space $V$. We can simultaneously diagonalize

$$\{\, \mathrm{d}\pi(X);\; X \in \mathfrak{t}\},$$

i.e. we seek $\lambda \in \mathfrak{h}^*$ and $v \in V$ such that

$$\mathrm{d}\pi(X)v = \lambda(X)v$$

for all $X \in \mathfrak{t}$. Then $\lambda$ is called *weight* and $v$ *weight vector*. The weights of the adjoint representation[1] are called *roots*. Let $\alpha$ be a root, i.e.

$$\operatorname{ad}_X X_\alpha = [X, X_\alpha] = \alpha(X)X_\alpha,$$

[1] More precisely, the weights of the complexification of Ad on $\mathfrak{g}^{\mathbb{C}}$.

for all $X \in \mathfrak{t}$. The eigenvector $X_\alpha$ is then called *root vector*. Thus we have

$$\mathrm{d}\pi(X)(\mathrm{d}\pi(X_\alpha)v) = \mathrm{d}\pi(X_\alpha)\,\mathrm{d}\pi(X)v + \mathrm{d}\pi([X, X_\alpha])v = (\lambda(X) + \alpha(X))\,\mathrm{d}\pi(X_\alpha)v,$$

implying that $\mathrm{d}\pi(X_\alpha)v$ is a weight vector[2] with weight $\lambda + \alpha$. Thus, the structure of the Lie algebra plays an important role in the theory of representations. In particular, it forces the weights to obey certain integrality conditions so that the weights have to lie in an *integral lattice* $\mathscr{I}$.

Another important role in this theory is played by the *Weyl group* $W(T)$, which by definition is the group of automorphisms of $T$. These arise from those inner automorphisms of $G$ leaving $T$ setwise invariant, i.e. the normalizer $N(T) = \{g \in G;\ \mathrm{Ad}_g T = T\}$ by taking the quotient

$$W(T) = N(T)/C(T),$$

where $C(T) = \{g \in G;\ \mathrm{Ad}_g h = h \text{ for all } h \in T\}$ denotes the centralizer of $T$. Now let $S \in W(T)$ be induced by $g \in G$, i.e.

$$S(h) = \mathrm{Ad}_g(h), \quad h \in T.$$

If then $\lambda$ is a weight for $\pi$, $v_l$ the corresponding weight vector and $X \in \mathfrak{t}$

$$\mathrm{d}\pi(X)\pi(g)v_l = \pi(g)\pi(S(X)^{-1})v_l = \lambda(S^{-1}(X))\pi(g)v_l = (S\lambda)(X)\pi(g)v_l,$$

with the obvious dual action of $W(T)$ on $\mathfrak{t}^*$. Thus, $\pi(g)v_l$ is a weight vector and $S\lambda$ is a weight, i.e. the weights of $\pi$ are left invariant by the action of $W(T)$. We can therefore consider the action of $W(T)$ on $\mathfrak{h}^*$, whose geometry is characterized by the following properties:

(i) $W(T)$ is generated by elements of order two which act on $\mathfrak{t}^*$ as reflections (with respect to the natural inner product on $\mathfrak{t}^*$ which comes from that on $\mathfrak{g}^*$) at hyperplanes.

(ii) Any $\lambda \in \mathfrak{t}^*$ left invariant by some non-trivial $S \in W(T)$ is left invariant by some element of order two acting as a reflection. Thus the set of invariant elements is a family of hyperplanes.

(iii) If these hyperplanes are removed from $\mathfrak{t}^*$, the remaining points are a union of open polyhedral cones whose closure are called *Weyl chambers*.

(iv) The Weyl chambers are images of each other under the action of the Weyl group and each non-trivial element of the Weyl group leaves no chamber setwise invariant. Thus the number of Weyl chambers is precisely the order of $W(T)$.

One chooses one chamber once and for all and calls it the *fundamental (Weyl) chamber*. Those elements of the weight lattice $\mathscr{I}$ which lie in the fundamental chamber are called the *dominant weights*, denoted by $\mathscr{I}_d$, and those in the interior of the fundamental chamber are the *strongly dominant weights*. The basic theorem is then

[2] if non-zero

**Theorem C.i.** *There is a one-to-one correspondence between irreducible (unitary) representations of* $\mathfrak{g}$ *and elements* $\lambda \in \mathscr{I}_d$*. The representation* $\pi_\lambda$ *corresponding to* $\lambda$ *is uniquely determined by the fact that* $\lambda$ *is a weight of* $\pi_\lambda$*, and among all weights it is the maximal in the* $\mathscr{I}_d$*-order. Moreover,* $\{v;\ v \text{ is a weight vector for } \lambda\}$ *is one-dimensional.*

$\lambda$ is called the *maximal weight* of $\pi_\lambda$. Recall, that $\mathscr{I}_d$ has an ordering which is defined by $\lambda > \mu$ if and only if $\lambda - \mu \in \mathscr{I}_d$.

# C.1 Coherent projections and maximal weight vectors

In this Section we want to collect some facts and analytical properties of *coherent projections*, i.e.

**Definition C.1.1.** Let $\mathscr{H}$ be a Hilbert space and $(X, \Sigma, \mu)$ a measure space. A family of coherent projections is a weakly measurable map $x \mapsto \mathrm{P}(x)$ from $X$ to the orthogonal projections on $\mathscr{H}$ such that

(i) $\dim \operatorname{ran} \mathrm{P}(x) = 1$ for all $x \in X$.

(ii) $\int_X \mathrm{P}(x)\, \mathrm{d}\mu(x) = 1$, in the sense that

$$\int_X \langle \varphi, \mathrm{P}(x)\psi \rangle\, \mathrm{d}\mu(x) = \langle \varphi, \psi \rangle$$

for all $\varphi,\ \psi \in \mathscr{H}$.

It is an immediate consequence of (ii) in the above Definition that

$$d = \dim \mathscr{H} = \int_X \mathrm{d}\mu(x)$$

if $\mathscr{H}$ is finite-dimensional. Furthermore, coherent projections always arise from coherent vectors:

**Proposition C.1.2.** *If* $\mathrm{P}(x)$ *is a family of coherent projections then there exists a measurable family* $\psi(x)$ *of unit vectors so that* $\mathrm{P}(x) = \langle \psi(x), \cdot \rangle \psi(x)$*. Moreover, the* $\psi(x)$ *are total.*

This immediately shows that in our case the coherent projections arise as the projections on the subspaces spanned by coherent states, see Section 3.6. Therefore we can define lower symbols

$$\sigma_{\mathrm{l}}[A](x) = \operatorname{tr}(A\mathrm{P}(x))$$

for any bounded operator $A$ on $\mathscr{H}$ and upper symbols $\sigma_{\mathrm{u}}[A]$ in an analogous way through

$$A = \int_X \mathrm{P}(x)\sigma_{\mathrm{u}}[A](x)\,\mathrm{d}\mu(x).$$

If we now study the properties of coherent projections we can obtain analytical properties for the upper and lower symbols. Let us first remark that

$$\|\sigma_{\mathrm{l}}[A]\|_\infty \leq \|A\|$$

and

$$\|A\| \leq \|\sigma_{\mathrm{u}}[A]\|_\infty,$$

which explains the names *upper* and *lower* symbol, since operators are always bounded from below by the norms of lower symbols and from above by the norms of upper symbols.

**Definition C.1.3.** We say that a family of coherent projections is *complete* if and only if the map $f \mapsto \int_X \mathrm{P}(x)f(x)\,\mathrm{d}\mu(x)$ is sequentially strongly dense[3] in the bounded operators on $\mathscr{H}$.

**Theorem C.1.4.** *Let $\mathrm{P}(x)$ be a family of coherent projections on a separable Hilbert space $\mathscr{H}$ and let $A \in \mathscr{I}_1$ be of trace class. Then $\sigma_{\mathrm{l}}[A] \in L^1(X, \mathrm{d}\mu)$ and*

$$\mathrm{tr}\left(A\int_X f(x)\mathrm{P}(x)\,\mathrm{d}\mu(x)\right) = \int_X f(x)\sigma_{\mathrm{l}}[A](x)\,\mathrm{d}\mu(x)$$

*holds for any $f \in L^\infty(X, \mathrm{d}\mu)$.*

*Furthermore, let $A \in \mathscr{I}_p$, which means that the singular values of $A$ are $p$-summable[4]. Then $\sigma_{\mathrm{l}}[A] \in L^p(X, \mathrm{d}\mu)$ and*

$$\int_X |\sigma_{\mathrm{l}}[A](x)|^p\,\mathrm{d}\mu(x) \leq \mathrm{tr}(|A|^p).$$

*If $f \in L^p(X, \mathrm{d}\mu)$ then for all $\varphi, \psi \in \mathscr{H}$*

$$\langle\varphi, \mathrm{op}[f]\psi\rangle = \int_X \langle\varphi, \mathrm{P}(x)\psi\rangle f(x)\,\mathrm{d}\mu(x)$$

*converges, and the corresponding operator $\mathrm{op}[f]$ lies in $\mathscr{I}_p$ and obeys*

$$\mathrm{tr}(|\,\mathrm{op}[f]|^p) \leq \int_X |f|^p(x)\,\mathrm{d}\mu(x).$$

[3] This means that it is dense in the strong operator topology, i.e. the weakest topology on the Banach space of operators such that the maps $E_\psi : \mathcal{L}(\mathscr{H}) \to \mathscr{H}$ defined by $E_\psi(T) = Tx$ are continuous for all $\psi \in \mathscr{H}$. In this topology a net $\{T_\alpha\}$ of operators converges to an operator $T$ if and only if $\|T_\alpha\psi - T\psi\| \to 0$ for all $\psi \in \mathscr{H}$.

[4] see [GVF01, Sim79, GK69]

It is important to know that a family of coherent projections is complete if the kernel of the lower symbol map is trivial in the trace class operators.

**Theorem C.1.5.** *A family* $\mathrm{P}(x)$ *of coherent projections is complete if and only if* $\ker\sigma_1 \cap \mathscr{I}_p = \{0\}$.

Let us add

**Remark C.1.6.** Note the following properties:

1. The notion of completeness is equivalent to norm density in the compact operators.
2. If $\dim(\mathscr{H}) < \infty$ all the topologies are equivalent and all subspaces are closed so that $\operatorname{ran}\operatorname{op}[\cdot] = \mathcal{L}(\mathscr{H})$ if and only if $\ker\sigma_1[\cdot] = \{0\}$.
3. In general $\ker\operatorname{op}[\cdot]$ is very large even if $\ker\sigma_1[\cdot]$ is trivial, i.e. upper symbols are highly non-unique.

We now turn to a special class of coherent projections: we consider $\mathrm{P}(g)$ with $g$ from a compact simple Lie group $G$. We want to calculate the upper symbol for a representation operator $\mathrm{d}\pi_\lambda(X)$, where $X \in \mathfrak{g}$ and $(\pi_\lambda, V_\lambda)$ a unitary irreducible representation with maximal weight $\lambda$. Let $\mathrm{d}\mu$ denote the Haar measure on $G$ which is normalized according to $\int_G \mathrm{d}\mu = \dim V_\lambda$ and let $\mathrm{P}(e)$ be the projection on the maximal weight weight vector for $\pi_\lambda$. Then

$$\mathrm{P}(g) = \pi(g)\mathrm{P}(e)\pi(g)^{-1}.$$

By Schur's lemma it follows that

$$\int_G \mathrm{P}(g)\,\mathrm{d}\mu(g) = \mathrm{id},$$

and we claim that

$$\mathrm{d}\pi_\lambda(X) = c\int \lambda(\mathrm{Ad}_{g^{-1}X})\mathrm{P}(g)\,\mathrm{d}\mu(g),$$

where $c = 1 + 2\frac{\langle\lambda,\delta\rangle}{\langle\lambda,\lambda\rangle}$ with $\delta$ the magic weight, i.e. the sum of the fundamental weights. We use that the lower symbol map based on this family of coherent projections has trivial kernel, see Proposition 3.7.2. According to the above remark this means that any $A \in \mathcal{L}(\mathscr{H})$ is of the form $\operatorname{op}[f]$ for some $f$. Now we have

**Theorem C.1.7.** *Let* $(\pi', V')$ *be a second irreducible representation of* $G$ *of dimension* $m$. *Let* $A_1, \ldots, A_m$ *be operators obeying*

$$\pi(g)A_i\pi(g)^{-1} = \sum_j V_{ji}(g)A_j.$$

*Then there exist functions* $f_1, \ldots, f_n$ *on* $G$ *such that*

$$A_i = \int_G f_i(g)\mathrm{P}(g)\,\mathrm{d}\mu(g)$$

*and*

$$f_i(h^{-1}g) = \sum_j V_{ji}(h) f_j(g).$$

Let us now consider the adjoint representation Ad and let $B$ be any map from $\mathfrak{g}$ to $\operatorname{End} V_\lambda$ that is linear and fulfills

$$\pi(g)B(X)\pi(g)^{-1} = B(\operatorname{Ad}_g X).$$

By the above Theorem and the Peter-Weyl theorem[5] we conclude that

$$B(X) = C_\eta(X) = \int_G \eta(\operatorname{Ad}_{g^{-1}} X)\mathrm{P}(g)\,\mathrm{d}\mu(x). \tag{C.1.1}$$

We have

**Theorem C.1.8.** $\mathrm{d}\pi_\lambda(X)$ *is of the form (C.1.1) for some* $\eta \in \mathfrak{g}^*$ *with*

$$\langle \eta, \lambda \rangle = \langle \lambda, \lambda \rangle + 2\langle \lambda, \delta \rangle.$$

It is important to note the following invariance property:

**Proposition C.1.9.** *Let* $C_\eta$ *be given by (C.1.1) and let* $g \in G$ *with* $\operatorname{Ad}^*_g \lambda = \lambda$. *Then*

$$C_{\operatorname{Ad}^*_g \eta} = C_\eta.$$

In particular, this result implies that if $\tau$ denotes the projection from $\mathfrak{g}^*$ to $\mathfrak{t}^*$, those elements of $\mathfrak{g}^*$ which are zero on $\mathfrak{t}^\perp$, then

$$C_\eta = C_{\tau\eta}.$$

These properties allow us to pull down the coherent projections from the group to the coadjoint orbit $G/G_\lambda$: let $\mathrm{P}(\lambda)$ denote the projector on the maximal weight space, then $\pi(g)\mathrm{P}(\lambda)\pi(g)^{-1} = \mathrm{P}(\lambda)$ if and only if $\operatorname{Ad}^*_g \lambda = \lambda$. Now let $\eta \in \mathcal{O}_\lambda \simeq G/G_\lambda$ and define

$$\mathrm{P}(\eta) = \pi(g)\mathrm{P}(\lambda)\pi(g)$$

for any $g \in G$ such that $\operatorname{Ad}^*_g \lambda = \eta$, so the above definition for $\mathrm{P}(\eta)$ is independent of the particular $g$ chosen. Moreover, we have

$$\int_{\mathcal{O}_\lambda} \mathrm{P}(\eta)\,\mathrm{d}\eta = \dim V_\lambda \int_G \pi(g)\mathrm{P}(\lambda)\pi(g)^{-1}\,\mathrm{d}\mu(g) = 1.$$

[5] which implies that the only functions on $G$ that transform under the left-regular representation as under the adjoint representation are of the form $\eta(\operatorname{Ad}^{-1}_{(\cdot)} X)$ for some $\eta \in \mathfrak{g}^*$

# Appendix D

# Relations for Poisson brackets of matrix valued functions

In this appendix we collect some relations for Poisson brackets of matrix valued functions on the phase space $\mathrm{T}^*\mathbb{R}^d$ that are needed in section 2.3.2. These relations are already stated in [EW96, GMMP97, Spo00] and can be verified by straightforward calculations.

Our convention for the Poisson bracket of smooth matrix valued functions $A, B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ is

$$\{A, B\} := \partial_\xi A \, \partial_x B - \partial_x A \, \partial_\xi B.$$

The first general relation then reads

$$A\{B, C\} - \{A, B\}C = \{AB, C\} - \{A, BC\}. \tag{D.1}$$

Furthermore, for projection matrices $P = PP$ one finds

$$P\{\lambda, P\}P = 0, \tag{D.2}$$

where $\lambda$ is any smooth scalar function on $\mathrm{T}^*\mathbb{R}^d$.

For $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ commuting with $P$ one then derives

$$P\{\lambda, B\}P = \{\lambda, PBP\} - [PBP, [P, \{\lambda, P\}]]. \tag{D.3}$$

In particular, using (D.1) for projection matrices one obtains

$$P\{P, B\} - \{P, P\}B = \{P, B\} - \{P, PB\}$$

and

$$B\{P, P\} - \{B, P\}P = \{BP, P\} - \{B, P\}.$$

Using these relations together with the condition $[B, P] = 0$ one obtains

$$P\big(\{B, P\} - \{P, B\}\big)P = [B, P\{P, P\}P].$$

Furthermore, for different projection matrices $P_\mu$ and $P_\nu$, with $P_\mu P_\nu = 0$ for $\nu \neq \mu$, the general relation (D.1) implies

$$P_\mu\{P_\nu, P_\nu\} = -\{P_\mu, P_\nu\}(1 - P_\nu)$$

and

$$\{P_\nu, P_\nu\}P_\mu = -(1 - P_\nu)\{P_\nu, P_\mu\}.$$

In the case $[P_\nu, B] = 0 = [P_\mu, B]$ one finds

$$\begin{aligned} P_\nu\{P_\mu, B\} - \{P_\nu, P_\mu\}B &= -\{P_\nu, P_\mu B\}, \\ B\{P_\mu, P_\nu\} - \{B, P_\mu\}P_\nu &= \{BP_\mu, P_\nu\}. \end{aligned}$$

These equations imply

$$P_\nu\big(\{B, P_\mu\} - \{P_\mu, B\}\big)P_\nu = -[B, P_\nu\{P_\mu, P_\mu\}P_\nu].$$

One can now apply the above relations to expressions of the type arising in section 2.3.2, i.e.,

$$\begin{aligned} &P_\mu\left(\frac{\partial}{\partial t}B(t) + \frac{1}{2}\Big(\{B(t), \lambda_\nu P_\nu\} - \{\lambda_\nu P_\nu, B(t)\}\Big) + \mathrm{i}[B(t), H_1]\right)P_\mu \\ &\quad= \frac{\partial}{\partial t}P_\mu B(t)P_\mu - \delta_{\nu\mu}\{\lambda_\nu, P_\mu B(t)P_\mu\} \\ &\qquad+ \left[\frac{\lambda_\nu}{2}(-1)^{\delta_{\nu\mu}}P_\mu\{P_\nu, P_\nu\}P_\mu - \delta_{\nu\mu}[P_\nu, \{\lambda_\nu, P_\nu\}] - \mathrm{i}\,P_\mu H_1 P_\mu, P_\mu B(t)P_\mu\right]. \end{aligned}$$

Therefore, the definition

$$\tilde{H}_1 := \mathrm{i}(-1)^{\delta_{\nu\mu}}\frac{\lambda_\nu}{2}P_\mu\{P_\nu, P_\nu\}P_\mu - \mathrm{i}\,\delta_{\nu\mu}[P_\nu, \{\lambda_\nu, P_\nu\}] + P_\mu H_1 P_\mu \tag{D.4}$$

allows us to conclude that

$$\frac{\partial}{\partial t}P_\mu BP_\mu - \delta_{\nu\mu}\{\lambda_\nu, P_\mu BP_\mu\} - \mathrm{i}[\tilde{H}_1, P_\mu BP_\mu] = 0. \tag{D.5}$$

# Appendix E

# Singular cohomology and Čech cohomology

## E.1 Singular cohomology

In the context of line bundles we will frequently encounter local constructions. The typical problem then is to patch the local constructions together in order to from a globally defined line bundle. This problem is concerned with global topological properties of the underlying manifold.

We start by considering the map

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0),$$

which embeds $\mathbb{R}^n$ naturally in $\mathbb{R}^{n+1}$. One can thus view each $\mathbb{R}^n$ as subspace of $\mathbb{R}^{n+1}$ and consider the union

$$\mathbb{R}^\infty = \bigcup_{n \geq 0} \mathbb{R}^n.$$

As in [BT82, Alv85] denote by $p_i$ the $i$-th standard basis vector in $\mathbb{R}^\infty$ and by $p_0$ the origin. The standard $q$-simplex is then given by

$$\Delta_q = \left\{ \sum_{j=0}^{q} t_j p_j ;\ \sum_{j=0}^{q} t_j = 1 \right\}.$$

Now if $X$ is a topological space, a *singular $q$-simplex* in $X$ is a continuous map $s : \Delta_q \to X$ and a *singular $q$-chain* in $X$ is a finite linear combination of singular $q$-simplices with integer coefficients. These $q$-chains form an abelian group $S_q(X)$. We define the $i$-th face map of the standard simplex to be the function

$$\partial_q^i : \Delta_{q-1} \to \Delta_q, \quad \partial_q^i \left( \sum_{j=0}^{q-1} t_j p_j \right) = \sum_{j=0}^{i-1} t_j p_j + \sum_{j=i+1}^{q} t_j p_j.$$

Then the graded group of chains

$$S_*(X) = \bigoplus_{q\geq 0} S_q(X)$$

can be made into a differential complex (see e.g. [Bre93] for a definition) with boundary operator

$$\partial : S_q(X) \to S_{q-1}(X), \quad \partial s = \sum_{i=0}^{q} (-1)^i s \circ \partial_q^i,$$

for which $\partial^2 = 0$. The homology of this complex is the singular homology with integer coefficients $\mathrm{H}_*(X, \mathbb{Z})$. By taking linear combinations of simplices with coefficients in an Abelian group $G$, we obtain the singular homology with coefficients in $G$, $\mathrm{H}_*(X, G)$.

Now a singular *$q$-cochain* on a topological space $X$ is a linear functional on the $\mathbb{Z}$-module $S_q(X)$ of singular $q$-chains. Thus the group of singular $q$-cochains is $S^q(X) = \mathrm{Hom}(S_q(X), \mathbb{Z})$, where the coboundary operator is defined by $(\mathrm{d}\omega)(c) = \omega(\partial c)$, under which the graded group $S^*(X) = \bigoplus S^q(X)$ becomes a differential complex. The cohomology of this complex is the singular cohomology of $X$ with integer coefficients. Again, by replacing $\mathbb{Z}$ with an general abelian group $G$ we obtain the singular cohomology $\mathrm{H}^*(X, G)$.

The relation between homotopy and singular homology for the first groups is given by

**Theorem E.1.1.** *Let $X$ be a path-connected space. Then $\mathrm{H}_1(X)$ is the Abelianization of $\pi_1(X)$, i.e.*

$$\mathrm{H}_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

In higher dimension, we have

**Theorem E.1.2 (Hurewicz Isomorphism Theorem).** *Let $X$ be a simply connected path-connected CW-complex. Then the first non-trivial homology and homotopy occur in the same dimension and are equal.*

For the definition of $CW$-complexes see [BT82, Bre93, SZ94].

## E.2 Čech cohomology

We use the brief presentation given in [SW76]. Let $M$ be a smooth manifold and let $U = \{U_i; i \in I\}$ be a fixed open contractible cover[1] of $M$.

A $k$-simplex in the sense of Čech is any $k+1$-tuple $(i_0, \ldots, i_k) \in I^{k+1}$ such that $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$. Let $G$ be an abelian group, then a $k$-cochain is any totally skew map

$$g : (i_0, i_1, \ldots, i_k) \mapsto g(i_0, \ldots, i_k) \in G,$$

[1] i.e. each of the intersections of the elements is either empty or contractible

from the set of $k$-simplices into $G$. The set of all $k$-cochains is denoted by $\mathscr{C}^k(U,G)$. We have a sequence of group homomorphisms

$$\delta_k : \mathscr{C}^k(U,G) \to \mathscr{C}^{k+1}(U,G)$$

defined by

$$\delta g(i_0,\ldots,i_{k+1}) = \sum_{j=0}^{k+1}(-1)^i(i_0,\ldots,\not{i}_j,\ldots,i_{k+1}),$$

with $\delta^2=0$. A cochain $g\in\mathscr{C}^k(U,G)$ such that $\delta g=0$ is called $k$-cocycle. If, in addition, $g=\delta h$ for some $h\in\mathscr{C}^{k-1}(U,G)$, $g$ is called a $k$-coboundary. The set of $k$-cocycles is denoted by $\mathrm{Z}^k(U,G)$. The $k$-coboundaries $\mathrm{B}^k(U,G)$ form a subgroup of $\mathrm{Z}^k(U,G)$ and the quotient $\check{\mathrm{H}}(U,G) := \mathrm{Z}^k(U,G)/\mathrm{B}^k(U,G)$ is called the $k$-th cohomology group. If $V$ is a refinement of $U$ one has a homomorphism $\check{\mathrm{H}}^k(U,G)\to\check{\mathrm{H}}^k(V,G)$, which enables us to take the inductive limit over all coverings. This limit is the Čech-cohomology group $\check{\mathrm{H}}(X,G)$, and for any particular covering $U$ we have a natural homomorphism $\check{\mathrm{H}}^k(U,G)\to\check{\mathrm{H}}(X,G)$, which becomes an isomorphism if $U$ is contractible. However, contractible coverings exist and every covering has a contractible refinement.

A generalization of the above procedure is given by allowing $g(i_0,\ldots,i_k)$ to be a smooth function on $U_{i_0}\cap\cdots\cap U_{i_k}$, where then all the operations have to be understood pointwise. The resulting cohomology groups are denoted by $\check{\mathrm{H}}(U,\underline{G})$.

**Example E.2.1.** We choose a triangulation of $X$ with vertices $\{x_i;\ i\in I\}$ and take $U$ to be the set of star neighbourhoods of the vertices, that is $U=\{U_i\}_{i\in I}$ where

$$U_i = \{x\in X;\ x \text{ lies in the interior of a simplex in the triangulation with vertices } x_i\}.$$

This then implies the following intersection relations:

$$x_i,x_j,\ldots,x_k \text{ are the vertices of a simplex in the triangulation } \iff U_i\cap U_j\cap\cdots\cap U_k\neq\emptyset.$$

Thus, the simplices making up the triangulation are in one-to-one correspondence with the Čech simplices of the covering $U$.

Now let $\alpha$ be a complex $k$-form. This defines a $k$-cochain relative to $U$ according to

$$\alpha(i_0,i_1,\ldots,i_k) = \int_{x_{i_0}x_{i_1}\cdots x_{i_k}}\alpha,$$

where $x_{i_0}x_{i_1}\cdots x_{i_k}$ is the simplex in the triangulation with vertices $x_{i_0}x_{i_1}\cdots x_{i_k}$. By Stoke's theorem we obtain

$$\delta\psi(i_0,\ldots,i_{k+1}) = (\,\mathrm{d}\psi)(i_0,\ldots,i_{k+1}).$$

This association of $k$-forms with $k$-cochains sets up an isomorphism between the $k$-th (complex) de Rham cohomology group and $\check{\mathrm{H}}^k(U,\mathbb{C})$.

This isomorphism exists for more general contractible covers:

**Theorem E.2.2** ([Wei52]). *There is an isomorphism* $w : \mathrm{H}^k_{\mathrm{dR}}(X, \mathbb{R}) \xrightarrow{\sim} \check{\mathrm{H}}^k(X, \mathbb{R})$ *between the de Rham cohomology of* $X$ *and the Čech cohomology of* $X$ *with real coefficients.*

Now suppose that $\pi : L \to X$ is a line bundle and that $\{U_i, s_i\}$ is a local system of sections. We may assume that $U = \{U_i\}$ is contractible[2]. Then the transition functions

$$c_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^\times$$

on non-empty intersections of the covering $U_i \cap U_j \neq \emptyset$ define a 1-cochain $c \in \mathscr{C}^1(U, \underline{\mathbb{C}}^\times)$. Furthermore, the relation

$$c_{ij}c_{jk}c_{ki} = 1 \quad \text{on } U_i \cap U_j \cap U_k \tag{E.2.1}$$

shows that $c$ is a 1-cocycle, thus $L$ determines an equivalence class $[c] \in \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times)$. In addition, if $L_1$ and $L_2$ are two equivalent line bundles over $X$, see e.g. [Ste51, Hus75] for a definition, the corresponding cohomology classes are equal: an equivalence of line bundles $\tau : L_1 \to L_2$ is given by the following commuting diagram,

$$\begin{tikzcd} L_1 \arrow[rr, "\tau"] \arrow[dr, "\pi_1"'] & & L_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

By choosing a sufficiently fine cover we have local systems $\{U_i, s_i^{(k)}\}_{i \in I, k=1,2}$ for $L_1$ and $L_2$ and can define a set of functions $g_i : U_i \to \mathbb{C}^\times$ according to

$$s_i^{(1)} = g_i s_i^{(2)}.$$

From these we obtain an induced relation for the transition functions

$$c_{ij}^{(1)} = g_i c_{ij}^{(2)} g_j^{-1},$$

such that the $g_i$ define a 0-cochain $g \in \mathscr{C}^0(U, \underline{\mathbb{C}}^\times)$. Hence the last equation can be rewritten as

$$c^{(1)} = \delta g c^{(2)}.$$

This shows explicitly, that $L_1$ and $L_2$ define the same equivalence class in $\check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times)$. Conversely, given $[c] \in \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times)$ we can construct a line bundle $L$ with transition functions in $[c]$: choose a map

$$c_{ij} : U_i \cap U_j \to \mathbb{C}^\times$$

on the Čech simplices such that the cocycle relation (E.2.1) is fulfilled and take $L$ to be the disjoint union

$$\bigcup_i U_i \times \mathbb{C},$$

[2] We can pass to a contractible refinement if necessary

factored by the equivalence relation

$$(x_1, z_1) \sim (x_2, z_2) \quad \Longleftrightarrow \quad x_1 = x_2 \text{ in } X \text{ and } z_1 = c_{i_1 i_2}(x) z_2. \tag{E.2.2}$$

Then, with the obvious projection, $L$ is a line bundle over $X$. We can construct a local system by defining

$$s_i : U_i \longrightarrow L, \ x \longmapsto [(x, 1)],$$

where $[(x, 1)]$ is the equivalence class of $(x, 1) \in U_i \times \mathbb{C}$ under the equivalence relation (E.2.2). The transition functions of this line bundle are then just the $c_{ij}$, and clearly a different choice of the representative $c$ for $[c]$ will lead to an equivalent line bundle. Thus, for a given contractible cover the set of equivalence classes of line bundles over $X$ for which it is possible to construct local systems of the form $\{U_i, s_i\}$ is in one to one correspondence with the set of cohomology classes $\check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times)$ [3].

We can give a more tractable description of the above correspondence if we consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0,$$

where $\mathbb{Z} \to \mathbb{C}$ is the inclusion and $\mathbb{C} \to \mathbb{C}^\times$ is given by $z \mapsto \mathrm{e}^{2\pi \mathrm{i} z}$. This sequence induces an exact sequence in cohomology,

$$\cdots \to \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}) \to \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times) \xrightarrow{\epsilon} \check{\mathrm{H}}^2(U, \underline{\mathbb{Z}}) \to \check{\mathrm{H}}^2(U, \underline{\mathbb{C}}) \to \cdots ,$$

where $\epsilon : \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times) \xrightarrow{\sim} \check{\mathrm{H}}^2(U, \underline{\mathbb{Z}})$ is the connecting isomorphism, see [GVF01]. Since $\mathbb{C}$ is contractible the cohomology classes $\check{\mathrm{H}}^1(U, \underline{\mathbb{C}})$ and $\check{\mathrm{H}}^2(U, \underline{\mathbb{C}})$ are trivial, and as a consequence we have an isomorphism

$$\check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times) \simeq \check{\mathrm{H}}^2(U, \underline{\mathbb{Z}}),$$

where $\check{\mathrm{H}}^2(U, \underline{\mathbb{Z}})$ is the same as $\check{\mathrm{H}}^2(U, \mathbb{Z})$ since smooth integer valued functions are necessarily trivial. In the case of a line bundle we can explicitly construct the map $\epsilon : \check{\mathrm{H}}^1(U, \underline{\mathbb{C}}^\times) \to \check{\mathrm{H}}^2(U, \underline{\mathbb{Z}})$ by defining

$$f_{ij} : U_i \cap U_j \to \mathbb{C}, \quad f_{ij} = \frac{1}{2\pi \mathrm{i}} \log c_{ij}.$$

Because of (E.2.1) we have $\exp(2\pi \mathrm{i}\, a_{ijk}) = 1$ for

$$a_{ijk} = f_{ij} - f_{ik} + f_{jk}.$$

But then $a_{ijk}$ has to be $\mathbb{Z}$-valued and therefore constant on $U_{ijk}$, so defining an element of $\mathscr{C}^2(U, \mathbb{Z})$. It clearly is a two-cocycle and thus defines an element $[a] \in \check{\mathrm{H}}^2(U, \mathbb{Z})$ which clearly is independent of the choice of logarithms and the of representative within an equivalence class of line bundles. We have

**Proposition E.2.3.** *The map*

$$\kappa : \mathscr{L} \longrightarrow \check{\mathrm{H}}^2(X, \mathbb{Z})$$

*from the equivalence classes of line bundles to* $\check{\mathrm{H}}^2(X, \mathbb{Z})$ *is bijective.*

[3] This correspondence becomes a group isomorphism when the tensor product is used to define the group structure on the equivalence classes of line bundles.

For a proof see [Kos70].

**Remark E.2.4.** A line bundle $L \to X$ is called trivial if it is equivalent to the product bundle $X \times \mathbb{C}$, i.e. if and only if it is possible to find a nowhere vanishing section, which is equivalent to the triviality of $\check{\mathrm{H}}^2(X, \mathbb{Z})$. One can regard $\kappa[L]$ as the obstruction to finding a nowhere vanishing section.

## E.3 Existence of a metalinear bundle

Suppose we are given a principal $\mathrm{GL}(n, \mathbb{C})$ bundle $\pi : B \to X$ over a manifold $X$. We want to investigate the question, when it is possible to find a principal $\mathrm{ML}(n, \mathbb{C})$ bundle $\tilde{\pi} : \tilde{B} \to X$ and a double covering $\tau : \tilde{B} \to B$ such that

(i) $\tau$ commutes with the the projection

$$\begin{tikzcd} \tilde{B} \arrow[rr, "\tau"] \arrow[dr, "\tilde{\pi}"'] & & B \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

(ii) $\tau$ commutes with right translations

$$\begin{tikzcd} \tilde{B} \times \mathrm{ML}(n, \mathbb{C}) \arrow[r] \arrow[d, "\tau\times\rho"'] & \tilde{B} \arrow[d, "\tau"] \\ B \times \mathrm{GL}(n, \mathbb{C}) \arrow[r] & B \end{tikzcd},$$

where the horizontal arrows denote group actions and $\rho : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{ML}(n, \mathbb{C})$ is the covering map.

We rephrase this questions in terms of transition functions

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$$

of the bundle $B \to X$, and ask if it is possible to find a set of maps

$$z_{ij} : U_i \cap U_j \to \mathbb{C}^{\times}$$

such that

$$\det g_{ij} = (z_{ij})^2 \quad \text{on } U_i \cap U_j$$

and

$$z_{ij} z_{jk} = z_{ik} \quad \text{on } U_i \cap U_j \cap U_k.$$

If this is possible the maps

$$\tilde{g}_{ij} : U_i \cap U_j \to \mathrm{ML}(n, \mathbb{C}), \quad x \mapsto \begin{pmatrix} g_{ij}(x) & 0 \\ 0 & z_{ij}(x) \end{pmatrix}$$

satisfy the cocycle relation and can be used to construct $\tilde{B}$. We start by taking $f_{ij} : U_i \cap U_j \to \mathbb{C}$ to be one of the square roots of $\det g_{ij}$. Since $U_i \cap U_j$ is assumed to be contractible, this is well defined and we obtain

$$(f_{ij} f_{jk} f_{ki})^2 = 1.$$

Thus, if $a_{ijk} = f_{ij} f_{jk} f_{ki}$, then

$$a : (i, j, k) \mapsto a_{ijk} \in \mathbb{Z}_2$$

is a cocycle and defines an equivalence class $[a] \in \check{\mathrm{H}}^2(X, \mathbb{Z}_2)$ which is independent of the choice of the $f_{ij}$. By construction $a$ is a cocycle in $\mathscr{C}^2(U, \underline{\mathbb{C}}^\times)$, but it will in general not be a coboundary in $\mathscr{C}^2(U, \mathbb{Z}_2)$. If $a$ is a coboundary in $\mathscr{C}^2(U, \mathbb{Z}_2)$ then

$$a_{ijk} = c_{ij} c_{jk} c_{ki}.$$

for some one-cochain $c \in \mathscr{C}^1(U, \mathbb{Z}_2)$, and the problem is solved by putting

$$z_{ij} = \frac{f_{ij}}{c_{ij}}.$$

Conversely, if $z_{ij}$'s can be found then, on making the choice $f_{ij} = z_{ij}$, it is clear that $[a]$ vanishes. Thus the construction of $\tilde{B}$ is possible if and only if $[a]$ vanishes in $\check{\mathrm{H}}^2(X, \mathbb{Z}_2)$. This is why $[a]$ is called the *obstruction cocycle*. If the obstruction vanishes, the only freedom available in the construction of $\tilde{B}$ lies in the choice of $c$, and the various possible equivalence classes of $\tilde{B}$ will be parameterized by the elements $[c] \in \check{\mathrm{H}}^1(X, \mathbb{Z}_2)$.

Since metaplectic structures are used to impose a metalinear structure on Lagrangian subbundles we note

**Theorem E.3.1.** *Let $(X, \omega)$ be a symplectic manifold and let $L \subset \mathrm{T}X$ a Lagrangian subbundle, then $\mathrm{T}X$ admits a metaplectic structure if and only if $L$ admits a metalinear structure.*

See [BW97] for a proof.

## E.4 Existence of (pre-)quantum structures

In this Section we will briefly show how to use cohomology and characteristic classes to obtain a criterion for the existence of prequantum structures: let $\omega \in \Lambda^2(M)$ be a real closed two-form and $[\omega] \in \mathrm{H}^2(M, \mathbb{R})$ its cohomology class. We associate an element in

$\check{H}(M, \mathbb{R})$ to $[\Omega]$. Let $\{U_l\}$ be an contractible covering of $M$ such that $\Omega|_{U_l} = \mathrm{d}\theta_l$. Then we have

$$\mathrm{d}\theta_l = \mathrm{d}\theta_j$$

on $U_{lj} = U_l \cap U_j$ and thus $\theta_j - \theta_l = \mathrm{d}f_{lj}$. On $U_{ljk}$ we have $\mathrm{d}f_{lj} + \mathrm{d}f_{jk} - \mathrm{d}f_{lk} = 0$, thus $f_{lj} + f_{jk} - f_{lk} = \alpha_{ljk}$ is constant on $U_{ljk}$ and the map $(U_l, U_j, U_k) \mapsto \alpha_{jkl}$ is a Čech cochain associated with the cohomology class $[\omega]$. We can construct a map

$$\mathrm{H}^2(M, \mathbb{R}) \to \check{H}^2(M, \mathbb{R}), \quad [\omega] \mapsto [a]$$

since $\mathrm{d}a = 0$. Furthermore, the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a morphism

$$\epsilon : \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathbb{R})$$

where $[\omega]$ (resp. $[a]$) is the image under $\epsilon$ if and only if there is a contractible covering $\{U_l\}$ such that

$$f_{lj} + f_{jk} - f_{lk} \in \mathbb{Z}.$$

Now let $\omega$ be the curvature form of a connection on the complex line bundle $L \to M$ with hermitian metric $h$. Let $\{U_l, s_l\}$ be a trivialization of the bundle with $h(s_l, s_l) = 1$. Then the restrictions $\omega_l$ are real and

$$(\omega_j - \omega_l)_{U_{lj}} = \mathrm{d}f_{lj} = \frac{1}{2\pi\,\mathrm{i}} \frac{\mathrm{d}c_{lj}}{c_{lj}}$$

where $c_{lj}$ denote the transition functions in the above trivialization. For this we have the cocycle relation $c_{lj}c_{jk} = c_{lk}$. Since $2\pi\,\mathrm{i}\, f_{lj} = \log c_{lj}$ it follows that $\log c_{lj} + \log c_{jk} - \log c_{lk}$ is an integer multiple of $2\pi\,\mathrm{i}$ and hence the cohomology class $[\omega]$ is integer. On the other hand, if $[\omega]$ is integer we define $c_{lj} = \mathrm{e}^{2\pi\,\mathrm{i}\, f_{lj}}$ to be the transition functions for a complex line bundle in a certain trivialization which also gives a hermitian structure. Then we have

**Theorem E.4.1.** *A necessary and sufficient condition for $\omega$ to be the curvature form of a hermitian connection on a line bundle $L \to M$ endowed with a hermitian metric is that the cohomology class $[\omega] \in \mathrm{H}^2(M, \mathbb{R})$ is integer, i.e. the Čech cohomology class canonically associated with $[\omega]$ belongs to the image of the morphism $\epsilon : \check{\mathrm{H}}^2(M, \mathbb{Z}) \to \check{\mathrm{H}}^2(M, \mathbb{R})$.*

The possible prequantum structures can be parameterized according to the following

**Theorem E.4.2.** *Let $M$ be a differentiable manifold and $\omega$ an integer real closed two-form. The group $\check{\mathrm{H}}^1(M, \mathrm{S}^1)$ acts freely and transitively on the set of equivalence classes of complex line bundles with hermitian connection with curvature $\omega$.*

*In particular, if $M$ is simply connected then $\check{\mathrm{H}}^1(M, \mathrm{S}^1)$ is trivial and there is a unique line bundle with the above property.*

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