Terminal wealth problems in illiquid markets under a drawdown constraint

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Contents

1. Introduction

1.1. Motivation

In classical portfolio allocation problems an investor may invest his wealth in a financial market, which usually consists of a finite number of risky assets and a riskless asset. Thereby he wants to find the “best“ investment policy. In order to determine such a policy, he first of all needs to choose a performance criteria. A very popular one is, for example, the maximization of the expected utility of terminal wealth.
Originally, this problem was studied by Robert C. Merton in Merton (1969). In the framework of a Black-Scholes market and a CRRA utility function, he found out that the optimal investment policy can be described by a fixed proportion of wealth, which should be put in the risky asset. Furthermore, he showed that this proportion admits an explicit expression. However, while applying this policy the investor has to adjust his portfolio continuously, since his wealth is changing at all times.
Moreover, in Merton (1969) it is assumed that the market is perfectly liquid, i.e. the investor's transactions do not influence the asset price and they are executed immediately. Obviously, most markets are not perfectly liquid, since at least one of these assumptions fails. Hence, it is more realistic to take the liquidity risk into account.
In the literature there are many ways to model liquidity risk. An overview of the most common approaches is given in the following:

- *Transaction costs:* In Davis & Norman (1990) the liquidity risk is measured by means of transaction costs. In principal, the investor can trade whenever he wants, but high frequency trading is impossible due to large transaction costs.

- *Price impacts due to transactions:* Subramamian & Jarrow (2001) and Cetin et al. (2004) use an approach to model liquidity risk by price impacts due to transactions. Such a kind of liquidity risk can for for example be observed when considering large traders.

- *Restriction of trading times:* In Rogers (2001), Rogers & Zane (2002), Matsumoto (2003, 2006, 2007, 2009), Pham & Tankov (2008, 2009) and Gassiat et al. (2011) the liquidity risk is modeled by assuming that trading is only possible at some exogenous random times, which are given by the jump times of a Poisson process. Such a situation occurs for example in over-the-counter markets. In those markets transactions may not be executed immediately, due to the lack of a counter party. Thus the investor has to wait until his transaction takes place.

In the current work, we will use the last approach to take the liquidity risk into account. During the last years, one could observe a rather volatile behavior in the financial markets. When the housing bubble busted, stock exchanges rode roller coasters and investors had big problems to control their risks. In such uncertain times, it is reasonable to save parts of made gains immediately, since otherwise they may be gone a moment later. Therefore, we introduce in our model a drawdown constraint, which restricts the policies in such a way, that the investor's terminal wealth is always greater than or equal to a certain percentage of his maximal observed wealth. Hence, if he makes money, the wealth process rises and he saves parts thereof. Portfolio optimization problems with such a drawdown constraint are, among others, discussed in Grossman & Zhou (1993), Cvitanic & Karatzas (1995), Elie & Touzi (2008) and Elie (2008).

1.2. Overview and contributions of this thesis

In the sequel we present a short survey of literature concerned with portfolio optimization problems in illiquid markets, where the liquidity risk is modeled by exogenous random times, at which trading is possible:

- In the work of Rogers & Zane (2002), they consider a Black-Scholes market and an investor, who can invest his money in a riskless bank account and a stock. Thereby he can only transfer his money between the assets at the jump times of a homogeneous Poisson process. The investor wants to maximize the expected utility of consumption over an infinite horizon. Furthermore, it is assumed that the investor has a CRRA utility function.

- Pham & Tankov (2008, 2009) extend the investment consumption problem of Rogers & Zane (2002) to a more general market and a larger class of utility functions.

- Matsumoto (2003, 2006) considers terminal wealth problems with a finite horizon and a CRRA utility function in a market, which coincides with the one in Rogers & Zane (2002). This approach is extended to a more general market and a larger class of utility functions in the present work.

Table 1.1 on the next page gives an overview over the above mentioned works and shows, how this work contributes to that research area.

In the present work we consider terminal wealth problems in an illiquid jump market with a general utility function. In contrast to the above mentioned works, we assume that the random times, at which trading is possible, are given by the jump times of an inhomogeneous Poisson process. This approach has the advantage that we are able to model time periods with different levels of liquidity risk.
Moreover, we include in our model a drawdown constraint, which guarantees that the wealth process does not fall under a fixed percentage of the investor's maximal observed wealth. This enables the investor to save parts of the gains, which have been made

	Investment / Consumption problem	Terminal wealth problem
Black-Scholes market and CRRA utility function	Rogers & Zane (2002)	Matsumoto (2003, 2006)
Jump markets and general utility functions	Pham & Tankov (2008, 2009)	solved in the present work as a special case

Table 1.1.: Overview of existing research without a drawdown constraint

during the investment period. That additional feature makes the objective of our portfolio optimization problem more interesting for risk sensitive investors. If we set the fixed percentage equal to zero, then the drawdown constraint vanishes and we are dealing with classical terminal wealth problems, which are mentioned as a special case in Table 1.1.

By following the ideas of Bäuerle & Rieder (2009), we show that the terminal wealth problem with the mentioned drawdown constraint can be reduced to a contracting Markov Decision Process. The benefit of that reduction is given in the opportunity to apply the general results of MDP-Theory to that problem. Because of that consideration, we are able to show that there exists an optimal portfolio and that the value function can be characterized as the unique fixed point of the maximal reward operator. Moreover, Howard's policy improvement algorithm is valid and can be used to compute an optimal policy.

The work of Gassiat et al. (2011) plays a special role, since they are considering a non-standard objective. More precisely, they assume an inhomogeneous Poisson process with an increasing intensity process such that the jump times (τ_n) of the process converge increasingly to the finite horizon. Then they maximize the following objective

$$\mathbb{E}\Big[U\big(\lim_{n\to\infty} X_{\tau_n}\big)\Big] ,$$

where U denotes the investor's utility function and X_{τ_n} his wealth at time τ_n. We will extend their model by introducing the same drawdown constraint as above and reduce the problem to a limsup Markov Decision Process. Then, we will solve that limsup MDP by applying a Structure Theorem, which we develop for those limsup MDPs. It turns out that the solution of the limsup MDP is close to the solution of the contracting MDP from above. Furthermore, under mild assumptions we can show, that one can approximate the value function of the limsup MDP by the value function of a contracting MDP. Since such an approximation can also be shown for the optimal policy, the limsup MDP can be considered as a limit case of the contracting MDP.

1.3. Outline of this thesis

The remaining parts of this work are organized as follows: In Chapter 2 we introduce inhomogeneous Lévy processes, which will be used to model the returns of the assets prices. We show, that they are additive processes and semimartingales. Further, we introduce an assumption under which an inhomogeneous Lévy processes is a special semimartingale and develop its canonical representation. This assumption also implies a finite exponential moment, which will be used frequently in the subsequent chapters. Finally, we study the stochastic exponential of an inhomogeneous Lévy processes and proof some auxiliary results. At the end we have a closer look at a special case, the inhomogeneous Poisson processes, and derive some of their properties.

Chapter 3 is concerned with the introduction of the terminal wealth problem. We setup an illiquid financial market, in which we have a risky asset driven by an inhomogeneous Lévy process, a riskless asset and exogenous random times, at which trading is possible. The random times are given by the jump times of an inhomogeneous Poisson process. Then an investor is introduced and we determine his wealth process and establish the drawdown constraint as well as the class of admissible policies. At the end, we formulate the investor's terminal wealth problem and discuss two major cases of the underlying inhomogeneous Poisson process.

In Chapter 4 we solve the investor's terminal wealth problem under the assumption of a bounded intensity process of the inhomogeneous Poisson process. Thereby we will show, that the terminal wealth problem can be reduced to discrete-time optimization problem - a contracting Markov Decision Process - by means of which we can compute the value function and the optimal policy. In the main results (Theorem 4.10) we proof that the value function can be characterized by the unique fixed point of the maximal reward operator and that there exists an optimal stationary policy. Since the Markov Decision Process is contracting Howard's policy improvement algorithm (Theorem 4.11) is valid and can be used to approximate an optimal policy. Last but not least, we derive a separation ansatz for the value function under the assumption of a CRRA utility function.

In Chapter 5 we deal with a terminal wealth problem under the assumption of an unbounded intensity process of the inhomogeneous Poisson process. Due to that assumption the exogenous random times converge increasingly to the investor's finite horizon and hence the investor can not observe his wealth at the horizon. However, since the left sided limit of the wealth process exists at the end of the investment period and is observable, we may consider that limit as the investor's terminal wealth. As in Chapter 4, the considered optimization problem can be reduced to a discrete-time problem, a so called limsup Markov Decision Process, by means of which we can compute the value function and the optimal policy. The main results (Theorem 5.12) cover the following: The value function can be characterized by the unique fixed point of the maximal reward operator, which satisfies some additional conditions, and there exists an optimal stationary policy. Moreover, there is also a separation ansatz for the value function under a CRRA utility function.

In Chapter 6 we show under a mild assumption that we may approximate a terminal wealth problem with an unbounded intensity process by a terminal wealth problem with a bounded intensity process. This approximation includes convergence of the value function as well as convergence of the optimal policy.

Finally, we illustrate the results of the previous chapters in Chapter 7. Therefore we assume a classical Black-Scholes market and a *Power Utility* function. Then we make use of the separation ansatz and solve the terminal wealth problem for different intensities and different levels of the drawdown constraint. Moreover, these examples suggest to use a generalized Merton policy, which simply neglects the intensity process. This policy approximates the optimal policies very well, hence we are able to make a recommendation for practitioners.

2. Inhomogeneous Lévy processes

In this chapter we introduce inhomogeneous Lévy processes by following (Kluge, 2005, §1.3). For the sake of completeness, we will also give some of the short proofs. In the following chapters these inhomogeneous Lévy processes will be used to model the returns of the assets prices.

Inhomogeneous Lévy processes are a subclass of additive processes, which include all Lévy processes. This subclass enlarges the class of Lévy processes in such a way that the condition of stationary increments is no longer required. This enlargement has the advantage that the deterministic characteristics of such processes, which describe the local behavior, may now depend on time. Among others, this enables us to model a seasonal asset price dynamic. Such a seasonal dynamic can for example be observed in energy markets. For more details on energy markets and an extensive treatment of price dynamics in those markets we refer to (Benth et al., 2008, Section 1.5).
Further, inhomogeneous Lévy processes have a absolutely continuous characteristics, which implies that they are semimartingales and hence the powerful tool of stochastic integration can be used. Finally, the generalization of Lévy processes does not come at a high price, since inhomogeneous Lévy processes are still good to handle.
We also introduce inhomogeneous Poisson processes. These processes arise as special cases of the inhomogeneous Lévy processes, if the corresponding intensity process is bounded.

This chapter is organized as follows: In the first section we introduce inhomogeneous Lévy processes and derive some of their properties. In Section 2.2 we formulate an assumption, under which an inhomogeneous Lévy process has a finite exponential moment and is a special semimartingale. Moreover, we derive the canonical representation of an inhomogeneous Lévy process under that assumption. That the stochastic exponential of an inhomogeneous Lévy process is a Markov process is proven in Section 2.3. The next section presents some auxiliary results, which are needed in the following chapters. In the last section we introduce inhomogeneous Poisson processes and derive some of their properties.

2.1. Definition and properties

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. It is assumed that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions. In the following, all stochastic processes are defined on that complete stochastic basis and they are assumed to be $\mathbb{R}$-valued.

Definition 2.1
An $\mathbb{F}$-adapted càdlàg process (L_t) is an inhomogeneous Lévy process, if the following holds:

1. (L_t) has independent increments, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t < \infty$.
2. For every $t \in [0, \infty)$, the law of L_t is characterized by the characteristic function

$$\mathbb{E}\big[e^{iuL_t}\big] = \exp \int_0^t \left[iub_s - \frac{1}{2}u^2 c_s + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})F_s(dx) \right] ds \ , \tag{2.1}$$

where $b : [0, \infty) \to \mathbb{R}$ and $c : [0, \infty) \to [0, \infty)$ are measurable functions and $(F_s)_{s\geq 0}$ is a kernel, such that F_s is a Lévy measure for each s. It is further assumed that

$$|b_s| + |c_s| + \left| \int_{\mathbb{R}} (x^2 \wedge 1)F_s(dx) \right| \leq C_L^1 \ , \qquad \forall s \geq 0 \ ,$$

for some positive constant C_L^1.

Furthermore, we call $(b, c, F) := (b_s, c_s, F_s)_{s\geq 0}$ the characteristics of the inhomogeneous Lévy process (L_t).

Since Lévy processes are infinitely divisible, we can fall back to them to show that inhomogeneous Lévy processes are also infinitely divisible. Later, this property of the inhomogeneous Lévy processes enables us to use the well-known results of infinitely divisible distributions when handling such processes.

Proposition 2.2
Let (L_t) be an inhomogeneous Lévy process. Then the distribution of L_t is infinitely divisible with Lévy-Khintchine triplet (b, c, F), where

$$b := \int_0^t b_s ds \ , \quad c := \int_0^t c_s ds \ , \quad F(dx) := \int_0^t F_s(dx)ds \ .$$

Proof:

Fix $t \in [0, \infty)$. By applying monotone convergence, we can show the σ-additivity of

$$F(dx) := \int_0^t F_s(dx)ds$$

and conclude that F is a positive Borel measure on $\mathbb{R}$. Moreover, we have

$$F(A) = \int_0^t \int_{\mathbb{R}} \mathbb{1}_A F_s(dx)ds \ , \qquad \forall A \in \mathcal{B}(\mathbb{R}) \ ,$$

and so

$$\int_{\mathbb{R}} (x^2 \wedge 1) F(dx) = \int_0^t \int_{\mathbb{R}} (x^2 \wedge 1) F_s(dx) ds \leq C_L^1 \cdot t < \infty ,$$

as well as $F(\{0\}) = 0$. Then the claim follows by (2.1) and (Applebaum, 2009, Theorem 1.2.14). □

Theorem 2.3
An inhomogeneous Lévy process (L_t) is an additive process.

Proof:

First, we check the properties of Definition B.9 to show that we have an additive process in law:

Property 1: (L_t) has independent increments by definition.

Property 2: This property follows with the characteristic function of L_0.

Property 3: Let $0 \leq s \leq t < \infty$. Using the independent increment property of (L_t) yields

$$\begin{aligned} \mathbb{E}\big[e^{iuL_t}\big] &= \mathbb{E}\big[e^{iu(L_t - L_s + L_s)}\big] \\ &= \mathbb{E}\big[\cos(u(L_t - L_s + L_s))\big] + i\mathbb{E}\big[\sin(u(L_t - L_s + L_s))\big] \\ &= \mathbb{E}\big[e^{iu(L_t - L_s)}\big]\mathbb{E}\big[e^{iuL_s}\big] . \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathbb{E}\big[e^{iu(L_t - L_s)}\big] &= \frac{\mathbb{E}\big[e^{iuL_t}\big]}{\mathbb{E}\big[e^{iuL_s}\big]} \\ &= \exp \int_s^t \Big(iub_v - \frac{1}{2}c_v u^2 + \int_{\mathbb{R}} \big(e^{iux} - 1 - iux\mathbb{1}_{\{|x| \leq 1\}}\big) F_v(dx)\Big) dv . \end{aligned}$$

By (Elstrodt, 1999, Folgerung VII.4.12 a) & Satz VII.4.14) the exponent in the last line is continuous, thus $L_t - L_s \xrightarrow{s \to t} 0$ in distribution. This implies that $L_t - L_s \xrightarrow{s \to t} 0$ in probability. Hence (L_t) is stochastically continuous.

Now the claim follows using the fact that every inhomogeneous Lévy process is càdlàg by definition. □

Now we are able to derive some properties of inhomogeneous Lévy processes from additive processes. However, since there are processes with independent increments which are not semimartingales (see (Jacod & Shiryaev, 2003, Chapter II, §4c))), we do not know if the framework for stochastic integration and stochastic differential equations remains valid for the introduced inhomogeneous Lévy processes. This issue will be answered in the following theorem by means of Theorem B.4.

Theorem 2.4
An inhomogeneous Lévy process (L_t) is a semimartingale.

Proof:

Consider the function

$$t \to \int_0^t \left[iub_s - \frac{1}{2}u^2 c_s + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})F_s(dx) \right] ds \; .$$

By (Elstrodt, 1999, Folgerung VII.4.12b), Satz VII.4.14) this is a function of finite variation and so is

$$t \to \exp \int_0^t \left[iub_s - \frac{1}{2}u^2 c_s + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})F_s(dx) \right] ds \; .$$

Then the claim follows by Theorem B.4. □

Remark 2.5
Note that a complex valued process is of finite variation, if its real and purely imaginary parts are of finite variation, see (Jacod & Shiryaev, 2003, page 86).

A semimartingale can be described by its characteristics. Therefore we derive here the characteristics of an inhomogeneous Lévy process.

Proposition 2.6
The semimartingale characteristics of an inhomogeneous Lévy process (L_t) associated with the truncation function $\mathbb{1}_{\{|x|\leq 1\}}$, are given by

$$B_t = \int_0^t b_s ds \; , \quad C_t = \int_0^t c_s ds \; , \quad \nu([0,t]\times A) = \int_0^t \int_A F_s(dx) ds \quad (A \in \mathcal{B}(\mathbb{R})) \; .$$

Proof:

Consider

$$\begin{aligned} A(u)_t &:= iuB_t - \frac{1}{2}C_t u^2 + \int_0^t \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})\nu(ds,dx) \\ &= \int_0^t iub_s ds - \frac{1}{2}\int_0^t c_s u^2 ds + \int_0^t \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})F_s(dx) ds \; . \end{aligned}$$

As before this is a continuous function in t, which has finite variation. Thus

$$\mathcal{E}[A(u)]_t = \exp A(u)_t = \mathbb{E}\big[e^{iuL_t}\big] \; .$$

The independent increment property yields

$$\frac{\mathbb{E}[e^{iuL_t}|\mathcal{F}_s]}{\mathbb{E}[e^{iuL_t}]} = \frac{\mathbb{E}[e^{iu(L_t-L_s+L_s)}|\mathcal{F}_s]}{\mathbb{E}[e^{iuL_t}]} = \frac{\mathbb{E}[e^{iu(L_t-L_s)}|\mathcal{F}_s]e^{iuL_s}}{\mathbb{E}[e^{iuL_t}]} = \frac{e^{iuL_s}}{\mathbb{E}[e^{iuL_s}]} .$$

Hence

$$\frac{e^{iuL_t}}{\mathbb{E}[e^{iuL_t}]}$$

is a martingale and with (Jacod & Shiryaev, 2003, Corollary II.2.48) the claim follows. □

2.2. Exponential moment and canonical representation

In this work we assume that every arising inhomogeneous Lévy process has a finite exponential moment. For that purpose we require the following assumption, which stands in force for the rest of this chapter.

Assumption 2.7
There is a positive constant C_L^2 such that

$$\int_{|x|\geq 1} e^x F_s(dx) \leq C_L^2 , \qquad s \geq 0 .$$

Under this assumption, we are now able to proof that the exponential moment of an inhomogeneous Lévy process is finite.

Theorem 2.8
Fix $t \in [0, \infty)$. For an inhomogeneous Lévy process (L_t), we have

$$\mathbb{E}\left[e^{L_t}\right] < \infty .$$

More precisely, $\mathbb{E}\left[e^{L_t}\right] = e^{\psi(-i)}$, where

$$\psi(u) = iu \int_0^t b_s ds - \frac{1}{2}u^2 \int_0^t c_s ds + \int_0^t \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})F_s(dx)ds .$$

Proof:

Fix $t \in [0, \infty)$. Let $(\tilde{L}_t)$ be a Lévy process, such that $L_t \sim \tilde{L}_1$. Then its generating triple is given by (b, c, F) as introduced in Proposition 2.2. Moreover,

$$\int_{\{|x|\geq 1\}} e^x F(dx) = \int_0^t \int_{\{|x|\geq 1\}} e^x F_s(dx)ds \leq C_L^2 \cdot t < \infty , \qquad \forall t \in [0, \infty) ,$$

which yields with (Sato, 2005, Theorem 25.17)

$$\mathbb{E}\big[e^{\tilde{L}_1}\big] < \infty\ .$$

Further, we conclude that

$$\mathbb{E}\big[e^{\tilde{L}_1}\big] = e^{\psi(-i)}\ ,$$

where ψ is the characteristic exponent of the Lévy process $(\tilde{L}_t)$. Now the claim follows since $\tilde{L}_1 \sim L_t$. □

Corollary 2.9

Fix $0 \le s \le t < \infty$. Then it holds for an homogeneous Lévy process (L_t)

$$\mathbb{E}\big[e^{L_t - L_s}\big] = \exp \int_s^t \Big(b_u + \frac{1}{2}c_u + \int_{\mathbb{R}} \big(e^x - 1 - x\mathbb{1}_{\{|x|\le 1\}}\big)F_u(dx)\Big)du\ .$$

Proof:

Using the independent increment property of (L_t) yields

$$\mathbb{E}\big[e^{L_t}\big] = \mathbb{E}\big[e^{L_t - L_s + L_s}\big] = \mathbb{E}\big[e^{L_t - L_s}\big]\mathbb{E}\big[e^{L_s}\big]\ .$$

Therefore

$$\mathbb{E}\big[e^{L_t - L_s}\big] = \frac{\mathbb{E}\big[e^{L_t}\big]}{\mathbb{E}\big[e^{L_s}\big]} = \exp \int_s^t \Big(b_u + \frac{1}{2}c_u + \int_{\mathbb{R}} \big(e^x - 1 - x\mathbb{1}_{\{|x|\le 1\}}\big)F_u(dx)\Big)du\ .$$

□

We have already shown in Theorem 2.4 that each inhomogeneous Lévy process is a semimartingale. Yet, under Assumption 2.7, the following stronger result can be shown.

Proposition 2.10

An inhomogeneous Lévy process (L_t) is a special semimartingale, if $\inf\{\Delta L_t, t > 0\} > -1$.

Proof:

Since $F_s(dx)$ is a positive measure,

$$t \to \int_0^t \int_{\mathbb{R}} (x^2 \wedge |x|)F_s(dx)ds$$

is an increasing, continuous, deterministic process with finite variation. Furthermore

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} (x^2 \wedge |x|) F_s(dx) ds &= \int_0^t \int_{|x|\leq 1} x^2 F_s(dx) ds + \int_0^t \int_{|x|>1} |x| F_s(dx) ds \\ &\leq \int_0^t \int_{|x|\leq 1} x^2 F_s(dx) ds + \int_0^t \int_{|x|>1} e^x F_s(dx) ds \\ &\leq \int_0^t C_L^1 ds + \int_0^t C_L^2 ds = (C_L^1 + C_L^2) t < \infty \ .\end{aligned}$$

Hence, by (Jacod & Shiryaev, 2003, Prop. II.2.29 a)) we conclude that (L_t) is a special semimartingale. □

The following canonical representation of an inhomogeneous Lévy process (L_t) simplifies the handling of those processes considerably.

Theorem 2.11
Let (L_t) be an inhomogeneous Lévy process such that $\inf\{\Delta L_t, t > 0\} > -1$. Then we have the following canonical representation

$$L_t = \int_0^t b_s^{'} ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx) \ ,$$

where $b_s^{'} := b_s + \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x|\leq 1\}}) F_s(dx)$. Moreover, we have

$$|b_s^{'}| \leq C_L^1 + C_L^2 \ .$$

Proof:

Let (L_t) be an inhomogeneous Lévy process with semimartingale characteristic (B, C, ν). Due to Proposition 2.6 and Theorem B.7 we get

$$L_t = L_0 + A_t + L_t^c + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx) \ ,$$

where

$$A_t = \int_0^t b_s ds + \int_0^t \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x|\leq 1\}}) F_s(dx) ds \ .$$

Note that L^c is a continuous local martingale due to (Jacod & Shiryaev, 2003, Prop. I.4.27). Because

$$\langle L^c, L^c \rangle = \int_0^t c_s ds \ ,$$

we obtain with (Jacod & Shiryaev, 2003, Theorem II.4.4)

$$L_t^c = \int_0^t \sqrt{c_s} dW_s \ .$$

Finally we have

$$L_t = \int_0^t b_s^{'} ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx) \ ,$$

where $b_s^{'} := b_s + \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x|\le 1\}}) F_s(dx)$.

Moreover,

$$\begin{aligned} |b_s^{'}| &\le |b_s| + \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x|\le 1\}}) F_s(dx) \le C_L^1 + \int_{x\ge 1} x F_s(dx) \\ &\le C_L^1 + \int_{x \ge 1} e^x F_s(dx) \le C_L^1 + C_L^2 \ . \end{aligned}$$

□

2.3. The Markov property of the stochastic exponential

With (Protter, 2005, Theorem V.32) it follows that the stochastic exponential of a Lévy process is a Markov process. This remains true for an inhomogeneous Lévy process, if the process has only jumps greater that minus one.

Theorem 2.12
Let (L_t) be an inhomogeneous Lévy process such that $\inf\{\Delta L_t, t > 0\} > -1$. Then the stochastic exponential $\mathcal{E}(L)_t$ of (L_t) is a Markov process.

Proof:

Let $S_t := \mathcal{E}(L)_t$. We have to show that for $s \le t$

$$\mathbb{E}\big[g(S_t)|\mathcal{F}_s\big] = \mathbb{E}\big[g(S_t)|\sigma(S_s)\big] \ ,$$

for any bounded measurable function g. Therefore consider

$$\mathbb{E}\big[g(S_t)|\mathcal{F}_s\big] = \mathbb{E}\big[g(\tfrac{S_t}{S_s} \cdot S_s)|\mathcal{F}_s\big] \ , \quad \mathbb{E}\big[g(S_t)|\sigma(S_s)\big] = \mathbb{E}\big[g(\tfrac{S_t}{S_s} \cdot S_s)|\sigma(S_s)\big] \ .$$

Since

$$\begin{aligned} \frac{S_t}{S_s} &= \frac{e^{L_t - \frac{1}{2}\int_0^t c_v dv} \prod_{0\le u\le t} \big[(1+\Delta L_u)e^{-\Delta L_u}\big]}{e^{L_s - \frac{1}{2}\int_0^s c_v dv} \prod_{0\le u\le s} \big[(1+\Delta L_u)e^{-\Delta L_u}\big]} \\ &= e^{L_t - L_s - \frac{1}{2}\int_s^t c_v dv} \prod_{s<u\le t} \big[(1+\Delta L_u)e^{-\Delta L_u}\big] \end{aligned}$$

is independent of $\mathcal{F}_s$, we obtain for a constant $y > 0$

$$\mathbb{E}\big[g(\frac{S_t}{S_s} \cdot y)|\mathcal{F}_s\big] = \mathbb{E}\big[g(\frac{S_t}{S_s} \cdot y)\big] =: h(y) \ ,$$

for some measurable function h. Because of that and since S_s is $\mathcal{F}_s$ measurable we conclude

$$\mathbb{E}\big[g(\frac{S_t}{S_s} \cdot S_s)|\mathcal{F}_s\big] = h(S_s) \ .$$

Analogously, it follows that

$$\mathbb{E}\big[g(\frac{S_t}{S_s} \cdot S_s)|\sigma(S_s)\big] = h(S_s)$$

and thus

$$\mathbb{E}\big[g(S_t)|\mathcal{F}_s\big] = h(S_s) = \mathbb{E}\big[g(S_t)|\sigma(S_s)\big] \ .$$

□

For more details on Markov solutions of stochastic differential equations we refer to Protter (1977).

2.4. Auxiliary results

Finally we show some auxiliary results, which will be frequently used in the following chapters.

Proposition 2.13
Let (L_t) be an inhomogeneous Lévy process such that $\inf\{\Delta L_t, t \geq 0\} > -1$. Moreover, let $(S_t) := \mathcal{E}(L)_t$. Then we have for $0 \leq s \leq t < \infty$

$$\mathbb{E}\left[\left|\frac{S_t - S_s}{S_s}\right|\right] \leq 1 + e^{(C_L^1 + C_L^2)(t-s)} \ .$$

Proof:

- $(S_t) > 0$ by Theorem B.2. Thus we may introduce

$$\begin{aligned}
\frac{S_t - S_s}{S_s} &= \frac{S_t}{S_s} - 1 \\
&= \frac{e^{L_t - \frac{1}{2}\int_0^t c_v dv} \prod_{0 \leq u \leq t} \left[(1 + \Delta L_u)e^{-\Delta L_u}\right]}{e^{L_s - \frac{1}{2}\int_0^s c_v dv} \prod_{0 \leq u \leq s} \left[(1 + \Delta L_u)e^{-\Delta L_u}\right]} - 1 \\
&= e^{L_t - L_s - \frac{1}{2}\int_s^t c_v dv} \prod_{s < u \leq t} \left[(1 + \Delta L_u)e^{-\Delta L_u}\right] - 1
\end{aligned}$$

- The function $h : (-1,\infty) \to \mathbb{R}_+$ with $h(x) := (1+x)e^{-x}$ attains its maximum on the domain $(-1,\infty)$ at $x = 0$ with value 1. Hence, we have

$$\left[(1+\Delta L_u)e^{-\Delta L_u}\right] \in [0,1] .$$

Since the number of jump times in the interval $(s,t]$ is at most countable, see (Applebaum, 2009, Theorem 2.9.2), we may write

$$\prod_{s<u\leq t} \left[(1+\Delta L_u)e^{-\Delta L_u}\right] = \prod_{n=1}^{\infty} \left[(1+\Delta L_{u_n})e^{-\Delta L_{u_n}}\right] ,$$

where $(u_n, n \geq 1)$ are the jump times in $(s,t]$. Further we obtain for the partial sum

$$\prod_{n=1}^{k} \left[(1+\Delta L_{u_n})e^{-\Delta L_{u_n}}\right] \in [0,1] , \quad k \in \mathbb{N} ,$$

and it follows that

$$\prod_{s<u\leq t} \left[(1+\Delta L_u)e^{-\Delta L_u}\right] \in [0,1] .$$

This in turn yields

$$\left|\frac{S_t - S_s}{S_s}\right| \leq e^{L_t - L_s - \frac{1}{2}\int_s^t c_v dv} + 1 .$$

- Applying Corollary 2.9 gives

$$\begin{aligned}
\mathbb{E}\left[\left|\tfrac{S_t-S_s}{S_s}\right|\right] &\leq 1 + \exp\int_s^t \big(b_u + \frac{1}{2}c_u + \int_{\mathbb{R}} \big(e^x - 1 - x\mathbb{1}_{\{|x|\leq 1\}}\big)F_u(dx)\big)du - \frac{1}{2}\int_s^t c_v dv \\
&= 1 + \exp\int_s^t \big(b_u + \int_{\mathbb{R}} \big(e^x - 1 - x\mathbb{1}_{\{|x|\leq 1\}}\big)F_u(dx)\big)du \\
&\leq 1 + \exp\,(C_L^1 + C_L^2)(t-s) .
\end{aligned}$$

□

Proposition 2.14
Let (L_t) be an inhomogeneous Lévy process such that

- $\inf\{\Delta L_t\ , t > 0\} > -1$,
- For $l \in \mathbb{R}$ it holds
$$\int_{(-1,\infty)} \sup_{\pi\in[0,1]} \Big((1+\pi y)^l - 1 - l\pi y\Big) F_s(dy) \le C\ , \quad \forall s \in [0,T]\ ,$$
for some constants $C > 0$ and $T > 0$.

Moreover, let $\pi : \Omega \times \mathbb{R}_+ \to [0,1]$ be an adapted càglàd process and let Y be the unique semimartingale, which solves
$$Y_t = x + \int_0^t \pi_s Y_{s-} dL_s\ ,$$
for some positive $x \in \mathbb{R}$. Then
$$\mathbb{E}\big[(Y_t)^l\big] \le x^l C_l\ , \quad t \in [0,T]\ ,$$
where C_l is a positive constant depending on l.

Proof:

Applying Itô's formula with $f(x) = x^l$ yields

$$\begin{aligned}
(Y_t)^l &= x^l + \int_0^t l(Y_{s-})^{l-1} dY_s + \frac{1}{2}\int_0^t l(l-1)(Y_{s-})^{l-2} d[Y^c, Y^c]_s \\
&\quad + \sum_{0<s\le t} \Big\{(Y_s)^l - (Y_{s-})^l - l(Y_{s-})^{l-1}\Delta Y_s\Big\} \\
&= x^l + \int_0^t l(Y_{s-})^{l-1} dY_s + \frac{1}{2}\int_0^t l(l-1)(Y_{s-})^{l-2} Y_{s-}^2 \pi_s^2 c_s ds \\
&\quad + \sum_{0<s\le t} \Big\{(Y_s)^l - (Y_{s-})^l - l(Y_{s-})^{l-1}\Delta Y_s\Big\} \\
&= x^l + \int_0^t l(Y_{s-})^l \pi_s dL_s + \frac{1}{2}\int_0^t l(l-1)(Y_{s-})^l \pi_s^2 c_s ds \\
&\quad + \int_0^t \int_{-1}^\infty (Y_{s-})^l \big[(1+\pi_s y)^l - 1\big] \mu^L(ds, dy) \\
&\quad - \int_0^t \int_{-1}^\infty l(Y_{s-})^l \pi_s y \mu^L(ds, dy)\ .
\end{aligned}$$

Combining the *local martingales* yields

$$\begin{aligned}
(Y_t)^l &= x^l + \int_0^t (Y_{s-})^l \big(l\pi_s b_s' + \tfrac{l(l-1)}{2} c_s \pi_s^2\big) ds \\
&\quad + \int_0^t \int_{-1}^{\infty} (Y_{s-})^l \big[(1+\pi_s y)^l - 1 - l\pi_s y\big] F_s(dy) ds \\
&\quad + \textit{local martingale} \, .
\end{aligned}$$

Now let $T_n^{'}$ be a fundamental sequence of stopping times for the *local martingale* above and define

$$T_n := T_n^{'} \wedge \inf\{s : (Y_s)^l \geq n\} \, .$$

If $(Y_s)^l$ is bounded, then $T_n = T_n^{'}$ for large n and so $T_n \nearrow \infty$. If on the other hand $(Y_s)^l$ is unbounded, then

$$\inf\{s : (Y_s)^l \geq n\} \nearrow \infty \, , \quad (n \to \infty) \, ,$$

since $(Y_s)^l$ is a càdlàg function which implies that it is bounded on each closed interval. Therefore $T_n \nearrow \infty$. Because of $(Y_s)^l < n$ on $[0, T_n)$, we also have $(Y_{s-})^l \leq n$ on $[0, T_n]$. Further for $t \in [0,T]$ we get

$$\begin{aligned}
\mathbb{E}[(Y_{t\wedge T_n})^l] &= x^l + \mathbb{E}\Big[\int_0^{t\wedge T_n} (Y_{s-})^l \Big\{ \Big(l\pi_s b_s' + \tfrac{l(l-1)}{2} c_s \pi_s^2\Big) \\
&\quad + \int_{-1}^{\infty} \Big(\big(1+\pi_s y\big)^l - 1 - l\pi_s y\Big) F_s(dy) \Big\} ds\Big] \\
&\leq x^l + \mathbb{E}\Big[\int_0^{t\wedge T_n} (Y_{s-})^l \Big\{ \Big(|l\pi_s b_s'| + \tfrac{|l(l-1)|}{2} c_s \pi_s^2\Big) \\
&\quad + \int_{-1}^{\infty} \sup_{\pi\in[0,1]} \Big(\big(1+\pi y\big)^l - 1 - l\pi y\Big) F_s(dy) \Big\} ds\Big] \\
&\leq x^l + \mathbb{E}\Big[\int_0^{t\wedge T_n} (Y_{s-})^l \Big\{ \Big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\Big) + C \Big\} ds\Big] \\
&\leq x^l + \mathbb{E}\Big[\int_0^{t\wedge T_n} n \Big\{ \Big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\Big) + C \Big\} ds\Big] \\
&\leq \tilde{C}_l \, ,
\end{aligned}$$

where $\tilde{C}_l$ is a positive constant depending on l. Since we are dealing with a path-by-path Lebesgue integral, we almost surely have

$$\begin{aligned}
&\int_0^{t\wedge T_n} (Y_{s-})^l \Big\{ \big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\big) + C \Big\} ds \\
&= \int_0^{t\wedge T_n} (Y_s)^l \Big\{ \big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\big) + C \Big\} ds \\
&\leq \int_0^{t} (Y_{s\wedge T_n})^l \Big\{ \big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\big) + C \Big\} ds \, ,
\end{aligned}$$

and hence

$$\mathbb{E}[(Y_{t\wedge T_n})^l] \leq x^l + \mathbb{E}\Big[\int_0^t (Y_{s\wedge T_n})^l \Big\{ \big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\big) + C \Big\} ds\Big] \, .$$

Now we can apply Fubini's Theorem and obtain

$$\mathbb{E}[(Y_{t\wedge T_n})^l] \le x^l + \int_0^t \mathbb{E}\big[(Y_{s\wedge T_n})^l\big]\Big\{\big(|lb'_s| + \tfrac{|l(l-1)|}{2}c_s\big) + C\Big\}ds \ .$$

Since $\mathbb{E}[(Y_{t\wedge T_n})^l] \le \tilde{C}_l$ for $t \in [0,T]$ we can further make use of Gronwall's inequality to get

$$\mathbb{E}[(Y_{t\wedge T_n})^l] \le x^l C_l \ , \quad t \in [0,T] \ ,$$

where C_l is a positive constant depending on l. Using Fatou's Lemma, we finally conclude

$$\mathbb{E}[(Y_t)^l] \le x^l C_l \ , \quad t \in [0,T] \ .$$

□

2.5. A special case: The inhomogeneous Poisson processes

In this section we introduce inhomogeneous Poisson processes. These processes generalize the standard (homogeneous) Poisson process in such a way, that the intensity is not necessarily constant. Hence, this leads to an intensity process (λ_t), which indicates the time-varying intensity of the jumps.

Now let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, satisfying

$$\int_0^t \lambda_s ds < \infty \ , \ \forall t \ge 0 \qquad \text{and} \qquad \int_0^\infty \lambda_s ds = \infty \ .$$

Definition 2.15

An $\mathbb{F}$-adapted counting process (N_t) with independent increments is called an inhomogeneous Poisson process with intensity process (λ_t), if for $s < t$ it holds:

$$\mathbb{P}(N_t - N_s = n) = e^{-\Lambda(s,t)}\frac{(\Lambda(s,t))^n}{n!} \ ,$$

where $\Lambda(s,t) = \int_s^t \lambda_u du$.

In the following we will summarize some important properties of just introduced inhomogeneous Poisson processes, which we will use in the next chapters.

The characteristic function of an inhomogeneous Poisson process is given by

$$\mathbb{E}(e^{iuN_t}) = e^{((e^{iu}-1)\Lambda(0,t))} \ .$$

Hence, if the intensity process (λ_t) is bounded, the inhomogeneous Poisson process is an inhomogeneous Lévy process in the sense of Definition 2.1.

Let $(\tau_n)_{n\in\mathbb{N}_0}$ be the sequence of successive jump times of (N_t). Then it holds

$$\mathbb{P}(\tau_n \le t) = \frac{1}{(n-1)!}\int_0^t e^{-\Lambda(0,s)}(\Lambda(0,s))^{n-1}d\Lambda(0,s)\ .$$

To construct an inhomogeneous Poisson process we can use the following result:

Let $(\hat{N}_t)$ be an homogeneous Poisson process with constant intensity equal to 1 and let $\Lambda(t) = \int_0^t \lambda_u du$. Then

$$N_t = \hat{N}_{\Lambda(t)}$$

is an inhomogeneous Poisson process with intensity process (λ_t).

Since we are dealing with discrete-time models, we are interested in the conditional distribution of successive jump times.

Proposition 2.16
Let (N_t) be an inhomogeneous Poisson process with positive intensity process $\lambda : \mathbb{R}_+ \to (0,\infty)$, then the conditional probability of successive jump times is given by

$$\mathbb{P}(\tau_{n+1} \in [u,v] \mid \tau_n = u) = \int_u^v \lambda_s e^{-\int_u^s \lambda_y dy} ds\ , \quad 0 \le u \le v < \infty\ , \quad n \in \mathbb{N}_0\ .$$

Proof:

Let $\Lambda : [0,\infty) \to [0,\infty)$ with

$$\Lambda(t) := \int_0^t \lambda_s ds\ .$$

First note that the function Λ is a bijection and both Λ and the inverse of Λ, denoted by Λ^{-1}, are continuous. Furthermore, we have

$$N_t = \tilde{N}_{\Lambda(t)}\ ,$$

where $(\tilde{N}_t)$ is a homogeneous Poisson process with intensity equal to 1.

Now let $(\tilde{\tau}_n)$ denote the jump times of $(\tilde{N}_t)$ and (τ_n) the jump times of (N_t). Then it holds

$$\Lambda^{-1}(\tilde{\tau}_n) = \tau_n$$

and

$$\mathbb{P}(\tilde{\tau}_{n+1} \in [u,v] | \tilde{\tau}_n = u) = \int_0^{v-u} e^{-s} ds = \int_u^v e^{-(s-u)} ds\ , \qquad 0 \le u \le v < \infty\ .$$

Using substitution finally yields

$$\begin{aligned}\mathbb{P}(\tau_{n+1} \in [u,v] | \tau_n = u) &= \mathbb{P}(\tilde{\tau}_{n+1} \in [\Lambda(u), \Lambda(v)] | \tilde{\tau}_n = \Lambda(u)) \\ &= \int_{\Lambda(u)}^{\Lambda(v)} e^{-(s-\Lambda(u))} ds \\ &= \int_u^v \lambda_s e^{-\int_u^s \lambda_u du} ds \ ,\end{aligned}$$

for $0 \leq u \leq v < \infty$. □

3. The portfolio optimization problem

In this chapter we build the foundation for the following chapters. We start by introducing a financial market with liquidity risk in which we consider an investor. The market consists of two assets, a risky asset (S_t), called stock, and a riskless asset (B_t), called bond. The liquidity risk in this financial market is modeled by restricting the observation and trading times. Hence the investor can only observe and trade the assets at exogenous random times $(\tau_n)_{n\in\mathbb{N}_0}$, which are given by the jump times of an inhomogeneous Poisson process (N_t). The investor puts his initial wealth in this financial market and then aims to maximize the expected utility of his terminal wealth at time $0 < T < \infty$. Because of that we assume that he may also observe the market and trade the assets at time T. On the other hand, we assume that the investor is risk-averse in the sense that he wants to save a certain percentage of his maximal observed wealth. Therefore, we require that the investor's observed wealth does not fall under a certain percentage of its running maximum.

This chapter is organized as follows: In Section 3.1 we introduce the financial market. Then, in the following section, we consider an investor in this market and define his wealth process as well as a class of admissible policies. In Section 3.3 we formulate the investor's portfolio optimization problem - a terminal wealth problem. In the last section we consider two different intensity processes of the inhomogeneous Poisson process, for which we will solve the terminal wealth problem in the subsequent chapters.

3.1. Financial Market

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. It is assumed that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions and that all stochastic processes are defined on that complete stochastic basis.

Now we consider a financial market, consisting of a stock, a bond and exogenous random times. We assume that the stock price (S_t) is given by

$$S_t = \mathcal{E}(L)_t \ , \quad t \geq 0 \ ,$$

where $\mathcal{E}$ is the stochastic exponential operator. Furthermore (L_t) is an adapted, inhomogeneous Lévy process with characteristics (b, c, F), which satisfy

$$\inf\{\Delta L_t, t > 0\} > -1 \ , \qquad \text{and} \qquad \int_{|x|\geq 1} e^x F_s(dx) \leq C_L^2 \ , \quad s \geq 0 \ ,$$

for some positive constant C_L^2.

Moreover, we suppose that the price of the bond (B_t) is constant equal to 1, i.e.

$$B_t = 1 \; , \quad \forall t \geq 0 \; .$$

Finally we model the exogenous random times $(\tau_n)_{n\in\mathbb{N}_0}$ by the jump times of an adapted, inhomogeneous Poisson process (N_t). This process is assumed to be independent of the stock price (S_t) and to have a deterministic intensity process (λ_t).

Example 3.1

If we choose the characteristics of (L_t) to be $(\mu, \sigma^2, 0)$ where μ and $\sigma > 0$ are constants, then (L_t) is a Brownian motion with drift. Thus (S_t) is a geometric Brownian motion with coefficients μ and σ, which coincides with the well-known Black-Scholes model.

Example 3.2

If we choose the characteristics of (L_t) to be (μ, σ^2, ν), where μ and $\sigma > 0$ are constants and ν is a Lévy measure, then (L_t) is an ordinary Lévy process. Since the ordinary exponential of a Lévy process can be represented by the stochastic exponential of another Lévy process, the exponential-Lévy model is included in the introduced financial market.

The inhomogeneous Poisson process (N_t) is suitable for modeling the exogenous random times (τ_n), since this process has the following desirable properties:

- independent increments, i.e. the future liquidity risk of the stock does not depend on the past,

- a non constant intensity process, i.e. it is possible to model time periods in which the liquidity risk is low or high.

At this time we want to emphasize, that choosing the bond price (B_t) constant one does not restrict the model, since the bond can always be chosen as a numéraire. However, the given framework has the advantage that some computational issues become simpler. For a detailed discussion of the change of numéraire technique we refer to Chesney et al. (2009) and Delbaen & Schachermayer (2006).

Remark 3.3

The assumption $\inf\{\Delta L_t, t > 0\} > -1$ guarantees that the stock price (S_t) is strictly positive.

3.2. Wealth process and admissible policies

In the following we consider an investor, who starts to invest his initial capital $x > 0$ at time $t = 0$ in the introduced financial market. We assume that he observes and trades his assets only at the exogenous random times $(\tau_n)_{n\in\mathbb{N}_0}$ with $\tau_0 = 0$.

Remark 3.4
In financial markets with liquidity risk it is not guaranteed that a continuous observation of the stock price (S_t) is possible. Examples for situations in which continuous observation is not possible are the OTC markets. However, if we would assume that the investor observes the stock price (S_t) continuously but trades the assets only at the exogenous random times (τ_n), then he would not improve his situation, since the stock price (S_t) is a Markov process and so the additional information is useless.

Consider again our investor in the financial market. We assume that his information is given by the filtration $\mathbb{G}$, where $\mathbb{G} = (\mathcal{G}_n)_{n\in\mathbb{N}_0}$ with

$$\mathcal{G}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{G}_n = \sigma\{(\tau_k, S_{\tau_k}) : 0 \le k \le n\}, \quad n \ge 1 .$$

Hence, $\mathcal{G}_n$ denotes the information, which the investor can access at time τ_n. By means of the filtration $\mathbb{G}$, we define an investment policy as a $\mathbb{R}$-valued and $\mathbb{G}$-adapted process

$$\pi = (a_n)_{n\in\mathbb{N}_0} ,$$

where a_n is the amount invested in the stock over the period $(\tau_n, \tau_{n+1}]$ after observing the stock price (S_t) at time τ_n.

Now let

$$\tilde{\pi}_t := \sum_{n=0}^{\infty} \frac{a_n}{S_{\tau_n}} \cdot \mathbb{1}_{\{\tau_n < t \le \tau_{n+1}\}} , \quad 0 \le t < \infty ,$$

be the number of stocks, which the investor owns at time t. Furthermore, we restrict the class of policies to the self-financing ones. Hence the investor's wealth process (X_t^π) with respect to the policy π is given by

$$X_t^\pi = x + \int_0^t \tilde{\pi}_s dS_s . \tag{3.1}$$

A self-financing policy means, that the changes in the wealth process are exclusively due to the price changes in the stock price (S_t). Hence, there is no removal or injection of cash after the initial set-up.

Because of the liquidity risk, the number of stocks between two exogenous random times is constant and so $\tilde{\pi}$ is a simple, predictable process. Therefore we can simplify the stochastic integral in (3.1). To do so we first introduce the return of the stock:

Definition 3.5 Let $0 \leq t \leq u < \infty$. Then the random variable $Z_{t,u}$, which is given by

$$Z_{t,u} := \frac{S_u - S_t}{S_t} ,$$

is called the return of the stock. We denote the distribution of $Z_{t,u}$ by $p(t,u,dz)$.

Proposition 3.6
The distribution $p(t,u,dz)$ of $Z_{t,u}$ is a stochastic transition kernel.

Proof:

We have to show that the mapping

$$H : ([0,\infty) \times [0,\infty), \mathcal{B}([0,\infty)) \otimes \mathcal{B}([0,\infty)) \to ([0,1], \mathcal{B}([0,1]))$$

defined by

$$H(t,v) = \mathcal{P}(Z_{t,t+v} \in A) = p(t, t+v, A)$$

is measurable for each $A \in \mathcal{B}((-1,\infty))$.

Since $Z_{t,u}$ is a combination of càdlàg functions, it is itself càdlàg, i.e.

$$Z_{t_n,t_n+v_n} \to Z_{t_0,t_0+v_0} ,$$

if $(t_n, t_n + v_n) \searrow (t_0, t_0 + v_0)$.

Then we can show analogously to the proof of (Karatzas & Shreve, 2005, Proposition 1.13) that

$$Z_{t,t+v} : ([0,\infty) \times [0,\infty) \times \Omega, \mathcal{B}([0,\infty)) \otimes \mathcal{B}([0,\infty)) \otimes \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is a measurable mapping.

Hence, by Fubini Theorem (Klenke, 2008, Theorem 14.16), it follows that H is a measurable mapping. $\square$

Now we are able to evaluate the wealth process (X_t^π) at an exogenous random time τ_n, which yields

$$\begin{aligned} X_{\tau_n}^\pi &= x + \int_0^{\tau_n} \tilde{\pi}_s dS_s = x + \sum_{k=0}^{n-1} \frac{a_k}{S_{\tau_k}} (S_{\tau_{k+1}} - S_{\tau_k}) = x + \sum_{k=0}^{n-1} a_k Z_{\tau_k,\tau_{k+1}} \\ &= x + \sum_{k=0}^{n-1} a_k Z_{k+1} , \end{aligned}$$

where

$$Z_{k+1} := \frac{S_{\tau_{k+1}} - S_{\tau_k}}{S_{\tau_k}} .$$

Figure 3.1 below illustrates a possible evolution of the stock price (S_t) and possible exogenous random times for observing and trading. For simplicity, the investor's horizon T equals 1. As we can easily see, on average the investor makes money by investing in

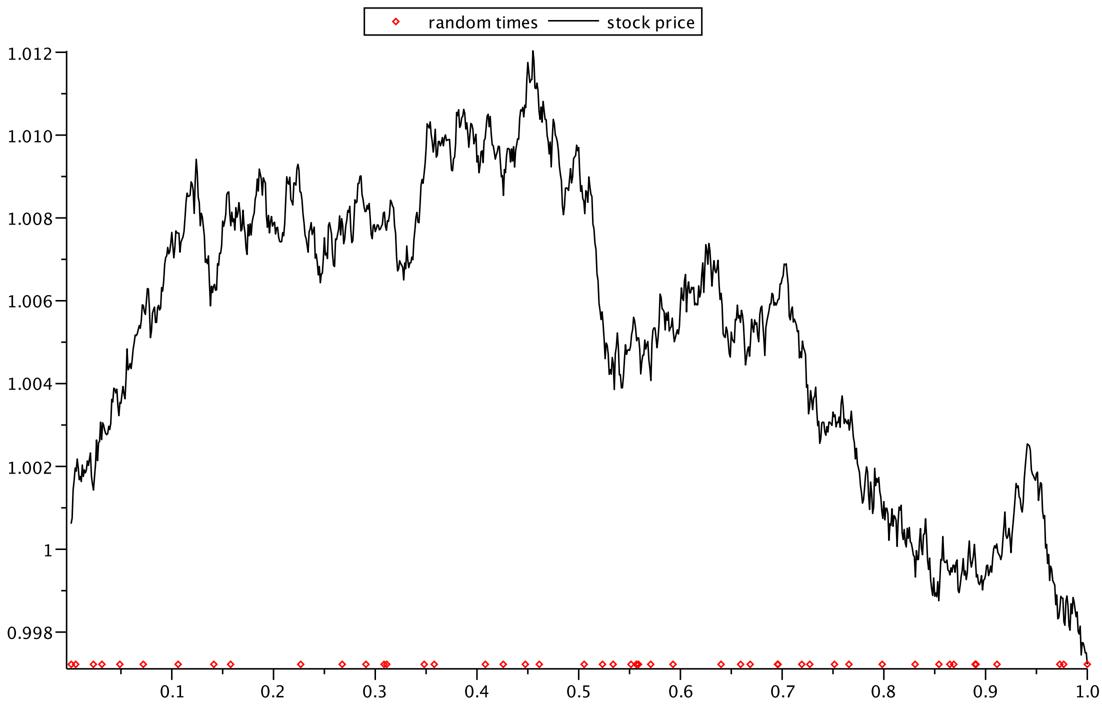

Figure 3.1.: Example of a stock price (S_t) and exogenous random times τ_n.

the stock until time $t \approx 0.46$. After that time, if he continuous to invest in the stock, he will lose money. Hence, on average the investor's fortune is biggest at time $t \approx 0.46$. We introduce now a drawdown constraint to guarantee, that the investor's wealth process does not fall under a certain percentage of his maximal observed wealth.

Therefore let $\beta \in [0, 1)$ be the fixed model parameter determining the percentage of wealth, which will be guaranteed by the drawdown constraint. In order to put this constraint on a formal basis, we first introduce the running maximum of the investor's observed wealth process.

Definition 3.7
We define the process (M_t) by

$$M_0 = m_0 \,, \qquad M_t = \max\{m_0, X^\pi_{\tau_1}, ..., X^\pi_{\tau_n}\} \ \text{ if } \tau_n \le t < \tau_{n+1} \,,$$

where $m_0 \in (0, \frac{x}{\beta})$ is fixed. If $\beta = 0$, then we set $\frac{x}{\beta} = \infty$. In the following the process (M_t) is called the running maximum.

Observe that the running maximum (M_t) carries the information about the investor's maximal observed wealth up to now. Note further that (M_t) is an increasing process, which starts at a strictly positive level. With that information we are able to define admissible policies, for which the associated wealth process satisfies the drawdown constraint.

Definition 3.8
A policy $\pi = (a_0, a_1, a_2, ...)$ is called admissible, if the following holds:

- The observed wealth process $(X^\pi_{\tau_n})$ satisfies the following constraint:

$$X^\pi_{\tau_n} \geq \beta M_{\tau_n} \quad \forall n \in \mathbb{N}_0 \,. \tag{3.2}$$

- For each $n \in \mathbb{N}_0$, there exists a measurable function $f_n : (\tilde{E})^{(n+1)} \to \mathbb{R}$, such that

$$a_n = f_n((\tau_0, X^\pi_{\tau_0}, M_{\tau_0}), ..., (\tau_n, X^\pi_{\tau_n}, M_{\tau_n}))\;,$$

  where $\tilde{E} := \{(t, x, m) : t \in [0, \infty), x \in (0, \infty), m \in (0, \frac{x}{\beta})\}$.

Remark 3.9
Note that condition (3.2) is equivalent to: $\quad X^\pi_{\tau_n} \geq \beta M_{\tau_{n-1}}\;, \quad \forall n \in \mathbb{N}_0$.

The next proposition shows how the drawdown constraint restricts the amount of wealth, which may be invested in the stock. On the one hand the stock price (S_t) is unbounded above. Therefore, short sales of the stock are not allowed, since unbounded losses might occur by selling the stock short and so the constraint (3.2) could not be guaranteed. On the other hand, due to the dynamics of the stock, the investor may lose almost all his money invested in the stock. Hence at time τ_n only the wealth $X^\pi_{\tau_n}$ minus the guaranteed wealth βM_{τ_n} may be invested in the stock.

Proposition 3.10
An admissible policy $\pi = (a_0, a_1, ...)$ satisfies the following condition:

$$0 \leq a_n \leq X^\pi_{\tau_n} - \beta M_{\tau_n} \quad \forall n \in \mathbb{N}_0.$$

Proof:

Let π be an admissible policy. Then for $n \in \mathbb{N}_0$, it holds $X^\pi_{\tau_{n+1}} = X^\pi_{\tau_n} + a_n Z_{n+1}$. Moreover, it follows for $n \in \mathbb{N}_0$:

$$X^\pi_{\tau_{n+1}} = X^\pi_{\tau_n} + a_n Z_{n+1} \geq \beta M_{\tau_n} \qquad \Leftrightarrow \qquad \begin{cases} a_n \text{ arbitrary} & , \text{ if } Z_{n+1} = 0, \\ a_n \geq \frac{-(X^\pi_{\tau_n} - \beta M_{\tau_n})}{Z_{n+1}} & , \text{ if } Z_{n+1} > 0, \\ a_n \leq \frac{(X^\pi_{\tau_n} - \beta M_{\tau_n})}{|Z_{n+1}|} & , \text{ if } Z_{n+1} < 0\,. \end{cases}$$

Since Z_n has support on $(-1,\infty)$, we get

$$X^\pi_{\tau_n} + a_n Z_{n+1} \geq \beta M_{\tau_n} \quad \Leftrightarrow \quad 0 \leq a_n \leq X^\pi_{\tau_n} - \beta M_{\tau_n} \ .$$

□

It is a straight forward consequence of Proposition 3.10 that the constraint (3.2) is fulfilled at all times and not only at the exogenous random times. We summarize this observation in the following corollary.

Corollary 3.11
Let π be an admissible policy. Then the wealth process (X^π_t) is always above the process (βM_t), i.e.

$$X^\pi_t \geq \beta M_t \ , \quad \forall t \geq 0 \ .$$

In particular, Corollary 3.11 implies that

$$X^\pi_T \geq \beta \max\{m, X^\pi_{\tau_1}, X^\pi_{\tau_2}, ..., X^\pi_{\tau_n}, X^\pi_T\} \ , \qquad \text{for some } n \in \mathbb{N}_0 \ .$$

This means, that the terminal wealth X^π_T is always greater than or equal to β times the maximal observed wealth during the investment period $[0,T]$.

3.3. The terminal wealth problem

In the given financial market, we so far introduced the investor's wealth process (X^π_t), which can be influenced by the underlying admissible policy. Now we introduce a performance criterion - the expected utility of the terminal wealth - which the investor is going to maximize by choosing the best admissible policy.

More precisely, let $T > 0$ be a finite horizon and assume that the investor may observe and trade in the financial market at time T. Then the investor aims to maximize his expected utility over all admissible policies, i.e. he is interested in

$$V(y) := \sup_{\pi\in\mathcal{A}(y)} \mathbb{E}_y\big[U(X^\pi_T)\big] \tag{3.3}$$

for $y \in E := \{(t,x,m) : t \in [0,T], x \in (0,\infty), m \in (0, \frac{x}{\beta})\}$, where

- $\mathbb{E}_y$ denotes the expectation of an investor, who observes the financial market at time t and invests his wealth x at the same time.
- $\mathcal{A}(y)$ denotes the set of admissible policies of an investor, who invests his wealth x at time t in the financial market and the start level of his running maximum equals m.

Moreover $U : (0, \infty) \to \mathbb{R}$ denotes the investor's utility function, which is assumed to be strictly increasing, strictly concave, in C^1 and to satisfy the following growth conditions:

(i) There exist constants $p \in (0, 1)$ and $C_U > 0$ such that

$$U^+(x) \leq C_U(1 + x^p) , \qquad x > 0 ,$$

(ii) If $U(0) := U(0+) = -\infty$, then there exist constants $p' < 0$ and $C'_U > 0$ such that

$$U^-(x) \leq C'_U(1 + x^{p'}) , \qquad x > 0 .$$

If $U(0) > -\infty$, then we set $p' = 0$.

In particular these assumptions for the utility function are satisfied in the following examples:

- *Power Utility:* $U(x) = \frac{x^\alpha}{\alpha}$ with $\alpha < 1$, $\alpha \neq 0$. These utility functions are called Constant Relative Risk Aversion (CRRA), since the relative risk aversion $-x\frac{U''(x)}{U'(x)}$ equals $(1 - \alpha)$.
- *Logarithmic Utility:* $U(x) = \log(x)$. This utility function is again CRRA and it can be seen as the limit case of the *Power Utility* function, since $\frac{x^\alpha - 1}{\alpha} \to \log(x)$ when $\alpha \to 0$.
- *Exponential Utility:* $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$ with $\alpha > 0$. In this case U has Constant Absolute Risk Aversion (CARA), since the absolute risk aversion $\frac{-U''(x)}{U'(x)}$ equals α.

3.4. Intensity processes

At the end of this chapter, we have a closer look on the intensity process (λ_t) of the inhomogeneous Poisson process (N_t). In particular, its specification plays a crucial role in the terminal wealth problem (3.3). Because of that, we are going to differentiate two major cases:

(i) The intensity process (λ_t) is bounded by some positive constant and $\int_0^\infty \lambda_t dt = \infty$.

(ii) The intensity process (λ_t) is unbounded, $\int_0^u \lambda_t dt < \infty$ for $u \in [0, T)$ and

$$\int_0^T \lambda_t dt = \infty .$$

Example 3.12

- The intensity process $\lambda : [0, \infty) \to [1, 2]$ with $\lambda_t := 1 + e^{-\frac{t}{10}}$ satisfies the conditions of item (i).
- The intensity process $\lambda : [0, T) \to (0, \infty)$ with $\lambda_t := \frac{1}{T-t}$ satisfies the conditions of item (ii).

In the following chapters we will solve the terminal wealth problem (3.3) under the assumptions of the above considered intensity processes. Furthermore, we will show how these terminal wealth problems are related and how one may be approximated by the other.

Remark 3.13

If we assume an intensity process (λ_t) as in (i), then it is possible that the investor sets up his initial portfolio at time $t = 0$ and that the first exogenous random time τ_1 occurs after the investor's horizon T, i.e. $\tau_1 > T$. In this case the investor has no chance to adjust his portfolio during the investment period $[0, T]$. To avoid such a scenario, we want to consider an intensity process, such that the investor may adjust his portfolio at least once during his investment period. For that purpose, consider the probability that the first jump time τ_1 is in $[0, T]$, which is given by

$$\mathbb{P}(\tau_1 \leq T) = \int_0^T e^{-\int_0^s \lambda_u du} \lambda_s ds = 1 - e^{-\int_0^T \lambda_u du} .$$

This yields that the investor adjusts his portfolio at least once if and only if

$$\int_0^T \lambda_u du = \infty.$$

This is precisely the assumption required in item (ii).

4. A terminal wealth problem with a bounded intensity process

In this chapter we will solve the investor's terminal wealth problem under the assumption of a bounded intensity process (λ_t). The considered portfolio optimization problem is basically a problem in continuous-time. However, since the investor observes and trades only at the exogenous random times (τ_n), he neglects all the information about what happens in the market between those random times. Therefore he controls his wealth process only on a discrete-time basis.
In what follows we will show that the terminal wealth problem can be reduced to a discrete-time optimization problem - a contracting Markov Decision Process - by means of which we can compute the value function and the optimal policy. The main results are the following:

- The value function can be characterized by the unique fixed point of the maximal reward operator.
- There is an optimal stationary policy.

Moreover, Howard's policy improvement algorithm is valid and can be used to approximate an optimal policy. If the algorithm yields no improvement, then we have found an optimal policy. Last but not least, we derive a separation ansatz for the value function under a CRRA utility function.

The outline of this chapter is the following: We start by formalizing our assumptions and derive some characteristics of the considered terminal wealth problem. Then in Section 4.2, we will show that there exists a contracting Markov Decision Process by means of which we can solve the terminal wealth problem. In Section 4.3 we state and proof the solution of the MDP in Theorem 4.10. In the subsequent section, we present Howard's policy improvement algorithm. In the last section we derive a separation ansatz for the value function under a CRRA utility function.

4.1. Assumptions and characteristics

As mentioned above, we assume a bounded intensity process in this chapter:

Assumption 4.1
The intensity process (λ_t) of the inhomogeneous Poisson process (N_t) satisfies

- $\lambda : [0,\infty) \to (0, C_\lambda]$, for some positive constant C_λ,
- $\int_0^\infty \lambda_t dt = \infty$.

Due to that assumption the investor observes and adjusts his portfolio only finitely many times during his investment period.

Proposition 4.2
The exogenous random times (τ_n) converge increasingly to ∞.

Proof:

The inhomogeneous Poisson process (N_t) may be written as

$$N_t = \tilde{N}_{\int_0^t \lambda_u du} ,$$

where $(\tilde{N}_t)$ is a homogeneous Poisson process with constant intensity $\lambda = 1$. Then it follows by (Cont & Tankov, 2004, Section 2.5.5) and the time change that the jump times of (N_t) converge increasingly to infinity. □

The following proposition shows that the considered portfolio optimization problem is well-defined, since all arising expectations are well-defined.

Proposition 4.3

$$\sup_{\pi \in \mathcal{A}(y)} \mathbb{E}_y\big[U^+(X_T^\pi)\big] < \infty , \quad \forall y \in E . \tag{4.1}$$

Proof:

Because of the growth condition on U^+, we have for $\pi \in \mathcal{A}(y)$

$$\mathbb{E}_y\big[U^+(X_T^\pi)\big] \leq \tilde{C}_U\big(1 + \mathbb{E}_y\big[X_T^\pi\big]\big) ,$$

for some constant $\tilde{C}_U > 0$. Since

$$X_t^\pi = x + \int_0^t \tilde{\pi}_s dS_s = x + \int_0^t \frac{\tilde{\pi}_s S_{s-} X_{s-}^\pi}{S_{s-} X_{s-}^\pi} dS_s ,$$

we get by using the stochastic logarithm

$$X_t^\pi = x + \int_0^t \tilde{\pi}_s' X_{s-}^\pi dL_s ,$$

where $\tilde{\pi}'$ is some adapted càglàd process valued in $[0,1]$. Using Proposition 2.14 with $l = 1$ yields $\mathbb{E}\big[X_T^\pi\big] < xC_1$ for some positive constant C_1. Hence $\mathbb{E}_y\big[X_T^\pi\big] < xC_1$ and so the claim follows. □

At the end we state a technical assumption on the inhomogeneous Lévy process (L_t), which guarantees an integrability condition for U^-. In the next section this assumption is needed to show, that the Markov Decision Process is well-defined. Therefore the following assumption stands in force for the rest of this chapter.

Assumption 4.4
If $U(0) = -\infty$, then there exists a constant $C_L^3 > 0$ such that

$$\int_{(-1,\infty)} \sup_{\pi\in[0,1]} \Big((1+\pi y)^{p'} - 1 - p'\pi y \Big) F_s(dy) \le C_L^3 \,, \quad \forall s \in [0,T]\,.$$

For convenience we state now a proposition, which provides a sufficient condition that Assumption 4.4 is fulfilled.

Proposition 4.5
Assumption 4.4 is satisfied, if the following holds

$$\int_{(-1,\infty)} \Big((1+y)^{p'} - 1 - p'y \Big) F_s(dy) \le C_L^3 \,, \quad \forall s \in [0,T]\,.$$

Proof:

Since $f(\pi) := 1 + \pi y$ is a convex, monotone function and $g(x) := x^{p'}$ is a convex function, it follows that the composition $h := g \circ f$ is a convex function. Because a convex function attains its maximum at the boundary we get

$$\int_{(-1,\infty)} \sup_{\pi\in[0,1]} \Big((1+\pi y)^{p'} - 1 - p'\pi y \Big) F_s(dy) \le \int_{(-1,\infty)} \Big((1+y)^{p'} - 1 - p'y \Big) F_s(dy)\,,$$

using $(1+y)^{p'} - 1 - p'y \ge 0$ for all $y > -1$. Now the claim follows. □

4.2. Solution via Contracting Markov Decision Process

In what follows we apply the ideas of Bäuerle & Rieder (2009) to show that there exists a contracting Markov Decision Process (short: MDP) by means of which we can compute the value function V and an optimal policy of the continuous-time terminal wealth problem (3.3).

We start with considering the following contracting Markov Decision Process with infinite horizon:

Contracting Markov Decision Process with infinite horizon

- State space $E = \{(t,x,m) : t \in [0,T], x \in (0,\infty), m \in (0,\frac{x}{\beta})\}$ endowed with the Borel σ-algebra $\mathcal{B}(E)$, where t is the exogenous random time, x the current wealth and m the current value of the running maximum. Moreover, the state process is denoted by $Y_n = (T_n, X_{T_n}, M_{T_n})$ and there is a cemetery state $\Delta \notin E$ such that Y_n equals Δ if $T_n > T$.
- Action space $A = [0,\infty)$ endowed with the Borel σ-algebra $\mathcal{B}(A)$.
- The possible state-action combinations are given by

$$D = \{(y,a) : y \in E\ , a \in [0, x - \beta m]\} \cup \{(\Delta, 0)\}\ ,$$

  and the admissible actions are given by

$$\begin{aligned} D(y) &= \{a \in \mathbb{R} \mid (y,a) \in D\} = [0, x-\beta m]\ , \quad \forall y \in E\ , \\ D(\Delta) &= 0\ . \end{aligned}$$

- For $B \in \mathcal{B}(E)$, $y \in E$ and $a \in D(y)$ the transition probability Q is given by

$$Q(B \mid (y,a)) = \int_t^T \int_{(-1,\infty)} \mathbb{1}_B(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du\ .$$

  Moreover for $y \in E$ and $a \in D(y)$ we have

$$Q(\Delta \mid (y,a)) = 1 - Q(E \mid (y,a))\ , \qquad Q(\Delta \mid (\Delta, 0)) = 1\ .$$

- One-stage reward $r : D \to \mathbb{R}$ given by

$$\begin{aligned} r(y,a) &= e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x+az) p(t,T,dz)\ , \quad \forall y \in E\ , \\ r(\Delta, 0) &= 0\ . \end{aligned}$$

For more details on Markov Decision Processes we refer to Bäuerle & Rieder (2011), Bertsekas & Shreve (1978) and Hinderer (1970).

Proposition 4.6
Q is a stochastic transition kernel from D to E.

Proof:

First note that it is enough to show that Q is a substochastic transition kernel from $D \setminus (\Delta, 0)$ to E.

Since $\lambda_u e^{-\int_t^u \lambda_s ds} \mathbb{1}_{\{u \geq t\}}$ is the density of a probability measure on the measure space $\big[[0,\infty), \mathcal{B}([0,\infty))\big]$ with parameter t, it follows that

$$\kappa : (t, A) \to \int_t^\infty \mathbb{1}_A \lambda_u e^{-\int_t^u \lambda_s ds} du \ , \quad A \in \mathcal{B}([0,\infty)) \ ,$$

is a stochastic transition kernel from $\big[[0,\infty), \mathcal{B}([0,\infty))\big]$ to $\big[[0,\infty), \mathcal{B}([0,\infty))\big]$. Because $p(t, u, \cdot)$ is also a stochastic transition kernel, we get by (Klenke, 2008, Theorem 14.22) that

$$\tilde{Q} : (t, B) \to \int_t^\infty \int_{(-1,\infty)} \mathbb{1}_B(u, z) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \ ,$$

is a stochastic transition kernel for $B \in \mathcal{B}([0,\infty)) \otimes \mathcal{B}((-1,\infty))$. By (Bertsekas & Shreve, 1978, Proposition 7.29) it follows that

$$(y, a) \to \int_t^T \int_{(-1,\infty)} \mathbb{1}_B(u, x + az, \max\{m, x + az\}) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du$$

is measurable and so a substochastic transition kernel from $D \setminus (\Delta, 0)$ to E. □

Since the transition probability depends on the admissible action a, we are able to control the Markov Decision Process. As usual in MDP theory, the admissible action will be chosen by a decision rule, which is a measurable mapping $f : E \cup \Delta \to A$, such that $f(y) \in D(y)$, $\forall y \in E$ and $f(\Delta) = 0$. Moreover, we define a Markovian policy π as a sequence of decision rules, i.e.

$$\pi := (f_0, f_1, f_2, f_3, ...) \ ,$$

where f_k is a decision rule for each k.

By using such a Markovian policy π, we can now define the gain corresponding to π with start $y \in E$ by

$$V_{1,\pi}(y) := \mathbb{E}_y^\pi \left[\sum_{k=0}^\infty r(Y_k, f_k(Y_k)) \right] .$$

Thereby $\mathbb{E}_y^\pi$ denotes the expectation under the Markovian policy π and initial value $y \in E$ of the MDP.

Furthermore, we define the value function of the MDP by

$$V_1(y) := \sup_{\pi \in \Pi} V_{1,\pi}(y) \quad \forall y \in E \ , \tag{4.2}$$

where Π is the set of all Markovian policies.

In the following theorem we proof that the value function V of the terminal wealth problem (3.3) coincides with the value function V_1 of the MDP. Moreover, an optimal Markovian policy of the MDP is an optimal policy for (3.3).

Theorem 4.7

a) For a Markovian policy π, it holds:

$$\mathbb{E}_y\big[U(X_T^\pi)\big] = V_{1,\pi}(y)\ , \quad \forall y \in E\ .$$

b) Moreover, we have: $V(y) = V_1(y)\ , \quad \forall y \in E\ .$

Proof:

a) To avoid heavy notation in this proof, we define $\tilde{Y}_k := (\tau_k, X_{\tau_k}^\pi, M_{\tau_k})$. First note that $\pi \in \mathcal{A}(y)$. By denoting the one point measure at time T by $\tilde{\mu}$ and using (Hinderer, 1972, Satz 19.8) as well as Proposition 4.3, we have

$$\mathbb{E}_y\big[U(X_T^\pi)\big] = \mathbb{E}_y\Big[\int_t^\infty U(X_u^\pi)\tilde{\mu}(du)\Big] = \sum_{k=0}^\infty \mathbb{E}_y\Big[\int_{[\tau_k,\tau_{k+1})} U(X_u^\pi))\tilde{\mu}(du)\Big]\ ,$$

where $\tau_0 = t$ and τ_k is the k-th exogenous random time after time t. Moreover, we get

$$\begin{aligned}
&\mathbb{E}_y\big[U(X_T^\pi)\big] = \\
&= \sum_{k=0}^\infty \mathbb{E}_y\Bigg[\mathbb{E}_y\Bigg[\mathbb{E}_y\Bigg[\int_{[\tau_k,\tau_{k+1})} U(X_{\tau_k}^\pi + f_k(\tilde{Y}_k)\tfrac{S_u - S_{\tau_k}}{S_{\tau_k}})\tilde{\mu}(du)\Bigg|\mathcal{G}_k \vee \sigma(\tau_{k+1})\Bigg]\Bigg|\mathcal{G}_k\Bigg]\Bigg] \\
&= \sum_{k=0}^\infty \mathbb{E}_y\Bigg[\int_{\tau_k}^\infty \int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \le T < u\}} \cdot U(X_{\tau_k}^\pi + f_k(\tilde{Y}_k)z)p(\tau_k, T, dz)\lambda_u e^{-\int_{\tau_k}^u \lambda_s ds} du\Bigg] \\
&= \sum_{k=0}^\infty \mathbb{E}_y\Bigg[e^{-\int_{\tau_k}^T \lambda_u du} \int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \le T\}} \cdot U(X_{\tau_k}^\pi + f_k(\tilde{Y}_k)z)p(\tau_k, T, dz)\Bigg] \\
&= \sum_{k=0}^\infty \mathbb{E}_y\Big[r(\tilde{Y}_k, f_k(\tilde{Y}_k))\Big]\ .
\end{aligned}$$

Now fix some $k \in \mathbb{N}$. Since the function $r(\tilde{Y}_k, f_k(\tilde{Y}_k))$ may be written as some measurable function $g(\tilde{Y}_k)$, it follows

$$\mathbb{E}_y\left[\mathbb{E}_y\left[g(\tilde{Y}_k)\middle|\mathcal{G}_{k-1}\right]\right] = \mathbb{E}_y\left[\int_{\tau_{k-1}}^{T}\int_{(-1,\infty)} g(u, X^{\pi}_{\tau_{k-1}} + f_{k-1}(\tilde{Y}_{k-1})z, \max\{M_{\tau_{k-1}}, X^{\pi}_{\tau_{k-1}} + f_{k-1}(\tilde{Y}_{k-1})z\})p(\tau_{k-1}, s, dz)\lambda_u e^{-\int_{\tau_{k-1}}^{u}\lambda_s ds} du\right]$$
$$= \mathbb{E}_y\left[\int_E g(y_k)Q(dy_k|\tilde{Y}_{k-1}, f_{k-1}(\tilde{Y}_{k-1}))\right] ,$$

where we used the definition of Q for the last equality. Inductively we get

$$\mathbb{E}_y\left[g(\tilde{Y}_k)\right] = \int_E \dots \int_E g(y_k)Q(dy_k|y_{k-1}, f_{k-1}(y_{k-1}))\dots Q(dy_1|y, f_0(y)) = \mathbb{E}_y^{\pi}\left[g(Y_k)\right] ,$$

where we used (Bertsekas & Shreve, 1978, Proposition 7.45) for the last equality. Hence

$$\mathbb{E}_y\left[U(X_T^{\pi})\right] = \mathbb{E}_y^{\pi}\sum_{k=0}^{\infty} r(Y_k, f_k(Y_k)) ,$$

where again (Hinderer, 1972, Satz 19.8) was used.

b) Let $\pi \in \mathcal{A}(y)$. Then, there exists a measurable function $f_k : E^{(k+1)} \to \mathbb{R}$ such that

$$a_k = f_k(Y_0, ..., Y_k) , \qquad \forall k : \tau_k \le T .$$

Since the state process of the MDP is Markovian, by (Bäuerle & Rieder, 2011, Remark 7.1.3) the maximal expected gain cannot be improved by history dependent policies. Therefore we obtain

$$V(y) = \sup_{\pi\in\Pi} V_{1,\pi}(y) = V_1(y) .$$

□

From now on, we concentrate on the introduced MDP to solve the considered terminal wealth problem. The following proposition shows that the MDP has a bounding function.

Proposition 4.8
Define the function $b : E \to \mathbb{R}_+$ by

$$b(t, x) := e^{\gamma(T-t)}(1 + x + x^{p'})$$

with a fixed $\gamma > 0$. If γ is large enough, then $b(t, x)$ is a bounding function, i.e. there are constants $C_r > 0$ and $0 < C_\gamma < 1$ such that

a) $|r(y, a)| \le C_r b(t, x)$,

b) $\int_t^T \int_{(-1,\infty)} b(u, x + az)p(t, u, dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \le C_\gamma b(t, x)$,

for all $(y, a) \in D$.

Proof:

a) Consider

$$\begin{aligned}
&\int_{(-1,\infty)} U^+(x+az)p(t,T,dz) \leq \tilde{C}_U \int_{(-1,\infty)} (1+x+az)p(t,T,dz) \\
&\leq \left[\tilde{C}_U(1+x)\int_{(-1,\infty)}(1+\frac{a}{1+x}|z|)p(t,T,dz)\right] \leq \left[\tilde{C}_U(1+x)(2+e^{(C_L^1+C_L^2)(T-t)})\right] \\
&\leq \tilde{C}_U(1+x)3e^{(C_L^1+C_L^2)(T-t)} \leq 3\tilde{C}_U b(t,x) \; ,
\end{aligned}$$

for some constant $\tilde{C}_U > 0$.

Then it follows

$$\begin{aligned}
|r(y,a)| &\leq \int_{(-1,\infty)} U^+(x+az) + U^-(x+az)p(t,T,dz) \\
&= \int_{(-1,\infty)} U^+(x+az)p(t,T,dz) + \int_{(-1,\infty)} U^-(x+az)p(t,T,dz) \\
&\leq 3\tilde{C}_U b(t,x) + \int_{(-1,\infty)} U^-(x+az)p(t,T,dz) \; .
\end{aligned}$$

Hence, if $U(0) > -\infty$, then $U^-(x)$ is bounded above and the claim of part a) follows.

If $U(0) = -\infty$, then

$$\begin{aligned}
\int_{(-1,\infty)} U^-(x+az)p(t,T,dz) &\leq C'_U\left(1+\int_{(-1,\infty)}(x+az)^{p'}p(t,T,dz)\right) \\
&= C'_U\left(1+\mathbb{E}\big[(x+aZ_{T,t})^{p'}\big]\right) .
\end{aligned}$$

Now we introduce the càdlàg process (Y_t) by

$$Y_t := x + a\frac{S_t - S_s}{S_s} \; , \quad t \geq 0 \; , \qquad \text{for fixed } s \in [0,T] \; .$$

Note that

$$Y_{t-} = x + a\frac{S_{t-} - S_s}{S_s} = (x-a) + a\frac{S_{t-}}{S_s} > 0 \; ,$$

since $\frac{S_{t-}}{S_s} > 0$. Because of that and the stochastic logarithm, we get

$$\begin{aligned}
Y_T &= x + aZ_{T,t} = x + a\frac{S_T - S_t}{S_t} = x + \frac{a}{S_t}\int_t^T \frac{S_{u-}Y_{u-}}{S_{u-}Y_{u-}}dS_u \\
&= x + \int_t^T \pi_u Y_{u-}\frac{1}{S_{u-}}dS_u = x + \int_t^T \pi_u Y_{u-}dL_u \\
&= x + \int_0^T \mathbb{1}_{\{u>t\}}\pi_u Y_{u-}dL_u \; ,
\end{aligned}$$

where $\pi_u := \frac{aS_{u-}}{S_t Y_{u-}} \in [0,1]$.

Applying Proposition 2.14 yields $\mathbb{E}\big[(Y_T)^{p'}\big] \leq x^{p'} C_{p'}$ for some positive constant $C_{p'}$. This implies

$$\mathbb{E}\big[(x + aZ_{T,t})^{p'}\big] \leq x^{p'} C_{p'} \ .$$

Hence there exists a constant $C_r > 0$ such that $|r(y,a)| \leq C_r b(t,x)$.

b)

$$\begin{aligned}
&\int_t^T \int_{(-1,\infty)} b(u, x+az) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&= \int_t^T \int_{(-1,\infty)} e^{\gamma(T-u)} (1 + x + az + (x+az)^{p'}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&\leq \int_t^T e^{\gamma(T-u)} (1+x) \int_{(-1,\infty)} \left[1 + \frac{a}{1+x}|z|\right] p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&\quad + \int_t^T e^{\gamma(T-u)} \int_{(-1,\infty)} (x+az)^{p'} p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&\leq \int_t^T e^{\gamma(T-u)} (1+x)(2 + e^{(C_L^1 + C_L^2)(u-t)}) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&\quad + \int_t^T e^{\gamma(T-u)} \int_{(-1,\infty)} (x+az)^{p'} p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
&\leq 3C_\lambda (1+x) \int_t^T e^{\gamma(T-u)} e^{(C_L^1 + C_L^2)(u-t)} du + x^{p'} C_{p'} C_\lambda \int_t^T e^{\gamma(T-u)} du \\
&\leq 3C_\lambda (1+x) \frac{1}{\gamma - C_L^1 - C_L^2} e^{\gamma(T-t)} + x^{p'} C_{p'} C_\lambda \frac{1}{\gamma} e^{\gamma(T-t)} \\
&\leq \underbrace{\left[\frac{3C_\lambda}{\gamma - C_L^1 - C_L^2} + \frac{C_{p'} C_\lambda}{\gamma}\right]}_{=:C_\gamma} b(t,x) \ .
\end{aligned}$$

Hence b is a bounding function and for large γ we have $0 < C_\gamma < 1$.

□

By using the bounding function b with a large γ, such that $0 < C_\gamma < 1$, it directly follows that the MDP is well-defined, since both *Integrability Assumption (A)* and *Convergence Assumption (C)* are satisfied. For more details on that topic see (Bäuerle & Rieder, 2011, Remark 7.3.2 b)).

4.3. Main Results

In this section, we present the main results of this chapter. To do so, we first have to introduce some notations.

Let

$$\mathbb{B}_b := \{v : E \to \mathbb{R} \mid v \text{ is measurable}, \exists\, C \in \mathbb{R}_+ : |v(t,x,m)| \le Cb(t,x)\}\ ,$$

be the set of functions, consisting of all measurable functions, which are bounded by the bounding function b. Further, we define a metric d on $\mathbb{B}_b$ by

$$d(v,w) := \sup_{y \in E} \frac{|v(y) - w(y)|}{b(t,x)}\ .$$

By following the proof of (Werner, 2007, Beispiel (b) on page 3), it can be easily shown that $(\mathbb{B}_b, d)$ is a complete metric space.

Moreover, we define for $v \in \mathbb{B}_b$ the operator (L_1) by

$$\begin{aligned}(L_1 v)(y,a) := {} & e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x+az) p(t,T,dz) \\ & + \int_t^T \int_{(-1,\infty)} v(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du\ ,\end{aligned}$$

for a decision rule f the operator $(\mathcal{T}_{1,f})$ by

$$(\mathcal{T}_{1,f} v)(y) := (L_1 v)(y, f(y))\ ,$$

and the operator $(\mathcal{T}_1)$ by

$$(\mathcal{T}_1 v)(y) := \sup_{a \in [0, x - \beta m]} (L_1 v)(y,a)\ .$$

$\mathcal{T}_1$ is called maximal reward operator. Now, we introduce the subset $\mathbb{M} \subset \mathbb{B}_b$:

Definition 4.9
Let $\mathbb{M}$ be the subset of $\mathbb{B}_b$, such that for each $v \in \mathbb{M}$ the following conditions hold:

(i) $U(x) \le v(y)$ for $y \in E$.

(ii) $v(t,\cdot,\cdot)$ is concave on $\mathcal{D} := \{(x,m) : x \in (0,\infty), m \in (0, \frac{x}{\beta})\}$, for fixed t.

(iii) $v(t,x,\cdot)$ is decreasing on $(0, \frac{x}{\beta})$ for fixed t and fixed x.

(iv) $v(t,\cdot,m)$ is increasing on $(\beta m, \infty)$ for fixed t and fixed m.

(v) The function

$$s \to v(t, x+s, m+s)$$

is increasing on $[0,\infty)$, for fixed $y \in E$.

Finally, we are able to formulate the main results of this chapter.

Theorem 4.10

a) $V = V_1 \in \mathbb{M}$ and V_1 is the unique fixed point of $\mathcal{T}_1$ in $\mathbb{M}$.

b) Let $g \in \mathbb{M}$. Then the following error estimation holds

$$d(V, \mathcal{T}_1^n g) \leq \frac{C_\gamma^n}{1 - C_\gamma} d(\mathcal{T}_1 g, g) \ .$$

c) There exists a maximizer f^* of V_1, i.e. there exists a decision rule f^* such that

$$\mathcal{T}_{1,f^*} V_1 = \mathcal{T}_1 V_1 \ ,$$

and each maximizer f^* of V_1 defines an optimal stationary policy $\pi := (f^*, f^*, f^*, ...)$.

Before we proof the main results, we first want to discuss them:

(i) Since $V \in \mathbb{M}$, we know that the value function V is increasing in the wealth and decreasing in the start level of the running maximum. This is exactly what we would expect, since a large initial wealth yields a large expected terminal wealth and a stronger restriction on the policies yields a smaller expected terminal wealth.

(ii) In general, the value function is not decreasing in time. To understand that, let us consider the following example: Consider two investors with the same initial wealth. One of them, called investor A, starts to invest his money in the market at time $t_1 \in [0, T)$ and the other investor, called investor B, starts to invest his money at time $t_2 \in (t_1, T)$. Due to the illiquid market, investor A is not necessarily able to trade at time t_2 and so he might not be able to conserve his initial wealth in the bond until time t_2 and then duplicate the policy of investor B. Hence, it is not guaranteed that investor A can do as well as investor B.

Proof of Theorem 4.10:

We are going to proof the statements by applying the *Structure Theorem* for contracting MDPs, see (Bäuerle & Rieder, 2011, Theorem 7.3.5). Since it is not guaranteed that $0 \in \mathbb{M}$, we first have to consider a larger subset of functions $\tilde{\mathbb{M}}$ with $\mathbb{M} \subset \tilde{\mathbb{M}} \subset \mathbb{B}_b$, for which we can apply the theorem. $\tilde{\mathbb{M}}$ is defined in the following way:

Let $\tilde{\mathbb{M}}$ be the subset of $\mathbb{B}_b$ such that for each $v \in \tilde{\mathbb{M}}$ the following conditions hold:

(i) $v(t, \cdot, \cdot)$ is concave on $\mathcal{D} := \{(x, m) : x \in (0, \infty), m \in (0, \frac{x}{\beta})\}$, for fixed t.

(ii) $v(t,x,\cdot)$ is decreasing on $(0,\frac{x}{\beta})$ for fixed t and fixed x.

(iii) $v(t,\cdot,m)$ is increasing on $(\beta m,\infty)$ for fixed t and fixed m.

(iv) The function
$$s \to v(t,x+s,m+s)$$
is increasing on $[0,\infty)$, for fixed $y \in E$.

Step 1:

Obviously $0 \in \tilde{\mathbb{M}}$.

Step 2:

We show: For $v \in \tilde{\mathbb{M}}$, there exists a maximizer of v, i.e. there exists a decision rule f such that
$$\mathcal{T}_{1,f}v = \mathcal{T}_1 v\ .$$

Proof:

Let $v \in \tilde{\mathbb{M}}$. Now we proof that $(L_1 v)$ is an upper semicontinuous function on $[0, x-\beta m]$ for fixed (t,x,m). To do so, let $(c_n)_{n\in\mathbb{N}_0}$ be a convergent sequence in $[0,x-\beta m]$ with limit c_0. Because of the assumptions on the utility function U, we have
$$U(x+c_n z) \le U^+(x+c_n z) \le \tilde{C}_U(1+x+c_n z) \le \tilde{C}_U(1+x+x|z|)$$
for some constant $\tilde{C}_U > 0$. Moreover, for $v \in \tilde{\mathbb{M}}$ it follows that
$$v(u,x+c_n z,\max\{m,x+c_n z\}) \le Cb(u,x+c_n z)\ .$$
Hence, we can apply Fatou's Lemma, which yields
$$\begin{aligned}
&\limsup_{n\to\infty}\Bigg[e^{-\int_t^T\lambda_u du}\int_{(-1,\infty)} U(x+c_n z)p(t,T,dz)\\
&\quad+\int_t^T\int_{(-1,\infty)} v(u,x+c_n z,\max\{m,x+c_n z\})p(t,u,dz)\lambda_u e^{-\int_t^u\lambda_s ds}du\Bigg]\\
&\le e^{-\int_t^T\lambda_u du}\int_{(-1,\infty)}\limsup_{n\to\infty} U(x+c_n z)p(t,T,dz)\\
&\quad+\int_t^T\int_{(-1,\infty)}\limsup_{n\to\infty} v(u,x+c_n z,\max\{m,x+c_n z\})p(t,u,dz)\lambda_u e^{-\int_t^u\lambda_s ds}du\ .
\end{aligned}$$
Due to the concavity of v, it follows that v is continuous on $\mathcal{D}$. Further
$$(x+c_0 z,\max\{m,x+c_0 z\}) \in \mathcal{D}\ .$$

Therefore

$$\begin{aligned}
&e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} \limsup_{n\to\infty} U(x + c_n z)p(t,T,dz) \\
&\quad + \int_t^T \int_{(-1,\infty)} \limsup_{n\to\infty} v(u, x + c_n z, \max\{m, x + c_n z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\
&= e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x + c_0 z)p(t,T,dz) \\
&\quad + \int_t^T \int_{(-1,\infty)} v(u, x + c_0 z, \max\{m, x + c_0 z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \ .
\end{aligned}$$

It follows that $(L_1 v)$ is upper semicontinuous on $[0, x - \beta m]$ for fixed $y \in E$. Since

$$\begin{aligned}
(L_1 v)(y,a) = e^{-\int_t^T \lambda_u du} &\int_{(-1,\infty)} U(x(1 + \frac{a}{x} z))p(t,T,dz) \\
&+ \int_t^T \int_{(-1,\infty)} v(u, x(1 + \frac{a}{x} z, \max\{m, x(1 + \frac{a}{x} z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du
\end{aligned}$$

and $\frac{a}{x} \in [0,1]$, we can find by (Bertsekas & Shreve, 1978, Proposition 7.33) a decision rule f, which is a maximizer of v.

Step 3:

We show: If $v \in \tilde{\mathbb{M}}$, then $\mathcal{T}_1 v$ is well-defined and $\mathcal{T}_1 v \in \tilde{\mathbb{M}}$.

Proof:

Let $v \in \tilde{\mathbb{M}}$. Then

- $(L_1 v)$ is a measurable function on D. Plugging in the maximizer f of v yields

  $$\mathcal{T}_1 v(t,x,m) = \mathcal{T}_{1,f} v(t,x,m) = L_1(t,x,m,f(t,x,m)) \ ,$$

  where the right hand side is a measurable function on E.

- By using Proposition 4.8, we get

  $$\begin{aligned}
  &|\mathcal{T}_1 v(t,x,m)| \\
  &\le \sup_{a\in[0,x-\beta m]} \Bigg[\left| e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x + az)p(t,T,dz) \right| \\
  &\qquad + \int_t^T \int_{(-1,\infty)} |v(u, x + az, \max\{m, x + az\})| p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \Bigg] \\
  &\le \sup_{a\in[0,x-\beta m]} \left[C_r b(t,x) + C \int_t^T \int_{(-1,\infty)} b(u, x + az)p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \right] \\
  &\le \sup_{a\in[0,x-\beta m]} \left[C_r b(t,x) + C C_\gamma b(t,x) \right] \le C' b(t,x) \ ,
  \end{aligned}$$

  for some constant $C' > 0$.

- We fix $t \in [0,T]$ and define the convex set

$$G := \{(x,m,a) : x \in (0,\infty), m \in (0, \tfrac{x}{\beta}), a \in [0, x-\beta m]\} .$$

By using the definition of concavity and the following inequality

$$\max\{\gamma a + (1-\gamma)b, \gamma c + (1-\gamma)d\} \le \gamma \max\{a,c\} + (1-\gamma)\max\{b,d\}$$

for some constants $a,b,c,d \in \mathbb{R}$ and $\gamma \in [0,1]$, it follows that

$$U(x+az) \quad \text{and} \quad v(u, x+az, \max\{m, x+az\})$$

are concave functions on G. Hence $(L_1 v)$ is concave on G due to the linearity of the integral. Then by (Rockafellar, 1972, Theorem 5.7) we get that $\mathcal{T}_1 v(t,\cdot,\cdot)$ is a concave function on $\mathcal{D}$ for fixed t.

- Now we show that $\mathcal{T}_1 v(t, x+s, m+s)$ is increasing in s. Therefore consider

$$\begin{aligned} v(u, x+s+az, \max\{m+s, x+s+az\}) &= v(u, x'+s, \max\{m+s, x'+s\}) \\ &= v(u, x_1'+s, m'+s\}) , \end{aligned}$$

where $x' := x+az$ and $m' := \max\{x', m\}$. Then it follows, that $L_1 v(t, x+s, m+s)$ is increasing in s for fixed a. Furthermore

$$[0, x+s-\beta(m+s)] = [0, x-\beta m + (1-\beta)s] ,$$

expands when s rises and so

$$\mathcal{T}_1 v(t, x+s, m+s)$$

is increasing in s.

- Obviously $\mathcal{T}_1 v$ is decreasing in m and increasing in x.

Step 4:

We show: $\tilde{\mathbb{M}}$ is closed in $(\mathbb{B}_b, d)$.

Proof:

Let $\{v_n \in \tilde{\mathbb{M}}, n \ge 0\}$ be a convergent sequence in $\mathbb{B}_b$. We have to show, that

$$\lim_{n\to\infty} v_n \in \tilde{\mathbb{M}} .$$

Since uniform convergence imply pointwise convergence, we obtain by (Rockafellar, 1972, Theorem 10.8) that

$$\lim_{n\to\infty} v_n$$

is a concave function on $\mathcal{D}$ for fixed t. Since

$$v_n(t,x,m) \ge v_n(t,x,m') ,$$

when $m' \geq m$, it follows that $\lim_{n\to\infty} v_n$ is decreasing in m. Repeating exactly the same steps, the remaining properties of $\lim_{n\to\infty} v_n$ follow.

Hence $\tilde{\mathbb{M}}$ is closed.

Step 5:

By (Bäuerle & Rieder, 2011, Theorem 7.3.5) we have

$$V_1 = \lim_{n\to\infty} \mathcal{T}_1^n g \qquad g \in \tilde{\mathbb{M}} \ .$$

Since $|U(x)| \leq C_U(1+x^p) + C'_U(1+x^{p'}) \leq Cb(t,x)$ for some constant $C > 0$, we get $U \in \tilde{\mathbb{M}}$ and hence

$$V_1 = \lim_{n\to\infty} \mathcal{T}_1^n g \qquad g \in \mathbb{M} \ .$$

Moreover, $\mathcal{T}_1 g \geq \mathcal{T}_1 U \geq U$ for $g \in \mathbb{M}$. This yields $\mathcal{T}_1^n g \geq U$, which implies $V_1 \in \mathbb{M}$.

□

4.4. Howard's policy improvement algorithm

Due to Theorem 4.10 it is enough to maximize $V_{1,\pi}$ over all stationary policies. Therefore, it satisfies to find the best decision rule. For that purpose we can make use of Howard's policy improvement algorithm. This algorithm improves in each run an arbitrary decision rule. If there is no further improvement, then we have found a decision rule, such that the corresponding stationary policy is an optimal one.

Now let f be a decision rule and $\pi := (f, f, ...)$ be the corresponding stationary policy. Then we have

$$V_{1,\pi}(y) = \lim_{n\to\infty} (\mathcal{T}_{1,f}^n U)(y) =: J_f(y) \quad \forall y \in E \ .$$

Note that the function J_f is well-defined by (Bäuerle & Rieder, 2011, Lemma 7.3.3).

We state now the theorem, on which Howard's policy improvement algorithm is based:

Theorem 4.11 Let f and g be two decision rules. For $y \in E$, we define

$$D(y,f) := \left\{ a \in D(y) \;\middle|\; L_1 J_f(y,a) > J_f(y) \right\} \ .$$

a) If it holds for some measurable subset $E_0 \subset E$
   - $g(y) \in D(y,f)$, for $y \in E_0$,
   - $g(y) = f(y)$, for $y \notin E_0$,

   then we have

– $J_g \geq J_f$, for $y \in E$,

– $J_g(y) > J_f(y)$, for $y \in E_0$.

The decision rule g is then called an improvement of f.

b) If $D(y,f) = \emptyset$ for all $y \in E$, then $J_f = V_1 = V$ and $\pi := (f, f, f, ...)$ is an optimal stationary policy.

A proof of Theorem 4.11 can be found in Bäuerle & Rieder (2009) or Bäuerle & Rieder (2011).

Howard's policy improvement algorithm:

1. Choose an arbitrary decision rule f_0 and set $k = 0$.

2. Compute J_{f_k} and determine $D(y, f_k)$ for all $y \in E$.

3. – If $D(y, f_k) = \emptyset$ for all $y \in E \Rightarrow$ STOP: $V = V_1 = J_{f_k}$ and the stationary policy $\pi := (f_k, f_k, ...)$ is optimal.

   – If $D(y, f_k) \neq \emptyset$ for some $y \in E \Rightarrow$ Compute an improvement f_{k+1} of f_k and set $k = k + 1$ and go to step 2.

Furthermore, if Howard's policy improvement algorithm does not terminate, then the algorithm generates a sequence of decision rules f_k, such that

$$\lim_{k \to \infty} J_{f_k} = V_1 = V \ .$$

For a proof see (Bäuerle & Rieder, 2011, Corollary 7.5.3).

4.5. Separation ansatz for CRRA utility functions

In this section we show, that we may separate the value function V_1 in the case of CRRA utility functions.

Proposition 4.12
In the case of a *Power Utility* function, there exists a function $F : [0, T] \times (0, \frac{1}{\beta}) \to \mathbb{R}$ such that $V_1(y) = U(x) \cdot F(t, \frac{m}{x})$ for $y \in E$.

Proof:

First, we show inductively that $(\mathcal{T}_1^n U)(y) = U(x) \cdot F_n(t, \frac{m}{x})$ for some function

$$F_n : [0,T] \times (0, \tfrac{1}{\beta}) \to \mathbb{R} \ .$$

Now let $v_n(y) := (\mathcal{T}_1^n U)(y)$ for $y \in E$. Then we get

$$\begin{aligned}
v_1 = (\mathcal{T}_1 U)(y) = & \sup_{a \in [0, x-\beta m]} e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x+az) p(t,T,dz) \\
& \qquad + \int_t^T \int_{(-1,\infty)} U(x+az) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
= U(x) & \sup_{a \in [0, 1-\frac{\beta m}{x}]} e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} (1+az)^\alpha p(t,T,dz) \\
& \qquad + \int_t^T \int_{(-1,\infty)} (1+az)^\alpha p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
=: U(x) & \cdot F_1(t, \tfrac{m}{x}) \ .
\end{aligned}$$

The induction hypothesis yields

$$\begin{aligned}
v_{n+1} = (\mathcal{T}_1 v_n)(y) = & \sup_{a \in [0, x-\beta m]} e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} U(x+az) p(t,T,dz) \\
& + \int_t^T \int_{(-1,\infty)} v_n(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
= U(x) & \sup_{a \in [0, 1-\frac{\beta m}{x}]} e^{-\int_t^T \lambda_u du} \int_{(-1,\infty)} (1+az)^\alpha p(t,T,dz) \\
& + \int_t^T \int_{(-1,\infty)} (1+az)^\alpha F_n(u, \tfrac{\max\{m/x, 1+az\}}{1+az}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\
=: U(x) & \cdot F_{n+1}(t, \tfrac{m}{x}) \ .
\end{aligned}$$

Since v_n converges to V_1, it follows that F_n converges to a function F such that

$$V_1(y) = U(x) \cdot F(t, \tfrac{m}{x})$$

for $y \in E$. □

By following the proof of Proposition 4.12, we can also show the following corollary.

Corollary 4.13
In the case of a *Logarithmic Utility* function, there exists a function $F : [0,T] \times (0, \frac{1}{\beta}) \to \mathbb{R}$ such that $V_1(y) = U(x) + F(t, \frac{m}{x})$ for $y \in E$.

Remark 4.14 Note that in the case of a CRRA utility function, the maximizer of V_1 depends only on the time t and the ratio $\frac{m}{x}$.

5. A terminal wealth problem with an unbounded intensity process

In this chapter, we solve the terminal wealth problem (3.3) under the assumption of an unbounded intensity process (λ_t). Under this assumption, it turns out that the terminal wealth is not well-defined at time T. However, since the stock price (S_t) is stochastically continuous, we can solve this issue by considering the left sided limit of the wealth process at time T. As in the chapter above, the considered optimization problem can be reduced to a discrete-time problem, a so called limsup Markov Decision Process, by means of which we can compute the value function and the optimal policy. The main results of this chapter are the following:

- The value function can be characterized by the unique fixed point of the maximal reward operator, which satisfies some additional conditions.
- There is an optimal stationary policy.

Similar to the results of the previous chapter, there exists also a separation ansatz for the value function under a CRRA utility function.

The outline of this chapter is the following: We start by formalizing our assumptions and develop some properties of the considered terminal wealth problem. Then, in Section 5.2, we describe the optimization problem with the aid of a limsup-MDP. In Section 5.3, we state and proof the main results which show how to solve the original problem by means of the limsup-MDP. In the last section, we derive a separation ansatz for the value function under a CRRA utility function.

5.1. Assumptions and properties of the model

All assumptions stated in this section will stand in force for the rest of this chapter. We start by assuming an unbounded intensity process (λ_t):

Assumption 5.1
The intensity process (λ_t) of the inhomogeneous Poisson process (N_t) satisfies

- $\lambda : [0, T) \to (0, \infty)$,
- $\int_0^t \lambda_u du < \infty, \ \forall t < T$, $\qquad \int_0^T \lambda_u du = \infty$.

Because of that, the investor observes and adjusts his portfolio always infinitely many times during his investment period.

Proposition 5.2
The exogenous random times (τ_n) converge increasingly to T, i.e. $\tau_n \nearrow T$.

Proof:

Let $\Lambda : [0, T) \to [0, \infty)$ with

$$\Lambda(t) := \int_0^t \lambda_u du \ .$$

Note that the function Λ is a bijection and both Λ and the inverse of Λ, denoted by Λ^{-1}, are continuous. According to Section 2.5, we have

$$N_t = \tilde{N}_{\int_0^t \lambda_u du} = \tilde{N}_{\Lambda(t)} \ , \qquad 0 \le t < T \ ,$$

where $(\tilde{N}_t)$ is a homogeneous Poisson process with intensity equals to 1. Let $\tilde{\tau}_n$ be the n-th jump time of the homogeneous Poisson process $(\tilde{N}_t)$. Hence, the inhomogeneous Poisson process (N_t) jumps at time $\Lambda^{-1}(\tilde{\tau}_n) = \tau_n$. Since the jump times of the homogeneous Poisson process $\tilde{N}$ converge increasingly to infinity, it follows that τ_n is increasing and

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \Lambda^{-1}(\tilde{\tau}_n) = \Lambda^{-1}(\lim_{n\to\infty} \tilde{\tau}_n) = T \ .$$

□

Since the observation and trading times always lie in the interval $[0, T)$, we may restrict the time parameters of the stochastic processes to $[0, T]$. This is done in the following, hence we always consider the restricted time parameter $t \in [0, T]$.

Now, let π be an admissible policy and

$$\tilde{\pi}_t = \sum_{n=0}^{\infty} \frac{a_n}{S_{\tau_n}} \cdot \mathbb{1}_{\{\tau_n < t \le \tau_{n+1}\}}$$

be the number of stocks, which the investor owns at time t. Since the exogenous random times (τ_n) converge increasingly to T, the càglàd process $(\tilde{\pi}_t)$ has domain $[0, T)$ and so the wealth process (X_t^π) is only determined for $t \in [0, T)$. Hence, it is not obvious whether the terminal wealth X_T^π is well-defined.

The next step is concerned with a solution of that problem. Since the stock price (S_t) is stochastically continuous, it follows that $S_T = S_{T-}$. On the other hand the stochastic integral with respect to (S_t) jumps if and only if (S_t) jumps, thus we may write

$$\int_0^T \tilde{\pi}_s dS_s = \lim_{t \to T} \int_0^t \tilde{\pi}_s dS_s \ ,$$

whenever the right hand side exists. This justifies to define

$$X_T^{\pi} := x + \int_0^T \tilde{\pi}_s dS_s = x + \lim_{t \to T} \int_0^t \tilde{\pi}_s dS_s \ .$$

Because of that, we have to guarantee the existence of $\lim_{t \to T} \int_0^t \tilde{\pi}_s dS_s$. This can be done by introducing an assumption. Yet, before we can state this assumption, we first have to remember that one can represent the inhomogeneous Lévy process (L_t) in the following way

$$L_t = \int_0^t b'_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx) \ .$$

Assumption 5.3

The characteristics of the inhomogeneous Lévy process (L_t) are determined in such a way that $c_s > 0 \ , \forall s \geq 0$ and

$$\frac{(b'_s)^2}{c_s} ds \leq C_L^{NA} \ , \qquad \forall s \geq 0 \ .$$

for some positive constant C_L^{NA}.

Theorem 5.4

Let (X_t^{π}) be the wealth process under the policy $\pi \in \mathcal{A}(y)$, which is given by

$$X_t^{\pi} = x + \int_0^t \tilde{\pi}_u dS_u \ , \qquad 0 \leq t < T \ .$$

Then X_T^{π} exists and we have

$$X_T^{\pi} = \lim_{n \to \infty} X_{\tau_n}^{\pi} \ .$$

Proof:

We begin by considering the canonical representation of (L_t):

$$L_t = L_0 + B_t + L_t^c + \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} (\mu^L - \nu)(ds, dx) + \sum_{s \leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > 1\}} \ .$$

By Theorem B.15

$$L_0 + B_t + L_t^c \quad \text{and} \quad \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} (\mu^L - \nu)(ds, dx) + \sum_{s \leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > 1\}}$$

are independent. Since $\int_0^t \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x| \leq 1\}}) F_s(dx) ds$ is deterministic, it follows that

$$L_0 + \int_0^t b'_s ds + \int_0^t \sqrt{c_s} dW_s \quad \text{and} \quad \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx) \ ,$$

are independent. Furthermore,

$$\int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(ds, dx)$$

is a local martingale.

Now we define an equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(- \int_0^T \frac{b'_s}{\sqrt{c_s}} dW_s - \frac{1}{2} \int_0^T \frac{b'^2_s}{c_s} ds \right) .$$

Under the measure $\mathbb{Q}$,

$$L_0 + \int_0^t b'_s ds + \int_0^t \sqrt{c_s} dW_s$$

is a martingale and

$$\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} (\mu^L - \nu)(ds, dx)$$

remains to be a local martingale, since it is independent of W. This shows, that (L_t) is a local martingale under the equivalent measure $\mathbb{Q}$.

Let $\pi \in \mathcal{A}(y)$. Then the associated wealth process (X_t^π) is given by

$$X_t^\pi = x + \int_0^t \tilde{\pi}_s dS_s \ , \qquad t < T \ .$$

It follows that (X_t^π) is a local martingale under the equivalent probability measure $\mathbb{Q}$, since (L_t) being a local martingale implies that (S_t) is one, too. Because (X_t^π) is positive, we have by (Pham, 2009, Proposition 1.1.7), that (X_t^π) is a supermartingale, which converges to a finite limit X_∞^π by (Jacod & Shiryaev, 2003, Th. I.1.39). On the other hand $\tau_n \nearrow T$ when n goes to infinity, yields

$$X_{\tau_n}^\pi \longrightarrow X_\infty^\pi \ , \qquad (n \to \infty) \ .$$

Due to $\mathbb{P} \sim \mathbb{Q}$, we finally obtain under $\mathbb{P}$

$$\lim_{n \to \infty} X_{\tau_n}^\pi = \lim_{t \nearrow T} X_t^\pi = X_\infty^\pi \ .$$

□

At the end of this section, we state the some technical assumptions on the utility function U and the inhomogeneous Lévy process (L_t), which will be needed in the course of this chapter. More precisely, they ensure that the value function is uniformly integrable.

Assumption 5.5
The Fenchel-Legendre transform $\tilde{U}$ of the utility function U has domain $(0, \infty)$, i.e.

$$dom(\tilde{U}) = (0, \infty) \ .$$

Note that Assumption 5.5 is satisfied, if U satisfies the Inada conditions, i.e.

$$U'(0+) = \infty \qquad \text{and} \qquad U'(\infty) = 0 \ .$$

Assumption 5.6

(i) There exist some constants $C_L^3 > 0$ and $q > 1$ such that

$$\int_{(-1,\infty)} \sup_{\pi\in[0,1]} \big[(1+\pi x)^q - 1 - q\pi x\big] F_s(dx) \le C_L^3 \ , \quad \forall s \in [0,T] \ .$$

(ii) If $U(0) = -\infty$, then in addition to part (i), there exists a $r < p'$ such that

$$\int_{(-1,\infty)} \sup_{\pi\in[0,1]} \big[(1+\pi x)^r - 1 - r\pi x\big] F_s(dx) \le C_L^3 \ , \quad \forall s \in [0,T] \ ,$$

Analogously to Proposition 4.5, we can derive sufficient conditions, under which Assumptions 5.6 (i) and (ii) are satisfied.

Proposition 5.7

a) Assumption 5.6 (i) is satisfied, if

$$\int_{(-1,\infty)} \Big((1+y)^q - 1 - qy\Big) F_s(dy) \le C_L^3 \ , \quad \forall s \in [0,T] \ .$$

b) Assumption 5.6 (ii) is satisfied, if

$$\int_{(-1,\infty)} \Big((1+y)^r - 1 - ry\Big) F_s(dy) \le C_L^3 \ , \quad \forall s \in [0,T] \ .$$

5.2. Solution via limsup Markov Decision Process

As in the previous chapter, the terminal wealth problem (3.3) can be regarded as a optimization problem in discrete-time. Since

$$\mathbb{E}_y\Big[U\big(X_T^\pi\big)\Big] = \mathbb{E}_y\Big[\lim_{n\to\infty} U\big(X_{\tau_n}^\pi\big)\Big] \tag{5.1}$$

we can reformulate the problem as a limsup Markov Decision Process (short: limsup-MDP).

In what follows, we formulate a limsup MDP by means of which we can compute the value function V and the optimal policy of the continuous-time terminal wealth problem (3.3). For more details on limsup models, we refer to (Schäl, 1990, §18).

Let us now consider the following limsup-MDP:

limsup Markov Decision Process

- State space $E = \{(t,x,m) : t \in [0,T), x \in (0,\infty), m \in (0, \frac{x}{\beta})\}$ endowed with the Borel σ-algebra $\mathcal{B}(E)$, where t will be the exogenous random time, x the current wealth and m the current value of the process M. Moreover, the state process is denoted by $Y_n = (T_n, X_{T_n}, M_{T_n})$.
- Action space $A = [0,\infty)$ endowed with the Borel σ-algebra $\mathcal{B}(A)$.
- The possible state-action combinations are given by

  $$D = \{(y,a) : y \in E\ , a \in [0, x-\beta m]\} \subset E \times A$$

  and the admissible actions are given by

  $$D(y) = \{a \in A \mid (y,a) \in D\} = [0, x-\beta m]\ , \quad \forall y \in E\ .$$

- The stochastic transition kernel Q from D to E is given by

  $$Q(B \mid (y,a)) = \int_t^T \int_{(-1,\infty)} \mathbb{1}_B(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du\ ,$$

  where $y \in E$, $a \in D(y)$ and $B \in \mathcal{B}(E)$.
- Terminal reward $g : E \to \mathbb{R}$ with $g(y) = U(x)$.

Using a Markovian policy π as in Chapter 4, we can now define the gain corresponding to π with start in the state $y \in E$ by

$$V_{2,\pi}(y) := \mathbb{E}_y^\pi\Big[\lim_{n\to\infty} U\big(X_{T_n}\big)\Big]\ .$$

Furthermore, the value function of the limsup-MDP is given by

$$V_2(y) = \sup_{\pi \in \Pi} V_{2,\pi}(y) \quad \forall y \in E \; , \tag{5.2}$$

where Π is the set of all Markovian policies.

By the same arguments as in Chapter 4, we can show that V and V_2 coincide and that the optimal policy of the limsup-MDP is an optimal policy for (3.3).

Theorem 5.8
Let $y \in E$. Then we have

a) For a Markovian policy $\pi = (f_0, f_1, f_2, ..) \in \mathcal{A}(y)$ it holds:

$$\mathbb{E}_y\big[U(X_T^\pi)\big] = V_{2,\pi}(y) \; , \quad y \in E \; .$$

b) Moreover, we have: $\quad V(y) = V_2(y) \; , \quad y \in E.$

Due to that theorem, we can from now on concentrate on the limsup-MDP, which we are going to solve by using the *Structure Theorem* (A.1) for limsup-MDP's.

5.3. Main Results

In this section we present the solution of the terminal wealth problem, which we consider in this chapter. To proof these results we will follow the ideas of Gassiat et al. (2011). Yet, before we can state the main results, we have to introduce some notations. We begin with the function h.

Definition 5.9
We define $h : [0, T] \times (0, \infty) \to \mathbb{R}$ by

$$h(t, x) := \inf_{y>0} \Big\{ \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy \Big\} \; ,$$

where

$$Y_{t,T} := e^{-\int_t^T \frac{b'_u}{\sqrt{c_u}} dW_u - \frac{1}{2}\int_t^T \frac{b'^2_u}{c_u} du} \; ,$$

(W_t) is a standard Brownian Motion, $\tilde{U}$ is the Fenchel-Legendre transform of the utility function U and (b'_s), c_s are determined form the canonical representation of the inhomogeneous Lévy process (L_t).

Proposition 5.10

Let h be the function defined in Definition 5.9.

a) Let $b(t,x) = e^{\gamma(T-t)}(1 + x + x^{p'})$. Then there exists a positive constant C and a $\gamma > 0$ such that

$$U(x) \leq h(t,x) \leq Cb(t,x) \quad \forall (t,x) \in [0,T] \times (0,\infty) \ .$$

b) $h(t,\cdot)$ is concave on $(0,\infty)$ for fixed t.

c) For $y \in E$, it holds

$$\sup_{a\in[0,x-\beta m]} \int_t^T \int_{(-1,\infty)} h(u, x + az)p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \leq h(t,x) \ .$$

d) $\lim_{t\nearrow T, x'\to x} h(t,x') = U(x)$.

<u>Proof:</u>

a) We have

$$\int_t^T \frac{b'_u}{\sqrt{c_u}} dW_u = \int_0^{T-t} \frac{b'_{t+u}}{\sqrt{c_{u+t}}} d(W_{u+t} - W_t + W_t) = \int_0^{T-t} \frac{b'_{t+u}}{\sqrt{c_{u+t}}} dW_u \ ,$$

since $u \to W_{u+t} - W_t$ is a Brownian Motion for fixed $t \geq 0$. Hence,

$$\int_t^T \frac{b'_u}{\sqrt{c_u}} dW_u$$

is normal distributed with mean 0 and variance $\int_t^T \frac{b'^2_u}{c_u} du$. This yields $\mathbb{E}[Y_{t,T}] = 1$. Using Jensen's inequality, we obtain

$$\begin{aligned} h(t,x) &= \inf_{y>0} \left\{ \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy \right\} \geq \inf_{y>0} \left\{ \tilde{U}(y\mathbb{E}[Y_{t,T}]) + xy \right\} \\ &= \inf_{y>0} \left\{ \tilde{U}(y) + xy \right\} = U(x) \ . \end{aligned}$$

For $y > 0$ consider now

$$\tilde{U}(y) = \sup_{x>0} \big\{ U(x) - xy \big\} \leq \sup_{x>0} \big\{ C_U(1 + x^p) - xy \big\} \ .$$

Computing the right hand side yields

$$\tilde{U}(y) \leq C'(1 + y^{-\frac{p}{1-p}}) \ ,$$

for some large constant C'. Hence

$$\mathbb{E}\big[\tilde{U}(Y_{t,T})\big] \le C'(1 + \mathbb{E}(Y_{t,T})^{-\frac{p}{1-p}}) \ .$$

Using the moment generating function of a normal distribution, we get

$$\begin{aligned}
\mathbb{E}(Y_{t,T})^{-\frac{p}{1-p}} &= \mathbb{E}\left[e^{\frac{p}{1-p}\int_t^T \frac{b'_u}{\sqrt{c_u}}dW_u + \frac{p}{2(1-p)}\int_0^T \frac{b'^2_u}{c_u}du}\right] \\
&= e^{\frac{p}{2(1-p)}\int_t^T \frac{b'^2_u}{c_u}du}\mathbb{E}\left[e^{\frac{p}{1-p}\int_t^T \frac{b'_u}{\sqrt{c_u}}dW_u}\right] \\
&= e^{\frac{p}{2(1-p)}\int_t^T \frac{b'^2_u}{c_u}du}e^{\frac{p^2}{2(1-p)^2}\int_t^T \frac{b'^2_u}{c_u}du} \\
&= e^{\frac{p}{2(1-p)}(1+\frac{p}{1-p})C_L^{NA}(T-t)} \ .
\end{aligned}$$

It follows

$$\begin{aligned}
h(t,x) = \inf_{y>0}\left\{\mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy\right\} &\le \mathbb{E}\big[\tilde{U}(Y_{t,T})\big] + x \le C'(1 + \mathbb{E}(Y_{t,T})^{-\frac{p}{1-p}}) + x \\
&\le C'(1 + e^{\frac{p}{2(1-p)}(1+\frac{p}{1-p})C_L^{NA}(T-t)}) + x \le 2C'e^{\gamma(T-t)}(1 + x + x^{p'}) = 2C'b(t,x) \ ,
\end{aligned}$$

for some large $\gamma > 0$.

b) Using (Rockafellar, 1972, Theorem 5.5), we easily see, that $b(t,\cdot)$ is a concave function for fixed t.

c) Now fix $0 \le t \le u \le T$ and $x \in (0,\infty)$. Furthermore, fix $y > 0$ and $a \in [x - \beta m]$. By definition of h, we have

$$\begin{aligned}
\mathbb{E}\big[h(u, x + aZ_{t,u})\big] &\le \mathbb{E}\big[\tilde{U}(yY_{t,u}Y_{u,T}) + x(1 + \tfrac{a}{x}Z_{t,u})yY_{t,u}\big] \\
&= \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + \mathbb{E}\big[x(1 + \tfrac{a}{x}Z_{t,u})yY_{t,u}\big] \\
&= \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy\mathbb{E}\big[Y_{t,u} + \tfrac{a}{x}Z_{t,u}Y_{t,u}\big] \\
&= \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy\big[\underbrace{\mathbb{E}Y_{t,u}}_{=1} + \tfrac{a}{x}\mathbb{E}(Z_{t,u}Y_{t,u})\big] \ .
\end{aligned}$$

Since (S_t) is a $\mathbb{Q}$ supermatingale, we get by (Klebaner, 2005, Theorem 10.10)

$$\begin{aligned}
\mathbb{E}\big[Z_{t,u}Y_{t,u}\big] &= \mathbb{E}\Big[\mathbb{E}\big[Z_{t,u}Y_{t,u}|\mathcal{F}_t\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[Z_{t,u}\tfrac{Y_{0,u}}{Y_{0,t}}|\mathcal{F}_t\big]\Big] = \mathbb{E}\Big[\mathbb{E}_{\mathbb{Q}}\big[Z_{t,u}|\mathcal{F}_t\big]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}_{\mathbb{Q}}\big[\frac{S_u - S_t}{S_t}|\mathcal{F}_t\big]\Big] \le 0 \ .
\end{aligned}$$

Therefore

$$\mathbb{E}\big[h(u, x + aZ_{t,u}))\big] \le \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] + xy \ .$$

It follows that

$$\mathbb{E}\big[h(u, x + aZ_{t,u})\big] = \int_{(-1,\infty)} h(u, x + az)p(t,u,dz) \le h(t,x)$$

and so

$$\int_t^T \int_{(-1,\infty)} h(u, x + az) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \leq h(t, x) \; .$$

This yields

$$\sup_{a \in [0, x - \beta m]} \int_t^T \int_{(-1,\infty)} h(u, x + az) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \leq h(t, x) \; .$$

d) Fix a $y > 0$. Then we have for a large constant C'

$$\tilde{U}^+(y) \leq C'(1 + y^{-\frac{p}{1-p}}) \; .$$

Moreover,

$$\tilde{U}(y) \geq \sup_{x>0} \big\{ - C'_U(1 + x^{p'}) - xy \big\} \; .$$

Computing the right hand side yields

$$\tilde{U}(y) \geq -C''(1 + y^{\frac{p'}{p'-1}})$$

for some large constant C'' and so

$$\tilde{U}^-(y) \leq C''(1 + y^{\frac{p'}{p'-1}}) \; .$$

With

$$\mathbb{E}(yY_{t,T})^z = y^z e^{-\frac{z}{2}\int_t^T \frac{b_u'^2}{c_u} du} e^{\frac{z^2}{2}\int_t^T \frac{b_u'^2}{c_u} du} \leq y^z e^{\frac{|z|}{2} C_L^{NA} T + \frac{z^2}{2} C_L^{NA} T} < \infty \; ,$$

for $z \in \mathbb{R}$, we finally get

$$\sup_{0 \leq t < T} \mathbb{E}\big[|\tilde{U}(yY_{t,T})|^2\big] < \infty \; .$$

By (Protter, 2005, Th. I.11) $\big(\tilde{U}(yY_{t,T})\big)_{t\in[0,T)}$ is uniformly integrable, which yields

$$\lim_{t \nearrow T} \mathbb{E}\big[\tilde{U}(yY_{t,T})\big] = \mathbb{E}\big[\tilde{U}(\lim_{t \nearrow T} yY_{t,T})\big] = \tilde{U}(y) \; .$$

Using the definition of h, we get

$$U(x) \leq \liminf_{t \nearrow T, x' \to x} h(t, x') \leq \limsup_{t \nearrow T, x' \to x} h(t, x') \leq \tilde{U}(y) + xy \; , \quad \forall y > 0 \; .$$

Hence

$$U(x) = \lim_{t \nearrow T, x' \to x} h(t, x') \; .$$

$\square$

Now we introduce the set of functions

$$\mathbb{B}_b := \{v : E \to \mathbb{R} \mid v \text{ is measurable}, \exists C \in \mathbb{R}_+ : |v(t,x,m)| \le Cb(t,x)\} ,$$

consisting of all measurable functions, which are bounded by the bounding function b.

Moreover, we define for $v \in \mathbb{B}_b$ the operator $(L_2 v) : D \to \mathbb{R}$ by

$$(L_2 v)(y,a) := \int_t^T \int_{(-1,\infty)} v(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du ,$$

for a decision rule f the operator $\mathcal{T}_{2,f} : E \to \mathbb{R}$ by

$$(\mathcal{T}_{2,f} v)(y) := (L_2 v)(y, f(y)) ,$$

and the terminal reward operator $\mathcal{T}_2 : E \to \mathbb{R} \cup \{\infty\}$ by

$$(\mathcal{T}_2 v)(y) := \sup_{a \in [0, x-\beta m]} (L_2 v)(y,a) .$$

In the following theorem we will show, that the value function is included in the subsequent set of of functions.

Definition 5.11
Let $\mathbb{M}'$ be the subset of $\mathbb{B}_b$, such that for each $v \in \mathbb{M}'$ the following conditions hold:

(i) $U(x) \le v(y) \le h(t,x)$, $\quad \forall y \in E$.

(ii) $v(t,\cdot,\cdot)$ is concave on $\mathcal{D} := \{(x,m) : x \in (0,\infty), m \in (0, \frac{x}{\beta})\}$, for fixed t.

(iii) $v(t,x,\cdot)$ is decreasing on $(0, \frac{x}{\beta})$ for fixed t and fixed x.

(iv) $v(t,\cdot,m)$ is increasing on $(\beta m, \infty)$ for fixed t and fixed m,

(v) The function
$$s \to v(t, x+s, m+s)$$
is increasing on $[0,\infty)$ for fixed $y \in E$.

Now we are able to present and proof the main results of this chapter.

Theorem 5.12

a) $V = V_2 \in \mathbb{M}'$ and V_2 is the unique fixed point of $\mathcal{T}_2$ in $\mathbb{M}'$, which satisfies
   - $\lim_{n\to\infty} V_2(Y_n) = \limsup_{n\to\infty} g(Y_n) \qquad \mathbb{P}_y^\pi - (a.s.)$.
   - $\big(V_2(Y_n)\big)_{n\geq 0}$ is $\mathbb{P}_y^\pi$-uniformly integrable for all $\pi \in \Pi$ and $y \in E$.

b) $V_2 = \lim_{n\to\infty} \mathcal{T}_2^n g$ for $g \in \mathbb{M}'$.

c) There exists a maximizer f^* of V_2 and each maximizer of V_2 defines an optimal stationary policy $\pi := (f^*, f^*, f^*, ...)$.

Proof:

We will proof the theorem by using the *Structure Theorem* A.1. Therefore, the first part consists of showing that the assumptions (i) to (v) of Theorem A.1 are fulfilled for $\mathbb{M}' \subset \mathbb{B}_b'$.

(i): $U \in \mathbb{M}'$ is obviously satisfied.

(iii): Let $v \in \mathbb{M}'$. We show that $(L_2 v)$ is an upper semicontinuous function on $[0, x - \beta m]$ for fixed y. To do so, let $(c_n)_{n\in\mathbb{N}}$ be a convergent sequence in $[0, x-\beta m]$ with limit c_0. Since $v \leq h \leq b$, we can apply Fatou's Lemma and get, by using the continuity of v on $\mathcal{D}$,

$$\begin{aligned}&\limsup_{n\to\infty} \int_t^T \int_{(-1,\infty)} v(u, x + c_n z, \max\{m, x + c_n z\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq \int_t^T \int_{(-1,\infty)} v(u, x + c_0 z, \max\{m, x + c_0 z\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \ .\end{aligned}$$

It follows that $(L_2 v)$ is an upper semicontinuous function on $[0, x - \beta m]$ for fixed y. Since

$$\begin{aligned}&(L_2 v)(y, a) \\ &= \int_t^T \int_{(-1,\infty)} v(u, x(1 + \frac{a}{x} z), \max\{m, x(1 + \frac{a}{x} z\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du\end{aligned}$$

and $\frac{a}{x} \in [0,1]$, we can find by (Bertsekas & Shreve, 1978, Proposition 7.33) a decision rule f, which is a maximizer of v.

(ii): Let $v \in \mathbb{M}'$.

Step 1: For $y \in E$ we have $\mathcal{T}_2 v(y) \leq \mathcal{T}_2 h(t,x) \leq h(t,x)$.

Step 2: For $y \in E$ we have

$$\mathcal{T}_2 v(y) \geq \int_t^T \int_{(-1,\infty)} U(x) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du = U(x) \ .$$

Step 3: The remaining properties of $(\mathcal{T}_2 v)$ follow analogously to *Step 2* of the proof of Theorem 4.10.

(iv): ***Step 1:*** Let $(v_m)_{m\in\mathbb{N}_0}$ be recursively defined by

$$v_0 := U \ , \quad v_{m+1} := \mathcal{T}_2 v_m \quad \forall m \geq 0 \ .$$

Now we show that

$$v_m \leq v_{m+1} \leq h \ , \quad m \geq 0 \ .$$

Therefore we first proof by induction that $v_m \geq v_{m-1}$. Let $m = 1$. Then

$$v_1(y) = \sup_{a\in[0,x-\beta m]} \int_t^T \int_{(-1,\infty)} U(x+az)p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \geq U(x) \ .$$

Now let $m \geq 1$ be arbitrary. By the induction hypothesis we have $v_m \geq v_{m-1}$ and hence we get

$$v_{m+1} = \mathcal{T}_2 v_m \geq \mathcal{T}_2 v_{m-1} = v_m \ .$$

Next we show by induction, that

$$v_m \leq h \ , \quad \forall m \geq 0 \ .$$

Therefore let $m = 0$. From Proposition 5.10 we know that $U \leq h$. Next let $m \geq 1$ be arbitrary. Then we have, using the induction hypothesis,

$$v_{m+1} = \mathcal{T}_2 v_m \leq \mathcal{T}_2 h \leq h \ .$$

Step 2: Let $(\tilde{v}_m)_{m\in\mathbb{N}_0}$ be recursively defined by

$$\tilde{v}_0 := h \ , \quad \tilde{v}_{m+1} := \mathcal{T}_2 \tilde{v}_m \quad \forall m \geq 0 \ .$$

Now we show that

$$\tilde{v}_m \geq \tilde{v}_{m+1} \geq U \ , \quad m \geq 0 \ .$$

Therefore we first proof that $\tilde{v}_m \leq \tilde{v}_{m-1}$ by induction. Let $m = 1$, then

$$\begin{aligned} \tilde{v}_1(y) = &\sup_{a\in[0,x-\beta m]} \int_t^T \int_{(-1,\infty)} h(u, x+az)p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq h(t,x) \ . \end{aligned}$$

Now let $m \geq 1$ be arbitrary. By the induction hypothesis we have $\tilde{v}_m \leq \tilde{v}_{m-1}$ and so we get

$$\tilde{v}_{m+1} = \mathcal{T}_2 \tilde{v}_m \leq \mathcal{T}_2 \tilde{v}_{m-1} = \tilde{v}_m \ .$$

Next we show by induction, that

$$\tilde{v}_m \geq U \ , \quad \forall m \geq 0 \ .$$

Again choose first $m = 0$. From Proposition 5.10 we know that $U \leq h$. Now let $m \geq 1$ be arbitrary. By the induction hypothesis we obtain

$$\tilde{v}_{m+1} = \mathcal{T}_2 \tilde{v}_m \geq \mathcal{T}_2 U \geq U \ .$$

Step 3: Due to the monotonicity of v_n, we know that $v_n(y)$ is an increasing sequence for fixed $y \in E$. Since $v_n \leq h$, v_n converges pointwise to a function $v_\infty : E \to \mathbb{R}$. Analogously $\tilde{v}_n$ converges pointwise to a function $\tilde{v}_\infty : E \to \mathbb{R}$.

Step 4: In this step we show that $v_\infty, \tilde{v}_\infty \in \mathbb{M}'$. Since

$$v_\infty = \lim_{n\to\infty} v_n \ ,$$

v_∞ is measurable and $U \leq v_\infty \leq h$. According to (Rockafellar, 1972, Th. 10.8) the limit of finite concave functions is concave. Hence $v_\infty(t, \cdot, \cdot)$ is concave on $\mathcal{D}$. Now consider

$$v_\infty(t,x,m) = \lim_{n\to\infty} v_n(t,x,m) \geq \lim_{n\to\infty} v_n(t,x,m') = v_\infty(t,x,m') \ ,$$

i.e. v_∞ is deceasing in m. Repeating exactly the same steps, the remaining properties of v_∞ follow. Thus $v_\infty \in \mathbb{M}'$. The same arguments also show, that $\tilde{v}_\infty \in \mathbb{M}'$.

Step 5: We show that for fixed $(y, a) \in D$

$$\int_{(-1,\infty)} U^-(x+az) p(t,u,dz) < \infty \ ,$$

for $u \geq t$. Since $U^-(x+az) \leq C'_U(1 + (x+az)^{p'})$, we get

$$\mathbb{E}\big[U^-(x+az)\big] \leq C'_U(1 + \mathbb{E}\big[(x+az)^{p'}\big]) \ .$$

Because we can show, similarly to the proof of Proposition 4.8, that

$$\mathbb{E}(x+az)^r \leq x^r \cdot C_r < \infty \ ,$$

it follows that $\mathbb{E}(x+az)^{p'} < \infty$. Then the claim follows.

Step 6: Now we show that v_∞ is a fixed point of $\mathcal{T}_2$. Let $(y,a) \in D$. Then

$$v_{m+1}(y) \geq \int_t^T \int_{(-1,\infty)} v_m(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \ .$$

Due to $v_m \geq -U^-$ and *Step 5* we can use monotone convergence and obtain

$$v_\infty(y) \geq \int_t^T \int_{(-1,\infty)} v_\infty(u, x+az, \max\{m, x+az\}) p(t,u,dz) \lambda_u e^{-\int_t^u \lambda_s ds} du$$

and so $v_\infty(y) \geq \mathcal{T}_2 v_\infty(y)$. For the revised inequality choose $\epsilon > 0$. Then

$$v_\infty(y) - \epsilon \leq v_{m+1}(y)$$

for some m large enough and we obtain

$$v_\infty(y) - \epsilon \le \mathcal{T}_2 v_m(y) \le \mathcal{T}_2 v_\infty(y) \ .$$

Hence

$$v_\infty(y) \le \mathcal{T}_2 v_\infty(y) \ ,$$

which yields

$$v_\infty(y) = \mathcal{T}_2 v_\infty(y) \ .$$

Step 7: $\tilde{v}_\infty$ is also a fixed point of $\mathcal{T}_2$. To show that, let $(y, a) \in D$. Then

$$\tilde{v}_{m+1}(y) \le \int_t^T \int_{(-1,\infty)} \tilde{v}_m(u, x + az, \max\{m, x + az\}) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du \ .$$

Due to $\tilde{v}_m \le h$, we can use monotone convergence and get

$$\tilde{v}_\infty(y) \le \int_t^T \int_{(-1,\infty)} \tilde{v}_\infty(u, x + az, \max\{m, x + az\}) p(t, u, dz) \lambda_u e^{-\int_t^u \lambda_s ds} du$$

and hence $\tilde{v}_\infty(y) \le \mathcal{T}_2 \tilde{v}_\infty(y)$. For the revised inequality let $\epsilon > 0$. Then

$$\tilde{v}_\infty(y) + \epsilon \ge \tilde{v}_{m+1}(y)$$

for some m large enough and we obtain

$$\tilde{v}_\infty(y) + \epsilon \ge \mathcal{T}_2 \tilde{v}_m(y) \ge \mathcal{T}_2 \tilde{v}_\infty(y) \ .$$

Hence

$$\tilde{v}_\infty(y) \ge \mathcal{T}_2 \tilde{v}_\infty(y) \ ,$$

which yields

$$\tilde{v}_\infty(y) = \mathcal{T}_2 \tilde{v}_\infty(y) \ .$$

Step 8: Because of $U(x) \le v_\infty(y) \le h(t, x)$, we obtain with Proposition 5.10, that

$$\lim_{t \nearrow T, x \to x_0} v_\infty(t, x, m) = U(x_0) \ ,$$

which implies that $\lim_{n\to\infty} v_\infty(Y_n) = \lim_{n\to\infty} U(Y_n) \quad \mathbb{P}_x^\pi - (a.s.)$. The same arguments show that $\lim_{n\to\infty} \tilde{v}_\infty(Y_n) = \lim_{n\to\infty} U(Y_n) \quad \mathbb{P}_x^\pi - (a.s.)$.

Step 9: Let $y \in E$ and π be a Markovian policy. Because of the stochastic logarithm we get

$$\begin{aligned} X_u^\pi &= x + \int_0^u \tilde{\pi}_s dS_s = x + \int_0^u \tilde{\pi}_s \frac{S_{s-} X_{s-}^\pi}{S_{s-} X_{s-}^\pi} dS_s = x + \int_0^u \pi_s \frac{X_{s-}^\pi}{S_{s-}} dS_s \\ &= x + \int_0^u \pi_s X_{s-}^\pi dL_s \quad \forall u \in [0, T) \ , \end{aligned}$$

where $\pi_s = \frac{\tilde{\pi} S_{s-}}{X_{s-}}$ is an adapted càglàd process valued in $[0,1]$ and $\pi_s = 0$ on $[0,t]$. Now we need to extend Proposition 2.14 to stopping times. Therefore we apply Itô's formula with $f(x) = x^l$, where $l \in \{q, r\}$ and q, r were given in Assumption 5.6. This yields

$$\begin{aligned}
(X_u^\pi)^l = x^l &+ \int_0^u (X_{s-}^\pi)^l \big(l\pi_s b_s' + \tfrac{l(l-1)}{2} c_s \pi_s^2\big) ds \\
&+ \int_0^u \int_{-1}^\infty (X_{s-}^\pi)^l \big[(1+\pi_s y)^l - 1 - l\pi_s y\big] F_s(dy) ds \\
&+ \textit{local martingale} .
\end{aligned}$$

Now let $T_n^{'}$ be a fundamental sequence of stopping times for the local martingale above. Define

$$T_n := T_n^{'} \wedge \{\inf s : (X_s^\pi)^l \geq n\} .$$

Then we have $T_n \nearrow T$. Fix a stopping time τ with value in $[0,T)$. By Assumption 5.6 we obtain for $0 \leq t < T$

$$\begin{aligned}
\mathbb{E}[(X_{u\wedge\tau\wedge T_n}^\pi)^l] &\leq x^l + \mathbb{E}\bigg[\int_0^{u\wedge\tau\wedge T_n} (X_{s-}^\pi)^l \bigg\{\bigg(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s\bigg) ds + C_L^3 ds\bigg\}\bigg] \\
&\leq x^l + C \cdot T ,
\end{aligned}$$

for some constant $C > 0$. Using Fubini's Theorem, we obtain

$$\mathbb{E}[(X_{u\wedge\tau\wedge T_n}^\pi)^l] \leq x^l + \int_0^u \mathbb{E}\big[(X_{s\wedge\tau\wedge T_n}^\pi)^l\big] \big(|lb_s'| + \tfrac{|l(l-1)|}{2} c_s + C_L^3\big) ds .$$

Applying Gronwall's inequality yields

$$\mathbb{E}[(X_{u\wedge\tau\wedge T_n}^\pi)^l] \leq x^l \cdot C_l < \infty \quad \forall u \in [0,T) ,$$

where C_l is a constant depending on l. Now, using Fatou's Lemma, we finally get

$$\mathbb{E}[(X_\tau^\pi)^l] \leq x^l \cdot C_l < \infty ,$$

for each stopping time τ with value in $[0,T)$. Hence we have for $n \geq 0$

$$\mathbb{E}_y^\pi[(X_n)^l] = \mathbb{E}_y[(X_{\tau_m}^\pi)^l] \leq x^l \cdot C_l < \infty ,$$

for some $m \in \mathbb{N}_0$. Then by choosing $l = q$, $\big\{(X_n)_{n\geq 0}\big\}$ is $\mathbb{P}_y^\pi$-uniformly integrable by (Rogers & Williams, 2003, Lemma 20.5). Moreover, it follows that $h^+(T_n, X_n)$ is also uniformly integrable, since by (Gassiat et al., 2011, Lemma 3.3)

$$0 \leq h^+(T_n, X_n) \leq C(1 + X_n) .$$

If $U(0+) > -\infty$, then $\big\{U^-(X_n)\big\}$ is bounded and so uniformly integrable.

On the other hand, if $U(0+) = -\infty$, then we set $l = r$ and get $\big\{(X_n^{p'})_{n\geq 0}\big\}$ is $\mathbb{P}_y^\pi$-uniformly integrable. Since

$$0 \leq U^-(X_n) \leq C_U'(1 + X_n^{p'}) ,$$

it follows that $U^-(X_n)$ is uniformly integrable. Due to

$$0 \le |v_\infty| \le v_\infty^+ + v_\infty^- \le h^+ + U^- \quad \text{and} \quad 0 \le |\tilde{v}_\infty| \le \tilde{v}_\infty^+ + \tilde{v}_\infty^- \le h^+ + U^- ,$$

v_∞ and $\tilde{v}_\infty$ are $\mathbb{P}_y^\pi$-uniformly integrable.

Now it follows with Theorem A.1 that

- $V_2 \in \mathbb{M}'$ and V_2 is the unique fixed point of $\mathcal{T}_2$ in $\mathbb{M}'$, which satisfies
  (i) $\lim_{n\to\infty} V_2(Y_n) = \limsup_{n\to\infty} g(Y_n) \qquad \mathbb{P}_y^\pi - (a.s.)$.
  (ii) $\big(V_2(Y_n)\big)_{n\ge 0}$ is $\mathbb{P}_y^\pi$-uniformly integrable for all $\pi \in \Pi$ and $y \in E$.

  Moreover, $V_2 = \lim_{n\to\infty} \mathcal{T}^n U$.
- There exists a maximizer f^* of V_2 and each maximizer of V_2 defines an optimal stationary policy $\pi = (f^*, f^*, f^*, ...)$.

Since V_2 is the unique fixed point of $\mathcal{T}_2$ satisfying (i) and (ii), it follows that

$$V_2 = v_\infty = \tilde{v}_\infty .$$

Moreover, for $g \in \mathbb{M}'$, we have

$$\mathcal{T}_2^n U \le \mathcal{T}_2^n g \le \mathcal{T}_2^n h \qquad \forall n \ge 0 ,$$

which yields $V_2 = \lim_{n\to\infty} \mathcal{T}_2^n g$. □

5.4. Separation ansatz for CRRA utility functions

As in the previous chapter, we may separate the value function V_2 in the case of a CRRA utility function. Since the proofs can be done analogously to Section 4.5, we omit them.

Corollary 5.13
In the case of a *Power Utility* function, there exists a function $F : [0,T] \times (0, \frac{1}{\beta}) \to \mathbb{R}$ such that $V_2(y) = U(x) \cdot F(t, \frac{m}{x})$ for $y \in E$.

Corollary 5.14
In the case of a *Logarithmic Utility* function, there exists a function $F : [0,T] \times (0, \frac{1}{\beta}) \to \mathbb{R}$ such that $V_2(y) = U(x) + F(t, \frac{m}{x})$ for $y \in E$.

Remark 5.15 Note that in the case of a CRRA utility function, the maximizer of V_2 depends only on the time t and the ratio $\frac{m}{x}$.

6. Convergence Results

In this chapter we want to approximate the terminal wealth problem with an unbounded intensity process by a terminal wealth problem with a bounded one. The crucial idea on which this approximation is based, is the following: Consider a terminal wealth problem with an unbounded intensity process (λ_t). In Chapter 4, we have shown that $X_T = X_{T-}$. This means that the wealth process is almost surely continuous at time T. Therefore, if we consider a shortened horizon $T - \epsilon$ for some small $\epsilon > 0$, $X_{T-\epsilon}$ is close to X_T.
If the unbounded intensity process (λ_t) is bounded on $[0, T - \epsilon]$, then by considering the shortened horizon $T - \epsilon$, we are facing a terminal wealth problem in the framework of Chapter 4, which we can solve. When ϵ tends to zero, the approximation of X_T by $X_{T-\epsilon}$ becomes more precise and hence we can finally approximate the value function of the terminal wealth problem with an unbounded intensity process (λ_t) by the value function of a terminal wealth problem with a bounded intensity process. Additionally, this procedure enables us to approximate an optimal policy of the terminal wealth problem with an unbounded intensity process.

This chapter is organized as follows. We begin with some assumptions, under which we will proof the convergence results. Then, in Section 6.2, we will show that the value function of the terminal wealth problem with an unbounded intensity process can be approximated by the value function of a terminal wealth problem with a bounded intensity process. In the last section we will determine a sequence of policies, which converge to an optimal policy of the terminal wealth problem with an unbounded intensity process.

6.1. Assumptions

We introduce now some additional assumptions, which will be needed to proof the convergence results. Therefore, they will stand in force for the rest this chapter.

In addition to Chapter 5, we assume that the intensity process (λ_t) of the inhomogeneous Poisson process (N_t) satisfies the following assumption:

Assumption 6.1
For each $t \in [0, T)$ there exists a constant $C_t > 0$ depending on t, such that the intensity process (λ_t) satisfies $\lambda_s \in (0, C_t]$, $\forall s \in [0, t]$.

Moreover, we require assumptions on the financial market and the utility function U.

Assumption 6.2
The utility function U and the financial market are set up in such a way, that the function

$$u \to \int_{(-1,\infty)} U(x+az)p(t,u,dz)$$

is non-decreasing on $[t,T]$ for fixed $t \in [0,T]$, $x \in (0,\infty)$ and $a \in [0,x]$.

Remark 6.3 Assumption 6.2 ensures that the investor invests his money in a profitable financial market, since his expected utility is non-decreasing in the time parameter.

By using Itô's formula we may check Assumption 6.2. This is demonstrated this in the next proposition.

Proposition 6.4
Let $U(x) = \frac{x^\alpha}{\alpha}$ for $\alpha \in (0,1)$. Further, we assume that L has no jumps, i.e. that the compensator ν of L is zero. Then Assumption 6.2 is fulfilled, if the following holds:

$$2b_u \geq (1-\alpha)c_u \ , \quad \forall u \in [0,T] \ ,$$

Proof:

As in the proof of Proposition 4.8, we have

$$\int_{(-1,\infty)} U(x+az)p(t,u,dz) = \mathbb{E}\Big[U\Big(x + a\frac{S_u - S_t}{S_t}\Big)\Big] = \mathbb{E}\big[U(Y_u)\big] \ ,$$

where $Y_u := x + a\frac{S_u - S_t}{S_t}$. Similarly to Proposition 2.14 we can show that

$$\begin{aligned} \frac{(Y_u)^\alpha}{\alpha} &= \frac{x^\alpha}{\alpha} + \int_0^u (Y_s)^\alpha \mathbb{1}_{\{s>t\}} \pi_s dL_s + \frac{1}{2}\int_0^u (\alpha-1)(Y_s)^\alpha \mathbb{1}_{\{s>t\}} \pi_s^2 c_s ds \\ &= \frac{x^\alpha}{\alpha} + \int_0^u (Y_s)^\alpha \mathbb{1}_{\{s>t\}} \pi_s b_s ds + \underbrace{\int_0^u (Y_s)^\alpha \mathbb{1}_{\{s>t\}} \pi_s \sqrt{c_s} dW_s}_{martingale} \\ &\quad + \frac{1}{2}\int_0^u (\alpha-1)(Y_s)^\alpha \mathbb{1}_{\{s>t\}} \pi_s^2 c_s ds \ . \end{aligned}$$

Hence

$$\mathbb{E}\Big[\frac{(Y_u)^\alpha}{\alpha}\Big] = \frac{x^\alpha}{\alpha} + \int_t^u (Y_s)^\alpha \pi_s \Big[b_s + \frac{1}{2}(\alpha-1)\pi_s c_s\Big] ds \ ,$$

and $\mathbb{E}\big[\frac{(Y_u)^\alpha}{\alpha}\big]$ is non-decreasing in u, if $b_s \geq \frac{1-\alpha}{2} c_s \quad \forall s \in [0,T]$.

□

6.2. Convergence of the value function

Let us now consider a terminal wealth problem with an unbounded intensity process. In the following we will denote the value function of that portfolio problem by V_2.

Furthermore, let $\{T_n \in (0,T), n \geq 0\}$ be a sequence of horizons, which converge increasingly to T. Then we consider a terminal wealth problem with fixed horizon T_n for some n. Obviously this is a terminal wealth problem with a bounded intensity process, which we can solve according to the results from Chapter 4. From now on we will denote the value function of that portfolio problem by V_{1,T_n}.

Next we state the main result of this section.

Theorem 6.5
It holds: $\lim_{n\to\infty} V_{1,T_n}(y) = V_2(y)\ , \quad y \in E.$

Proof:

Let $\pi = (a_0, a_1, a_2, ...) \in \mathcal{A}(y)$, such that $a_k = 0$ if $\tau_k \geq T_n$.

By following the proof of Theorem 4.7, we can show that

$$\begin{aligned} \mathbb{E}_y\Big[U(X_T^\pi)\Big] &= \mathbb{E}_y\Big[\sum_{k=0}^{\infty} U(X^\pi_{\tau_{k+1}})\mathbb{1}_{\{\tau_k \leq T_n < \tau_{k+1}\}}\Big] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_y\Big[\int_{\tau_k}^{T}\int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \leq T_n < u\}} U(X^\pi_{\tau_k} + a_k z) p(\tau_k, u, dz)\lambda_u e^{-\int_{\tau_k}^{u}\lambda_s ds} du\Big]\ , \end{aligned}$$

where $\tau_0 = t$ and τ_k is the k-th exogenous random time after time t. Moreover, we have

$$\begin{aligned} V_2(y) &\geq \sum_{k=0}^{\infty} \mathbb{E}_y\Big[\int_{\tau_k}^{T}\int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \leq T_n < u\}} U(X^\pi_{\tau_k} + a_k z) p(\tau_k, u, dz)\lambda_u e^{-\int_{\tau_k}^{u}\lambda_s ds} du\Big] \\ &\geq \sum_{k=0}^{\infty} \mathbb{E}_y\Big[\int_{T_n}^{T}\int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \leq T_n\}} U(X^\pi_{\tau_k} + a_k z) p(\tau_k, T_n, dz)\lambda_u e^{-\int_{\tau_k}^{u}\lambda_s ds} du\Big] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_y\Big[e^{-\int_{\tau_k}^{T_n}\lambda_s ds} \cdot \int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \leq T_n\}} U(X^\pi_{\tau_k} + a_k z) p(\tau_k, T_n, dz)\Big]\ . \end{aligned}$$

As shown in the proof of Theorem 4.7, we have

$$V_{1,T_n}(y) = \sup_{\pi \in \mathcal{A}(y)} \sum_{k=0}^{\infty} \mathbb{E}_y\Big[e^{-\int_{\tau_k}^{T_n}\lambda_s ds} \cdot \int_{(-1,\infty)} \mathbb{1}_{\{\tau_k \leq T_n\}} U(X^\pi_{\tau_k} + a_k z) p(\tau_k, T_n, dz)\Big]\ ,$$

hence $V_2(y) \geq V_{1,T_n}(y)$ and so $\limsup_{n\to\infty} V_{1,T_n} \leq V_2$.

One the other hand, let T_n converge increasingly to T. Then we have for an arbitrary admissible policy $\pi \in \mathcal{A}(y)$

$$\mathbb{E}_y\big[U(X^\pi_{T_n})\big] \leq V_{1,T_n}(y)\ .$$

Since $U(X^\pi_{T_n})$ is uniformly integrable under π, we get

$$\mathbb{E}_y\big[U(X^\pi_T)\big] = \lim_{n\to\infty} \mathbb{E}_y\big[U(X^\pi_{T_n})\big] \leq \liminf_{n\to\infty} V_{1,T_n}(y)\ .$$

Hence $V_2(y) \leq \liminf_{n\to\infty} V_{1,T_n}(y)$, which yields

$$V_2 = \lim_{n\to\infty} V_{1,T_n}\ .$$

□

6.3. Convergence of the optimal policy

Again, let $\{T_n \in (0,T), n \geq 0\}$ be a sequence of horizons, which converge increasingly to T. Then consider a sequence of maximizers f_{T_n} of V_{1,T_n} for $n \geq 0$. Since there exists a convergent subsequence of $\{f_{T_n}(y), n \geq 0\}$ for each fixed $y \in E$, we may define

$$f_1^*(y) := \limsup_{n\to\infty} f_{T_n}(y) \quad \text{and} \quad f_2^*(y) := \liminf_{n\to\infty} f_{T_n}(y)\ , \quad \text{for } y \in E\ .$$

Theorem 6.6
If $V_2 \geq 0$, then the stationary policies

$$\pi_1 := (f_1^*, f_1^*, ...) \qquad \text{and} \qquad \pi_2 := (f_2^*, f_2^*, ...)$$

are optimal stationary policies for the terminal wealth problem with an unbounded intensity process.

Proof:

As already shown, it holds

$$V_2 = \lim_{n\to\infty} V_{1,T_n} \qquad \text{and} \qquad V_{1,T_n} \leq V_2\ .$$

Moreover, we have

$$V_{1,T_n} = \mathcal{T}_{1,n} V_{1,T_n}\ ,$$

where the operator $\mathcal{T}_{1,n}$ is defined by

$$(\mathcal{T}_{1,n}v)(y) := \sup_{a\in[0,x-\beta m]} \Bigg(e^{-\int_t^{T_n} \lambda_u du} \int_{(-1,\infty)} U(x+az)p(t,T_n,dz) \\ + \int_t^{T_n} \int_{(-1,\infty)} v(u,x+az,\max\{m,x+az\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \Bigg) .$$

Now let f_{T_n} be a maximizer of V_{1,T_n}. Then we get for $y \in E$

$$\begin{aligned} V_2(y) &= \lim_{n\to\infty} V_{1,T_n}(y) = \lim_{n\to\infty} \mathcal{T}_{1,n}V_{1,T_n}(y) \\ &= \lim_{n\to\infty} \Bigg(e^{-\int_t^{T_n} \lambda_u du} \underbrace{\int_{(-1,\infty)} U(x+f_{T_n}(y)z)p(t,T_n,dz)}_{bounded} + \\ &\quad \int_t^{T_n} \int_{(-1,\infty)} V_{1,T_n}(u,x+f_{T_n}(y)z,\max\{m,x+f_{T_n}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \Bigg) \\ &= \lim_{n\to\infty} \int_t^{T_n} \int_{(-1,\infty)} V_{1,T_n}(u,x+f_{T_n}(y)z,\max\{m,x+f_{T_n}(y)z\})\lambda_u e^{-\int_t^u \lambda_s ds} p(t,u,dz)du \, . \end{aligned}$$

Because $f_{T_n}(y)$ is a sequence in $[0, x-\beta m]$, there exists an accumulation point and a subsequence $f_{T_{n_k}}(y)$, which converges to that accumulation point. Therefore, by using Fatou's Lemma, we get

$$\begin{aligned} &V_2(y) \\ &= \lim_{k\to\infty} \int_t^{T_{n_k}} \int_{(-1,\infty)} V_{1,T_{n_k}}(u,x+f_{T_{n_k}}(y)z,\max\{m,x+f_{T_{n_k}}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq \lim_{k\to\infty} \int_t^{T_{n_k}} \int_{(-1,\infty)} V_2(u,x+f_{T_{n_k}}(y)z,\max\{m,x+f_{T_{n_k}}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq \lim_{k\to\infty} \int_t^{T} \int_{(-1,\infty)} V_2(u,x+f_{T_{n_k}}(y)z,\max\{m,x+f_{T_{n_k}}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq \int_t^{T} \int_{(-1,\infty)} \limsup_{k\to\infty} V_2(u,x+f_{T_{n_k}}(y)z,\max\{m,x+f_{T_{n_k}}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &= \int_t^{T} \int_{(-1,\infty)} V_2(u,x+\lim_{k\to\infty} f_{T_{n_k}}(y)z,\max\{m,x+\lim_{k\to\infty} f_{T_{n_k}}(y)z\})p(t,u,dz)\lambda_u e^{-\int_t^u \lambda_s ds} du \\ &\leq V_2(y) \, . \end{aligned}$$

Hence the policy

$$f_1^*(y) := \limsup_{n\to\infty} f_{T_n}(y)$$

is a maximizer of V_2 and so the policy $\pi_1 := (f_1^*, f_1^*, f_1^*, ...)$ is an optimal policy for the terminal wealth problem with an unbounded intensity process. The same holds for the policy $\pi_2 := (f_2^*, f_2^*, f_2^*, ...)$. □

7. Numerical examples with a Power Utility function

In this chapter we present numerical computations of optimal policies as well as value functions of the terminal wealth problems considered in the previous chapters. Thereby we assume different intensity processes (λ_t) and use Howard's policy improvement algorithm, which was introduced in Section 4.4. If facing an unbounded intensity process, the convergence results of Chapter 6 are used to simplify the computations. As basis for the numerical examples, we fix the following model:

- The finite horizon equals 1, i.e. $T = 1$,
- We are considering the popular Black-Scholes market with coefficients $\mu = 4\%$ and $\sigma = 33\%$,
- We assume a *Power Utility* function $U(x) = \frac{x^\alpha}{\alpha}$ with parameter $\alpha = \frac{1}{3}$.

Since we are assuming a *Power Utility* function, there exists, due to Proposition 4.12, a separation ansatz for the value function, such that

$$V(y) = U(x) \cdot F(t, \tfrac{m}{x}) \ , \quad \forall y \in E \ ,$$

for some function $F : [0, T] \times (0, \frac{1}{\beta}) \to \mathbb{R}$. Therefore, to determine the value function, it is enough to determine the function F. Moreover, this separation ansatz directly implies that

- an optimal policy depends only on the time t and the ratio $v := \frac{m}{x}$,
- an optimal policy indicates the **fraction** of wealth, which should be invested in the stock.

In the following the initial policy for Howard's policy improvement algorithm is given by the generalized Merton ratio f_0, where

$$f_0(t, v) = \min\{\pi^*, 1 - \beta v\} \ , \quad (t, v) \in [0, T] \times (0, \tfrac{1}{\beta})$$

and $\pi^* = \frac{\mu}{(1-\alpha)\sigma^2} = 0.55096$ denotes the classical Merton ratio.

Remark 7.1
Since we are always dealing with stationary policies, we will in the following denote a stationary policy $\pi = (f, f, f, ...)$ only by its decision rule f.

7.1. Bounded intensity processes

In this section we are considering bounded intensity processes (λ_t). Hence we are dealing with a terminal wealth problem, which was solved in Chapter 4.

7.1.1. Constant intensity process

First we choose the intensity process equal to 1, i.e.

$$\lambda_t := 1 , \qquad \forall t \in [0, \infty) .$$

We approximate an optimal policy using Howard's policy improvement algorithm. After the first improvement of the initial policy f_0 it turns out, that we already obtain a very good approximation of the optimal policy. Figure 7.1 shows this computed approximation f^* for different β. Moreover, this policy only depends on the time t and the ratio $v = \frac{m}{x}$. As we can see, if the ratio v is small, then we invest a large fraction of wealth in the stock, and if the ratio v is large, then we invest less in the stock. This can be explained in the following way: If the ratio v is small, then the wealth x is far away from the lower bound βm. Hence we can invest a larger amount in the stock without risking to fall below the lower bound. If the ratio v is large, then the wealth x is close to the lower bound βm. So we may invest only a small amount in the stock, such that the wealth does not fall below the lower bound.

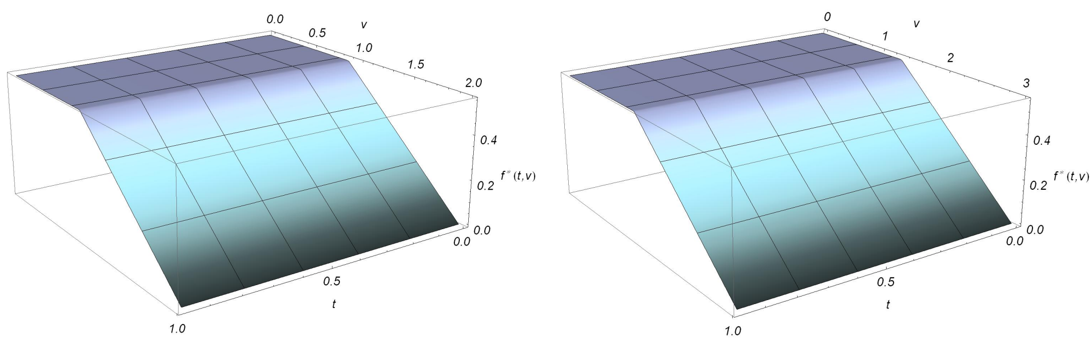

Figure 7.1.: Policy f^* - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

To obtain more insight into the micro-structure of the policy f^*, the slice planes $t \to f^*(t, 0.5)$ and $v \to f^*(0.5, v)$ are shown in Figure 7.2 and 7.3 below.

In Figure 7.2 and 7.3 we can see that the policy f^* is not constant in time and very close to the Merton ratio π^* for small values of v. Furthermore, we see that the policy f^* is constant as a function of the ratio v for small values thereof and then decreases linear to zero with slope $(-\beta)$.

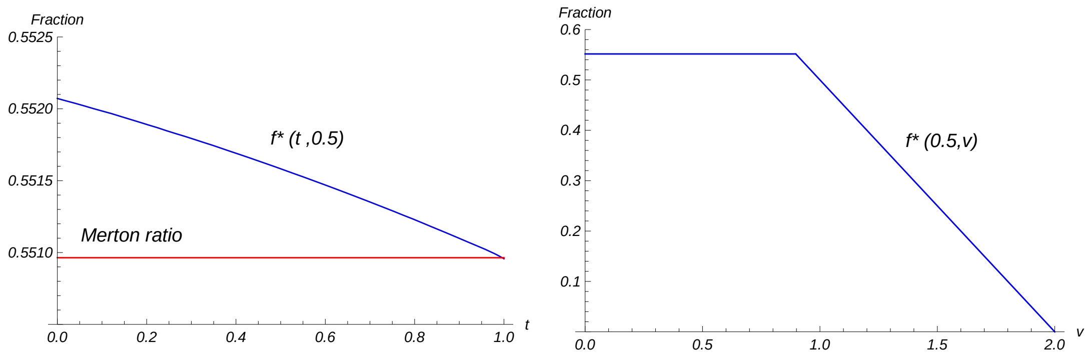

Figure 7.2.: Slice planes for $\beta = \frac{1}{2}$

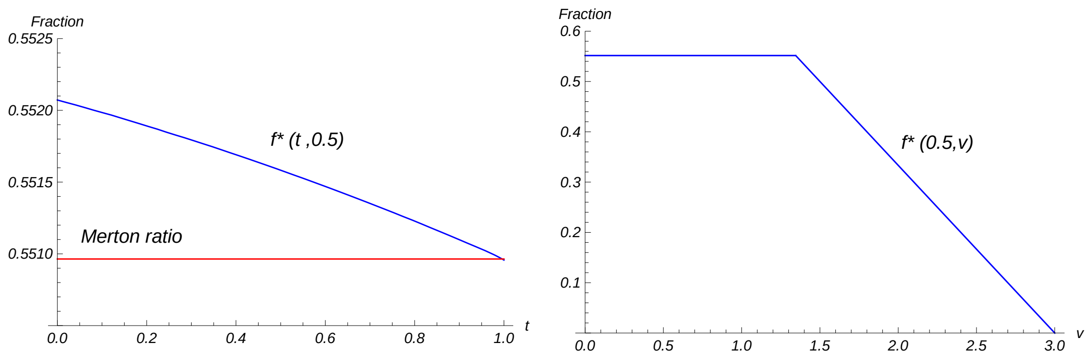

Figure 7.3.: Slice planes for $\beta = \frac{1}{3}$

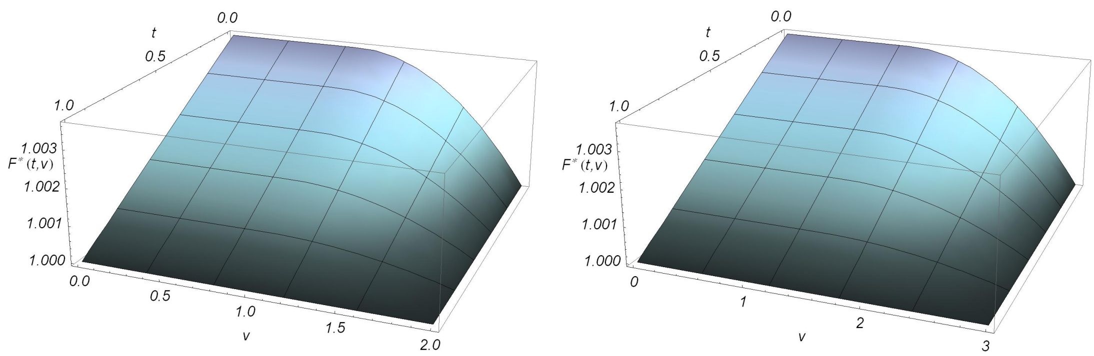

Figure 7.4.: Function F^* of the separation ansatz - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

Finally, we can summarize the following micro-structure of the policy f^*:

- The policy f^* is time dependent.
- The bend in Figure 7.1 runs in a line from
  - $(t, v) = (0, 0.8958)$ to $(t, v) = (1, 0.8981)$, if $\beta = \frac{1}{2}$,
  - $(t, v) = (0, 1.3437)$ to $(t, v) = (1, 1.3471)$, if $\beta = \frac{1}{3}$.
- The policy is constant as a function of the ratio v to the left of the bend and then decreases linearly to zero with slope $(-\beta)$.

At the end we present the function F^* of the separation ansatz for different β. Thereby Banach's Fixed Point Theorem is used to compute them recursively.

As we can see, the function F^* decreases in $v \in [0.89, 2]$ for $\beta = \frac{1}{2}$ and in $v \in [1.34, 3]$ for $\beta = \frac{1}{3}$. This is exactly the region, in which the drawdown constraint effects the policy f^* and therefore diminishes the value function. Furthermore,

- $F^*(0, 0) = 1.00364$, if $\beta = \frac{1}{2}$,
- $F^*(0, 0) = 1.00366$, if $\beta = \frac{1}{3}$.

Finally, if we compute the expected reward $V_{f_0} = U \cdot F_0$ under the initial policy f_0, we obtain $F_0 = F^*$ for both values of β. The lack of a measurable difference arises from the numerical computations. However, it shows that the policy f_0 is a very good approximation of an optimal policy for both values of β.

7.1.2. Time-varying intensity process

In this section, we assume a time-varying intensity process $\lambda_t = 1 + 5(t-1)^2$, which is illustrated in Figure 7.5 below:

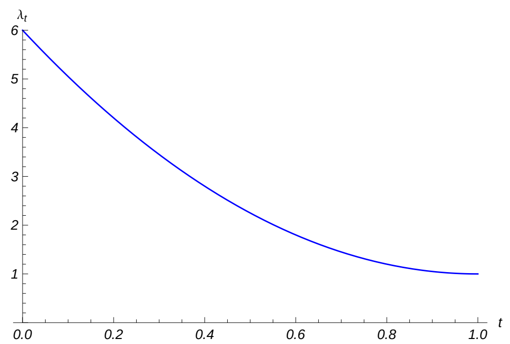

Figure 7.5.: Intensity process $\lambda_t = 1 + 5(t-1)^2$.

Below, we proceed as in the previous example and compute an approximation f^* of an optimal policy by using Howard's policy improvement algorithm and the function F^*. Figure 7.6 shows the computed approximation for different β. In this case it turned out, that the second improvement of the initial policy f_0 is a very good approximation. As one can see, the structure of the policy f^* is very close to the one in the example above. Hence, the time-varying intensity process has only a small influence on the optimal policy.

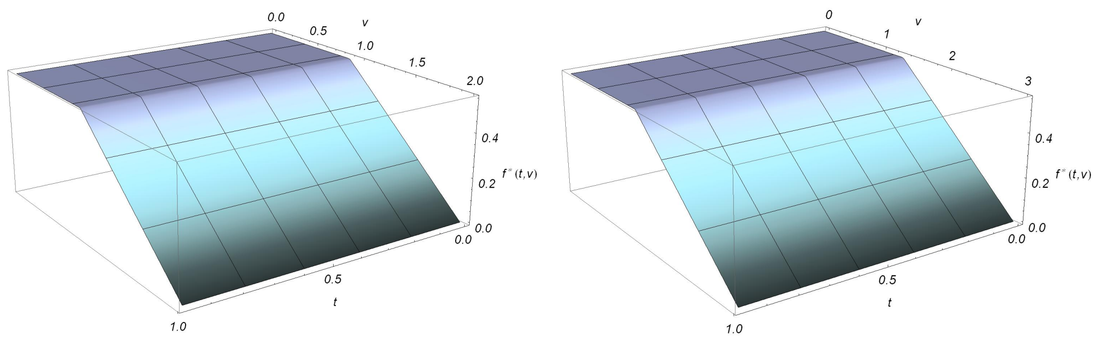

Figure 7.6.: Policy f^* - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

In Figure 7.7 and 7.8 on the next page, we can see that the policy f^* is again not constant in time and very close to the Merton ratio π^* for small values of v. Furthermore, we see that the policy f^* is constant as a function of the ratio v for small values thereof and then decreases linear to zero with slope $(-\beta)$.

Finally, we can summarize the micro-structure of the policy f^* similar to the previous example:

- The policy f^* is time dependent.
- The bend in Figure 7.6 runs in a line from
  - $(t, v) = (0, 0.8965)$ to $(t, v) = (1, 0.8981)$, if $\beta = \frac{1}{2}$,
  - $(t, v) = (0, 1.3448)$ to $(t, v) = (1, 1.3471)$, if $\beta = \frac{1}{3}$.
- The policy is constant as a function of the ratio v to the left of the bend and then decreases linearly to zero with slope $(-\beta)$.

In this example we observe, that the investor always invests less in the stock compared to the optimal behaviour in the previous example. This is exactly what one would expect, since, due to large values of the intensity process (λ_t), we have a lower liquidity risk. Hence one invests closer a fraction of wealth to the Merton ratio π^*, which is the optimal proportion in a completely liquid market.

For completeness we also illustrate the function F^* of the separation ansatz in Figure 7.9.

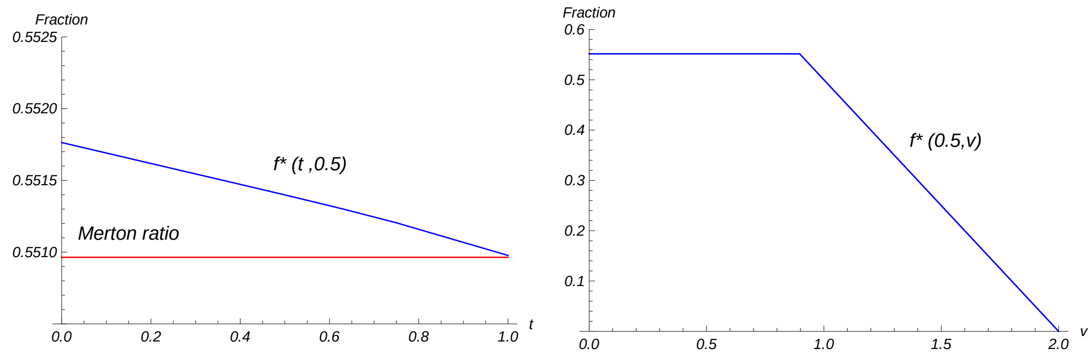

Figure 7.7.: Slice planes for $\beta = \frac{1}{2}$

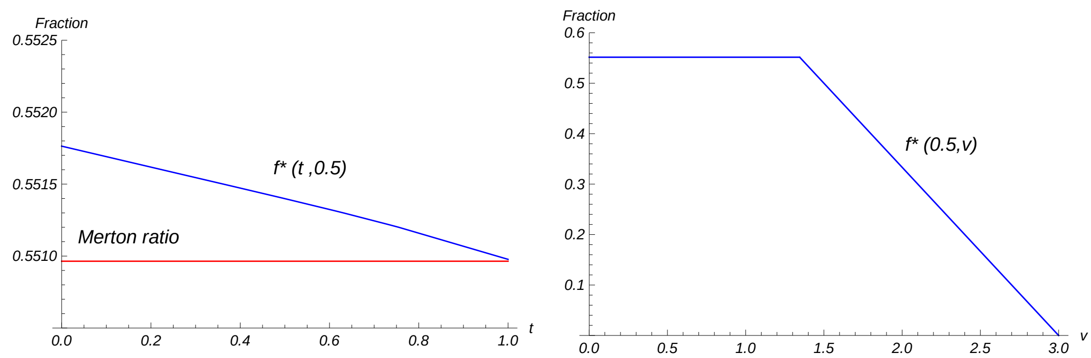

Figure 7.8.: Slice planes for $\beta = \frac{1}{3}$

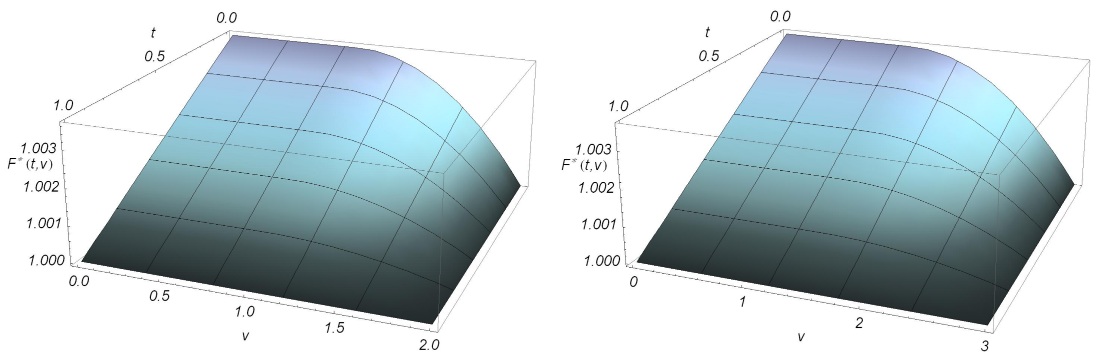

Figure 7.9.: Function F^* of the separation ansatz - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

As we can see, we obtain the same structure as in the previous example. Furthermore, we have

- $F^*(0,0) = 1.00369$, if $\beta = \frac{1}{2}$,
- $F^*(0,0) = 1.00371$, if $\beta = \frac{1}{3}$.

Again, the expected reward $V_{f_0} = U \cdot F_0$ of the initial policy f_0 equals the expected reward of the policy f^* due to numerical computations. Hence, a similar conclusion can be made and we get that the policy f_0 is a very good approximation of an optimal policy for both values of β.

7.2. Unbounded intensity process

In this section we consider an unbounded intensity process (λ_t), which is given by

$$\lambda_t = \tfrac{1}{1-t}\ , \qquad \forall t \in [0,1)\ .$$

Now we are able to apply the convergence results from Chapter 6 to approximate the solution of this terminal wealth problem with an unbounded intensity process. As in the previous examples, we compute an approximation of an optimal policy using Howard's policy improvement algorithm with the initial policy f_0. We start with plotting this approximation, denoted by f^*, in Figure 7.10 for different β.

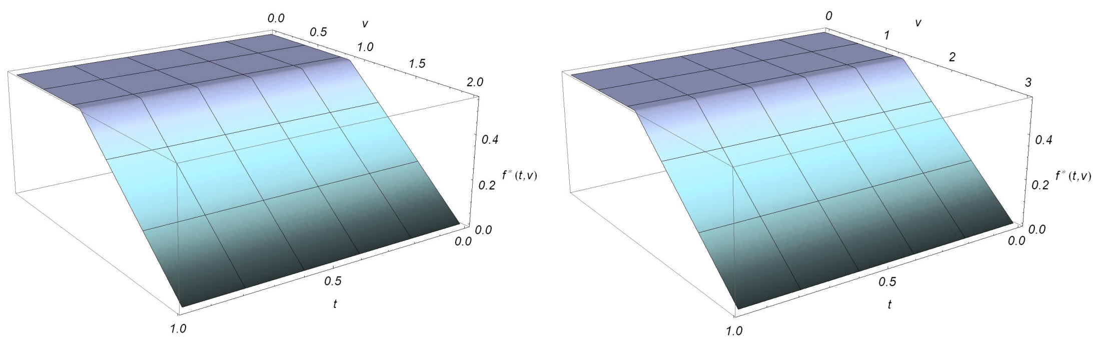

Figure 7.10.: Policy f^* - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

We observe a similar structure of the policy f^* as above, where we assumed a bounded intensity process:

- The policy f^* is time dependent.
- The bend in Figure 7.10 runs in a line from

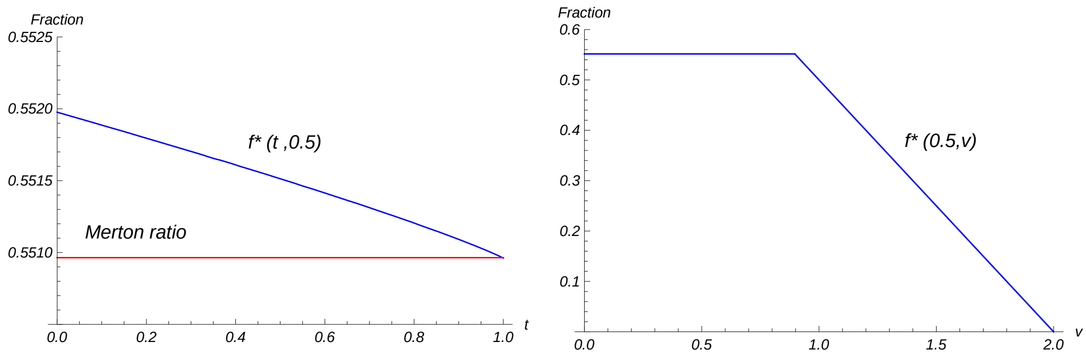

Figure 7.11.: Slice planes for $\beta = \frac{1}{2}$

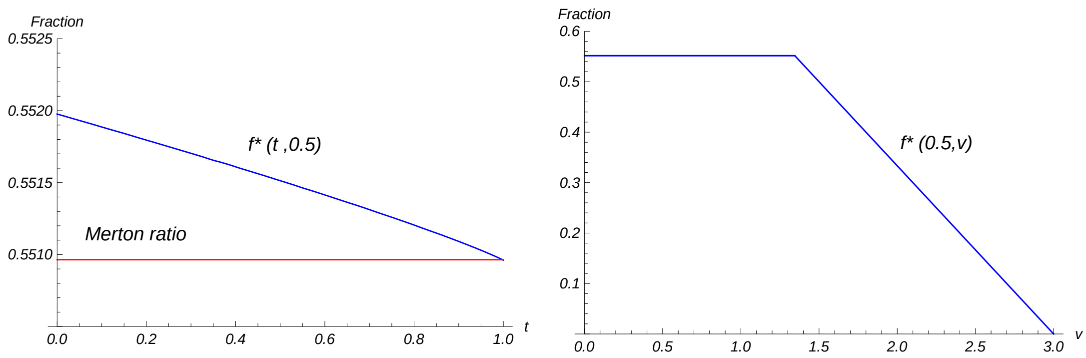

Figure 7.12.: Slice planes for $\beta = \frac{1}{3}$

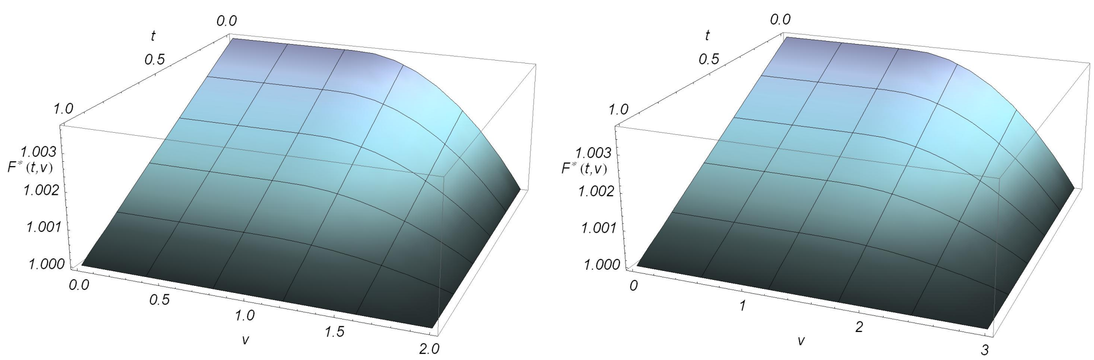

Figure 7.13.: Function F^* of the separation ansatz - on the left hand side for $\beta = \frac{1}{2}$ and on the right hand side for $\beta = \frac{1}{3}$.

  - $(t, v) = (0, 0.8960)$ to $(t, v) = (1, 0.8981)$, if $\beta = \frac{1}{2}$,
  - $(t, v) = (0, 1.3440)$ to $(t, v) = (1, 1.3471)$, if $\beta = \frac{1}{3}$.

- The policy is constant as a function of the ratio v to the left of the bend and then decreases linearly to zero with slope $(-\beta)$.

In this example the investor invests similarly to the one example with a constant intensity process. This is due to the fact that the intensity process in this section starts at the level 1 and only rises significantly during the last third of the investment horizon. Hence the influence of this unbounded intensity process is limited.

At the end, we depict the function F^* of the separation ansatz in Figure 7.13. Again, we obtain the same structure as above and

- $F^*(0, 0) = 1.00365$, if $\beta = \frac{1}{2}$,
- $F^*(0, 0) = 1.00367$, if $\beta = \frac{1}{3}$.

The expected reward $V_{f_0} = U \cdot F_0$ of the initial policy f_0 equals the expected reward of the policy f^* due to numerical computations. Hence, the policy f_0 already constitutes a very good approximation of an optimal policy for both values of β.

7.3. Conclusion for practitioners

In the examples above we have seen, that the intensity processes (λ_t) has only a small influence on the optimal policies. Therefore, we recommend for practitioners to neglect the intensity process and use the generalized Merton ratio f_0, which was given by

$$f_0(t, v) = \min\{\pi^*, 1 - \beta v\} , \quad (t, v) \in [0, T] \times (0, \tfrac{1}{\beta}) ,$$

where $\pi^* = \frac{\mu}{(1-\alpha)\sigma^2}$ denotes the Merton ratio.

This policy can only be improved with great effort while being nevertheless easily computed due to the explicit representation.

A. Structure Theorem for limsup Markov Decision Processes

In this Appendix we investigate structure assumptions under which we can characterize the solution of a limsup Markov Decision Process. For that, consider a Markov Decision Process with tuple (E, A, D, Q, g). As usual,

- E is the state space, endowed with a σ-algebra $\mathscr{E}$.
- A is the action space, endowed with a σ-algebra $\mathscr{A}$.
- $D \subset E \times A$ is a measurable subset of $E \times A$ and denotes the admissible state-action combinations. It is assumed that D contains the graph of a measurable mapping. Moreover, the set $D(x) = \{a \in A | (x, a) \in D\}$ is the set of admissible actions.
- Q is a stochastic transition kernel from D to E, called the transition law.
- $g : E \to \mathbb{R}$ is a measurable function, called the terminal reward.

A Markovian policy $\pi := (f_0, f_1, f_2, ...)$ is a sequence of decision rules (f_n), where $f_n : E \to A$ is a measurable mapping such that $f_n(x) \in D(x)$. The set of all Markovian policies is denoted by Π.

For a Markovian policy $\pi \in \Pi$, we define the reward with respect to π by

$$V_\pi(x) := \mathbb{E}_x^\pi \big[\limsup_{n\to\infty} g(X_n) \big] \ , \qquad x \in E \ , \pi \in \Pi \ .$$

Here, $\mathbb{E}_x^\pi$ is the expectation with respect to the conditional probability

$$\mathbb{P}_x^\pi(\cdot) := \mathbb{P}^\pi(\ \cdot \mid X_0 = x) \ .$$

The aim is to maximize the reward over all Markovian policies, i.e.

$$V(x) := \sup_{\pi\in\Pi} V_\pi(x) \ , \qquad x \in E \ . \tag{A.1}$$

In the following, we will proof a structure theorem, which gives us conditions under which optimization problem (A.1) has a solution and how this solution can be characterized. To do so we introduce the following notations:

Let $\mathbb{M}(E) := \{v : E \to \mathbb{R} \mid v \text{ is measurable }\}$. Then we define for $v \in \mathbb{M}(E)$ the operators

$$(Lv)(x,a) := \int_E v(x')Q(x'|x,a) \;, \qquad \forall (x,a) \in D \;,$$

whenever the integral exists, and

$$(\mathcal{T}v)(x) := \sup_{a \in D(x)} (Lv)(x,a) \;, \qquad x \in E \;.$$

Theorem A.1 *(Structure Theorem)*
If there exists a subset $\mathbb{M} \subset \mathbb{M}(E)$ such that

(i) $g \in \mathbb{M}$.

(ii) If $v \in \mathbb{M}$, then $\mathcal{T}v$ is well-defined and $\mathcal{T}v \in \mathbb{M}$.

(iii) If $v \in \mathbb{M}$, then there exists a maximizer of v, i.e. there exists a decision rule f^* such that

$$\mathcal{T}_{f^*}v = \mathcal{T}v \;.$$

(iv) For each fixed $x \in E$

$$\exists\, v_\infty(x) := \lim_{n\to\infty} (\mathcal{T}^n g)(x) \;,$$

and $v_\infty \in \mathbb{M}$. Further v_∞ satisfies the following conditions:

(1) $v_\infty = \mathcal{T}v_\infty$,

(2) $\lim_{n\to\infty} v_\infty(X_n) = \limsup_{n\to\infty} g(X_n) \qquad \mathbb{P}^\pi_x - (a.s.)$,

(3) $\big(v_\infty(X_n)\big)_{n\geq 0}$ is $\mathbb{P}^\pi_x$-uniformly integrable for all $\pi \in \Pi$ and $x \in E$.

holds. Then we have

a) $V \in \mathbb{M}$ and V is the unique fixed point of $\mathcal{T}$ in $\mathbb{M}$ which satisfies (2) and (3). Moreover,

$$V = \lim_{n\to\infty} \mathcal{T}^n g \;.$$

b) There exists a maximizer f^* of V and each maximizer of V defines an optimal stationary policy $\pi = (f^*, f^*, f^*, \ldots)$.

Proof:

1. Let $x \in E$ and $\pi := (f_0, f_1, \ldots)$ be a Markovian policy. Consider now $v_\infty \in \mathbb{M}$. It follows by the tower property of the conditional expectation that

$$\mathbb{E}^\pi_x\big[v_\infty(X_{n+1}) \mid \sigma(X_n)\big] = Lv_\infty(X_n, f_n(X_n)) \leq Tv_\infty(X_n) = v_\infty(X_n) \;.$$

Hence $v_\infty(X_n)$ is a supermartingale. Since $v_\infty(X_n)$ is uniformly integrable, we get by (Klenke, 2008, Theorem 6.25)

$$\begin{aligned} V(x) &= \sup_\pi \mathbb{E}_x^\pi\big[\limsup_{n\to\infty} g(X_n)\big] = \sup_\pi \mathbb{E}_x^\pi\big[\lim_{n\to\infty} v_\infty(X_n)\big] \\ &= \sup_\pi \lim_{n\to\infty} \mathbb{E}_x^\pi\big[v_\infty(X_n))\big] \le v_\infty(x) \ . \end{aligned}$$

2. Let $x \in E$ and $\hat\pi := (f^*, f^*, ...)$ be a stationary policy, where f^* is a maximizer of v_∞. By the same arguments as before, we get

$$\mathbb{E}_x^{\hat\pi}\big[v_\infty(X_{n+1}) \mid \sigma(X_n)\big] = Lv_\infty(X_n, f^*(X_n)) = \mathcal{T} v_\infty(X_n) = v_\infty(X_n) \ .$$

   Hence $v_\infty(X_n)$ is a martingale and so

$$V_{\hat\pi}(x) = \mathbb{E}_x^{\hat\pi}\big[\limsup_{n\to\infty} g(X_n)\big] = \mathbb{E}_x^{\hat\pi}\big[\lim_{n\to\infty} v_\infty(X_n)\big] = \lim_{n\to\infty} \mathbb{E}_x^{\hat\pi}\big[v_\infty(X_n)\big] = v_\infty(x) \ .$$

   This and Step 1 yields $V = v_\infty$ and that the stationary policy $\hat\pi$ is an optimal one.

3. Now, we assume that there are two functions $v_\infty \in \mathbb{M}$ and $\tilde v_\infty \in \mathbb{M}$ satisfying (1), (2) and (3). By Step 1 and Step 2, it follows that $V = v_\infty$ and $V = \tilde v_\infty$. Because of that v_∞ is the unique function in $\mathbb{M}$ satisfying (1), (2) and (3).

$\square$

B. Stochastic processes

In this Appendix we state definitions, properties and theorems of stochastic processes which are used throughout this work. For a more detailed treatment and proofs see Jacod & Shiryaev (2003), Revuz & Yor (1999) and Sato (2005). In what follows, all processes are assumed to be $\mathbb{R}$-valued.

B.1. Semimartingales and stochastic exponential

In this work the powerful tool of stochastic integration is needed. A large class of processes for which stochastic integration for general predictable integrands works, is the class of semimartingales.

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. It is assumed that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions.

Definition B.1

(i) A semimartingale is a process (X_t) of the form $X_t = X_0 + M_t + A_t$ where X_0 is a finite-valued and $\mathcal{F}_0$-measurable, where (M_t) is a local martingale such that $M_0 = 0$, and where (A_t) is an adapted, càdlàg process with $A_0 = 0$ and whose each path has finite variation over each finite interval $[0, t]$.

(ii) A special semimartingale is a process (X_t) which admits a decomposition $X_t = X_0 + M_t + A_t$ as in part (i), with a process (A_t) that is predictable. This decomposition is unique and therefore called canonical decomposition of (X_t).

With such a semimartingale (X_t), we can construct a new process $\mathcal{E}(X)_t$, called the stochastic exponential or Doléans-Dade exponential.

Theorem B.2
Let (X_t) be a semimartingale. Then the equation

$$Y_t = 1 + \int_0^t Y_{s-} dX_s$$

has one and only one (up to indistinguishability) càdlàg adapted solution. This solution is a semimartingale, denoted by $\mathcal{E}(X)_t$, and is given by

$$\mathcal{E}(X)_t = e^{X_t - X_0 - \frac{1}{2}\langle X^c, X^c\rangle_t} \times \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} ,$$

where the (possibly) infinite product is absolutely convergent. Furthermore

(i) If (X_t) has finite variation, then so has $\mathcal{E}(X)_t$.

(ii) If $(X)_t$ is a local martingale, then so is $\mathcal{E}(X)_t$.

(iii) Let $\xi = \inf\{t : \Delta X_t = -1\}$. Then $\mathcal{E}(X)_t \neq 0$ on $[0, \xi)$, and $\mathcal{E}(X)_{t-} \neq 0$ on $[0, \xi]$, and $\mathcal{E}(X)_t = 0$ on $[\xi, \infty)$.

In financial applications, it is often assumed, that the price process of an asset is given by the stochastic exponential of its return process. This has two advantages:

- If we restrict the jumps size of the returns to be strictly greater than minus 1, then we get a strictly positive process. This is necessary for realistic asset prices.
- Usually, it is simpler to model the return process than the asset price.

Another desirable feature of asset returns is the independent increment property, i.e. the returns on non overlapping time intervals are independent. This leads to processes with independent increments.

Definition B.3

A process with independent increments (in short: PII), is a càdlàg adapted process (X_t) such that $X_0 = 0$ and that for all $0 \leq s \leq t$ the variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$.

The following two theorems give a connection between semimartingales and PIIs. We emphasize that this is not clear, since there are stochastic processes with independent increments which are not semimartingales. For a example see (Jacod & Shiryaev, 2003, II, §4c)).

Theorem B.4

Let (X_t) be a PII. Then (X_t) is also a semimartingale if and only if, for each $u \in \mathbb{R}$, the function

$$t \to \mathbb{E}\left[e^{iuX_t}\right] , \qquad t \in [0, \infty) ,$$

has finite variation over finite intervals.

Theorem B.5
Let (X_t) be a semimartingale with $X_0 = 0$. Then it is a PII if and only if there is a version (B, C, ν) of its characteristics that is deterministic.

The canonical representation of semimartingales and special semimartingales is often helpful handling such processes.

Theorem B.6
Let (X_t) be a semimartingale, with characteristics (B, C, ν) relative to the truncation function $\mathbb{1}_{\{|x|\leq 1\}}$, and with the measure μ^X associated to its jumps. Then the following representation, called canonical representation, holds:

$$X_t = X_0 + B_t + X_t^c + \int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|\leq 1\}}(\mu^X - \nu)(ds, dx) + \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}} .$$

Theorem B.7
Let (X_t) be a special semimartingale, with characteristics (B, C, ν) and μ^X the measure associated to its jumps. If $X_t = X_0 + M_t + A_t$ is its canonical decomposition, then

$$X_t = X_0 + A_t + X_t^c + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu)(ds, dx) .$$

Theorem B.8
Let (X_t) be a special semimartingale, with deterministic characteristics (B, C, ν) relative to the truncation function $\mathbb{1}_{\{|x|\leq 1\}}$. Then the canonical decomposition $X_t = X_0 + M_t + A_t$ satisfies

$$A_t = B_t + \int_0^t \int_{\mathbb{R}} (x - x\mathbb{1}_{\{|x|\leq 1\}})\nu(ds, dx) .$$

B.2. Additive processes

In this section, we focus on additive processes. These processes are a generalization of the well known Lévy processes, since the condition of stationary increments is not required.

In the following we assume a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition B.9
A process (X_t) is an additive process in law, if

1. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$ the random variables
$$X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$$
are independent (independent increment property).
2. $X_0 = 0$.
3. (X_t) is stochastically continuous.

Such as Lévy processes, additive processes in law are Markov processes and they are infinitely divisible.

Theorem B.10
Let (X_t) be an additive process in law. Then (X_t) is a Markov process with transition function
$$\mathbb{P}_{s,t}(x, B) = \mathbb{P}[X_t - X_s \in B - x] \text{ for } 0 \leq s \leq t \;,$$
where $B - x = \{y - x : y \in B\}$ and starting point 0.

Theorem B.11
If (X_t) is an additive process in law, then the distribution of X_t is infinitely divisible for every t.

Because asset prices are usually càdlàg processes, additive processes will come to the fore.

Definition B.12
An additive process in law (X_t) is called an additive process, if there is a $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

The following theorem shows, that we can get an additive process from an additive process in law by choosing the appropriate modification.

Theorem B.13
Let (X_t) be an additive process in law. Then there exists a modification of (X_t) which is an additive process.

The following theorem is the main theorem in this section, since it shows, how to construct additive processes in law.

Theorem B.14

(i) Suppose that (X_t) is an additive process in law. Let (μ_t, A_t, v_t) be the generating triplet of the infinitely divisible distribution $\mathbb{P}_{X_t}$ for $t \geq 0$. Then the following conditions are satisfied.

  (1) $A_0 = 0$, $v_0 = 0$, $\mu_0 = 0$.

  (2) If $0 \leq s \leq t < \infty$, then $A_s \leq A_t$ and $v_s(B) \leq v_t(B)$ for $B \in \mathcal{B}(\mathbb{R})$.

  (3) As $s \to t$ in $[0, \infty)$, $\mu_s \to \mu_t$, $A_s \to A_t$ and $v_s(B) \to v_t(B)$ for $B \in \mathcal{B}(\mathbb{R})$ with $B \subset \{x : |x| > \epsilon\}$, $\epsilon > 0$.

(ii) Let $(\Gamma_t)_{t\geq 0}$ be a system of infinitely divisible probability measures on $\mathbb{R}$ with generating triplet (μ_t, A_t, v_t) satisfying the conditions (1) - (3). Then there exists, uniquely up to identity in law, an additive process in law (X_t) such that $\mathbb{P}_{X_t} = \Gamma_t$ for $t \geq 0$.

The last theorem in this section tells us, that an additive process (X_t) can be decomposed in a continuous and a discontinuous part, where the parts are independent.

Theorem B.15

Let (X_t) be an additive process with generating triplet (μ_t, A_t, v_t). Then

$$X_t = X_t^1 + X_t^2 \, ,$$

where (X_t^1) and (X_t^2) are independent, X_t^1 is a continuous additive process with generating triplet $(\mu_t, A_t, 0)$ and X_t^2 is an additive process with generating triplet $(0, 0, v_t)$.

C. Fenchel-Legendre transform

In this Appendix we introduce the Fenchel-Legendre transform, which is used in Chapter 5. Moreover, we present the Fenchel-Legendre transforms of popular utility functions which are introduced in Section 3.3.

Definition C.1 *(Fenchel-Legendre transform)*
Let $u : (0, \infty) \to \mathbb{R}$ be an increasing and concave function. Then we define the Fenchel-Legendre transform $\tilde{u} : (0, \infty) \to \mathbb{R} \cup \{\infty\}$ of u by

$$\tilde{u}(y) = \sup_{x>0} \big[u(x) - xy \big] , \quad \forall y > 0 ,$$

and the domain of $\tilde{u}$ by

$$dom(\tilde{u}) = \{ y > 0 : \tilde{u}(y) < \infty \} .$$

Example C.2

- The Fenchel-Legendre transform $\tilde{U}$ of the *Power Utility* function $U(x) = \frac{x^\alpha}{\alpha}$ with $\alpha < 1$, $\alpha \neq 0$ is given by

$$\tilde{U}(y) = \tfrac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}} .$$

- The Fenchel-Legendre transform $\tilde{U}$ of the *Logarithmic Utility* function $U(x) = \log(x)$ is given by

$$\tilde{U}(y) = -\log(y) - 1 .$$

- The Fenchel-Legendre transform $\tilde{U}$ of the *Exponential Utility* function $U(x) = -\frac{1}{\alpha} e^{-\alpha x}$ with $\alpha > 0$ is given by

$$\tilde{U}(y) = \frac{y(\ln(y) - 1)}{\alpha} .$$

Proposition C.3
Let $u : (0, \infty) \to \mathbb{R}$ be an increasing and concave function and $\tilde{u}$ its Fenchel-Legendre

transform. Then we have

- $\tilde{u}$ is a decreasing convex function on $(0, \infty)$.
- The following conjugate relation holds:

$$u(x) = \inf_{y>0} \left[\tilde{u}(y) + xy\right], \; x > 0.$$

- $\tilde{u}(0) := \lim_{y \downarrow 0} \tilde{u}(y) = u(\infty) := \lim_{x \to \infty} u(x)$.

Under the additional assumptions of

- u is strictly concave on $(0, \infty)$,
- u is differentiable on $(0, \infty)$,
- $u'(0) = \infty$ and $u'(\infty) = 0$,

we have $dom(\tilde{u}) = (0, \infty)$.

A proof can for example be found in (Pham, 2009, Section B.2).

Bibliography

Applebaum, D. (2009). *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2nd edition.

Bäuerle, N. & Rieder, U. (2009). MDP algorithms for portfolio optimization problems in pure jump markets. *Finance and Stochastics*, 13, 591–611.

Bäuerle, N. & Rieder, U. (2011). *Markov Decision Processes with Applications in Finance*. Springer-Verlag Berlin Heidelberg New York.

Benth, F. E., Benth, J. S., & Koekebakker, S. (2008). *Stochastic Modeling of Electricity and Related Markets*. World Scientific.

Bertsekas, D. P. & Shreve, S. E. (1978). *Stochastic Optimal Control: The Discrete-Time Case*. Academic Press, Inc.

Cetin, U., Jarrow, R. A., & Protter, P. (2004). Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8, 311–341.

Chesney, M., Jeanblanc, M., & Yor, M. (2009). *Mathematical Methods for Financial Markets*. Springer-Verlag London.

Cont, R. & Tankov, P. (2004). *Financial Modelling With Jumps Processes*. Chapman & Hall/CRC.

Cvitanic, J. & Karatzas, I. (1995). On portfolio optimization under drawdown constraints. *IMA Lecture Notes in Mathematics & Applications*, 65, 77–88.

Davis, M. & Norman, A. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15, 676–713.

Delbaen, F. & Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer-Verlag Berlin Heidelberg.

Elie, R. (2008). Finite time merton strategy under drawdown constraint: A viscosity solution approach. *Applied Mathematics & Optimization*, 58, 411–431.

Elie, R. & Touzi, N. (2008). Optimal lifetime consumption and investment under a drawdown constraint. *Finance and Stochastics*, 12, 299–330.

Elstrodt, J. (1999). *Maß- und Integrationstheorie*. Springer-Verlag Berlin Heidelberg New York, 2nd edition.

Gassiat, P., Pham, H., & Sîrbu, M. (2011). Optimal investment in finite horizon with random discrete order flow in illiquid markets. *International Journal of Theoretical and Applied Finance*, 14, 17–40.

Grossman, S. & Zhou, Z. (1993). Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3, 241–276.

Hinderer, K. (1970). *Foundations of Non-stationary Dynamic Programming with Discrete Time Parameter*. Springer-Verlag Berlin Heidelberg.

Hinderer, K. (1972). *Grundbegriffe der Wahrscheinlichkeitstheorie*. Springer-Verlag Berlin Heidelberg.

Jacod, J. & Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer-Verlag Berlin Heidelberg, 2nd edition.

Karatzas, I. & Shreve, S. E. (2005). *Brownian Motion and Stochastic Calculus*. Springer Science+Business Media, LLC, 2nd edition.

Klebaner, F. (2005). *Introduction To Stochastic Calculus With Applications*. Imperical College Press, 2nd edition.

Klenke, A. (2008). *Probability Theory*. Springer-Verlag London.

Kluge, W. (2005). *Time-inhomogeneous Lévy processes in interest rate and credit risk models*. PhD thesis, Albert-Ludwigs-Universität Freiburg im Breisgau.

Matsumoto, K. (2003). Optimal portfolio of low liquid assets with a power utility function. *Journal of Mathematical Sciences, The University of Tokyo*, 10, 687–726.

Matsumoto, K. (2006). Optimal portfolio of low liquid assets with a log-utility function. *Finance and Stochastics*, 10, 121–145.

Matsumoto, K. (2007). Portfolio insurance with liquidity risk. *Asia-Pacific Financial Markets*, 14, 363–386.

Matsumoto, K. (2009). Optimal growth rate in random trade time. *Advances in Mathematical Economics*, 12, 129–152.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51, 247–257.

Pham, H. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer-Verlag Berlin Heidelberg.

Pham, H. & Tankov, P. (2008). A model of optimal consumption under liquidity risk with random trading times. *Mathematical Finance*, 18, 613–627.

Pham, H. & Tankov, P. (2009). A coupled system of integrodifferential equations arising in liquidity risk model. *Applied Mathematics & Optimization*, 59, 147–173.

Protter, P. E. (1977). Markov solutions of stochastic differential equations. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 41, 39–58.

Protter, P. E. (2005). *Stochastic Integration and Differential Equations*. Springer-Verlag Berlin Heidelberg, 2nd edition.

Revuz, D. & Yor, M. (1999). *Continuous Martingales and Brownian Motion*. Springer-Verlag Berlin Heidelberg, 3rd edition.

Rockafellar, R. T. (1972). *Convex Analysis*. Princeton University Press.

Rogers, L. (2001). The relaxed investor and parameter uncertainty. *Finance and Stochastics*, 5, 131–154.

Rogers, L. & Williams, D. (2003). *Diffusions, Markov Processes and Martingales: Foundations*. Cambridge University Press, Cambridge, 2nd edition.

Rogers, L. & Zane, O. (2002). A simple model of liquidity effects. *Advances in Finance and Stochastics*, (pp. 161–176).

Sato, K.-I. (2005). *Lévy Processes and Infinite Divisible Distributions*. Cambridge University Press, Cambridge.

Schäl, M. (1990). *Markoffsche Entscheidungsprozesse*. Vieweg+Teubner.

Subramamian, A. & Jarrow, R. A. (2001). The liquidity discount. *Mathematical Finance*, 11, 447–474.

Werner, D. (2007). *Funktionalanalysis*. Springer-Verlag Berlin Heidelberg, 6th edition.

Zusammenfassung

In der klassischen Portfolio-Optimierung investiert ein Investor sein Vermögen in einen Finanzmarkt, welcher aus endlich vielen risikobehafteten Anlagen und einer risikolosen Anlage besteht. Dabei interessiert er sich für die beste Anlagestrategie. Um diese identifizieren zu können muss er jedoch zunächst ein Bewertungskriterium für die Strategien festlegen. Ein solches Kriterium ist zum Beispiel die Maximierung des Endnutzens.
Das zugehörige Portfolio-Optimierungsproblem wurde erstmals im Jahre 1969 von Robert C. Merton in Merton (1969) gelöst. Er betrachtete dabei einen klassischen Black-Scholes Markt sowie eine CRRA Nutzenfunktion. Darüber hinaus wurde jedoch auch angenommen, dass der Investor sein Portfolio jederzeit anpassen kann. Da diese Annahme von eher theoretischer Natur ist, werden wir in dieser Arbeit einen illiquiden Markt zugrunde legen, in welchem der Investor nur an zufälligen Zeitpunkten sein Portfolio anpassen kann.
Seit der Finanzkrise 2008 gehören starke Kursschwankungen zum Alltag an den Börsen dieser Welt. Deshalb sind Investoren heutzutage mehr und mehr an risikobewussten Anlagestrategien interessiert. Um dieses Verhalten in den Finanzmarkt integrieren zu können, führen wir in dieser Arbeit eine Verlustschranke ein. Diese garantiert einerseits ein Minimum an Vermögen und andererseits einen festgelegten Prozentsatz der erwirtschafteten Gewinne.

Im 2. Kapitel dieser Arbeit führen wir zunächst inhomogene Lévy Prozesse ein, da diese bei der Modellierung der Rendite der risikobehafteten Anlage eine zentrale Rolle spielen. Wir zeigen dabei, dass inhomogene Lévy Prozesse sowohl zur Gruppe der additiven Prozesse als auch zu den Semimartingalen zählen. Des Weiteren werden wir eine Einschränkung vornehmen, so dass das exponentielle Moment dieser Prozesse existiert. Unter diesen Voraussetzungen zeigen wir auch, dass die betrachteten inhomogenen Lévy Prozesse sogar spezielle Semimartingale sind. Darüber hinaus studieren wir das stochastische Exponential eines solchen Prozesses und beweisen weitere Hilfsresultate für die nachfolgenden Kapitel. Zum Schluss betrachten wir inhomogene Poisson Prozesse, welche einen Spezialfall der eingeführten Prozesse darstellen.

In Kapitel 3 führen wir das Endnutzenmaximierungsproblem ein. Dazu wird zunächst ein illiquider Markt aufgestellt. Dieser besteht aus einer risikolosen Anlage, einer risikobehafteten Anlage und zufälligen Zeitpunkten, an welchen das Handeln der Anlagen möglich ist. Diese zufälligen Handelszeitpunkte sind durch die Sprungzeitpunkte eines inhomogenen Poisson Prozesses gegeben. Dann betrachten wir einen Investor in diesem Markt und führen seinen Vermögensprozess, die zugehörige Verlustschranke sowie die Menge der

zulässigen Strategien ein. Am Ende des Kapitels formulieren wir noch das Endnutzenmaximierungsproblem und diskutieren zwei unterschiedliche Klassen des zugrundeliegenden inhomogenen Poisson Prozesses.

Dieses Endnutzenmaximierungsproblem lösen wir unter der Annahme eines beschränkten Intensitätsprozesses des inhomogenen Poisson Prozesses in Kapitel 4. Dabei reduzieren wir zunächst das ursprünglich zeitstetige Optimierungsproblem auf ein zeitdiskretes Problem – ein sogenanntes Markoffsches Entscheidungsmodell – und lösen dieses. In Theorem 4.10 beweisen wir, dass die Wertfunktion der eindeutige Fixpunkt des maximalen Gewinnoperators ist und dass eine optimale stationäre Strategie existiert. Zum Schluss zeigen wir noch, dass man unter der Annahme einer CRRA Nutzenfunktion die Wertfunktion separieren kann.

In Kapitel 5 lösen wir dann das Endnutzenmaximierungsproblem unter der Annahme eines unbeschränkten Intensitätsprozesses des inhomogenen Poisson Prozesses. Dabei ist zu beachten, dass die zufälligen Handelszeitpunkte in diesem Fall monoton steigend gegen den endlichen Horizont des Investors konvergieren. Deshalb kann der Investor sein Endvermögen am Horizont nicht beobachten. Es existiert jedoch der linksseitige Grenzwert des Vermögensprozesses in diesem Zeitpunkt und dieser kann vom Investor beobachtet werden. Wie in Kapitel 4 können wir nun auch dieses ursprünglich zeitstetige Optimierungsproblem auf ein zeitdiskretes Problem – ein sogenanntes limsup Markoffsches Entscheidungsmodell – reduzieren und dieses lösen. In Theorem 5.12 beweisen wir, dass die Wertfunktion der eindeutige Fixpunkt des maximalen Gewinnoperators ist und dass es eine optimale stationäre Strategie gibt. Unter der Annahme einer CRRA Nutzenfunktion gibt es, wie auch im vorhergehenden Kapitel, einen Separationsansatz für die Wertfunktion.

Schließlich zeigen wir, dass es unter einer kleinen Einschränkung möglich ist das Endnutzenmaximierungsproblem mit unbeschränktem Intensitätsprozess mit Hilfe von Endnutzenmaximierungsproblemen mit beschränkten Intensitätsprozessen zu approximieren. Diese Approximation umfasst sowohl die Konvergenz der Wertfunktionen als auch die Konvergenz der optimalen Strategien.

Im letzten Kapitel illustrieren wir die Ergebnisse der vorhergehenden Kapitel an Beispielen. Dazu nehmen wir einen klassischen Black-Scholes Markt und eine Potenznutzenfunktion an. Folglich können wir den Separationsansatz der Wertfunktion verwenden und anschließend das Problem für verschiedene Intensitätsprozesse und Prozentsätze der Sicherung lösen. Darüber hinaus ist zu erkennen, dass eine verallgemeinerte Merton Strategie, welche den Intensitätsprozess vernachlässigt, eine sehr gute Approximation der optimalen Strategie ergibt.

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Declaration

I hereby declare that this thesis was performed and written on my own and that references and resources used within this work have been explicitly indicated.

I am aware that making a false declaration may have serious consequences.

Ulm, September 2011

(Signature)