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Tractable multi-firm default models based on discontinuous processes

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Preface

The aim of this thesis is to contribute to the ongoing research in structural credit-risk models based on jump-diffusion processes. It concludes my doctorate research which was carried out at the *Institute of Mathematical Finance* at the *Universität Ulm* in the period spring 2005 to spring 2007. My work was supervised by Professor Rüdiger Kiesel and Professor Ulrich Stadtmüller, both *Universität Ulm*.

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Chapter 1

Introduction

1.1 The history of the bond market

The origin of organized markets dates back to the 12th century France and 13th century Belgium. In France, the first brokers traded in debts of agricultural communities. In Bruges, commodity traders met inside the house of *Van der Beurse*. These meetings were then institutionalized as *Brugse Beurse*. Later, organized markets for debts, stocks and commodities were founded in Italy and the Netherlands. The *Amsterdam Stock Exchange* is considered to be the first modern market, allowing continuous trade in options and other derivatives already in the 17th century. Today, corporate and government bonds are traded on every stock exchange in large volume. According to a recent study of the *ECMI*¹, the outstanding volume of corporate bonds in the US, Eurozone and Japan amounts to about 2.2, 1.2 and 0.6 trillion Euro in 2004, respectively. For government bonds, an outstanding volume of about 4.0, 5.0 and 4.6 trillion Euro, respectively, is reported for the same year.

Compared to the long history of bond markets, the global market of credit derivatives is surprisingly young. It arose in the early 1990s in London and New York and grew from virtually nothing to an outstanding notional amount of 26.0 trillion US\$, as reported by *Bloomberg News* on September 19, 2006. Today, the largest market share is still occupied by single-name instruments², but the importance of multi-name derivatives such as collateralized debt obligations grew significantly over the last years. Without doubt, the volume and growth of this market explain and justify the scientific interest in sophisticated credit-risk models.

¹ *European capital market institute*: Statistical package, 2006 edition.

² August 31, 2006: The *Wall Street Journal* reports the notional in credit default swaps (CDS) to exceed 17.0 trillion US\$.

1.2 Credit risk: Definition and models

Today, debt is viewed as an instrument which allows companies to pursue economic activities they could not finance from their own funds. Debtor and creditor agree on the standard of deferred payments, which typically includes the principal sum plus interest. This interest³ is interpreted as the price of debt which has to be determined based on economic considerations. Substantial influence on the amount of interest is founded by the creditworthiness of the obligor. More abstractly, we follow Schönbucher (2003) and define credit risk as *"the risk that an obligor does not honour his payment obligations."*⁴ Over the last decades, several credit-risk models have been set up to quantify this credit risk and to price bonds and credit derivatives. The vast majority of all modern credit-risk models is based on one of the following principles: The structural approach or the reduced-form approach.

Univariate structural default models

Structural default models aim to explain the economic cause of credit default of a company. More precisely, default is assumed to be the consequence of insufficient financial strength of a company. Solvency is linked to the ratio of the firm's assets and liabilities via the assumption that default is triggered when the value of the firm falls below a certain threshold⁵. Consequently, the model of the firm-value process implicitly specifies the term structure of default probabilities. Therefore, this process plays the pivotal role in structural default models. Corporate bonds and credit derivatives are then priced based on this implied term structure of default probabilities.

A natural criterion to distinguish structural default models is to classify them according to the underlying firm-value process. This classification is closely related to the historical development, as the model of the firm-value process has been generalized over the years. The first structural default model was published by Black and Scholes (1973), it relies on a geometric Brownian motion as firm-value process. Originally, this model was designed to describe stock prices rather than the value of a firm. Then, the observation *"It is not generally realized, that corporate liabilities other than warrants may be viewed as options."*⁶ transformed their

³ A well written article about the history and criticism of interest can be found in: *DIE ZEIT*, June 2003, "Ein paar Prozent Streit", <http://www.zeit.de/2003/06/Zinsgeschichte>.

⁴ Schönbucher (2003), page 1.

⁵ The company's total liabilities are often used as default threshold. Other popular interpretations are weighted averages of short- and long-term liabilities, *KMV*: Crosbie and Bohn (2003), or a minimum firm value which is required to operate the company, compare Black and Cox (1976).

⁶ Black and Scholes (1973), *Journal of Political Economy* 81, page 649.

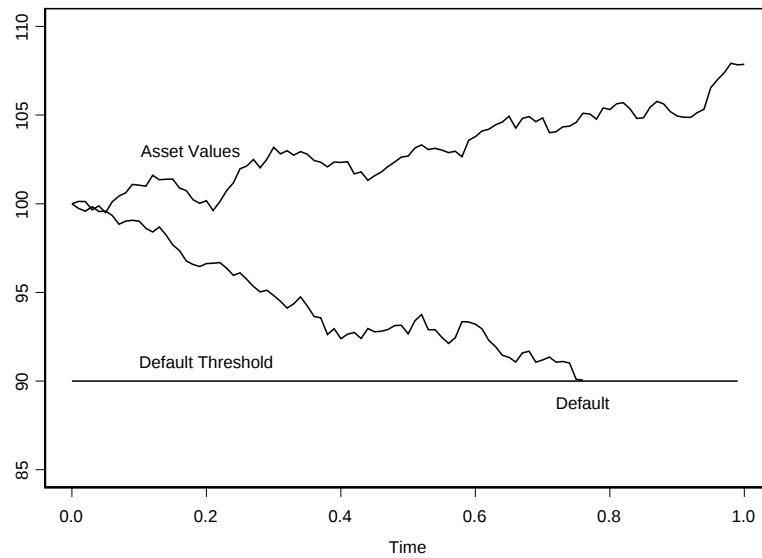


Figure 1.1: Two paths of the firm's asset process, the lower with default.

stock price model into the first structural default model. Their idea was worked out in detail by Merton (1974), who slightly changed the underlying stochastic differential equation of the firm-value process to include dividends and interest payments. However, the solution of this equation is still a geometric Brownian motion. Moreover, as in Black and Scholes (1973), default is only possible at maturity. This shortcoming was corrected by Black and Cox (1976), who criticize the original model as follows: *"Furthermore, it assumes that the fortunes of the firm may cause its value to rise to an arbitrary high level or dwindle to nearly nothing without any sort of reorganization occurring in the firm's financial arrangement. More generally, there may be both lower and upper boundaries at which the firm's securities must take on specific values."*⁷ To correct this unrealistic assumption, they propose to continuously test for default and define the time of default as the first-passage time of the firm-value process below a given barrier. Further generalizations of the model address the economic framework, allowing coupon bonds and bond indenture provisions as in Geske (1977), or include stochastic interest rates as in Longstaff and Schwartz (1995).

Still, all these models suffer from the same defect. In pure diffusion models, the time of default, if defined as a first-passage time of the firm-value process, is a predictable⁸ stopping time with respect to the filtration generated by the Brownian motion. This property turns out to imply vanishing credit spreads for bonds with short maturities, which contradicts the empirical observation that credit spreads

⁷ Black and Cox (1976), *The Journal of Finance* 31, page 352.

⁸ Mathematically, this means that there exists an increasing sequence of stopping times which converges a.s. to the time of default. Intuitively, this means that the time of default is announced.

have a positive limit at the short end of the term structure. This problem can be approached from two sides. First of all, it is possible to reduce or blur the filtration available to all investors. Alternatively, one can relax the assumption of a continuous firm-value process. In either case, the aim is to modify the model such that the time of default is no longer announced in advance.

Duffie and Lando (2001) address the first approach as follows: *"In practice, it is typically difficult for investors in the secondary market for corporate bonds to observe a firm's assets directly, because of noisy or delayed accounting reports, or barriers to monitoring by other means."*⁹ Their model is still based on a geometric Brownian motion, but default probabilities are obtained conditional on noisy accounting data and survivorship of the firm. They show that in this scenario, the default time admits an intensity process and the limit of credit spreads at time zero is positive. Subsequently, several authors considered the problem of incomplete information. Let us mention Giesecke (2006, with Goldberg (2004)), leaving the investor uncertain about the default threshold, Kusuoka (1999), who allows bond investors to receive noisy asset reports by observing some process whose drift is a function of the firm's value process and Çetin et al. (2002), who model the cash-flow process of a company and only allow investors to observe the sign of this process.

In contrast to Duffie and Lando, Zhou (2001a) does not change the filtration but suggests modeling the firm-value process as the superposition of a diffusion and a jump component instead, the latter with normally distributed jumps. He also presents a simple Monte Carlo algorithm to evaluate bond prices within his model. Moreover, he shows that the limit of credit spreads as implied by the model is positive. In this thesis, we further generalize Zhou's single-firm model and present two tractable pricing routines for bonds and CDS contracts. Furthermore, we derive several new theoretical results within this framework.

Finally, Leland (1994, with Toft (1996)) proposed a framework in which the default threshold is not exogenously given. Instead, shareholders are free to choose the default threshold such that the value of the firm's equity is maximized¹⁰. Mathematically, this translates in an optimal stopping problem. Generalizations of this approach have been proposed by Hilberink and Rogers (2002), allowing downward jumps in the firm-value process, and recently by Chen and Kou (2005), Acar (2006) and Dao and Jeanblanc (2006) to jump-diffusion processes with two-sided exponentially distributed jumps.

⁹ Duffie and Lando (2001), *Econometrica* 69, page 634.

¹⁰ Anderson and Sundaresan (1996) consider this approach with a game theoretical perspective.

Univariate reduced-form models

Unlike structural default models, reduced-form models (or intensity-based models) do not intend to explain the default of a company by means of an economic construction. Instead, the time of default is exogenously given and assumed to agree with the jump time of some one-jump stochastic process. The distribution of this totally inaccessible random variable depends on its default-intensity process, for which models with different complexity exist. Recent models often allow this default-intensity process to depend on a vector of state variables. Another important issue is the amount of information based on which bond and derivative prices are derived. Typical examples are the filtration generated by the default indicator process, the filtration of the state variables, or some given filtration enlarged by the default indicator. Each investor then calculates default probabilities conditional on the available information. The focus of this thesis lies in the forthcoming of structural default models. Therefore, we refer the interested reader to the original papers of Pye (1974), Ramaswamy and Sundaresan (1986), Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999a) for more information on reduced-form models. Comprehensive summaries of reduced-form models including further references are given in the books of Lando (2004) and Bielecki and Rutkowski (2002).

Correlated default

The growing popularity of derivatives on credit portfolios, such as collateralized debt obligations or n^{th} -to default contracts, gives rise to research in models which explain the default correlation among different companies. Intuitively, we mean by default correlation the tendency of different companies to default jointly. Another important phenomenon which dependence models try to capture are default clusters, i.e. short time periods with several defaults. Ultimately, pricing derivatives on a credit portfolio translates in the mathematical problem of modeling the term structure of portfolio losses. This loss distribution is specified by the term structure of individual default probabilities, combined with the dependence among the companies. Just as for univariate models, the same categorization in structural and reduced-form models applies to most dependence models.

Multivariate structural default models

The generalization of univariate to multivariate structural default models is quite intuitive. For each company, a stochastic process representing the firm's assets is considered. Default is again triggered by insufficient asset values of the respec-

tive company. In such a model, default correlation is implicitly modeled through dependence of the individual firm-value processes. It is difficult to distinguish existing models according to their approach of introducing dependence, as most models incorporate more than one concept. Therefore, we list and briefly explain the popular concepts of coupling the individual firm-value processes.

In a multivariate extension of Merton's model, a straightforward approach is to assume correlated Brownian motions. This assumption is proposed by Zhou (2001b) and others. Allowing the interpretation of common macroeconomic factors, the resulting dependence structure is referred to by Giesecke (2004) as *cyclical default correlation*. However, this first approach does not match the empirical observation of default clusters, which arise from the fact that the default of a company may substantially increase the default probability of affiliated companies. Giesecke (2004) terms this effect *contagion default correlation* and proposes to include this property by making the default thresholds unobservable to investors or by changing the parameters of the model if one of the companies defaults.

Factor models interpret the asset process of a firm as the superposition of an individual factor and one (or more) common factor(s)¹¹. The first factor model was Vasicek's asymptotic single factor model¹². This model extends Merton's model to several firms and assumes all idiosyncratic factors and the common market factor to be normally distributed¹³. In the sequel, several generalizations of Vasicek's model have been proposed, most of them are concerned with relaxing its assumptions or proposing different distributions for the factors. Among others, Hull and White (2004) propose a t -distribution framework, Kalemianova et al. (2005) implement factors with NIG distribution and Albrecher et al. (2006) present a general one-factor Lévy framework. Let us also mention that most factor models are static models in the sense that they do not test continuously for default. This concession is often necessary to achieve numerical tractability.

Multivariate reduced-form models

Multivariate reduced-form models introduce correlation via dependent intensity processes. Conversely, this assumption implies that defaults are independent conditional on the sample paths of the underlying intensity processes. Focusing on structural models, we only give a brief overview of the popular approaches to add correlation to the default intensity processes.

¹¹ These models often attach economic interpretations to their factors, from which estimation methods for the respective factors based on fundamental data are derived.

¹² The original reference is Vasicek (1987).

¹³ Vasicek's model is often referred to as Gaussian-copula model, due to the dependence structure the assumption of normally distributed factors implies.

Quite intuitive is the class of *conditionally independent default* models, where all individual intensity processes are functions of some common variables. These models can be generalized by allowing common jumps in the default intensity processes or by considering events which may cause several defaults simultaneously, compare Duffie and Singleton (1999) and Kijima (2000). To incorporate the contagion effect, it is reasonable to increase the default intensity of a firm if an affiliated company defaults. These models are studied by Davis and Lo (1999) and Jarrow and Yu (2001). Finally, it is possible to use copula functions to link independent survival probabilities or default thresholds, as proposed by Li (1999) and Schönbucher and Schubert (2001).

1.3 Our contribution and aim of this thesis

The aim of this thesis is to thoroughly study structural default models based on jump-diffusion processes. As noted earlier, jump-diffusion models were first proposed by Zhou (2001a), who also showed that these models have several desirable properties, most important, positive short-term spreads. On the other side, these models turned out to be complicated to implement, as the distribution of the running minimum of a jump-diffusion process is not known. We show that an efficient implementation of jump-diffusion models is possible and even fast enough to allow a calibration to market data. Moreover, we derive new theoretical results and generalize the model to jointly consider different companies.

1.3.1 Our findings in Zhou's univariate model

To incorporate sudden defaults, Zhou (2001a) suggested superposing the traditional diffusion model by a compound Poisson component. Obviously, this extension is a realistic generalization, since purely continuous firm-value processes are often not adequate. Unfortunately, this model is not as tractable as a continuous model, as the distribution of first-passage times of a jump-diffusion process is unknown. Therefore, simulation and approximation techniques are required to derive prices in this framework. The first algorithmic implementation of the bond pricing formula was suggested by Zhou (2001a). This approach relied on simulations of trajectories of the firm-value process on a small grid, which is straightforward to implement. However, this algorithm is slow and suffers from a massive discretization bias. What we propose is a Monte Carlo simulation which simulates as little as possible and calculates the remainder. More precisely, the idea and theoretical justification of our algorithm is to express the pricing formula of a bond in terms of a nested conditional expectation. Conditioned on the number and location of possible jumps, and the firm-value process at these jump times, we present a closed-form expression of bond prices. The distributions of all random variables on which we condition are known and straightforward to simulate. Our Algorithm 4.3.1 simulates these quantities in each Monte Carlo step. Then, the corresponding expected payoff is computed. This leads to an unbiased and extremely fast algorithm, which is not restricted to a specific jump-size distribution.

In terms of speed, we could further improve our Monte Carlo simulation by approximating an integral which occurs in the conditional payoff of the bond. This approximation is presented in Theorem 4.3.3. Our result is an improved variant of an approximation which was originally suggested by Metwally and Atiya (2002) in the context of option pricing. In Section 4.8.2, we present several numerical exper-

iments which illustrate the improvement in precision if our approximation is used. Also, this section provides an extensive analysis of all available pricing algorithms.

Considering credit default swaps (CDS), Algorithm 4.4.1 is closely related to the bond-pricing simulation before. We designed this Monte Carlo simulation to price CDS contracts in the same jump-diffusion framework. Again, the idea of this algorithm is to rewrite the expected discounted premium and default leg of a CDS in terms of nested conditional expectations.

When we used our algorithm to perform numerical experiments with different sets of parameters, we noticed that the limit of credit spreads at the short end of the term structure is positive and independent of the parameters of the diffusion component. The observation of a positive limit of spreads was predicted by Zhou (2001a). Moreover, independence of the limit of spreads of the diffusion component seems natural, since this limit is zero in a continuous model. We combined both observations in Theorem 4.3.4, where we present an exact formula of the limit of credit spreads in an arbitrary jump-diffusion model. This limit is found to be the product of the local default rate and the fractional loss at default. In Theorem 4.2.1, we further show that the local default rate is the product of the jump intensity and the probability of a jump to exceed the distance to default.

In the following sections, we present several generalizations of Zhou's model. First of all, we show how the random undershot of the firm-value process below the default threshold is used to endogenously define the recovery rate. Up to this point, the recovery rate was one of the deterministic parameters of the model. More realistically, however, is the assumption of a random variable which is drawn at the time of default. Since structural default models explicitly describe the firm-value process of a company, it seems natural to use the value of this process at the time of default to specify the recovery rate. This approach is redundant in a continuous model, as in this model the firm-value process necessarily agrees with the deterministic default threshold at default. In contrast, jumps allow the firm-value process to fall below the default threshold. This unknown undershot is used to induce a random recovery rate. While this idea is straightforward to implement in a simulation relying on complete trajectories of the firm-value process, we show in Section 4.5.1 how the same feature is embedded in our algorithm. Our generalization of Section 4.5.2 relaxes the assumption of a flat term-structure of interest rates. In this section, we explain how short-rate models are included in our framework. Exemplarily, we implement Vasicek's short-rate model and the CIR model. Our next generalization of the structural default model is based on the observation that the difference of two jump-diffusion processes is again a jump-diffusion process. This result is used in Section 4.5.3 to include a stochastic default threshold. An important special case of this generalization of the model is the popular exam-

ple of an exponential default threshold. Finally, we present pricing routines based on reduced filtrations in Section 4.5.4. This extension of the model is inspired by relaxing the unrealistic assumption of being able to continuously observe the firm-value process. Instead, we provide updates of the firm-value process on a periodic schedule, which are interpreted as accounting reports. The expected payoffs of bonds and CDS contracts, relative to different reduced filtrations, are derived. These results are interpreted and illustrated using a numerical experiment. Also, we present closed-form expressions of bonds and CDS contracts in a pure diffusion framework, relative to different reduced filtrations.

Up to this point, all available pricing routines relied on different Monte Carlo simulations. Given a set of parameters, we showed that such a routine provides reliable prices in a very short time. However, a calibration of the model requires the minimization of some distance of model to market prices over the parameter space. Typically, this leads to hundreds of evaluations of the objective function. Also, an efficient minimization routine requires the numerical approximation of the gradients of the objective function. For neither of them, a Monte Carlo estimation is suitable. We could overcome this problem by specifying the jump-size distribution to be a two-sided exponential distribution. In this scenario, the Laplace transform of first-passage times was derived by Kou and Wang (2003). This Laplace transform involves the roots of a quartic polynomial and is only known on the positive axis. Therefore, these first-passage probabilities can not be inverted explicitly from the transform. However, we can recover them numerically using the Gaver-Stehfest algorithm. Based on these approximated survival probabilities, we introduce an algorithm to price bonds and CDS contracts in fractions of seconds with high precision. This elegant approach is worked out in detail in Section 4.6. Moreover, the implementation of the resulting Algorithm 4.6.1 is explained and illustrated using several numerical examples.

Given this efficient pricing routine, we were able to calibrate the model to bond prices and CDS quotes. The first part of Section 4.9 explains a calibration of the model to bond prices of *DaimlerChrysler* and *GM*. Then, we calibrate the firm-value process of each of the 125 companies of the *iTraxx* CDS portfolio. This is done for 16 trading days, which amounts to 2,000 calibrations. The same calibration is performed in a pure diffusion model. Given these results, we conclude that the jump-diffusion model is capable to fitting observed prices with excellent accuracy. The pure diffusion model was outperformed by far in terms of fitting capability, especially for contracts with small maturities. Finally, we used our Laplace-pricing algorithm for an extensive study of the analytical properties of the model. More precisely, in Section 4.7 we present a sensitivity analysis of model prices with respect to changes in the parameters. Again, we conclude that the model is extremely flexible in producing different term structures of credit spreads.

1.3.2 Our new multidimensional model

Our next step was to generalize the model to a portfolio model consisting of several dependent firms. This model is then used to price multi-firm credit derivatives such as CDOs and n^{th} -to default contracts. Our objective was to present a mechanism of coupling the individual firm-value processes based on a firm economic rationale. This is achieved by partially replacing the Brownian motion of each individual firm-value process by a Brownian motion of the market. This common factor is interpreted as an indicator of the current macroeconomic situation. Moreover, we introduce dependence through common jumps in some of the firm-value processes. For this, we define a ticker process which reports the arrival of unexpected information. Jumps of the ticker process then induce jumps in some of the individual firm-value processes. Additionally, we present a variant of the model which is based on different industry sectors. From a firm-individual perspective, our construction of introducing dependence preserves the marginal default probabilities.

In a continuous model, the correlation of two firm-value processes is easily expressed in terms of their interdependence with the common market factor. We generalize this formula in Theorems 5.2.1 and 5.2.2 for our jump-diffusion model. Moreover, we perform several numerical investigations to determine the influence of common jumps on the default correlation of two firms. Compared to the possibility of simultaneous jumps, we conclude that a common market factor requires significantly more time to produce a relevant default correlation.

The calibration of our model to CDO quotes requires a fast method of computing the respective model spreads. We present a Monte Carlo simulation which is able to price CDOs within minutes on a standard notebook. This algorithm exploits the integral approximation of Theorem 4.3.3 and other results of the univariate model to achieve the required speed. Several numerical experiments are performed to illustrate the sensitivity of model spreads with respect to changes in the parameters of the dependence structure.

An important property of our multidimensional generalization is the option of calibrating the model in two steps. This is possible, as the parameters of the individual default probabilities are separated from the parameters of the dependence structure. Such a calibration is explained in detail in Section 5.3.4. At first, we calibrate the parameters of each firm-value process to CDS or bond quotes of the respective firm. In a second step, we use CDO quotes to calibrate the dependence parameters of the model. Our calibration of the model to 16 trading days shows that the model is able to match observed prices with high precision. At the same time, the model explains the term structure of CDS spreads of each company of the portfolio. Considering this, the fitting capability of the model is even more impres-

sive. Moreover, our model is designed to describe the term structure of portfolio losses, which enables us to price CDOs with different maturities in a consistent framework.

Up to this point, we primarily focused on the pricing of CDOs. However, our algorithm is easily adapted to the pricing of other portfolio derivatives. As an example, we show how n^{th} -to default contracts are priced using our model. Finally, we analyze the sensitivity of n^{th} -to default spreads with respect to changes in the parameters of the model.

1.3.3 Organization of this thesis

Following this introduction, Chapter 2 reviews the fundamental concepts of Lévy processes and the Laplace transform. In doing so, the notations of this thesis are introduced. Chapter 3 is devoted to the products we aim to price based on our default model. Explained are pricing issues, contractual terms and the payment streams of all contractual partners. The univariate default model of Chapter 4 is used to price bonds and single-name credit derivatives. Our multidimensional generalization allows the pricing of portfolio derivatives, as explained in Chapter 5. The content and organization of both chapters goes along with the description of our contribution. Finally, the Appendix contains several proofs, lists of tables and figures and the bibliography.

Chapter 2

Technical background

2.1 An introduction to Lévy processes

The fundamental object of our structural default model is the class of jump-diffusion processes, a subclass of the set of Lévy processes. In this section, we introduce the definition of a Lévy process and list all properties of Lévy processes used in this thesis. Then, we focus on the class of jump-diffusion processes, explaining their construction and deriving all relevant results for the context of credit risk. A well motivated introduction to Lévy processes and their applications in finance is presented in Cont and Tankov (2004), more technical details and results can be found in Sato (1999).

2.1.1 The probabilistic framework

Throughout this thesis, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Our sample space Ω is equipped with the σ -algebra $\mathcal{F}$, that is a family of subsets¹ which is stable under countably many set operations. We use a stochastic process² $V = \{V_t\}_{t \geq 0}$ to model the value of a company over time. At time zero, an investor only knows the initial value of the process V . As time goes on, the firm-value process is gradually revealed. The model for this dynamic information flow is called filtration and is denoted by $\mathbb{F}$. More precisely, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of σ -algebras, meaning that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t$. In our framework, this filtration is said to be generated by the

¹ The richness of $\mathcal{F}$ is specified below.

² A stochastic process is a family of random variables, in our context indexed by the time variable $t \geq 0$.

firm-value process, i.e.

$$\mathcal{F}_t = \sigma(V_s : 0 \leq s \leq t) \vee \mathcal{N} \quad \forall t \geq 0,$$

completed by all null sets $\mathcal{N}$ of $\mathbb{P}$. One can interpret $\mathcal{F}_t$ as the information about the history of the firm-value process up to time t . Including the set $\mathcal{N}$ already at time zero corresponds to knowing right from the beginning which evolutions of the firm-value process are virtually impossible. Moreover, we assume $\mathbb{F}$ to be right continuous, that is $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t > 0$.

We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote all natural numbers, all integers, all rational numbers and all complex numbers, respectively. Moreover, we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the positive elements of $\mathbb{Q}$ and $\mathbb{R}$, respectively. All random variables used in this thesis are real valued. Given the distribution $\mathbb{P}_Y$ of a random variable Y , where $\mathbb{P}_Y(A) := \mathbb{P}(Y^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R})$ ³, we denote by $F_Y(x) := \mathbb{P}_Y((-\infty, x])$ the cumulative distribution function of Y .

Given their existence, we denote the expectation and variance of Y by $\mathbb{E}[Y] := \int_{\Omega} Y d\mathbb{P}$ and $\text{Var}(Y) := \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$, respectively. The covariance of the random variables Y and Z is given by $\text{Cov}(Y, Z) := \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z]$.

Distribution functions and their abbreviations

In this chapter, we briefly introduce all distribution functions which are needed to construct our structural default model. Also, we list their density functions and moments. To begin with, let us remark that the distribution of a random variable Y is uniquely specified by its characteristic function $\Phi_Y : \mathbb{R} \rightarrow \mathbb{C}$, which we denote by

$$\Phi_Y(z) := \mathbb{E}[e^{izY}] = \int_{-\infty}^{\infty} e^{izu} dF_Y(u) \quad z \in \mathbb{R}.$$

Further, we denote by Φ the cumulative distribution function of a standard normal distributed random variable, i.e.

$$\Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

Other required distributions include the Poisson distribution with intensity parameter $\lambda > 0$, which is abbreviated as $\text{Poi}(\lambda)$. The probability-mass function of a

³ $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -algebra, which is generated by the set of open intervals.

Poisson distributed random variable X satisfies

$$\mathbb{P}(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad n \in \mathbb{N}_0.$$

A Bernoulli experiment is a yes/no experiment with success probability $p \in [0, 1]$. The number of successes X in a sequence of n i.i.d. Bernoulli experiments is referred to as binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$. For brevity, we write $X \sim B(n, p)$. The probability-mass function of X satisfies

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n.$$

The non-standard normal distribution with mean μ and standard deviation $\sigma > 0$ is abbreviated as $\mathcal{N}(\mu, \sigma^2)$. Its density function is given by

$$\varphi_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right) \quad \mu \in \mathbb{R}, \sigma > 0.$$

The continuous uniform distribution on $[a, b]$ with density function $f_{a,b}$ is abbreviated as $\text{Uni}[a, b]$ ⁴. For this distribution, the density function satisfies

$$f_{a,b}(x) = \frac{1}{b-a} \mathbf{1}_{\{a \leq x \leq b\}}.$$

The exponential distribution with parameter $\lambda > 0$ is abbreviated as $\text{Exp}(\lambda)$. The density function of an exponential distribution has the form

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}} \quad \lambda > 0.$$

Finally, the two-sided exponential distribution with parameters $p \in [0, 1]$, $\lambda_{\oplus} > 0$ and $\lambda_{\ominus} > 0$ is abbreviated as $2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$. In this case, the density function is given by

$$f(x) = p\lambda_{\oplus}e^{-\lambda_{\oplus}x} \mathbf{1}_{\{x \geq 0\}} + (1-p)\lambda_{\ominus}e^{\lambda_{\ominus}x} \mathbf{1}_{\{x < 0\}}.$$

Let us remark that the distribution of a random number drawn from a two-sided exponential distribution can be interpreted as the combination of an initial Bernoulli experiment with success probability p , determining the sign of the random number, followed by an independent draw of an exponential distribution with parameter $\lambda_{\oplus}$ or $\lambda_{\ominus}$, respectively.

For the reader's convenience, we summarize the first two moments of these distributions in Table 2.1.

⁴ If the endpoints are excluded, we abbreviate the uniform distribution on (a, b) as $\text{Uni}(a, b)$.

Distribution	Mean	Variance	Second moment
$\text{Poi}(\lambda)$	λ	λ	$\lambda(\lambda + 1)$
$\text{B}(n, p)$	np	$np(1 - p)$	$np(1 - p + np)$
$\mathcal{N}(\mu, \sigma^2)$	μ	σ^2	$\sigma^2 + \mu^2$
$\text{Uni}[a, b]$	$\frac{1}{2}(a + b)$	$\frac{1}{12}(b - a)^2$	$\frac{1}{3}(a^2 + ab + b^2)$
$\text{Exp}(\lambda)$	λ^{-1}	λ^{-2}	$2\lambda^{-2}$
$2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$	$\frac{p}{\lambda_{\oplus}} - \frac{1-p}{\lambda_{\ominus}}$	$\frac{p(2-p)}{\lambda_{\oplus}^2} + \frac{1-p^2}{\lambda_{\ominus}^2} + \frac{2p(1-p)}{\lambda_{\oplus}\lambda_{\ominus}}$	$\frac{2p}{\lambda_{\oplus}^2} + \frac{2(1-p)}{\lambda_{\ominus}^2}$

Table 2.1: Distributions and their first two moments.

2.1.2 Stopping times and martingales

A stochastic process $X = \{X_t\}_{t \geq 0}$ is said to be adapted to the filtration $\mathbb{F}$, if X_t is $\mathcal{F}_t$ -measurable for all $t \geq 0$. For instance, the firm-value process V of the structural default model of Chapter 4 is adapted to its natural filtration by construction. An important class of stochastic processes is the set of all processes whose sample paths are almost surely right continuous with left limits. Such processes are called càdlàg, the french acronym for continue à droite, limites à gauche. For a càdlàg process X , we define $X_{t-} := \lim_{s \nearrow t} X_s$ and $\Delta X_t := X_t - X_{t-}$, where we set $X_{0-} := X_0$.

In the context of a structural credit-risk model, the time of default is typically defined to be the first time the firm-value process passes a certain barrier. More precisely, we define the first-passage times of the process X below, respectively above, the barrier b , respectively $\tilde{b}$, as

$$\tau_b := \inf \{t \geq 0 : X_t \leq b\}, \quad \tau^{\tilde{b}} := \inf \{t \geq 0 : X_t \geq \tilde{b}\}, \quad b < X_0 < \tilde{b}, \quad (2.1)$$

with the convention $\inf \emptyset := \infty$. One can show⁵ that first-passage times of càdlàg processes are stopping times⁶ with respect to the filtration $\mathbb{F}$.

A càdlàg process X is said to be a martingale with respect to $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, if X is adapted to the filtration $\mathbb{F}$, $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$ and $\mathbb{E}[X_s | \mathcal{F}_t] = X_t$ whenever $s > t$. An intuitive interpretation of this property is that the best prediction for the future is the present level. This defining property of martingales is

⁵ Compare Rogers and Williams (1994), pages 182 and 186.

⁶ That is, a non-negative, real-valued random variable τ , such that the event $\{\tau \leq t\}$ is $\mathcal{F}_t$ -measurable for all $t \geq 0$.

generalized to stopping times by the so called sampling theorem. More precisely, given a martingale X and stopping times $0 \leq \tau_1 \leq \tau_2 \leq T$ for some fixed terminal time horizon $T < \infty$, we have⁷ $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] = X_{\tau_1}$. A well known example for a martingale is the Brownian motion, sometimes referred to as Wiener process, which we define below.

Definition 2.1.1 (Brownian motion)

An adapted, real-valued process $W = \{W_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $W_0 = 0$, is called Brownian motion, if W satisfies the following properties.

1. W has independent increments, i.e. whenever $0 = t_0 < t_1 < \dots < t_n$, the increments $W_{t_1} - W_{t_0}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent random variables.
2. W has stationary, normally distributed increments, i.e.

$$\forall t \geq 0 : W_{t+h} - W_t \sim \mathcal{N}(0, h) \quad \forall h > 0.$$

3. W has almost surely continuous paths, that is the function $t \mapsto W_t(\omega)$ is continuous for almost all $\omega \in \Omega$.

From this definition, it is easily checked that W is a martingale. Let us further remark that a process of the form $X_t = \gamma t + \sigma W_t$ is usually referred to as diffusion or Brownian motion with drift.

2.1.3 General properties of Lévy processes

Definition 2.1.2 (Lévy process)

An adapted càdlàg process $X = \{X_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $X_0 = 0$, is a real-valued Lévy process, if it satisfies the following properties.

1. X has independent increments, i.e. whenever $0 = t_0 < t_1 < \dots < t_n$, the increments $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent random variables.
2. X has stationary increments, that is the distribution of $X_{t+h} - X_t$ does not depend on t for all $t \geq 0$, $h > 0$.
3. X is stochastically continuous, that is

$$\forall t \geq 0, \forall \epsilon > 0 : \lim_{h \searrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0.$$

⁷ The stopping time σ -algebra of τ is defined by $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \in [0, T]\}$.

Let us remark that a stochastically continuous process can have discontinuous sample paths, as we shall see below. Moreover, the jump structure of the Lévy process X is specified by its jump measure J_X or by the corresponding Lévy measure ν . This random measure $J_X : \Omega \times \mathcal{B}(\mathbb{R}_0^+ \times \mathbb{R}) \rightarrow \mathbb{R}_0^+$, $(\omega, C) \mapsto J_{X(\omega)}(C)$, is defined by

$$J_X(A \times B) := \sum_{t \geq 0: \Delta X_t > 0} \mathbf{1}_{\{t \in A\}} \mathbf{1}_{\{\Delta X_t \in B\}} \quad \forall A \times B \in \mathcal{B}(\mathbb{R}_0^+ \times \mathbb{R}).$$

Based on the jump measure J_X , the Lévy measure $\nu : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}_0^+$ is defined by

$$\nu(B) := \mathbb{E}[J_X([0, 1] \times B)] \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

The Lévy measure of a set $B \in \mathcal{B}(\mathbb{R})$ can be interpreted as the expected number of jumps within a unit-time interval such that the jump size of X falls into the set B . We are now equipped with all definitions which are required to state the Lévy-Itô decomposition. A proof of this characterization is given in Sato (1999), pages 120 and 125.

Theorem 2.1.1 (Lévy-Itô decomposition of a Lévy process)

Given a Lévy process X with jump measure J_X and Lévy measure ν . Then, there exists $\gamma_1 \in \mathbb{R}$, $\sigma \geq 0$ and a Brownian motion W such that

$$X_t = \gamma_1 t + \sigma W_t + X_{1,t} + \lim_{\epsilon \searrow 0} \tilde{X}_t^\epsilon, \quad (2.2)$$

where

$$\begin{aligned} X_{1,t} &:= \int_{|x| > 1, s \in [0, t]} x J_X(ds \times dx), \\ \tilde{X}_t^\epsilon &:= \int_{\epsilon \leq |x| \leq 1, s \in [0, t]} x (J_X(ds \times dx) - \nu(dx)ds). \end{aligned}$$

In Equation (2.2), the processes W , X_1 and $\lim_{\epsilon \searrow 0} \tilde{X}^\epsilon$ are mutually independent.

However, this representation is not unique. The integral $\int_{|x| > 1} |x| \nu(dx)$ being not necessarily finite requires to truncate large jumps, which is done in Equation (2.2) via the truncation function $\mathbf{1}_{\{|x| \leq 1\}}$. Instead of truncating at the level one, every other positive constant can be used to truncate large jumps⁸. To emphasize the dependence of the Lévy-Itô decomposition on the choice of truncation function, we introduced the subscript "1" in the definition of γ_1 and X_1 . Nevertheless, as soon as a truncation function is chosen, the distribution of a Lévy process is

⁸ Different alternative truncation functions have been proposed in the literature. For instance, Paul Lévy suggested $\frac{1}{1+x^2}$.

fully specified by the triplet γ_1 , σ^2 and ν . This observation motivates the next definition.

Definition 2.1.3 (Lévy (or characteristic) triplet)

Given a Lévy process X , the triplet $(\gamma_1, \sigma^2, \nu)_1$, where γ_1 , σ^2 and ν are given as in Theorem 2.1.1, is called Lévy or characteristic triplet of X with respect to the truncation function $\mathbf{1}_{\{|x| \leq 1\}}$. If large jumps do not have to be truncated, that is if $\int_{|x| > 1} |x| \nu(dx)$ is finite, then $\gamma_c := \gamma_1 + \int_{|x| > 1} x \nu(dx)$ is called the center of the process X and $(\gamma_c, \sigma^2, \nu)_c$ is the corresponding Lévy triplet without truncation.

Another characterization of a Lévy process relies on the characteristic function of X_t for all $t > 0$. More precisely, the following theorem holds.

Theorem 2.1.2 (Lévy-Khinchin representation)

Given a Lévy process X with characteristic triplet $(\gamma_1, \sigma^2, \nu)_1$, the characteristic function of X_t satisfies $\Phi_{X_t}(z) = \exp(t\psi(z))$ for all $z \in \mathbb{R}$, where

$$\psi(z) := -\frac{1}{2}\sigma^2 z^2 + i\gamma_1 z + \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz\mathbf{1}_{\{|u| \leq 1\}}) \nu(du). \quad (2.3)$$

If $\int_{|x| > 1} |x| \nu(dx)$ is finite, Equation (2.3) simplifies to

$$\psi(z) := -\frac{1}{2}\sigma^2 z^2 + i\gamma_c z + \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz) \nu(du).$$

2.1.4 Building a jump-diffusion process

In what follows, we introduce the notion of a jump-diffusion process. We already defined the Brownian motion in Definition 2.1.1. Below, we introduce the compound Poisson process as a second building block of jump-diffusion processes and list some of its properties. If no other reference is given, proofs of the following results can be found in Cont and Tankow (2004).

Definition 2.1.4 (Poisson process, compound Poisson process)

Given a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of independent, $\text{Exp}(\lambda)$ -distributed random variables, we define $T_n := \sum_{i=1}^n \tau_i$. Then, the process $N_t := \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}}$ is called Poisson process with intensity λ . Given a Poisson process $N = \{N_t\}_{t \geq 0}$ and a sequence of independent random variables $\{Y_i\}_{i \in \mathbb{N}}$ with distribution $\mathbb{P}_Y$, the process $M_t := \sum_{i=1}^{N_t} Y_i$ is called compound Poisson process with intensity $\lambda > 0$ and jump-size distribution $\mathbb{P}_Y$.

Let us remark that N_t follows a $\text{Poi}(\lambda t)$ distribution, as shown for instance in Billingsley (1995), page 299. Additionally, we need the following result on thinning and superposing Poisson processes. A proof of Lemma 2.1.1 is given in Durrett (1999), page 140.

Lemma 2.1.1 (Thinning and superposing Poisson processes)

Assume as given a Poisson process N with intensity λ and a sequence $\{B_j\}_{j \in \mathbb{N}}$ of independent Bernoulli distributed random variables with success probability $b \in (0, 1]$. Each jump time τ_j of N is then associated with the outcome of the corresponding Bernoulli experiment B_j . The thinned-out Poisson process $N_t(b)$ is defined as the sum of all successful Bernoulli experiments up to time t . For this process, it holds that

- $N(b) = \{N_t(b)\}_{t \geq 0}$ is again a Poisson process with intensity $b\lambda$.

Given m independent Poisson processes $N^1, \dots, N^m$ with intensity $\lambda^1, \dots, \lambda^m$, respectively. Then, it holds that

- the superposition $\sum_{i=1}^m N^i$ is again a Poisson process with intensity $\sum_{i=1}^m \lambda^i$.

Definition 2.1.5 (Jump-diffusion process)

The superposition of a diffusion and a compound Poisson process is called jump-diffusion process. More precisely, with the notations of Definitions 2.1.1 and 2.1.4, the stochastic process $X = \{X_t\}_{t \geq 0}$ is called jump-diffusion process with linear drift $\gamma \in \mathbb{R}$, volatility of the diffusion component $\sigma > 0$, jump-intensity $\lambda > 0$ and jump-size distribution $\mathbb{P}_Y$, where

$$X_t := \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i.$$

All random variables of this definition are mutually independent.

Lemma 2.1.2 (The Lévy triplet of a jump-diffusion process)

All building blocks of a jump-diffusion process are Lévy processes. For a diffusion, this follows immediately from the definition of a Brownian motion. The required calculations in the case of a compound Poisson process are given in Cont and Tankov (2004), page 71. Due to independence of diffusion and jump component, the characteristic function of a jump-diffusion process is easily obtained, and so is its Lévy triplet.

1. A Brownian motion has the Lévy triplets

$$(0, 1, 0)_1, \quad (0, 1, 0)_c.$$

2. With respect to $\mathbf{1}_{\{|x| \leq 1\}}$, the Lévy triplet of a compound Poisson process with intensity λ and jump-size distribution $\mathbb{P}_Y$ is given by

$$\left(\lambda \int_{\mathbb{R}} u \mathbf{1}_{\{|u| \leq 1\}} \mathbb{P}_Y(du), 0, \lambda \mathbb{P}_Y(du) \right)_1.$$

If $\mathbb{E}[Y]$ exists, then large jumps do not have to be truncated, and we obtain

$$\left(\lambda \int_{\mathbb{R}} u \mathbb{P}_Y(du), 0, \lambda \mathbb{P}_Y(du) \right)_c.$$

3. A jump-diffusion process, compare Definition 2.1.5, has the Lévy triplet

$$\left(\gamma + \lambda \int_{\mathbb{R}} u \mathbf{1}_{\{|u| \leq 1\}} \mathbb{P}_Y(du), \sigma^2, \lambda \mathbb{P}_Y(du) \right)_1.$$

If $\mathbb{E}[Y]$ exists, then its Lévy triplet without truncation is given by

$$\left(\gamma + \lambda \int_{\mathbb{R}} u \mathbb{P}_Y(du), \sigma^2, \lambda \mathbb{P}_Y(du) \right)_c.$$

Lemma 2.1.3 (Moments of a jump-diffusion process)

Given a jump-diffusion process X as in Definition 2.1.5, we obtain the following results by using the definition of a Brownian motion and by conditioning on N_t .

1. If the jump-size distribution $\mathbb{P}_Y$ is integrable, then

$$\mathbb{E}[X_t] = t(\gamma + \lambda \mathbb{E}[Y_1]) \quad \forall t \geq 0.$$

2. If the jump-size distribution $\mathbb{P}_Y$ is square integrable, then

$$\text{Var}(X_t) = t(\sigma^2 + \lambda \mathbb{E}[Y_1^2]) \quad \forall t \geq 0.$$

Lemma 2.1.4 (Itô's formula for jump-diffusion processes)

Let $X = \{X_t\}_{t \in [0, T]}$ be a jump-diffusion process as in Definition 2.1.5 with additional starting value X_0 and finite time horizon T . Further, let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$

be a $C^{1,2}$ function⁹. Then, the process $Z_t := f(t, X_t)$ satisfies

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left(f_t(s, X_s) + \gamma f_x(s, X_s) + \frac{\sigma^2}{2} f_{xx}(s, X_s) \right) ds \\ &\quad + \int_0^t \sigma f_x(s, X_s) dW_s + \sum_{\{i \in \mathbb{N}, T_i \leq t\}} (f(T_i, X_{T_i-} + Y_i) - f(T_i, X_{T_i-})). \end{aligned}$$

2.2 The Laplace transform

The Laplace transform is an important tool for analyzing stochastic variables and is used several times within this thesis. Therefore, we briefly summarize its definition and all properties which will be used. A detailed introduction to the theory of the Laplace transform and the proofs of Lemma 2.2.1 are given in the textbook of Schiff (1999).

2.2.1 Definition and basic properties

Definition 2.2.1 (The Laplace transform of a function)

Given a real or complex-valued function f of the time variable t , we define

$$c_f := \inf \left\{ \alpha \in \mathbb{R} : \int_0^\infty e^{-\alpha t} f(t) dt \text{ exists} \right\}, \quad U_f := \{s \in \mathbb{C} : \operatorname{Re}(s) > c_f\}.$$

Then, if $U_f \neq \emptyset$, the Laplace transform of f is defined by¹⁰

$$\mathcal{L}(f) : U_f \rightarrow \mathbb{C}, \quad (\mathcal{L}(f))(s) := \int_0^\infty e^{-st} f(t) dt.$$

The Laplace inverse $\mathcal{L}^{-1}$ is the inverse function of the Laplace transform, i.e. the inverse Laplace transform of $\mathcal{L}(f(t)) = F(s)$ is denoted by

$$\mathcal{L}^{-1}(F(s)) = f(t) \quad t \geq 0.$$

Lemma 2.2.1 (Properties of the Laplace transform)

For the reader's convenience, we summarize several important properties of the Laplace transform and its inverse in this lemma.

⁹ $f \in C^{1,2}$ means that f is continuously differentiable with respect to its first argument and twice continuously differentiable with respect to its second argument.

¹⁰ The symbol $\mathcal{L}$ is the Laplace transformation. This operator assigns the new function $F(s) := \mathcal{L}(f(t))$ to the function $f = f(t)$.

1. Linearity

The Laplace transformation is linear. Given the functions f and g with Laplace transforms defined on U_f and U_g , respectively. Then, for arbitrary constants $\beta_1, \beta_2 \in \mathbb{C}$, on $\tilde{U}_{f+g} := \{s \in \mathbb{C} : \operatorname{Re}(s) > \max\{c_f, c_g\}\}$ it holds that

$$\mathcal{L}(\beta_1 f + \beta_2 g) = \beta_1 \mathcal{L}(f) + \beta_2 \mathcal{L}(g).$$

2. The shift theorem

Assume as given a function f with Laplace transform defined on U_f . Then, for all $a \in \mathbb{R}$, the Laplace transform of $e^{-at}f(t)$ is defined on $\tilde{U}_f := \{s \in \mathbb{C} : \operatorname{Re}(s) > c_f - a\}$ and is given by

$$(\mathcal{L}(e^{-at}f(t)))(s) = (\mathcal{L}(f))(s + a).$$

3. The Laplace transform takes convolutions into products

Given the functions f and g with Laplace transforms defined on U_f and U_g , respectively. Then, the Laplace transform of $f \star g$ ¹¹ is defined on $\tilde{U}_{f \star g} := \{s \in \mathbb{C} : \operatorname{Re}(s) > \max\{c_f, c_g\}\}$ and is given by

$$(\mathcal{L}(f \star g))(s) = (\mathcal{L}(f))(s) \cdot (\mathcal{L}(g))(s).$$

4. The Laplace inverse

The Laplace inverse is given as the following complex integral

$$f(t) = \frac{1}{2\pi i} \int_{y-i\infty}^{y+i\infty} e^{ts} (\mathcal{L}(f))(s) ds, \quad (2.4)$$

where all singularities of $(\mathcal{L}(f))(s)$ are to the left of $y \in \mathbb{R}$.

Let us remark that the integral representation of Equation (2.4) establishes that $\mathcal{L}^{-1}$ is a linear operator.

2.2.2 The Gaver-Stehfest algorithm

Lemma 2.2.1 presents a closed-form expression of the Laplace inverse as a complex integral. Inverting the Laplace transform becomes necessary in Section 4.6, where we obtain the Laplace transform of certain default probabilities. Unfortunately, Equation (2.4) is not applicable in this context, as we only know the Laplace transform of first-passage times on the positive axis. To overcome this

¹¹ The convolution of f and g is defined as $(f \star g)(t) := \int_{-\infty}^{\infty} f(t-s)g(s)ds$.

problem we introduce the algorithm of Gaver and Stehfest, which only requires the Laplace transform of a function at certain positive values. Other advantages and disadvantages of this algorithm, and the following lemma on which this method is based on, are described in Abate and Whitt (1992).

Lemma 2.2.2 (Gaver (1966) and Stehfest (1970))

For a bounded and real-valued function f , continuous at t , it holds that

$$f(t) = \lim_{n \rightarrow \infty} \frac{\log(2)}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^n (-1)^k \binom{n}{k} (\mathcal{L}(f)) \left(\frac{\log(2)(n+k)}{t} \right).$$

In what follows, we abbreviate the sequence of functions inside the limit by

$$\tilde{F}_n(t) := \frac{\log(2)}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^n (-1)^k \binom{n}{k} (\mathcal{L}(f)) \left(\frac{\log(2)(n+k)}{t} \right).$$

Another sequence of weights was suggested by Stehfest. He showed that it is possible to approximate $f(t)$ via

$$\hat{F}_N(t) := \sum_{k=1}^N w(k, N) \tilde{F}_k(t), \quad w(k, N) := \frac{(-1)^{N-k} k^N}{k!(N-k)!}.$$

Application: Laplace approximation of first-passage times

Relevant for pricing bonds and credit derivatives is the distribution of the time of default τ . In a structural default model, this translates in finding the probability of the firm-value process not to pass a certain barrier over a given period of time. In Chapter 4, we express these probabilities in terms of first-passage times τ_b of a jump-diffusion process X . In this context, the threshold b is linked to the firm's initial leverage ratio d/v_0 via the relation $b = \log(d/v_0)$. If the jump-size distribution of X is assumed to be two-sided exponential, then it is possible to compute the Laplace transform of $\mathbb{P}(\tau_b \leq t)$, as shown by Kou and Wang (2003). This Laplace transform involves the roots of a function $G(x) - \alpha$, which is related to the moment-generating function of X_t . At this point, we omit further details and refer the reader to Section 4.6.

Algorithm 2.2.1 (Laplace approximation of first-passage times)

We assume as given a jump-diffusion process X with $\mathbb{P}_Y = 2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$. The stopping time τ_b is defined as in Equation (2.1). Based on Lemma 2.2.2, we

then approximate $\mathbb{P}(\tau_b \leq t)$ via

$$\hat{\mathbb{P}}_N(\tau_b \leq t) := \sum_{k=1}^N w(k, N) \tilde{F}_{k+B}(t) \approx \mathbb{P}(\tau_b \leq t).$$

For more stability, Kou and Wang (2003) propose to skip some of the first values of $\tilde{F}_k$. They suggest to choose $B = 2$ or $B = 3$ as burning-out number, on which we comment below.

The approximation of Algorithm 2.2.1 converges very quickly. In fact, we found that N chosen from $\{7, \dots, 10\}$ provides the best results for inverting transforms of first-passage times. However, the algorithm is extremely sensitive with respect to the precision of which the roots of $G(x) - \alpha$ are calculated. For more numerical stability, we suggest rewriting this problem in terms of a quartic polynomial. An algebraic expression of the roots of a quartic polynomial is provided by Ferrari's algorithm, which is presented in Section 6.2. If instead a numerical routine is used, one has to carefully choose the terminating condition to guarantee a high precision. We found it difficult to perpetuate a sufficient precision for $N \geq 12$, compare the last row of Table 2.2. Moreover, the results of several of our experiments do not support a better performance if the burning out number $B = 2$ is used. We found that if $B > 0$, the algorithm provides better results for small values of N . On the other side, larger values of B imply an increasing sensitive of the algorithm with respect to the precision of the roots. The result are massive rounding errors for larger values of N .

	$B = 0$		$B = 1$		$B = 2$		$B = 5$	
N	Polyn.	Pegas.	Polyn.	Pegas.	Polyn.	Pegas.	Polyn.	Pegas.
1	.769618	.769618	.791044	.791044	.799781	.799781	.809496	.809496
3	.819645	.819645	.816698	.816698	.815467	.815467	.815259	.815259
5	.820272	.820272	.820084	.820084	.819701	.819701	.818776	.818776
7	.820214	.820214	.820215	.820215	.820185	.820185	.819915	.819925
8	.820211	.820211	.820212	.820213	.820210	.820208	.820279	.820211
9	.820210	.820211	.820219	.820212	.820190	.820211	.827487	.818163
10	.820232	.820213	.820149	.820210	.820025	.820147	.502319	.850449
11	.820031	.820206	.819783	.820045	.823876	.822291	6.73095	.473994
12	.819259	.819782	.830194	.825787	.938046	.779469	-43.289	3.40704

Table 2.2: $\mathbb{P}(\tau > 5)$ for different values of N .

Table 2.2 compares the results of Algorithm 2.2.1 for different values of N , different burning-out numbers B and two different methods of finding the required roots of $G(x) - \alpha$. More precisely, we implemented the Pegasus algorithm as an example

for a numerical approach and Ferrari's algebraic solution of a quartic polynomial. For this example, the parameters of the process X are $\gamma = 0.025$, $\sigma = 0.05$, $\lambda = 2$, $\mathbb{P}_Y = 2\text{-Exp}(20, 20, \frac{1}{2})$ and $d/v_0 = 80\%$. A Monte Carlo simulation with Algorithm 4.2.1, based on 100,000 runs, estimates $\mathbb{P}(\tau > 5)$ by 0.820281.

Chapter 3

Products and pricing issues

3.1 Corporate bonds

Bonds are debt securities which are traded on fixed income markets. As compensation for granting a loan to the issuer of the bond, the bondholder receives periodic premium payments, called coupons, and the principal of the bond at maturity. Bonds are often classified according to their issuer in government and corporate bonds. However, when the fair price of a bond has to be assessed, the relevant classification criteria is the credit quality. While government bonds, such as German *Bundesschatzbriefe* or American *Treasure notes*, are considered to be default-free investments, corporate bonds and government bonds of emerging markets are subject to the possibility of credit default of the issuer. Consequently, the holder of a defaultable bond demands a higher interest rate for bearing this credit risk. This additional interest, called credit spread, primarily¹ depends on the market's view of the creditworthiness of the issuer. In what follows, we focus on the analysis and pricing of corporate bonds.

If a company defaults, its management loses control and the remaining assets are liquidated. The revenues of the liquidation are then distributed pro rata² in accordance to the invested principal of each bondholder. This fraction of the principal is referred to as recovery rate and abbreviated as $R \in [0, 1]$ in our model. In what follows, we concentrate on the pricing of defaultable zero-coupon bonds³, as any coupon bond can be replicated using a portfolio of zero-coupon bonds⁴.

¹ Other factors, such as liquidity risk, play a subsidiary role in explaining credit spreads.

² If no priority rule was specified.

³ The payment schedule of a zero-coupon bond only consists of the final payment, which we standardize to one unit of the respective currency.

⁴ The converse, separating coupons from final payment, is called *stripping of a bond*.

More precisely, we use the following lemma, which is a direct consequence of the no-arbitrage condition.

Lemma 3.1.1 (Replicating a coupon bond)

The payoff of a coupon bond with face value⁵ F and coupon payments $q_1, \dots, q_n$ at the times $0 < t_1 < \dots < t_n = T$ agrees with the payoff of a portfolio consisting of q_j zero-coupon bonds with maturity t_j for $j \in \{1, \dots, n\}$ and F zero-coupon bonds with maturity T .

Therefore, we focus on the analysis of zero-coupon bonds, even if most corporate bonds are bonds with promised coupon payments. In what follows, we assume as given a pricing measure $\mathbb{P}$ and discount all future payments using the flat interest rate $r \geq 0$ ⁶. The time of default is denoted by τ . This random variable and its distribution are specified in Chapter 4 by our structural default model. The fair price of a defaultable zero-coupon bond is then given as the expectation, given the investor's information $\mathbb{F}$, of its discounted payoff with respect to the pricing measure $\mathbb{P}$. Without loss of generality, we assume a unit face value. If the company survives until maturity, the payoff of the zero-coupon bond is one. Otherwise, the investor receives the fraction R at the time of default.

Lemma 3.1.2 (Price of a zero-coupon bond)

We denote by $\phi(t, T)$ the fair price of a defaultable zero-coupon bond at time t with maturity T . As long as $\tau > t$, this price satisfies

$$\phi(t, T) = e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{F}_t) + \mathbb{E} \left[e^{-r(\tau-t)} R \mathbf{1}_{\{t < \tau \leq T\}} \middle| \mathcal{F}_t \right]. \quad (3.1)$$

The credit spread which corresponds to $\phi(0, T)$ is denoted by η_T . It is defined as the real number that solves the equation

$$\phi(0, T) = \exp(-(r + \eta_T)T). \quad (3.2)$$

Alternative recovery schemes

Let us briefly remark that sometimes other recovery schemes are used to model the payoff at τ . In Equation (3.1), we assumed *fractional recovery of face value*, which is the scheme used by *Moody's* and *Standard & Poor's*. Here, bondholders

⁵ The term face value is used synonym to the term principal of a bond. It is the promised final payment at maturity.

⁶ The interest rate r is understood as a continuously compounded interest rate. For instance, a payment at maturity T is discounted by $\exp(-rT)$.

receive the fraction R of the bond's face value at the time of default. The scheme *fractional recovery of treasury value* defers the same payoff to the maturity of the bond. This assumption is unrealistic, but simplifies the evaluation of the pricing formula in first-passage models. *Fractional recovery of market value* implies that bondholders receive some fraction of the bond price immediately prior to default, i.e. $R \cdot \phi(\tau-, T)$, which is often used in intensity-based models. Finally, structural models suggest the scheme *fractional recovery of firm value*, where the recovery rate at τ is a function of the firm value V_τ . We generalize our model to allow for this scheme in Section 4.5.1.

3.2 Credit default swaps

Intuitively, credit default swaps (CDS) are insurance policies against the credit risk of some reference company. More precisely, these are contracts between a protection buyer and a protection seller, whereby the protection buyer makes periodic premium payments over a predetermined number of years and the protection seller commits to make a payment in the event of credit default of the reference entity. This contractual structure suggests that the term structure of default probabilities of the reference entity is the crucial factor in pricing CDS. In our context, we implicitly define the term structure of default probabilities by means of a structural default model.

CDS are the most important credit derivatives, in both market activity and notional amount⁷. In addition to the original idea of buying or selling default risk, CDS are interesting building blocks for different portfolio strategies and complex credit derivatives. For instance, CDS are often used to build a short position in credit risk. A comprehensive introduction to the common use and to contractual variants of CDS is presented in Bomfim (2005).

Equipped with the structure of an insurance policy against the default risk of a reference entity, pricing CDS is sometimes interpreted as an application of the actuarial principle of equivalence under the pricing measure, meaning that the premium is chosen such that the expected discounted payoffs of both contractual parties agree. A common simplification is to assume that the insurance buyer continuously⁸ pays the spread c as long as the reference entity is solvent, whereas the insurance seller indemnifies the insurance buyer by paying the difference of face value minus recovery in the event of credit default.

We again discount all future payments using the flat interest rate $r \geq 0$ and assume as given the pricing measure $\mathbb{P}$. With τ , denoting the time of default of the reference entity, we obtain the following expression for the price of a contract⁹ with face value one, continuous premium payments c and maturity T .

$$\begin{aligned} CDS(0, T) &= \mathbb{E} \left[e^{-r\tau} (1 - R) \mathbf{1}_{\{\tau \leq T\}} - \int_0^T c e^{-rt} \mathbf{1}_{\{\tau > t\}} dt \right] \\ &= (1 - R) \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t) - c \int_0^T e^{-rt} \mathbb{P}(\tau > t) dt. \end{aligned} \quad (3.3)$$

⁷ According to the *British Bankers Association*, compare Bomfim (2005), page 20.

⁸ This is easily generalized to periodic premium payments by replacing the integral with a sum over all premium dates in the second integral of Equation (3.3).

⁹ This formula reflects the view of the insurance buyer, the insurance seller uses the same formula with opposite signs.

Market prices for CDS contracts are typically not quoted as in Equation (3.3). Instead, the spread which allows both parties to enter the contract at zero cost is quoted¹⁰. This spread is called par spread¹¹ and obtained from solving Equation (3.3) for c . The par spread of a CDS with maturity T is considered as a function of the maturity and given by

$$c_T = \frac{(1 - R) \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t)}{\int_0^T e^{-rt} \mathbb{P}(\tau > t) dt}. \quad (3.4)$$

¹⁰ More details about *iTraxx* quotes are given in Section 4.9.2.

¹¹ Synonymously, the terms *fair spread* or *CDS spread* are used.

3.3 Collateralized debt obligations

Belonging to the class of asset-backed securities, the idea behind collateralized debt obligations (CDOs) is to pool financial assets which are subject to credit risk¹² and resell the resulting portfolio in several tranches with different seniority. CDOs first appeared at around 1990 and are now among the fastest growing credit derivatives. From a mathematical point of view, a CDO contract is of special interest, as its price is determined to large extent by the correlation among the pooled companies. A capacious introduction to CDOs has been published by one of the important market participants: *JP Morgan*, compare Lukas (2001). Besides the rationale of CDOs, this handbook covers legal issues, taxation and accounting questions.

CDOs can be classified, by purpose of the issuing party, in balance sheet and arbitrage CDOs. Balance sheet CDOs are typically issued by commercial banks or insurance companies who want, or have to, transfer some risk from their balance sheet. This is achieved by issuing debt against the CDO portfolio. Arbitrage CDOs aim to profit from reselling the securitized assets to investors who search for tranches that suit their personal risk-profile. Often, senior tranches¹³ of the portfolio are sold and riskier tranches are kept.

The most subordinate tranche in a CDO is usually called equity tranche and sustains all credit losses as long as the overall portfolio loss does not exceed the nominal value of this first tranche. Then, losses affect the second tranche, and so on. Investors who expose themselves to the credit risk of a certain tranche are compensated by receiving periodic premium payments until the maturity of the CDO. Those premium payments depend on the riskiness and the remaining nominal of the respective tranche. This construction of seniority, sometimes illustrated as a cascading waterfall, implies that premium payments are strictly decreasing in the seniority of a tranche.

The credit risk of each tranche is affected by the following components: Idiosyncratic default probabilities¹⁴ of the pooled assets and the default correlation among them. While markets have developed a deep understanding and a vast toolbox for the description of single-name derivatives, the valuation of CDO contracts challenges us with finding a realistic model for the dependence structure among the

¹² Prevailing assets are corporate bonds, commercial loans and mortgages. In the first two cases, the terms CBO and CLO are used synonym to CDO. A different variant is the class of synthetic CDOs which consist of a portfolio of CDS contracts. This construction does not require the issuer of the CDO to own the securitized assets.

¹³ The term senior tranches refers to the safest slices of the CDO portfolio.

¹⁴ Including the individual default severity.

securitized assets. At this point, we hope to learn from the fast growing market of synthetic CDOs. Whereas most traditional CDOs are traded over the counter, the launch of indices such as *iTraxx* and *DJ CDX* made it possible for us to implicitly observe the dependence structure among the listed companies.

In what follows, we introduce the basic notations of CDOs and describe the payment streams of a synthetic CDO. For simplicity, we assume the CDO contract to be newly issued today. We further assume that our portfolio consists of I CDS contracts, indexed by i , and is segmented in J tranches, which we index by j .

Name	Description
CDS^i	A CDS contract written on a bond of company i .
τ^i	The default time of company i .
N^i	The nominal value of CDS^i within the CDO.
R^i	The recovery rate of company i .
l^j, u^j	The lower and upper attachment point of tranche j .
M	The total nominal of the CDO.
M_t^j	The remaining nominal of tranche j at time t .
L_t	The cumulated loss of the CDO up to time t .
L_t^j	The cumulated loss in tranche j up to time t .
s^j	The annualized spread of tranche j .

Table 3.1: The basic notations related to CDOs.

Obviously, the total nominal value of the CDO is given by $M = \sum_{i=1}^I N^i$. This nominal value is segmented using the upper and lower attachment points of each tranche. More precisely, we define the partition¹⁵ $0 = l^1 < u^1 = l^2 < \dots < u^{J-1} = l^J < u^J = M$. Today, the market standard is defined by different synthetic portfolios of the *International Index Company*, called *iTraxx*, and the portfolio managed by *Dow Jones*, called *DJ CDX*. Table 3.2 lists their respective tranches.

$I = 125$ companies		<i>iTraxx</i>		<i>DJ CDX</i>	
Tranche	j	l^j in %	u^j in %	l^j in %	u^j in %
Equity	1	0%	3%	0%	3%
Junior mezzanine	2	3%	6%	3%	7%
Senior mezzanine	3	6%	9%	7%	10%
Senior	4	9%	12%	10%	15%
Super senior	5	12%	22%	15%	30%
	6	22%	100%	30%	100%

Table 3.2: The *iTraxx* and *DJ CDX* segmentation.

¹⁵ Instead of absolute values, these attachment points are often converted in percentages of the portfolio, as done in Table 3.2. However, all definitions of Table 3.1 are in absolute values.

The premium leg of tranche j

The premium leg of tranche j is calculated as follows. In the beginning, a payment schedule $0 < t_1 < \dots < t_n = T$ is specified. Most CDOs are based on quarterly premium payments. This implies that the payment frequency is set to $\eta = \frac{1}{4}$, and $t_k = k\eta$ for $k = 1, \dots, 4T$. At each payment date, the protection buyer is committed to pay the product of the remaining nominal value and the spread for this tranche corresponding to the length of the preceding period. Hence, the premium leg of tranche j is given by

$$PL^j = \sum_{k=1}^n s^j \Delta t_k M_{t_k}^j = \sum_{k=1}^n s^j \Delta t_k (u^j - l^j - L_{t_k}^j). \quad (3.5)$$

The corresponding expected discounted premium leg, based on a flat interest rate $r \geq 0$ and a given pricing measure $\mathbb{P}$, is then given by

$$EDPL^j = \sum_{k=1}^n s^j \Delta t_k e^{-rt_k} (u^j - l^j - \mathbb{E}[L_{t_k}^j]). \quad (3.6)$$

The default leg of tranche j

The default or protection leg of tranche j allows payments at any time up to maturity. A default payment becomes due if some company defaults, say company i , and the overall loss immediately before time τ^i is either within tranche j , i.e. $L_{\tau^i-} \in [l^j, u^j]$, or vaults into this tranche, i.e. $L_{\tau^i-} < l^j < L_{\tau^i}$. Formally, this is described as follows¹⁶

$$\begin{aligned} DL^j = & \sum_{i=1}^I \min \{ (1 - R^i) N^i, u^j - L_{\tau^i-} \} \mathbf{1}_{\{\tau^i \leq T\}} \mathbf{1}_{\{L_{\tau^i-} \in [l^j, u^j]\}} + \\ & \sum_{i=1}^I (L_{\tau^i} - l^j) \mathbf{1}_{\{\tau^i \leq T\}} \mathbf{1}_{\{L_{\tau^i-} < l^j < L_{\tau^i}\}}. \end{aligned} \quad (3.7)$$

The first sum collects all payments that occur while the overall loss is within tranche j , the minimum limits the last loss of this tranche to the remaining nominal value of the tranche. The second sum consists of at most one non-zero summand, which is the default that increases the overall loss into tranche j . In order to simplify calculations, it is often assumed that default payments are deferred to the next

¹⁶ At this point, we assume that the default of a single company can not cause a loss which exceeds the entire tranche j .

premium payment date. This assumption simplifies the default leg to

$$DL^j = \sum_{k=1}^n (L_{t_k}^j - L_{t_{k-1}}^j). \quad (3.8)$$

The expected discounted default leg corresponding to Equation (3.7) is given by

$$\begin{aligned} EDDL^j &= \sum_{i=1}^I \mathbb{E} \left[e^{-r\tau^i} \min \{ (1 - R^i) N^i, u^j - L_{\tau^i-} \} \mathbf{1}_{\{\tau^i \leq T\}} \mathbf{1}_{\{L_{\tau^i-} \in [l^j, u^j]\}} \right] + \\ &\quad \sum_{i=1}^I \mathbb{E} \left[e^{-r\tau^i} (L_{\tau^i} - l^j) \mathbf{1}_{\{\tau^i \leq T\}} \mathbf{1}_{\{L_{\tau^i-} < l^j < L_{\tau^i}\}} \right]. \end{aligned} \quad (3.9)$$

The fair spread of tranche j

Similar to CDS contracts, market prices for CDO tranches are quoted in terms of their fair spread. This fair spread is chosen such that the expected discounted default leg agrees with the expected discounted premium leg of the same tranche, which results in the following formula for the annualized fair spread of tranche j

$$s_f^j = \frac{EDDL^j}{\sum_{k=1}^n \Delta t_k e^{-rt_k} (u^j - l^j - \mathbb{E}[L_{t_k}^j])}.$$

The upfront payment

It has become market practice to modify the premium stream of the equity tranche using a fixed spread of 500 basis points¹⁷. To adjust for this artificial spread, which is typically below the fair spread of the equity tranche, an additional upfront payment is introduced which has to be paid as soon as the contract is settled. The amount of upfront payment is quoted in percent of the nominal of the first tranche. The fair upfront payment therefore satisfies the relation

$$(\text{upfront in } \%) \cdot (u^1 - l^1) + \sum_{k=1}^n 0.05 \Delta t_k e^{-rt_k} (u^1 - l^1 - \mathbb{E}[L_{t_k}^1]) = EDDL^1.$$

¹⁷ One basis point (bp) is equal to one hundredth of one percent.

Including accrued interest for defaulted companies

Depending on the terms of contract, accrued interest for defaulted companies is often stipulated. This means that if company i defaults within the premium payment dates t_{k-1} and t_k , i.e. $\tau^i \in (t_{k-1}, t_k)$, and the total loss at τ^i is within tranche j , then accrued interest for tranche j in the amount of $s^j \cdot (\tau^i - t_{k-1}) \cdot N^i$ has to be paid at t_k , additionally to the usual premium payment.

3.4 Portfolio CDS

Portfolio CDS¹⁸ are often introduced as a tranche of a synthetic CDO which covers the complete portfolio, that is $l^1 = 0\%$ and $u^1 = 100\%$. Indeed, depending on the terms of contract, this definition often holds. More precisely, the default leg compensates the insurance buyer for all losses from defaulted names in the portfolio, the premium leg is paid on the remaining nominal M of the portfolio. In what follows, we adopt the notations of Section 3.3 and assume a unit initial notional $M_0 = 1$ and I equally weighted companies, that is $N^i = 1/I$. Then, the expected discounted default leg is given by

$$EDDL = \mathbb{E} \left[\sum_{i=1}^I e^{-r\tau^i} N^i (1 - R^i) \mathbf{1}_{\{0 \leq \tau^i \leq T\}} \right].$$

Depending on the terms of contract, the default of CDS^i reduces the remaining nominal M of the portfolio by either $1/I$ or $(1 - R^i)/I$. The first alternative corresponds to the *iTraxx* convention, the second alternative corresponds to the interpretation of the portfolio CDS as a special tranche of a CDO¹⁹. Given the annualized spread s , the expected discounted premium leg satisfies

$$EDPL = \mathbb{E} \left[\sum_{k=1}^n s e^{-rt_k} \Delta t_k M_{t_k} \right].$$

Finally, the fair spread s_f is obtained by equating both legs and solving for s .

¹⁸ Also, the term credit index is often used.

¹⁹ As long as the number of defaulted companies is small, both assumptions yield very similar prices. This holds, as premium payments are relative to the large proportion of non-defaulted companies.

3.5 n^{th} -to default contracts

An n^{th} -to default contract is a portfolio derivative with payment streams depending on the time of the n^{th} default in the portfolio. Given the default times $\tau^1, \dots, \tau^I$, we denote their order statistics by $\tau_{(1)} \leq \dots \leq \tau_{(n)} \leq \dots \leq \tau_{(I)}$. Premium payments are due on the schedule $0 < t_1 < \dots < t_m = T$. However, these premium payments are conditional to fewer than n defaults. The amount of each premium payment is the product of the nominal value N of the contract, the time since the preceding premium payment in years²⁰, and the annualized spread $s^{(n)}$ of the n^{th} -to default contract. Also, it is possible to agree on accrued interest for the time $\tau_{(n)} - t_{k-1}$, where $t_{k-1} < \tau_{(n)} < t_k$. At the premium payment date after the n^{th} default, a default payment has to be paid in the amount of $(1 - R)$ times the nominal value N , where R is a pre-specified recovery rate. Hence, the expected discounted premium and default legs of an n^{th} -to default contract, without accrued interest payments, are given by

$$\begin{aligned}
 EDPL^{(n)} &= \mathbb{E} \left[\sum_{k=1}^m N s^{(n)} \Delta t_k e^{-rt_k} \mathbf{1}_{\{\tau_{(n)} > t_k\}} \right] \\
 &= N s^{(n)} \sum_{k=1}^m \Delta t_k e^{-rt_k} \mathbb{P}(\tau_{(n)} > t_k), \\
 EDDL^{(n)} &= \mathbb{E} \left[(1 - R) N \sum_{k=1}^m e^{-rt_k} \mathbf{1}_{\{t_{k-1} < \tau_{(n)} \leq t_k\}} \right] \\
 &= (1 - R) N \sum_{k=1}^m e^{-rt_k} \mathbb{P}(t_{k-1} < \tau_{(n)} \leq t_k).
 \end{aligned}$$

Finally, the fair spread $s_f^{(n)}$, which allows both parties to enter the contract at zero cost, is obtained from equating both legs and solving for $s^{(n)}$. This yields

$$s_f^{(n)} = \frac{(1 - R) \sum_{k=1}^m e^{-rt_k} \mathbb{P}(t_{k-1} < \tau_{(n)} \leq t_k)}{\sum_{k=1}^m \Delta t_k e^{-rt_k} \mathbb{P}(\tau_{(n)} > t_k)}.$$

²⁰ Typically, premium payments are made periodically with payment frequency $\eta = \frac{1}{4}$ years.

Chapter 4

The univariate model

In the preceding chapter, we introduced the economic fundamentals as well as the payment streams of corporate bonds and CDS contracts. To assess prices to these single-name contracts we have to specify the distribution of the time of default τ . In what follows, we explain τ by means of a first-passage model and derive algorithms to evaluate the respective pricing formulas.

To begin with, we introduce all objects of our structural default model in Section 4.1. Section 4.2 is then concerned with default probabilities and the local-default rate of τ , as implied by the model. An efficient Monte Carlo algorithm to evaluate bond and CDS prices is given in Sections 4.3 and 4.4, respectively. Several generalizations of the model are worked out in Section 4.5. The restriction to two-sided exponentially distributed jumps allows the approximation of bond and CDS prices based on the Laplace transform of first-passage times. This approach is explained in Section 4.6. The resulting algorithm enables us to present a detailed analysis of sensitivities with respect to the parameters of the model in Section 4.7. Both algorithms are compared in Section 4.8 to the algorithm of Zhou (2001a) in terms of run time and precision. It turns out that our algorithm is fast enough to implement a calibration of the model to bond and CDS data. The results of this calibration, as well as a comparison to a calibration of a pure diffusion model, are given in Section 4.9. Finally, Chapter 4 is summarized in Section 4.10.

4.1 Model description

We assume the value of the modeled company to start at some initial level $v_0 > 0$ and to evolve stochastically over time. Moreover, we want to incorporate small and unsystematic changes of the value process, resulting from daily business activities,

as well as sudden jumps due to unexpected events. Finally, we assume the total value of all assets to remain positive. Therefore, we model the value of a company as a stochastic process $V = \{V_t\}_{t \geq 0}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where

$$V_t = v_0 \exp(X_t) \quad v_0 > 0, \forall t \geq 0.$$

The process $X = \{X_t\}_{t \geq 0}$ is a jump-diffusion process as introduced in Definition 2.1.5 and given as

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i. \quad (4.1)$$

Throughout this chapter, we assume as given the pricing measure $\mathbb{P}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denotes the natural filtration of the firm-value process, i.e.

$$\mathcal{F}_t = \sigma(V_s : 0 \leq s \leq t) \vee \mathcal{N} = \sigma(X_s : 0 \leq s \leq t) \vee \mathcal{N},$$

augmented to satisfy the usual conditions of completeness and right continuity. To exclude degenerated cases, we impose the constraints $\sigma > 0$, $\lambda > 0$ and $\mathbb{P}_Y \neq \delta_0$ ¹. Following Black and Cox (1976), we define τ as the first passage of the firm-value process below the debt level of the company, which we denote by d . Formally, the time of default is defined by²

$$\tau := \inf\{t > 0 : V_t \leq d\}.$$

All model parameters and a brief interpretation are listed in the table below.

Parameter	Interpretation
$\gamma \in \mathbb{R}$	The linear trend of the diffusion component.
$\sigma \in \mathbb{R}^+$	The volatility of the diffusion component.
$\lambda \in \mathbb{R}_0^+$	The jump intensity.
$\mathbb{P}_Y$	The jump-size distribution.
$v_0 \in \mathbb{R}^+$	The initial value of the company.
$d \in \mathbb{R}^+$	The default threshold, satisfying $d < v_0$.

Table 4.1: The parameters of our default model.

¹ δ_x denotes the Dirac measure, i.e. $\delta_x(A) = \mathbf{1}_{\{x \in A\}}$ for all $A \in \mathcal{B}(\mathbb{R})$.

² In this scenario, τ is a stopping time with respect to $\mathbb{F}$, compare Section 2.1.2.

4.2 First-passage times

Formulating a first-passage model raises the natural question of calculating the probability for the firm-value process to remain above the default threshold for some interval of time. In our scenario, we first observe that this problem can be formulated in terms of the infimum of the process X , as

$$\mathbb{P}(\tau > t) = \mathbb{P}\left(\inf_{0 \leq s \leq t} V_s > d\right) = \mathbb{P}\left(\inf_{0 \leq s \leq t} X_s > \log(d/v_0)\right).$$

In what follows, we use the process $x_t := -\log(d/V_t)$ as distance to default for X ³. While the distribution of the running minimum of a Brownian motion is known explicitly, closed-form expressions are not available in a jump-diffusion scenario. To overcome this problem, we present an unbiased and extremely fast Monte-Carlo simulation in Section 4.2.3. This algorithm requires the distribution of the running minimum of a Brownian bridge, which is given below. The local default rate of τ is computed in Section 4.2.2 for a pure diffusion model, in Section 4.2.4 for a jump-diffusion model.

4.2.1 First-passage times in a pure diffusion scenario

In a pure diffusion scenario, the firm-value process simplifies to $V_t = v_0 \exp(\gamma t + \sigma W_t)$, which corresponds to $\lambda = 0$ in Equation (4.1). The pure diffusion model is not only a degenerated case of our model, the main reason for studying its properties is the idea of conditioning on the number, the location and size of possible jumps. As soon as these quantities are fixed, the remaining problem is to find the distribution of the running minimum of a Brownian motion and a Brownian bridge. The distribution of these functionals is known and given below.

The running minimum of a Brownian motion with drift is inverse Gaussian distributed, and so are first-passage times in a pure diffusion model. More precisely, we obtain the following result.

Lemma 4.2.1 (The minimum of a Brownian motion)

Let $X_t = \gamma t + \sigma W_t$ denote a Brownian motion with drift over the interval $[t_0, t_1]$, starting at X_{t_0} . Further, let $b \in \mathbb{R}$ denote an arbitrary barrier and define $\Delta t :=$

³ Let us remark that in a pure diffusion scenario, the distance to default is usually measured in terms of standard deviations. Here, the volatility of the process X does not only depend on σ , but also on the jump-measure of X . Therefore, we do not normalize the distance to default.

$t_1 - t_0$. Then, we obtain for the probability of X to remain above the threshold b

$$\begin{aligned}\Phi_{b,\gamma,\sigma}^{BM}(X_{t_0}, \Delta t) &:= \mathbb{P}\left(\min_{t_0 \leq s \leq t_1} (\sigma W_s + \gamma s) > b \mid X_{t_0}\right) \\ &= \mathbf{1}_{\{X_{t_0} > b\}} \Phi\left(\frac{X_{t_0} - b + \gamma \Delta t}{\sigma \sqrt{\Delta t}}\right) - \\ &\quad \mathbf{1}_{\{X_{t_0} > b\}} e^{-2\gamma(X_{t_0} - b)\sigma^{-2}} \Phi\left(\frac{-(X_{t_0} - b) + \gamma \Delta t}{\sigma \sqrt{\Delta t}}\right),\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution.

A proof of Lemma 4.2.1 is given in Musiela and Rutkowski (2004), page 61. With $x_0 = -\log(d/v_0)$ and $v_0 > d$ we thus obtain

$$\mathbb{P}(\tau > t) = \Phi\left(\frac{x_0 + \gamma t}{\sigma \sqrt{t}}\right) - e^{-2\gamma x_0 \sigma^{-2}} \Phi\left(\frac{-x_0 + \gamma t}{\sigma \sqrt{t}}\right). \quad (4.2)$$

The probability of a Brownian bridge not crossing a certain barrier b is calculated by Metwally and Atiya (2002). We present a simplified expression which is more convenient to work with. Similar results for the maximum of a Brownian bridge can be found in Borodin and Salminen (1996), page 61, or in Karatzas and Shreve (1997), page 265.

Lemma 4.2.2 (The minimum of a Brownian bridge)

Let $X = \{X_t\}_{t \in [t_0, t_1]}$ denote a Brownian bridge⁴ over $[t_0, t_1]$ with volatility σ , pinned at X_{t_0} and X_{t_1} . Let $b \in \mathbb{R}$ denote an arbitrary barrier. Then, we define

$$C_t := \{\omega \in \Omega : \{X_s(\omega)\}_{t_0 \leq s \leq t_1} \text{ passes } b \text{ for the first time in } [t, t + dt]\}$$

and obtain for $t \in (t_0, t_1]$

$$\begin{aligned}g(t)dt &:= \mathbb{P}(C_t \in dt \mid X_{t_0}, X_{t_1}) \\ &= \mathbf{1}_{\{X_{t_0} > b\}} \frac{X_{t_0} - b}{2y\pi\sigma^2(t - t_0)^{3/2}(t_1 - t)^{1/2}} \cdot \\ &\quad \exp\left(-\frac{(X_{t_1} - b)^2}{2(t_1 - t)\sigma^2} - \frac{(X_{t_0} - b)^2}{2(t - t_0)\sigma^2}\right) dt,\end{aligned} \quad (4.3)$$

where y is defined by

$$y := \frac{1}{\sqrt{2\pi\sigma^2(t_1 - t_0)}} \exp\left(-\frac{(X_{t_1} - X_{t_0})^2}{2\sigma^2(t_1 - t_0)}\right).$$

⁴ A definition and some properties of a Brownian bridge are given in Durrett (1999), page 245.

By integration, we obtain the probability of X falling below the barrier b

$$\begin{aligned}\tilde{\Phi}_{b,\sigma}^{BB}(X_{t_0}, X_{t_1}, \Delta t) &:= \mathbb{P} \left(\min_{t_0 \leq s \leq t_1} X_s \leq b \middle| X_{t_0}, X_{t_1} \right) \\ &= \mathbf{1}_{\{X_{t_0} \leq b \text{ or } X_{t_1} \leq b\}} + \\ &\quad \mathbf{1}_{\{X_{t_0} > b \text{ and } X_{t_1} > b\}} \exp \left(-\frac{2(X_{t_0} - b)(X_{t_1} - b)}{(t_1 - t_0)\sigma^2} \right).\end{aligned}$$

Finally, we define $\Phi_{b,\sigma}^{BB}(X_{t_0}, X_{t_1}, \Delta t) := 1 - \tilde{\Phi}_{b,\sigma}^{BB}(X_{t_0}, X_{t_1}, \Delta t)$ to be the probability of the Brownian bridge to remain above the threshold b within the interval $[t_0, t_1]$.

4.2.2 The local default rate in a pure diffusion scenario

It is a common property of first-passage models that the probability for the firm to default within h units of time tends to zero in h . What distinguishes a jump-diffusion model from a pure diffusion model is the rate of convergence. We shall see that the limit of credit spreads at the short end of the term structure essentially depends on the local default rate, which is defined below.

Definition 4.2.1 (The local default rate of τ)

The local default rate of τ is abbreviated as LDR_τ and defined by

$$LDR_\tau := \lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(\tau \leq h).$$

In pure diffusion models, the local default rate of τ is zero. More precisely, we apply Lemma 4.2.1, l'Hospital's rule and some algebraic manipulations to find the following result for a solvent company, for which $x_0 = -\log(d/v_0) = -b > 0$.

$$\begin{aligned}LDR_\tau &= \lim_{h \searrow 0} \frac{1}{h} (1 - \Phi_{b,\gamma,\sigma}^{BM}(0, h)) \\ &= \lim_{h \searrow 0} \frac{1}{h} \left(1 - \Phi \left(\frac{-b + \gamma h}{\sigma \sqrt{h}} \right) + e^{2\gamma b \sigma^{-2}} \Phi \left(\frac{b + \gamma h}{\sigma \sqrt{h}} \right) \right) \\ &= \lim_{h \searrow 0} \frac{-b}{\sqrt{2\pi}\sigma h^{3/2}} \exp \left(-\frac{1}{2} \left(\frac{-b + \gamma h}{\sigma \sqrt{h}} \right)^2 \right) \\ &= 0.\end{aligned}\tag{4.4}$$

It turns out that this fact forces credit spreads of zero-coupon bonds in a pure diffusion model to tend to zero in the maturity, as shown by Duffie and Lando (2001). Later, we show that allowing negative jumps in the firm-value process

results in a positive local default rate and a positive limit of credit spreads for short maturities. This positive limit of credit spreads coincides with empirical observations of bond and CDS spreads for contracts with short maturities.

4.2.3 First-passage times in a jump-diffusion scenario

As mentioned earlier, closed-form expressions of the distribution of first-passage times are not known in a jump-diffusion scenario. A probabilistic approach to estimating survival probabilities is to perform a Monte Carlo simulation. A naïve approach would require one to sample the firm-value process on a discrete grid and test for default on this grid. Not only is this computationally expensive, it also implies a systematic discretisation bias, as a possible default in between two grid points is not considered.

The algorithm we propose is a variant of an algorithm for pricing barrier options. A description in this context can be found in Cont and Tankov (2004), page 176, or in Metwally and Atiya (2002). The idea of our algorithm is as follows. To efficiently estimate passage probabilities of the jump-diffusion process X over the interval $[0, T]$ it is sufficient to simulate the number of jumps N_T , their location $0 < \tau_1 < \dots < \tau_{N_T} < T$ and the process X , respectively its left limit X_- , at those times. In a second step, we calculate the probability for Brownian bridges that connect those jumps not to fall below the passage threshold b . More precisely, we rewrite the survival probability of the process X conditioned on the number of jumps. This gives

$$\mathbb{P}\left(\inf_{0 \leq s \leq T} X_s > b\right) = \sum_{k=0}^{\infty} \mathbb{P}\left(\inf_{0 \leq s \leq T} X_s > b \mid N_T = k\right) \mathbb{P}(N_T = k). \quad (4.5)$$

Knowing the number of jumps $N_T = k$ allows us to further rewrite $\mathbb{P}(\inf_{0 \leq s \leq T} X_s > b \mid N_T = k)$ by conditioning on the location of the jumps $0 < \tau_1 < \dots < \tau_k < T$, the size of the jumps $y_1, \dots, y_k$ and the increments of the pure diffusion in between two jumps $x_1, \dots, x_k$. Conditioned on the number of jumps, the jump times are distributed as order statistics of $N_T = k$ independent $\text{Uni}(0, T)$ distributed random variables, compare Sato (1999), page 17. Jump sizes are assumed to be i.i.d. with distribution $\mathbb{P}_Y$ and the increments of the pure diffusion are normally distributed with mean $\gamma \Delta \tau_j$ and variance $\sigma^2 \Delta \tau_j$. For $k \geq 1$, this yields for the conditional survival probability of Equation (4.5)

$$\int_{\substack{(\tau_1, \dots, \tau_k) \\ \in [0, T]^k}} \int_{\substack{(x_1, \dots, x_k) \\ \in (-\infty, \infty)^k}} \int_{\substack{(y_1, \dots, y_k) \\ \in (-\infty, \infty)^k}} \Phi_{b, \gamma, \sigma}^{BM}(X_{\tau_k}, T - \tau_k) \cdot \prod_{j=1}^k \Phi_{b, \sigma}^{BB}(X_{\tau_{j-1}}, X_{\tau_j}, \Delta \tau_j).$$

$$\prod_{j=1}^k \mathbb{P}_Y(dy_j) \cdot \prod_{j=1}^k \varphi_{\gamma\Delta\tau_j, \sigma^2\Delta\tau_j}(x_j) dx_j \cdot \mathbf{1}_{\{0 < \tau_1 < \dots < \tau_k < T\}} \frac{k!}{T^k} d(\tau_1, \dots, \tau_k),$$

where $\tau_0 = 0$, $\Delta\tau_j = \tau_j - \tau_{j-1}$ and φ_{μ, σ^2} denotes the density of a normal distribution with the respective parameters. This artificial reformulation of the survival probability does not only motivate the following Monte Carlo simulation, it also shows that the algorithm is unbiased.

Algorithm 4.2.1 (Monte Carlo simulation of first-passage times)

Repeat the following steps K times and calculate the average over the resulting conditional survival probabilities $\{SP_n\}_{n=1, \dots, K}$. We then obtain the estimate

$$\mathbb{P}(\tau_b > T) \approx \frac{1}{K} \sum_{n=1}^K SP_n.$$

1. Simulate the compound Poisson process and the diffusion component of X .

- (a) Simulate the number of jumps N_T from a $\text{Poi}(\lambda T)$ distribution.
- (b) Simulate the jump times $0 < \tau_1 < \dots < \tau_{N_T} < T$. As mentioned before, conditional on N_T , these jump times are distributed as order statistics of $\text{Uni}(0, T)$ distributed random variables. Therefore, it is sufficient to simulate N_T independent $\text{Uni}(0, T)$ distributed random numbers and rearrange them in increasing order.
- (c) Generate two series of random numbers $x_1, \dots, x_{N_T}$, the increments of the diffusion in between two jumps, and $y_1, \dots, y_{N_T}$, the jump sizes of the compound Poisson process. All random numbers are mutually independent and independent of N_T and $\tau_1, \dots, \tau_{N_T}$. Further, they are distributed as follows

$$\begin{aligned} x_j &\sim \mathcal{N}(\gamma\Delta\tau_j, \sigma^2\Delta\tau_j), \\ y_j &\sim \mathbb{P}_Y. \end{aligned}$$

- (d) Calculate successively $X_0, X_{\tau_1-}, X_{\tau_1}, X_{\tau_2-}, \dots, X_{\tau_{N_T}}$ by

$$\begin{aligned} X_{\tau_0} &= 0, \\ X_{\tau_j-} &= X_{\tau_{j-1}} + x_j, \quad \forall j \in \{1, \dots, N_T\}, \\ X_{\tau_j} &= X_{\tau_j-} + y_j, \quad \forall j \in \{1, \dots, N_T\}. \end{aligned}$$

2. Calculate each conditional survival probability SP_n .

(a) Let $\mathcal{F}^*$ be given by

$$\mathcal{F}^* = \sigma \{N_T; 0 < \tau_1 < \dots < \tau_{N_T} < T; X_0, \dots, X_{\tau_j-}, X_{\tau_j}, \dots, X_{N_T}\}.$$

(b) Calculate the probability of each Brownian bridge connecting $X_{\tau_{j-1}}$ with X_{τ_j-} and the Brownian motion starting at $X_{\tau_{N_T}}$ not to fall below the threshold b . This yields

$$\begin{aligned} SP_n &:= \mathbb{P} \left(\inf_{0 \leq s \leq T} X_s > b \middle| \mathcal{F}^* \right) \\ &= \Phi_{b, \gamma, \sigma}^{BM}(X_{\tau_{N_T}}, T - \tau_{N_T}) \cdot \prod_{j=1}^{N_T} \Phi_{b, \sigma}^{BB}(X_{\tau_{j-1}}, X_{\tau_j-}, \Delta\tau_j). \end{aligned}$$

4.2.4 The local default rate in a jump-diffusion scenario

In this section, we derive the local default rate of τ in a jump-diffusion environment. The first version of this result is given in Scherer (2005) for the special case $\mathbb{P}_Y = 2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$. Then, we derived a generalization to continuous jump-size distributions as described in Example 4.2.1. Finally, a further generalization to arbitrary jump-size distributions was worked out with Johannes Ruf. A detailed version of the proof of Theorem 4.2.1 is given in Ruf (2006).

Theorem 4.2.1 (The local default rate of τ)

Let F_Y denote the cumulative jump-size distribution function. At time zero, the distance to default for X is given by $x_0 = -\log(d/v_0)$. We then obtain

$$LDR_{\tau} = \lambda F_Y((-x_0)-) + \frac{\lambda}{2} \mathbb{P}(Y = -x_0).$$

If the jump-size distribution is absolutely continuous, this simplifies to

$$LDR_{\tau} = \lambda F_Y(-x_0) = \nu((-\infty, -x_0]),$$

where ν denotes the Lévy measure of X . This shows that the local default rate is determined by the Lévy measure of the logarithm of the firm-value process and the distance to default.

Proof: We condition on the number N_h of jumps which occurred up to time h and denote the first jump time by $\tau(h)$. We obtain

$$\lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(\tau \leq h)$$

$$\begin{aligned}
&= \lim_{h \searrow 0} \frac{1}{h} \sum_{n=0}^{\infty} \mathbb{P}(N_h = n) \mathbb{P} \left(\inf_{0 \leq s \leq h} X_s \leq -x_0 \middle| N_h = n \right) \\
&= \lim_{h \searrow 0} \frac{e^{-\lambda h}}{h} \mathbb{P} \left(\inf_{0 \leq s \leq h} (\gamma s + \sigma W_s) \leq -x_0 \right) + \\
&\quad \lim_{h \searrow 0} \lambda e^{-\lambda h} \mathbb{P} \left(\inf_{0 \leq s \leq h} (\gamma s + \sigma W_s + \mathbf{1}_{\{s \geq \tau(h)\}} Y_1) \leq -x_0 \right) + \\
&\quad \lim_{h \searrow 0} \frac{1}{h} \sum_{n=2}^{\infty} \frac{e^{-\lambda h} (\lambda h)^n}{n!} \mathbb{P} \left(\inf_{0 \leq s \leq h} \left(\gamma s + \sigma W_s + \sum_{j=1}^{N_s} Y_j \right) \leq -x_0 \middle| N_h = n \right).
\end{aligned} \tag{4.6}$$

The first limit, representing a pure diffusion setup, is zero by l'Hospital's rule⁵. Considering the last limit, a dominated convergence argument allows us to interchange limit and summation, establishing that this limit also equals zero. We now examine the second limit, the case of exactly one jump. Writing $B_s := \gamma s + \sigma W_s$, $A_t(x) := \{\omega \in \Omega : \inf_{0 \leq s < t} B_s(\omega) \leq x\}$ and $A_t^C(x) := \Omega \setminus A_t(x)$ for brevity, we obtain by conditioning

$$\begin{aligned}
&\mathbb{P} \left(\inf_{0 \leq s \leq h} (B_s + \mathbf{1}_{\{s \geq \tau(h)\}} Y_1) \leq -x_0 \right) \\
&= \mathbb{P} (A_{\tau(h)}(-x_0)) + \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{A_{\tau(h)}^C(-x_0) \cap \tilde{A}_{h-\tau(h)}(-x_0 - Y_1 - B_{\tau(h)})} \middle| B_{\tau(h)}, Y_1 \right] \right],
\end{aligned}$$

where $\tilde{A}_t(x)$ is defined as $A_t(x)$ with B being replaced by the Brownian motion $\tilde{B}_s := B_{\tau(h)+s} - B_{\tau(h)}$. Since $\tau(h) \leq h$ holds, the limit of the first term tends to zero with h . If $Y_1 > -x_0$, the conditional expectation tends to zero, since $\mathbb{P}(B_{\tau(h)} \leq -x_0 - y)$ decreases to zero for all $y > -x_0$ for h tending to zero, due to the continuity of the diffusion part. If $Y_1 < -x_0$, the conditional expectation tends to one, since so does $\mathbb{P}(B_{\tau(h)} \leq -x_0 - y)$ for all $y < -x_0$ for h tending to zero. If $Y_1 = -x_0$ then the conditional expectation tends to zero if $B_{\tau(h)} > 0$, and to one if $B_{\tau(h)} \leq 0$ with h . $\diamond$

This result can be interpreted as follows. If a negative jump exceeds the distance to default with positive probability, that is $F_Y((-x_0)-) > 0$, then the local default rate is positive. Based on this local default rate we are later able to calculate the exact limit of credit spreads as maturity decreases to zero.

Example 4.2.1 (Continuous jump-size distribution)

This example is designed to illustrate the final step of the proof of Theorem 4.2.1. For simplicity, we assume a continuous jump-size distribution, for instance $\mathbb{P}_Y = 2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$ or $\mathbb{P}_Y = \mathcal{N}(0, \hat{\sigma}^2)$. This simplification allows us to present an alternative proof, which we feel is more intuitive. We proceed as in Equation (4.6),

⁵ Compare Equation (4.4).

except for the case of exactly one jump, where we additionally condition on whether this jump is negative or not. Obviously, only the case of a negative jump is of interest. We let $B_t := \gamma t + \sigma W_t$ and find

$$\begin{aligned} \mathbb{P}(B_h + Y_1 \leq -x_0 | Y_1 < 0) \\ \leq \mathbb{P}\left(\inf_{0 \leq s \leq h} (B_s + \mathbf{1}_{\{s \geq \tau_1\}} Y_1) \leq -x_0 \mid N_h = 1, Y_1 < 0\right) \\ \leq \mathbb{P}\left(\inf_{0 \leq s \leq h} B_s + Y_1 \leq -x_0 \mid Y_1 < 0\right). \end{aligned}$$

The sequence of events $A_h := \{\omega \in \Omega : \inf_{0 \leq s \leq h} B_s + Y_1 \leq -x_0, Y_1 < 0\}$ is decreasing in h . Therefore, by the continuity of the probability measure, we obtain the following result for the limit of upper bounds

$$\lim_{h \searrow 0} \mathbb{P}(A_h) = \mathbb{P}(A_0). \quad (4.7)$$

From Equation (4.7), it follows that

$$\lim_{h \searrow 0} \mathbb{P}\left(\inf_{0 \leq s \leq h} B_s + Y_1 \leq -x_0 \mid Y_1 < 0\right) = \mathbb{P}(Y_1 \leq -x_0 | Y_1 < 0).$$

Showing that this limit agrees with the limit of lower bounds is straightforward if $B_h + Y_1$ conditioned on $Y_1 < 0$ has a closed-form expression⁶. In general, this can be shown as follows. For arbitrary $c > 0$, we have

$$\begin{aligned} \liminf_{h \searrow 0} \mathbb{P}(B_h + Y_1 \leq -x_0 | Y_1 < 0) \\ \geq \liminf_{h \searrow 0} \mathbb{P}\left(B_h + Y_1 \leq -x_0, B_h \leq \gamma h + c\sqrt{h\sigma^2} \mid Y_1 < 0\right) \\ \geq \liminf_{h \searrow 0} \mathbb{P}\left(Y_1 \leq -x_0 - \gamma h - c\sqrt{h\sigma^2} \mid Y_1 < 0\right) \mathbb{P}\left(B_h \leq \gamma h + c\sqrt{h\sigma^2}\right) \\ = \mathbb{P}(Y_1 \leq -x_0 | Y_1 < 0) \Phi(c). \end{aligned}$$

Since $c > 0$ was chosen arbitrarily, we are allowed to let c tend to infinity and obtain the result. $\diamond$

Resuming with our initial example, if $\mathbb{P}_Y = 2\text{-Exp}(\lambda_\oplus, \lambda_\ominus, p)$ or $\mathbb{P}_Y = \mathcal{N}(0, \hat{\sigma}^2)$, the local default rate of τ is given by $\lambda(1-p)\exp(-x_0\lambda_\ominus)$ and $\lambda\Phi(-x_0/\hat{\sigma})$, respectively.

⁶ This holds for instance if $\mathbb{P}_Y = 2\text{-Exp}(\lambda_\oplus, \lambda_\ominus, p)$, compare Scherer (2005).

4.3 Pricing corporate bonds

The fair price of a zero-coupon bond is given in terms of its expected discounted cash flow with respect to the pricing measure $\mathbb{P}$, as motivated in Equation (3.1). The crucial factor of this formula is the distribution of the time of default τ . In the following sections, we provide different methods to evaluate the pricing formula of a zero-coupon bond, depending on the respective choice of firm-value process.

4.3.1 Pricing in a pure diffusion scenario

If the firm-value process is the exponential of a diffusion it is possible to explicitly evaluate Equation (3.1). To begin with⁷, we assume $t = 0$ and rewrite the pricing formula in terms of a discounted survival probability and an integral with respect to the distribution of τ . We obtain

$$\phi(0, T) = e^{-rT} \mathbb{P}(\tau > T) + R \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t). \quad (4.8)$$

This equation is simplified in Theorem 4.3.1 below.

Theorem 4.3.1 (Zero-coupon bond prices in a pure diffusion model)

The price of a defaultable zero-coupon bond with maturity T is given by

$$\phi(0, T) = e^{-rT} \Phi_{b, \gamma, \sigma}^{BM}(0, T) + R e^{-b(\tilde{\gamma} - \gamma)\sigma^{-2}} (1 - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(0, T)), \quad (4.9)$$

where $\tilde{\gamma} = \sqrt{\gamma^2 + 2r\sigma^2}$ and $b = \log(d/v_0)$.

Proof: In this scenario, the distribution of τ is explicitly known and given in Equation (4.2). This leads to the following formula for the default probability of the modeled company

$$\mathbb{P}(\tau \leq t) = \Phi\left(\frac{b - \gamma t}{\sigma\sqrt{t}}\right) + e^{2\gamma b\sigma^{-2}} \Phi\left(\frac{b + \gamma t}{\sigma\sqrt{t}}\right).$$

To evaluate the Riemann-Stieltjes integral of Equation (4.8) we make use of a lemma which is presented in Bielecki and Rutkowski (2002), page 74. For the reader's convenience, we restate it as Lemma 4.3.1 below.

Lemma 4.3.1

⁷ Generalizations are presented in Theorem 4.5.2.

For real numbers a , b and c , satisfying $b < 0$ and $c^2 > a$, we have

$$\int_0^y e^{ax} d\Phi\left(\frac{b-cx}{\sqrt{x}}\right) = \frac{d+c}{2d}g(y) + \frac{d-c}{2d}h(y),$$

with the abbreviations $d = \sqrt{c^2 - 2a}$ and

$$g(y) = e^{b(c-d)}\Phi\left((b-dy)y^{-1/2}\right), \quad h(y) = e^{b(c+d)}\Phi\left((b+dy)y^{-1/2}\right).$$

Finally, a rather tedious than complicated calculation using Lemma 4.3.1 leads to

$$\begin{aligned} \int_0^T e^{-rt} d\Phi\left(\frac{b-\gamma t}{\sigma\sqrt{t}}\right) &= \frac{1}{2}\left(1 + \frac{\gamma}{\tilde{\gamma}}\right) e^{b(\gamma-\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b-\tilde{\gamma}T}{\sigma\sqrt{T}}\right) + \\ &\quad \frac{1}{2}\left(1 - \frac{\gamma}{\tilde{\gamma}}\right) e^{b(\gamma+\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b+\tilde{\gamma}T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Moreover, using the same result, it holds that

$$\begin{aligned} e^{2\gamma b\sigma^{-2}} \int_0^T e^{-rt} d\Phi\left(\frac{b+\gamma t}{\sigma\sqrt{t}}\right) &= \frac{1}{2}\left(1 - \frac{\gamma}{\tilde{\gamma}}\right) e^{b(\gamma-\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b-\tilde{\gamma}T}{\sigma\sqrt{T}}\right) + \\ &\quad \frac{1}{2}\left(1 + \frac{\gamma}{\tilde{\gamma}}\right) e^{b(\gamma+\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b+\tilde{\gamma}T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Combined, this yields

$$R \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t) = R \left(e^{b(\gamma-\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b-\tilde{\gamma}T}{\sigma\sqrt{T}}\right) + e^{b(\gamma+\tilde{\gamma})\sigma^{-2}} \Phi\left(\frac{b+\tilde{\gamma}T}{\sigma\sqrt{T}}\right) \right).$$

Some rearrangements based on Lemma 4.2.1 complete the proof. $\diamond$

4.3.2 Pricing in a jump-diffusion scenario

Zhou (2001a) suggested an algorithm for calculating bond prices in a jump-diffusion framework. His idea is to discretize the time to maturity and to sample trajectories of the firm-value process on this grid. Then, on each grid point it is checked whether the company defaults or not. A detailed description of Zhou's algorithm is given in Section 6.3. However, we show in Section 4.8.2 that this algorithm produces biased bond prices and is very time-consuming. In what follows, we therefore propose a different algorithm.

We present an algorithm which not only produces unbiased results, but is also

significantly faster than Zhou's algorithm. The principal idea of our algorithm is to condition on the number of jumps of the firm-value process up to time T , the location of the jump times and the values of the jump-diffusion process at these times. We further generalize the algorithm to include stochastic recovery rates in Section 4.5.1 and to allow a stochastic short-rate process in Section 4.5.2. We begin with a theorem which motivates the algorithm which we introduce later. This theorem is a reformulation of the pricing formula in terms of conditional expectations.

Theorem 4.3.2 (Price of a zero-coupon bond)

The zero-coupon bond price of Equation (3.1) can be expressed as

$$\phi(0, T) = \mathbb{E} [\mathbb{E} [\mathbf{1}_{\{\tau > T\}} e^{-rT} + R \mathbf{1}_{\{\tau \leq T\}} e^{-r\tau} | \mathcal{F}^*]] \quad (4.10)$$

$$= \sum_{k=0}^{\infty} \int_{\substack{(\tau_1, \dots, \tau_k) \\ \in [0; T]^k}} \int_{\substack{(x_1, \dots, x_{k+1}) \\ \in (-\infty; \infty)^{k+1}}} \int_{\substack{(y_1, \dots, y_k) \\ \in (-\infty; \infty)^k}} \mathbb{E} [\mathbf{1}_{\{\tau > T\}} e^{-rT} + R \mathbf{1}_{\{\tau \leq T\}} e^{-r\tau} | \mathcal{F}^*] \cdot \\ \prod_{j=1}^k \mathbb{P}_Y(dy_j) \cdot \prod_{j=1}^{k+1} \varphi_{\gamma \Delta \tau_j, \sigma^2 \Delta \tau_j}(x_j) dx_j \cdot \\ \mathbf{1}_{\{0 < \tau_1 < \dots < \tau_k < T\}} \frac{k!}{T^k} d(\tau_1, \dots, \tau_k) \cdot \frac{(\lambda T)^k}{k!} e^{-\lambda T}, \quad (4.11)$$

where

$$\mathcal{F}^* := \sigma \{N_T; 0 < \tau_1 < \dots < \tau_{N_T} < T; X_{\tau_1-}, X_{\tau_1}, \dots, X_{\tau_i-}, X_{\tau_i}, \dots, X_T\} \quad (4.12)$$

is the σ -algebra representing the information from the number of jumps, their location and the values of X immediately before the jump times, at the jump times and at maturity. The function $\varphi_{\gamma \Delta \tau_j, \sigma^2 \Delta \tau_j}$ represents the density function of the normal distribution with mean $\gamma(\tau_j - \tau_{j-1})$ and variance $\sigma^2(\tau_j - \tau_{j-1})$, where $\tau_0 = 0$ and $\tau_{N_T+1} = T$. With $b = \log(d/v_0)$, the conditional expectation satisfies

$$\begin{aligned} & \mathbb{E} [\mathbf{1}_{\{\tau > T\}} e^{-rT} + R \mathbf{1}_{\{\tau \leq T\}} e^{-r\tau} | \mathcal{F}^*] \\ &= R \sum_{i=1}^U \prod_{j=1}^{i-1} \Phi_b^{BB}(j) \int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds + \\ & \quad R \mathbf{1}_{\{I \neq 0\}} e^{-r\tau_I} \prod_{j=1}^I \Phi_b^{BB}(j) + \mathbf{1}_{\{I=0\}} e^{-rT} \prod_{j=1}^{N_T+1} \Phi_b^{BB}(j), \end{aligned} \quad (4.13)$$

where

$$I := \min \{i \in \{1, \dots, N_T\} : X_{\tau_i} \leq b\}, \quad \min \emptyset := 0,$$

denotes the index of the first jump time such that X_{τ_I} crosses the barrier and

$$U := \begin{cases} I & \text{if } I \neq 0, \\ N_T + 1 & \text{if } I = 0. \end{cases}$$

Finally, $\Phi_b^{BB}(j) := \Phi_{b,\sigma}^{BB}(X_{\tau_{j-1}}, X_{\tau_j-}, \tau_j - \tau_{j-1})$ represents the probability of the company not defaulting within the interval (τ_{j-1}, τ_j) and $g_i(t)$ is defined as in Equation (4.3), with X_{t_0} and X_{t_1} replaced by $X_{\tau_{i-1}}$ and X_{τ_i-} , respectively.

Proof: Equation (4.10) is obtained from applying the tower property of conditional expectation on Equation (3.1) with $t = 0$. Considering the densities in Equation (4.11), the number of jumps up to time T follows a $\text{Poi}(\lambda T)$ distribution, conditioned on which the jump times are distributed as order statistics of $\text{Uni}(0, T)$ distributed random variables, compare Sato (1999), page 17. Finally, the increments of the diffusion component in between two jumps as well as the jump sizes are independent of each other and distributed $\mathcal{N}(\gamma \Delta \tau_j, \sigma^2 \Delta \tau_j)$ and $\mathbb{P}_Y$, respectively.

In what follows, we explain the three summands in Equation (4.13). The firm-value process is only sampled at τ_i and τ_i- , which opens three possibilities for the time of default. First of all, default is possible at some τ_i by jump, which corresponds to $X_{\tau_i-} > b$ and $X_{\tau_i} \leq b$. Then, it is possible that the diffusion component in between τ_{i-1} and τ_i- declines below the default threshold, that is $X_{\tau_{i-1}} > b$ and $X_{\tau_i-} \leq b$. Finally, we have to consider an unobserved default in between two jump times, which happens when the diffusion component starts and ends above b but declines below this threshold somewhere in (τ_{i-1}, τ_i) .

The case $I = 0$ corresponds to no default at some τ_i . However, we have to multiply the resulting discounted payoff by the probability of no default by diffusion prior to T , which equals $\prod_{j=1}^{N_T+1} \Phi_b^{BB}(j)$. Otherwise, default is caused by the I^{th} jump at time τ_I , which corresponds to a payoff of R , discounted by $e^{-r\tau_I}$, conditional on survivorship up to τ_I . Default by diffusion is considered via the first summand in Equation (4.13). Unobserved defaults have to be considered up to an observed default or up to maturity, whichever occurs first. The variable U is set accordingly. The conditional density of τ within (τ_{i-1}, τ_i) , given the start and endpoint of the Brownian bridge connecting $X_{\tau_{i-1}}$ with X_{τ_i-} , is denoted by g_i and defined as in Equation (4.3). Conditional on the firm's survivorship up to τ_{i-1} , the integral $\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds$ takes into account all possible recovery payments for defaults within (τ_{i-1}, τ_i) . $\diamond$

Finally, we notice that Equation (4.13) simplifies remarkably if the recovery scheme *fractional recovery of treasury value* is used, which corresponds to deferring the re-

covery payment to the bond's maturity T . In this case, $e^{-r\tau}$ is replaced by e^{-rT} and $\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds$ by $\Phi_b^{BB}(i)$ in Equation (4.13).

Based on Theorem 4.3.2, we now formally introduce our Brownian-bridge pricing algorithm.

Algorithm 4.3.1 (Brownian-bridge pricing algorithm)

Choose the number of simulation runs K and approximate $\phi(0, T)$ by

$$\phi(0, T) \approx \frac{1}{K} \sum_{j=1}^K \phi_j(0, T),$$

where each $\phi_j(0, T)$ is calculated by the following steps.

1. Simulate the number of jumps N_T from a $Poi(\lambda T)$ distribution.
2. Simulate the jump times $0 < \tau_1 < \tau_2 < \dots < \tau_{N_T} < T$. Conditioned on N_T , these jump times are distributed as order statistics of $Uni(0, T)$ distributed random variables on $[0, T]$.
3. Generate two series of mutually independent random numbers $x_1, \dots, x_{N_T+1}$ and $y_1, \dots, y_{N_T}$, independent from N_T and $\tau_1, \dots, \tau_{N_T}$, with

$$\begin{aligned} x_i &\sim \mathcal{N}(\gamma(\tau_i - \tau_{i-1}), \sigma^2(\tau_i - \tau_{i-1})), \\ y_i &\sim \mathbb{P}_Y. \end{aligned}$$

4. Calculate successively $X_0, X_{\tau_1-}, X_{\tau_1}, X_{\tau_2-}, \dots, X_{\tau_{N_T}}, X_{\tau_{N_T}+1-} = X_{\tau_{N_T}+1}$ by

$$\begin{aligned} X_{\tau_0} &= 0, \\ X_{\tau_i-} &= X_{\tau_{i-1}} + x_i, \quad \forall i \in \{1, \dots, N_T + 1\}, \\ X_{\tau_i} &= X_{\tau_i-} + y_i, \quad \forall i \in \{1, \dots, N_T\}. \end{aligned}$$

5. Determine I , U and b as in Theorem 4.3.2.

6. Calculate

$$\phi_j(0, T) = \mathbb{E} [\mathbf{1}_{\{\tau > T\}} e^{-rT} + R \mathbf{1}_{\{\tau \leq T\}} e^{-r\tau} | \mathcal{F}^*]$$

as described in Equation (4.13) of Theorem 4.3.2.

The run time of this algorithm strongly depends on the expected number of jumps λT . The larger the jump intensity λ , the more samples have to be drawn and the more integrals have to be calculated. We illustrate the performance of Algorithm

4.3.1 in Section 4.8, where we compare the run time of this algorithm using different sets of parameters.

4.3.3 Accelerating the algorithm

The most time-consuming step of Algorithm 4.3.1 is the computation of the integrals $\int e^{-rs} g_i(s) ds$. Metwally and Atiya (2002) suggest an approximation of these integrals which we improve in Theorem 4.3.3. The idea of this approximation is to compute the Laplace transform of the integral, which can be represented as the convolution of two functions. Then, this Laplace transform is expanded into a Taylor series in r . In the next step, the Laplace inverse of the second-order approximation is obtained. The reason for reexamining the original result was to check whether applying the inverse Laplace transform increases the error of the Taylor approximation or not. We showed that this is not the case, but we also found a slightly different result for the inverse Laplace transform compared to the approximation of Metwally and Atiya (2002). However, we ran several numerical experiments and found that our approximation is more accurate. The results of these experiments are presented in Section 4.8.2. The proof of Theorem 4.3.3 is based on joint work with Johannes Ruf. An outline of this proof is given in the Appendix, a very detailed version can be found in Ruf (2006).

Theorem 4.3.3 (Approximation of the integral)

We assume that the firm-value process starts above the default threshold, that is $X_{\tau_{i-1}} > b$. The integral of Equation (4.13) can then be approximated by⁸

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds = e^{-r\tau_{i-1}} \left(\exp \left(-\frac{2(X_{\tau_{i-1}} - b)(X_{\tau_i} - b)}{\Delta\tau_i \sigma^2} \right) + \frac{r(X_{\tau_{i-1}} - b)}{4\sigma} (A_1 + C_1 B) \right) + \mathcal{O}(r^3) \quad (4.14)$$

if $X_{\tau_i} > b$ and by

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds = e^{-r\tau_{i-1}} \left(1 + \frac{r(X_{\tau_{i-1}} - b)}{4\sigma} (A_2 + C_2 B) \right) + \mathcal{O}(r^3) \quad (4.15)$$

if $X_{\tau_i} \leq b$, where $\Delta\tau_i := \tau_i - \tau_{i-1}$, $\Delta X_i := X_{\tau_i} - X_{\tau_{i-1}}$ and

$$A_1 := -\frac{r}{\sigma} \Delta\tau_i \Delta X_i \exp \left(-\frac{2(X_{\tau_{i-1}} - b)(X_{\tau_i} - b)}{\Delta\tau_i \sigma^2} \right),$$

⁸ The function $f : \mathbb{R} \rightarrow \mathbb{R}$ belongs to $\mathcal{O}(g(x))$, if $f(x)/g(x)$ is bounded.

$$\begin{aligned}
C_1 &:= -\sqrt{2\pi\Delta\tau_i} \exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right) \Phi\left(\frac{2b - X_{\tau_i-} - X_{\tau_{i-1}}}{\sqrt{\Delta\tau_i}\sigma}\right), \\
B &:= 4 - r\Delta\tau_i - \frac{r}{\sigma^2}\Delta X_i (X_{\tau_i-} + X_{\tau_{i-1}} - 2b), \\
A_2 &:= \frac{r}{\sigma}\Delta\tau_i (X_{\tau_i-} + X_{\tau_{i-1}} - 2b), \\
C_2 &:= -\sqrt{2\pi\Delta\tau_i} \exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right) \Phi\left(\frac{\Delta X_i}{\sqrt{\Delta\tau_i}\sigma}\right).
\end{aligned}$$

4.3.4 The limit of credit spreads for short maturities

Zhou (2001a) presents an intuitive argument why the limit of credit spreads in a jump-diffusion model should be positive. He argues: *"Because a diffusion process is almost unlikely to cause a default in a short period of time, the defaults of short-term bonds are usually caused by the jump component of the firm value."*⁹ This argument is derived from comparing the vanishing probability of a default by diffusion with the probability of a default caused by a single jump. We made his argument precise in Theorem 4.2.1, where we showed that the local default rate of τ does not depend on the diffusion component of X . In what follows, we present the exact limit of credit spreads at time zero. A similar result for the limit of CDS spreads is presented in Theorem 4.4.2.

Theorem 4.3.4 (Credit spreads at time zero)

The limit of credit spreads at time zero is given by

$$\lim_{h \searrow 0} \eta_h = (1 - R)LDR_\tau.$$

Proof: We let $t = 0$ in Equation (3.1) of Lemma 3.1.2 and notice that the second discount factor is bounded by $e^{-rh} \leq e^{-r\tau} \leq 1$. Thus, a lower bound for the bond price $\phi(0, h)$ is given by

$$\tilde{\phi}(0, h) := e^{-rh} (\mathbb{P}(\tau > h) + R \cdot \mathbb{P}(\tau \leq h)),$$

an upper bound by

$$\hat{\phi}(0, h) := e^{-rh} \mathbb{P}(\tau > h) + R \cdot \mathbb{P}(\tau \leq h).$$

The functions $\tilde{\phi}$ and $\hat{\phi}$ can both be interpreted as bonds with different recovery schemes. The bond $\tilde{\phi}(0, h)$, respectively $\hat{\phi}(0, h)$, pays in the case of a default the fraction R , respectively Re^{rh} , of the bond's face value at maturity. It turns out

⁹ Zhou (2001a), *Journal of Banking and Finance* 25, page 2027.

that both recovery schemes imply the same limit of credit spreads. This is shown for the lower bound $\tilde{\phi}$, the result for the upper bound $\hat{\phi}$ is obtained similarly. By the definition of credit spreads, we find

$$\begin{aligned} \lim_{h \searrow 0} \tilde{\eta}_h &= \lim_{h \searrow 0} -\frac{1}{h} \ln \left(\tilde{\phi}(0, h) \right) - r \\ &= - \frac{\partial}{\partial h} \ln \left(\mathbb{P}(\tau > h) + R \cdot \mathbb{P}(\tau \leq h) \right) \Big|_{h=0} \\ &= \text{LDR}_\tau (1 - R). \end{aligned}$$

As credit spreads are monotone decreasing functions of the bond value, we obtain $\lim_{h \searrow 0} \eta_h$. ◇

We found that the limit of credit spreads at the short end of the term structure is the product of the local default rate of τ and the fractional loss at default. This is economically reasonable, as the potential loss at default is decreasing in the recovery rate, which implies smaller credit spreads. Moreover, the local default rate of τ approximates the probability of default within small intervals of time. Therefore, credit spreads of bonds with small maturities merely depend on the probability of a sudden default. In other words, their credit spreads are increasing in the local default rate.

4.4 Pricing CDS contracts

Pricing CDS bears close analogy to pricing corporate bonds. Again, it requires the evaluation of an expectation which depends on the distribution of τ . To begin with, we use the integration by parts formula and find

$$\begin{aligned} \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t) &= e^{-rT} \mathbb{P}(\tau \leq T) + r \int_0^T \mathbb{P}(\tau \leq t) e^{-rt} dt \\ &= e^{-rT} (1 - \mathbb{P}(\tau > T)) + r \int_0^T (1 - \mathbb{P}(\tau > t)) e^{-rt} dt \\ &= 1 - e^{-rT} \mathbb{P}(\tau > T) - r \int_0^T \mathbb{P}(\tau > t) e^{-rt} dt. \end{aligned}$$

Using this, Equation (3.3) is rearranged as follows

$$\begin{aligned} CDS(0, T) &= \left(1 - R + \frac{c}{r}\right) \int_0^T e^{-rt} d\mathbb{P}(\tau \leq t) - \frac{c}{r} (1 - e^{-rT} \mathbb{P}(\tau > T)) \end{aligned} \quad (4.16)$$

$$= ((R - 1)r - c) \int_0^T e^{-rt} \mathbb{P}(\tau > t) dt + (1 - R) (1 - e^{-rT} \mathbb{P}(\tau > T)). \quad (4.17)$$

Still, these formulas all depend on the distribution of τ . In what follows, we present a closed-form expression of CDS spreads in a pure diffusion model, compare Theorem 4.4.1. The general case is again approached by means of the Monte Carlo simulation in Section 4.4.2. Alternatively, Algorithm 4.6.1 provides an approximation of CDS prices for the case $\mathbb{P}_Y = 2\text{-Exp}(\lambda_{\oplus}, \lambda_{\ominus}, p)$.

4.4.1 Pricing CDS in a pure diffusion scenario

Without jumps, we again rely on Equation (4.2) to evaluate Equation (4.16) explicitly. The required calculation is rather long than complicated. Nevertheless, it can be simplified considerably using Lemma 4.3.1. Finally, we obtain the following result.

Theorem 4.4.1 (Pricing CDS in a pure diffusion model)

We consider a CDS contract with continuous spread c and unit notional. At time zero, the price of this contract with maturity T satisfies

$$CDS(0, T) = \left(1 - R + \frac{c}{r}\right) A - \frac{c}{r} (1 - e^{-rT} B), \quad (4.18)$$

where $b = \log(d/v_0)$, $\tilde{\gamma} = \sqrt{\gamma^2 + 2r\sigma^2}$ and

$$\begin{aligned} A &= e^{b\sigma^{-2}(\gamma-\tilde{\gamma})} \Phi\left(\frac{b-\tilde{\gamma}T}{\sigma\sqrt{T}}\right) + e^{b\sigma^{-2}(\gamma+\tilde{\gamma})} \Phi\left(\frac{b+\tilde{\gamma}T}{\sigma\sqrt{T}}\right) \\ &= e^{b\sigma^{-2}(\gamma-\tilde{\gamma})} (1 - \Phi_{b,\tilde{\gamma},\sigma}^{BM}(0, T)), \end{aligned} \quad (4.19)$$

$$\begin{aligned} B &= \Phi\left(\frac{-b+\gamma T}{\sigma\sqrt{T}}\right) - e^{2\gamma b\sigma^{-2}} \Phi\left(\frac{b+\gamma T}{\sigma\sqrt{T}}\right) \\ &= \Phi_{b,\gamma,\sigma}^{BM}(0, T). \end{aligned} \quad (4.20)$$

The premium that allows both parties to enter the contract at par is given by

$$c_T = \frac{r(1-R)A}{1 - e^{-rT}B - A}. \quad (4.21)$$

Proof: We combine Equation (4.16) with the explicit formula of $\int_0^T e^{-rt} d\mathbb{P}(\tau \leq t)$ as computed in the proof of Theorem 4.3.1. $\diamond$

4.4.2 Pricing CDS in a jump-diffusion scenario

The idea of efficiently estimating the price of a CDS contract by means of a Monte Carlo simulation is similar to the idea of the respective bond-pricing algorithm. We again condition the pricing formula, here Equation (4.16), on the information $\mathcal{F}^*$ which is defined as in Equation (4.12) before. Then, we find

$$\mathbb{E}\left[e^{-r\tau}(1-R)\mathbf{1}_{\{\tau \leq T\}} - \int_0^T ce^{-rt}\mathbf{1}_{\{\tau > t\}}dt \middle| \mathcal{F}^*\right] = \left(1-R+\frac{c}{r}\right)A - \frac{c}{r}(1-e^{-rT}B),$$

where we define the abbreviations

$$A := \sum_{i=1}^U \left(\prod_{j=1}^{i-1} \Phi_b^{BB}(j) \right) \int_{\tau_{i-1}}^{\tau_i} e^{-rt} g_i(t) dt + \mathbf{1}_{\{I \neq 0\}} e^{-r\tau_I} \prod_{j=1}^I \Phi_b^{BB}(j), \quad (4.22)$$

$$B := \mathbf{1}_{\{I=0\}} \prod_{j=1}^{N_T+1} \Phi_b^{BB}(j). \quad (4.23)$$

The variables I , U and $\Phi_b^{BB}(j)$ are defined as in Theorem 4.3.2.

Algorithm 4.4.1 (Monte Carlo pricing of CDS)

We consider a CDS contract with continuous spread c and unit notional. The

initial price of this contract with maturity T is estimated as follows. Choose the number of simulation runs K and estimate $CDS(0, T)$ via

$$CDS(0, T) \approx \left(1 - R + \frac{c}{r}\right) \bar{A}_K - \frac{c}{r} (1 - e^{-rT} \bar{B}_K),$$

where we let

$$\bar{A}_K := \frac{1}{K} \sum_{n=1}^K A_n, \quad \bar{B}_K := \frac{1}{K} \sum_{n=1}^K B_n.$$

In each step of the simulation, A_n and B_n are calculated as described in Equations (4.22) and (4.23) from a new set of simulated jumps, with default threshold $b = \log(d/v_0)$. The par spread c_T is then estimated via

$$c_T \approx \frac{r(1 - R)\bar{A}_K}{1 - e^{-rT}\bar{B}_K - \bar{A}_K}.$$

4.4.3 The limit of CDS spreads for short maturities

Previously, we already worked out that in pure diffusion models it is virtually impossible for solvent companies to default within a small interval of time, due to the continuity of their firm-value processes. Moreover, just as for bond spreads, we show that par spreads of CDS tend to zero in the maturity of the contract. To prove this claim, we consider the abbreviations A and B from Equations (4.19) and (4.20) as functions of T . Then, we observe that as long as the company is solvent or, in other words, as long as the distance to default $x_0 = -\log(d/v_0)$ is positive, we obtain the following limits

$$\lim_{T \searrow 0} A(T) = \lim_{T \searrow 0} A'(T) = \lim_{T \searrow 0} B'(T) = 0, \quad \lim_{T \searrow 0} B(T) = 1.$$

L'Hospital's rule finally establishes

$$\lim_{T \searrow 0} c_T = \lim_{T \searrow 0} \frac{r(1 - R)A'(T)}{re^{-rT}B(T) - e^{-rT}B'(T) - A'(T)} = 0. \quad (4.24)$$

In a jump-diffusion model, the limit of CDS par spreads at the short end of the term structure is positive and can be found using the local default rate of τ .

Theorem 4.4.2 (The limit of CDS spreads with jumps)

Within the setup of our jump-diffusion model, the limit of CDS spreads at the short end of the term structure agrees with the limit of bond spreads. Hence, it is given by

$$\lim_{T \searrow 0} c_T = (1 - R)LDR_\tau.$$

Proof: At first, we rewrite c_T as

$$c_T = \frac{\frac{1}{T}(1-R) \left(1 - e^{-rT} \mathbb{P}(\tau > T) - r \int_0^T \mathbb{P}(\tau > t) e^{-rt} dt \right)}{\frac{1}{T} \int_0^T e^{-rt} \mathbb{P}(\tau > t) dt}.$$

Using Tonelli's theorem, we find

$$\frac{1}{T} \int_0^T e^{-rt} \mathbb{P}(\tau > t) dt = \frac{1}{T} \mathbb{E} \left[\int_0^{\tau \wedge T} e^{-rt} dt \right]. \quad (4.25)$$

Chen and Kou (2005) consider a model with endogenously given default threshold and $\mathbb{P}_Y = 2\text{-Exp}(\lambda_\oplus, \lambda_\ominus, p)$. However, we can take over their argument from Appendix C and conclude that if T tends to zero, the limit in Equation (4.25) is one. Equipped with this limit and the local default rate of τ from Theorem 4.2.1, we find that in our model

$$\begin{aligned} \lim_{T \searrow 0} c_T &= \lim_{T \searrow 0} \frac{\frac{1}{T}(1-R) \left(1 - e^{-rT} \mathbb{P}(\tau > T) - r \int_0^T \mathbb{P}(\tau > t) e^{-rt} dt \right)}{\frac{1}{T} \int_0^T e^{-rt} \mathbb{P}(\tau > t) dt} \\ &= \lim_{T \searrow 0} (1-R) \left(\frac{1 - e^{-rT}}{T} + \frac{e^{-rT} \mathbb{P}(\tau \leq T)}{T} - \frac{r}{T} \int_0^T \mathbb{P}(\tau > t) e^{-rt} dt \right) \\ &= (1-R)(r + 1 \cdot \text{LDR}_\tau - r). \end{aligned}$$

◇

4.5 Generalizations of the model

In this section, we present several generalizations of the model and explain how these extensions are included into our Monte Carlo pricing algorithm for zero-coupon bonds. However, the same generalizations apply to the problem of pricing CDS, with the obvious changes in the respective algorithm. To begin with, we show how jump-diffusion models endogenously define a stochastic recovery rate and how this feature is implemented in Algorithm 4.3.1. Then, we relax the assumption of a flat risk-free interest rate in Section 4.5.2 by allowing short-rate processes which imply closed-form expressions of default-free zero-coupon bonds. Another assumption of our initial default model is a constant default threshold, which is relaxed in Section 4.5.3. Finally, the question of what information about the firm-value process are available to an ordinary investor is examined in Section 4.5.4. Here, we present formulas for bond and CDS prices based on reduced filtrations. Before we proceed, we note that Sections 4.5.1 and 4.5.3 were worked out in cooperation with Johannes Ruf.

4.5.1 Including a stochastic recovery rate

In the event of credit default, a company is liquidated and the remaining assets are distributed among the bondholders. So far, we assumed the bondholders to receive the constant and predetermined fraction R of their invested principal. However, it is more realistically to assume the recovery rate to be a random variable which is drawn at the time of default. This economic consideration translates in an $\mathcal{F}_\tau$ -measurable random variable in our mathematical model. For structural models, we feel that a natural determinant of the recovery rate is the firm-value process at the time of default, as this value represents the remaining assets of the company. At this point, another advantage of allowing jumps in the firm-value process is revealed. In contrast to pure diffusion models, where the value of the company at time τ necessarily agrees with the default threshold, the possibility of passing the default threshold by jump suggests a natural model of the default severity of the company. More precisely, we can use the random undershot $d - V_\tau$ to endogenously specify the recovery rate by the model. This idea was first proposed by Zhou (2001a), we slightly alter his definitions and show how this approach is embedded into our Monte Carlo algorithm.

In what follows, we model the recovery rate as an $\mathcal{F}_\tau$ -measurable random variable which is determined by the ratio of the firm-value process and the default threshold at the time of default, that is $R_\tau = w(V_\tau/d)$, where the function $w : [0, 1] \rightarrow [0, 1]$

is non-decreasing¹⁰. Reconsidering Formula (4.13), we observe that this expression can be viewed as a weighted sum of three summands with the following interpretation. The first summand corresponds to the payoff of the bond in the event of default by diffusion, the second to the event of default by jump. The last summand corresponds to the bond's payoff in the event of no default. The weights are the conditional probabilities for the respective events to occur. These weights depend on the simulated jumps and Brownian increments within each simulation run. The recovery rate in the event of default by diffusion is given by $R_\tau = w(1)$, as this case corresponds to $V_\tau = d$. In our algorithm, the variable I is used to indicate whether a jump forced the company to default or not. If $I > 0$, the recovery rate at $\tau = \tau_I$ is given by $R_\tau = w(V_{\tau_I}/d)$. Based on these considerations, we alter Formula (4.13) and the respective step of Algorithm 4.3.1 as follows

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\{\tau > T\}} e^{-rT} + w(V_\tau/d) \mathbf{1}_{\{\tau \leq T\}} e^{-r\tau} \middle| \mathcal{F}^* \right] \\ &= w(1) \sum_{i=1}^U \prod_{j=1}^{i-1} \Phi_b^{BB}(j) \int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds + \\ & \quad w(V_{\tau_I}/d) \mathbf{1}_{\{I \neq 0\}} e^{-r\tau_I} \prod_{j=1}^I \Phi_b^{BB}(j) + \mathbf{1}_{\{I=0\}} e^{-rT} \prod_{j=1}^{N_T+1} \Phi_b^{BB}(j). \end{aligned} \quad (4.26)$$

Function w	Parameter	Interpretation
$w_0(x) = R$	$R \in [0, 1]$	Constant recovery
$w_1(x) = Rx$	$R \in [0, 1]$	Linear dependence on V_τ/d
$w_2(x) = \max\{r_0 + r_1 x, 1\}$	$r_0, r_1 \in [0, 1]$	Linear, with lower bound r_0

Table 4.2: Different recovery functions w .

4.5.2 Including a short-rate model

In this section, the flat risk-free interest rate is replaced by an $\mathbb{F}$ -adapted, time-homogeneous short-rate process $r = \{r_t\}_{t \geq 0}$. This process is also modeled under the pricing measure $\mathbb{P}$. At time t , the price of a default-free zero-coupon bond with maturity T is given in terms of its expected discounted payoff with respect to $\mathbb{P}$. Therefore, we define

$$\varphi(t, T) := \mathbb{E} \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \quad (4.27)$$

¹⁰ This assumption is not only reasonable from an economic perspective, it also guarantees that $w(V_\tau/d)$ is measurable with respect to $\mathcal{F}_\tau$.

If the distribution of $\exp(-\int_t^T r_s ds)$ is analytically tractable, then it is possible to express the price of a default-free zero-coupon bond as a function of the parameters of the respective short-rate process. For instance, such an expression is available for Vasicek's (V) model¹¹, where the short-rate process is modeled as

$$dr_t = k(\theta - r_t)dt + \hat{\sigma}d\hat{W}_t \quad r_0 > 0, \quad (4.28)$$

and the Cox, Ingersoll and Ross (CIR) model¹², where the short-rate dynamics evolve according to the stochastic differential equation

$$dr_t = k(\theta - r_t)dt + \hat{\sigma}\sqrt{r_t}d\hat{W}_t \quad r_0 > 0. \quad (4.29)$$

In both examples, the structure of $\varphi(t, T)$ is given by

$$\varphi(t, T) = A(t, T)e^{-B(t, T)r_t}, \quad (4.30)$$

where the functions A and B of the respective model are defined as follows

$$\begin{aligned} A^V(t, T) &:= \exp \left\{ \left(\theta - \frac{\hat{\sigma}^2}{2k^2} \right) (B^V(t, T) - T + t) - \frac{\hat{\sigma}^2}{4k} B^V(t, T)^2 \right\}, \\ B^V(t, T) &:= \frac{1}{k} (1 - e^{-k(T-t)}), \\ A^{CIR}(t, T) &:= \left(\frac{2he^{(k+h)(T-t)/2}}{2h + (k+h)(e^{(T-t)h} - 1)} \right)^{2k\theta/\hat{\sigma}^2}, \\ B^{CIR}(t, T) &:= \frac{2e^{(T-t)h} - 2}{2h + (k+h)(e^{(T-t)h} - 1)}, \\ h &:= \sqrt{k^2 + 2\hat{\sigma}^2}. \end{aligned}$$

A proof of these formulas is given in Brigo and Mercurio (2001), Chapter 3.2.

Including the stochastic short-rate in the pricing formula of the bond

The assumption that the Brownian motions of the short-rate processes in Equations (4.28) and (4.29) are independent of the Brownian motion of the firm-value process allows us to replace the discount factor $\exp(-rs)$ by $\varphi(0, s)$ in Theorem 4.3.2 and Algorithm 4.3.1. An implementation of this generalized Monte

¹¹ In Vasicek's model, the short-rate process is assumed to follow an Ornstein-Uhlenbeck process with constant coefficients. The parameter $\theta > 0$ can be interpreted as the mean-reverting level of the process, $k > 0$ specifies how fast the process returns to this long-term mean and $\hat{\sigma} > 0$ is the volatility of the Brownian component.

¹² Again, the parameters θ , k and $\hat{\sigma}$ are positive. The additional factor $\sqrt{r_t}$ combined with the condition $2k\theta > \hat{\sigma}^2$ implies that the process r remains positive, which is not guaranteed in Vasicek's model.

Carlo pricing algorithm requires to numerically evaluate integrals of the form $\int \varphi(0, s)g_i(s)ds$ instead of $\int e^{-rs}g_i(s)ds$. Computationally, this is only slightly more expensive, due to the closed form of $\varphi(0, s)$ as given in Equation (4.30). To illustrate the effect of including a short-rate model, we initially calculated corporate bond prices based on a flat interest rate, then using the short-rate models of Equations (4.28) and (4.29). The results after 50,000 simulation runs are presented in Table 4.3¹³. The parameter setup of this experiment is $r = 0.03$, $T = 5$, $R = 42\%$, $\gamma = 0.045$, $\sigma = 0.05$, $\lambda = 2$, $\mathbb{P}_Y = 2\text{-Exp}(20, 20, 0.5)$, $d/v_0 = 85\%$ and $\theta^V = \theta^{CIR} = r_0^V = r_0^{CIR} = 0.035$, $k^V = k^{CIR} = 0.1$, $\hat{\sigma}^V = 0.01$, $\hat{\sigma}^{CIR} = 0.05$.

Short rate r	Corporate bond	Default-free bond	Spread
Constant r	$\phi(0, T) = 0.7717$	$\exp(-rT) = 0.8607$	218 bp
Vasicek model	$\phi(0, T) = 0.7546$	$\varphi(0, T) = 0.8407$	216 bp
CIR model	$\phi(0, T) = 0.7554$	$\varphi(0, T) = 0.8405$	214 bp

Table 4.3: Bond prices using a short-rate model.

Our choice of risk-neutral parameters implies a premium for the risk of stochastic interest rates. Therefore, it is not surprising that the resulting default-free bonds trade below the default-free bond in an environment with a flat term-structure of interest rates. However, we were surprised by the fact that including a stochastic short-rate process affects defaultable and default-free bonds alike. This is reflected in credit spreads which remain at about the same level if computed relative to the corresponding default-free bond.

Let us conclude this section with the remark that if the firm-value process and the short-rate process have correlated Brownian motions, then one can still use the Monte Carlo pricing algorithm of Zhou (2001a). This algorithm samples trajectories of the processes r and V on a fine grid and computes the respective payoff of the bond in each simulation run. Again, such a Monte Carlo simulation produces biased bond prices and is very time consuming.

4.5.3 Including a stochastic default threshold

Up to this point, a constant default threshold was assumed. However, a simple calculation shows that this restriction can easily be relaxed to default thresholds that are modeled using a second jump-diffusion process. More precisely, the constant

¹³ In this table, spreads are calculated relative to the default-free bond of the same row.

threshold d is replaced by an $\mathbb{F}$ -adapted process $D = \{D_t\}_{t \geq 0}$ of the form

$$D_t = d_0 \exp\{d_t\}, \quad d_t = \tilde{\gamma}t + \tilde{\sigma}\tilde{W}_t + \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i \quad 0 < d_0 < v_0.$$

With respect to $\mathbb{P}$, the process $d = \{d_t\}_{t \geq 0}$ is also a jump-diffusion process with jump-size distribution $\mathbb{P}_{\tilde{Y}}$. The Brownian motion of the firm-value process and the Brownian motion of the default threshold are correlated with coefficient $\rho \in (-1, 1)$. All other random variables are assumed to be mutually independent. The time of default is then given by

$$\tilde{\tau} = \inf\{t > 0 : V_t \leq D_t\} = \inf\{t > 0 : X_t - d_t \leq \log(d_0/v_0)\},$$

which differs only slightly from the time of default which was used earlier, namely

$$\tau = \inf\{t > 0 : V_t \leq d\} = \inf\{t > 0 : X_t \leq \log(d/v_0)\}.$$

In fact, to show that the parameters of the model can be chosen such that both times of default agree in distribution, and therefore imply the same bond and CDS prices, it is enough to show that the difference of two jump-diffusion processes is again a jump-diffusion process. This result is given in the theorem below.

Theorem 4.5.1 (The difference of two jump-diffusion processes)

We consider the difference of the jump-diffusion processes X and d , given by

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad d_t = \tilde{\gamma}t + \tilde{\sigma}\tilde{W}_t + \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i, \quad (4.31)$$

with $\text{Cov}(X_t, d_t) = \sigma\tilde{\sigma}\text{Cov}(W_t, \tilde{W}_t) = \sigma\tilde{\sigma}\rho t$, where $\rho \in (-1, 1)$. All other random variables of Equation (4.31) are mutually independent. Moreover, let the intensity of N and $\tilde{N}$ be denoted by λ and $\tilde{\lambda}$, respectively, the jump-size distribution of Y_i and $\tilde{Y}_i$ by $\mathbb{P}_Y$ and $\mathbb{P}_{\tilde{Y}}$, respectively. Then, the difference $X - d$ is again a jump-diffusion process which agrees in distribution with the process

$$\hat{X}_t = \hat{\gamma}t + \hat{\sigma}\hat{W}_t + \sum_{i=1}^{\hat{N}_t} \hat{Y}_i,$$

where $\hat{\gamma} = \gamma - \tilde{\gamma}$, $\hat{\sigma} = \sqrt{\sigma^2 + \tilde{\sigma}^2 - 2\rho\sigma\tilde{\sigma}}$, $\hat{W}$ is a Brownian motion, $\hat{N}$ is a Poisson process with intensity $\hat{\lambda} = \lambda + \tilde{\lambda}$ and the jump-size distribution of $\hat{Y}_i$ is given by $\mathbb{P}_{\hat{Y}} = \lambda/(\lambda + \tilde{\lambda})\mathbb{P}_Y + \tilde{\lambda}/(\lambda + \tilde{\lambda})\mathbb{P}_{\tilde{Y}}$.

Proof: The deterministic part is clear. From the definition of a Brownian motion and the stability of the normal distribution under convolution, we obtain that

$$\hat{W}_t =_d \frac{1}{\sqrt{\sigma^2 + \tilde{\sigma}^2 - 2\rho\sigma\tilde{\sigma}}} \left(\sigma W_t - \tilde{\sigma} \tilde{W}_t \right) \sim \mathcal{N}(0, t).$$

Being independent, the characteristic function of the sum of the compound Poisson processes $M_t := \sum_{i=1}^{\hat{N}_t} \hat{Y}_i$ and $\tilde{M}_t := \sum_{i=1}^{\tilde{N}_t} (-\tilde{Y}_i)$ is the product of the two individual characteristic functions. Hence, we find

$$\begin{aligned} \Phi_{M_t + \tilde{M}_t}(z) &= \exp \left(t \int_{\mathbb{R}} (e^{iuz} - 1) \left(\lambda \mathbb{P}_Y(du) + \tilde{\lambda} \mathbb{P}_{-\tilde{Y}}(du) \right) \right) \\ &= \exp \left(t \hat{\lambda} \int_{\mathbb{R}} (e^{iuz} - 1) \left(\frac{\lambda}{\hat{\lambda}} \mathbb{P}_Y(du) + \frac{\tilde{\lambda}}{\hat{\lambda}} \mathbb{P}_{-\tilde{Y}}(du) \right) \right). \end{aligned}$$

This last expression is the characteristic function of a compound Poisson process with intensity $\hat{\lambda}$ and jump-size distribution $\mathbb{P}_{\hat{Y}}$. $\diamond$

4.5.4 Pricing based on reduced information

Pricing corporate bonds and CDS based on the filtration $\mathbb{F}$, the filtration generated by the firm-value process, is convenient from a mathematical point of view. However, the justification of this model from an economic perspective is questionable, as this assumption requires investors to observe the firm-value process continuously. While quotes of stocks and other liquidly traded objects are available continuously, the firm-value process is not a traded object and is generally not observable at all. However, the assumption that investors get periodic information about the value of the firm in terms of balance sheets (on a fixed schedule $0 = t_0 < t_1 < \dots < t_n = T$) is realistic. Additionally, we assume that all investors are informed whether or not the company has defaulted so far. Motivated by those economic considerations, we define the filtrations

$$\begin{aligned} \mathcal{H}_t &:= \sigma(V_s : 0 \leq s \leq t_i, t_i \leq t < t_{i+1}) \vee \mathcal{N}, \\ \mathcal{G}_t &:= \mathcal{H}_t \vee \sigma(\tau \leq t), \\ \mathcal{F}_t &:= \sigma(V_s : 0 \leq s \leq t) \vee \mathcal{N}. \end{aligned}$$

$\mathcal{H}_t$ is the history of the firm-value process up to time t_i , the time of the last update, which can be interpreted as the last published balance sheet. $\mathcal{G}_t$ additionally contains the information whether the company defaulted up to time t . Finally, $\mathcal{F}_t$ contains the complete history of the firm value process up to time t . At first,

we observe that $\mathcal{H}_t \subset \mathcal{G}_t \subset \mathcal{F}_t$ for all $t \geq 0$. We then notice that the σ -algebras coincide at the dates when the filtrations $\mathbb{H}$ and $\mathbb{G}$ are updated. It is now possible to compute default probabilities conditioned on the respective information, as done in a continuous model by Jeanblanc and Valchev (2005). We follow their arguments and obtain for $t_i \leq t < t_{i+1}$

$$\begin{aligned} \mathbb{P}(\tau > T | \mathcal{H}_t) &= \mathbf{1}_{\{\tau > t_i\}} \mathbb{P}(\tau > T | \mathcal{F}_{t_i}) \\ &= \mathbf{1}_{\{\tau > t_i\}} \mathbb{P} \left(\inf_{t_i \leq s \leq T} X_s > \log(d/v_0) \middle| \mathcal{F}_{t_i} \right), \end{aligned} \quad (4.32)$$

$$\begin{aligned} \mathbb{P}(\tau > T | \mathcal{G}_t) &= \mathbf{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T | \mathcal{F}_{t_i})}{\mathbb{P}(\tau > t | \mathcal{F}_{t_i})} \\ &= \mathbf{1}_{\{\tau > t\}} \frac{\mathbb{P} \left(\inf_{t_i \leq s \leq T} X_s > \log(d/v_0) \middle| \mathcal{F}_{t_i} \right)}{\mathbb{P} \left(\inf_{t_i \leq s \leq t} X_s > \log(d/v_0) \middle| \mathcal{F}_{t_i} \right)}, \end{aligned} \quad (4.33)$$

$$\mathbb{P}(\tau > T | \mathcal{F}_t) = \mathbf{1}_{\{\tau > t\}} \mathbb{P} \left(\inf_{t \leq s \leq T} X_s > \log(d/v_0) \middle| \mathcal{F}_t \right). \quad (4.34)$$

The price process of a zero-coupon bond in a reduced filtration

Let us fix a filtration $\mathbb{J} \in \{\mathbb{H}, \mathbb{G}, \mathbb{F}\}$. We then evaluate the pricing formula of a zero-coupon bond with maturity T conditioned on the σ -algebra $\mathcal{J}_t$. We find

$$\begin{aligned} \phi_{\mathbb{J}}(t, T) &:= \mathbb{E} \left[e^{-r(T-t)} \mathbf{1}_{\{\tau > T\}} + R e^{-r(\tau-t)} \mathbf{1}_{\{t \leq \tau \leq T\}} \middle| \mathcal{J}_t \right] \\ &= e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{J}_t) + R \int_t^T e^{-r(s-t)} d\mathbb{P}(\tau \leq s | \mathcal{J}_t) \\ &= e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{J}_t) (1 - R) + R \left(1 - r \int_t^T e^{-r(s-t)} \mathbb{P}(\tau > s | \mathcal{J}_t) ds \right). \end{aligned} \quad (4.35)$$

The last formula is convenient for numerical implementations. Given $\mathcal{J}_t$, an implementation only requires the evaluation of a Riemann integral. The respective default probabilities are estimated within a jump-diffusion model by means of the Monte Carlo simulation of Section 4.2.3 or approximated via the Laplace method of Section 4.6, if two-sided exponentially distributed jumps are assumed. In a pure diffusion scenario it is even possible to explicitly evaluate the pricing formulas of zero-coupon bonds and CDS contracts. The required calculations are of moderate difficulty but extremely lengthy. Comparable calculations can be found in a slightly different setup in Jeanblanc and Valchev (2005). Within our model, we obtain the following formulas for the zero-coupon bond of Equation (4.35) given the respective filtration.

Theorem 4.5.2 (Pricing given $\mathbb{J} \in \{\mathbb{H}, \mathbb{G}, \mathbb{F}\}$ in a pure diffusion scenario)
In all formulas, let $b = \log(d/v_0)$ and $\tilde{\gamma} = \sqrt{\gamma^2 + 2r\sigma^2}$. The price process of a zero-coupon bond, conditioned on the respective filtration, is then given as follows.

$\mathbb{H}$: For $t_i < t < t_{i+1}$ and $\tau > t_i$, we have

$$\begin{aligned} \phi_{\mathbb{H}}(t, T) &= e^{-r(T-t)} \Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, T - t_i) + Re^{-r(t_i-t)} e^{(X_{t_i}-b)(\tilde{\gamma}-\gamma)\sigma^{-2}} \cdot \\ &\quad (\Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, t - t_i) - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, T - t_i)). \end{aligned}$$

$\mathbb{G}$: For $t_i < t < t_{i+1}$ and $\tau > t$, we have

$$\begin{aligned} \phi_{\mathbb{G}}(t, T) &= e^{-r(T-t)} \frac{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, T - t_i)}{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, t - t_i)} + \frac{Re^{-r(t_i-t)} e^{(X_{t_i}-b)(\tilde{\gamma}-\gamma)\sigma^{-2}}}{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, t - t_i)} \cdot \\ &\quad (\Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, t - t_i) - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, T - t_i)). \end{aligned}$$

$\mathbb{F}$: For $\tau > t$, we have

$$\begin{aligned} \phi_{\mathbb{F}}(t, T) &= e^{-r(T-t)} \Phi_{b, \gamma, \sigma}^{BM}(X_t, T - t) + \\ &\quad Re^{(X_t-b)(\tilde{\gamma}-\gamma)\sigma^{-2}} (1 - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_t, T - t)). \end{aligned}$$

Sketch of the proof: To begin with, we use Lemma 4.2.1 to express the probabilities $\mathbb{P}(\tau > T | \mathcal{J}_t)$ of Equations (4.32), (4.33) and (4.34) in terms of survival probabilities of a Brownian motion with drift, for $\mathcal{J}_t \in \{\mathcal{H}_t, \mathcal{G}_t, \mathcal{F}_t\}$. The next step is to evaluate the respective Riemann-Stieltjes integrals of Equation (4.35) using Lemma 4.3.1, which yields extremely long terms. However, most of them allow to be expressed in terms of survival probabilities of a Brownian motion with some suitably adjusted drift. $\diamond$

Interpretation of Theorem 4.5.2 and a numerical experiment

If we examine the formulas of Theorem 4.5.2 from the perspective of how an investor uses the available information about the firm-value process to assess the price of a bond, we observe that all knowledge about the company occurs in the respective pricing formula as expected. Given $\mathbb{H}$, the best information about the firm-value process is the value as revealed with the latest update. As pricing based on $\mathbb{H}$ also involves uncertainty about whether the company is still solvent or not, the price conditioned on $\mathbb{H}$ is always below the price conditioned on $\mathbb{G}$. Of course, this only holds as long as the company is solvent. Based on $\mathbb{G}$, the pricing formula contains the last available information about the value of the firm and additionally

the default status of the company. Finally, pricing based on $\mathbb{F}$ is done based on the current observation of the firm-value process and the history of this process which determines the default status.

To illustrate the effect of reduced information on bond prices, we simulated a sample path of the firm-value process on the time interval $[0, 10]$. Then, we computed the respective prices $\phi_{\mathbb{J}}(t, 10)$ given the full information $\mathbb{F}$ and the sub-filtration $\mathbb{G}$. The parameters of this experiment are $\gamma = 0$, $\sigma = 0.05$, $\lambda = 0.5$, $\mathbb{P}_Y = 2\text{-Exp}(20, 20, 0.5)$, $r = 0.02$, $R = 42\%$ and $d/v_0 = 85\%$. The results of this experiment are presented in Figure 4.1, the left-hand side showing the distance to default of X as seen given $\mathbb{F}$ and $\mathbb{G}$, the right-hand side exhibits the corresponding price processes $\phi_{\mathbb{J}}(t, 10)$.

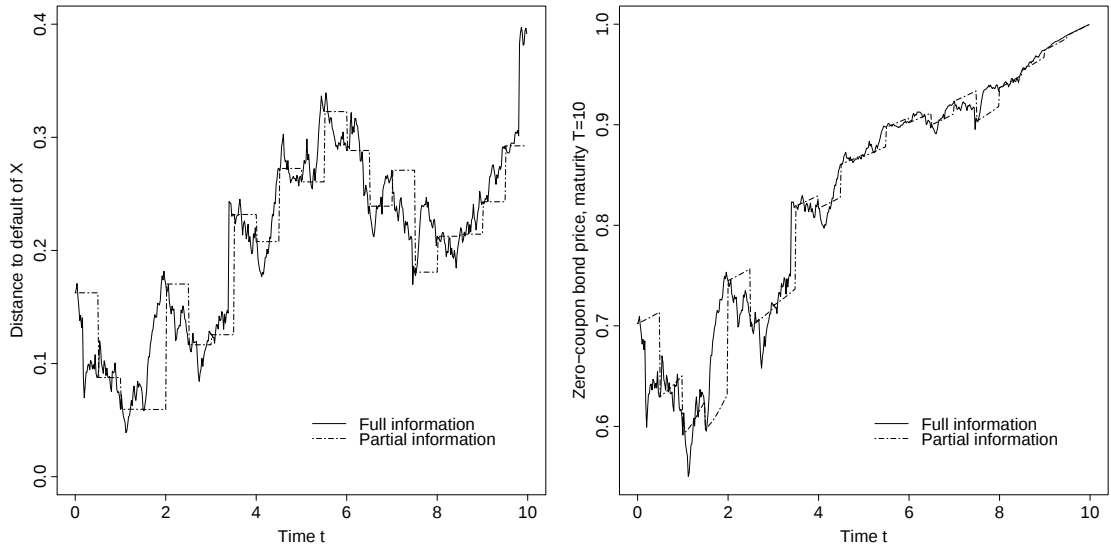


Figure 4.1: A sample path of $X_t - b$ and bond prices based on $\mathbb{G}$ and $\mathbb{F}$.

Examining Figure 4.1, we observe that if the distance to default widened since the last update, then bond prices based on $\mathbb{F}$ are above the corresponding prices based on $\mathbb{G}$, and vice versa. This is reasonable, as the widened distance to default is instantaneously considered given $\mathbb{F}$. Moreover, whenever the filtration $\mathbb{G}$ is updated, the price process based on $\mathbb{G}$ jumps to match the price process computed under $\mathbb{F}$. This is also natural, as the filtrations coincide at the times of each update.

The price process of a CDS in a reduced filtration

We conclude this section with the corresponding results for a CDS with unit notional. The price process of this CDS, conditioned on $\mathbb{J} \in \{\mathbb{H}, \mathbb{G}, \mathbb{F}\}$, satisfies

$$\begin{aligned}
 CDS_{\mathbb{J}}(t, T) &= \mathbb{E} \left[(1 - R)e^{-r(\tau-t)} \mathbf{1}_{\{t \leq \tau \leq T\}} - \int_t^T c \mathbf{1}_{\{\tau > s\}} e^{-r(s-t)} ds \middle| \mathcal{J}_t \right] \\
 &= (1 - R) (1 - e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{J}_t)) + \\
 &\quad ((R - 1)r - c) \int_t^T e^{-r(s-t)} \mathbb{P}(\tau > s | \mathcal{J}_t) ds \\
 &= \left(1 - R + \frac{c}{r}\right) \int_t^T e^{-r(s-t)} d\mathbb{P}(\tau \leq s | \mathcal{J}_t) - \\
 &\quad \frac{c}{r} (1 - e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{J}_t)). \tag{4.36}
 \end{aligned}$$

In a pure diffusion model, we computed Equation (4.36) as before, given the filtration $\mathbb{J} \in \{\mathbb{H}, \mathbb{G}, \mathbb{F}\}$. These results are presented in Theorem 4.5.3.

Theorem 4.5.3 (CDS prices given $\mathbb{J} \in \{\mathbb{H}, \mathbb{G}, \mathbb{F}\}$, pure diffusion scenario)

In all formulas, we use the abbreviations $b = \log(d/v_0)$, $\tilde{\gamma} = \sqrt{\gamma^2 + 2r\sigma^2}$ and $A = (1 - R + \frac{c}{r}) \exp\{(X_{t_i} - b)(\tilde{\gamma} - \gamma)\sigma^{-2}\}$. The CDS price process, conditioned on the respective filtration, is then given as follows.

$\mathbb{H}$: For $t_i < t < t_{i+1}$ and $\tau > t_i$, we have

$$\begin{aligned}
 CDS_{\mathbb{H}}(t, T) &= Ae^{-r(t_i-t)} (\Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, t - t_i) - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, T - t_i)) - \\
 &\quad \frac{c}{r} (1 - e^{-r(T-t)} \Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, T - t_i)).
 \end{aligned}$$

$\mathbb{G}$: For $t_i < t < t_{i+1}$ and $\tau > t$, we have

$$\begin{aligned}
 CDS_{\mathbb{G}}(t, T) &= Ae^{-r(t_i-t)} \frac{(\Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, t - t_i) - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_{t_i}, T - t_i))}{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, t - t_i)} - \\
 &\quad \frac{c}{r} \left(1 - e^{-r(T-t)} \frac{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, T - t_i)}{\Phi_{b, \gamma, \sigma}^{BM}(X_{t_i}, t - t_i)} \right).
 \end{aligned}$$

$\mathbb{F}$: For $\tau > t$, we have

$$\begin{aligned}
 CDS_{\mathbb{F}}(t, T) &= \left(1 - R + \frac{c}{r}\right) e^{(X_t - b)(\tilde{\gamma} - \gamma)\sigma^{-2}} (1 - \Phi_{b, \tilde{\gamma}, \sigma}^{BM}(X_t, T - t)) - \\
 &\quad \frac{c}{r} (1 - e^{-r(T-t)} \Phi_{b, \gamma, \sigma}^{BM}(X_t, T - t)).
 \end{aligned}$$

Let us remark that the corresponding par-spread processes $c_{\mathbb{J},T-t}$ are easily obtained by solving $CDS_{\mathbb{J}}(t,T) = 0$ for c .

Sketch of the proof: We again omit the exact calculations involved in Theorem 4.5.3, as they are simple but extremely long. The relevant terms of Equation (4.36) are closely related to the terms of Equation (4.35), which we already simplified in the proof of Theorem 4.5.2. $\diamond$

4.6 The two-sided exponential distribution

In this section, we specify the jump-size distribution of the jump-diffusion process X to be a two-sided exponential distribution, that is $\mathbb{P}_Y = 2\text{-Exp}(\lambda_\oplus, \lambda_\ominus, p)$. For the reader's convenience, let us recall that the density of this distribution is given by

$$f(x) = p\lambda_\oplus e^{-\lambda_\oplus x} \mathbf{1}_{\{x>0\}} + (1-p)\lambda_\ominus e^{\lambda_\ominus x} \mathbf{1}_{\{x<0\}}. \quad (4.37)$$

The parameters of this distribution allow the following interpretation. A jump is positive with probability $p \in [0, 1]$ and negative with probability $1-p$. Positive and negative jumps are exponentially distributed with parameters $\lambda_\oplus > 0$ and $\lambda_\ominus > 0$, respectively. This jump-size distribution along with the corresponding jump-diffusion process was introduced to the financial literature as a model for stock prices by Kou (2002).

In the context of credit risk, we feel that this jump-size distribution has several desirable properties. First of all, the tails of this distribution are semi-heavy, and therefore heavier as in the model of Zhou (2001a), which relies on normally distributed jumps¹⁴. Moreover, the two-sided exponential distribution is leptokurtic, which implies that a larger fraction of the variance is due to infrequent extreme deviations, compared to the normal distribution. Another desirable property is the possibility to choose the parameters of this distribution such that jumps are asymmetric, which allows to intensify downside jumps. Finally, it turns out that this choice of jump-size distribution allows the computation of the Laplace transform of first-passage times. Based on this result, we derive an extremely fast and accurate approximation of bond and CDS prices. This algorithm makes the structural default model tractable for applications such as a calibration to market quotes, as presented in Section 4.9.

4.6.1 Basic properties of Kou's stock-price model

According to Lemma 2.1.2, the Lévy density of the jump-diffusion process X factorizes to $\nu(dx) = \lambda f(x)dx$. The n^{th} -absolute moment of X_t exists for some $t > 0$ or, equivalently for all $t \geq 0$, if and only if $\int_{|x| \geq 1} |x|^n \nu(dx) < \infty$. This is guaranteed for all $n \in \mathbb{N}$ by the exponential tails of the jump-size distribution. We define

$$\mu_c := \gamma + \lambda \left(\frac{p}{\lambda_\oplus} - \frac{1-p}{\lambda_\ominus} \right) \quad (4.38)$$

¹⁴ Consequently, the jump-size distribution of the returns in Zhou's firm-value model is a log-normal distribution.

to be the center of the process X and obtain from Lemma 2.1.3

$$\mathbb{E}[X_t] = t\mu_c, \quad \text{Var}(X_t) = t \left(\sigma^2 + 2\lambda \left(\frac{p}{\lambda_\oplus^2} + \frac{1-p}{\lambda_\ominus^2} \right) \right). \quad (4.39)$$

For $\theta \in (-\lambda_\ominus, \lambda_\oplus)$, the moment-generating function of the jump-size distribution $\mathbb{P}_Y$ exists and is given by

$$\mathbb{E}[e^{\theta Y}] = \frac{p\lambda_\oplus}{\lambda_\oplus - \theta} + \frac{(1-p)\lambda_\ominus}{\lambda_\ominus + \theta}. \quad (4.40)$$

From Equation (4.40), the moment-generating function of X_t is deduced as follows. The diffusion component is normally distributed with well-known moment-generating function

$$\mathbb{E}[e^{\theta(\gamma t + \sigma W_t)}] = \exp \left(t \left(\gamma\theta + \frac{\theta^2 \sigma^2}{2} \right) \right) \quad \theta \in \mathbb{R}. \quad (4.41)$$

As the diffusion and jump component are independent, the moment-generating function of X_t is easily obtained from Equations (4.40) and (4.41). For $\theta \in (-\lambda_\ominus, \lambda_\oplus)$, it holds that $\mathbb{E}[e^{\theta X_t}] = e^{G(\theta)t}$, where $G : \mathbb{R} \setminus \{\lambda_\oplus, -\lambda_\ominus\} \rightarrow \mathbb{R}$ is defined as

$$G(x) := x\gamma + \frac{1}{2}x^2\sigma^2 + \lambda \left(\frac{p\lambda_\oplus}{\lambda_\oplus - x} + \frac{(1-p)\lambda_\ominus}{\lambda_\ominus + x} - 1 \right). \quad (4.42)$$

For the structural firm-value model, this implies that the n^{th} moment of $V_t = v_0 e^{X_t}$ exists, if and only if the corresponding exponential moment of X_t is finite. This is fulfilled¹⁵ as long as $\lambda_\oplus > n$. Given their existence, the first two moments of V_t satisfy

$$\begin{aligned} \mathbb{E}[V_t] &= v_0 \exp \left(\gamma t + \frac{1}{2}\sigma^2 t + \lambda t \left(\frac{p\lambda_\oplus}{\lambda_\oplus - 1} + \frac{(1-p)\lambda_\ominus}{\lambda_\ominus + 1} - 1 \right) \right), \\ \text{Var}(V_t) &= v_0^2 (e^{tG(2)} - e^{2tG(1)}). \end{aligned}$$

Given the moment-generating function of X_t from Equation (4.42) it is straightforward to derive the closely related Lévy-Khinchin representation of the process X . This characterization is given by $\mathbb{E}[e^{izX_t}] = e^{t\Psi(z)}$ for $z \in \mathbb{R}$, with characteristic exponent

$$\Psi(z) := iz\gamma - \frac{1}{2}z^2\sigma^2 + \lambda \left(\frac{p\lambda_\oplus}{\lambda_\oplus - iz} + \frac{(1-p)\lambda_\ominus}{\lambda_\ominus + iz} - 1 \right). \quad (4.43)$$

¹⁵ Degenerated cases such as $p = 0$ or $\lambda = 0$ may require weaker conditions.

As the jump-size distribution is integrable, we can compute the Lévy triplet of X without truncating large jumps, see Cont and Tankov (2004), page 83. Using Lemma 2.1.2 and the abbreviation

$$\begin{aligned}\mu_1 &:= \gamma + \int_{|x| \leq 1} xf(x)dx \\ &= \gamma + \lambda \left(\frac{(1-p)e^{-\lambda_\ominus}}{\lambda_\ominus} (1 + \lambda_\ominus - e^{\lambda_\ominus}) + \frac{pe^{-\lambda_\oplus}}{\lambda_\oplus} (-1 - \lambda_\oplus + e^{\lambda_\oplus}) \right),\end{aligned}$$

we obtain the following Lévy triplets of X

$$(\mu_1, \sigma^2, \lambda f(x)dx)_1, \quad (\mu_c, \sigma^2, \lambda f(x)dx)_c,$$

computed relative to $\mathbf{1}_{\{|x| \leq 1\}}$ and no truncation, respectively.

4.6.2 The Laplace transform of first-passage times

Previously, we observed that pricing corporate bonds and credit derivatives requires the distribution of first-passage times, compare Equations (3.1) and (3.3). We also pointed out that for an arbitrary jump-diffusion process a closed-form expression of this distribution is not known. However, due to the memoryless property of the exponential distribution it is possible to explicitly calculate the Laplace transform¹⁶ of first-passage times if two-sided exponentially distributed jumps are assumed. To begin with, we recall the definitions

$$\tau_b := \inf \{t \geq 0 : X_t \leq b\}, \quad \tau^{\tilde{b}} := \inf \{t \geq 0 : X_t \geq \tilde{b}\}, \quad b < X_0 < \tilde{b}.$$

To derive an approximation of default probabilities, we proceed as follows. Given the Laplace transform of $\mathbb{P}(\tau_b \leq t)$, which is derived in Theorem 4.6.1, the second step is to numerically recover $\mathbb{P}(\tau_b \leq t)$ from this transform. To do so, we implemented the Gaver-Stehfest algorithm which is explained in Section 2.2.2.

The Laplace transform of $\mathbb{P}(\tau_b \leq t)$

For brevity, we define $\varphi(\alpha) := (\mathcal{L}(\mathbb{P}(\tau_b \leq t))) (\alpha)$. Using integration by parts, we rewrite the Laplace transform of $\mathbb{P}(\tau_b \leq t)$ as the following expectation

$$\varphi(\alpha) = \int_0^\infty e^{-\alpha t} \mathbb{P}(\tau_b \leq t) dt = \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} d\mathbb{P}(\tau_b \leq t) = \frac{1}{\alpha} \mathbb{E}[e^{-\alpha \tau_b}]. \quad (4.44)$$

¹⁶ The definition of this transform and a collection of its properties was given in Section 2.2.

This expectation admits an analytical solution which is presented in Theorem 4.6.1. To derive this result, we need the following lemma about the function $G(x)$ of Equation (4.42).

Lemma 4.6.1 (Kou and Wang (2003): The roots of $G(x) - \alpha$)

For $\alpha > 0$, the function $G(x) - \alpha$ has exactly four roots. These roots are denoted by $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$. Moreover, all roots are real and satisfy

$$0 < \beta_{1,\alpha} < \lambda_{\oplus} < \beta_{2,\alpha} < \infty, \quad 0 < \beta_{3,\alpha} < \lambda_{\ominus} < \beta_{4,\alpha} < \infty.$$

Kou and Wang (2003) expressed the Laplace transform of $\mathbb{P}(\tau^b \leq t)$ in terms of these roots. They showed that $\mathbb{E} \left[\exp(-\alpha \tau^b) \right] = A_1 e^{-\tilde{b}\beta_{1,\alpha}} + B_1 e^{-\tilde{b}\beta_{2,\alpha}}$, where

$$A_1 = \frac{\lambda_{\oplus} - \beta_{1,\alpha}}{\lambda_{\oplus}} \frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}, \quad B_1 = \frac{\beta_{2,\alpha} - \lambda_{\oplus}}{\lambda_{\oplus}} \frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}.$$

We adapt their proof to obtain the corresponding Laplace transforms of $\mathbb{P}(\tau_b \leq t)$.

Theorem 4.6.1 (The Laplace transform of $\mathbb{P}(\tau_b \leq t)$)

Let $\alpha > 0$ and $b < 0$. Then

$$\mathbb{E}[e^{-\alpha \tau_b}] = A_2 e^{b\beta_{3,\alpha}} + B_2 e^{b\beta_{4,\alpha}}, \quad (4.45)$$

where the factors A_2 and B_2 are defined as

$$A_2 := \frac{\lambda_{\ominus} - \beta_{3,\alpha}}{\lambda_{\ominus}} \frac{\beta_{4,\alpha}}{\beta_{4,\alpha} - \beta_{3,\alpha}}, \quad B_2 := \frac{\beta_{4,\alpha} - \lambda_{\ominus}}{\lambda_{\ominus}} \frac{\beta_{3,\alpha}}{\beta_{4,\alpha} - \beta_{3,\alpha}}.$$

The Laplace transform of $\mathbb{P}(\tau_b \leq t)$ is then obtained from Equation (4.44).

Sketch of the proof: This proof is based on the corresponding proof of Kou and Wang (2003) for the running maximum. For brevity, we write $\beta_i = \beta_{i,\alpha}$ and define

$$u(x) := \begin{cases} 1 & : x \leq b, \\ A_2 e^{\beta_3(b-x)} + B_2 e^{\beta_4(b-x)} & : x > b. \end{cases}$$

After some lengthy algebraic manipulations, we find

$$-\alpha u(x) + (\mathcal{G}(u))(x) = 0 \quad \forall x > b,$$

where the infinitesimal generator $\mathcal{G}$ of the jump-diffusion process X is given by

$$(\mathcal{G}(v))(x) = \frac{1}{2}\sigma^2 v''(x) + \gamma v'(x) + \lambda \int_{-\infty}^{\infty} (v(x+y) - v(x)) f(y) dy,$$

with density function f from Equation (4.37), for all $v \in C^2$. We approximate u using a sequence $\{u_n\}_{n \in \mathbb{N}}$ of C^2 functions with properties $u_n = u$ on $x \geq b$, $u_n \equiv 1$ on $x \leq b - 1/n$ and $u_n \leq 2$. For all $x > b$, this gives

$$(\mathcal{G}(u_n))(x) = \alpha u(x) + \lambda \int_{b-x-1/n}^{b-x} u_n(x+y) f(y) dy - \lambda \int_{b-x-1/n}^{b-x} u(x+y) f(y) dy,$$

which we use to establish

$$| -\alpha u_n(x) + (\mathcal{G}(u_n))(x) | \leq \frac{\lambda \lambda_{\ominus}}{n} \quad \forall x > b.$$

An application of Lemma 2.1.4 (Itô's formula for jump-diffusion processes) gives

$$\begin{aligned} e^{-\alpha(t \wedge \tau_b)} u_n(X_{t \wedge \tau_b}) &= u_n(X_0) + \\ &\int_0^{t \wedge \tau_b} e^{-\alpha s} (-\alpha u_n(X_s) + (\mathcal{G}(u_n))(X_s)) ds + \int_0^{t \wedge \tau_b} e^{-\alpha s} \sigma u'_n(X_s) dW_s + \\ &\int_0^{t \wedge \tau_b} e^{-\alpha s} \int_{-\infty}^{\infty} (u_n(X_{s-} + y) - u_n(X_{s-})) (J_X(ds \times dy) - \lambda f(y) dy ds), \end{aligned}$$

from which it follows that

$$M_t^n := e^{-\alpha(t \wedge \tau_b)} u_n(X_{t \wedge \tau_b}) - \int_0^{t \wedge \tau_b} e^{-\alpha s} (-\alpha u_n(X_s) + (\mathcal{G}(u_n))(X_s)) ds$$

is a local martingale starting at $u_n(0) = u(0)$. By dominated convergence, M^n is even a martingale. Hence, we have

$$\mathbb{E}[M_t^n] = \mathbb{E} \left[e^{-\alpha(t \wedge \tau_b)} u_n(X_{t \wedge \tau_b}) - \int_0^{t \wedge \tau_b} e^{-\alpha s} (-\alpha u_n(X_s) + (\mathcal{G}(u_n))(X_s)) ds \right] = u(0).$$

By uniform convergence, we observe that the second summand vanishes as n tends to infinity. This gives

$$\begin{aligned} u(0) &= \mathbb{E} [e^{-\alpha(t \wedge \tau_b)} u(X_{t \wedge \tau_b})] \\ &= \mathbb{E} [e^{-\alpha(t \wedge \tau_b)} u(X_{t \wedge \tau_b}) \mathbf{1}_{\{\tau_b < \infty\}}] + \mathbb{E} [e^{-\alpha(t \wedge \tau_b)} u(X_{t \wedge \tau_b}) \mathbf{1}_{\{\tau_b = \infty\}}]. \end{aligned}$$

Finally, we let t tend to infinity and use that u is bounded and $u(X_{\tau_b}) = 1$ on the set $\{\tau_b < \infty\}$. We conclude

$$u(0) = \mathbb{E} [e^{-\alpha\tau_b} u(X_{\tau_b})] = \mathbb{E} [e^{-\alpha\tau_b}] ,$$

establishing the claim. $\diamond$

4.6.3 Bond and CDS pricing using the Laplace transform

Given the explicit expression of the Laplace transform of $\mathbb{P}(\tau_b \leq t)$ from Equation (4.45), we need an algorithm that numerically recovers this probability. In what follows, our implementations are based on the Gaver-Stehfest algorithm, as this algorithm has the advantage of working purely on the real line. This feature is important, as the Laplace transform of $\mathbb{P}(\tau_b \leq t)$ is only derived for positive values of α . As a result of several numerical experiments, we found that the precision of this inversion algorithm strongly depends on the precision of the roots of Lemma 4.6.1. Also, we are interested in a fast implementation of finding these roots. We found that instead of numerically solving $G(x) - \alpha = 0$, a better performance in terms of precision and speed is achieved if the expression $G(x) - \alpha = 0$ is rewritten in terms of a quartic polynomial. Using Ferrari's algorithm, it is then possible to find these roots algebraically, as illustrated in Section 6.2. Alternatively, we obtained a comparable performance in terms of precision and speed using the Pegasus algorithm, which is explained in Engeln-Müllges and Reuter (1991), page 34. The advantage of Ferrari's algorithm is obvious, the roots are given by algebraic formulas. However, these formulas are very long and require a complex arithmetic, which possibly leads to an accumulation of round-off errors that exceed the precision of an approximation algorithm with given terminal condition. Therefore, it is often more convenient to implement the real-valued Pegasus algorithm.

Given the approximation of first-passage probabilities, the idea of our Laplace-pricing algorithm is to approximate the Riemann-Stieltjes integral of Equation (3.1) as a Riemann-Stieltjes sum. The integrator is evaluated at the points of the partition using the inverse Laplace method. Considering CDS prices, we apply the same approach to Equation (4.17).

Algorithm 4.6.1 (Bond and CDS pricing using the Laplace transform)

Choose $K, N \in \mathbb{N}$, where K denotes the number of subintervals of $[0, T]$ and N denotes the precision of the inverse Laplace algorithm¹⁷.

¹⁷ As a rule of thumb, we suggest $K = 50T$ and $N = 9$.

1. Partition the interval $[0, T]$ into K equidistant subintervals. Denote the endpoints of these subintervals by $t_j := jT/K$ for $j \in \{0, \dots, K\}$.
2. Approximate the required default probabilities using the Gaver-Stehfest algorithm. With the notations of Algorithm 2.2.1, compute for $j \in \{1, \dots, K\}$

$$\hat{\mathbb{P}}_N(\tau \leq t_j) = \sum_{k=1}^N w(k, N) \tilde{F}_{k+2}(t_j).$$

3. Approximate the zero-coupon bond price $\phi(0, T)$ by

$$\begin{aligned} \phi_{K,N}(0, T) &:= e^{-rT} \left(1 - \hat{\mathbb{P}}_N(\tau \leq T)\right) + \\ &\quad R \sum_{j=1}^K \exp\left(-r \frac{t_{j-1} + t_j}{2}\right) \hat{\mathbb{P}}_N(t_{j-1} < \tau \leq t_j). \end{aligned}$$

4. Approximate the CDS price $CDS(0, T)$ and the par spread c_T by

$$\begin{aligned} CDS_{K,N}(0, T) &:= ((R-1)r - c) D_K + (1-R) \left(1 - e^{-rT} \hat{\mathbb{P}}_N(\tau > T)\right), \\ c_T^{K,N} &:= \frac{(1-R) \left(1 - e^{-rT} \hat{\mathbb{P}}_N(\tau > T) - r D_K\right)}{D_K}, \end{aligned}$$

where the integral $\int_0^T \mathbb{P}(\tau > t) e^{-rt} dt$ is approximated via

$$D_K := \frac{T}{K} \sum_{j=1}^K \hat{\mathbb{P}}_N\left(\tau > \frac{t_{j-1} + t_j}{2}\right) \exp\left(-r \frac{t_{j-1} + t_j}{2}\right).$$

The vector $\left(\hat{\mathbb{P}}_N(\tau > (t_{j-1} + t_j)/2)\right)_{j=1}^K$ is computed as in Step 2.

In Section 4.8, we provide a comparison of Algorithm 4.6.1 with the Monte Carlo simulation of Zhou (2001a) and our Monte Carlo approach of Section 4.3.2 based on different fictitious scenarios. This analysis shows that Algorithm 4.6.1 is highly accurate and by far the fastest pricing algorithm in a framework with two-sided exponentially distributed jumps.

4.7 Sensitivity of the model parameters

In this section, we analyze the sensitivity of bond and CDS spreads in a structural jump-diffusion model with respect to changes in the parameters of the model. Throughout this paragraph, two-sided exponentially distributed jumps are assumed, as this choice of jump-size distribution allows us to use Algorithm 4.6.1 to obtain the presented graphs. The risk-free interest rate and the recovery rate are assumed to equal $r = 0.03$ and $R = 40\%$, respectively, the maturity varies within $T \in [0, 5]$ years. We present two graphs for each set of firm-value parameters. The graph on the left-hand side of the corresponding figure exhibits credit spreads of a zero-coupon bond, the graph on the right-hand side presents par spreads of a CDS contract written on this bond. The standard case for this analysis is a company with parameters¹⁸ as given in Table 4.4. In each of the following subsections, we fix all but one parameter. This parameter is then varied within an appropriate range.

γ_i	σ_i	$\mathbb{P}_Y$	λ	d/v_0
0.025	0.05	2-Exp(20, 20, 0.5)	2.0	80.0%

Table 4.4: The standard case of Section 4.7.

4.7.1 Sensitivity with respect to the drift

To begin with, we vary the drift of X within $\{-0.01, 0.01, 0.03, 0.05, 0.07\}$. Intuitively, the drift specifies the systematic growth of the modeled company. We observe that the resulting term structures of spreads are decreasing in γ , meaning that the highest curves in both graphs of Figure 4.2 correspond to the smallest gamma, and vice versa. This result is as expected, since default probabilities are obviously decreasing in γ . Also, we observe that the influence of the drift is more pronounced for long maturities. Finally, we find that the limit of spreads at the short end of the term structure does not depend on the drift of the diffusion. This property was predicted by Theorems 4.3.4 and 4.4.2, which state that the limit of spreads does not depend on the diffusion component of X .

Conclusion 1

Spreads are decreasing in γ . However, the impact of γ strongly depends on the time to maturity. In our example, spreads corresponding to maturities $T < \frac{1}{2}$ hardly changed with γ , whereas for $T > 1$ a strong dependence was observed.

¹⁸ Based on the results of our calibration in Section 4.9.2, we feel that these parameters are realistic for a speculative-grade company.

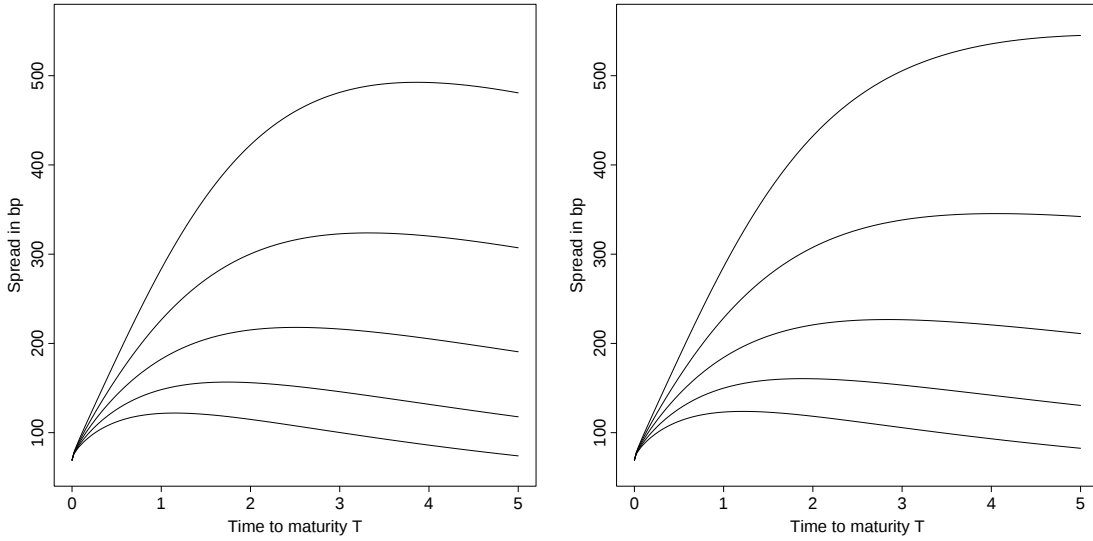


Figure 4.2: Bond and CDS spreads depending on γ .

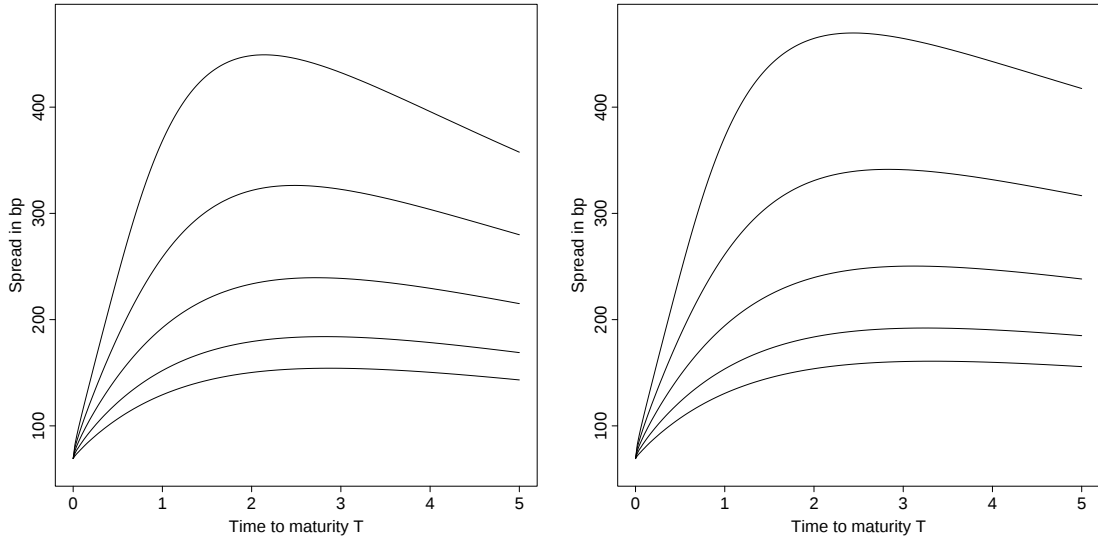
4.7.2 Sensitivity with respect to the diffusion volatility

We now return to the original drift $\gamma = 0.025$ and vary the parameter $\sigma > 0$ within $\{0.01, 0.03, 0.05, 0.07, 0.09\}$. The volatility of X_t linearly grows in σ^2 , which implies increasing default probabilities in this parameter¹⁹. As a result, spreads are increasing in σ . Consequently, larger values of σ correspond to higher curves in both graphs of Figure 4.3. Moreover, we observe that the limit of spreads as maturity decreases to zero does not depend on the volatility of the diffusion component, which is again a consequence of Theorems 4.3.4 and 4.4.2. Another interesting observation is that increasing values of σ imply that the diffusion component starts dominating the overall behavior of the process X_t . As a result, the typical hump-size structure of spreads, as implied by pure diffusion models, is more pronounced.

Conclusion 2

For realistic firm-value parameters, spreads are increasing in σ . The limit of spreads at time zero is positive and independent of the diffusion component of X . In our examples, contracts maturing within one or three years showed the strongest dependence on σ .

¹⁹ This statement holds in our standard case, where the drift is positive and jumps are symmetric around zero. However, if the drift or the expectation of the jump-size distribution are extremely negative, then it might happen that larger values of σ reduce the default probability up to some maturity. We constructed such examples, but the required parameters are extremely unrealistic for the firm-value process of a company.

Figure 4.3: Bond and CDS spreads depending on σ .

4.7.3 Sensitivity with respect to the jump intensity

In this section, we show that the influence of the jump intensity λ on credit and CDS spreads depends on the jump-size distribution $\mathbb{P}_Y$. To begin with, let us recall that the expectation and variance of X_t are given by $t(\gamma + \lambda \mathbb{E}[Y])$ and $t(\sigma^2 + \lambda \mathbb{E}[Y^2])$, respectively. Hence, the sign of $\mathbb{E}[Y]$ determines whether the center of the process X is increasing or decreasing in λ . In contrast, the variance of X_t is always increasing in λ . If the expectation of Y is less than or equal to zero, then spreads are obviously increasing in λ . More ambiguous is a positive expectation of Y , since both the center of the process X and the variance of X_t are increasing in λ . Whichever effect dominates depends on the jump-size distribution and the time to maturity. For short maturities, spreads are close to being linearly increasing in λ , compare Theorems 4.3.4 and 4.4.2. For longer maturities, additional jumps with positive expectations possibly reduce spreads. To illustrate this claim, we construct two sets of parameters. The first example is $\mathbb{P}_Y = 2\text{-Exp}(20, 20, \frac{1}{2})$, which corresponds to an expectation of zero. Our second choice is $\mathbb{P}_Y = 2\text{-Exp}(10, 30, \frac{4}{5})$, corresponding to a positive expectation²⁰. Then, we fix the diffusion and leverage ratio d/v_0 as in Table 4.4 and vary λ in $\{0, 0.5, 1, 2, 3\}$. As a result, we obtain Figures 4.4 and 4.5. The different curves of these four graphs can easily be distinguished according to their values at time zero, which are increasing in λ .

Conclusion 3

For short maturities, spreads are close to being linearly increasing in λ . The long-

²⁰ The obvious case of a negative expectation of Y is omitted.

term effect of λ depends on the sign of $\mathbb{E}[Y]$. However, if the expectation of Y is not extremely positive, then spreads are increasing in λ for all maturities. In our experiments, spreads of contracts maturing within one or three years showed the strongest dependence on λ .

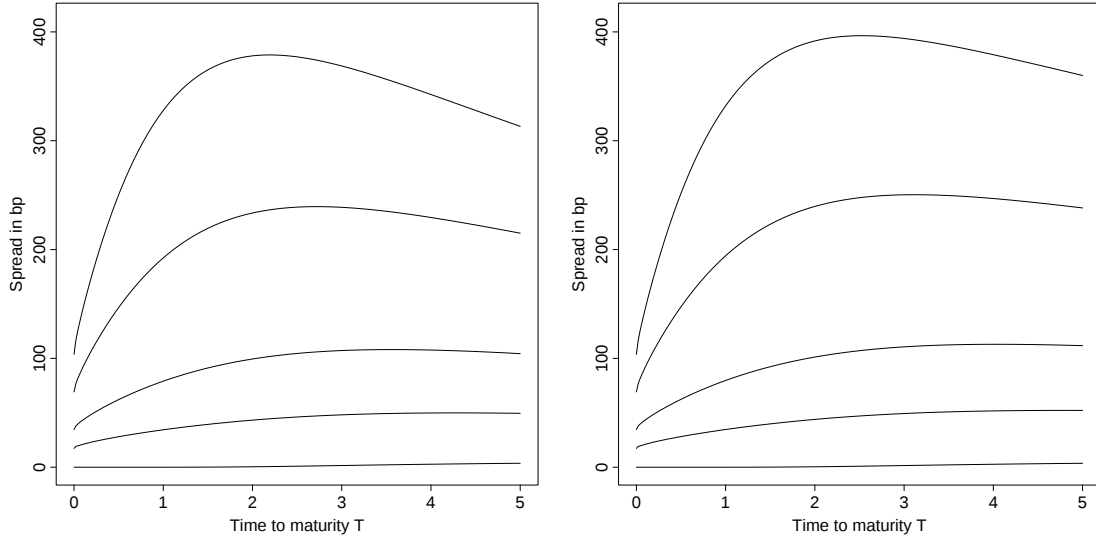


Figure 4.4: Bond and CDS spreads depending on λ , $\mathbb{P}_Y = 2\text{-Exp}(20, 20, \frac{1}{2})$.

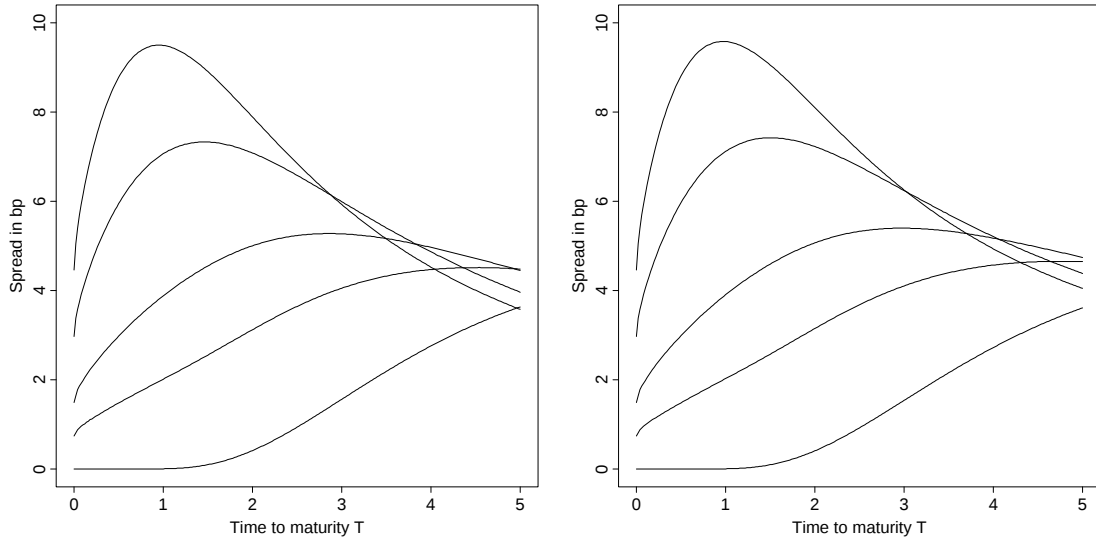


Figure 4.5: Bond and CDS spreads depending on λ , $\mathbb{P}_Y = (10, 30, \frac{4}{5})$.

4.7.4 Sensitivity with respect to the influence of jumps

Equation (4.39) allows us to determine what percentage of $\text{Var}(X_t)$ is explained by jumps. Given $\mathbb{P}_Y$ as in Table 4.4, the compound Poisson component contributes

the fraction $2\lambda/(\lambda_{\oplus}^2\sigma^2+2\lambda)$ to the overall variance of X_t . We fix $\text{Var}(X_t) = t(0.1)^2$ and choose σ and λ such that $\text{Var}(X_t)$ is explained to 0%, 25%, 50%, 75% and 95% by jumps. To distinguish the curves of Figure 4.6, let us remark that at time zero the spread is strictly increasing in λ . We also observe that especially for small maturities, an increasing probability of defaulting by jump results in significantly larger spreads, which is again a consequence of Theorems 4.3.4 and 4.4.2. However, it turns out that the key-factor for long maturities is the volatility of the diffusion component.

Conclusion 4

For short maturities, spreads primarily depend on the frequency and size of possible jumps. Spreads of contracts with long maturities depend to a large extent on the parameters of the diffusion component of X .

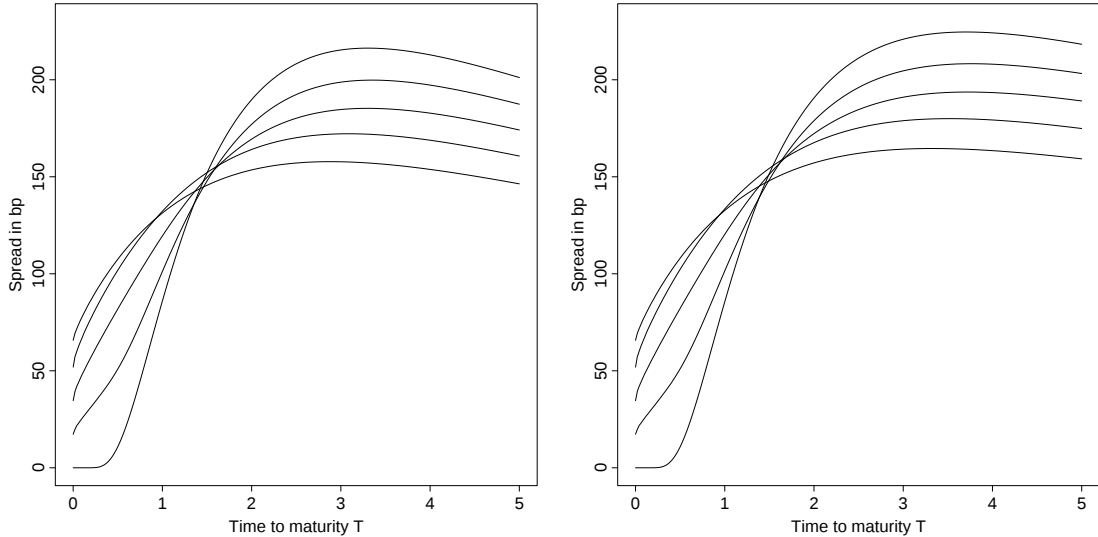


Figure 4.6: Bond and CDS spreads depending on the source of variance.

4.7.5 Sensitivity with respect to the leverage ratio

Finally, we fix all parameters but the leverage ratio as in Table 4.4. We then vary d/v_0 in $\{75\%, 80\%, 85\%, 90\%, 92.5\%\}$. Obviously, spreads are increasing in d/v_0 , as so are default probabilities. Surprising is how sensitive spreads of highly leveraged companies are, compare the tall peaks in Figure 4.7. Also, we notice that spreads of highly leveraged companies exhibit the typical hump-size structure. In contrast, the term structure of spreads of companies with more equity capital is upward sloping, at least on the interval $T \in [0, 5]$. Both observations are supported by empirical evidence.

Conclusion 5

Spreads are increasing in the leverage ratio of the company. For highly leveraged companies, the hump-size structure of spreads is more pronounced.

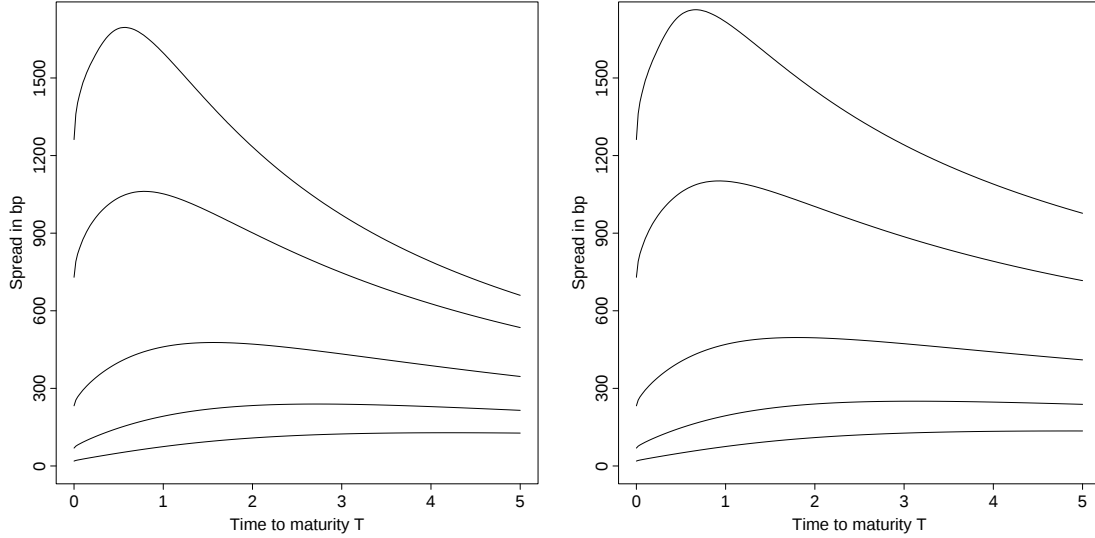


Figure 4.7: Bond and CDS spreads depending on the ratio d/v_0 .

4.7.6 Summary

At first, we conclude that spreads react to changes in the parameters of the model as expected. For reasonable parameters, spreads are decreasing in the drift and increasing in the volatility of the diffusion component of X . A larger jump intensity increases the volatility of X_t , which typically implies increasing spreads²¹. Finally, spreads are increasing in the leverage ratio of the company. The property of being consistent with our intuition is a very important feature for practical applications, as it allows us to attach an interpretation to each of the parameters. Given such interpretations, the mechanisms of the model are easily understood and accepted by practitioners.

Our next observation is the extreme flexibility of the model in explaining different term structures of spreads. This flexibility enables the model to match prices of a vast spectrum of companies. The drawbacks of pure diffusion models, i.e. vanishing spreads and the restriction to the typical hump-size structure, are overcome by allowing negative jumps. For most real companies, we observed the term structure of spreads to be increasing with positive limit at the short end, which is matched by the model using appropriate parameters.

²¹ As long as the jump-size distribution is not essentially located on the positive axis.

As a result of our investigations, we finally suggest the following rule of thumb. While the jump component essentially explains the short-term behavior of spreads, the long-term structure is explained to a large extent by the diffusion component. This rule is useful if the parameters of the model are calibrated to market quotes.

4.8 A comparison of the different algorithms

In this section, we provide a numerical comparison of all algorithms mentioned earlier. We implemented all algorithms in C, using the NAG-software library²² to generate the required random numbers. We worked on a *Sun* computer equipped with an UltraSPARC-III+ processor with 900MHz. To provide a benchmark of the run time, the output user time of the Unix command `timex` was chosen.

4.8.1 Run time and precision

Concerning Zhou's algorithm, we used two different discretizations. The number of grid points was set to $12T$ and $250T$, where T denotes the maturity in years, which corresponds to checking whether the bond defaulted once per month or once per trading day, respectively. As parameters, we chose $r = 0.04$, $\gamma = 0.045$, $\sigma = 0.05$ and $T = 5$. Jump sizes are assumed to be two-sided exponentially distributed with $p = 0.5$ and different choices of $\lambda_{\oplus} = \lambda_{\ominus}$. The leverage ratio d/v_0 is set to 80%. We performed all simulations in four different scenarios. In the first three scenarios, the recovery rate is kept constant, that is $w_0(x) \equiv 40\%$ with the notation of Section 4.5.1. In the scenario entitled *Low*, we expect only $\lambda = 0.5$ jumps per year, but they are expected to be large, so we choose $\lambda_{\ominus} = \lambda_{\oplus} = 10$. The scenario *Middle* corresponds to $\lambda = 2$ and $\lambda_{\ominus} = \lambda_{\oplus} = 20$. In the scenario *High*, $\lambda = 8$ jumps per year are expected with $\lambda_{\ominus} = \lambda_{\oplus} = 40$. Finally, the scenario *Stochastic* has the same jump structure as the scenario *Middle*, but the recovery rate is stochastic with $w_1(x) = 0.5x$. It can be shown that the volatility and expectation of the underlying Lévy process X remain the same in all scenarios. We performed one million simulations per Monte Carlo algorithm. Since the Brownian-bridge pricing technique generates an unbiased price, we additionally performed ten million runs of this algorithm and interpreted the result as the correct price.

We did two parts of simulations. In the first part, each Monte Carlo algorithm was run one million times individually and the respective run time was taken. In the second part, we measured the accuracy of each algorithm. In order to obtain a comparison of the Brownian bridge algorithm with both of its approximations which is not based on different samples, we initially generated one million samples of $\mathcal{F}^*$, compare Equation (4.12). Then, we computed the resulting output of the three algorithms based on the same samples of $\mathcal{F}^*$. The results of this experiment are presented in Table 4.5 below.

²² See <http://www.nag.co.uk> for details.

		<i>Low</i>	<i>Middle</i>	<i>High</i>	<i>Stochastic</i>
Zhou's algorithm, Algorithm 6.3.1 ($K=12T$)	Spread in bp	104.52	116.98	125.14	97.07
	Rel. error in %	7.3486	9.8142	11.1032	9.5424
	h:min:sec	0:07:32	0:07:57	0:10:17	0:8:11
Zhou's algorithm, Algorithm 6.3.1 ($K=250T$)	Spread in bp	109.60	125.77	136.19	103.01
	Rel. error in %	2.8455	3.0375	3.2535	4.0071
	h:min:sec	2:33:09	2:31:48	2:34:49	2:32:48
Brownian-bridge pricing technique, Algorithm 4.3.1	Spread in bp	112.65	129.80	140.96	107.35
	Rel. error in %	0.1418	-0.0694	-0.1350	-0.0373
	h:min:sec	0:06:39	0:19:05	0:58:47	0:19:08
Original integral approximation of Metwally and Atiya	Spread in bp	113.48	130.30	141.10	107.97
	Rel. error in %	-0.5939	-0.4549	-0.2344	-0.6150
	h:min:sec	0:00:44	0:02:28	0:9:24	0:2:28
Our integral approximation, Theorem 4.3.3	Spread in bp	112.54	129.69	140.85	107.22
	Rel. error in %	0.2393	0.0154	-0.0568	0.0839
	h:min:sec	0:00:45	0:02:25	0:9:17	0:2:25
Laplace pricing, Algorithm 4.6.1 ($N=4, K=50T$)	Spread in bp	113.49	130.24	141.33	n.a.
	Rel. error in %	-0.6028	-0.4049	-0.3991	n.a.
	h:min:sec	0:00:00	0:00:00	0:00:00	n.a.
Laplace pricing, Algorithm 4.6.1 ($N=7, K=50T$)	Spread in bp	112.81	129.64	140.85	n.a.
	Rel. error in %	0.0012	0.0524	-0.0535	n.a.
	h:min:sec	0:00:00	0:00:00	0:00:00	n.a.
Laplace pricing, Algorithm 4.6.1 ($N=10, K=50T$)	Spread in bp	112.83	129.72	140.90	n.a.
	Rel. error in %	-0.0172	-0.0061	-0.0940	n.a.
	h:min:sec	0:00:00	0:00:00	0:00:00	n.a.
Brownian bridge, 10^7 runs, in bp		112.81	129.71	140.77	107.31

Table 4.5: Comparison of the algorithms.

Table 4.5 contains, beside the run time and the generated credit spread of each algorithm and scenario, the relative error of the credit spread, which we define as $(\text{spread} - \text{generated spread})/\text{spread}$ ²³.

The data set shows that Zhou's algorithm produces a significant bias. When simulating with only 12 grid points per year, the relative error exceeds 7%. Even with 250 grid points per year, the relative error is at least 2.8%, which is still above bid-ask spreads. Examining the run time of the Brownian-bridge pricing technique shows that it depends strongly on the expected number of jumps. The reason for this is the dependence of the number of random numbers that have to be drawn and the number of integrals which have to be calculated on the number of jumps. We also observe that the approximation of the integrals significantly reduces the

²³ While *spread* denotes the credit spread obtained from the Brownian-bridge simulation with ten million runs, *generated spread* represents the credit spread from the corresponding algorithm.

run time, without producing a relevant bias.

In terms of speed, the Laplace algorithm outperforms every Monte Carlo simulation by far. Algorithm 4.6.1 is able to price bonds in fractions of seconds, it is therefore the best choice for applications such as a calibration, where the pricing formula has to be evaluated several times. Another advantage of this approximation compared to a Monte Carlo simulation is the following. While two Monte Carlo runs with the same parameters most likely provide different results, it is possible to interpret the output of Algorithm 4.6.1 as a real valued function of the parameter space. Hence, it is possible to numerically approximate its partial derivatives, which is required in most numerical optimization routines. However, the drawback of this approach is that it is only available for two-sided exponentially distributed jumps. Also, generalizations to stochastic recovery rates or stochastic interest rates are not available.

4.8.2 A closer comparison of both integral approximations

Our approximation of $\int e^{-rs} g_i(s) ds$ differs slightly from the approximation of Metwally and Atiya (2002). They obtain the additional factor $\exp(-\Delta\tau_i)$ and evaluate Φ at different points²⁴. Since the error terms in Equations (4.14) and (4.15) are negative, we expect bond prices generated by both approximations to be slightly larger than real prices. Said differently, we expect that the corresponding credit spreads are slightly lower. In order to compare the accuracy of both approximations, we performed two kinds of simulations. Firstly, we compared the approximations for randomly generated parameters. Secondly, we compared bond prices obtained by both approximations.

A comparison based on random numbers

We generated random numbers for r , σ , τ_1 , b and $X_{\tau_1} - b$. More precisely, in every step we draw a uniformly distributed random number on $[0, 0.1]$ (resp. $[0.1, 0.5]$, $[0.5, 2.0]$, $[-0.2, -0.01]$, $[-0.2, 0.2]$) for r (resp. σ , τ_1 , b , $X_{\tau_1} - b$). We chose those ranges as they are realistic for our bond-pricing problem. For every such parameter set, we calculated the relative error of the approximation suggested by Metwally and Atiya (2002) and of our approximation of $\int_0^{\tau_1} e^{-rs} g_1(s) ds$. We compared those results with a highly accurate²⁵ numerical approximation of the

²⁴ To be more precise, in Metwally and Atiya (2002) the second term in the sums of Equations (4.14) and (4.15) is multiplied by $e^{-\Delta\tau_i}$. Moreover, Φ is evaluated at $(2b - X_{\tau_i} - X_{\tau_{i-1}})/\sqrt{2\Delta\tau_i\sigma^2}$ and $\Delta X_i/\sqrt{2\Delta\tau_i\sigma^2}$ in C_1 and C_2 , respectively.

²⁵ We used the NAG routine `d01ajc` with parameter `epsabs` = 10^{-8} for the absolute error.

integral, which we interpreted as the correct value. After 500,000 simulations, the average relative error of the original and our approximation was found to be 1.553% and 0.001%, respectively.

The effect of both approximations on bond prices

To test the impact of the improved and the original approximation on resulting bond prices, we implemented Algorithm 4.3.1 using both approximations and calculated bond prices for different parameter sets and different interest rates. Our simulations show that our approximation almost always implies a lower relative pricing error²⁶. While the pricing error of the original algorithm increases significantly in r , the pricing error of our modification remains small when the interest rate increases. Our formulas always produce a bias as expected, namely bond prices which are slightly too high. In contrast, the original formulas imply bond prices which are too low.

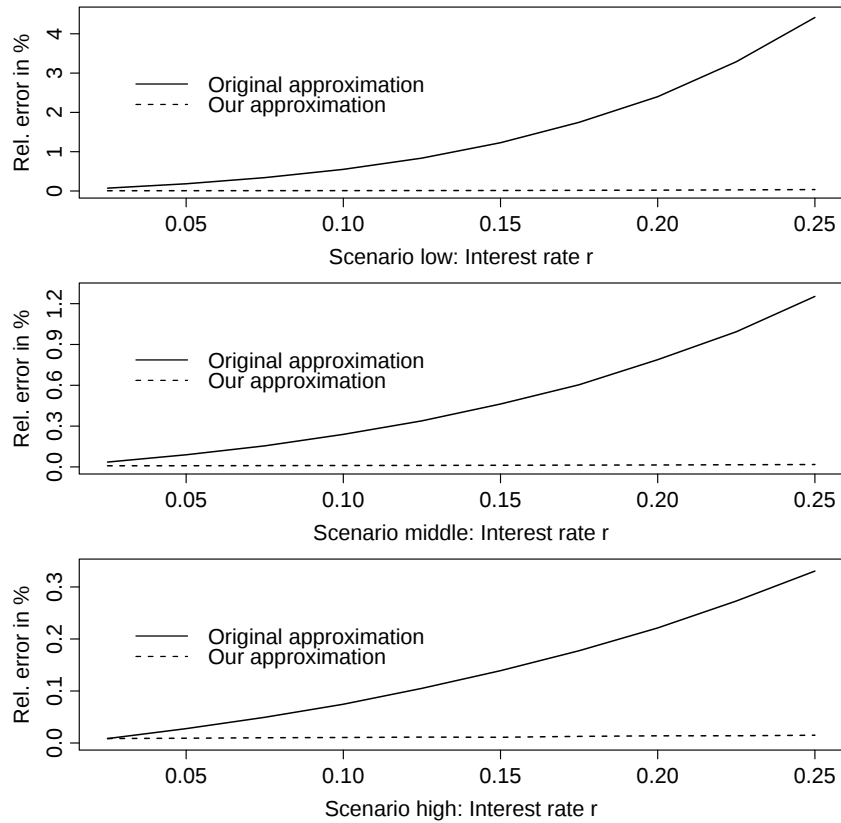


Figure 4.8: The relative pricing error of the original and our approximation.

²⁶ Except for the scenario *High*, when r equals 2.5%.

Figure 4.8 displays the absolute values of the relative pricing errors of both approximations in different scenarios. More precisely, it shows $|(p_u - p_a)/p_u|$, where p_u and p_a represent the unbiased and approximated bond price, respectively. We calculated the pricing error in the first three scenarios of Table 4.5 with parameters $\gamma = 0.025$, $\sigma = 0.05$ and two-sided exponentially distributed jumps with $p = 0.5$. The recovery rate was set constant to $R = 40\%$ and $T = 5$ was chosen as maturity. For each scenario and interest rate, we performed ten million simulations. For the scenario *Low*, the relative pricing error of the original Taylor approximation increases to 4.41%, when the interest rate tends to 25%²⁷. In the scenario *Middle*, the pricing error increases to 1.25%, and even in the scenario *High*, the pricing error still exceeds 0.33%. In contrast, applying our formulas, the pricing error remains relatively small, namely around 0.036%, 0.018% and 0.015%, respectively. The exact numbers underlying this graph are given in Table 4.6.

	Low		Middle		High	
r	Original	Our	Original	Our	Original	Our
0.025	0.072560	-0.006450	0.035636	-0.008159	0.008500	-0.008627
0.050	0.183640	-0.007161	0.089252	-0.008701	0.027612	-0.009191
0.075	0.339830	-0.008382	0.154848	-0.009499	0.049408	-0.010083
0.100	0.550475	-0.009074	0.239234	-0.010398	0.074507	-0.010582
0.125	0.836835	-0.010999	0.338649	-0.011148	0.105069	-0.011337
0.150	1.228662	-0.012840	0.462425	-0.012113	0.139050	-0.010996
0.175	1.747642	-0.016792	0.603955	-0.013367	0.177395	-0.012765
0.200	2.400023	-0.021394	0.788351	-0.014252	0.221226	-0.013789
0.225	3.288684	-0.028168	0.993531	-0.015914	0.272949	-0.013913
0.250	4.414416	-0.036117	1.253171	-0.017627	0.330520	-0.014954

Table 4.6: The relative error of the original and our approximation in %.

²⁷ Of course, $r = 0.25$ is not realistic. Nevertheless, simulations with high interest rates illustrate how fast the original approximation becomes inaccurate. However, even for $r = 0.05$, the relative pricing error of the original approach is 0.18%, which exceeds the relative pricing error of 0.007% in our approximation by far.

4.9 Calibration

In this section, we present an approach on how parameter estimates for the firm-value process of a company may be obtained from market data. One problem we have to overcome is that the value of the modeled company is only observable at the dates when the firm's balance sheet is published. This snapshot on the firm's financial situation is typically published at the end of each fiscal year, which excludes statistical estimation techniques based on the complete history of the firm-value process. Therefore, instead of past trajectories of the firm-value process we use market quotes of bonds and CDS contracts to calibrate the model. Nevertheless, our objective is to calibrate the firm-value process with respect to the pricing measure $\mathbb{P}$. Therefore, using up-to-date market quotes is rather a natural approach than just a makeshift. Most data used in Section 4.9 was provided by *Bloomberg L.P.*. This data service is available at the *Universität Ulm* due to the generous support of the *Landesbank Baden-Württemberg (LBBW)*.

4.9.1 Calibration to corporate bonds

The assets of all major companies are financed to a large proportion by bonds with different maturities. The quoted prices of these bonds reflect the market's view on the creditworthiness of the respective company. In what follows, we attempt to choose the parameters of our model such that the model implies the same term structure of default probabilities as the market. Said differently, the model implies the same bond prices.

In what follows, we denote the bonds of a company by $B_1, \dots, B_n$. For these bonds, we obtain model prices $B_1^M, \dots, B_n^M$ from Equation (3.1) and Lemma 3.1.1. These model prices depend on the parameters of the underlying firm-value process. To begin with, we specify the jump-size distribution $\mathbb{P}_Y$ to be a two-sided exponential distribution, as this choice allows the use of Algorithm 4.6.1. Hence, the firm-value process is fully specified by the parameters $(\gamma, \sigma, \lambda, \lambda_\oplus, \lambda_\ominus, p)$. The set of market prices which we seek to match with our model is denoted by $B_1^R, \dots, B_n^R$. Our next concern is a suitable measure of distance of model to market prices. Popular examples for this error functional are the sum of absolute, relative or squared distances. Within this section, we choose to minimize the sum of squared distances of model to market prices. Formally, we obtain the minimization problem

$$\left(\hat{\gamma}, \hat{\sigma}, \hat{\lambda}, \hat{\lambda}_\oplus, \hat{\lambda}_\ominus, \hat{p} \right) = \operatorname{argmin} \sum_{i=1}^n (B_i^R - B_i^M)^2,$$

where $\gamma \in \mathbb{R}$, $\sigma, \lambda, \lambda_{\oplus}, \lambda_{\ominus} \in \mathbb{R}^+$ and $p \in [0, 1]$.

The companies *DaimlerChrysler* (DCX) and *General Motors* (GM)

We decided to exemplarily fit the parameters of the firm-value processes of *DaimlerChrysler* (DCX) and *General Motors* (GM). Both companies have issued several liquidly traded bonds with different maturities. While DCX is considered to be an investment-grade company, GM is rated as sub investment-grade company at the time of the calibration. We slightly modify Formula (3.1), replacing the constant risk-free interest rate by a deterministic risk-free yield curve. For the respective day, we obtained this yield curve from market prices of *Bundesanleihen*, as published by *Stuttgart's stock exchange*²⁸. Both companies publish their debt-to-value ratio every quarter. These numbers are obtained from *Bloomberg*. The last parameter of our model which has to be specified is the recovery rate. Here, we rely on historical numbers as published by Altman and Kishore (1996). All industry sectors combined, they report an average recovery rate of about 41%. Considering the business structure of DCX and GM and reported recoveries for different industry sectors, we chose the recovery rate for DCX and GM a notch higher, that is $R = 42\%$. Before we present the results of our calibration in Tables 4.7, 4.8 and 4.9, let us first add some remarks on the numerical implementation.

Numerical details

Our optimization problem is to minimize a function of several variables with unknown partial derivatives. Moreover, the parameters show functional dependence and different local minima complicate the optimization. The required time to evaluate our objective function, which is the sum of squared differences of model to market prices given a set of parameters, strongly depends on the accuracy of the Laplace inversion in Algorithm 2.2.1. We obtained a good performance by gradually increasing this accuracy from $N = 3$ to $N = 9$. The minimization procedure of our choice is the routine `nag_opt_bounds_no_deriv` of the NAG-software library. According to the documentation, this procedure is implemented as follows. In each iteration step, the gradient at the current position is estimated by means of a finite-difference approximation. Further, an approximation of the Hesse-matrix is also obtained and a search direction is computed. The objective function is then minimized along the search direction and all variables are updated for the next step. A more detailed description of this algorithm is given in the documentation of the NAG-software library.

²⁸ Available online for maturities $T \in \{0, \dots, 10\}$, compare <http://www.boerse-stuttgart.de>. Non-integer maturities are obtained from a linear interpolation.

Setup and results for DCX and GM

We used bond quotes as of August 25, 2005 from *Stuttgart's stock exchange*. These prices include accrued interest, which we considered in our pricing formula. Considering the other input parameters, we used a debt-to-value ratio of $d/v_0 = 82.3913\%$ for DCX and $d/v_0 = 94.47131\%$ for GM. These ratios are obtained from the respective balance sheets of the second quarter of 2005. The risk-free yield curve is obtained from *Bundesanleihen*. The lists of bonds (currency of all bonds: Euro) which were used for the calibration are given in Table 4.8 and 4.9. Using this setup, we obtained the following results.

	$\hat{\gamma}$	$\hat{\sigma}$	$\hat{p}$	$\hat{\lambda}$	$\hat{\lambda}_{\oplus}$	$\hat{\lambda}_{\ominus}$	$\sum(B_i^M - B_i^R)^2$
DCX	.00445	.02026	.47689	.85278	35.791	28.512	0.39046
GM	-.00391	.01231	.50553	.37310	96.599	38.350	2.49747

Table 4.7: Calibrated parameters for DCX and GM, August 25, 2005.

WKN	Coupon	Maturity	B^M	B^R	Rel. error
369293	4.625	10.03.2006	103.306861	103.19	-0.1132%
611867	6.125	21.03.2006	104.712257	104.67	-0.0404%
689080	5.625	06.07.2006	103.498493	103.42	-0.0759%
A0DB7Z	2.000	05.09.2006	101.547979	101.54	-0.0079%
907882	3.750	02.10.2006	104.799059	104.73	-0.0659%
829942	5.625	16.01.2007	107.656132	107.46	-0.1825%
A0DHP3	2.475	16.03.2007	101.017892	101.22	0.1997%
851890	6.125	27.03.2007	108.038199	107.80	-0.2210%
A0BD90	2.608	02.07.2007	100.416606	100.78	0.3606%
A0DZP6	3.125	10.03.2008	102.317418	102.39	0.0709%
765013	3.750	04.06.2008	103.311009	103.19	-0.1173%
A0ACD4	4.125	23.01.2009	106.146911	106.19	0.0406%
611868	7.000	21.03.2011	121.048224	120.84	-0.1723%
A0DDFR	4.250	04.10.2011	107.844546	108.02	0.1624%

Table 4.8: Model and market prices of *DaimlerChrysler*.

In this table, the relative error is calculated as $(B^R - B^M)/B^R$.

Interpretation of the results

At first, we observe the excellent fitting capability of the model, both for small and long maturities. For all bonds, the differences of model to market prices are far

below bid-ask spreads. The estimated parameters for DCX and GM correspond to $\mathbb{E}[X_1] = 0.000166$ and -0.006771 , respectively. The variances of X_1 are given by $\text{Var}(X_1) = 0.002143$ and 0.000443 , respectively. In absolute values, these variances seem to be extremely small. However, the initial distance to default of the process X is given by $x_0 = -\log(d/v_0)$, which is also very small. If we divide this distance to default by the root of $\text{Var}(X_1)$, we observe the process X to be only 4.18 and 2.70 standard deviations above default, respectively.

Another interesting observation is that the variance of X_t is only explained to 19.15% and 34.22% by the diffusion component, respectively, the jump component accounts for the remainder. This indicates that including jumps in a traditional diffusion model is not just a realistic, but also a necessary generalization, if a fit to bonds with different maturities is required. While the diffusion component explains the long-term behavior of credit spreads, the jump component corrects unrealistic small spreads for bonds with short maturities. Finally, we observe that in both examples negative jumps are pronounced, that is $\hat{\lambda}_\ominus < \hat{\lambda}_\oplus$.

WKN	Coupon	Maturity	B^M	B^R	rel. error
183098	7.000	15.11.2005	106.142388	106.21	0.0637%
291815	4.000	09.02.2006	102.234274	102.48	0.2398%
610260	5.750	14.02.2006	103.897810	104.15	0.2421%
776306	4.174	03.03.2006	102.130616	102.22	0.0874%
A0BC23	2.625	14.06.2006	99.430427	99.43	-0.0004%
908510	4.375	26.09.2006	104.190749	104.50	0.2959%
748413	6.000	16.10.2006	107.077676	107.15	0.0675%
A0DACL	3.040	15.02.2007	99.598528	98.60	-1.0127%
850892	6.125	15.03.2007	105.091735	105.23	0.1314%
A0E8A5	4.125	02.06.2007	99.988892	100.45	0.4590%
A0E7D3	5.625	13.07.2007	102.130365	102.16	0.0290%
819413	4.750	16.07.2007	100.433439	100.22	-0.2130%
A0DCTX	2.923	14.09.2007	98.939518	99.27	0.3329%
A0DG6B	3.674	03.12.2007	99.648395	99.45	-0.1995%
894454	6.000	03.07.2008	101.925706	101.32	-0.5978%
905302	3.920	12.09.2008	98.760729	98.76	-0.0007%
A0BEAR	3.429	30.06.2009	90.925046	91.51	0.6392%
A0DCTY	4.750	14.09.2009	98.718413	98.60	-0.1201%
908511	5.750	27.09.2010	100.522887	100.24	-0.2822%
A0AWBL	5.375	06.06.2011	93.247923	93.43	0.1949%

Table 4.9: Model and market prices of *General Motors*.
The relative error is calculated as $(B^R - B^M)/B^R$.

4.9.2 Calibration to *iTraxx* CDS quotes

Bonds are highly standardized products which are typically traded on public markets. In contrast, most CDS are traded over the counter and are based on different terms of contract. Therefore, it is extremely difficult to obtain and compare CDS quotes. In order to make the CDS market more transparent, several leading investment banks founded the joint venture *International Index Company*²⁹ (IIC), which launched the first standardized CDS index in 2004. Since then, different series of selected CDS contracts have been issued. These quotes provide us with the market's view on the term structure of default probabilities of the listed companies. What makes this set of data even more valuable is the fact that for some series, not only CDS quotes, but also prices of portfolio derivatives are listed. These quotes are used in Section 5.3.4 to calibrate our multidimensional generalization of the model.

The data set

We used CDS quotes of the fifth European *iTraxx* series. This inhomogeneous portfolio contains 125 companies from six business sectors³⁰, their ratings vary from AA+ to BBB-. CDS spreads are quoted for each firm for contracts maturing in one, three, five, seven and ten years, respectively. All contracts are computed based on a recovery rate of 40%. Unfortunately, not all contracts are traded liquidly enough to provide daily quotes for all maturities. More concrete, daily quotes for contracts maturing in three, five or ten years are available for most companies. On the other side, about half of the CDS quotes for contracts with maturities of one or seven years are not available in *Bloomberg*. In this case, we extrapolated or interpolated empty cells of our database using the average slope of all available companies, attached to the listed CDS contract with the closest maturity of the same company³¹. Besides CDS quotes, we obtained the term structure of default-free interest rates in the Eurozone from *Bloomberg*. These interest rates are available day-to-day for integer-valued maturities, other maturities are obtained from a linear interpolation. This deterministic term structure of interest rates is used to replace the flat interest rate in the CDS pricing formula.

²⁹ <http://www.indexco.com>

³⁰ Auto 8%, Consumer 24%, Energy 16%, Financial 20%, Industrial 16% and TMT 16%.

³¹ For instance, if the one-year spread of a company was not available, then this value was extrapolated starting at the three-year quote of the same firm. For this, the average slope of all other companies with available one and three-year quote was computed.

The calibration algorithm

In what follows, we present our calibration method which is used to calibrate each firm-value process of the 125 companies of the *iTraxx* portfolio. We again assume two-sided exponentially distributed jumps. Though, we fix $p = \frac{1}{2}$ and $\lambda_{\oplus} = \lambda_{\ominus}$ to reduce the dimension of the parameter space. This reduction of dimension is done for two reasons. First of all, it significantly accelerates the convergence of the minimization. Moreover, it accounts for the fact that our data set only contains five data points per company, which makes a calibration of more parameters very unstable. Unlike bonds which trade around their par value, CDS spreads for different maturities are not around the same level. Typically, CDS spreads are increasing in the maturity of the contract. For instance, we observed the ten-year CDS spread of several companies to exceed the corresponding one-year spread by more than factor ten. Therefore, we changed our objective function to be the sum of relative differences, that is

$$\left(\hat{\gamma}, \hat{\sigma}, \hat{\lambda}, \hat{\lambda}_{\oplus} = \hat{\lambda}_{\ominus}\right) = \operatorname{argmin} \sum_{t \in \{1,3,5,7,10\}} \left| \frac{c_t^R - c_t^M}{c_t^R} \right|, \quad (4.46)$$

where c_t^M is the model spread depending on the set of parameters and c_t^R is the market quote for the respective maturity. Using this construction it is guaranteed that all maturities are considered to equal parts in the minimization. The initial leverage ratio was obtained from the last available balance sheet of each company, as reported by *Bloomberg*.

Again, we used the procedure `nag_opt_bounds_no_deriv` of the NAG-software library to perform the required minimization. Our first approach was to use identical initial parameters for all companies. This approach was not satisfying, as the minimization routine often stopped in some local minimum close to the initial value. The reason therefor is the inhomogeneity of the companies in the portfolio. We fixed this problem using the following two-step approach.

1. **For each company, find an appropriate initial position**

To begin with, we define a coarse grid on the parameter space. Then, we perform a naïve search for initial parameters by evaluating the objective function at each of these points. The position of the minimum on this grid is stored.

2. **Start the actual minimization at this position**

The minimization routine `nag_opt_bounds_no_deriv` is then started at the optimal grid point of the previous step. It turned out that the minimization routine converged extremely fast from this initial position.

Results of the calibration

We fitted the model to data of an eight week period, beginning May 2006. The calibration is performed using *iTraxx* CDS quotes of each Tuesday and Thursday of this period. This corresponds to 2,000 individual calibrations, resulting from 125 companies at 16 different dates.

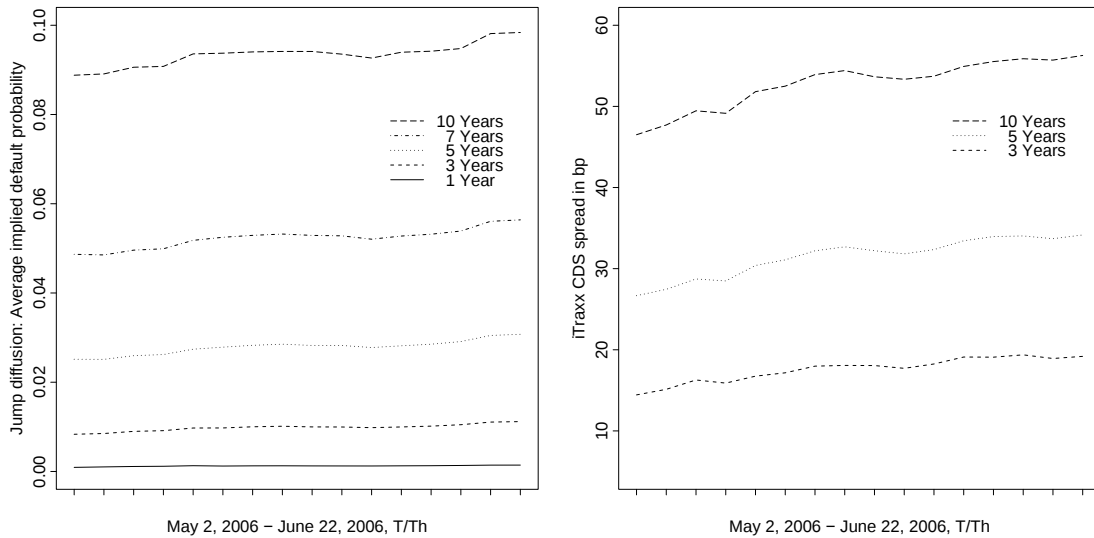


Figure 4.9: Implied average default probabilities and portfolio CDS spreads.

The left-hand side of Figure 4.9 displays the average implied default probability over a period of one, three, five, seven and ten years, respectively, as obtained from the calibration. The right-hand side shows quoted portfolio CDS spreads for a contract maturing in three, five and ten years, respectively. We notice that within the observed period, the average implied default probabilities are slightly increasing, but do not exhibit sudden changes. This result is realistic, as for the same period we observe increasing portfolio CDS spreads, which essentially depend on the average default probability over the respective period.

Figure 4.10 exhibits the average pricing error for each day of the calibration, in absolute values on the left-hand side, in relative values on the right-hand side. Considering that the companies within the portfolio are extremely inhomogeneous, we conclude that the fitting capability of the model is excellent³². For all days and maturities, the average pricing error is below three basis points, which is within the bid-ask interval for almost all companies. In contrast, if we fit a pure diffusion

³² This fitting capability is further improved if asymmetric jumps are allowed or a finer grid for the search of initial parameters is used. This was computationally too expensive for 2,000 calibrations.

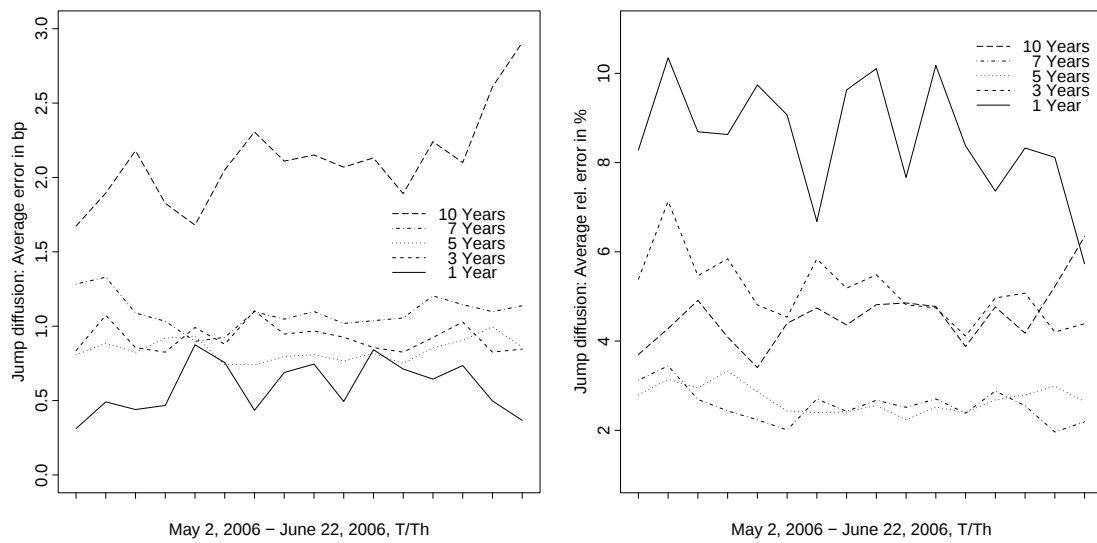


Figure 4.10: Average pricing error.

model to the same data, we observe that such a model is not able to match a realistic term structure of spreads. The typical hump-size structure of spreads, as implied by a pure diffusion model, and the zero limit of spreads for short maturities lead to a massive underpricing of contracts with maturities of one and ten years, and a slight overpricing of contracts with maturities of three, five and seven years, respectively. Figure 4.11 illustrates that the implied spreads of CDS maturing in one and ten years, respectively, are far off a realistic range. In absolute values, the average pricing error in a continuous model is about eight times larger for CDS maturing in one and ten years, respectively. For the remaining maturities, the average pricing error is still reduced by about 50% if jumps are allowed.

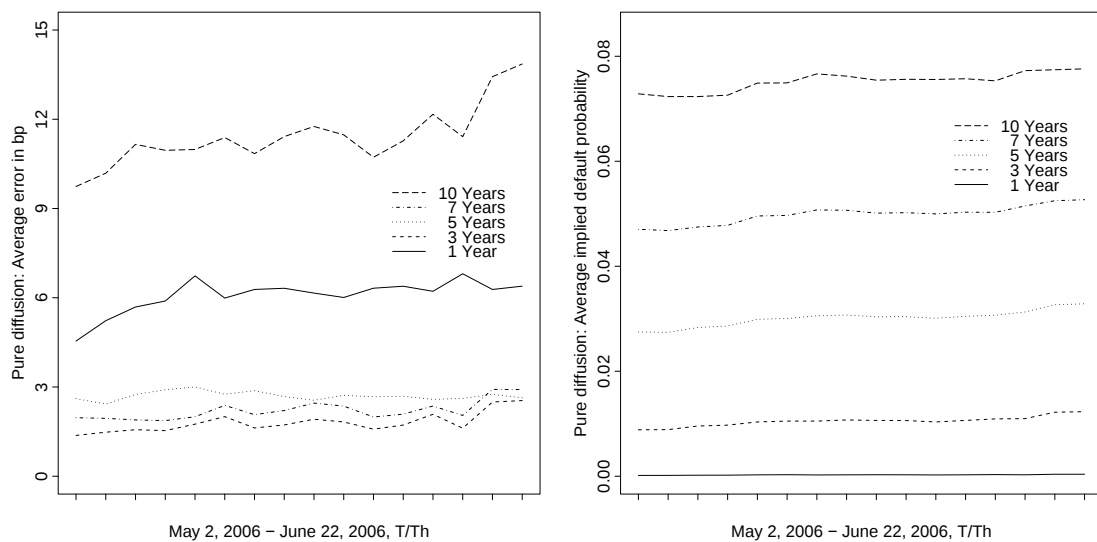


Figure 4.11: Calibration in a pure diffusion model.

4.10 Summary of the univariate model

Structural default models are based on the economic interpretation of default as a result of insufficient asset values. In daily practice, most structural default models still rely on a geometric Brownian motion as model for the firm-value process, which is rather a concession to its analytical tractability than to its capability of matching empirical facts. The major drawback of this approach is that the continuity of the Brownian motion allows investors to predict the time of default, contradicting the sudden bankrupt of several prominent companies³³ over the last decades. Consequently, it is virtually impossible for a solvent company to default within a short amount of time, which forces credit spreads to tend to zero in the time to maturity.

It was Zhou (2001a) who first proposed to overcome this shortfall by superposing the diffusion of the asset value process with a compound Poisson process. In such a model, default is possible by diffusion and by jump, the latter of which is unpredictable. This possibility of a negative jump, prevailing even in a short amount of time, leads to a positive limit of spreads, which we succeeded to compute in closed form. However, the distribution of first-passage times in a general jump-diffusion model is still an open mathematical problem. The pricing of bonds and single-name derivatives is therefore usually done by means of a Monte Carlo simulation, for which we presented an unbiased and efficient approach. Another feature of jump-diffusion models is the possibility of using the random undershot in the event of default by jump to explain the recovery rate as an endogenously specified random variable. This property is also included in our algorithm. Moreover, we generalized the model to random default thresholds, short-rate processes and different filtrations.

If two-sided exponentially distributed jumps are assumed, then the Laplace transform of first-passage times is known, due to Kou and Wang (2003). Combined with an inverse Laplace algorithm, this result suggests an extremely fast and accurate method of approximating the pricing formula of single-name derivatives. We worked out this approach in detail and compared it to different Monte Carlo simulations. As an application of this algorithm, we presented a calibration of the model to market data and an analysis of the sensitivity of model prices with respect to changes in the parameters of the firm-value process. We found that the term structure of spreads, as induced by a structural default model based on a jump-diffusion process, is extremely flexible and able to match observed prices far better than a continuous model.

³³ For instance, let us mention the bankrupts of *PanAM* (1991), *Enron* (2001), *Sabena* (2001), *WorldCom* (2002) and *Swissair* (2002).

Chapter 5

The multidimensional model

So far, we were primarily concerned with the pricing of single-name credit derivatives. More generally, we considered contracts with payment streams depending on the creditworthiness of a single obligor. In this chapter, we generalize the structural default model of Chapter 4 to a multidimensional model. This model is able to simultaneously¹ explain the individual default probabilities of different firms and their default correlation. These two variables determine the loss distribution of a credit portfolio. Our model is then used to price multi-name credit derivatives such as CDOs and n^{th} -to default contracts. Also, our intention is to present a numerically tractable model which allows for a calibration to market quotes. Before we formally introduce our model, let us first collect some empirical observations which we want to incorporate.

The first property we want to match is the dependence of each firm-value process on the macroeconomic situation. This sensitivity to common macroeconomic factors differs from company to company, depending on the business structure of the respective firm. If translated in joint default probabilities, this phenomenon is often referred to as *cyclical default correlation*.

The second property we want to incorporate is the phenomenon of *default clusters*, i.e. time periods with several defaults. For a jump-diffusion model, we feel that it is natural to specify the model such that common jumps are possible with positive probability. This idea also accounts for the fact that different companies simultaneously respond to unexpected events with jumps in their firm-value processes. The consequence of allowing common jumps is a larger default correlation, compared to a continuous model with a single market factor. In such a factor model, companies are independent conditioned on this variable.

¹ Most portfolio models consider individual default probabilities as given input variables and focus on modeling the default correlation alone.

Our last concern is to generalize the model without changing the structure of the individual firm-value processes. Such a generalization allows us to transfer most of the properties and algorithms which we developed in the previous chapter. Moreover, it provides the possibility of using a large pool of data² to calibrate the marginal default probabilities of the model. Being consistent with single-name derivatives, it remains to calibrate the dependence structure such that prices of multi-name derivatives are matched.

5.1 The multivariate firm-value model

In the sequel, we assume a credit portfolio consisting of $I \in \mathbb{N}$ firms which we index by $i \in \{1, \dots, I\}$. As in Chapter 4, our model for each individual firm-value process is the exponential of a jump-diffusion process. In what follows, we develop different concepts on how dependence may be introduced to these processes. Also, we show how derivatives on a credit portfolio are priced based on the respective generalization of the model.

5.1.1 A common market factor and common jumps

In this first approach, we introduce dependence through a common market factor and via jumps that are triggered by a joint Poisson process. The common market factor accounts for the observation that most companies are sensitive to business cycles. The current macroeconomic situation is modeled using a Brownian motion of the market, which is denoted by $W^M = \{W_t^M\}_{t \geq 0}$. The individual Brownian motion $W^i = \{W_t^i\}_{t \geq 0}$ of company i is then replaced by the weighted sum

$$a_i W^M + \sqrt{1 - a_i^2} W^i,$$

where the factor $a_i \in (-1, 1)$ assesses the degree of systematic dependence of company i on the market. Our mechanism of triggering common jumps is motivated by the following economic interpretation. Intuitively, jumps of the firm-value process of some company are triggered if unexpected information or events are revealed. Mathematically, this translates into a Poisson process $N = \{N_t\}_{t \geq 0}$, whose jumps are interpreted as the arrival of new information. Admittedly, not all information is relevant for company i . Therefore, we introduce the factor $b_i \in (0, 1]$ to represent the probability of company i to respond with a jump in its firm-value process to a jump of the ticker process N . This construction corresponds to a

² Compare Section 4.9, where we presented a calibration to bond and CDS quotes.

thinned-out Poisson process with intensity λ , the intensity of the trigger process N , and thinning probability $(1 - b_i)$. In what follows, this object is denoted by $N(b_i) = \{N_t(b_i)\}_{t \geq 0}$. Finally, we obtain the following model for the firm-value process of company i

$$V_t^i = v_0^i \exp \{X_t^i\}, \quad X_t^i = \gamma_i t + \sigma_i \left(a_i W_t^M + \sqrt{1 - a_i^2} W_t^i \right) + \sum_{j=1}^{N_t(b_i)} Y_j^i. \quad (5.1)$$

In Equation (5.1), the Brownian motions W^M and W^i , as well as all random variables defining the jump component, are assumed to be mutually independent. Moreover, all firm-individual parameters v_0^i , γ_i and σ_i and the jump-size distribution $\mathbb{P}_{Y^i}$ obey the same restrictions as in Chapter 4. The first consequence of this construction is the following lemma.

Lemma 5.1.1 (Original jump-diffusion model for all margins)

In distribution, the firm-value process of company i agrees with v_0^i times the exponential of a jump-diffusion process with diffusion volatility σ_i , jump intensity $b_i \lambda$ and jump-size distribution $\mathbb{P}_{Y^i}$.

Proof: W_t^M and W_t^i being independent, their weighted sum $a_i W_t^M + \sqrt{1 - a_i^2} W_t^i$ agrees in distribution with a Brownian motion at time t . This is easily checked from the properties of a Brownian motion and the properties of the normal distribution. Moreover, it is deduced from the definition that the thinned-out Poisson process $N(b_i)$ agrees in distribution with a regular Poisson process with intensity $b_i \lambda$, compare Lemma 2.1.1. $\diamond$

5.1.2 A common market factor and dependent jumps

So far, we assumed all jumps to be mutually independent. However, it seems reasonable that most news affect different firms in a similar manner. Therefore, we propose to classify new information as being good or bad for the economy. Even if there might exist examples for information which are positive for some and negative for other companies, a well accepted phenomenon is that markets are highly correlated in extreme events. Based on this observation, we relax the assumption of independent jumps and assume that jumps at the same jump time have a common sign. However, they do not have identical jump sizes. The sign of all jumps at some jump time τ_l is determined by an initial Bernoulli experiment with success probability $p = 0.5$. The outcome of this experiment specifies whether an information is considered to be good or bad. In order to not changing the

individual default probabilities, we have to restrict the set of possible jump-size distributions of each individual firm-value process using the condition

$$\mathbb{P}_{Y^i}(Y_l^i > 0) = \mathbb{P}_{Y^i}(Y_l^i < 0) = 0.5 \quad \forall i \in \{1, \dots, I\}, \forall l \in \mathbb{N}.$$

The implementation of this variant of the model simplifies considerably if a jump-size distribution is chosen where up and downward jumps are easy to distinguish. For instance, we implemented the model using two-sided exponentially distributed jumps with $\mathbb{P}_{Y^i} = 2\text{-Exp}(\lambda_{\oplus}^i, \lambda_{\ominus}^i, \frac{1}{2})$ and normally distributed jumps with zero mean, i.e. $\mathbb{P}_{Y^i} = \mathcal{N}(0, (\tilde{\sigma}^i)^2)$.

Consequences of dependent jumps

Our numerical investigations of Section 5.2 show that the assumption of dependent jumps significantly increases the default correlation among the firms. The effect of this augmented correlation leads to larger spreads in the senior tranches of a CDO, as illustrated in Tables 5.3 and 5.4. A sample plot of three dependent firm-value processes is given in Figure 5.1.

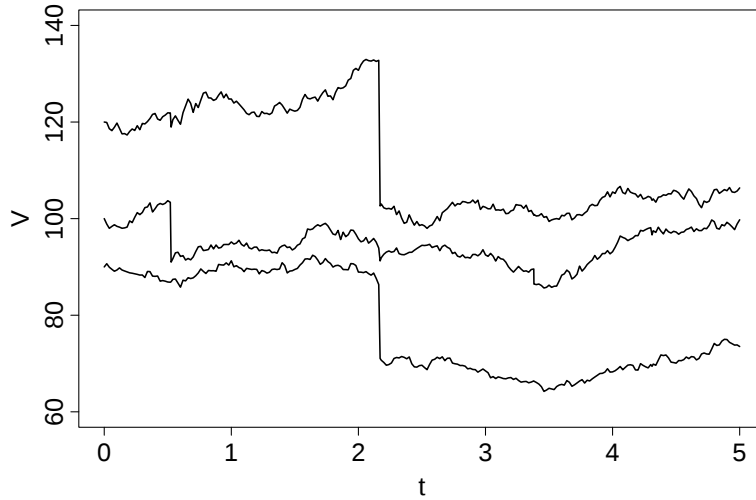


Figure 5.1: The sample paths of three dependent firm-value processes.

In Figure 5.1, the first jump at $\tau_1 = 0.53$ only affects a single firm. In contrast, the second jump at $\tau_2 = 2.17$ affects the other two firms. The jumps at these τ_i have the same direction but different sizes. The diffusion components are coupled to the market using $a_i = 0.5$ for all $i \in \{1, 2, 3\}$. Following each negative jump, it is likely that several firms default simultaneously, which we interpret as a default cluster.

5.1.3 Segmentation by industry sector

This approach on coupling the individual firm-value processes is inspired by mapping each company to a specific branche. For instance, each company of the *European iTraxx* portfolio is assigned to one of the six branches: *Auto*, *Consumer*, *Energy*, *Financial*, *Industrial* and *TMT*. The idea for this variant of the model came up in a discussion³ with Wim Schoutens, *K.U. Leuven*, whom I want to thank at this point.

More precisely, we assume as given $S \in \mathbb{N}$ different industry sectors, indexed by $s \in \{1, \dots, S\}$. Further, the mapping of each company to the set of industry sectors is well-defined. Considering common factors, we introduce a common factor of the market $W^M = \{W_t^M\}_{t \geq 0}$ and a common factor of each industry sector $W^s = \{W_t^s\}_{t \geq 0}$. Then, the Brownian motion of each company is replaced by a weighted sum consisting of its individual Brownian motion $W^i = \{W_t^i\}_{t \geq 0}$, the Brownian motion of the respective industry sector W^s and the Brownian motion of the market W^M . These processes are assumed to be mutually independent. As abbreviation, we introduce

$$\tilde{W}_t^i(a_i, c_i) := a_i W_t^M + c_i W_t^s + \sqrt{1 - a_i^2 - c_i^2} W_t^i, \quad a_i, c_i \in (-1, 1), \quad a_i^2 + c_i^2 \leq 1.$$

In this framework, our concept of incorporating dependence via jumps combines the idea of using common factors with the idea of supporting common jumps from Section 5.1.1. More precisely, we assume that some information is relevant to all companies, others affect a specific industry sector. Finally, some news are only relevant to an individual company. The ticker processes which report these pieces of information are independent Poisson processes which we denote by $N^M = \{N_t^M\}_{t \geq 0}$, $N^s = \{N_t^s\}_{t \geq 0}$ and $N^i = \{N_t^i\}_{t \geq 0}$, respectively, where $s \in \{1, \dots, S\}$ and $i \in \{1, \dots, I\}$. Their intensities are denoted by λ^M , λ^s and λ^i , respectively. For company i , relevant news are reported by the superposition of market, sector and individual Poisson process. This superposition is abbreviated as

$$\tilde{N}_t^i := N_t^M + N_t^s + N_t^i.$$

The model of the firm-value process of company i is then given as

$$V_t^i = v_0^i \exp \{X_t^i\}, \quad X_t^i = \gamma_i t + \sigma_i \tilde{W}_t^i(a_i, c_i) + \sum_{j=1}^{\tilde{N}_t^i} Y_j^i, \quad (5.2)$$

³ At the conference on credit risk in Ulm, September 2005.

where we impose the same restrictions on the individual firm-value parameters as in Chapter 4. Again, we find that the univariate margins agree with the single-firm model of Chapter 4. More precisely, we obtain the following lemma.

Lemma 5.1.2 (Original jump-diffusion model for all margins)

The firm-value process of company i , belonging to sector s , agrees in distribution with v_0^i times the exponential of a jump-diffusion process with diffusion volatility σ_i , jump intensity $\lambda_i = \lambda^M + \lambda^s + \lambda^i$ and jump-size distribution $\mathbb{P}_{Y^i}$.

Proof: Using basic properties of the normal distribution and independence of the Brownian motions W_t^M , W_t^s and W_t^i , it is easily checked that the weighted sum $\tilde{W}_t^i(a_i, c_i)$ agrees in distribution with a Brownian motion at time t . Lemma 2.1.1 establishes that the superposition of independent Poisson processes is again a Poisson process. Moreover, the intensity of the superposition is the sum of the intensities of the summands. $\diamond$

In Section 5.3, we introduce an algorithm for the pricing of CDOs within the models of Sections 5.1.1 and 5.1.2. Altering this algorithm for an implementation of the model with different industry sectors only requires minor changes. However, we focus on the first two versions of the model, as we do not have sufficient market data to fit this latter variant of the model.

5.1.4 Properties and applications of the model

One important feature of our model is the possibility of modeling different firms with different sets of firm-value parameters. This distinguishes our approach from models which accept the simplification of a homogeneous portfolio. Also, we model the evolution of each firm-value process over time. In this section, we briefly comment on consequences and possible applications of these properties.

1. Sensitivity and hedging

If a company defaults, it is very likely that this company was rated below the average rating of the portfolio. In a model with identical companies, CDO spreads of the remaining portfolio remain unchanged. In our model, the average default probability of the remaining portfolio decreases, if one of the substandard companies defaults. The result is that spreads of a newly issued CDO contract, based on the remaining companies, are decreasing. We interpret this as the relief of the market that one of the substantial risk factors is removed from the portfolio. The opposite holds, if a company defaults which was considered to be a safe investment. Then, the average default

probability of the remaining portfolio increases, and so do spreads of a new CDO contract on the remaining portfolio.

More generally, these considerations suggest an analysis of the sensitivity of CDO spreads with respect to single companies. For instance, hedging a CDO using single-name CDS requires such results.

2. Creating sub-portfolios

Given identical companies, the loss distribution of each sub-portfolio only depends on its size. In contrast, a sub-portfolio in our model is automatically equipped with a realistic default structure. An important application of this property is the pricing of CDS sector indices, that are sub-portfolios of the *iTraxx* CDO portfolio consisting of companies of the same industry sector.

3. Modifying the portfolio

The composition of the *iTraxx* portfolio changes about twice per year. In this case, several companies are delisted from the portfolio and replaced by new firms. The default structure of the updated portfolio obviously depends on the relative creditworthiness of the new companies compared to the old firms. Capturing this feature also requires heterogeneous companies.

4. Simultaneously describing single and multi-name derivatives

Our model is designed to realistically describe the term structure of default probabilities of each company in the portfolio. While this property is presumably the major advantage compared to pure dependence models, when it comes to a calibration of the model, it turns out to be a burden and an advantage at the same time. A burden, as a large number of parameters have to be adjusted. An advantage, as it allows us to use a vast quantity of market information as input for the calibration. We will comment on this in detail in Section 5.3.4.

5. A time consistent framework

Another important aspect is the consistency of our model with respect to the time, as we explicitly model the evolution of each firm-value process. Therefore, the model specifies the complete term structure of default probabilities of each company and the overall portfolio loss. This allows us to price portfolio derivatives with different maturities within a consistent framework.

5.1.5 Existing structural portfolio models

At the moment, most structural portfolio models are static models in the sense that the underlying firm-value processes are only considered at the premium payment

dates. Typically, these models are generalizations of Vasicek's model⁴ and relax some of its assumptions or change the embedded distributions. A list of structural models which are comparable to our approach is given below.

Hull et al. (2005) model the asset process of each company as a geometric Brownian motion, where dependence is introduced using one or more common factors which partially replace the Brownian motion of each individual firm-value process. The model is implemented via a Monte Carlo simulation, default is only tested on a grid. As a result, their prices are close to prices in Vasicek's model. Compared to our approach, this model is the special case of Equation (5.1) where jumps are not allowed⁵. Moreover, we continuously test for default.

Willemann (2005) models each firm-value process as Zhou (2001a), assuming a jump-diffusion model with normally distributed jumps. He splits the diffusion component of each company into a common and an idiosyncratic factor. Jumps in this model are triggered by a common Poisson process, which implies an identical jump intensity for all companies. The combination of normally distributed jumps and the simplification that default is only possible at the premium payment dates allows to obtain default probabilities as infinite series. This is achieved by conditioning on the number of jumps of the trigger process. Willemann presents a calibration method and concludes that his model allows a better fit to market data than Vasicek's model. However, he reports an extremely small implied jump intensity. Willemann's model corresponds to choosing $b_i = 1$ and $\mathbb{P}_{Y^i} = \mathcal{N}(\mu_i, (\hat{\sigma}_i)^2)$ for all i in Equation (5.1) of our model. Moreover, our model generalizes this model in the sense that we continuously test for default.

Finally, two models relying on specific choices of Lévy processes have been proposed by Moosbrucker (2006) and Kassberger (2006). Moosbrucker presents several generalizations of the model of Luciano and Schoutens (2005), where the value process of each firm is the exponential of a variance-gamma process. He introduces dependence using correlated Brownian motions, correlated gamma processes or through a common time shift, which is interpreted as business time. The resulting CDO spreads of the different tranches exhibit a correlation smile and a good fit to market quotes. Kassberger develops a general Lévy setup and implements a model with firm-value processes following NIG distributions. To achieve a good fit to portfolio CDS quotes, the default thresholds in this model are time dependent. Kassberger derives a good fit to CDO quotes with a relatively small number of parameters. The model is implemented using a Monte Carlo simulation.

⁴ Compare Section 6.4.

⁵ That is, we set $\lambda = 0$, $b_i = 0$ or $\mathbb{P}_{Y^i} = \delta_0$ for all i .

5.2 Default, asset and implied correlation

At several points in this thesis, the term *default correlation* was used without a formal introduction. In this section, we state a mathematical definition and explain the dependence of this quantity on the parameters of our multivariate model. Also, we explicitly derive the asset correlation of two companies in the models of Sections 5.1.1 and 5.1.2. Finally, we present some numerical investigations considering the implied correlation of our model.

5.2.1 Default correlation

Definition 5.2.1 (Default correlation)

Let the default status of company $i \in \{1, \dots, I\}$ at time t be explained by the indicator variable $D^i = \{D_t^i\}_{t \geq 0}$, where

$$D_t^i := \mathbf{1}_{\{\tau^i \leq t\}}.$$

The default correlation of two companies i, j up to time $t > 0$ is then defined as

$$\rho_t^D := \text{Corr}(D_t^i, D_t^j) = \frac{\mathbb{E}[D_t^i D_t^j] - \mathbb{E}[D_t^i] \mathbb{E}[D_t^j]}{\sqrt{\text{Var}(D_t^i) \text{Var}(D_t^j)}}.$$

Being Bernoulli distributed random variables, the expectation and variance of D_t^i can easily be expressed in terms of default probabilities⁶. More delicate is the calculation of $\mathbb{E}[D_t^i D_t^j]$, which is the probability of a default of company i and j up to time t . This expectation is difficult to obtain, as for non-degenerated choices of model parameters, the firm-value processes of both companies are dependent random variables, and so are their running minimums. However, in a purely continuous model it is possible to express the default correlation of two companies in terms of a double integral of Bessel functions. This result was derived by Zhou (2001b) using several results of Rebholz (1994). In the presence of jumps, we have to rely on a Monte Carlo simulation to estimate these quantities. Before we present our findings, let us remark that such a Monte Carlo simulation requires a large number of runs to produce reliable results, since multiple defaults are rare events. Our Monte Carlo simulation is a simple modification of the first part of Algorithm 5.3.1, we allow ourselves to omit the details. We controlled the accuracy of this algorithm by reproducing some of the tables of Zhou (2001b).

⁶ Without jumps, these default probabilities are known in closed-form. In the presence of jumps, these default probabilities are estimated or approximated as shown in Sections 4.2.3 and 4.6.

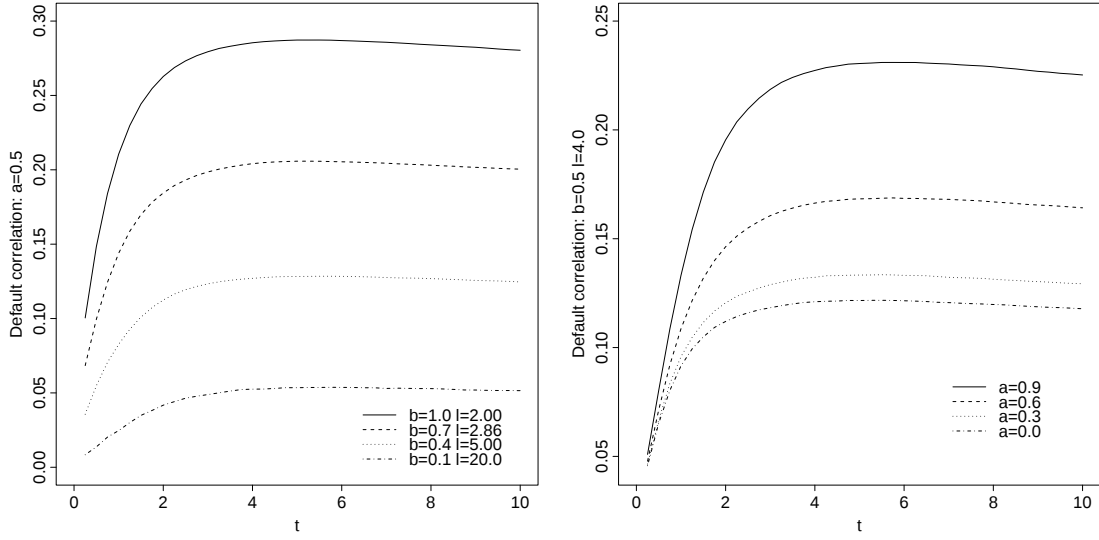


Figure 5.2: Implied default correlations for fixed a and b , respectively.

Figure 5.2 exhibits simulated default correlations of two identical companies with parameters $\gamma = 0$, $\sigma = 0.05$, $\mathbb{P}_Y = 2\text{-Exp}(20, 20, \frac{1}{2})$ and $d/v_0 = 85\%$. This experiment is performed in the framework of Section 5.1.2 with dependent jumps. On the left-hand side, we fix $a = 0.5$ and vary b ⁷. The figure on the right-hand side is calculated based on a fixed level of $b = 0.5$, $\lambda = 4$ and different levels of a .

Our first remark is that our model has two parameters to adjust the default correlation, contrasting pure diffusion models with a single common factor. Moreover, supporting common jumps produces simultaneous defaults already for small maturities. This differs from the situation in a continuous model, where defaults within the first year are extremely rare events, and so are multiple defaults. The fact that a continuous model requires more time to generate a relevant default correlation becomes even more evident in Figure 5.3. This figure is produced using the same setup as before⁸, the three scenarios differ by the influence of jumps.

More precisely, the largest default correlation was implied for all maturities by the model of Section 5.1.2 with dependent jumps. Anticipating Theorem 5.2.1, adding independent jumps with zero expectation to a continuous model decreases the asset-value correlation. Still, the model of Section 5.1.1 with independent jumps implies a larger default correlation than a continuous model for small maturities. On a longer time horizon, the opposite holds. The phenomenon that continuous

⁷ To keep the individual default probabilities constant, we fix the individual jump intensity, which is the product $b\lambda$.

⁸ Parameters: $\gamma = 0$, $\sigma = 0.05$, $\mathbb{P}_Y = 2\text{-Exp}(20, 20, \frac{1}{2})$, $d/v_0 = 85\%$, $a = 0.4$, $b = 0.5$.

models require some time to produce a relevant default correlation should be kept in mind, if the objective is to price portfolio derivatives with short maturities.

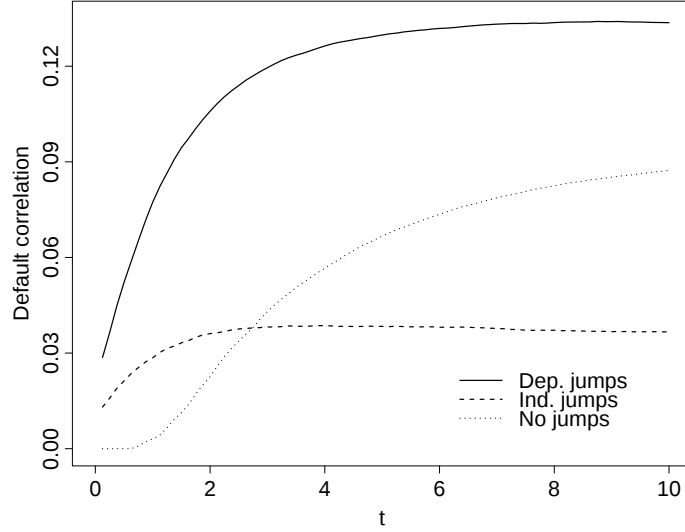


Figure 5.3: Default correlations with independent, dependent and no jumps.

Conclusion 6

Common jumps imply a significant default correlation already for small maturities. In contrast, a common Brownian motion requires some time to induce multiple defaults. For short maturities, a larger asset-value correlation does not necessarily imply a larger default correlation.

5.2.2 Asset-value correlation

In the previous section, we had to rely on simulation techniques to estimate the default correlation of two companies $i \neq j$. In this section, we present a closed-form expression of the correlation of X_t^i and X_t^j , the exponents of the respective firm-value processes. If both processes are continuous, that is

$$X_t^l = \gamma_l t + \sigma_l \left(a_l W_t^M + \sqrt{1 - a_l^2} W_t^l \right) \quad l \in \{i, j\},$$

then $\text{Cov}(X_t^i, X_t^j) = \sigma_i \sigma_j a_i a_j t$ is deduced from independence of all Brownian motions and basic properties of the covariance. Therefore, the correlation of X_t^i and X_t^j is the product of the factors a_l , i.e.

$$\text{Corr}(X_t^i, X_t^j) = a_i a_j. \quad (5.3)$$

The continuous model being a special case of the jump-diffusion framework, a general result necessarily boils down to Equation (5.3) if $\lambda = 0$, $b_i = b_j = 0$ or $\mathbb{P}_Y^i = \mathbb{P}_Y^j = \delta_0$. This property is easily checked to hold in Theorems 5.2.1 and 5.2.2.

Theorem 5.2.1 (Asset-value correlation, independent jumps)

Given the model of Section 5.1.1 with square integrable jump-size distributions $\mathbb{P}_Y^i$ and $\mathbb{P}_Y^j$, the processes X^i and X^j satisfy

$$\rho^X := \text{Corr}(X_t^i, X_t^j) = \frac{\sigma_i \sigma_j a_i a_j + \lambda b_i b_j \mathbb{E}[Y^i] \mathbb{E}[Y^j]}{\sqrt{\sigma_i^2 + \lambda b_i \mathbb{E}[(Y^i)^2]} \sqrt{\sigma_j^2 + \lambda b_j \mathbb{E}[(Y^j)^2]}}.$$

Proof: By independence of both diffusion and jump components, and by using the same arguments as in the pure diffusion case, we obtain

$$\begin{aligned} \text{Cov}(X_t^i, X_t^j) &= \sigma_i \sigma_j \text{Cov}(a_i W_t^M, a_j W_t^M) + \text{Cov}(CP_t^i, CP_t^j) \\ &= \sigma_i \sigma_j a_i a_j t + \lambda t b_i b_j \mathbb{E}[Y^i] \mathbb{E}[Y^j], \end{aligned} \quad (5.4)$$

where the abbreviation $CP_t^l := \sum_{k=1}^{N_t(b_l)} Y_k^l$ for $l \in \{i, j\}$ is used. To justify Equation (5.4), we have to derive the covariance of the jump components. This is achieved by first conditioning on the number of information $N_t = k$, then on the number of jumps $N_t(b_i) = l^i$ and $N_t(b_j) = l^j$, respectively. Given the $\text{Poi}(\lambda t)$ distributed random variable N_t , the number of jumps of the thinned-out Poisson processes $N_t(b_i)$ and $N_t(b_j)$ follow binomial distributions with respective parameters. We identify the sums involved in the following computation as the expectation of a binomial distribution and the second moment of a Poisson distribution. Therefore, we find

$$\begin{aligned} \mathbb{E}[CP_t^i CP_t^j] &= \mathbb{E}[\mathbb{E}[CP_i CP_j | \sigma(N_t, N_t(b_i), N_t(b_j))]] \\ &= \mathbb{E}\left[\sum_{k=0}^{\infty} \left(\sum_{l^i=0}^k \sum_{l^j=0}^k \left(\sum_{l=1}^{l^i} Y_l^i \right) \left(\sum_{l=1}^{l^j} Y_l^j \right) \cdot \right. \right. \\ &\quad \left. \left. \binom{k}{l^i} b_i^{l^i} (1-b_i)^{k-l^i} \binom{k}{l^j} b_j^{l^j} (1-b_j)^{k-l^j} \right) \frac{(\lambda t)^k e^{-\lambda t}}{k!} \right] \\ &= b_i b_j \mathbb{E}[Y^i] \mathbb{E}[Y^j] \sum_{k=0}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= b_i b_j \mathbb{E}[Y^i] \mathbb{E}[Y^j] ((\lambda t)^2 + \lambda t). \end{aligned}$$

The claim follows, since $\mathbb{E}[CP_t^l] = t \lambda b_l \mathbb{E}[Y^l]$ for $l \in \{i, j\}$ and the variances of the jump-diffusion processes X_t^l are given by $\text{Var}(X_t^l) = t (\sigma_l^2 + \lambda b_l \mathbb{E}[(Y^l)^2])$ for

$l \in \{i, j\}$, respectively. ◇

The assumption of independent jumps considerably simplified the main computation of this proof. If jumps in the same direction are imposed instead, then one has to additionally condition on the number of jumps of both companies in either direction. Under some mild conditions on the symmetry of the jump-size distributions, we were able to obtain the corresponding result in the context of Section 5.1.2.

Theorem 5.2.2 (Asset-value correlation, dependent jumps)

We now consider the model of Section 5.1.2 with jumps in the same direction. As this model requires jumps in a pre-specified direction, we define $Y_{\oplus}^l$ and $Y_{\ominus}^l$ for $l \in \{i, j\}$ to be the size of a positive and negative jump, respectively, given the sign of Y^l . More precisely, the distributions of $Y_{\oplus}^l$ and $Y_{\ominus}^l$ are given for $x > 0$ by $\mathbb{P}(Y_{\oplus}^l \leq x) = \mathbb{P}(Y^l \leq x | Y^l > 0)$ and $\mathbb{P}(Y_{\ominus}^l \leq x) = \mathbb{P}(-Y^l \leq x | Y^l < 0)$, respectively. Then, we again assume square integrable jump-size distributions $\mathbb{P}_{Y^l}$ for $l \in \{i, j\}$. Further, we impose the following assumption on the symmetry of the jump-size distributions

$$\mathbb{P}_{Y^l}(Y^l > 0) = \mathbb{P}_{Y^l}(Y^l < 0) = 0.5 \quad l \in \{i, j\}.$$

Moreover, we need

$$\mathbb{E}[Y_{\oplus}^i] = \mathbb{E}[Y_{\ominus}^i] = \mathbb{E}[|Y^i|], \quad \mathbb{E}[Y_{\oplus}^j] = \mathbb{E}[Y_{\ominus}^j] = \mathbb{E}[|Y^j|].$$

Then, the processes X^i and X^j satisfy

$$\rho^X := \text{Corr}(X_t^i, X_t^j) = \frac{\sigma_i \sigma_j a_i a_j + \lambda b_i b_j \mathbb{E}[|Y^i|] \mathbb{E}[|Y^j|]}{\sqrt{\sigma_i^2 + \lambda b_i \mathbb{E}[(Y^i)^2]} \sqrt{\sigma_j^2 + \lambda b_j \mathbb{E}[(Y^j)^2]}}.$$

Proof: Large parts of this proof are similar to the proof of Theorem 5.2.1. The main difference is the delicate computation of the expectation of the product of the jump components, i.e. $\mathbb{E}[CP_t^i CP_t^j]$. The evaluation of this expectation is complicated, as jump sizes at common jump times are forced to have the same sign.

To begin with, we again condition on the amount of information up to time t . By construction, this equals the maximum number of jumps of the processes X^i and X^j . Following a $\text{Poi}(\lambda t)$ distribution, we have $\mathbb{P}(N_t = k) = (\lambda t)^k e^{-\lambda t} / k!$. Given $N_t = k$, we additionally condition on how much of this news is positive.

This yields

$$\mathbb{P}(l|k) := \mathbb{P}(l \text{ positive news} | N_t = k) = \binom{k}{l} 0.5^l (1 - 0.5)^{k-l} \quad 0 \leq l \leq k.$$

Given k news, from which l are classified as good, the number of bad news is $k - l$. The conditional probabilities of X^i to have exactly $l_{\oplus}^i$ upward and $l_{\ominus}^i$ downward jumps are therefore given and abbreviated as

$$\begin{aligned} \mathbb{P}(l_{\oplus}^i | l) &:= \binom{l}{l_{\oplus}^i} b_i^{l_{\oplus}^i} (1 - b_i)^{l - l_{\oplus}^i} \quad 0 \leq l_{\oplus}^i \leq l, \\ \mathbb{P}(l_{\ominus}^i | l) &:= \binom{k-l}{l_{\ominus}^i} b_i^{l_{\ominus}^i} (1 - b_i)^{k-l-l_{\ominus}^i} \quad 0 \leq l_{\ominus}^i \leq k-l. \end{aligned}$$

The probabilities $\mathbb{P}(l_{\oplus}^j | l)$ and $\mathbb{P}(l_{\ominus}^j | l)$ are defined similarly for X^j . Using these abbreviations, we rewrite $\mathbb{E}[CP_t^i CP_t^j]$ as

$$\mathbb{E} \left[\sum_{k=0}^{\infty} \left(\sum_{l=0}^k \text{IS}^i(l, k) \text{IS}^j(l, k) \mathbb{P}(l|k) \right) \mathbb{P}(N_t = k) \right], \quad (5.5)$$

where the inner sums $\text{IS}^i(l, k)$ and $\text{IS}^j(l, k)$ are defined as

$$\begin{aligned} \text{IS}^i(l, k) &:= \sum_{l_{\oplus}^i=0}^l \sum_{l_{\ominus}^i=0}^{k-l} \left(\sum_{h=1}^{l_{\oplus}^i} Y_{\oplus_h}^i - \sum_{h=1}^{l_{\ominus}^i} Y_{\ominus_h}^i \right) \mathbb{P}(l_{\oplus}^i | l) \mathbb{P}(l_{\ominus}^i | l), \\ \text{IS}^j(l, k) &:= \sum_{l_{\oplus}^j=0}^l \sum_{l_{\ominus}^j=0}^{k-l} \left(\sum_{h=1}^{l_{\oplus}^j} Y_{\oplus_h}^j - \sum_{h=1}^{l_{\ominus}^j} Y_{\ominus_h}^j \right) \mathbb{P}(l_{\oplus}^j | l) \mathbb{P}(l_{\ominus}^j | l). \end{aligned}$$

We take the expectation inside in Equation (5.5) and use that $\mathbb{E}[|Y^i|] = \mathbb{E}[Y_{\oplus}^i] = \mathbb{E}[Y_{\ominus}^i]$. Using the expectation of a binomial distribution with l , respectively $k-l$, experiments and success probability b_i , we find

$$\begin{aligned} \mathbb{E}[\text{IS}^i(l, k)] &= \sum_{l_{\oplus}^i=0}^l \sum_{l_{\ominus}^i=0}^{k-l} (l_{\oplus}^i \mathbb{E}[|Y^i|] - l_{\ominus}^i \mathbb{E}[|Y^i|]) \mathbb{P}(l_{\oplus}^i | l) \mathbb{P}(l_{\ominus}^i | l) \\ &= (2l - k) b_i \mathbb{E}[|Y^i|]. \end{aligned}$$

Similarly, it holds that $\mathbb{E}[\text{IS}^j(l, k)] = (2l - k) b_j \mathbb{E}[|Y^j|]$. We further observe that

$$\sum_{l=0}^k (2l - k)^2 b_i b_j \mathbb{E}[|Y^j|] \mathbb{E}[|Y^i|] \mathbb{P}(l|k) = b_i b_j \mathbb{E}[|Y^j|] \mathbb{E}[|Y^i|] k.$$

This holds, since the sum allows the interpretation of being $4b_i b_j \mathbb{E}[|Y^j|] \mathbb{E}[|Y^i|]$ times the variance of a binomial distribution with k experiments and success probability 0.5. Finally, the outer sum is $b_i b_j \mathbb{E}[|Y^j|] \mathbb{E}[|Y^i|]$ times the expectation of a $\text{Poi}(\lambda t)$ distribution. Therefore, we find

$$\mathbb{E}[CP_t^i CP_t^j] = \lambda t b_i b_j \mathbb{E}[|Y^j|] \mathbb{E}[|Y^i|].$$

At the same time, this is the covariance of the jump components, since $\mathbb{E}[CP_t^i] = \mathbb{E}[CP_t^j] = 0$, by the initial assumptions on the symmetry of $\mathbb{P}_{Y^i}$ and $\mathbb{P}_{Y^j}$. $\diamond$

Let us finish this section with a brief remark on the results of Theorems 5.2.1 and 5.2.2. First of all, the postulated correlations are within the required range of $[-1, 1]$, due to the Cauchy-Schwarz inequality. Moreover, both results contain the pure diffusion model as a special case. The fact that allowing common jumps does not necessarily increase the asset correlation is remarkable. For instance, if the expectations of the jump-size distributions of two companies have opposite signs, then even a negative correlation of X^i and X^j is possible. However, modeling jumps using common signs implies a positive correlation of the respective jump components.

5.2.3 Implied correlations

If a new option-pricing model is proposed, it is common practice to compare it with the model of Black and Scholes (1973). For derivatives on a credit portfolio, the standard benchmark is *Vasicek's asymptotic single factor model*⁹. This model is chosen as reference for various reasons. First of all, it is easy to implement. It even allows for an analytical solution of most portfolio derivatives. Secondly, the correlation among the companies is adjusted using the single parameter ρ . This construction allows to invert prices of derivatives for the parameter ρ , which is then interpreted as *implied correlation*. Finally, the model is still used by many market participants.

Implied correlations of CDO tranches

Quoted CDO spreads provide us with the market's view on the correlation structure of the respective portfolio. Given the price of some portfolio derivative, the idea behind the notion of implied correlations is to invert Vasicek's model for the correlation. This means, the correlation is adjusted such that the market quote of

⁹ This model is also known as *Gaussian-copula* model. A description of Vasicek's model is postponed to Section 6.4 of the Appendix.

the respective derivative is matched by the model. If a CDO is considered, this corresponds to matching the market spread of a certain tranche¹⁰.

From a mathematical perspective, we first observe that this notion is not necessarily well defined. While spreads of the equity tranche are decreasing in the correlation, spreads of the senior tranches are typically increasing. Both stem from the fact that adding correlation increases the probability for multiple defaults, but also the probability of a small number of defaults. In contrast, the reaction of the mezzanine tranches on increasing correlation is ambiguous, especially the spread of the junior mezzanine tranche is typically not monotone in ρ . This complicates the interpretation and computation of implied correlations, as the equation

$$\text{market price} = \text{model price}(\rho) \quad (5.6)$$

may be solved by several values of ρ . We therefore define the implied correlation of CDO tranches as in Definition 5.2.2 below. Implied correlations of other derivatives are defined similarly.

Definition 5.2.2 (Implied correlation of CDO tranches)

Given tranche $j \in \{1, \dots, J\}$ of a CDO with maturity T , we define

$$\rho_T^{I,j} := \inf \{ \rho \in (-1, 1) : \text{market quote of } s_f^j = \text{model price of } s_f^j(\rho) \}.$$

If no solution of Equation (5.6) is found, that is $\inf \emptyset$, then $\rho_T^{I,j}$ is defined to be the smallest minimizer of the difference of model to market prices.

The correlation skew

Examining market data suggests that implied correlations are not constant across different tranches. More precisely, the implied correlations of the equity and senior tranche are typically observed to be larger than the implied correlations of mezzanine tranches. This observation is known as *correlation skew*, a detailed description of this phenomenon is given in Lehnert et al. (2005). Also, it is observed that the implied correlation of a tranche is typically not constant for contracts with different maturities.

Anticipating the numerical investigations of Section 5.3.3, we conclude that our model produced very realistic implied correlations. More precisely, we used our

¹⁰ Alternatively, instead of considering a single tranche of the CDO, it is possible to consider some fraction $[0, x\%]$ of the portfolio. The resulting implied correlations of this approach are known as *base correlations*.

model to derive CDO spreads based on several fictitious portfolios and different assumptions on the correlation structure. Then, we interpreted these spreads as real prices and computed the implied correlations. The results of this experiment are given in Tables 5.3 and 5.4. As expected, we observe that implied correlations are increasing in the parameters a and b . It is surprising by how much the implied correlation increases if the model is run with jumps in the same direction instead of independent jumps. Also, and this coincides with market quotes, a correlation skew is implied by our model in most examples of the experiment.

5.3 Pricing CDOs via Monte Carlo simulation

In this section, we introduce a Monte Carlo simulation which is designed to price CDOs in a very general setting. Most notations are already given in Section 3.3, some more are introduced below. So far, we did not specify the individual default times τ^i . Again, we define the time of default of company i as the first-passage time of its firm-value process below its default threshold, which is denoted by d^i . This definition seems natural, since it agrees with the time of default in the univariate model. Nevertheless, let us again stress the fact that most structural models only test for default at the times of the premium payments. This concession is typically accepted to avoid computational difficulties.

The algorithm we introduce estimates the expected discounted payment streams as given in Equations (3.6) and (3.9). Later, for a calibration to CDO quotes, we need some adjustments of the algorithm to match the exact contractual terms of the *iTraxx* portfolio. However, these adjustments are rather tedious to implement than mathematically challenging. Our algorithm is based on the models of Sections 5.1.1 and 5.1.2, a model with different industry sectors as in Section 5.1.3 is implemented similarly.

Algorithm 5.3.1 (Monte Carlo estimation of CDO prices)

For simplicity, we assume periodic premium payments with frequency η years. Depending on the terms of contract, this algorithm might require minor changes.

Within each simulation run, perform the following steps.

I) Simulate the required random variables

1. *Simulate the number of information arriving within $[0, T]$. This corresponds to simulating a realization of N_T from a $Poi(\lambda T)$ -distribution.*
2. *Simulate the location of these news $0 < \tau_1 < \dots < \tau_{N_T} < T$. Conditioned on N_T , these random times are distributed as order statistics of $Uni(0, T)$ distributed random variables on $[0, T]$.*
3. *Create an equidistant grid on $[0, T]$ with mesh¹¹ κ . This grid needs to be fine enough to contain all premium payment dates, as the computation of the premium leg requires the expected loss at these times¹². Therefore, we need $\eta = \kappa k$ for some $k \in \mathbb{N}$.*

¹¹ For an equidistant grid, the mesh is the difference of any two consecutive points.

¹² If $\eta = \frac{1}{4}$, we propose to choose a monthly grid, that is $\kappa = \frac{1}{12}$.

4. Combine this grid with $\tau_1 < \dots < \tau_{N_T}$ to a refined partition of $[0, T]$. Denote the attachment points of this partition by $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$. This partition is no longer equidistant, but contains the jump times of all companies and the times of all premium payments. Store the information whether t_l is a jump of N or not.
5. Simulate a realization of the Brownian motion of the market W^M , sampled at the partition above. More precisely, generate a series of independent random variables $x_1, \dots, x_{n+1}$, where $x_j \sim \mathcal{N}(0, \Delta t_l)$. Then, inductively compute $W_{t_l}^M$ via

$$W_{t_0}^M = 0, \quad W_{t_l}^M = W_{t_{l-1}}^M + x_l \quad \forall l \in \{1, \dots, n+1\}.$$

6. Simulate realizations of the exponent of each firm-value process at, that is X_{t_l} , and immediately before, that is X_{t_l-} , the points of the partition.

First, determine the jumps of each firm-value process. Simulate I times a series of N_T independent Bernoulli experiments, where the success probabilities in the i^{th} series are given by b_i . The outcomes of these experiments are denoted by B_l^i and specify the jump times of each firm-value process. More precisely, a success in experiment l of series i , i.e. $B_l^i = 1$, corresponds to a jump of V^i at position τ_l .

The next step depends on whether we assume independent jump sizes, that corresponds to the model of Section 5.1.1, or jumps with the same sign, which corresponds to the variant of Section 5.1.2.

- (a) If jumps are assumed to be independent, simulate I times N_T independent random numbers, denoted by y_l^i , where

$$y_l^i \sim \begin{cases} \mathbb{P}_{Y^i} & : B_l^i = 1, \\ \delta_0 & : B_l^i = 0. \end{cases}$$

- (b) If the model of Section 5.1.2 is used, perform N_T independent Bernoulli experiment with success probability $p = 0.5$ to determine the common sign at each τ_l . Then, simulate each y_l^i as above, conditioned on the respective sign.

Finally, simulate the increments of the diffusion components in between the points of the grid. This corresponds to simulating I times $n+1$ independent random numbers x_l^i , where $x_l^i \sim \mathcal{N}(0, \Delta t_l)$, and the construction below.

For each company, inductively compute $X_{t_0}^i, X_{t_1-}^i, X_{t_1}^i, \dots, X_{t_{n+1}}^i$ by

$$X_{t_0}^i = 0,$$

$$\begin{aligned}
X_{t_l-}^i &= X_{t_{l-1}}^i + \gamma_i \Delta t_l + \sigma_i \left(a_i \Delta W_{t_l}^M + \sqrt{1 - a_i^2} x_l^i \right) \quad \forall l \in \{1, \dots, n+1\}, \\
X_{t_l}^i &= \begin{cases} X_{t_l-}^i + y_l^i & : \exists j \in \{1, \dots, N_T\} : t_l = \tau_j, \\ X_{t_l-}^i & : t_l \neq \tau_j, \forall j \in \{1, \dots, N_T\}, \end{cases} \\
&\quad \forall l \in \{1, \dots, n+1\}.
\end{aligned}$$

7. Define $\mathcal{F}^*$ as the information of all firm-value processes on the grid, i.e.

$$\mathcal{F}^* = \sigma \{X_{t_l-}^i, X_{t_l}^i, t_l : l \in \{0, \dots, n+1\}, i \in \{1, \dots, I\}\}.$$

8. Calculate conditional survival probabilities of each company up to each point of the grid. These probabilities are easily expressed in terms of the distribution of the running minimum of a Brownian bridge, compare Lemma 4.2.2.

$$\begin{aligned}
\mathbb{P}\mathbb{P}_l^i &:= \mathbb{P}(\tau^i \geq t_l | \mathcal{F}^*) \\
&= \mathbb{P}\left(\inf_{0 \leq s < t_l} X_s^i > \log\left(\frac{d^i}{v_0^i}\right) \middle| \mathcal{F}^*\right) \\
&= \prod_{j=1}^l \Phi_{b, \sigma_i}^{BB}(X_{t_{j-1}}^i, X_{t_j-}^i, \Delta t_j), \quad b = \log(d^i/v_0^i).
\end{aligned}$$

Moreover, we also need the conditional probability of each firm not to default within $[t_{l-1}, t_l)$, which is given by

$$\begin{aligned}
\mathbb{P}_l^i &:= \mathbb{P}(\tau^i \notin [t_{l-1}, t_l) | \mathcal{F}^*) \\
&= \mathbb{P}\left(\inf_{t_{l-1} \leq s < t_l} X_s^i > \log\left(\frac{d^i}{v_0^i}\right) \middle| \mathcal{F}^*\right) \\
&= \Phi_{b, \sigma_i}^{BB}(X_{t_{l-1}}^i, X_{t_l-}^i, \Delta t_l), \quad b = \log(d^i/v_0^i).
\end{aligned}$$

II) Estimate the expected loss at each t_{l-} and t_l

1. Initialize $(\hat{L}_{t_0-}, \dots, \hat{L}_{t_{n+1}-})$, as well as $(\hat{L}_{t_0}, \dots, \hat{L}_{t_{n+1}})$, by $(0, \dots, 0)$.
2. The following computations have to be performed for $i \in \{1, \dots, I\}$. For company i , let $l \in \{0, \dots, n\}$ loop through each point of the grid and consider the following exclusive cases.

$$(a) \quad X_{t_{l+1}-}^i > \log(d^i/v_0^i) \text{ and } X_{t_{l+1}}^i > \log(d^i/v_0^i).$$

These conditions prevent company i from defaulting exactly at time t_{l+1} , which excludes a default by jump at t_{l+1} . Nevertheless, even if the firm-value process at time t_{l+1} is above d^i , the probability of its

running minimum to touch this level is given by $1 - \mathbb{P}_{l+1}^i$. We consider this probability of overseeing such a default by increasing $\hat{L}_{t_{l+1}-}$ by

$$\mathbb{P}\mathbb{P}_l^i (1 - \mathbb{P}_{l+1}^i) N^i (1 - R^i).$$

In short, the factor $\mathbb{P}\mathbb{P}_l^i (1 - \mathbb{P}_{l+1}^i)$ is the conditional probability given $\mathcal{F}^*$ of company i to survive up to t_l- and default in $[t_l, t_{l+1})$. The resulting loss is the nominal N^i times the fractional loss at default $(1 - R^i)$.

$$(b) \quad X_{t_{l+1}-}^i > \log(d^i/v_0^i) \text{ and } X_{t_{l+1}}^i \leq \log(d^i/v_0^i).$$

In this case, a jump causes company i to default at time t_{l+1} . The conditional probability of no prior default is given by $\mathbb{P}\mathbb{P}_{l+1}^i$, we therefore increase $\hat{L}_{t_{l+1}}$ by

$$\mathbb{P}\mathbb{P}_{l+1}^i N^i (1 - R^i).$$

Still, we must not ignore a possible default of company i on $[t_l, t_{l+1})$. We account for this possibility by increasing $\hat{L}_{t_{l+1}-}$ by

$$\mathbb{P}\mathbb{P}_l^i (1 - \mathbb{P}_{l+1}^i) N^i (1 - R^i),$$

which is explained as in the previous case.

$$(c) \quad X_{t_{l+1}-}^i \leq \log(d^i/v_0^i).$$

Here, default is caused within the interval $[t_l, t_{l+1})$ by diffusion, the probability of no default up to time t_l is given by $\mathbb{P}\mathbb{P}_l^i$. Hence, we increase $\hat{L}_{t_{l+1}-}$ by

$$\mathbb{P}\mathbb{P}_l^i N^i (1 - R^i).$$

3. So far, we calculated the losses which occurred exactly at, or in the interval before each point of the partition. These losses are now aggregated to obtain the cumulative loss up to t_l and at t_l . To do so, we loop $l = 0, \dots, n$ and increase $\hat{L}_{t_{l+1}-}$ by $\hat{L}_{t_l}$ and $\hat{L}_{t_{l+1}}$ by $\hat{L}_{t_{l+1}-}$, respectively.

III) Estimate the expected discounted premium leg of each tranche

1. Initialize the estimate $EDPL^j$ by zero for all tranches $j \in \{1, \dots, J\}$.
2. For each tranche $j \in \{1, \dots, J\}$, let t_l^p loop through all premium payment dates $\{t_1^p, \dots, t_n^p\}$ ¹³. At each date, compute the estimated expected discounted premium of tranche j given $\mathcal{F}^*$. This premium depends on the time since the last premium payment $\Delta t_l^p = t_l^p - t_{l-1}^p$, the discount factor $\exp(-rt_l^p)$ and the expected remaining nominal of tranche j at time t_l^p , that is $\hat{M}_{t_l^p}^j$ ¹⁴.

¹³ That is $t_l^p \in \{\eta, 2\eta, \dots, T\}$ for a new contract with payment frequency η -years.

¹⁴ Given the estimate $\hat{L}_{t_l^p}$, we obtain $\hat{M}_{t_l^p}^j = u^j - l^j - \min\{\max\{0, \hat{L}_{t_l^p} - l^j\}, u^j - l^j\}$.

Moreover, if the market value of a running contract has to be assessed, then the spread of this tranche s^j is known and also a factor. In this case, increase $EDPL^j$ by

$$e^{-rt_l^p} \Delta t_l^p s^j \left(u^j - l^j - \min \left\{ \max \left\{ 0, \hat{L}_{t_l^p} - l^j \right\}, u^j - l^j \right\} \right).$$

However, if we are interested in the fair spread s_f^j , increase $EDPL^j$ by

$$e^{-rt_l^p} \Delta t_l^p \left(u^j - l^j - \min \left\{ \max \left\{ 0, \hat{L}_{t_l^p} - l^j \right\}, u^j - l^j \right\} \right).$$

3. If the contract demands for accrued interest for defaulted companies, we can approximate this quantity as follows. Given the premium payment dates t_{l-1}^p and t_l^p and the corresponding losses $\hat{L}_{t_{l-1}^p}^j \leq \hat{L}_{t_l^p}^j$ within tranche j , we have to assess where these losses occurred, which corresponds to assessing the accrued interest. To do so, we let t_k loop through all points of the grid, starting at the grid point after t_{l-1}^p and ending at t_l^p . For each t_k , we then increase $EDPL^j$ by

$$e^{-rt_l^p} \left(\frac{t_k + t_{k-1}}{2} - t_{l-1}^p \right) \left(\hat{L}_{t_k}^j - \hat{L}_{t_{k-1}}^j \right),$$

or alternatively by s^j times this number, depending on whether s_f^j has to be found or not. This increment consists of the discount factor corresponding to the premium payment date t_l^p , the distance between the midpoint of t_{k-1} and t_k to the previous premium payment date t_{l-1}^p , and the loss within tranche j in between t_{k-1} and t_k .

IV) Estimate the expected discounted default leg of each tranche

1. Initialize the estimate $EDDL^j$ by zero for all tranches $j \in \{1, \dots, J\}$.
2. Find the index of the first and the last time of the partition where the overall loss given $\mathcal{F}^*$, respectively its left limit, is within tranche j . Define

$$k^j := \begin{cases} 0 & : \hat{L}_{t_l} < l^j, \forall l = 1, \dots, n+1, \\ \min \{ l : \hat{L}_{t_l} \in [l^j, u^j] \} & : \text{otherwise,} \end{cases}$$

$$m^j := \begin{cases} 0 & : k^j = 0, \\ \max \{ l : \hat{L}_{t_l} \in [l^j, u^j] \} & : \text{otherwise.} \end{cases}$$

Similarly define k_-^j and m_-^j , with $\hat{L}_{t_l}$ replaced by $\hat{L}_{t_l-}$.

3. A loss at time t_{k^j} , respectively $t_{k^j}-$, only affects tranche j with the fraction which exceeds the remaining nominal of the preceding tranche, the same holds for the loss at time t_{m^j+1} , respectively $t_{m^j+1}-$. Here, the subsequent tranche is affected by the loss exceeding tranche j .
4. Losses in tranche j occur from time t_{k^j} , respectively $t_{k^j}-$, to t_{m^j+1} , respectively $t_{m^j+1}-$, depending on whether the first or the last loss of tranche j was caused by jump or by diffusion.

For each $l = \max\{k^j, 1\}, \dots, \min\{m^j + 1, n + 1\}$ do the following.

- (a) Define the correction factors $x_{t_l}^j$ and $x_{t_l-}^j$ as

$$x_{t_l}^j := \begin{cases} 1 & : l = k^j \text{ and } k^j = k_-^j, \\ \frac{\hat{L}_{t_l} - l^j}{\hat{L}_{t_l} - \hat{L}_{t_l-}} & : l = k^j \text{ and } k^j < k_-^j, \\ 0 & : l = m^j + 1 \text{ and } m^j = m_-^j, \\ \frac{u^j - \hat{L}_{t_l-}}{\hat{L}_{t_l} - \hat{L}_{t_l-}} & : l = m^j + 1 \text{ and } m^j < m_-^j, \\ 1 & : \text{otherwise,} \end{cases}$$

$$x_{t_l-}^j := \begin{cases} \frac{\hat{L}_{t_l-} - l^j}{\hat{L}_{t_l-} - \hat{L}_{t_{l-1}}} & : l = k^j \text{ and } k^j = k_-^j, \\ 0 & : l = k^j \text{ and } k^j < k_-^j, \\ \frac{u^j - \hat{L}_{t_{l-1}}}{\hat{L}_{t_l-} - \hat{L}_{t_{l-1}}} & : l = m^j + 1 \text{ and } m^j = m_-^j, \\ 1 & : l = m^j + 1 \text{ and } m^j < m_-^j, \\ 1 & : \text{otherwise.} \end{cases}$$

- (b) Loop through all companies $i \in \{1, \dots, I\}$ and do the following.

Check, whether a jump causes company i to default at t_l . If so, increase $EDDL^j$ by the discounted loss caused by company i at this point, corrected by the probability of no prior default and the correction factor. This corresponds to

$$x_{t_l}^j \mathbb{P} \mathbb{P}_{t_l}^i e^{-rt_l} N^i (1 - R^i).$$

If default was not caused by a jump at t_l , increase $EDDL^j$ by

$$x_{t_l-}^j \mathbb{P} \mathbb{P}_{t_{l-1}}^i \int_{t_{l-1}}^{t_l} e^{-rs} g_l^i(s) ds N^i (1 - R^i),$$

where g^i is the density of the first-passage time τ^i conditioned on $\mathcal{F}^*$, compare Equation (4.3). The justification of this step is done by considering the same three cases as in the calculation of the expected loss.

V) Summarizing each Monte Carlo run

1. After each Monte Carlo run, store the expected discounted premium and default leg of all tranches $j \in \{1, \dots, J\}$, indexed by the number of the current run.
2. Reinitialize all variables and proceed with the next run.
3. After the final run, calculate the average of all expected discounted premium and default legs of each tranche, as simulated in the different runs. These quantities are then used to estimate the expected fair spread of the corresponding tranche or to assess the market value of the contract, whichever is of interest.

5.3.1 Discussion of the pricing algorithm

It is typical for numerical routines to face a tradeoff between speed and accuracy. In what follows, we explain how the algorithm is efficiently implemented. Also, we address some numerical pitfalls which have to be considered. A common configuration of CDO portfolios (*iTraxx* convention) sets the number of companies to $I = 125$, which emphasizes the importance of a fast implementation of this high-dimensional problem.

1. To begin with, let us discuss on how many random numbers the algorithm requires in each Monte Carlo run. In total, we have to simulate I firm-value processes on a grid. On average, this grid consists of $\mathbb{E}[N_T] = \lambda T$ random times and $T\kappa^{-1} + 1$ systematic points. For each increment Δt_l , a realization of a normally distributed random variable is required for each company and the Brownian motion of the market. At each τ_l it is further required to check for jumps via a Bernoulli experiment and to simulate a jump size if the respective experiment succeeds. In total, about $\mathcal{O}(T(3\lambda + \kappa^{-1})I)$ simulations of random numbers are required¹⁵. In our implementation, we use the random-number generators of the NAG-software library.
2. For each company, it is further required to compute the conditional survival probabilities $\mathbb{P}\mathbb{P}_l^i$ and $\mathbb{P}_l^i$. To do so, the iteration $\mathbb{P}\mathbb{P}_{l+1}^i = \mathbb{P}\mathbb{P}_l^i \cdot \mathbb{P}_{l+1}^i$ is useful. Moreover, as soon as $\mathbb{P}\mathbb{P}_l^i = 0$ for some l , this iteration is stopped and all following $\mathbb{P}\mathbb{P}_k^i$ are set to zero. The remaining $\mathbb{P}_k^i$ are not required any more.
3. A large fraction of the overall run time is needed for the evaluation of integrals of the form $\int e^{-rs} g^i(s) ds$. We showed that it is possible to approximate these integrals with high precision by either the approach of Metwally and Atiya

¹⁵ For a typical 5-year contract in an *iTraxx* framework, this number is around 20,000.

(2002) or our approximation from Section 4.3.3. Such an approximation significantly increases the speed of the algorithm.

4. An important concern is the mesh κ of the systematic grid of $[0, T]$, as our algorithm implies a small discretization bias with respect to the common factor W^M . In between two points of the grid, we take into account the individual probability of a company to default unobserved. What we do not control is the evolution of the common factor W^M , which is possibly responsible for multiple defaults. Therefore, we slightly underestimate the default correlation of the model. However, several numerical experiments have shown that a monthly grid is fine enough for pricing CDOs. In these experiments, a finer grid did not change spreads below the noise of the Monte Carlo simulation.

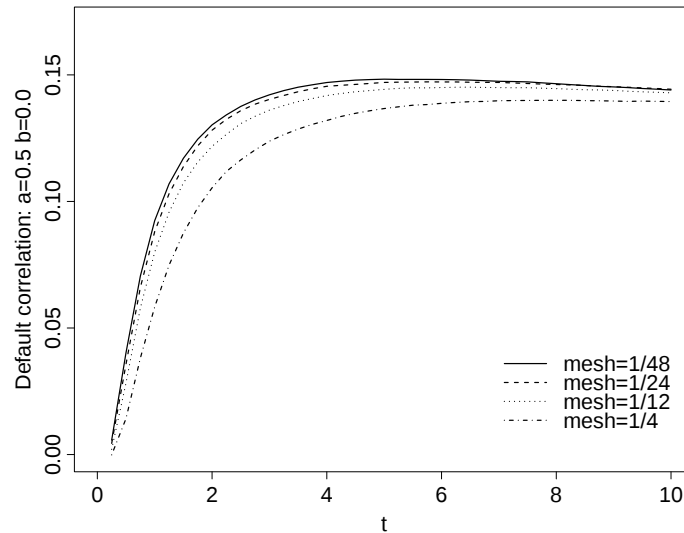


Figure 5.4: Implied default correlation for different κ .

The implications of different κ on the default correlation are illustrated in Figure 5.4, which illustrates the default correlation of two identical companies with parameters $\gamma = 0$, $\sigma = 0.1$ and $d/v_0 = 85\%$ in a model without jumps, based on ten million Monte Carlo runs. This setup represents in some sense the worst-case scenario for the discretization bias, as the default correlation is fully explained by the common market factor. We find that even in this worst-case scenario a mesh of $\kappa = \frac{1}{24}$ is acceptable, and a larger mesh does not significantly increase the default correlation.

5. If a CDO contract is already on the run, the time until the first premium payment does not agree with the time between every other two premium payments. In this case, an equidistant grid typically does not contain all

premium payment dates. Therefore, we have to insert one additional small interval, for instance, $\Delta t_1 < \kappa$ could be chosen appropriately.

6. Let us finish this paragraph with a remark on an infrequent but possible event, which has to be considered in an implementation of the algorithm. One feature of our model is to produce simultaneous defaults by allowing common jumps. When we experimented with portfolios containing very risky companies, it occurred that at some point in time the portfolio loss increased by more than the nominal of a complete tranche j . The result was that the variables k^j and m^j were not well defined. This is considered in our implementation by controlling the increments of the portfolio loss before the variables k^j and m^j are set. If the loss exceeds the complete tranche j at time t_l , this loss affects the default leg of tranche j only at t_l , and the correction factors of the adjoining tranches are set appropriately.

5.3.2 The convergence of estimated CDO spreads

Written in C, our implementation of Algorithm 5.3.1 is able to perform about 60,000 simulation runs per hour on a standard notebook¹⁶. Our numerical experiments suggest that CDO spreads of the equity and junior tranche converge fast, but senior tranches require significantly more simulation runs until their spreads stabilize. This is not surprising, as losses in senior tranches are rare events, while losses in subordinate tranches occur in almost every simulation run. More precisely, Figure 5.5 exhibits the spread of each tranche after the n^{th} Monte Carlo run. The setup of this experiment is described in Section 5.3.3. It is the fictitious portfolio M with *medium* correlation and dependent jumps. We observe that for this scenario, about 10,000 Monte Carlo runs are enough to capture the spreads of the first three tranches within a band of $\pm 2.5\%$ around their terminal value after 60,000 runs. To obtain a similar precision in tranche 4 and 5, we propose to use at least 30,000 runs. Finally, the super senior tranche requires at least 50,000 simulation runs to attain stability. Still, this estimate should be handled with caution.

¹⁶ Apple iBook G4, equipped with a 1.2 GHz processor and 512 MB Ram.

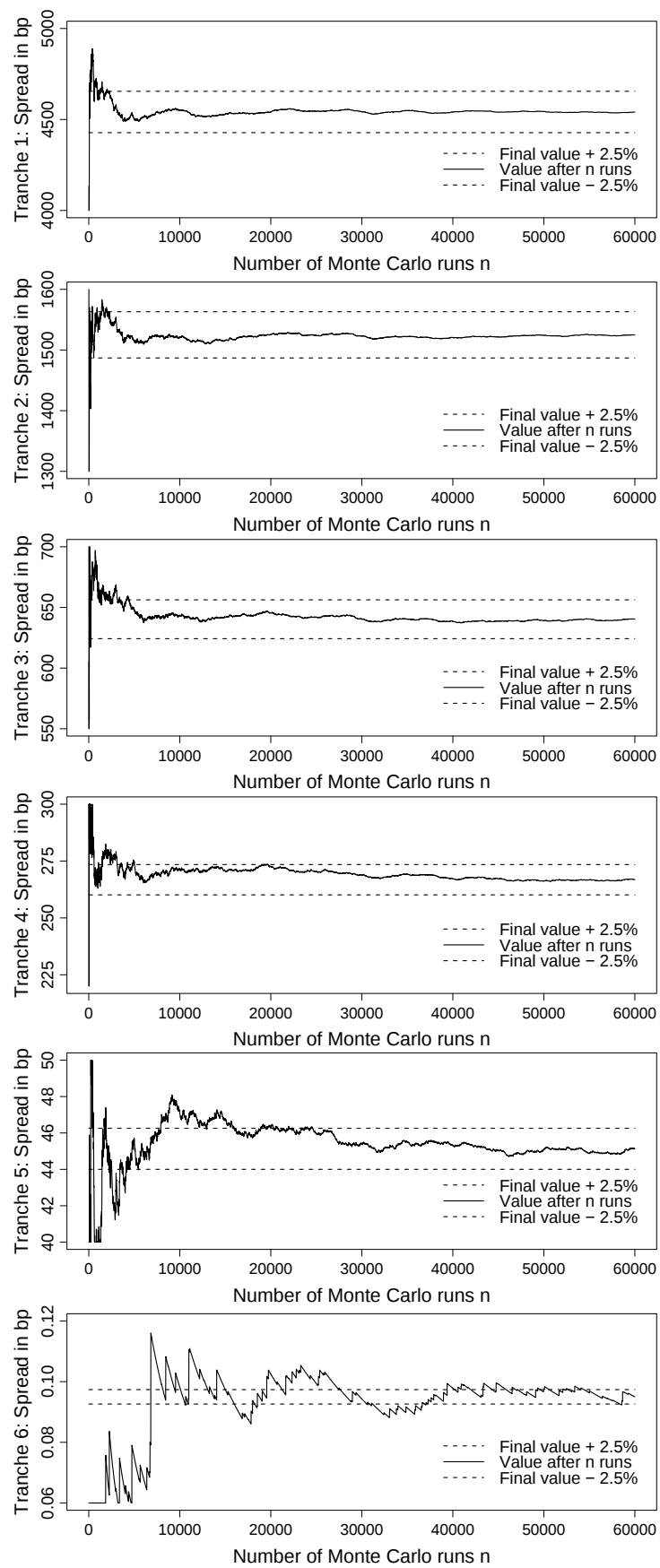


Figure 5.5: The speed of convergence of the different tranches.

5.3.3 Numerical experiments with fictitious portfolios

We construct three fictitious portfolios to analyze our model and Algorithm 5.3.1. The first portfolio (IG) is designed to represent a portfolio containing 125 investment-grade companies, the second portfolio (SG) is set up to match a portfolio consisting of speculative-grade companies. Finally, the last portfolio (M) is inhomogeneous, it contains 75 investment-grade and 50 speculative-grade companies.

General setup and individual default probabilities

We assume a risk-free interest rate of $r = 0.03$ and a recovery rate of $R^i = 40\%$ for each company. Each portfolio contains $I = 125$ companies, all of which contribute $N^i = 1,000$ to the overall nominal of the CDO. Premium payments are made on a quarter-yearly basis, i.e. $\eta = \frac{1}{4}$, the contract matures in five years. The attachment points of each tranche are chosen as in a standard *iTraxx* portfolio. The individual jump-size distributions are assumed to be $\mathbb{P}_{Y^i} = 2\text{-Exp}(\lambda_{\oplus}^i, \lambda_{\ominus}^i, \frac{1}{2})$, the firm-value parameters of each company are randomly drawn from uniform distributions¹⁷.

Portfolio	γ	σ	$\lambda_{\oplus}$	$\lambda_{\ominus}$	d/v_0	$\mathbb{P}(\tau \leq 1)$	$\mathbb{P}(\tau \leq 5)$
IG	0.035	0.044	30.27	30.50	75.0%	0.067%	0.722%
SG	0.020	0.060	22.51	22.49	84.9%	4.656%	21.830%
M	0.029	0.051	27.30	27.07	79.2%	1.985%	9.507%

Table 5.1: Average parameters and default probabilities of the portfolios.

Different scenarios for the dependence

We measure the influence of the common market factor and common jumps by calculating CDO spreads based on three scenarios for each portfolio. These scenarios correspond to high, medium and low dependence. For simplicity, we assume all companies to have the same parameters $a \equiv a_i$ and $b \equiv b_i$. These parameters are chosen as described in Table 5.2 below. Depending on the scenario, we have to adjust the intensity of the ticker process N to keep the intensity $b_i\lambda$ of each individual firm-value processes constant, which implies the same individual default probabilities for all experiments. As well as varying the parameters a and b ,

¹⁷ Their ranges are specified such that the resulting one and five-year default probabilities agree with historical default rates of investment-grade (respectively speculative-grade) companies from 1983-2000, as reported by *Moody's Investors Service*, Gupton (2001), Exhibit 41.

we additionally change the dependence structure by running each scenario initially with independent jumps, and later with jumps in the same direction, corresponding to the models of Sections 5.1.1 and 5.1.2, respectively.

Scenario	$a_i^2 = b_i^2$	$a_i = b_i$	λ
<i>High</i> dependence	0.50	0.707	1.414
<i>Medium</i> dependence	0.25	0.500	2.000
<i>Low</i> dependence	0.05	0.224	4.472

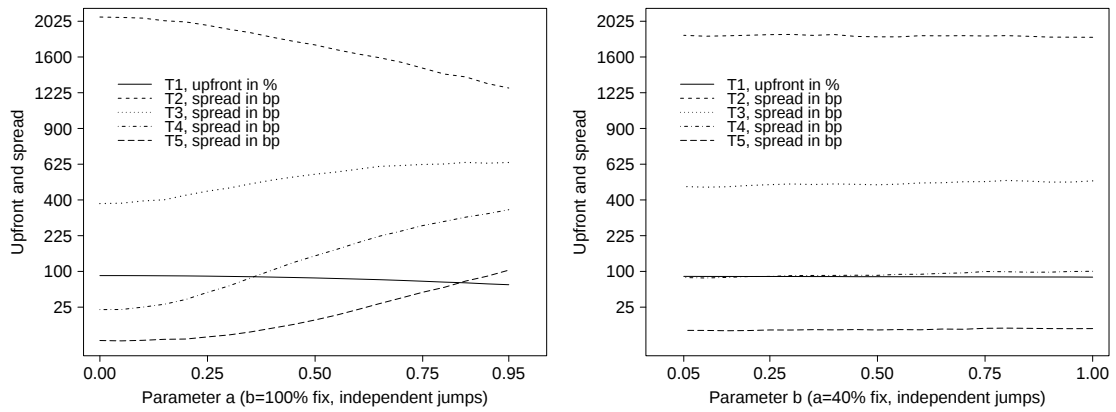
Table 5.2: The three scenarios.

Results of the experiment

The results of our experiment are given in Tables 5.3 and 5.4. From an economic perspective, we observe that the model reacts to changes in the correlation structure as expected. Spreads of senior tranches are increasing in correlation, the opposite is true for junior tranches. An important observation is the large difference of prices if independent jumps are replaced by jumps in the same direction. Considering the implied correlations, we first observe that spreads of equity tranches correspond to the largest implied correlation. We also find that in most examples implied correlations are skewed towards the mezzanine tranches.

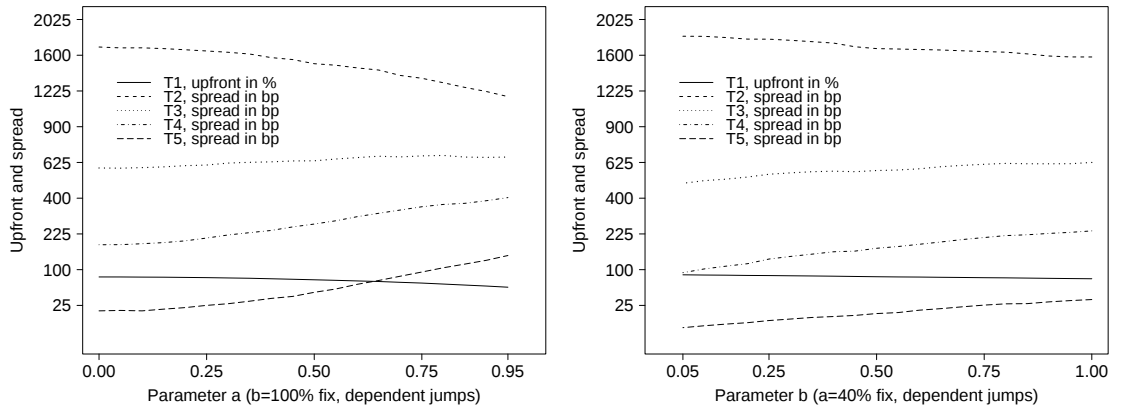
Scenario		<i>High</i>		<i>Medium</i>		<i>Low</i>	
CDO	j	$\hat{s}_f^j$ in bp	$\rho_T^{T,j}$	$\hat{s}_f^j$ in bp	$\rho_T^{T,j}$	$\hat{s}_f^j$ in bp	$\rho_T^{T,j}$
IG	1	292.65	0.500	307.20	0.450	312.33	0.430
	2	10.34	0.440	2.77	0.350	0.15	0.255
	3	1.20	0.435	0.06	0.330	0.00	n.a.
	4	0.28	0.440	0.00	n.a.	0.00	n.a.
	5	0.02	0.430	0.00	n.a.	0.00	n.a.
	6	0.00	n.a.	0.00	n.a.	0.00	n.a.
SG	1	8875.76	0.485	12053.79	0.385	16004.76	0.270
	2	3927.42	0.450	5074.60	0.340	6522.83	0.205
	3	2345.13	0.445	2883.59	0.330	3740.90	0.195
	4	1520.71	0.430	1726.57	0.315	2104.04	0.185
	5	653.19	0.465	576.57	0.350	412.58	0.200
	6	23.44	0.460	8.20	0.335	0.30	0.185
M	1	4706.73	0.405	5928.61	0.315	7443.69	0.225
	2	1523.44	0.330	1734.36	0.250	2009.67	0.165
	3	621.37	0.510	555.66	0.325	399.45	0.190
	4	251.61	0.380	145.65	0.275	30.93	0.170
	5	40.49	0.350	10.16	0.260	0.21	0.160
	6	0.09	0.335	0.0 0	n.a.	0.00	n.a.

Table 5.3: Independent jumps: Monte Carlo simulation with 100,000 runs.

Figure 5.6: Portfolio M, spreads depending on $a \equiv a^i$ or $b \equiv b^i$.

Scenario		<i>High</i>		<i>Medium</i>		<i>Low</i>	
CDO	j	$\hat{s}_f^j$ in bp	$\rho_T^{I,j}$	$\hat{s}_f^j$ in bp	$\rho_T^{I,j}$	$\hat{s}_f^j$ in bp	$\rho_T^{I,j}$
IG	1	259.88	0.605	291.24	0.515	313.97	0.440
	2	28.80	0.570	13.87	0.465	1.57	0.325
	3	8.09	0.565	1.60	0.445	0.01	0.295
	4	2.57	0.555	0.38	0.445	0.00	n.a.
	5	0.38	0.540	0.02	0.435	0.00	n.a.
	6	0.00	n.a.	0.00	n.a.	0.00	n.a.
SG	1	5669.98	0.625	8342.23	0.510	13314.19	0.350
	2	2977.33	0.575	3850.05	0.460	5590.83	0.295
	3	1972.45	0.555	2332.98	0.445	3150.86	0.285
	4	1406.25	0.525	1523.11	0.430	1827.61	0.275
	5	721.90	0.645	667.22	0.490	530.45	0.300
	6	49.46	0.600	24.47	0.465	4.26	0.285
M	1	3413.05	0.520	4521.08	0.420	6414.60	0.290
	2	1340.74	0.430	1510.49	0.340	1814.22	0.225
	3	689.10	0.505	637.06	0.510	518.57	0.275
	4	369.13	0.555	265.39	0.395	108.38	0.245
	5	102.61	0.470	45.06	0.360	5.17	0.235
	6	0.91	0.430	0.10	0.340	0.00	n.a.

Table 5.4: Dependent jumps: Monte Carlo simulation with 100,000 runs.

Figure 5.7: Portfolio M, spreads depending on $a \equiv a^i$ or $b \equiv b^i$.

5.3.4 Calibration to *iTraxx* quotes

*"The effects of correlation can be most simply thought of in terms of a long, thin balloon: if you squeeze the middle, it will swell at some other point."*¹⁸

A calibration of our structural model seems to be extremely difficult, since the model contains a large number of parameters. On the other side, we designed the model to simultaneously explain single and multi-name derivatives, which allows the use of a vast quantity of market quotes as input variables. The key observation for fitting the model is hidden in Lemma 5.1.1. This results shows that the parameter a_i does not affect the distribution of τ^i . Also, as long as the product $b_i\lambda$ is kept constant, one can adjust b_i without altering the term structure of default probabilities of company i . Summarized, this suggest to initially fit the individual firm-value parameters, followed by a calibration of the parameters of the dependence structure. In what follows, we proceed according to the two-step scheme presented below.

1. Fitting each firm-value process individually

In this first step, we fit the parameters γ_i , σ_i , $\lambda^i = b_i\lambda$, d^i/v_0^i and $\mathbb{P}_{Y^i}$ to single-name derivatives of company i . For instance, CDS spreads and bond quotes are possible input variables for this calibration. For more numerical tractability, we suggest to obtain d^i/v_0^i from balance-sheet data and to choose a jump-size distribution with at most two parameters. In our calibration, we proceed as described in Section 4.9.2 of the univariate models, assuming $\mathbb{P}_{Y^i} = 2\text{-Exp}(\lambda_{\oplus}^i, \lambda_{\ominus}^i, \frac{1}{2})$, with the constraint $\lambda_{\oplus}^i = \lambda_{\ominus}^i$.

2. Adjusting the dependence structure

To adjust the dependence among the companies, we have to rely on quoted CDO spreads. To this day, these are the only liquidly traded portfolio derivatives with available market quotes which depend on the correlation among the companies. Theoretically, other portfolio derivatives, such as n^{th} -to default contracts, are further input variables for the fit of the dependence structure.

The data set

We use the same set of CDS data as in Section 4.9.2. Additionally, *iTraxx* provides portfolio CDS spreads and spreads for the different tranches of the CDO. Further input parameters are the term structures of interest rates of default-free bonds and the last available debt-to-value ratio of each company, both are obtained from

¹⁸ Paul J. Davies, "Market rally cuts riskiest CDO spreads", *Financial Times*, Nov. 9, 2006.

Bloomberg. These term structures of interest rates are used to replace the flat interest rate in all pricing formulas.

Fitting each firm-value process to CDS spreads

This step is closely related to the fitting approach presented in Section 4.9.2. Additionally, the condition $\lambda \geq \max_{i \in \{1, \dots, I\}} \hat{\lambda}^i$ is required, as the implied jump intensities $\hat{\lambda}^i$ of the pooled companies are not identical and jumps are triggered by a common ticker process with intensity λ . Therefore, we sacrifice some fitting capability and restrict each parameter λ^i by some artificial upper bound λ^{max} in Equation (4.46). We found that most companies of the *iTraxx* portfolio have an implied jump intensity of less than one, so $\lambda^{max} = 2$ is a restriction which does not decrease the fitting capability of the model. In the search for the implied parameters of the dependence structure, the initial value for b_i is set to $b_i = \hat{\lambda}^i / \lambda^{max}$, the initial intensity λ of the ticker process N is set to λ^{max} .

The calibration to CDO spreads

At first, we have to specify a measure of distance, for which we describe two alternatives. In practice and for our calibration, the first approach is preferred.

1. The first approach is to choose the parameters of the dependence structure such that the quoted equity tranche is perfectly matched by the model. Then, the sum of absolute distances of model to market spreads over all remaining tranches is used as a measure of the fitting capability of the model. Most models imply the spread of the equity tranche to be a continuous and monotone function of the parameters adjusting the dependence. Therefore, a perfect fit of the equity tranche is possible. If the dependence is adjusted by more than a single parameter, then it is typically possible to match the first tranche by several combinations of these parameters. Restricted to these combinations, the fit of the remaining tranches is optimized. An argument for this approach is the level of the spread of the equity tranche, which exceeds the remaining spreads by far¹⁹, compare Tables 5.3 and 5.4. Therefore, the constraint of a perfect fit of the first tranche is reasonable.
2. Alternatively, it is possible to choose the dependence parameters such that some measure of distance of market to model prices is minimized. This

¹⁹ This property is often overseen if the first tranche is quoted in terms of an upfront payment, as this quotation hides the true spread of the equity tranche. However, the running spread of 500 bp alone exceeds the spread of any other tranche by far.

approach suffers from the fact that the first tranche is usually quoted in terms of an upfront payment, whereas all other tranches are quoted in basis points. This upfront payment can not be converted into basis points without introducing some implicit assumptions on the default and premium leg of the first tranche. As these assumptions vary from investor to investor, we suggest to use the sum of relative distances as a measure of distance. This measure does not require such a transformation and puts equal weight on all tranches.

The CDO pricing algorithm for our model is a Monte Carlo simulation. Therefore, sophisticated search routines based on estimated gradients are not applicable. Instead, we implement a naïve search on a grid over the parameter space of the model. To make this approach numerically tractable, we have to reduce the dimension of this space. To do so, we assume a homogeneous correlation of all firms to the market factor, i.e. $a \equiv a_i$ for all $i \in \{1, \dots, I\}$. At this point, recall that changing the parameter a_i does not affect the distribution of τ^i . Adjusting the parameters b_i is done conditional on the constraint $\hat{\lambda}^i = b_i \lambda$ for all $i \in \{1, \dots, I\}$, which is required for preserving the previously calibrated individual default probabilities. Therefore, we gradually increase λ and adjust each b_i appropriately. The implied dependence is obviously decreasing in λ . More precisely, with λ^{max} as described in the calibration of each individual firm-value process, we define

$$\lambda(x) := \frac{\lambda^{max}}{x}, \quad b_i(x) := \frac{\hat{\lambda}^i x}{\lambda^{max}}, \quad x \in (0, 1]. \quad (5.7)$$

This construction guarantees a constant jump intensity of $b_i(x)\lambda(x) \equiv \hat{\lambda}^i$ for each firm and all $x \in (0, 1]$. Given this construction, we proceed as follows.

1. Define a grid on $[0, 1) \times (0, 1]$. In our calibration, we rely on a grid consisting of 30×30 points, where $a \in \{0, 1/30, \dots, 29/30\}$ and $x \in \{1/30, \dots, 1\}$.
2. Derive CDO spreads using Algorithm 5.3.1 for each point of the grid.
3. Compute the required measure of distance for each point of the grid.
4. Find the minimal distance of model to market prices on the grid.
5. Use Equation (5.7) to retrieve $\hat{b}_i$ from $\hat{x}$, λ^{max} and $\hat{\lambda}^i$.

Adjusting d/v_0 to match portfolio CDS spreads

Up to this point, we only used individual CDS and CDO tranche quotes to fit the model. Additionally, *iTraxx* provides portfolio CDS quotes of the respective

portfolio for the maturities three, five, seven and ten years. The pricing formula for this derivative does not depend on the correlation, due to the linearity of the expectation. Therefore, one can use quoted spreads of portfolio CDS to check whether the combined individual default probabilities agree with the market's expectation. In our calibration, we observed that these prices are matched relatively well. Typically, a deviation of less than two basis points was observed for the five year spread²⁰. To incorporate these prices, it is possible to adjust the individual default probabilities of all companies until a perfect fit to observed portfolio CDS spreads is achieved. To do so, we suggest multiplying each leverage ratio d^i/v_0^i by an appropriate correction term, which is determined prior to the calibration to CDO quotes. In our calibration, a correction of less than 1.5% was usually sufficient. Let us add the warning that adjusting the individual default probabilities goes along with sacrificing some precision in matching individual CDS contracts. Therefore, this adjustment should not be used if individual and portfolio derivatives are priced simultaneously. Also, in our calibration we considered the portfolio CDS as a single tranche of the CDO which covers the complete portfolio. This assumption differs slightly from the *iTraxx* convention, but is already implemented as a special case of Algorithm 5.3.1. As long as the overall portfolio loss is small, we feel that this simplification is acceptable.

Results of the calibration 1: Contracts maturing in five years

Before presenting our results, we want to thank Dr. Heike Koch-Beuttenmüller and Michael Lehn of the *KIZ*²¹ and *UZR*²², respectively, for providing the required computation time on their machines. All calculations are based on 100,000 simulation runs, which is time consuming but ensures a high precision. At first, we fitted our model to CDOs maturing in five years, as these are the most liquidly traded contracts. For each day, we run the fitting procedure with independent, dependent and without jumps, respectively. Also, we performed each calibration with an initial adjustment to portfolio CDS quotes. The results are listed in Tables 5.6 - 5.10. The results of a continuous model without initial adjustment to portfolio CDS prices are omitted, as these results are far from leading to realistic prices. Each of the following tables contains the parameters that correspond to the minimal distance²³. Then, the upfront payment and the model spreads of tranche 2-5 are reported. The columns d_1 , d_{2-5} and d_{CDS} contain the absolute pricing errors for the upfront payment, the sum of distances of tranche 2-5, and the portfolio CDS, respectively.

²⁰ In the observed period, this spread traded around 31 bp.

²¹ <http://www.kiz.uni-ulm.de>

²² <http://www.informatik.uni-ulm.de/uzwr>

²³ For this, we accepted a deviation of the upfront payment of 0.1, which is below bid-ask spreads. Then, we minimized over the sum of distances of the remaining tranches.

Day	up	s^2	s^3	s^4	s^5	CDS
2.05	16.750	46.750	12.500	4.250	2.832	26.669
4.05	17.126	43.750	13.000	4.877	2.832	27.450
9.05	18.141	45.708	13.833	5.167	3.000	28.727
11.05	19.344	51.563	14.906	5.875	3.167	28.485
16.05	20.344	53.906	16.375	6.313	3.583	30.396
18.05	21.183	57.333	17.500	6.438	3.708	31.085
23.05	23.438	66.125	20.000	7.625	4.083	32.200
25.05	24.031	68.719	21.000	8.063	4.167	32.690
30.05	21.813	61.250	17.667	6.417	3.500	32.210
1.06	22.985	68.125	19.313	7.500	3.917	31.821
6.06	22.047	68.875	18.750	6.931	3.688	32.349
8.06	22.891	73.313	19.938	7.438	3.917	33.438
13.06	24.174	78.250	21.500	7.875	4.333	33.955
15.06	24.480	80.417	21.625	7.917	4.438	34.032
20.06	24.502	81.250	22.000	8.345	4.415	33.695
22.06	24.424	81.313	22.188	8.470	4.415	34.156

Table 5.5: CDO market quotes (5 years).

Day	$\hat{a}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	d_{CDS}
2.05	0.27	17.15	60.08	3.72	0.24	0.01	0.40	28.95	0.05
4.05	0.30	17.16	76.51	6.45	0.67	0.02	0.03	46.33	0.26
9.05	0.33	17.70	102.45	13.25	1.74	0.08	0.44	63.68	0.26
11.05	0.27	19.58	71.38	4.76	0.33	0.01	0.23	38.67	0.04
16.05	0.30	20.88	97.77	8.99	0.97	0.03	0.54	60.14	0.30
18.05	0.30	21.67	103.21	9.96	1.09	0.04	0.49	62.43	0.32
23.05	0.30	22.93	112.69	11.49	1.28	0.05	0.51	65.46	0.32
25.05	0.30	23.60	116.66	12.03	1.33	0.05	0.43	67.77	0.30
30.05	0.33	21.80	130.54	18.85	2.76	0.14	0.01	77.49	0.27
1.06	0.30	22.43	110.01	10.88	1.11	0.05	0.55	60.59	0.34
6.06	0.33	21.86	132.55	19.40	2.86	0.13	0.19	71.96	0.24
8.06	0.33	23.13	141.81	21.46	3.28	0.14	0.24	77.95	0.26
13.06	0.33	23.84	146.56	22.50	3.47	0.15	0.34	77.90	0.22
15.06	0.30	24.97	128.04	13.95	1.50	0.07	0.49	66.09	0.37
20.06	0.30	24.72	124.25	13.19	1.43	0.07	0.22	63.07	0.38
22.06	0.33	24.06	147.64	22.78	3.51	0.16	0.37	76.13	0.20

Table 5.6: Fitted CDO prices (5 years): continuous model, adjusted d/v_0 .

Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	d_{CDS}
2.05	0.77	0.97	16.83	119.64	28.81	9.81	1.85	0.08	95.74	2.69
4.05	0.77	0.80	17.07	115.39	27.46	8.93	1.53	0.06	91.45	1.90
9.05	0.90	0.03	18.11	114.11	34.81	15.70	4.58	0.03	101.49	1.62
11.05	0.80	0.37	19.31	113.96	27.17	9.45	1.95	0.03	79.45	2.18
16.06	0.83	0.63	20.42	129.59	33.48	13.17	2.99	0.08	100.24	1.75
18.05	0.83	0.03	21.11	131.06	37.79	15.59	4.10	0.07	103.56	1.56
23.05	0.70	0.53	23.46	127.02	24.92	6.97	0.96	0.02	69.59	0.97
25.05	0.73	0.33	23.96	126.35	24.49	6.91	1.18	0.07	65.26	0.79
30.05	0.87	0.07	21.84	127.87	35.16	15.13	4.27	0.03	93.60	0.90
1.06	0.77	0.27	22.98	125.67	28.10	8.68	1.56	0.00	69.87	1.27
6.06	0.77	0.43	22.12	124.50	28.32	9.24	1.55	0.07	69.64	0.19
8.06	0.80	0.07	22.87	124.04	30.39	10.64	2.16	0.02	66.14	0.40
13.06	0.73	0.13	24.12	124.72	25.90	7.81	1.01	0.05	54.26	0.48
15.06	0.73	0.50	24.53	135.00	26.80	7.58	1.08	0.05	63.45	0.13
20.06	0.83	0.47	24.60	154.88	40.02	15.33	3.75	0.10	99.30	2.09
22.06	0.90	0.07	24.51	153.82	46.23	20.65	6.08	0.09	110.39	1.92

Table 5.7: Fitted CDO prices (5 years): independent jumps.

Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	d_{CDS}
2.05	0.63	0.07	16.80	68.01	9.93	2.15	0.22	0.05	28.54	0.00
4.05	0.67	0.33	17.17	81.94	13.85	3.44	0.46	0.04	42.85	0.00
9.05	0.73	0.40	18.20	91.73	17.20	5.13	0.79	0.06	51.64	0.02
11.05	0.67	0.07	19.37	76.65	11.40	2.30	0.23	0.03	35.11	0.02
16.06	0.77	0.17	20.41	103.51	22.02	6.95	1.27	0.07	58.20	0.02
18.05	0.73	0.10	21.12	109.09	23.19	6.74	1.03	0.06	60.43	0.00
23.05	0.67	0.20	23.42	107.35	18.37	4.49	0.49	0.02	49.58	0.01
25.05	0.70	0.10	24.11	110.82	19.41	5.11	0.65	0.08	50.16	0.01
30.05	0.80	0.07	21.89	116.37	27.62	10.02	2.02	0.08	70.16	0.00
1.06	0.70	0.13	22.99	104.75	18.98	4.89	0.59	0.00	42.89	0.02
6.06	0.80	0.03	22.03	119.82	29.88	11.00	2.40	0.02	67.43	0.00
8.06	0.80	0.30	22.94	131.80	32.71	11.78	2.42	0.05	77.10	0.01
13.06	0.73	0.23	24.21	132.11	28.18	8.26	1.21	0.04	64.05	0.01
15.06	0.77	0.13	24.57	129.38	28.28	9.12	1.62	0.09	59.64	0.01
20.06	0.77	0.07	24.44	124.51	27.12	8.59	1.45	0.06	51.59	0.00
22.06	0.70	0.73	24.49	135.52	24.84	6.46	0.68	0.07	62.60	0.00

Table 5.8: Fitted CDO prices (5 years): independent jumps, adjusted d/v_0 .

Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	d_{CDS}
2.05	0.80	0.13	16.77	115.05	31.50	12.73	2.72	0.02	95.89	2.69
4.05	0.67	0.47	17.12	120.77	25.36	6.68	0.90	0.01	93.11	1.91
9.05	0.73	0.40	18.24	125.03	27.58	8.02	1.22	0.10	97.70	1.63
11.05	0.70	0.37	19.36	121.88	24.11	6.11	0.86	0.02	82.06	2.17
16.05	0.80	0.23	20.39	130.73	32.00	11.17	2.32	0.05	98.57	1.74
18.05	0.70	0.33	21.16	139.10	30.54	8.72	1.29	0.02	99.51	1.55
23.05	0.63	0.27	23.37	129.84	21.91	4.54	0.49	0.07	72.30	0.96
25.05	0.73	0.10	24.09	131.72	27.37	8.17	1.36	0.06	72.29	0.77
30.05	0.80	0.17	21.83	138.88	35.07	12.72	2.79	0.02	102.05	0.90
1.06	0.77	0.10	23.01	130.77	30.42	9.96	1.80	0.03	78.33	1.26
6.06	0.70	0.27	22.02	133.16	26.25	6.69	0.91	0.03	74.80	0.19
8.06	0.80	0.03	22.88	124.62	30.19	10.97	2.42	0.01	66.59	0.43
13.06	0.70	0.13	24.08	129.00	25.01	7.12	1.10	0.09	58.25	0.49
15.06	0.77	0.07	24.52	135.21	31.13	10.10	2.09	0.04	68.83	0.13
20.06	0.83	0.17	24.48	164.30	43.68	17.19	4.12	0.02	113.87	2.08
22.06	0.83	0.17	24.37	165.11	44.10	16.92	3.97	0.05	114.60	1.93

Table 5.9: Fitted CDO prices (5 years): dependent jumps.

Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	d_{CDS}
2.05	0.63	0.03	16.69	69.95	10.45	2.23	0.23	0.06	29.87	0.00
4.05	0.63	0.13	17.20	79.52	11.65	2.48	0.24	0.07	42.11	0.01
9.05	0.67	0.27	18.12	95.01	15.49	3.40	0.40	0.02	55.33	0.01
11.05	0.67	0.03	19.34	79.18	11.37	2.28	0.19	0.00	37.73	0.01
16.05	0.77	0.07	20.29	103.83	21.43	6.93	1.41	0.05	57.77	0.01
18.05	0.63	0.23	21.15	110.18	18.16	3.90	0.35	0.03	59.40	0.01
23.05	0.67	0.10	23.41	113.58	19.35	4.77	0.52	0.03	54.52	0.03
25.05	0.70	0.03	24.09	112.24	20.40	5.38	0.68	0.06	50.29	0.02
30.05	0.57	0.50	21.83	134.92	21.16	3.81	0.28	0.02	82.99	0.02
1.06	0.67	0.10	22.98	110.64	18.79	4.30	0.47	0.00	49.69	0.01
6.06	0.70	0.20	21.99	123.52	23.69	5.99	0.87	0.06	63.34	0.03
8.06	0.70	0.27	22.90	138.93	28.63	7.87	0.97	0.01	77.69	0.03
13.06	0.67	0.23	24.16	139.21	26.78	6.65	0.74	0.01	71.06	0.02
15.06	0.63	0.27	24.55	137.79	22.27	4.79	0.50	0.07	65.08	0.01
20.06	0.73	0.07	24.53	125.25	24.82	7.26	1.01	0.03	51.31	0.00
22.06	0.60	0.37	24.48	142.77	23.09	4.33	0.39	0.06	70.52	0.00

Table 5.10: Fitted CDO prices (5 years): dependent jumps, adjusted d/v_0 .

Interpretation of the results

First of all, we notice that a jump-diffusion model produces significantly better results than a continuous model. This observation is not surprising, since the continuous model is a special case of the jump-diffusion model. It is more difficult to determine the reason why our model performs better. For this, we feel that the improved fit to individual default probabilities, compare Section 4.9.2, and the additional factor to adjust the dependence structure are responsible to equal parts. The improved fit to individual default probabilities becomes apparent when the continuous model is run without initial adjustment to portfolio CDS prices. In this case, the resulting prices turned out to be far off real CDO and portfolio CDS quotes. More precisely, an adjustment of the initial leverage ratios of up to 15% was required to obtain the prices of Table 5.6. Therefore, it was not possible for us to simultaneously describe single-firm derivatives and portfolio derivatives within a continuous model. On a portfolio level, including common jumps creates the possibility of multiple defaults, which increases spreads of the mezzanine and senior tranches. Relative to the continuous model, this rise was impressive. In absolute values, however, the jump-diffusion model still slightly underprices these tranches.

Focusing on the fitting capability of the jump-diffusion model, we notice that the fit of the second tranche is not satisfying. As long as a perfect fit to the equity tranche is required, the second tranche is systematically overpriced by the model. The same phenomenon is observed by all models that work with an embedded normal distribution²⁴. In our model, this overpricing is reduced by the presence of jumps, but is not eliminated completely. Important for us is the fact that our model is able to produce realistic CDO prices without an adjustment of the initial leverage ratios. This means that all individual contracts, as well as all tranches of the CDO and the portfolio CDS, are simultaneously explained by the model. We feel that this property distinguishes our model from most of the other approaches. If a fit to individual contracts is not required, then it is possible to further improve the fit to CDO quotes by adjusting the firm's leverage ratios. These ratios are adjusted such that the observed portfolio CDS spread is matched.

Finally, let us stress the fact that independent jumps implied a better fit than dependent jumps for most days of the observed period. This property was surprising to us, as the interpretation of jumps in the same direction seems very reasonable from an economic perspective.

²⁴ In Vasicek's model, this property is referred to as correlation skew.

Results of the calibration 2: The term structure of CDO spreads

A new issue in modeling CDOs is the problem of explaining the term structure of CDO spreads. Most existing models are designed to match CDO spreads of a certain maturity. In contrast, our dynamic model allows to simultaneously price contracts with different maturities. Unfortunately, portfolio CDS spreads are not available for contracts maturing in seven years. Therefore, all further results are computed without an initial adjustment of d/v_0 to portfolio CDS spreads. To begin with, we present the results of a calibration of our model to contracts maturing in five, seven and ten years, respectively. These results are presented in Table 5.12, the corresponding market quotes are given in Table 5.11. Computationally, this calibration is more expensive than the calibration before. Therefore, we considered only two trading days.

Day	up	s^2	s^3	s^4	s^5	T
1.06	22.985	68.125	19.313	7.500	3.917	5
1.06	41.219	184.500	47.375	25.313	8.500	7
1.06	50.323	520.875	119.875	55.250	19.906	10
6.06	22.047	68.875	18.75	6.931	3.688	5
6.06	40.048	185.063	45.500	24.438	8.188	7
6.06	49.829	510.750	119.063	52.500	20.063	10

Table 5.11: CDO market quotes.

In a second step, we run the pricing formula for contracts maturing in five, seven and ten years, respectively. For this, we used the implied parameters $\hat{a}$ and $\hat{x}$ as obtained from the fit to contracts maturing in five years. We chose the five-year contracts as reference, as these are traded the most liquidly. The results of this experiment are presented in Table 5.13. Let us emphasize again that using these parameters, the model simultaneously describes the term structure of CDO spreads and the term structure of individual CDS spreads.

Interpretation of the results

First of all, we conclude from Table 5.12 that our model is able to match CDO spreads of all maturities based on identical term structures of default probabilities. Moreover, the implied parameters of the dependence structure, as obtained from fits to different maturities, are close to each other. In our example, the implied correlation of contracts maturing in seven years was slightly below the implied correlation of contracts maturing in five and ten years, respectively. Still, we conclude from Table 5.13 that our model is able to simultaneously produce realistic term structures of CDO and CDS spreads.

Independent jumps										
Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	T
1.06	0.77	0.27	22.98	125.67	28.10	8.68	1.56	0.00	69.87	5
1.06	0.67	0.27	41.38	316.14	84.32	26.55	4.38	0.16	173.94	7
1.06	0.77	0.10	50.37	522.41	220.11	104.29	29.86	0.05	160.76	10
6.06	0.77	0.43	22.12	124.50	28.32	9.24	1.55	0.07	69.64	5
6.06	0.67	0.27	40.23	301.65	80.39	25.56	4.03	0.18	156.76	7
6.06	0.77	0.13	49.64	513.07	215.33	102.36	29.35	0.19	157.73	10

Dependent jumps										
Day	$\hat{a}$	$\hat{x}$	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	T
1.06	0.77	0.10	23.01	130.77	30.42	9.96	1.80	0.03	78.33	5
1.06	0.63	0.10	41.30	317.17	81.69	23.62	3.58	0.08	173.60	7
1.06	0.70	0.13	50.32	543.73	228.18	102.18	26.12	0.00	184.30	10
6.06	0.70	0.27	22.02	133.16	26.25	6.69	0.91	0.03	74.80	5
6.06	0.67	0.07	40.17	304.53	81.74	26.72	4.36	0.12	161.82	7
6.06	0.70	0.13	49.79	534.14	225.14	101.37	25.64	0.04	183.91	10

Table 5.12: Fitted CDO prices for different maturities.

Independent jumps										
Day	a	x	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	T
1.06	0.77	0.27	22.98	125.67	28.10	8.68	1.56	0.00	69.87	5
1.06	0.77	0.27	38.27	304.59	96.38	38.87	9.18	2.95	183.34	7
1.06	0.77	0.27	49.87	523.61	220.87	103.83	29.38	0.46	161.79	10
6.06	0.77	0.43	22.12	124.50	28.32	9.24	1.55	0.07	69.64	5
6.06	0.77	0.43	37.42	304.10	97.48	38.82	8.91	2.63	186.12	7
6.06	0.77	0.43	49.02	517.43	221.72	106.30	31.13	0.81	174.20	10

Dependent jumps										
Day	a	x	$\hat{u}p$	$\hat{s}_f^2$	$\hat{s}_f^3$	$\hat{s}_f^4$	$\hat{s}_f^5$	d_1	d_{2-5}	T
1.06	0.77	0.10	23.01	130.77	30.42	9.96	1.80	0.03	78.33	5
1.06	0.77	0.10	37.87	311.06	101.31	41.70	9.66	3.34	198.04	7
1.06	0.77	0.10	48.91	520.33	225.67	108.53	31.39	1.41	171.11	10
6.06	0.70	0.27	22.02	133.16	26.25	6.69	0.91	0.03	74.80	5
6.06	0.70	0.27	36.67	314.59	96.65	34.23	6.51	3.37	192.15	7
6.06	0.70	0.27	47.03	518.53	231.25	112.04	31.92	2.80	191.37	10

Table 5.13: The term structure of model prices.

5.4 Pricing n^{th} -to default contracts via Monte Carlo simulation

In the following, we describe how n^{th} -to default contracts, as introduced in Section 3.5, are priced within our multivariate default model. Depending on the terms of contract, the presented algorithm may require minor adjustments.

5.4.1 Implementation of the pricing formula

Large parts of our Monte Carlo simulation agree with Algorithm 5.3.1, the implementation of the CDO pricing formula. For the pricing of n^{th} -to default contracts we can use the first step of Algorithm 5.3.1 to compute the expected portfolio loss at each payment date $t_1 < \dots < t_m$, given a set of simulated jumps $\mathcal{F}^*$. Given this, we derive the expected discounted premium and default legs of the current run. At the end, we compute the average premium and default legs to obtain the fair spread. A precise formulation of the algorithm is given below.

Algorithm 5.4.1 (Monte Carlo estimation of n^{th} -to default contracts)

Chose the number of simulation runs K . Within each run, compute the following Steps 1 to 4. Proceed with Step 5 afterwards.

1. *Compute step I.1-I.8 of Algorithm 5.3.1.*
2. *For each time $t_1 < \dots < t_m$, calculate the expected number of defaults, conditioned on $\mathcal{F}^*$. This number is given by*

$$X_{t_l}^* := \sum_{i=1}^I (1 - \mathbb{P}\mathbb{P}_{t_l}^i).$$

3. *Calculate the period in which $\tau_{(n)}$ is expected to fall. Define the time of the last premium payment as*

$$t^{(n)} := \max \{t_l \in \{t_1, \dots, t_m\} : X_{t_l}^* < n\}, \quad \max \emptyset := 0.$$

4. *Calculate the expected discounted premium and default legs, conditioned on $\mathcal{F}^*$. These payment streams are given by*

$$\widehat{EDPL}^{(n)} = \sum_{t_l \in \{t_1, \dots, t^{(n)}\}} N e^{-rt_l} \Delta t_l,$$

$$EDDL^{(n)} = \begin{cases} (1 - R)Ne^{-r(t^{(n)} + \Delta)} & : t^{(n)} < T, \\ 0 & : t^{(n)} = T, \end{cases}$$

where Δ denotes the time between the last premium date $t^{(n)}$ and the settlement date of the default leg.

5. Calculate the average $EDPL^{(n)}$ and $EDDL^{(n)}$ of the K simulation runs.
6. Use these averages to obtain the fair spread $s_f^{(n)}$.

5.4.2 Numerical examples

We use Algorithm 5.4.1 to compute fair spreads of n^{th} -to default contracts based on a portfolio consisting of ten identical companies, with parameters as described in Table 5.14. The jump-size distribution of each process X^i is chosen to be two-sided exponential. The parameters a and b , which adjust the dependence among the companies, correspond to the scenarios of Table 5.2. Also, we computed spreads given the model of Section 5.1.1 with independent, and the model of Section 5.1.2 with dependent jumps. The results are presented in Table 5.15, which also contains the asset value and default correlation between any two companies of the portfolio, which we computed using Theorems 5.2.1 and 5.2.2, or estimated based on ten million Monte Carlo runs. n^{th} -to default spreads are computed with 100,000 Monte Carlo runs.

Interpretation of the results

Obviously, spreads have to be decreasing in n , as $\tau_{(n)} \leq \tau_{(m)}$ for $n < m$, which holds for our examples. Further, we observe that spreads of large²⁵ values of n are increasing in correlation, the opposite is true for spreads of small values of n . Values in between do not behave monotonically. This observation is reasonable, since multiple defaults, or no defaults at all, are more likely if the correlation among the companies is large.

In Figures 5.8 and 5.9, we either fix b or a and vary the other parameter. Here, we observe that the limit of spreads for small maturities depends on b but not on a . Also, we observe a positive limit of spreads for short maturities. For longer maturities, spreads behave as expected, meaning that the spread of the first-to default contract is shifted to contracts for larger values of n if the correlation becomes

²⁵ In this context, small (resp. large) is meant to hold relative to the expected number of defaults.

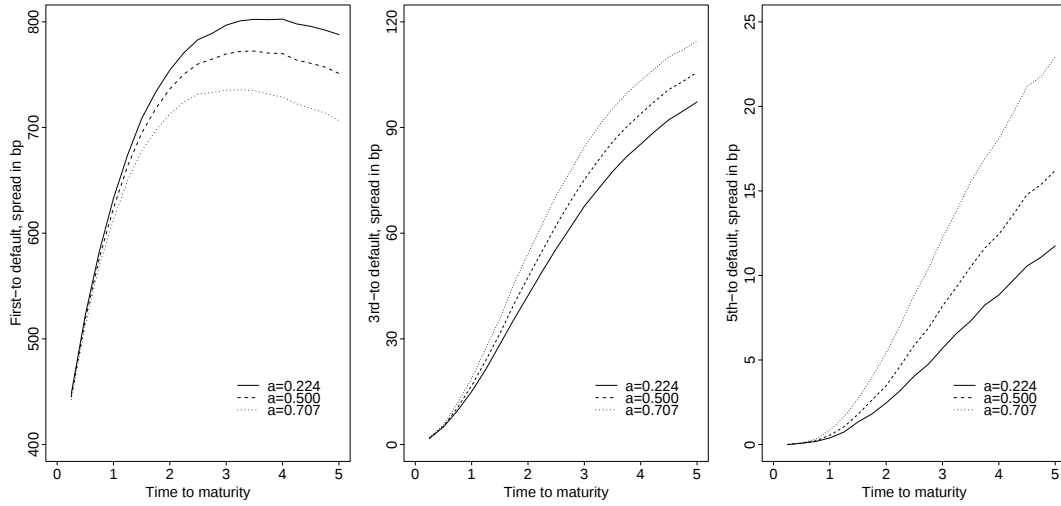
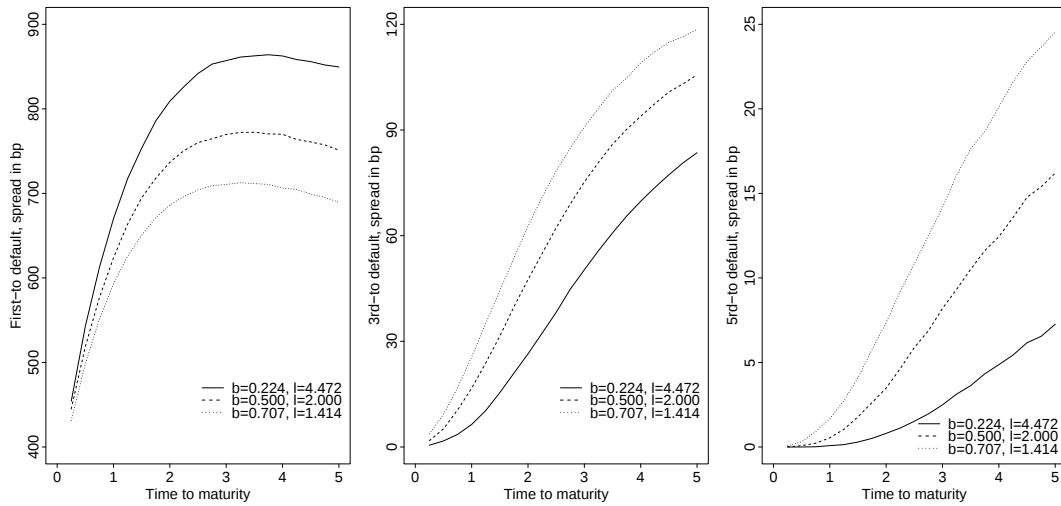
larger. This holds independently of whether the correlation is introduced through diffusion or common jumps.

Firm-value parameters						Dependence		Contract setup			
γ_i	σ_i	$\lambda_{\oplus}^i = \lambda_{\ominus}^i$	d^i/v_0^i	R^i	λ	a_i	b_i	I	r	T	Δt_l
0.025	0.05	20.0	80.0%	40.0%	2.0	Table 5.2		10	0.03	5	0.25

Table 5.14: The setup of the fictitious n^{th} -to default contract.

	<i>High</i>		<i>Medium</i>		<i>Low</i>	
n	IJ	DJ	IJ	DJ	IJ	DJ
1	815.15	643.61	890.82	748.68	955.64	878.30
2	256.36	257.30	250.41	260.62	236.12	246.59
3	88.72	124.46	71.15	104.96	53.86	71.69
4	28.81	62.53	17.48	40.76	8.92	17.39
5	8.64	31.35	3.81	16.10	1.15	4.09
6	2.14	14.49	0.62	5.75	0.06	0.87
7	0.42	6.43	0.06	1.68	0.00	0.14
8	0.08	2.41	0.02	0.47	0.00	0.02
9	0.01	0.75	0.00	0.06	0.00	0.01
10	0.00	0.15	0.00	0.00	0.00	0.00
ρ^X	13.06%	39.18%	8.33%	25.00%	2.64%	7.92%
ρ_T^D	5.47%	14.95%	2.93%	8.55%	0.85%	3.08%

Table 5.15: Fair spread in bp: Monte Carlo simulation with 100,000 runs.

Figure 5.8: n^{th} -to default spreads depending on a , $n \in \{1, 3, 5\}$.Figure 5.9: n^{th} -to default spreads depending on b , $n \in \{1, 3, 5\}$.

5.5 Summary of the multivariate model

We have shown that it is possible to couple univariate jump-diffusion processes to a multidimensional model based on a firm economic interpretation. An important feature of our approach is that the distribution of the individual firm-value processes remains unchanged. Hence, the term structure of marginal default probabilities is also retained and can therefore be fitted individually. In addition to the common approach of introducing a market factor to which all firm-value processes are correlated, we propose a mechanism to support common jumps based on a joint ticker process. The result is an additional factor to adjust the dependence among the companies. Moreover, it is possible to implement the model using dependent jumps, which further increases the default correlation among the modeled companies. We ran several experiments to test how the default correlation depends on the model parameters and found that especially for short maturities, the possibility of a joint default by jump significantly increases the default correlation, compared to a continuous model. Analytically, we succeeded in expressing the asset correlation of two companies in terms of the model parameters. This result contains the pure diffusion approach as a special case.

Given that the pricing of single-name credit derivatives based on jump-diffusion models is already delicate, one should not expect a pricing formula for multi-name derivatives in closed form. What makes the situation even more complicated is that a typical CDO portfolio consists of 125 companies, which makes an implementation of the pricing formula via a Monte Carlo simulation extremely time consuming. What we present are two Monte Carlo simulations for the pricing of CDOs and n^{th} -to default contracts which take advantage of several approximations and a clever implementation. Finally, these algorithms are fast enough to price both contracts on a standard computer.

We extensively tested these algorithms using several numerical examples based on different fictitious portfolios and scenarios. These experiments also provide valuable insight in the characteristics of our default model.

Computationally, the most challenging problem was to fit the model. Most CDO models assume a portfolio of identical companies and focus on explaining the tranches of the CDO alone. Our approach was to fit every single company and the CDO simultaneously. En passant, the model also explains CDS portfolio spreads with high precision. In Section 5.3.4, we presented a two-step procedure to fit the model. In a first step, each individual firm-value processes was calibrated to match the corresponding term structure of single-name CDS spreads. In the second step, the parameters of the dependence structure were adjusted to match observed CDO

tranche spreads. The result of a calibration to 16 trading days is that the model outperforms a continuous model by far, and is able to reproduce observed CDO spreads with high precision. Moreover, we addressed the new problem of explaining the term structure of CDO spreads. The results of this calibration were also very promising.

Chapter 6

Appendix

6.1 The proof of Theorem 4.3.3

The proof of Theorem 4.3.3 was developed together with Johannes Ruf. A more detailed version of this proof is given in Ruf's diploma thesis, compare Ruf (2006). Also, we would like to thank Dr. Ludwig Tomm, who helped us simplifying some of the involved identities.

Proof of Theorem 4.3.3: At first, we rewrite the integral $\int e^{-rs} g_i(s) ds$ as a convolution. More precisely, we have

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds = e^{-r\tau_{i-1}} \int_0^{\Delta\tau_i} f(x) h(\Delta\tau_i - x) dx,$$

where the functions f and h are given by

$$f(x) = \frac{e^{-rx} \Delta_0}{\sqrt{2\pi\sigma^2}} x^{-\frac{3}{2}} \exp\left(-\frac{\Delta_0^2}{2x\sigma^2}\right), \quad h(x) = \frac{x^{-\frac{1}{2}}}{\sqrt{2\pi\sigma^2}y} \exp\left(-\frac{\Delta_1^2}{2x\sigma^2}\right),$$

y is defined as in Lemma 4.2.2, $\Delta_0 := X_{\tau_{i-1}} - b$ and $\Delta_1 := X_{\tau_i} - b$. The Laplace transform of this convolution is the product of the Laplace transforms of f and h , compare Lemma 2.2.1, which can be obtained from tables of known Laplace transforms¹. Using these tables and the shift theorem, we find

$$(\mathcal{L}(f))(s) = \exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right), \quad (\mathcal{L}(h))(s) = \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{2s\sigma}y}.$$

¹ We used Oberhettinger and Badii (1973), page 41.

We now define $\alpha := (\sqrt{2}\sigma y)^{-1} = \sqrt{\pi\Delta\tau_i} \exp\left(-\frac{(\Delta X_i)^2}{2\sigma^2\Delta\tau_i}\right)$ and obtain

$$\begin{aligned} l_r(s) &= \left(\mathcal{L}\left(\int_0^t f(x)h(t-x)dx\right) \right)(s) \\ &= \frac{\alpha}{\sqrt{s}} \exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right) \cdot \exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right). \end{aligned}$$

This Laplace transform can now be interpreted as a function in r , which we develop into a Taylor series of the second order around $r_0 = 0$. We obtain

$$l_r(s) = l_0(s) + r \cdot l_0^1(s) + \frac{r^2}{2} \cdot l_0^2(s) + \frac{r^3}{6} \cdot l_{\tilde{r}_s}^3(s),$$

where $\tilde{r}_s \in (0, r)$ is depending on s and $l_r^1(s)$, $l_r^2(s)$ and $l_r^3(s)$ are given by

$$\begin{aligned} l_r^1(s) &= -\frac{\Delta_0\alpha}{\sqrt{2}\sigma} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right)}{\sqrt{s+r}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}}, \\ l_r^2(s) &= \frac{\Delta_0^2\alpha}{2\sigma^2} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right)}{s+r} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} + \\ &\quad \frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right)}{(s+r)^{\frac{3}{2}}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}}, \\ l_r^3(s) &= \sum_{j=3}^5 c_j \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r}\right)}{(s+r)^{\frac{j}{2}}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}}, \end{aligned}$$

for some constants c_3 , c_4 and c_5 . Linearity of the inverse Laplace transformation allows us to examine each summand separately. Considering the last summand, we show that the inverse Laplace transform is uniformly bounded in $\tilde{r}_s \in (0, s)$. This justifies that the error which is caused by truncating the Taylor series after the quadratic term remains of order $\mathcal{O}(r^3)$.

$$\begin{aligned} &\left| \left(\mathcal{L}^{-1} \left(\frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+\tilde{r}_s}\right)}{(s+\tilde{r}_s)^{\frac{j}{2}}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \right) \right)(t) \right| \leq \\ &\left| \int_{y-i\infty}^{y+i\infty} e^{st} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+\tilde{r}_s}\right)}{(s+\tilde{r}_s)^{\frac{j}{2}}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} ds \right| \leq \\ &e^{yt} \int_{y-i\infty}^{y+i\infty} \left| \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+\tilde{r}_s}\right)}{(s+\tilde{r}_s)^{\frac{j}{2}}} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \right| ds \leq \end{aligned}$$

$$e^{yt} \int_{y-i\infty}^{y+i\infty} \left| \frac{\exp\left(-\frac{\Delta_0}{\sigma} \sqrt{|s + \tilde{r}_s|}\right)}{(|s + \tilde{r}_s|)^{\frac{j}{2}}} \cdot \frac{\exp\left(-\frac{|\Delta_1|}{\sigma} \sqrt{|s|}\right)}{\sqrt{|s|}} \right| ds \leq$$

$$e^{yt} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{\Delta_0+|\Delta_1|}{\sigma} \sqrt{|x|}\right)}{|x|^{\frac{j+1}{2}}} dx < \infty,$$

for $j \in \{3, 4, 5\}$. We now derive the inverse Laplace transform of the second order Taylor approximation of $l_r(s)$. The Formulas (5.83), (5.89) and (5.92) of Oberhettinger and Badii (1973), page 258-259, and a lengthy calculation give

$$\begin{aligned} \left(\mathcal{L}^{-1} \left(\frac{\exp(-2a\sqrt{s})}{\sqrt{s}} \right) \right) (t) &= \frac{\exp\left(-\frac{a^2}{t}\right)}{\sqrt{\pi t}}, \\ \left(\mathcal{L}^{-1} \left(\frac{\exp(-2a\sqrt{s})}{s} \right) \right) (t) &= 2 \left(1 - \Phi \left(\frac{\sqrt{2}a}{\sqrt{t}} \right) \right), \\ \left(\mathcal{L}^{-1} \left(\frac{\exp(-2a\sqrt{s})}{s^{\frac{3}{2}}} \right) \right) (t) &= \frac{2\sqrt{t} \exp\left(-\frac{a^2}{t}\right)}{\sqrt{\pi}} - 4a \left(1 - \Phi \left(\frac{\sqrt{2}a}{\sqrt{t}} \right) \right), \\ \left(\mathcal{L}^{-1} \left(\frac{\exp(-2a\sqrt{s})}{s^2} \right) \right) (t) &= -\frac{2a\sqrt{t} \exp\left(-\frac{a^2}{t}\right)}{\sqrt{\pi}} + 2(t + 2a^2) \left(1 - \Phi \left(\frac{\sqrt{2}a}{\sqrt{t}} \right) \right). \end{aligned}$$

Another lengthy calculation involving the formulas above and $a = (\Delta_0 + |\Delta_1|)/(\sqrt{2}\sigma)$ establishes the approximation of the integral $\int e^{-rs} g_i(s) ds$. $\diamond$

6.2 The roots of a quartic polynomial

The Laplace transform of first-passage times requires the roots of $G(x) - \alpha$, compare Section 4.6, which is equivalent to finding the roots of the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

where $a = \sigma^2$, $b = 2\gamma - \sigma^2(\lambda_{\oplus} - \lambda_{\ominus})$, $c = -\sigma^2\lambda_{\oplus}\lambda_{\ominus} - 2\gamma(\lambda_{\oplus} - \lambda_{\ominus}) - 2\lambda - 2\alpha$, $d = -2\gamma\lambda_{\oplus}\lambda_{\ominus} - 2\lambda p(\lambda_{\oplus} + \lambda_{\ominus}) + 2\lambda\lambda_{\oplus} + 2\alpha(\lambda_{\oplus} - \lambda_{\ominus})$ and $e = 2\alpha\lambda_{\oplus}\lambda_{\ominus}$. Kou et al. (2005) adapt Ferrari's algorithm to the present situation. They present the following formulas

$$\beta_{1,\alpha} = -\frac{b}{4a} + \frac{p_1 - \tilde{p}_1}{2}, \quad \beta_{2,\alpha} = -\frac{b}{4a} + \frac{p_1 + \tilde{p}_1}{2},$$

$$\beta_{3,\alpha} = \frac{b}{4a} + \frac{p_1 - p_2}{2}, \quad \beta_{4,\alpha} = \frac{b}{4a} + \frac{p_1 + p_2}{2},$$

where $B_0 = c^2 - 3bd + 12ae$, $B_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace$, $B_2 = \sqrt{B_1^2 - 4B_0^3}$, $B_3 = \frac{b^2}{4a^2} - \frac{2c}{3a}$, $B_4 = \frac{b^2}{2a^2} - \frac{4c}{3a}$, $B_5 = \frac{4bc}{a^2} - \frac{8d}{a} - \frac{b^3}{a^3}$, $B_6 = \sqrt[3]{B_1 + B_2}$, $B_7 = \frac{\sqrt[3]{2}B_0}{3aB_6}$ and $B_8 = \frac{B_6}{3\sqrt[3]{2}a}$. Moreover, $p_1 = \sqrt{B_3 + B_7 + B_8}$, $p_2 = \sqrt{B_4 - B_7 - B_8 - \frac{B_5}{4p_1}}$ and $\tilde{p}_2 = \sqrt{B_4 - B_7 - B_8 + \frac{B_5}{4p_1}}$.

Let us remark that an implementation of these formulas requires a complex algebra. We implemented the algorithm using the standard C++ Class **Complex**. Figure 6.1 illustrates an example with $\gamma = 0.025$, $\sigma = 0.05$, $\lambda = 2$, $\mathbb{P}_Y = 2\text{-Exp}(30, 20, \frac{1}{2})$ and $\alpha = 2$.

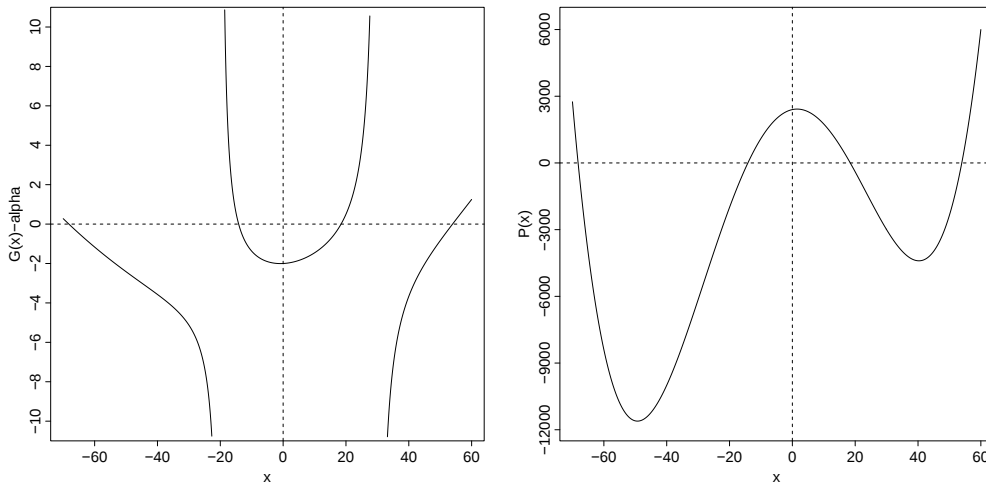


Figure 6.1: $G(x) - \alpha$ and the corresponding polynomial $P(x)$.

6.3 Zhou's bond-pricing algorithm

In Zhou (2001a), a simple Monte Carlo simulation for the pricing of corporate bonds is presented. The idea of this simulation is to discretize the interval $[0, T]$ into N equidistant bins. Then, several trajectories of the firm-value process are sampled on this grid. Default is only tested at each point of the grid, which implies biased bond prices, as the probability of a default in between two points of the grid is not considered. Another drawback of this algorithm is that it requires the simulation of a large number of random numbers², which makes this algorithm very slow. Using several numerical examples, we illustrated both shortfalls in Section 4.8. However, the algorithm is very intuitive and straightforward to implement.

Algorithm 6.3.1 (Zhou's bond-pricing algorithm)

Choose the number of simulation runs K and the number of grid points N . Specify the jump-size distribution $\mathbb{P}_Y$ and the recovery rate R , or the recovery function $w(x)$ of Section 4.5.1. Approximate the price $\phi(0, T)$ of a zero-coupon bond by

$$\phi(0, T) \approx \frac{1}{K} \sum_{n=1}^K \phi_n^N(0, T),$$

where in each simulation run $\phi_n^N(0, T)$ is calculated as follows.

1. Partition the interval $[0, T]$, i.e. define $t_i := i\frac{T}{N}$ for $i \in \{0, \dots, N\}$.
2. Sample the firm-value process on this grid, i.e. generate mutually independent random variables $x_i \sim \mathcal{N}(\gamma\frac{T}{N}, \sigma^2\frac{T}{N})$, $y_i \sim \mathbb{P}_Y$ and $\pi_i \sim B(1, \lambda\frac{T}{N})$ for $i \in \{1, \dots, N\}$.
3. Successively construct the sampled firm-value process $\hat{V}$.

$$\hat{V}_{t_0} := v_0, \quad \hat{V}_{t_i} := \hat{V}_{t_{i-1}} \exp(x_i + \pi_i y_i) \quad i \in \{1, \dots, N\}.$$

4. Find the first point of the grid, such that $\hat{V}_{t_i} \leq d$. If such an i exists, let³

$$\phi_n^N(0, T) = w(\hat{V}_{t_i}/d) \exp\left(-r \frac{t_i + t_{i-1}}{2}\right).$$

Otherwise, the company survives and $\phi_n^N(0, T)$ is set to be

$$\phi_n^N(0, T) = \exp(-rT).$$

² More precisely, with the notations of Algorithm 6.3.1, we need $3NK$ random numbers.

³ A constant recovery rate corresponds to $w(x) \equiv R$.

6.4 Vasicek's asymptotic single factor model

Vasicek's model⁴ is a multidimensional generalization of Merton's structural firm-value model. The different firm-value processes are assumed to evolve according to correlated geometric Brownian motions, default is tested at maturity only. The model for each individual firm-value process is given by

$$dV_t^i = V_t^i(\gamma_i dt + \sigma_i dW_t^i) \quad v_0 > 0.$$

We solve this stochastic differential equation using Itô's formula and rewrite the firm value of company i at maturity as

$$V_T^i = V_t^i \exp \left\{ \left(\gamma_i - \frac{\sigma_i^2}{2} \right) (T - t) + \sigma_i \sqrt{T - t} X_t^i \right\},$$

where $X_t^i := (W_T^i - W_t^i)/\sqrt{T - t}$ follows a standard normal distribution. To incorporate correlation among the companies, we partially explain X_t^i by the common market factor M_t and the idiosyncratic risk factors ϵ_t^i , i.e. we redefine X_t^i to

$$X_t^i := \rho M_t + \sqrt{1 - \rho^2} \epsilon_t^i \quad \rho \in (0, 1),$$

where $M_t, \epsilon_t^1, \dots, \epsilon_t^I$ are i.i.d. $\mathcal{N}(0, 1)$ distributed. First of all, we notice that this construction implies $\text{Corr}(X_t^i, X_t^k) = \rho^2$ for $i \neq k$. Moreover, each X_t^i is again distributed according to the standard normal law. The next observation is that the firm-value processes are independent, conditional on the market factor M_t . Consequently, conditional on M_t , all individual default probabilities are independent. We denote these conditional default probability by $p^i(M_t)$ and obtain

$$p^i(M_t) := \mathbb{P}(\tau^i < t | M_t) = \Phi \left(\frac{k_t^i - \rho M_t}{\sqrt{1 - \rho^2}} \right),$$

where

$$k_t^i := \frac{\log \left(\frac{d_t^i}{V_t^i} \right) - \left(\gamma_i - \frac{\sigma_i^2}{2} \right) (T - t)}{\sigma_i \sqrt{T - t}}.$$

The purpose of Vasicek's model is to explain the dependence among the companies, not the individual default probabilities. Therefore, we assume the term structure of individual default probabilities as given and set $\Phi^{-1}(p_t^i) := k_t^i$. The next simplification is to assume all companies to have identical default probabilities, the

⁴ A detailed description of this standard model, including all calculations which we omit, is presented in Elizalde (2005). The original reference is Vasicek (1987).

same portfolio weights and the same recovery rates. Therefore, we have

$$p(M_t) := p^i(M_t) = \Phi \left(\frac{\Phi^{-1}(p_t) - \rho M_t}{\sqrt{1 - \rho^2}} \right).$$

We now define Ω_t as the random variable that describes the fraction of defaults in the CDO portfolio up to time t . The distribution of Ω_t depends on two parameters. These parameters are the individual default probabilities p_t and the correlation ρ among the companies. In what follows, we denote the distribution function of Ω_t by

$$F_{p_t, \rho}(x) := \mathbb{P}(\Omega_t \leq x).$$

The last simplification is to assume the number of companies within the CDO portfolio to be large enough to justify the use of the strong law of large numbers to approximate $\mathbb{P}(\Omega_t \leq x)$ by $p(M_t)$. A straightforward calculation then establishes

$$F_{p_t, \rho}(x) \approx \Phi \left(\frac{\sqrt{1 - \rho^2} \Phi^{-1}(x) - \Phi^{-1}(p_t)}{\rho} \right).$$

This approximation is continuous and strictly increasing in x . As it further maps the unit interval onto itself it is a distribution function, too. The expected discounted premium and default legs are now given as⁵

$$\begin{aligned} EDPL^j &= \sum_{t \in \{\eta, 2\eta, \dots, T\}} \eta e^{-rt} s^j \mathbb{E}[u^j - l^j - L_t^j], \\ EDDL^j &= \sum_{t \in \{\eta, 2\eta, \dots, T\}} e^{-rt} \mathbb{E}[L_t^j - L_{t-1}^j]. \end{aligned}$$

The fair spread of tranche j is then found by equating the expected discounted payment legs and solving this relation for s^j . The last remaining problem is the evaluation of $\mathbb{E}[L_t^j]$. This expectation is derived by numerically evaluating the integral

$$\mathbb{E}[L_t^j] = \int_0^1 \min \{ (1 - R)xM, u^j \} - \min \{ (1 - R)xM, l^j \} dF_{p_t, \rho}(x).$$

⁵ Default payments within a period are deferred to the next payment date, premium payments are paid at the end of a period. The premium schedule is $0 < t_1 < \dots < t_n = T$, with payment frequency $\Delta t_k \equiv \eta$ years. The attachment points l^j and u^j are given in absolute values.

Chapter 7

Zusammenfassung

Das Ziel dieser Arbeit ist es, die Forschung von strukturellen Kreditrisikomodellen voranzutreiben. Dabei verlassen wir die übliche Klasse der stetigen Modelle, da diese Modelle Marktpreise von Unternehmensanleihen und Credit Default Swap (CDS) Verträgen nur unzureichend erklären, und widmen uns stattdessen Modellen, welche den Wert einer Firma als das Exponential einer Sprungdiffusion beschreiben. Zhou (2001a) formulierte als Erster solch ein Modell und beschrieb dessen Vorzüge. Allerdings gelang es ihm nicht, einen schnellen Algorithmus für die Auswertung der Preisformel von Null-Kupon Anleihen zu finden. Unser Hauptaugenmerk gilt demnach zunächst der Entwicklung eines solchen Algorithmus, wofür wir zwei Lösungen anbieten. Der zweite Teil dieser Arbeit trägt dem rasch wachsenden Markt an Derivaten auf ein Portfolio von Kreditverträgen Rechnung. Hier entwickeln wir eine elegante Verallgemeinerung des Modells auf mehrere Firmen. Diese Verallgemeinerung verändert nicht die Randverteilungen der einzelnen Firmen, was es wiederum ermöglicht, verschiedene Derivate gleichzeitig mit einem Modell zu beschreiben. Die Abhängigkeit zwischen den Firmen wird dabei durch gemeinsame Sprünge und korrelierte Brown'sche Bewegungen erzeugt. Diese Konstruktion kann darüberhinaus auch ökonomisch motiviert werden. Basierend auf diesem Modell erläutern wir, wie Preise für unterschiedliche Derivate kalkuliert werden. Im Folgenden geben wir eine detaillierte Zusammenfassung der einzelnen Kapitel dieser Arbeit.

Einleitung

Einleitend geben wir einen Abriss über die historische Entwicklung verschiedener Finanzmärkte. Speziell beschreiben wir das rasante Wachstum des Marktes für Kreditderivate und unterstreichen dessen Bedeutung durch aktuelle Handels-

volumina. Es wird auf das Ausfallrisiko einer Firma als wichtigste Kenngröße zur Bewertung von Kreditverträgen verwiesen und *Kreditrisiko* als abstrakter Begriff definiert. Zur Quantifizierung dieses Kreditrisikos existieren verschiedene Modelle, welche in *strukturelle-* und *intensitätsbasierende Modelle* unterteilt werden. Diese Arbeit fokussiert auf Modelle der ersten Klasse, daher beschränkt sich die Einleitung auf eine kurze Literaturübersicht der intensitätsbasierenden Modelle.

Strukturelle Ausfallmodelle, auch Firmenwertmodelle genannt, erklären die Insolvenz einer Firma als den ersten Zeitpunkt, an dem der Firmenwert die Verbindlichkeiten der Firma unterschreitet. Feiner klassifiziert werden diese Modelle anhand des stochastischen Prozesses, welcher den Firmenwert beschreibt. Seit Black und Scholes (1973) wurden diese Modelle mehrmals verallgemeinert, aktuell bedient man sich der Klasse der Lévy-Prozesse als stochastisches Hilfsmittel. Der Hauptkritikpunkt an den lange beliebten stetigen Modellen ist deren Eigenschaft des sich vorab abzeichnenden Ausfalls. Einerseits gab es in der Vergangenheit mehrere prominente Fälle von unerwarteten Insolvenzen, was gegen stetige Firmenwertprozesse spricht. Andererseits impliziert die Vorhersagbarkeit von Firmenausfällen sehr unrealistisch Preise für kurzlaufende Kreditverträge, da kurzfristige Risiken unterschätzt werden. Eine Lösungsmöglichkeit ist es, Sprünge im Firmenwert zuzulassen, welche dann die benötigten überraschenden Ausfälle erzeugen können. In einem größeren Absatz werden Derivate auf Portfolios von Kreditverträgen beschrieben. Die Bewertung solcher Produkte erfordert zusätzlich ein Modell der Abhängigkeitsstruktur von Firmenausfällen, wofür wir verschiedene Ansätze diskutieren.

Der letzte Absatz der Einführung grenzt unseren wissenschaftlichen Beitrag von der existierenden Literatur ab.

Technisches Rahmenwerk

In Kapitel 2 fassen wir die für diese Arbeit benötigten mathematischen Objekte zusammen und beschreiben deren wichtigste Eigenschaften. Nachdem das Wahrscheinlichkeitstheoretische Fundament aus Grundraum, σ -Algebra, Filtration und Wahrscheinlichkeitsmaß gelegt ist, werden stochastische Prozesse als Familien von Zufallsvariablen eingeführt. Speziell definieren wir dann die Klasse der Lévy-Prozesse. Das sind Prozesse, die stetig in Wahrscheinlichkeit sind, und sowohl unabhängige als auch stationäre Zuwächse besitzen. Repräsentanten aus der für uns wichtigen Unterklasse der Sprungdiffusionen können dabei höchstens endlich viele Sprünge auf einem beschränkten Zeitintervall besitzen. Lévy-Prozesse können mit Hilfe der Lévy-Itô-Zerlegung charakterisiert werden, was wiederum die Definition des Lévy-Triplets ermöglicht. Alternativ kann ein Lévy-Prozess mittels der berühmten

Lévy-Kinchin-Darstellung ihrer charakteristischen Funktion beschrieben werden. Wir zeigen weiter, dass Sprungdiffusionen stets aus einem linearen Drift, einer Brown'schen Bewegung sowie einer zufälligen Zahl an Sprüngen mit einer gewissen Sprunghöhenverteilung bestehen. Für diese Basisprozesse bestimmen wir das charakteristische Lévy-Triplet, die ersten Momente sowie eine spezielle Itô-Transformationsformel. Neben den stochastischen Hilfsmitteln benötigen wir an zwei wichtigen Stellen dieser Arbeit die Laplace-Transformation und ihre Inverse. Explizit wird in Section 4.6 die Laplace-Transformation von Firmenausfallwahrscheinlichkeiten berechnet. Deren Darstellung ist allerdings zu kompliziert, um in geschlossener Form invertiert zu werden. Daher präsentieren wir den Approximationsalgorithmus von Gaver und Stehfest, welcher später in den numerischen Implementierungen des Modells eingesetzt wird.

Finanzkontrakte und deren Eigenschaften

Nachdem das mathematische Rahmenwerk abgesteckt ist, werden nun die zu bewertenden Kreditverträge eingeführt. Dabei beginnen wir mit Unternehmensanleihen und deren vereinfachter Version ohne periodische Zinszahlungen, welche als Nullkuponanleihen bezeichnet werden. Die Zahlungsströme dieser Wertpapiere hängen im Wesentlichen von der Zahlungsfähigkeit des Emittenten und den genauen Vertragsbedingungen ab. Auf Modellebene übersetzt entspricht dies der Verteilung der Ausfallzeit und dem angenommenen Rückgewinnung-Schema im Schadensfall, für welches wir verschiedene Modellierungsansätze anbieten. Schließlich wird mit dem Zinsaufschlag von Unternehmensanleihen zu nicht ausfallbehafteten Staatsanleihen eine wichtige Kenngröße zur Beschreibung des Risikos eines Kredites definiert. Zur Absicherung gegen den Ausfall von Unternehmensanleihen gibt es *Credit Default Swap* (CDS) Verträge, die in ihrer Funktion vergleichbar mit Versicherungspolice sind. Dieses Kreditderivat wird mittlerweile in größerem Volumen als das Basisobjekt Anleihe gehandelt, weil damit unter anderem komplexe Handelsstrategien, wie das Verkaufen von Ausfallrisiko, möglich sind. Die Zahlungsflüsse eines CDS hängen wieder vom Ausfallzeitpunkt der zugrundeliegenden Firma ab.

Die Kreditderivate, die im Folgenden dargestellt werden, unterscheiden sich dadurch von Unternehmensanleihen und CDS-Verträgen, dass ihre Zahlungsströme von einem Portfolio an Einzelkrediten abhängen. Bei der Bewertung erfordert dies die zusätzliche Modellierung der Abhängigkeiten zwischen den Firmen, bzw. deren Ausfällen, da gerade in konjunkturell schlechten Zeiten Firmenausfälle gehäuft auftreten. Die von uns betrachteten Verträge werden *Collateralized Debt Obligation* (CDO) und *nth-to default* genannt. Bei einem CDO wird ein Portfolio von Krediten in kleine Tranchen unterschiedlicher Seniorität aufgeteilt, diese werden dann einzeln weiterverkauft. Fallen Kredite im Portfolio aus, so betreffen diese

Ausfälle zunächst die unterste Tranche, bis diese komplett aufgezehrt ist. Die folgenden Ausfälle werden von der nächsthöheren Tranche aufgefangen, und so weiter. Bei n^{th} -to default Verträgen wird Zins gezahlt, solange weniger als n Firmen des Portfolios ausgefallen sind. Dann muß der Versicherungsgeber den Versicherungsnehmer für den n -ten Ausfall entschädigen. Wir beschreiben die Zahlungsströme beider Derivate explizit und entwickeln in Kapitel 5 ein Modell, was sowohl die individuellen Ausfallrisiken als auch die gemeinsame Ausfallstruktur beschreibt.

Das Einfirmen-Modell

Kapitel 4 behandelt das strukturelle Ausfallmodell für einzelne Firmen. Der Firmenwert wird dabei als das Exponential einer Sprungdiffusion beschrieben, der Ausfall der Firma ist der erste Zeitpunkt an dem der Firmenwert eine vorab festgelegte Schranke passiert. Nachdem die Modellannahmen diskutiert sind, wird das Problem der Bestimmung von Ausfallwahrscheinlichkeiten in diesem Rahmenwerk angegangen. Dazu fassen wir zunächst die bekannten Resultate über die Verteilung des Minimums einer Brown'schen Bewegung bzw. einer Brown'schen Brücke zusammen und passen diese auf das Modell an. Für Sprungdiffusionen ist die Verteilung des Minimums nicht bekannt. Allerdings ist es möglich, diese in Form eines Mehrfachintegrals über bedingte Wahrscheinlichkeiten umzuschreiben, was die Konstruktion eines schnellen und unverzerrten Monte-Carlo-Schätzalgorithmus ermöglicht. Als wichtiges theoretisches Resultat berechnen wir die lokale Ausfallrate einer Sprungdiffusion, das ist quasi die Ableitung der Ausfallwahrscheinlichkeit zur Zeit Null. Diese lokale Ausfallrate ist im Wesentlichen das Produkt aus Sprungintensität und der Wahrscheinlichkeit, durch einen Sprung einen Firmenausfall zu erzeugen.

Preise von Anleihen und CDS Verträgen

Um nun den Preis einer Firmenanleihe zu bestimmen, genügt es, die Verteilung des Ausfallzeitpunktes zu kennen. Im stetigen Fall reduziert sich daher die Berechnung von Preisen für Firmenanleihen auf die Auswertung eines Riemann-Stieltjes Integrals. Im allgemeinen Fall sind wir wieder auf statistische Schätzverfahren angewiesen. Unser Vorschlag ist es, den Wert einer Firmenanleihe wiederum künstlich als ein Mehrfachintegral über einen bedingten Erwartungswert zu schreiben. Dabei integrieren wir über alle Möglichkeiten für die Anzahl an Sprüngen, deren Lage sowie den Firmenwert zu diesen Sprungzeiten. Sind diese Größen fixiert, so kann die erwartete Auszahlung der Anleihe explizit bestimmt werden. Diese Darstellung ist die Basis für einen schnellen Monte-Carlo-Algorithmus. In jedem Schritt des Algorithmus simulieren wir so wenig wie nötig - das sind die Anzahl und Lage der

Sprünge sowie der Firmenwert zu den Sprungzeiten - und berechnen so viel wie möglich, das ist die erwartete Auszahlung gegeben den simulierten Größen.

Dieser sehr schnelle Algorithmus kann noch zusätzlich beschleunigt werden, indem man ein in der bedingten Auszahlung auftretendes Integral approximiert. Diese Approximation wurde ursprünglich von Metwally und Atiya (2002) im Zusammenhang mit der Bewertung von Barrier-Optionen vorgeschlagen. Es gelang uns, diese Approximation weiter zu verbessern. Als weiteres theoretisches Resultat ermitteln wir den Grenzwert des Zinsaufschlags von Unternehmensanleihen für kurzlaufende Maturitäten. Dieser Grenzwert ist positiv und setzt sich faktoriell aus der zuvor bestimmten lokalen Ausfallrate und dem erwarteten Verlust beim Ausfall zusammen. Der dann folgende Abschnitt behandelt die Auswertung der CDS Preisformel, was in weiten Zügen analog zur Bewertung von Unternehmensanleihen erfolgt. Auch hier geben wir einen Simulationsalgorithmus an und berechnen den Grenzwert von CDS Beiträgen für kurzlaufende Maturitäten.

Verallgemeinerungen des Modells

Nun wenden wir uns verschiedenen Verallgemeinerungen des Modells zu und beschreiben, wie diese in unseren Simulationsalgorithmen berücksichtigt werden können. Die erste Erweiterung des Modells betrifft die Rückholrate beim Firmenausfall, welche bisher als konstant angenommen wurde. Da strukturelle Modelle aber gerade den Firmenwert modellieren, liegt es nahe, dessen Wert am Ausfallzeitpunkt als Determinante der Rückholrate einzubinden. Diese Modellannahme bringt bei stetigen Modellen mit konstanter Ausfallrate keine neue Erkenntnis, da dort der Firmenwert beim Ausfall notwendigerweise mit der Ausfallrate übereinstimmt. Werden hingegen Sprünge im Firmenwert zugelassen, so ist dieser Wert zum Ausfallzeitpunkt zufällig, da unklar ist, wie tief der Firmenwert bei einem plötzlichen Ausfall unter die Ausfallschranke springt. Da in unserem Algorithmus der Firmenwert aber gerade an den Sprungzeiten simuliert wird und bei einem Ausfall durch Diffusion mit der Ausfallschranke übereinstimmt, kann diese Verallgemeinerung elegant in unseren Algorithmus aufgenommen werden.

Bisher benutzten wir eine konstante Zinsrate, um zukünftige Zahlungsströme abzuzinsen. Im Folgenden wird diese Zinsrate durch einen stochastischen Prozess ersetzt, für welchen wir exemplarisch die Standardmodelle von Vasicek sowie von Cox, Ingersoll und Ross betrachten. Unter der Annahme der stochastischen Unabhängigkeit von Firmenwert und Zinsprozess zeigen wir, wie diese Erweiterung in unseren Monte-Carlo-Algorithmus eingebunden werden kann.

Die nächste Verallgemeinerung beruht auf der Tatsache, dass die Differenz zwei-

er Sprungdiffusionen wiederum eine Sprungdiffusion ist. Wir benutzen dieses unscheinbare Resultat, um die bisher konstante Ausfallrate durch das Exponential einer Sprungdiffusion zu ersetzen, was auch den populären Spezialfall einer exponentiell ansteigenden Ausfallrate beinhaltet.

Abschließend diskutieren wir die der Bewertung von Anleihen und CDS zugrundeliegende Basis an Informationen. Bisher erlaubten wir es allen Investoren, den Firmenwertprozess stetig zu beobachten, was in der Praxis selbst für Firmeninsider unrealistisch ist. Plausibler hingegen ist die Modellannahme, dass der Firmenwert in periodischen Abständen veröffentlicht wird, z.B. in Form einer Bilanz. Zusätzlich sind die Investoren zu jedem Zeitpunkt über den Ausfallstatus der Firma informiert. Für ein stetiges Modell können wir wieder geschlossene Preisformeln, basierend auf den jeweiligen Filtrationen, anbieten. In unstetigen Modellen muss auf die bekannte Simulationstechnik zurückgegriffen werden. Wir illustrieren den Einfluss dieser Filtrationsverkleinerung anhand eines numerischen Beispiels, welches sich sehr anschaulich interpretieren lässt.

Der Laplace Ansatz

Bis zu diesem Abschnitt vertrauten wir auf stochastische Simulationen, um Preise für Anleihen und CDS zu ermitteln. Dieser Ansatz ist schnell und präzise genug, wenn Preise zu gegebenen Modellparametern bestimmt werden sollen. Ist das Ziel hingegen die Kalibrierung des Modells, so erfordert dies einerseits das wiederholte Auswerten der Preisformel, andererseits arbeiten die üblichen Minimierungsalgorithmen mit numerisch approximierten Gradienten, was bei einer Monte-Carlo-Formel problematisch ist. Wählt man nun als Sprungverteilung zweiseitig exponentiell verteilte Sprünge, so lässt sich die Laplace-Transformation von Ausfall und Überlebenswahrscheinlichkeiten einer Sprungdiffusion bestimmen. Mittels des Laplace-Inversionsalgorithmus von Gaver und Stehfest lassen sich diese Überlebenswahrscheinlichkeiten mit hoher Präzision zurückgewinnen. Auf Basis dieser theoretischen Resultate konstruieren wir einen sehr schnellen Approximationsalgorithmus für Anleihenpreise und CDS-Verträge.

In Abschnitt 4.7 untersuchen wir die Reaktion der Preisstrukturkurven von Anleihen und CDS Verträgen auf Veränderungen in den Modellparametern. Dies führen wir unter der Annahme von zweiseitig exponentiell verteilten Sprüngen durch, was es uns ermöglicht, den Approximationsalgorithmus aus dem vorigen Abschnitt einzusetzen. Zunächst ändern wir den Drift bzw. die Volatilität der Diffusionskomponente. Dies führt zu dem erwarteten Ergebnis, dass Zinsaufschläge und CDS Prämien fallend im Drift und steigend in der Volatilität sind. Allerdings - und dies ist im Einklang mit unseren theoretischen Ergebnissen - ist der Grenzwert

für kurzlaufende Verträge unabhängig von den Parametern der Diffusionskomponente. Anschließend untersuchen wir den Einfluss der Sprungintensität, dem relativen Anteil an der Gesamtvarianz welcher durch Sprünge induziert wird sowie dem Verschuldungsgrad der Firma. Es zeigt sich, dass das Modell wie erwartet auf Veränderungen in den Parametern reagiert, was eine anschauliche Interpretation der Parameter zulässt. Eine weitere Beobachtung ist die große Bandbreite an Strukturkurven von Anleihen und CDS Preisen, die das Modell nachzubilden vermag.

Neben Zhou's Algorithmus stehen uns mit unserem Simulationsalgorithmus, dessen Integralapproximation sowie dem Laplace-Algorithmus mehrere Möglichkeiten zur Verfügung, um Unternehmensanleihen zu bewerten. Um die spezifischen Vor- und Nachteile der Algorithmen zu illustrieren, konstruieren wir vier Szenarien in denen wir Anleihenpreise berechnen. Es stellt sich heraus, dass Zhou's Algorithmus bei weitem der langsamste ist und selbst für ein sehr feines Grid verzerrte Preise ausgibt. Alle anderen Algorithmen zeichnen sich durch eine ähnlich hohe Präzision aus, der Laplace-Algorithmus ist zusätzlich in der Lage, Preise in Bruchteilen von Sekunden zu berechnen. In einem weiteren Abschnitt untersuchen wir unsere Integralapproximation mit der von Metwally und Atiya (2002). Basierend auf verschiedenen Szenarien stellt sich auch hier unser Ansatz als der deutlich genauere heraus.

Im abschließenden Abschnitt über das Einfirmen-Modell stellen wir uns der Herausforderung, das Modell zu kalibrieren. Gelingt uns dies, so ist einerseits der Nachweis erbracht, dass das Modell in der Lage ist, realistische Preisstrukturkurven zu erzeugen. Andererseits unterstreicht dies die praktische Relevanz von unserem Laplace-Algorithmus, da solch eine Kalibrierung bisher nicht möglich war. Der Firmenwertprozess selbst ist nicht beobachtbar, da nicht am Markt gehandelt. Daher wählen wir den Ansatz einer risikoneutralen Kalibrierung der Modellparameter. Dies bedeutet, dass gegeben den beobachteten Anleihen oder CDS einer Firma, die Parameter des Modells so gewählt werden, dass ein Fehlerfunktional von Marktpreisen zu Modellpreisen minimiert wird. Wie schon angeschnitten erfordert diese Minimierung über den Parameterraum das mehrmalige präzise Auswerten der jeweiligen Preisformel. Zunächst kalibrieren wir die jeweiligen Firmenwertprozesse von *DaimlerChrysler* und *General Motors* an deren Anleihen. Dann kalibrieren wir jede der 125 Firmen des *iTraxx* Index an deren CDS Quotierungen verschiedener Laufzeiten. Zum Vergleich führen wir die identische Kalibrierung in einem Modell ohne Sprünge durch. Es zeigt sich, dass das Sprungdiffusions-Modell in der Lage ist, die Strukturkurven von Anleihen und CDS unterschiedlichster Firmen nachzubilden. Die Präzision dieser Kalibrierung ist um Längen besser als in einem vergleichbaren stetigen Modell.

Das Mehrfirmen-Modell

In den letzten Jahren stieg die Marktkapitalisierung und damit auch das Interesse an Derivaten auf ein Portfolio von Kreditprodukten sprunghaft an. Um solche Derivate bewerten zu können, benötigen wir ein Modell, welches die Abhängigkeiten zwischen den Firmen beschreibt. Unsere Ansprüche an solch ein Modell sind sehr hoch. Einerseits soll die Abhängigkeit zwischen den Firmen durch einen gemeinsamen Marktfaktor beschrieben werden, was sich als Konjunkturzyklus interpretieren lässt. Andererseits soll das Modell kurze Perioden mit unerwartet vielen Ausfällen erzeugen können, sogenannte Ausfallcluster. Diese Ausfallcluster interpretieren wir als die Reaktion des Marktes auf unerwartete negative Ereignisse, z.B. den Ausfall eines großen Konzerns, welcher die Insolvenz vieler Zulieferer nach sich zieht. Darüber hinaus streben wir es im Gegensatz zu puren Abhängigkeitsmodellen an, innerhalb des Modells auch weiterhin realistische Preise für Derivate auf alle Einzelfirmen erzeugen zu können, was sehr ambitioniert ist. Da unser Einfirmen-Modell allen Ansprüchen auf firmenindividueller Ebene genügt, ist unser Ansatz dessen Verallgemeinerung auf mehrere Firmen. Das bedeutet, dass die einzelnen Firmenwertprozesse mittels neu eingeführter Variablen abhängig gemacht werden, ohne die individuellen Randverteilungen zu verändern. Dies wiederum sichert die Konsistenz von Derivatpreisen auf Kredite einzelner Firmen. Im Folgenden stellen wir drei Möglichkeiten vor, wie diese mehrdimensionale Verallgemeinerung erreicht werden kann. Darüber hinaus erläutern wir die jeweilige ökonomische Interpretation dieser Erweiterungen.

Die drei Ansätze für ein Portfolio Modell

Im ersten Ansatz modellieren wir die Abhängigkeit zwischen den Firmen durch einen gemeinsamen Marktfaktor und mittels gemeinsamer Sprünge. Dazu wird die Brown'sche Bewegung eines jeden Firmenwertprozesses in einen gemeinsamen Marktfaktor und einen individuellen Faktor aufgespalten. Der relative Anteil des Marktfaktors bestimmt dann den Grad der systematischen Abhängigkeit zwischen den Firmen. Zusätzlich definieren wir einen Zählprozess, welcher das Eintreffen wichtiger Informationen ankündigt. Jede Firma reagiert mit individueller Wahrscheinlichkeit mit einem Sprung in ihrem Firmenwertprozess auf das Eintreffen einer neuen Nachricht. Mathematisch entspricht diese Konstruktion einem ausgedünnten Poisson-Prozess. Mit steigender Wahrscheinlichkeit, auf neue Informationen mit einem Sprung im Firmenwert zu reagieren, wächst auch die Wahrscheinlichkeit für gemeinsame Sprünge und damit für simultane Firmenausfälle. Die durch gemeinsame Sprünge induzierte Abhängigkeit kann noch deutlich gesteigert werden, wenn Sprünge zu identischen Zeitpunkten das gleiche Vorzeichen besitzen. In dieser zweiten Variante des Modells wird jede Information zunächst als

gut oder schlecht für die Gesamtwirtschaft klassifiziert. Abhängig von dieser Einteilung werden dann an dem zugehörigen Zeitpunkt ausschließlich Aufwärtssprünge beziehungsweise Abwärtssprünge zugelassen. In der dritten Variante des Modells ordnen wir zunächst jeder Firma einen Sektor zu, etwa Automobil, Konsum, Telekommunikation, usw. Neben dem globalen Marktfaktor definieren wir einen Faktor für jeden Sektor und jede Firma. Die Brown'schen Bewegungen der individuellen Firmenwertprozesse werden dann in den Marktfaktor, den jeweiligen Faktor des Sektors sowie den firmenindividuellen Faktor aufgespalten. Analog gehen wir bei der Definition von Sprüngen vor. Wir betrachten einen Nachrichtenticker für den Gesamtmarkt, für jeden Sektor sowie für jede Firma. Die Superposition dieser Tickerprozesse bestimmt dann das Sprungverhalten der jeweiligen Firma.

Die Abhängigkeit zwischen den Firmen

Im nun folgenden Abschnitt untersuchen wir die durch das Modell implizierte Abhängigkeitsstruktur genauer. Dazu definieren wir zunächst den Begriff der Ausfallkorrelation bis zu einem Zeitpunkt, welche wir dann in verschiedenen Szenarien mittels einer Monte-Carlo-Simulation bestimmen. Es zeigt sich, dass gemeinsame Sprünge schon innerhalb von kurzen Zeiträumen eine relevante Ausfallkorrelation erzeugen, wohingegen ein stetiger Marktfaktor ein viel längeres Zeitfenster benötigt, um einen meßbaren Einfluß auf die Ausfallkorrelation aufzubauen. Im Anschluß daran leiten wir die Korrelation zweier Firmenwertprozesse zu einem gegebenen Zeitpunkt her. Im stetigen Fall ist dies gerade das Produkt der Anteile des Marktfaktors an den Firmenwertprozessen der betrachteten Firmen. Diese Formel kann elegant um den Einfluß gemeinsamer Sprünge erweitert werden. In einer aufwändigen Rechnung gelingt es uns sogar, das zugehörige Ergebnis bei Sprüngen in die gleiche Richtung herzuleiten. Wie erwartet ist die Korrelation in dieser Modellvariante höher als bei unabhängigen Sprüngen.

Der Algorithmus zur Bewertung von CDO-Verträgen

Im Mittelpunkt dieses Abschnitts steht die Herleitung eines Monte-Carlo-Algorithmus zur Bewertung der unterschiedlichen Tranchen eines CDO-Vertrags. Dieser Algorithmus bedient sich mehrerer schon im Einfirmen-Modell hergeleiteter Hilfsmittel. Wieder simulieren wir in jedem Schritt zunächst die Anzahl und Lage der Sprünge des Tickerprozesses. Anschließend simulieren wir den gemeinsamen Marktfaktor sowie die einzelnen Firmenwertprozesse zu diesen Zeitpunkten. Nun zeigt sich auch der Vorteil des Modells bei der Implementierung. Dadurch, dass ein gemeinsamer Zählprozess mögliche Sprünge in den einzelnen Firmenwerten vorgibt, ist sichergestellt, dass alle Firmenwertprozesse zwischen diesen Sprungzeiten

stetig verlaufen. Mittels Brown'scher Brücken können wir daher wieder deren Ausfallwahrscheinlichkeit, bedingt an die simulierten Größen, bestimmen. Diese Ausfallwahrscheinlichkeiten werden aggregiert, um damit die erwarteten Verluste im Portfolio zu jedem Zeitpunkt bestimmen zu können. Auf Basis dieser Portfolioverluste werden dann Prämien und Ausfallkompensationszahlungen aller Tranchen bestimmt. Um den numerischen Aufwand erträglich zu halten, werden Hilfsmittel wie die Integralapproximation eingesetzt. Dies ist unumgänglich, da der Berechnungsaufwand linear in der Anzahl der Firmen steigt und ein typisches *iTraxx* Portfolio 125 Firmen beinhaltet.

Im Anschluß an die formale Darstellung des Algorithmus gehen wir detailliert auf mögliche Schwierigkeiten bei dessen Implementierung ein. Wir ermitteln heuristisch, wie viele Monte-Carlo-Schritte benötigt werden, um verlässliche Ergebnisse erzielen zu können. Um die Mechanismen des Modells besser zu verstehen, konstruieren wir verschiedene Szenarien für die Kreditwürdigkeit sowie die Abhängigkeitsstruktur der Firmen innerhalb eines Portfolios. Für diese Szenarien berechnen wir dann Preise der unterschiedlichen Tranchen eines CDOs. Es zeigt sich, dass die vom Modell ermittelten Preise wie erwartet auf Veränderungen in der Abhängigkeitsstruktur reagieren.

Die Kalibrierung des mehrdimensionalen Modells

Eine Kalibrierung des mehrdimensionalen Modells ist ambitioniert, da das Modell sehr viele Parameter enthält. Zwei Tatsachen machen diese Kalibrierung letztendlich doch möglich. Zum einen kann das Modell die verschiedensten Wertpapiere beschreiben, was es ermöglicht eine Vielzahl an Marktinformationen in die Kalibrierung des Modells einfließen zu lassen. Zum anderen können die Parameter der Abhängigkeitsstruktur unabhängig von den individuellen Parametern, welche die firmenspezifischen Ausfallwahrscheinlichkeiten adjustieren, geschätzt werden. Wir kombinieren diese Beobachtungen zu folgendem zweistufigen Verfahren. Im ersten Schritt ermitteln wir die Parameter der einzelnen Firmenwertprozesse aus CDS-Quotierungen unterschiedlicher Laufzeit. Für den zweiten Schritt fassen wir zunächst die Firmen zu einem Portfolio zusammen. Dann verändern wir die Parameter der Abhängigkeitsstruktur, bis das Modell CDO-Preise möglichst optimal nachbildet. Diese Kalibrierung führen wir exemplarisch an 16 Tagen durch. Es zeigt sich, dass das Modell in der Lage ist, gleichzeitig die Preise von individuellen CDS, Portfolio-CDS sowie den verschiedenen Tranchen eines CDOs mit hoher Genauigkeit zu beschreiben.

Die Bewertung von n^{th} -to default Verträgen

Der letzte Abschnitt über das mehrdimensionale Modell illustriert, wie einfach unser Algorithmus auf die Bewertung anderer Portfolioderivate adaptierbar ist. Speziell beschreiben wir die Bewertung eines n^{th} -to default Vertrags. Auch für dieses Kreditderivat konstruieren wir verschiedene Beispiele, die verdeutlichen, wie in unserem Modell Preise zustande kommen.

Anhang

Im Anhang wird zunächst der sehr lange und technische Beweis der Approximation des Integrals über die abdiskontierte Ausfalldichte einer Brown'schen Brücke vorgestellt. Danach präsentieren wir Ferrari's Algorithmus zur Auflösung einer bi-quadratischen Gleichung, angepasst auf das Problem der Bestimmung der Laplace-Transformation von Ausfallwahrscheinlichkeiten. In einem weiteren Abschnitt wird der oft zitierte Algorithmus von Zhou (2001a) beschrieben. Schließlich erläutern wir Vasicek's Portfolio Modell, welches bis heute die Referenz bei der Bestimmung von impliziten Korrelationen ist.

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