Estimation and Optimization Problems in Power Markets

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften
der Universität Ulm

vorgelegt von
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Ulm, im Dezember 2011

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Tag der Promotion:	28. Februar 2012

Acknowledgement

First and foremost I would like to express my deepest gratitude to Professor Rüdiger Kiesel for his constant support and encouragement throughout the last three years. Professor Kiesel has not only made this doctoral thesis possible: Besides creating a great working atmosphere he gave me the opportunity to meet and collaborate with numerous people from different institutes and universities. In that way he encouraged the collaboration with the University of Oslo and particularly Professor Fred Espen Benth. I am deeply grateful to Professor Benth for hosting me and providing me with fruitful discussions and comments.

My very special thanks go to Professor Ulrich Rieder for being the co examiner of this thesis.

Furthermore, I would like to thank Richard Biegler-König and Stefan Ehrenfried for providing me with numerous comments, ideas and discussions that contributed to this work.

The great atmosphere at the DFG Research Training Group 1100 would not have been possible without Achim Gegler, Mario Rometsch, Kristina Steih and Christoph Wopperer to just mention a few. I greatly appreciate the financial support of the DFG.

My special thanks also go to the Institute of Financial Mathematics and the UZWR for hosting me in several coffee breaks.

Most importantly I would like to thank Sebastian Kestler, Mario Rometsch and my father Uwe Jensen for proofreading this thesis several times.

Last but not least everything would not have been possible without the constant love and support of my familiy. My father encouraged and supported my studies from the very beginning until the last days of this thesis. I deeply appreciate it.

Contents

Chapter 1

Introduction

1.1 The Liberalized Energy Market

The liberalization of energy markets has induced generators, suppliers and large-scale end users to trade actively on the market. Actors like energy utilities have a variety of trading relations for the purchase and sale of electricity that are about to abandon the (still) wide-spread long-term full supply and purchase contracts. An energy utility has (or is modeled by) a portfolio of purchase and supply contracts for electricity. Any market movement leads to a change of purchase and sales possibilities and thus to a change of the portfolio value. In that sense portfolio analysis is understood as the process of measuring and controlling the ratio of risk and return of the portfolio. An energy utility having different fuel supply sources in contrast to a long-term full supply contract faces different types of risks in the liberalized energy market. The sources of risk are wide-ranging, just to name the market price risk, fuel price risk, risk of investing in production capacities or the volume risk. Thus portfolio management is closely related to risk management and the plant managers need a tool to quantify these risks. Therefore it is necessary to employ techniques that accurately incorporate the **uncertain environment** in the portfolio and risk management process. Uncertainty in the electricity market is additionally evoked by a number of factors such as political changes, weather changes or plant outages. Looking at historical electricity spot price series clearly reflects that uncertain environment and sets them apart from stock prices or equity index values. The series show sudden increases in value (known as the electricity spikes) and high levels of volatility. Besides that, they show a tendence to revert to a long term mean level. Such a behavior is often referred to as the mean reverting property of electricity prices. Moreover, one detects a seasonal pattern. The **spot market** is a market, where goods are traded for "immediate" delivery (at the next day), also called the *day-ahead market*. The units of production are sold into cash markets and are not only traded on a forward basis. Consequently, spot bidding happens before all relevant quantities are known. Thus the realized cash-flows can be negative.

Another crucial factor of electricity markets is the fact that financial derivatives written on the underlying spot prices cannot be traded and valued in the conventional way. This is due to the (general) non-storability and non-existence of liquid markets for these financial products. However, with regard to risk management purposes these financial products written on the spot price of electricity play a crucial role. **Forward contracts** for example are used to hedge against spot price risks. A complication in the electricity market is the fact that forward contracts that deliver the underlying electricity at a fixed maturity time are not traded but so-called *swap contracts* delivering a continuous flow of electricity over a pre-specified future period of time (e.g. a month or a quarter).

After these introductionary remarks to quote Carmona & Durleman [CD03]: "The diversity of the statistical characteristics of the underlying indexes on which financial products are written together with the extreme complexity of the derivatives traded, makes the analysis of these markets an exciting challenge to mathematically inclined observers."

1.2 Objective of Chapter 2

Within such a challenging environment we take the view of an electricity utility such as a gas-fired power plant. Immediately a number of questions arise: *How to reflect the specific properties of electricity prices? Is it possible for such a complex market to make reliable predictions of future price developments in order to assess the environmentally-induced risks? Based on such a possible price forecasting model, can we deduce a plants' value and an optimal operating schedule?* These questions motivate the considerations elaborated throughout Chapter 2 of this thesis. It focuses on the presentation and discussion of an adequate model for the comovement of electricty and gas prices. Such a model is then used for the purpose of risk management and to calculate a gas fired stations' plant value by a series of spark spread options.

Power Plants and Spark Spread Options A power plant provides the owner the right but not the obligation to transform fuel into power. In that sense the main driver in the valuation of a gas-fired power plant is the difference between power output prices and gas input costs, the **spark spread**. The spark spread is known as the theoretical gross margin of a gas-fired power plant from selling one unit of electricity, having bought the fuel required to produce this unit of electricity. Applying pricing theory of derivatives to the option opportunities in "real-life" decisions is often termed real option valuation. Here we apply the real options approach in the sense that we model the instantaneous plant pay-off per unit of production as the pay-off of a spark spread option. Valuing a power plant using real options theory has two main purposes in competitive markets: "First, an investor who contemplates the purchase or sale of a power plant must accurately determine its value. Second, the real options theory facilitates the use of risk managememt tools developed for financial markets in order to hedge both asset value and earnings" (Gardner and Zhuang [GZ00]). Many authors (such as De Jong [DJW07]) argue, that for optimal management and realistic valuation the spark spread optionality needs to be modelled explicitly. Hence an adequate model for the comovement of electricity and gas prices must be found, since it builds the essential tool to address the involved price risk. The price of a spread option is given by an expectation over the sample paths of the solution of a system of stochastic differential equations describing the model dynamics. Thus our main objective within this chapter is the development of a bivariate spot price model that reflects the peculiar characteristics of electricty and gas prices at the same time.

From the beginning of this century **regime-switching models** have shown great potential in capturing the spot price dynamics of electricity. Within such a model the dynamics are modulated by a Markov chain representing different regime states. Typically the regime separation (that basically goes back to Hamilton [Ham90] in 1990) is used to reflect the systematic alternations between stable and unstable states of prices, referred to as the normal and the spike states. As many authors argued such alternations are based on imbalances between supply and demand. Especially the spikes can be initiated by extreme weather changes or generation outages. Hence, the subsequent sections provide a detailed description of the regime-switching model that forms the basis for managing and valuing a real asset - the gas-fired power plant. Usually in financial markets liquidly traded options are used to calibrate the pricing models. Often it is not necessary to analyse historical data extensively. However, in electricity markets prices of options are often not directly observable or those products are not standardized. So the need for a simulation model used to generate sample paths reflecting the time evolution of prices in a realistic way is much greater. The model we propose is thus fitted to historical market prices.

Outline of Chapter 2. We begin in Section 2.1 with introducing the two-factor regime-switching model goverened by a Markov chain having finite state space. Different model approaches are presented and a discrete version along with the probabilistic features of the underlying observation processes is derived. The complex calibration routine and the optimal parameter values are presented in detail throughout Section 2.2 up to Section 2.5. Particular care is taken on the implications provoked by the unobservability of the modulating Markov chain. Finally, in Section 2.6 the generated sample paths are used to calculate the plant value of a gas-fired power station by a series of spark spread options.

1.3 Objective of Chapter 3

The spark spread valuation does not account for certain operational constraints such as start-up costs, ramp-up constraints or capacity constraints among others. As argued by several authors (such as Gardner & Zhuang [GZ00] or Deng & Oren [DO03]) these constraints may have a significant impact on the plant value and an operating strategy. Again a number of questions arise: *How to determine the optimal operating schedule and plant value, when considering (at least a few) operational constraints? To what extend should forward contracts be used to hedge against spot price risks?* That leads to the central question addressed within Chapter 3: *How much of its generation capacity should a power plant devote to forward contracts and how much should it keep for bidding on the spot market?*

The path dependence of the involved sequential decision-making process immediately suggests to use a **stochastic dynamic programming** (SDP) representation of the problem. Hence, the focus of the third chapter is to set up and solve an SDP representation of the problem of optimally scheduling and valuing a power plant, when spot price risks are hedged by selling generation capacity through forward contracts.

We contribute by stating existence and uniqueness of an optimal solution to the **optimal allocation problem** introduced in terms of SDP. In contrast to Chapter 2 spot bidding is made under uncertainty. Thus the bidding on the spot market happens before the “day-ahead” spot price is known. With the added complexity of operational constraints, forward hedging and uncertain spot bidding the use of a univariate price model (in contrast to the complex two-factor modelling presented in Chapter 2) is reasonable. However, we provide the theoretical SDP model framework in such a general form, that the use of a model for the spread, i.e. the difference between the price of electricity and the price of the fuel required to produce that electricity, is straightforward. Due to the path dependence of the problem a one factor model is yet preferable especially with regard to a numerical analysis. Within such a dynamic programming ansatz usually a **utility function** representing the risk preferences of the individual decision maker(s) is incorporated. Since it is not clear which utility to choose, it is desirable to keep it in a general form. We solve the problem by backward induction techniques for a general class of utility functions, namely the class of concave, continuously differentiable utility functions. For that class we state uniqueness and existence of an optimal policy and the applicability of the iterative dynamic programming technique. Furthermore, a specific structure of the value function is derived.

To understand the impact of the market movements and the operational constraints and to understand the interaction of forward selling and spot bidding the path dependent problem is numerically analyzed by approximating the underlying price processes by **lattice structures**. The implications of different such market structures are discussed and compared by calculating the corresponding optimal policies and value functions solving the SDP problem. Finally, the so derived optimal company specific (sales) portfolio based on the underlying market, the imposed operational characteristics and the hedging strategy is used for plant valuation. Moreover, the classical tools to quantify and assess risks of the plants sales portfolio are applied.

Outline of Chapter 3. Chapter 3 is presented in three main parts: After motivating the economic side of the problem in Section 3.1 the allocation and valuation problem is set up and analyzed in mathematical terms in detail in Sections 3.2 up to 3.4. Existence and uniqueness of the solution along with the specific structure of the value function are stated in Section 3.5. Last but not least, Section 3.6 applies the model framework to the Cox Ross Rubinstein market. The derived values then form the benchmark for judging the values calculated based on a one factor model representation (consistent with market data and approximated by a trinomial tree structure). The one factor model is presented in Section 3.7 and its implications are discussed with regard to the benchmark in Section 3.8.

Conclusion of the Thesis. A summary of results and ideas gained on the one hand by fitting a two-factor regime-switching model for the comovement of electricity and gas prices to market data and by adopting these price forecasts for plant valuation and risk management purposes and on the other hand by setting up and solving a path-dependent SDP representation of the optimal plant operation and valuation problem, focused on hedging spot price risks through forward selling of generation capacity, is presented in the concluding Chapter 4.

Chapter 2

Regime Switching Model for the Comovement of Electricity and Gas Prices

2.1 Regime-Switching Two-Factor Model

For the pricing of physical and financial contracts and for the valuation of real assets related to power markets the peculiar characteristics of prices have induced the development of special electricity price models.

In mathematical finance the classical models belong to the class of semi-martingale processes. These processes imply the existence of an equivalent martingale measure, such that the discounted price series are martingales. If there exists a unique such measure one speaks of a complete market with no arbitrage possibilities. In electricity markets however the underlying spot contracts are not liquid in the sense, that they can be bought and sold at any time or held in a portfolio over time. Thus the electricity market is considered to be incomplete, where there does not exist a unique such measure but any probability measure $\mathbb{P}^*$ being equivalent to the real world measure $\mathbb{P}$ is an equivalent martingale measure. Most spot price models are defined to follow continuous time stochastic processes generating discrete observations, due to spot prices only being quoted on an hourly basis (compare [eex]).

(One of) the model(s) forming the basis for many subsequent approaches to model the spot electricity evolution is the famous Schwartz' model [Sch97]. At the heart of that classical commodity price model is the mean-reverting feature reflected by dynamics stemming from the exponential of an Ornstein Uhlenbeck (OU) process.

The Schwartz' model belongs to the class of geometric models. Arithmetic models in contrast represent price dynamics by a series of OU processes. An elaborate treatment of these models with applications can be found in [BŠBK08]. Turning back to the geometric model class it is worth mentioning the two-factor model of Lucia & Schwartz [LS00]. It extends the Schwartz' model in the sense that it has one non-stationary process for the long-term equilibrium price level and another possibly correlated mean-reverting process for the short-term fluctuations.

The inclusion of jumps in the stochastic differential equation (SDE) governing the dynamics of underlying asset prices was first proposed by Merton [Mer76] and Cox & Ross [CR76] in 1979. After several market crashes and in the scope of the extreme volatile behavior of electricity prices these models have gained new interest. In that sense Villaplana [Vil03] extended the Lucia-Schwartz model in 2004 to include jumps in the short-term level, where the long-term process is now mean reverting. The idea of including jumps to the process dynamics is that they account for the spikes occuring after imbalances in supply and demand, the Brownian motion then models the small variations in electricity prices when normal trading takes place. Differentiating between spikes and small variations directly leads to regime-switching models as natural candidates. With the application of regime-switching models to power markets a promising way arised to include the spikes in the dynamics of electricity prices.

One of the first papers dealing with Markov switching regressions in econometrics is the one of Goldfeld & Quandt from 1973 [GQ73]. Motivated by the dramatic breaks in economic time series that are associated with financial crisis or abrupt changes in government policy the regime change reflects fundamental changes usually between a normal and an abnormal state. The likelihood function for such Markov switching regressions was first calculated by Cosslett & Lee [CL85] in 1985. In 1990 and 1994 Hamilton [Ham90] then formulated the problem in a way that all objects of interest are calculable as a by-product of an iterative algorithm similar to the Kalman filter (compare Section 2.2). Hamilton helped in a great way to popularize regime-switching models, since he was one of the first to adopt them to the financial and econometric world.

Allowing the incorporation of the most prominent features of electricity spot prices, such as mean reversion and spikes, but still treating the spikes as an integral part of the whole process, regime-switching models have become succesful in power markets. Deng [Den00], Huisman & Mahieu [HM03], Bierbrauer et al. [BTW05], De Jong & Huisman [DJH02], Weron [Wer06] or Geman & Roncoroni [GR06] have proposed such models. The popular model of Geman & Roncoroni is again a combination of a mean-reverting process and a jump process, where the jump component is state-dependent. The model defines a fixed threshold to separate different states without introducing more factors. Although it is not a regime-switching model in the usual sense, the threshold separation closely relates it. A thorough study of these different regime-switching models (especially the ones of Huisman, De Jong, Geman et al.) with applications to power markets can be found in the paper of De Jong [Jon06]. He tests "the nature of power spikes in a number of different markets" and finds that regime-switching models are better able to capture the market dynamics than a Garch(1,1) or Poisson jump model. The main drawback, according to the author, of e.g. the Poisson model combined with a mean reversion component is the need to set the mean reversion speed unrealistic high in order to pull prices back to a stable level. He further argues that for a proper valuation of e.g. real assets (like the spark spread options we want to consider) one needs to know to what extend spikes can be treated as independent events. The requirement of stochastic jump arrival probabilities according to De Jong directly leads to regime-switching models as natural candidate: *"They allow for distinct time-series behavior in different periods of time. The primary change is that the probability of a jump is no longer fixed but dependent on the current regime that the process is in."* (De Jong [Jon06])

Inter alia in the spirit of his work we have chosen to model the spark spread or more precisely the comovement of electricity and gas prices by a regime-switching model. We investigate different two-factor extensions and describe the challenging problem of estimating such models on data. Last but not least we apply the model to the valuation and risk managemet of a gas-fired power plant.

We begin with introducing the Markov chain modulating the two-factor regime-switching model in Section 2.1.1. Different model approaches are presented and a discrete version along with the probabilistic features of the underlying observation processes is derived in Sections 2.1.2 up to 2.1.4. The complex calibration routine and the optimal parameter values are presented in detail throughout Section 2.2 up to Section 2.5. Particular care is taken on the implications provoked by the unobservability of the modulating Markov chain. Finally, in Section 2.6 the generated sample paths are used to calculate the plant value by a series of spark spread options.

2.1.1 Introduction of the Markov Chain

To begin with, we introduce the hidden Markov chain representing regimes with higher and lower demand and hence governing the parameters of the underlying spot price processes.

For that, we first introduce the underlying probability space equipped with a filtration, which will be specified throughout the next sections.

Definition 2.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space endowed with a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ *, where* $\mathbb{T} := \{0, 1, \ldots, T\}$ *are the so-called trading dates. Let* $K \in \mathbb{N}$ *such that* $0 < K < T < \infty$ *and* $u \in [0, T]$ *denotes the continuous time index.*

Let s_t^x and s_t^y be two random variables, which can assume only the values $\{1, 2\}$. A value of one then indicates that the market is quiet, i.e. the price process follows the dynamics of the "normal" regime state. A value of two indicates that the market is in an abnormal state. The indices x and y indicate to the electricity and gas market, respectively. Now, for $t \in \mathbb{T}$ let the adapted sequence of random variables s_t be a homogeneous Markov chain in discrete time having a finite state space $\{1, \ldots, J\}$. Then e.g. the realization $s_t = 1$ represents the normal regime state, when both the electricity $(s_t^x = 1)$ and the gas $(s_t^y = 1)$ prices show the "normal" mean reverting character. Later on, having introduced the model framework we will deal with the finite state space $\mathcal{A}^s = \{1, 2, 3, 4\}$ corresponding to the Markov chain $s := (s_t)_{t\in\mathbb{T}}$. The different possible regime states are specified as follows

$$s_t = \begin{cases} 1 & \text{if} \quad s_t^x = 1 \quad \text{and} \quad s_t^y = 1 \\ 2 & \text{if} \quad s_t^x = 2 \quad \text{and} \quad s_t^y = 1 \\ 3 & \text{if} \quad s_t^x = 1 \quad \text{and} \quad s_t^y = 2 \\ 4 & \text{if} \quad s_t^x = 2 \quad \text{and} \quad s_t^y = 2 \end{cases} . \tag{2.1}$$

In order to be able to extend the state space, which becomes necessary during the calibration procedure introduced in Section 2.2, additionally we want to state the general case. That is, the Markov chain s can assume values in the finite set $\{1, \ldots, J\}$. Then the probability that s_t equals some particular value $j \in \{1, 2, \ldots, J\}$ depends on the past only through the most recent value s_{t-1} , i.e. $\mathbb{P}(s_t = j | s_{t-1} = i, s_{t-2} = k, \ldots) = \mathbb{P}(s_t = j | s_{t-1} = i) = p_{ij}$. The $(J \times J)$ matrix $P := (p_{i,j})_{i,j=1,\ldots,J}$ containing the transition probabilities is then known as the transition matrix. For more details on Markov chains we refer the interested reader to the work of Hamilton [Ham94] or Bremaud [Bre01].

2.1.2 The Underlying Observation Process

Before introducing the specific form of the stochastic observation process underlying our different model approaches, we want to refer to the general class of (non-Gaussian) Ornstein Uhlenbeck (OU) processes and their probabilistic features. A profound study of OU processes related to the spot price modelling in electricity and related markets can be found in Benth, Benth & Koekebakker [BŠBK08]. They provide the necessary background in stochastic analysis for independent increment (II) processes. These semimartingale processes are then used to introduce and discuss the spot price modelling with OU processes and more specifically geometric models characterized by the exponential of a OU process. In the scope of our work we highly recommend their book for a collection of results about II processes, stochastic integration with respect to martingales, the Itô formula for semimartingales leading thereafter to a nice general representation of the geometric spot price model covering many other approaches. Moreover, they provide integrability conditions on the spot process to ensure well-defined moments for the class of exponential models.

Two-Factor Regime-Switching Model for the Spark Spread. For the purpose of managing and valuing a gas-fired power plant the introduction of a two-factor extension of a regime-switching model with applications to power markets is at the heart of our work. From an economic point of view the model should reflect the specific situation of energy markets, where one observes imbalances of supply and demand leading to sudden and extreme price changes. Such a behavior will be incorporated through the model parameters and their ability to switch between different regimes representing various levels of supply and demand. Consider a specific point of time and assume the state of the Markov chain is given. Then the explicit form of the price dynamics is fixed by the process parameter combination associated with that state. Loosely speaking, such a parameter combination determines whether e.g. the electricity price is modeled close to or far away from a long-term level, possibly interpreted as the cost of production. Hence, the specific parameters implied by the state of the Markov chain determine to which state of the market the actual price belongs.

The stochastic processes (representing the comovement of logarithmic electricity and gas spot prices in such a specific regime state) are then basically given by mean reverting OU processes where the stochastic driver may allow for jumps. (When including seasonality the traditional notion of stationary properties of OU processes breaks down. However, analogue to [BŠBK08] we will keep the terminology in order to refer to a dynamic model with certain mean reversion properties.) Hence, the model incorporates the most prominent features of power markets in allowing for mean reversion and possible jumps.

Stochastic Fluctuation. We begin by introducing the stochastic driver of the model dynamics.

The Brownian motions model the small variations in prices. We will use two independent two dimensional Brownian motions $W^M := (W_u^1, W_u^2)_{u\in[0,T]}$ and $W^S := (W_u^3, W_u^4)_{u\in[0,T]}$, where especially for all $k, j = 1,2,3,4$ the Brownian motions W_u^k and W_u^j are independent for all $u \in [0,T]$. Moreover, $W = \{W^M, W^S\}$ is adapted to the standard Brownian filtration $\mathcal{F}^W$. Then W^M is accounting for the fluctuation in the "normal" states and W^S is accounting for the (small) fluctuation in the "abnormal" state. The state-dependent (two-dimensional) Brownian motion for all $t \in \mathbb{T}$

$$B(s) = ((B_u^x(s_t), B_u^y(s_t))^\intercal)_{u\in[0,T]} \tag{2.2}$$

is given by

$$\begin{aligned} B_u^x(s_t) &= \begin{cases} M_u^x := W_u^1 & \text{if} \quad s_t \in \{1,2\} \\ S_u^x := W_u^3 & \text{if} \quad s_t \in \{3,4\} \end{cases}, \\ B_u^y(s_t) &= \begin{cases} M_u^y := \rho W_u^1 + \sqrt{1-\rho^2}\, W_u^2 & \text{if} \quad s_t = 1 \\ M_u^y := W_u^2 & \text{if} \quad s_t = 3 \\ S_u^y := W_u^4 & \text{if} \quad s_t \in \{2,4\} \end{cases}, \end{aligned}$$

for all $u \in [0,T]$ with $\rho \in [0,1]$. We expect a non-negative correlation, hence we assume $\rho \in [0,1]$. From now on, the index i is introduced. It is either replaced by x (referring to the electricity price process) or it is replaced by y (referring to the gas price process).

Jump Part. A popular way to introduce the spikes in the price dynamics is by so-called compound Poisson processes. Additionally, we allow for a constant mean level of jumps, such that the jump processes $Z^i = (Z_u^i)_{u\in[0,T]}$ satisfy

$$dZ_u^i := f_J^i(s_t)du + J^i dq_u^i(s_t), \tag{2.3}$$

where the constant level f_J^i and the Poisson process $q^i = (q_u^i)_{u\in[0,T]}$ with intensity λ^i are dependent on the regime state s_t at time $t \in \mathbb{T}$. For all $u \in [0,T]$ we define $q_u^i(s_t) \equiv 0$ for all such s_t implying $\lambda^i(s_t) = 0$. The absolute size of the m-th jump in the log-scenario is given by

$$J_m^i := Z_{\tau_m^i} - Z_{\tau_{m-}^i},$$

where τ_{m-}^i denotes the left limit and the variables τ_m^i model the random times of occurence of the m-th jump corresponding to the Poisson process q^i, i.e $q_u^i =$

$\sum_{m\geq 1} \mathbb{1}_{\{\tau^i_m \leq u\}}$ for all $u \in [0,T]$. The jump times for $m = 1, 2, \dots$ are

$$\tau^i_m = \sum_{n=1}^{m} T^i_n,$$

where the so-called interarrival times T^i_n are independent, identically exponential distributed with parameter λ^i. As in Merton's jump-diffusion model we assume the m-th jump size J^i_m to be a normally distributed random variable with mean μ^i_J and standard deviation σ^i_J, i.e. $J^i_m \sim \mathcal{N}(\mu^i_J, \sigma^i_J)$, such that the jump sizes $J^i_1, J^i_2, \dots$ are independent, identically normal distributed random variables.

Bivariate Geometric Spot Price Model. After specifying the three independent stochastic driving processes, namely, B^i, q^i and J^i, it is left to account for the spoken to mean reverting and seasonality property of energy prices.

The deterministic seasonal price level is modelled by the function $\Lambda = (\Lambda^x, \Lambda^y)^\intercal : [0,T] \times [0,T] \to (0,\infty) \times (0,\infty)$, which is assumed to be continuously differentiable. It captures the seasonality in mean electricity and gas log-prices and is referred to as the seasonal function. The vector of model parameters is $\theta(s)$, an unknown parameter in a bounded set $\Theta \subset \mathbb{R}^d$, modulated by the hidden Markov chain s.

Bringing together the mean reversion component and the stochastic drivers the two factor spot price model (representing the comovement of logarithmic electricity and gas spot prices) is given by the next definition.

Definition 2.2 (Spot Price Model). *For any $t \in \mathbb{T}$ let $r = 1, 2, \dots, K$ with $r < t$ be fixed. The stochastic bivariate* ***spot price process*** *$S(u)$ for all $t - r \leq u \leq t$ is defined as*

$$\ln S(u) = \ln \Lambda(u) + z(u),$$

where given $z_{t-r} = (x'_{t-r}, y'_{t-r})^\intercal$ the right continuous process with left limits $z = (z_u)_{u\in[t-r,t]} = ((x_u, y_u)^\intercal)_{u\in[t-r,t]}$ is given by the unique strong solution to the system of SDE's

$$\begin{cases} dx_u = \left[-\alpha^x(s_t)\,(x_u - f^x(s_t)) + f^x_J(s_t)\right] du + \sigma^x(s_t) dB^x_u(s_t) + J^x\, dq^x_u(s_t) \\ \\ dy_u = \left[-\alpha^y(s_t)\,(y_u - f^y(s_t)) + f^y_J(s_t)\right] du + \sigma^y(s_t) dB^y_u(s_t) + J^y\, dq^y_u(s_t) \end{cases}, \quad (2.4)$$

where the level f^i, speed of mean reversion α^i, jump level f^i_J, the volatility σ^i, the Poisson process $q^i = (q^i_u)_{u\in[t-r,t]}$ with intensity λ^i and the Brownian motion $B^i = (B^i_u)_{u\in[t-r,t]}$, given according to (2.2)*, are dependent on the regime state s_t at time t. The jump components J^i and q^i are given according to the jump process Z defined in* (2.3)*.*

In order for this definition to be well-posed we need to know whether there exists such a unique strong solution. This is indeed the case as will be shown by Theorem 2.1 (on page 18).

The mean reverting coefficients $\alpha^i(s)$, $\sigma^i(s)$ and $f^i(s)$ in the price dynamics are assumed to be constant for all different states of the Markov chain. The constant mean reversion levels f^i can assume negative values. Whereas the speeds of mean reversion $\alpha^i(s)$ are non-negative for all possible states of the Markov chain s . The same is true for the volatility parameter $\sigma^i(s)$ scaling the fluctuations of the Brownian motion $B = ((B_u^x, B_u^y)^\intercal)_{u\in[0,T]}$.

Remark 2.1. (i) Let today's state of the Markov chain be $s_t = j$. Then today's observed prices $z_t' = (x_t', y_t')^\intercal$ are assumed to be generated by the process dynamics that are modulated by the associated parameters $\theta(j)$.

(ii) Let $s_t = 1$, i.e. today's prices are assumed to stem from the mean reverting regime. Then we might face the issue of latent prices: *"If prices were in a spike yesterday, we do not know from what level they have to revert today"* (De Jong [Jon06]). We will deal with latent prices and the associated transition densities in more detail in Section 2.3.

(iii) We pose the following assumptions:

Assumption 2.1. *If at time $t \in \mathbb{T}$ $s_t = j$ for any $j \in \mathcal{A}^s$, then the corresponding process dynamics prevailed (at least) over the period $[t-r, t]$ for given $r = 1, 2, \dots, K$ with $r < t$.*

Assumption 2.2. *With probability one the (bivariate) spot price process starts in the "normal" regime at date 0 , i.e. $\mathbb{P}(s_0 = 1) = 1$, and there exists at least one $j \in \mathcal{S}$ with $j \neq 1$ and $p_{1,j} > 0$.*

The preceeding assumptions will be of particular use in the calculation of those likelihood functions involving latent prices. These likelihoods will include the (r -stage) transition densities $f(z_t' \,|\, \Omega_{t-1},\, s_t = \ell;\, \theta)$, where ℓ refers to a regime state inferring latent prices.

2.1.3 Different Model Approaches

After introducing the Markov chain s modulating the bivariate observation process z , the different approaches we want to dicuss can be identified by imposing certain assumptions on the model parameters, when observed in a certain regime state. Thus, the size d of the $(1 \times d)$ parameter vector $\theta(s)$ in the bounded set $\Theta \subset \mathbb{R}^d$ needs to be adapted for every approach.

The **first approach** is the most simple case we want to study. It excludes the jump term of the price processes. Hence, the price processes are assumed to follow the dynamics of a mean reverting process not accounting for spikes. We are aware of assuming an obvious restriction of generality. However, this basic first approach is a good tool to introduce the model framework. In this case the Markov chain s can only assume the integer value $\{1\}$. The **second approach** then is the necessary extension including the possibility of jumps in the dynamics of the price processes. Hence, the Markov chain underlying this approach can assume four different states as defined in (2.1). Finally, the **third approach** is also based on the four state Markov chain s. It is identified by the distinct assumptions imposed (with regard to *Markov model I*) on the jump term.

Why are we dealing with all these different approaches? Firstly, the different approaches reflect the progress made in the development of electricity spot price models (here extended to the multivariate case): Starting with the very simple first approach, which is well suited to serve as a benchmark, several extensions have been derived. Secondly, the results of the calibration and simulation studies should not contradict each other and we can test whether the different model assumptions lead to the results expected in the specific case. Last but not least, we show the flexibility of the general model framework presented in this chapter. Due to the specific features of electricity (and also gas) prices it is not a trivial task to find a model reflecting those features adequately. Hence, working with several variants seems to be the right ansatz.

Now, all assumptions on the model parameters identifying our different approaches are listed. The reader, who wishes a quick start can immediately go to the summarizing part 2.1.3.2, where Table 2.1 provides an overview concerning all parameter specifications.

2.1.3.1 Specifying the Model Parameters

Recall, for any $t \in \mathbb{T}$ and $r = 1, 2, \dots, K$ with $r < t$ the dynamics of the observation process $z = ((z_u)^\intercal)_{u \in [t-r,t]}$ (as introduced in Definition 2.2) for given $z_{t-r} = (x'_{t-r}, y'_{t-r})^\intercal$ are given by

$$\begin{cases} dx_u = \left[-\alpha^x(s_t)\left(x_u - f^x(s_t)\right) + f_J^x(s_t)\right] du + \sigma^x(s_t) dB_u^x(s_t) + J^x\, dq_u^x(s_t), \\ \\ dy_u = \left[-\alpha^y(s_t)\left(y_u - f^y(s_t)\right) + f_J^y(s_t)\right] du + \sigma^y(s_t) dB_u^y(s_t) + J^y\, dq_u^y(s_t), \end{cases}$$

where all previously made specifications are supposed to be valid. Then the parameters specifying the different model approaches will be introduced throughout the following paragraphs.

Parameter according to the Benchmark and Markov Model I. The state dependent parameters according to the first and second approach for all $t \in \mathbb{T}$ are given by

$$\alpha^i(s_t) := \begin{cases} \alpha^i & \text{if } s_t^i = 1 \\ 0 & \text{if } s_t^i = 2 \end{cases}, \ \sigma^i(s_t) := \begin{cases} \sigma^i & \text{if } s_t^i = 1 \\ 0 & \text{if } s_t^i = 2 \end{cases}, \ f^i(s_t) := \begin{cases} f^i & \text{if } s_t^i = 1 \\ 0 & \text{if } s_t^i = 2 \end{cases},$$

where for $i = x$ and $i = y$ the parameters α^i and σ^i are constant and non-negative, whereas the constant parameters f^i and f_J^i can assume negative values. Given s_t at date $t \in \mathbb{T}$ the Poisson process $q^i(s)$ for all $u \in [t-r, t]$ has the intensity $\lambda^i(s_t)$. Then the state dependent spike regime parameters are given by

$$\lambda^i(s_t) := \begin{cases} 0 & \text{if } s_t^i = 1 \\ \lambda^i & \text{if } s_t^i = 2 \end{cases}, \ f_J^i(s_t) := \begin{cases} 0 & \text{if } s_t^i = 1 \\ f_J^i & \text{if } s_t^i = 2 \end{cases},$$

where for $i = x$ and $i = y$ the parameter λ^i is constant and non-negative. Further, the Poisson process q^i for all $u \in [t-r, t]$ satisfies $q_u^i(s_t) \equiv 0$ for all such s_t implying $\lambda^i(s_t) = 0$. The Brownian motion $B(s_t)$ for all $u \in [t-r, t]$ is given according to (2.2).

Remark 2.2. The difference between the so-called *Benchmark model* and *Markov model I* is determined by restricting the state space of the Markov chain. Within the *Benchmark model* the state space of the random variable s_t^i is reduced to the value $\{1\}$ for all $t \in \mathbb{T}$. That means, at any date $u \in [0, T]$ the prices are simply fluctuating around the mean level. Whereas, *Markov model I* includes the possibility of jumps to the price dynamics, i.e. $s_t^i \in \{1, 2\}$ for all $t \in \mathbb{T}$.

Parameter according to Markov Model II. The state dependent parameters according to the third approach for all $t \in \mathbb{T}$ are given by

$$\alpha^i(s_t) := \begin{cases} \alpha^i & \text{if } s_t^i = 1 \\ \alpha^i & \text{if } s_t^i = 2 \end{cases}, \ \sigma^i(s_t) := \begin{cases} \sigma^i & \text{if } s_t^i = 1 \\ \sigma^i & \text{if } s_t^i = 2 \end{cases}, \ f^i(s_t) := \begin{cases} f^i & \text{if } s_t^i = 1 \\ f^i & \text{if } s_t^i = 2 \end{cases},$$

where for $i = x$ and $i = y$ the parameters α^i and σ^i are constant and non-negative and $f^i \in \mathbb{R}$. Given s_t at date $t \in \mathbb{T}$ the Poisson process $q^i(s)$ for all $u \in [t-r, t]$ has the intensity $\lambda^i(s_t)$. Then the state dependent spike regime parameters are given by

$$\lambda^i(s_t) := \begin{cases} 0 & \text{if } s_t^i = 1 \\ \lambda^i & \text{if } s_t^i = 2 \end{cases}, \ f_J^i(s_t) := \begin{cases} 0 & \text{if } s_t^i = 1 \\ 0 & \text{if } s_t^i = 2 \end{cases},$$

where for $i = x$ and $i = y$ the parameter λ^i is constant and non-negative. Further, the Poisson process q^i for all $u \in [t-r, t]$ satisfies $q^i_u(s_t) \equiv 0$ for all such s_t implying $\lambda^i(s_t) = 0$. The Brownian motion $B(s_t)$ for all $u \in [t-r, t]$ is given according to (2.2).

Remark 2.3. Last but not least, *Markov model II* explicitly includes a jump specification to the dynamics of both the electricity and gas spot prices. The diffusive part is modelled by mean reverting processes. Whereas the jump part is modelled by mean reverting dynamics plus a normal compound poisson process. When the process follows the dynamics of the "normal" regime we assume to have a jump intensity of zero (i.e. we fix $\lambda^i(s_t) = 0$ for $s^i_t = 1$). According to our assumptions then no explicit jump term is included to the model being in such a regime state (i.e. $q^i \equiv 0$).

2.1.3.2 Summary

At the end of this section we summarize all different model approaches by listing the specific state dependent parameter assumptions determining the explicit form of the underlying stochastic observation process in Table 2.1.

Deterministic Part	$\alpha^i(s_t)$		$f^i(s_t)$		$f^i_J(s_t)$			
s^i_t	1	2	1	2	1	2		
Benchmark Model	α^i	–	f^i	–	–	–		
Markov Model I	α^i	0	f^i	0	0	f^i_J		
Markov Model II	α^i	α^i	f^i	f^i	0	0		
Stochastic Part	$\sigma^i(s_t)$		$B^i(s_t)$		$q^i(s_t)$		$\lambda^i(s_t)$	
s^i_t	1	2	1	2	1	2	1	2
Benchmark Model	σ^i	–	M^i	–	–	–	–	–
Markov Model I	σ^i	0	M^i	S^i	0	q^i	0	λ^i
Markov Model II	σ^i	σ^i	M^i	S^i	0	q^i	0	λ^i

Table 2.1: Model parameters governed by the hidden Markov chain s fixing the process dynamics corresponding to the different model approaches. All requirements posed in the according definitions are supposed to be valid.

Besides the parameter combinations discussed above, we recall the definition of the logarithmic spot price process $S(u)$, which is according to Definition 2.2 for any $t \in \mathbb{T}$ and $r = 1, 2, \ldots, K$ with $r < t$ for all $u \in [t-r, t]$ given by

$$\ln S(u) = \ln \Lambda(u) + z(u),$$

where $z = ((z_u)^\intercal)_{u \in [t-r,t]}$ is given through $z_{t-r} = (x'_{t-r}, y'_{t-r})^\intercal$ and the SDE's

$$\begin{cases} dx_u = [-\alpha^x(s_t)\,(x_u - f^x(s_t)) + f^x_J(s_t)]\,du + \sigma^x(s_t) dB^x_u(s_t) + J^x\,dq^x_u(s_t), \\ \\ dy_u = [-\alpha^y(s_t)\,(y_u - f^y(s_t)) + f^y_J(s_t)]\,du + \sigma^y(s_t) dB^y_u(s_t) + J^y\,dq^y_u(s_t), \end{cases}$$

modulated by the state of the Markov chain s_t at date t. The Markov chain $s = (s_t)_{t \in \mathbb{T}}$ is equipped with the finite state space $\mathcal{A}^s$ specified by

$$s_t = \begin{cases} 1 & \text{if} \quad s^x_t = 1 \quad \text{and} \quad s^y_t = 1 \\ 2 & \text{if} \quad s^x_t = 2 \quad \text{and} \quad s^y_t = 1 \\ 3 & \text{if} \quad s^x_t = 1 \quad \text{and} \quad s^y_t = 2 \\ 4 & \text{if} \quad s^x_t = 2 \quad \text{and} \quad s^y_t = 2 \end{cases}.$$

We will call the first approach **Benchmark model** since it constitutes the most basic form not including a spike specification. Moreover, in [] we have conducted a thorough statistical analysis of such a model. In contrast the second and third approaches include Markov modulated dynamics. Hence we will call them **Markov model I and II** from now on.

To make the summary complete, we point out the intuition behind our modelling approach: *This work is based on a two-factor regime-switching model for the comovement of electricity and gas price processes. One can imagine the comovement to be governed by two sets of processes over a certain period of time. One set depicting the dynamics of the electricity price process and one set depicting the dynamics of the gas price process. Each process is explicitly defined by the choice of its parameters. Thus, at any point of time on basis of the hidden Markov chain a specific choice of the process parameters is inferred. In other words the Markov chain forms the instrument, which determines the process of switching between the different proposed dynamics belonging on the one hand to the set of electricity and on the other hand to the set of gas price processes.*

2.1.4 Discrete Version of the Observation Process

As already stated, in this work we do not want to focus on finding a deterministic function Λ_u, modeling the seasonal level of the electricity and gas prices. At this point we refer to earlier studies done by Benth et al. [BŠBK08], Schindlmayr [Sch05], or De Jong [Jon06]. During the implementation process later on, we use the suggestion made by De Jong [Jon06] to extract seasonality from the daily spot prices. The seasonal component $\Lambda_u = (\Lambda^x_u, \Lambda^y_u)$ is for $i = x$ or $i = y$ given by

$$\Lambda^i_u = \theta^i_0 + \sum_{j=1}^{8} \theta^i_j\, D_{j,u} + \theta^i_9 \sin(\theta^i_5 \frac{2\pi u}{365.25}) + \theta^i_{10} EWMA^{0.975}_{u-1},$$

where $(EWMA^{0.975})$ is an exponentially weighted moving average with decay factor 0.975. Note, although we refer to Λ_u as being a deterministic function, it is in fact not having a moving average component. For reasons of generality with respect to other possible seasonal functions we remain with this notation. The seasonality during a week, public holidays and so-called semi-holidays is incorporated by 8 different dummy variables $(D_{j,u})$. The seasonality over the year is included by just one sinusoidal term, characterized by a location and a size parameter. For further specifications and details on this special seasonal function we refer the interested reader to the corresponding paper of De Jong [Jon06].

2.1.4.1 The SDE driving the OU process

Having introduced the general model framework and the specific assumptions posed on the bivariate dynamics reflecting the comovement of electricity and gas prices, the next step before starting the calibration procedure is to derive an explicit analytical solution to the system of SDE's driving the mentioned dynamics for all $u \in [t-r, t]$

Theorem 2.1 (Unique Strong Solution)**.** *Given all properties of the jump processes Z^i (stated in (2.3)) along with all properties of the Brownian motions B^i (stated in (2.2)), the logarithm of the deseasonalized spot prices, i.e. the unique strong solution z_t given $z_{t-r} = z'_{t-r}$ to the system of SDEs (introduced in Definition 2.2), for all $t \in \mathbb{T}$ with $r < t$ and $r = 1, 2, \ldots, K$ is given by*

$$
\begin{aligned}
z_t = \ln \frac{S(t)}{\Lambda(t)} \quad &= \quad \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t-r}\, e^{-\alpha^x(s_t)r} \\ y_{t-r}\, e^{-\alpha^y(s_t)r} \end{pmatrix} \\
&+ \quad \begin{pmatrix} f^x(s_t)\,(1 - e^{-\alpha^x(s_t)r}) \\ f^y(s_t)\,(1 - e^{-\alpha^y(s_t)r}) \end{pmatrix} + \begin{pmatrix} f_J^x(s_t)\, r \\ f_J^y(s_t)\, r \end{pmatrix} \\
&+ \quad \begin{pmatrix} \sigma^x(s_t)\, e^{-\alpha^x(s_t)t} \int_{t-r}^t e^{\alpha^x(s_t)u}\, dB_u^x(s_t) \\ \sigma^y(s_t)\, e^{-\alpha^y(s_t)t} \int_{t-r}^t e^{\alpha^y(s_t)u}\, dB_u^y(s_t) \end{pmatrix} \\
&+ \quad \begin{pmatrix} \sum_{m=q_{t-r}^x(s_t)+1}^{q_t^x(s_t)} e^{-\alpha^x(s_t)(t-\tau_m^x)}\, J_m^x \\ \sum_{m=q_{t-r}^y(s_t)+1}^{q_t^y(s_t)} e^{-\alpha^y(s_t)(t-\tau_m^y)}\, J_m^y \end{pmatrix},
\end{aligned}
$$

where the parameter $\alpha^i(s_t)$, $f_J^i(s_t)$, $\sigma^i(s_t)$, the Brownian motions $B^i(s_t)$ and the intensity of the Poisson process $q^i(s_t)$ are determined by the state s_t of the Markov chain at date $t \in \mathbb{T}$.

Proof. Let $i = x$ such that

$$
dx_u = [-\alpha^x(s_t)\,(x_u - f^x(s_t)) + f_J^x(s_t)]\, du + \sigma^x(s_t) dB_u^x(s_t) + J^x\, dq_u^x(s_t) \tag{2.5}
$$

with $x_{t-r} = x'_{t-r}$. When $i = y$ we can proceed analogously.

Uniqueness of the solution. Let x_1 and x_2 be two solutions to (2.5) and define the process $Z(t) := x_1(t) - x_2(t)$. Then right continuity follows from the properties of x_1 and x_2. Appealing to their dynamics, we find

$$Z(t) = -\alpha^x(s_t) \int_{t-r}^{t} Z(u)du$$

with $Z(t-r) = x'_{t-r} - x'_{t-r} = 0$. Then with Itô's Formula follows $Z(t) = Z(t-r)\, e^{-\alpha^x(s_t)r}$ and hence $Z(t) = 0$ for all $t \geq t-r$ such that uniqueness of the solution is established.

Existence of a strong solution. It is basically done working pathwise by applying the Itô formula for jump processes (also called Itô-Doeblin formula given by Proposition A.1 of the Appendix) to the right-continuous sample path x_u of the stochastic process x. The solution will then be called a strong solution on our given probability space, where the SDE arises.

We distinguish two cases induced by the possible states of the Markov chain s at date t. First we look at all those $s_t \in \mathcal{A}^s$ implying $\alpha^x(s_t) \neq 0$. This immediately implies $f^x_J(s_t) = 0$ for all such s_t (compare Table 2.1). Hence, on an interval $[t-r, t]$ with $r < t$ the Itô-Doeblin formula applied to the function $h(t, x_t) := e^{\alpha^x(s_t)t} x_t$ for a fixed state s_t of the Markov chain s, yields

$$\begin{aligned} h(t,x_t) = e^{\alpha^x(s_t)t}x_t &= e^{\alpha^x(s_t)(t-r)}x_{t-r} + \int_{t-r}^{t} e^{\alpha^x(s_t)u}\alpha^x(s_t)x_u du + \int_{t-r}^{t} e^{\alpha^x(s_t)u} dx^c_u \\ &+ \sum_{t-r<u\leq t} [e^{\alpha^x(s_t)u}x_u - e^{\alpha^x(s_t)u-}x_{u-}] \\ &= x_{t-r}\, e^{\alpha^x(s_t)(t-r)} + f^x(s_t)\,(e^{\alpha^x(s_t)t} - e^{\alpha^x(s_t)(t-r)}) \\ &+ \sigma^x(s_t) \int_{t-r}^{t} e^{\alpha^x(s_t)u}\, dB^x_u(s_t) + \sum_{m=q^x_{t-r}+1}^{q^x_t} e^{\alpha^x(s_t)\tau^x_m}\, J^x_m, \end{aligned}$$

where x^c_u denotes the continuous part of the process x at time u. The absolute size of the m-th jump in the log-scenario is given by $J^x_m := x_{\tau^i_m} - x_{\tau^i_{m-}}$, where all specifications are given according to (2.3). Finally, we obtain a solution for all such $s_t \in \mathcal{A}^s$ on an interval $[t-r, t]$

$$\begin{aligned} x_t &= x_{t-r}\, e^{-\alpha^x(s_t)r} + f^x(s_t)\,(1 - e^{-\alpha^x(s_t)r}) \\ &+ \sigma^x(s_t)\, e^{-\alpha^x(s_t)t} \int_{t-r}^{t} e^{\alpha^x(s_t)u}\, dB^x_u(s_t) + \sum_{m=q^x_{t-r}+1}^{q^x_t} e^{-\alpha^x(s_t)(t-\tau^x_m)}\, J^x_m. \end{aligned} \tag{2.6}$$

Let us now consider the other case, i.e., $s_t \in \mathcal{A}^s$ implies $\alpha^x(s_t) = 0$. We can further specify this event to be equivalent with the situation when $s_t^x = 2$ according to *Markov model I.* Hence, in such a model state x satisfies the differential equation

$$dx_u = dZ_u^x = f_J^x\, du + J^x\, dq_u^x. \tag{2.7}$$

Applying the Itô-Doeblin formula to the function $g(x_t) := x_t$, with x_u being a right-continuous sample path of the process x given in (2.7), yields

$$\begin{aligned} g(x_t) = x_t &= x_{t-r} + \int_{t-r}^{t} dx_u^c + \sum_{t-r<u\le t} (x_u - x_{u-}) \\ &= x_{t-r} + f_J^x\, r + \sum_{m=q_{t-r}^x+1}^{q_t^x} J_m^x, \end{aligned}$$

satisfying all assumptions made in the first case. Hence, for all such $s_t \in \mathcal{A}^s$ on an interval $[t-r,t] \subseteq [1,K]$ we obtain the solution

$$x_t = x_{t-r} + f_J^x\, r + \sum_{m=q_{t-r}^x+1}^{q_t^x} J_m^x. \tag{2.8}$$

After all, we can write the strong solution in the form of Theorem 2.1 by combining (2.6) and (2.8). The compact form considers the different values of $\alpha^x(s_t)$ for different states of the Markov chain s at date $t \in \mathbb{T}$ as specified in the different approaches explained in Section 2.1.3.

□

2.1.4.2 Discretizing the Analytical Solution

Throughout the calibration process, that will be outlined in the subsequent Section 2.2, we need a discrete version of the system dynamics introduced in Definition 2.2. The first step to derive such a discrete version has been accomplished by calculating for all $t \in \mathbb{T}$ an analytical expression for the unique strong solution (over an interval $[t-r,t]$) to the involved system of SDEs. The second step is then to provide an approximation of the integral terms involved in the derived solution z_t. For that, we use the results of Benth, Erlwein & Mamon in [EBM10] or the more general study of Benth as stated in [Ben11]. Thereafter, the diffusion term - being the essential ingredient of the continuous part x_t^c of z_t - is reasonably approximated (in distribution) over an interval $[t-r,t]$ by

$$\sigma^i(s_t)\, e^{-\alpha^i(s_t)t} \int_{t-r}^{t} e^{\alpha^i(s_t)u}\, dB_u^i(s_t) \approx \underbrace{\sigma^i(s_t)\sqrt{\frac{1-e^{-2\alpha^i(s_t)r}}{2\alpha^i(s_t)}}}_{:=\sigma_r^i(s_t)} R_r^i(s_t)\,,$$

where we assume $R_r^i(s_t) := (B_t^i(s_t) - B_{t-r}^i(s_t)) \sim \mathcal{N}(0, \sqrt{r})$. The approximation is reasonable in the sense that the left hand side is a normally distributed random variable with mean 0 and standard deviation $\sigma_r^i(s_t)\sqrt{r}$. (For more details compare Benth [Ben11].)

Further, we can approximate the jump term over $[t-r, t]$ by

$$\sum_{m=q_{t-r}^i(s_t)+1}^{q_t^i(s_t)} e^{-\alpha^i(s_t)(t-\tau_m^i)} J_m^i \underbrace{=}_{\text{in distr.}} \sum_{m=1}^{q_r^i(s_t)} e^{-\alpha^i(s_t)(r-\tau_m^i)} J_m^i := J_r^i(s_t)\,, \tag{2.9}$$

where τ_m^i are jump times in the interval $(0, r]$.

The last step is then to formulate a discrete version of (2.2) to establish the underlying random system that evolves in discrete time. To account for the small variations in prices evolving in discrete time we will use four independent identically distributed sequences of normal random variables $N^S := \{N_t^1, N_t^2\}_{t\in\mathbb{T}}$ and $N^M := \{N_t^3, N_t^4\}_{t\in\mathbb{T}}$ having zero mean and variance equal to r for all $t \in \mathbb{T}$ with $r < t$ and $r = 1, 2, \ldots, K$. Especially for $k, j \in \{1, 2, 3, 4\}$ with $k \neq j$ the random variables N_t^j and N_t^k are independent for all $t \in \mathbb{T}$. Then N^M account for the small variations in the "normal" states and N^S for those in the "abnormal" states. The state-dependent random vector

$$R(s_t) = ((R_r^x(s_t), R_r^y(s_t))^\intercal)_{t\in\mathbb{T}} \tag{2.10}$$

for all $r = 1, 2, \ldots, K$ with $r < t$ is given by

$$R_r^x(s_t) := \begin{cases} N_t^1 & \text{if} \quad s_t \in \{1, 2\} \\ N_t^3 & \text{if} \quad s_t \in \{3, 4\} \end{cases},$$

$$R_r^y(s_t) := \begin{cases} \rho N_t^1 + \sqrt{1-\rho^2}\, N_t^2 & \text{if} \quad s_t = 1 \\ N_t^2 & \text{if} \quad s_t = 3 \\ N_t^4 & \text{if} \quad s_t \in \{2, 4\} \end{cases},$$

for all $t \in \mathbb{T}$ with $\rho \in [0, 1]$.

With these approximations at hand we are in the position to formulate a discrete version which is equal in distribution to the observation process z_t, stated in Theorem 2.1.

Corollary 2.1. *Let $t \in \mathbb{T}$ and $r = 1, 2, \ldots, K$ such that $r < t$ then the derived discrete version of the observation process is given by*

$$
\begin{aligned}
z_t &= \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t-r}\, e^{-\alpha^x(s_t)r} \\ y_{t-r}\, e^{-\alpha^y(s_t)r} \end{pmatrix} \\
&+ \begin{pmatrix} f^x(s_t)\,(1 - e^{-\alpha^x(s_t)r}) \\ f^y(s_t)\,(1 - e^{-\alpha^y(s_t)r}) \end{pmatrix} + \begin{pmatrix} f_J^x(s_t)\, r \\ f_J^y(s_t)\, r \end{pmatrix} \\
&+ \begin{pmatrix} \sigma_r^x(s_t)\, R_r^x(s_t) \\ \sigma_r^y(s_t)\, R_r^y(s_t) \end{pmatrix} \\
&+ \begin{pmatrix} J_r^x(s_t) \\ J_r^y(s_t) \end{pmatrix},
\end{aligned}
$$

where the vector $\sigma_r := (\sigma_r^x, \sigma_r^y)^\intercal$ is given by

$$\sigma_r^i(s_t) := \sigma^i(s_t)\,\sqrt{\frac{1 - e^{-2\alpha^i(s_t)r}}{2\alpha^i(s_t)}}$$

and the vector $J_r := (J_r^x, J_r^y)^\intercal$ is given by

$$J_r^i(s_t) := \sum_{m=1}^{q_r^i(s_t)} e^{-\alpha^i(s_t)(r-\tau_m^i)}\, J_m^i.$$

The random jump times τ_m^i assume values in the interval $(0, r]$ according to the jump intensity λ^i of the corresponding Poisson process q_r^i and the state dependent random variables $R_r^i(s_t)$ satisfy all assumptions made in (2.10).

2.1.4.3 Probabilistic properties of the discretized Solution

At this point let us consider the *Benchmark model*, where $s_t = 1$ for all $t \in \mathbb{T}$. Then the bivariate model dynamics are governed by two OU processes fluctuating around a long term level. In that case the comovement is primarily described by the correlation parameter ρ (compare (2.2)). A profound study inter alia on the probabilistic features can be found in Jensen [Jen09]. Here we restrict ourselves to comment, that such a model evolves according to a joint bivariate normal distribution conditioned on the previously observed prices. The corresponding parameter estimates consitute the benchmark values for our empirical analysis (compare Section 2.5).

With regard to the Markov modulated models, i.e. *Markov model I* and *Markov model II*, let us first consider the cases $s_t \neq 1$ all modulating independent stochastic processes x and y. Note, the verification of independent jumps is postponed until Section 2.4 and supposed to be satisfied for now. Hence, we can restrict our considerations to the one-dimensional case. Moreover, it is sufficient to consider the dynamics of the (electricity price) process x. Then x_t conditioned on x_{t-r} is given by

$$x_t = x_{t-r}\, e^{-\alpha^x(s_t)r} + f^x(s_t)(1 - e^{-\alpha^x(s_t)r}) + f^x_J(s_t) + \sigma^x_r(s_t)R^x_r(s_t) + J^x_r(s_t)$$

for all $t \in \mathbb{T}$. Thereafter, let us take a look at the probabilistic properties of the distinct Markov modulated model approaches.

To begin with, we consider the model specifications posed within *Markov model II*. More specifically, we consider the marginal mean reverting process dynamics being present as soon as $s_t = 3$. Then given x_{t-r} the density of the **continuous part** denoted by x^c_t conditioned on the event $\{s^x_{t-r} = 1\}$ at time t is given by

$$\Phi^x_t(\theta|\, s^x_{t-r} = 1) := \phi\left(\cdot;\, e^x_{t,r},\, \sqrt{v^x_r}\right), \tag{2.11}$$

where $\phi(\cdot;\, \text{mean}, \text{sd})$ denotes the marginal density of the normal distribution with mean $e^x_{t,r} = f^x(1 - e^{-\alpha^x r}) + x_{t-r}\, e^{-\alpha^x r}$ and standard deviation (sd) $\sqrt{v^x_r} = \sigma^x_r \sqrt{r}$. Moreover, let the conditional density be characterized by the parameter vector θ (compare Section 2.2).
According to Erlwein et.al. [EBM10] we have seen in (2.9) that the **jump part** x^J_t with regard to (2.3) can be approximated on an interval $[t-r, t]$ by the term

$$\sum_{m=1}^{q^x_r} e^{-\alpha^x(r-\tau^x_m)}\, J^x_m = e^{-\alpha^x(s_t)r}(Z^x_t - Z^x_{t-r})$$

for $\tau^x_m \in (0, r]$ and $q^x_r \neq 0$. By the stationarity of the compound poisson process it holds $(Z^x_t - Z^x_{t-r}) \sim Z^x_r := x^J$. Thus, according to (2.3) the density of x^J_t is given by

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda^x r}(\lambda^x r)^k}{k!}\, \phi(\cdot;\, \mu^{Jx}, \sigma^{Jx}),$$

where $\mu^{Jx} := \mu^x_J\, e^{-\alpha^x r} k$ and $\sigma^{Jx} := \sigma^x_J\, e^{-\alpha^x r}\sqrt{k}$. Thereafter, pursuant to the results of Hanson and Westman [HW02] the density of x_t can be deducted as the convolution of the density of the continuous part x^c_t and the density of the jump part x^J_t. Altogether, if the process x is governed by regime $s_t \in \{2, 4\}$ at date t (and we are dealing with the specifications of *Markov model II*), the transition density of x_t conditioned on x_{t-r} is given by

$$\mathcal{Z}^x_t(\theta) := \sum_{k=0}^{\infty} \frac{e^{-\lambda^x r}(\lambda^x r)^k}{k!}\, \phi\left(\cdot;\, e^x_{t,r} + \mu^{Jx},\, \sqrt{v^x_r} + \sigma^{Jx}\right). \tag{2.12}$$

After that, we look at the dynamics suggested by *Markov model I*. The marginal mean reverting process dynamics of x are equivalent to the dynamics of *Markov model II*. Hence, again the density of the **continuous part** denoted by x_t^c conditioned on the event $\{s_{t-r}^x = 1\}$ at time t is given by (2.11). Attention needs to be payed to the derivation of the probabilistic features of the dynamics governed by the spike regime state, i.e. $s_t^x = 2$. According to *Markov model I* the price x_t given x_{t-r} being observed in the spike regime state is given by

$$x_t = x_{t-r} + f_J^x\, r + \sum_{m=1}^{q_r^x} J_m^x.$$

Thus, if $s_t^x = 2$ the continuous part of x is given by $x_t^c = x_{t-r} + f_J^x\, r$ and the remaining term $x_t^J = \sum_{m=1}^{q_r^x} J_m^x$ constitutes the **jump part**. Clearly, the mean of the continuous part is $\mu_t^{cx} = x_t^c$ and the standard deviation is $\sigma_t^c = 0$.
Turning to the jump part according to (2.3) we have $\mu^{Jx} = \mu_J^x k$ and $\sigma^{Jx} = \sigma_J^x \sqrt{k}$, where k represents the number of jumps on an interval of length r. Analogously, we obtain the transition density of x_t conditioned on x_{t-r} when the process is governed by one of the spike regime states $s_t \in \{2, 4\}$ at date t, i.e.

$$\mathcal{Z}_t^x(\theta) := \sum_{k=0}^{\infty} \frac{e^{-\lambda^x r} (\lambda^x\, r)^k}{k!}\, \phi(\cdot;\, \mu_t^{cx} + \mu^{Jx}, \sigma^{Jx}). \tag{2.13}$$

For such an analysis let us stress the importance of considering specific process dynamics, that have prevailed at least over the period $[t - r, t]$ by Assumption 2.1. As mentioned before (compare Remark 2.1) the Markov modulated models involve the so-called issue of latent prices. The key to solving the issue of latent prices will be in the choice of r, i.e. the number of past periods considered. It is reasonable (compare Section 2.4) to assume the absence of latent prices in the spike regime states. Thus the issue of latent prices solely arises within the calculation of transition densities corresponding to regime states inferring mean reverting diffusion dynamics. Nevertheless, the calculation of transition densities corresponding to regime states inferring jump-diffusion dynamics involves the well-known pitfalls, when estimating jump-diffusion models as encountered e.g. by Ait-Sahalia [AS04]. We will discuss and provide ways to deal with these challenges in Section 2.3 (accounting for the issue of latent prices) and Section 2.4 (accounting for the pitfalls in estimating jump-diffusion models). First of all, we want to explain the general procedure of calibrating the proposed model to data observed on the market.

2.2 Calibration Procedure in the case of a Regime-Switching Model

Which model should be chosen? Or to say it more accurately: Which particular parameter specification most adequately captures the comovement of electricity and gas prices? These fundamental questions involve two views. Firstly, this refers to the qualitative structure, i.e. the particular form of the diffusion term or the jump feature. In our case the particular form is determined by the different approaches described in Section 2.1.2. Secondly, the quantitative aspect of the model consists in the particular choice of numbers for the parameters.

In order to estimate values of the model parameters, we combine different methods. On the one hand we use two types of algorithms developed by Hamilton and Kim to obtain a **Bayesian inference** about the posterior probability distribution for the state of the Markov chain at some fixed point of time. On the other hand, the **Expectation Maximization** algorithm (EM) developed by Dempster, Laird and Rubin is used for finding maximum likelihood estimates of the model parameters taking into account that the model depends on unobserved latent variables.

As already mentioned, during the different steps of this calibration procedure one faces some difficulties. Throughout the subsequent sections we want to explain the procedure and address these issues. At first, Section 2.2.1 provides the general procedure along with an introduction to the used methods. The procedure mainly describes how to deal with the fact, that the regime state variable is not observable. Standard Maximum Likelihood (ML) is therefore not applicable. Thus, Hamilton [Ham94] provided a method using Bayesian inference techniques to derive a probability distribution for the unobservable regime state variable s_t.

Secondly, the issue of latent prices will be covered by Section 2.3. We contribute by presenting a method to derive analytical expressions for those likelihood functions involving latent prices. The method uses the already derived probability distribution for the unobservable regime state variable along with Bayes theory.

Last but not least, another well-known difficulty is the simultane calculation of parameter estimates for the jump and diffusion part of the underlying (bivariate) observation process. In the empirical jump-diffusion literature, such models (basically going back to Merton's (1976) option pricing model) are usually estimated with standard ML. However, the estimation is not as easy as it might appear. For example, Kiefer (1978) [Kie78] shows, that the likelihood function for some parametric specifications is unbounded, which causes inconsistency of standard ML. Along with a more profound discussion of the jump part of the processes this topic will be covered by Section 2.4.

2.2.1 The Iterative Calibration Procedure

This section provides the general procedure we have chosen to identify the quantitative aspect of the proposed bivariate spot price model. Hence, we want to determine a particular choice of parameter values for each model approach. Let

$$\Omega_t = \{z'_0, z'_1, z'_2, \dots, z'_t\}$$

be a set of observations obtained through date t. If the process is governed by regime $s_t = j$ for any $j = 1, 2, \dots, J$ at date t, then the conditional density of $z_t = z'_t$ is assumed to be given by

$$f(z'_t|\Omega_{t-1}, s_t = j; \theta) = \mathbb{P}(z_t = z'_t|\Omega_{t-1}, s_t = j; \theta), \tag{2.14}$$

where θ is the vector of parameters characterizing the conditional density. Note, conditioning the density on Ω_{t-1} in abuse of notation is the same as conditioning on the event $\{z_0 = z'_0, z_1 = z'_1, \dots, z_{t-1} = z'_{t-1}\}$. Generally, we deal with J different regimes. Thus, there are J different densities represented by (2.14) for $j = 1, 2, \dots, J$. These densities will be collected in the $(J \times 1)$ vector η such that $\eta_t^{(j)}(\theta) := f(z'_t|\Omega_{t-1}, s_t = j; \theta)$ for all j and t. Introducing the general procedure of identifying particular parameter values we leave the exact calculation for later and assume for now (2.14) to be given for $j = 1, 2, \dots, J$.

The general calibration procedure is derived as an iterative process based on Bayesian inference methods proposed by Hamilton and another algorithm proposed by Kim (to be found in Chapter 22.4 of [Ham94]). We combine these methods with the *Expectation Maximization* algorithm. The main issue we want to address within this section is the unobservability of the regime state variable. The unobserved regime s_t is presumed to have been generated by some probability distribution. A tool for finding such a probability distribution has been introduced by Hamilton in [Ham94] and from now on will be referred to as the **Hamilton filter**.

Before starting the particular steps of the calibration procedure, we introduce separately the methodology of the used filter and algorithms. Being familiar with those tools the reader can immediately go to Subsection 2.2.1.4 where Algorithm 2.4 summarizes the steps of the calibration procedure proposed in this work.

2.2.1.1 The Hamilton Filter

In his work Hamilton [Ham94] developed a recursive estimation method for the probability that the market is in a certain regime state at time t or equivalently that the Markov chain s assumes a specific value at time t. Recall, that method will be referred to as the *Hamilton filter*.

It can be described the following way: Assuming the vector of population parameters θ is known with certainty, it is possible to make an inference about the unobservable variable s_t, where $s := (s_t)_{t\in\mathbb{T}}$ is a homogeneous Markov chain in discrete time (compare Section 2.1.1). Let $\mathbb{P}(s_t = j \mid \Omega_t;\, \theta)$ denote the inference about the value of s_t based on the observed data $\Omega_t = \{z_0, z_1, \ldots, z_t\}$ through date t and based on knowledge of θ. We will refer to this conditional probability as **posterior probability**. The **prior probability** is then denoted by $\mathbb{P}(s_t = j|\Omega_{t-1};\, \theta)$, which can be imagined as a forecast of how likely the process is in regime j in period t given observations obtained through date $t-1$.
Optimal estimates for the inference and forecast (i.e. for the posterior and prior probabilities) can now be found by the subsequent algorithm: To begin with collect the *posterior probabilities* in a vector

$$\xi_{t|t} = \begin{pmatrix} \mathbb{P}(s_t = 1 \mid \Omega_t;\, \theta) \\ \vdots \\ \mathbb{P}(s_t = J \mid \Omega_t;\, \theta) \end{pmatrix}$$

and the *prior probabilities* in a vector

$$\xi_{t|t-1} = \begin{pmatrix} \mathbb{P}(s_t = 1 \mid \Omega_{t-1};\, \theta) \\ \vdots \\ \mathbb{P}(s_t = J \mid \Omega_{t-1};\, \theta) \end{pmatrix}$$

for all $t \in \mathbb{T}$.

Based on these introductionary assumptions the corresponding iterative Algorithm 2.1 includes three different steps.

Algorithm 2.1 (Hamilton Filter). Start the algorithm with $t-1 = k$ for some date $k < T$ such that the estimate $\hat{\xi}_{k|k}$ is known, i.e.

1. Calculate the *prior probability* at time t according to the rule

$$\hat{\xi}_{t|t-1} = P\,\hat{\xi}_{t-1|t-1}$$

where P is the $(J \times J)$ transition matrix.

2. Update the *posterior probability* at time t according to the rule

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{\mathbf{1}^T(\hat{\xi}_{t|t-1} \odot \eta_t)},$$

where $\odot$ denotes element-wise multiplication and $\mathbf{1}$ is a $(J \times 1)$ vector containing only unity entries. Using componentwise notation, that is dividing the total probability of observing z_t in state j by the total probability of observing z_t, we have

$$\mathbb{P}(s_t = j|\Omega_t;\, \theta) = \frac{\mathbb{P}(s_t = j|\Omega_{t-1}; \theta)\, f(z_t'|\Omega_{t-1}, s_t = j; \theta)}{\sum_{\ell=1}^{J} \mathbb{P}(s_t = \ell|\Omega_{t-1}; \theta)\, f(z_t'|\Omega_{t-1}, s_t = \ell; \theta)}.$$

3. Go back to 1. with $t \to t+1$ as long as $t \leq T$.

By iterating through the different steps one generates a probability distribution for the unobserved regime s_t given a value for the population parameter vector θ. As a byproduct one obtains the likelihood of observing the data over a time horizon $\{k, k+1, \ldots, T\}$. At each point of time the likelihood function is the weighted sum of the likelihood conditioned on the different regimes, where the weights are the prior probabilities for the corresponding regime states obtained through the algorithm. Thus we have

$$L(\theta) = \prod_{t=k}^{T} \sum_{j=1}^{J} \mathbb{1}_{\{s_t=j\}} \mathbb{P}(s_t = j|\Omega_{t-1};\, \theta)\, f(z_t'|\Omega_{t-1}, s_t = j; \theta)$$

or equivalently

$$L(\theta) = \exp\left\{ \sum_{t=k}^{T} \sum_{j=1}^{J} \mathbb{1}_{\{s_t=j\}} \log\left[\mathbb{P}(s_t = j|\Omega_{t-1};\, \theta)\, f(z_t'|\Omega_{t-1}, s_t = j; \theta)\right] \right\}.$$

Finally, the **log-Likelihood function** can be obtained as

$$\log L(\theta) = \sum_{t=k}^{T} \sum_{j=1}^{J} \mathbb{1}_{\{s_t=j\}} \log\left[\mathbb{P}(s_t = j|\Omega_{t-1};\, \theta)\, f(z_t'|\Omega_{t-1}, s_t = j; \theta)\right]. \tag{2.15}$$

Having established a way to calculate a probability distribution for the unobserved regime s_t and having an explicit form for the *log-likelihood function* $\log L(\theta)$, we have produced the basic requirements to address the problem of finding optimal parameter estimates.

In statistics an *Expectation Maximization* (EM) algorithm is used for finding maximum likelihood estimates of parameters in probabilistic models, where the model depends on unobserved latent variables. Hence, it constitutes the right tool for our application.

2.2.1.2 The Expectation Maximization (EM) Algorithm

EM is an iterative method alternating between performing an expectation step, which computes an expectation of the log likelihood with respect to the current estimate of the distribution of the latent variables and a maximization step, which computes the parameters maximizing the expected log likelihood found on the expectation step. Applying EM to the *log-likelihood function* given in (2.15) yields the particular steps of Algorithm 2.2.

Algorithm 2.2 (EM Algorithm).

1. **E-Step**: Given observations Ω_T obtained through date T based on the current estimate $\theta^{(m)}$ of the population parameter vector the expectation of the likelihood is

$$\mathbb{E}[\log L(\theta) \mid \Omega_T;\, \theta^{(m)}] = \sum_{t=k}^{T} \sum_{j=1}^{J} \mathbb{P}(s_t = j|\Omega_T;\, \theta^{(m)}) \cdot \log\left[\mathbb{P}(s_t = j|\Omega_{t-1};\, \theta^{(m)})\, f(z_t'|\Omega_{t-1}, s_t = j; \theta)\right],$$

   where $\mathbb{P}(s_t = j|\Omega_T;\, \theta^{(m)})$ is the current estimate of the probability of the latent variable s_t, determined via the so-called **Kim Algorithm** explained in Subsection 2.2.1.3.

2. **M-Step**: Choose θ to maximize the expectation obtained during the previous step, i.e.

$$\theta^{(m+1)} = \arg\max_{\theta} \mathbb{E}[\log L(\theta) \mid \Omega_T;\, \theta^{(m)}]$$

The *E-Step* of the EM algorithm is based on the current estimate of the probability distribution of the latent variables. In our setting, that means we need to find estimates for $\xi_{t|T} = \mathbb{P}(s_t = j|\Omega_T;\, \theta)$. These quantities represent so-called **smoothed inferences** about the regime the process was in at date t based on data obtained through some later date T. A tool for determining such smoothed inferences has been developed by Kim in 1993.

2.2.1.3 The Kim Algorithm

Running through Algorithm 2.1 generates a vector of *posterior probabilities* $\hat{\xi}_{t|t}$ and a vector of *prior probabilities* $\hat{\xi}_{t|t-1}$ for all $t \in \mathbb{T}$. These probabilities are then used to calculate the *smoothed inferences* by an application of the *Kim algorithm.* In vector form the algorithm can be described by the subsequent steps.

Algorithm 2.3 (Kim Algorithm).

1. Start with $\hat{\xi}_{T|T}$, the vector of *posterior probabilities* at time T.
2. Then $\hat{\xi}_{t|T}$ can be found by iterating backwards for $t = T-1, T-2, \ldots, k, \ldots, 1$ according to the rule

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \{P^T[\hat{\xi}_{t+1|T} \oslash \hat{\xi}_{t+1|t}]\},$$

where $\oslash$ denotes element-wise division and recalling P to be the transition matrix introduced in Section 2.1.1.

This algorithm is only valid if s_t follows a first-order Markov chain, which in our setting (compare Section 2.1.1) is the case.

2.2.1.4 The Calibration Procedure

Up to now, we have arranged all tools required in the setting of the discussed model to calculate estimates for the involved parameters. The estimation method then is in fact a combination of Maximum Likelihood to derive optimal parameter estimates (where we use a variant of the EM algorithm) and Bayesian inference to derive an estimate for the distribution of the latent variables (where we use the *Hamilton Filter* for optimal inferences and the *Kim Algorithm* for smoothed inferences).

Summarizing all particular steps necessary to calculate optimal parameter estimates in the light of the considered regime-switching spot price model (introduced in Section 2.1.2) finally yields Algorithm 2.4. Note that the involved modulating (J-state) Markov chain s satisfies all specifications as of Section 2.1.1.

Algorithm 2.4. Suppose the value of the parameter vector θ is known, i.e. make an initial guess denoted by $\theta^{(0)}$.

1. Determine the vector of *transition densities* η_t for all $t \in \{k, k+1, \ldots, T\}$, where

$$\eta_t^{(j)}(\theta) = f(z_t'|\Omega_{t-1}, s_t = j; \theta)$$

for $j = 1, \ldots, J$.

2. Start the *Hamilton Filter*, where the initial probabilities $\pi = \hat{\xi}_{k|k}$ can be found according to

$$\pi = (A^T A)^{-1} A^T b$$

with $\pi = (\pi_1, \pi_2, \dots, \pi_J)^T$, $A = \begin{pmatrix} I-P \\ \mathbf{1} \end{pmatrix}$, I the $(J \times J)$ identity matrix, P the $(J \times J)$ transition matrix, $\mathbf{1}$ a $(1 \times J)$ vector having only unity entries and b a $((J+1) \times 1)$ vector taking on the value unity for the $(J+1)$ th entry and the value zero otherwise. Thus one obtains a probability distribution for the unobservable regime state s_t, i.e at each point of time within the time horizon $\{k, k+1, \dots, T\}$ we obtain for $j = 1, \dots, J$ a prior probability of the unobservable regime state

$$\hat{\xi}^{(j)}_{t|t-1} = \mathbb{P}(s_t = j | \Omega_{t-1};\, \theta^{(m)})$$

based on the knowledge of $\theta^{(m)}$. As a by-product one obtains a value of the *log-likelihood function* for the observed data Ω_T evaluated at the value of $\theta^{(m)}$, i.e. $\log L(\theta^{(m)})$, calculated according to (2.15).

3. Calculate the vector of *smoothed inferences* $\hat{\xi}_{t|T}$ (and $\hat{\xi}_{t-r|t}$ for all $r \in \{1, 2, \dots, K\}$ with $r < t$) for all $t \in \{k, k+1, \dots, T\}$ using the *Kim Algorithm*, where e.g.

$$\hat{\xi}^{(j)}_{t|T} = \mathbb{P}(s_t = j | \Omega_T;\, \theta^{(m)})$$

for $j = 1, \dots, J$.

4. During the *E-Step* calculate the expectation of the *log-Likelihood function* with respect to the current estimate of the distribution for the latent variables, i.e.

$$\begin{aligned} \mathbb{E}[\log L(\theta) \,|\, \Omega_T;\, \theta^{(m)}] = & \sum_{t=k}^{T} \sum_{j=1}^{J} \mathbb{P}(s_t = j | \Omega_T;\, \theta^{(m)}) \\ & \cdot \log \left[\mathbb{P}(s_t = j | \Omega_{t-1};\, \theta^{(m)})\, f(z'_t | \Omega_{t-1}, s_t = j; \theta) \right]. \end{aligned}$$

5. During the *M-Step* determine the parameter values, that maximize the expected log likelihood found on the *E-Step* under some constraints. The estimates can be found by forming the Lagrangian

$$G(\theta) = \mathbb{E}[\log L(\theta) \,|\, \Omega_T;\, \theta^{(m)}] + g_1(1 - \sum_{j=1}^{J} \pi_j) + \sum_{\nu=1}^{J} g_{\nu+1}(1 - \sum_{\mu=1}^{J} p_{\nu,\mu}),$$

where $g_1, \dots, g_{J+1}$ are the Lagrangian multipliers. Calculating the derivative with respect to θ, setting the gradient equal to zero and solving for θ leads to a system of non-linear equations. That system cannot be solved analytically for θ as a function of the observed data Ω_T. The different functions corresponding to each entry of θ depend further on the current value $\theta^{(m)}$.

6. Then use these functions to obtain a new value for the parameter vector θ denote it by $\theta^{(m+1)}$. Now let $m \rightarrow m+1$.

7. Use $\theta^{(m)}$ to repeat all steps until it holds

$$|\theta^{(m+1)} - \theta^{(m)}| < \epsilon$$

for some small value $\epsilon > 0$.

Obviously, in order to perform the *E-Step* of the introduced calibration procedure we need an explicit form of the involved transition densities $f(z_t'|\Omega_{t-1}, s_t = j; \theta)$. As mentioned earlier, the derivation of these densities is not straightforward and will be covered by the next section.

2.3 The Issue of Latent Prices

Having the intuitive picture of a bivariate regime-switching jump diffusion model in mind, throughout the calibration procedure the next question arises: Switching between mean reverting dynamics represented by autoregressive time series and jump diffusion dynamics represented by (autoregressive time series plus) compound poisson processes - how does that work?

Such an issue becomes obvious when performing the *E-Step* of the introduced calibration procedure. Here an explicit formula for the probability $\eta_t^{(j)}(\theta) := f(z_t'|\Omega_{t-1}, s_t = j; \theta)$ is required. That is the probability of observing z_t given Ω_{t-1}, if the process z is governed by regime state s_t at date t based on knowledge of θ. To make the point more precise, we look at a concrete example:

Example 2.1. According to Section 2.1.4.3 the probability density function (or transition density) inferred by regime state $s_t = 1$ is given by the bivariate normal distribution. The transition density of z_t at date t corresponding to those **stable dynamics** is then of the form

$$\begin{aligned} f(z_t'|\Omega_{t-1}, s_t = 1; \theta) = & \\ & \frac{1}{2\pi\sqrt{1-\rho^2}\,\sqrt{v_1^x\, v_1^y}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \cdot \right. \\ & \left. \left(\frac{(x_t - e_{t,1}^x)^2}{v_1^x} + \frac{(y_t - e_{t,1}^y)^2}{v_1^y} - 2\rho \frac{(x_t - e_{t,1}^x)(y_t - e_{t,1}^y)}{\sqrt{v_1^x\, v_1^y}} \right) \right\} \\ & := \Phi_t^{xy}(\theta), \end{aligned} \tag{2.16}$$

where e.g. $e^x_{t,1} := f^x(1 - e^{-\alpha^x}) + x_{t-1}\, e^{-\alpha^x}$ and $v^x_1 := (\sigma^x_1)^2$ denote the conditional moments on the given probability space. However, assume at date $t-1$ the state $s_{t-1} = 4$ has been present such that according to the specifications made in Section 2.1.4.3 the transition density of z_{t-1} is assumed to be given by the product

$$f(z'_{t-1}|\Omega_{t-2}, s_{t-1} = 4; \theta) := \mathcal{Z}^x_{t-1}(\theta) \cdot \mathcal{Z}^y_{t-1}(\theta) \tag{2.17}$$

and call the inferred process dynamics - **jump dynamics**. (The reasoning for the assumption of independent jumps will be established in Section 2.4.) Now, the conditional probability density function given in (2.16) does not apply. How to choose the density after that?

Within that specific scenario we face the following situation: Although, today at time t the price is the realization of a mean reverting process at the previous point of time $t-1$ the regime state variable s_{t-1} indicates, that the price has been the realization of a jump(-diffusion) process. Hence, the distribution of the observed price at time t cannot as usually be conditioned on the previous observation z_{t-1}. We want to propose a way to condition the distribution on the last price observed in a regime state inferring again a mean reverting price process such that the parameter modulating the stable dynamics are estimated exclusively based on those "normal" prices. The first (to our knowledge) who addressed this issue were De Jong & Huisman in their work [DJH02]. Thereafter, also Kosater & Mosler [KM05] accounted for latent prices in the specification of the transition density. However, in view of the bivariate model structure we deal with a more complex problem that needs to be addressed.

If we face a situation as generated by Example 2.1, the distribution of the price z_t at time t must be conditioned on the first price z_{t-r} at time $t - r \in \mathbb{T}$ with $r < t$ and $r = 1, 2, \ldots, K$, that is again a realization of the process driven by *stable dynamics*. Now K denotes the maximum time span for which the transition density possibly needs to be calculated. In fact, given one such point of time $t - r$ the bivariate transition density is of the form

$$\begin{aligned} & f(z'_t|\Omega_{t-1}, s_t = 1, s_{t-r} = 1; \theta) \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}\,\sqrt{v^x_r\, v^y_r}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \cdot \right. \\ & \left. \left(\frac{(x_t - e^x_{t,r})^2}{v^x_r} + \frac{(y_t - e^y_{t,r})^2}{v^y_r} - 2\rho \frac{(x_t - e^x_{t,r})(y_t - e^y_{t,r})}{\sqrt{v^x_r\, v^y_r}} \right) \right\} \\ &:= \Phi^{xy}_t(\theta|s_{t-r} = 1), \end{aligned} \tag{2.18}$$

where e.g. $e^x_{t,r} := \mathbb{E}[x_t|x_{t-r}]$ and $v^x_r := \mathrm{Var}[x_t|x_{t-r}]$ denote the involved conditional moments on the given probability space. The explicit form of these moments are provided in (2.11).

After all, we notice the unobservability of the regime state. In order to calculate the desired transition densities the introductionary question can now be specified: How to determine that number of periods r one needs to consider, such that at date $t-r$ the process is again driven by mean reverting dynamics? In the course of tackling that concern, we will distinguish the marginal from the bivariate scenario.

2.3.1 Extension of the Markov Chain State Space

The intuitive first step to address the issue of latent prices pictured with the above scenario is then to look at the preceeding regime state. In order to do so, we extend the Markov chain s such that one state of the extended Markov chain $\tilde{s}$ supplies information about the actual regime state and about the most recent regime state from the past.

Definition 2.3 (Extended State Space). *Let $\mathcal{B} = \{1, 2, \ldots, 16\}$ be a finite state space in discrete time $t \in \mathbb{T}$, where at time t the elements are specified by*

$$\tilde{s}_t = \begin{cases} 1 & \textit{if} \quad s_t = 1 \quad \textit{and} \quad s_{t-1} = 1 \\ 2 & \textit{if} \quad s_t = 2 \quad \textit{and} \quad s_{t-1} = 1 \\ 3 & \textit{if} \quad s_t = 1 \quad \textit{and} \quad s_{t-1} = 2 \\ 4 & \textit{if} \quad s_t = 2 \quad \textit{and} \quad s_{t-1} = 2 \\ 5 & \textit{if} \quad s_t = 1 \quad \textit{and} \quad s_{t-1} = 3 \\ 6 & \textit{if} \quad s_t = 2 \quad \textit{and} \quad s_{t-1} = 3 \\ 7 & \textit{if} \quad s_t = 1 \quad \textit{and} \quad s_{t-1} = 4 \\ 8 & \textit{if} \quad s_t = 2 \quad \textit{and} \quad s_{t-1} = 4 \\ 9 & \textit{if} \quad s_t = 3 \quad \textit{and} \quad s_{t-1} = 1 \\ 10 & \textit{if} \quad s_t = 4 \quad \textit{and} \quad s_{t-1} = 1 \\ 11 & \textit{if} \quad s_t = 3 \quad \textit{and} \quad s_{t-1} = 2 \\ 12 & \textit{if} \quad s_t = 4 \quad \textit{and} \quad s_{t-1} = 2 \\ 13 & \textit{if} \quad s_t = 3 \quad \textit{and} \quad s_{t-1} = 3 \\ 14 & \textit{if} \quad s_t = 4 \quad \textit{and} \quad s_{t-1} = 3 \\ 15 & \textit{if} \quad s_t = 3 \quad \textit{and} \quad s_{t-1} = 4 \\ 16 & \textit{if} \quad s_t = 4 \quad \textit{and} \quad s_{t-1} = 4. \end{cases}$$

Then, $\tilde{s}$ is defined as a homogeneous Markov chain with state space $\mathcal{B}$ fulfilling all requirements made on the Markov chain s introduced in Section 2.1.1. The *corresponding* (16×16) *transition matrix* $\tilde{P}$ *assembles to*

$$\tilde{P} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_1 & 0 & 0 \\ 0 & 0 & \tilde{P}_3 & \tilde{P}_3 \\ \tilde{P}_2 & \tilde{P}_2 & 0 & 0 \\ 0 & 0 & \tilde{P}_4 & \tilde{P}_4 \end{pmatrix} \tag{2.19}$$

where

$$\tilde{P}_1 = \begin{pmatrix} p_{11} & 0 & p_{11} & 0 \\ p_{12} & 0 & p_{12} & 0 \\ 0 & p_{21} & 0 & p_{21} \\ 0 & p_{22} & 0 & p_{22} \end{pmatrix}, \quad \tilde{P}_2 = \begin{pmatrix} p_{13} & 0 & p_{13} & 0 \\ p_{14} & 0 & p_{14} & 0 \\ 0 & p_{23} & 0 & p_{23} \\ 0 & p_{24} & 0 & p_{24} \end{pmatrix},$$

$$\tilde{P}_3 = \begin{pmatrix} p_{31} & 0 & p_{31} & 0 \\ p_{32} & 0 & p_{32} & 0 \\ 0 & p_{41} & 0 & p_{41} \\ 0 & p_{42} & 0 & p_{42} \end{pmatrix}, \quad \tilde{P}_4 = \begin{pmatrix} p_{33} & 0 & p_{33} & 0 \\ p_{34} & 0 & p_{34} & 0 \\ 0 & p_{43} & 0 & p_{43} \\ 0 & p_{44} & 0 & p_{44} \end{pmatrix}$$

and $p_{\ell j}$ *are the probabilities of transitioning from regime state* ℓ *to regime state* j *for all* $\ell, j = 1, \ldots, 4$ *.*

Thus, the observation process z is now supposed to be modulated by the Markov chain $\tilde{s}$, which contains information about the regime state both from today and from the previous point of time. To derive the vector of transition densities η, we define different associated subsets of the state space $\mathcal{B}$ denoted by $\mathcal{L} := \mathcal{L}^x \cup \mathcal{L}^y \cup \mathcal{L}^{xy}$ and $\mathcal{S} := \mathcal{S}^x \cup \mathcal{S}^y \cup \mathcal{S}^{xy}$. For any state within one of these sets the derivation of the desired transition densities then works analogously.

Regime State Sets. Let

$$\mathcal{L}^x = \{11, 15\} \quad \text{and} \quad \mathcal{L}^y = \{6, 8\}$$

be the set of all states $\tilde{s}_t \in \mathcal{B}$ the Markov chain $\tilde{s}$ assumes at date t, such that $s_t^i = 1$ and $s_{t-1}^i = 2$, called the **marginal latent state sets**. Let

$$\mathcal{L}^{xy} = \{3, 7, 5\}$$

be the set of all states $\tilde{s}_t \in \mathcal{B}$ the Markov chain $\tilde{s}$ assumes at date t, such that $s_t = 1$ and $s_{t-1} \neq 1$, called the **bivariate latent state set**. Let

$$\mathcal{S}^x = \{2, 4, 6, 8\} \quad \text{and} \quad \mathcal{S}^y = \{9, 11, 13, 15\}$$

be the set of all states $\tilde{s}_t \in \mathcal{B}$ the Markov chain $\tilde{s}$ assumes at date t, such that $s_t^i = 2$, called the **marginal spike state sets**. Let

$$\mathcal{S}^{xy} = \{10, 12, 14, 16\}$$

be the set of all states $\tilde{s}_t \in \mathcal{B}$ the Markov chain $\tilde{s}$ assumes at date t, such that $s_t = 4$ called the **bivariate spike state set**.

Regime State Parameters. Let θ_{MR}^i be the set collecting all parameter entries of θ, that correspond to the (marginal) mean reverting regime (i.e. "normal") states. Then

$$\theta_{MR}^i := \{\alpha^i, f^i, \sigma^i\}.$$

Let θ_J^i be the set collecting all parameter entries of θ, that correspond exclusively to the (marginal) spike regime (i.e. "abnormal") states. Then

$$\theta_J^i := \begin{cases} \{f_J^i, \mu_J^i, \sigma_J^i, \lambda^i\} & \text{in case of } \textit{Markov model I}, \\ \{\mu_J^i, \sigma_J^i, \lambda^i\} & \text{in case of } \textit{Markov model II}. \end{cases}$$

Vector of Transition Densities. Now, all transition densities conditioned on the current regime state are assembled in the (16×1) vector η, which for $\eta_t^{(\ell)}(\theta) = f(z_t'|\Omega_{t-1}, \tilde{s}_t = \ell; \theta)$ and $\ell \in \mathcal{B}$ for all $t \in \mathbb{T}$ is componentwise given by

$$\begin{cases} \eta_t^{(1)}(\theta) = \Phi_t^{xy}(\theta) & \\ \eta_t^{(2)}(\theta) = \mathcal{Z}_t^x(\theta)\, \Phi_t^y(\theta) & \\ \eta_t^{(\ell)}(\theta) = L_{\ell,t}^{xy}(\theta) & \text{if } \ell \in \mathcal{L}^{xy} \\ \eta_t^{(4)}(\theta) = \mathcal{Z}_t^x(\theta)\, \Phi_t^y(\theta) & \\ \eta_t^{(\ell)}(\theta) = \mathcal{Z}_t^x(\theta)\, L_{\ell,t}^y(\theta) & \text{if } \ell \in \mathcal{L}^y \\ \eta_t^{(9)}(\theta) = \Phi_t^x(\theta)\, \mathcal{Z}_t^y(\theta) & \\ \eta_t^{(\ell)}(\theta) = \mathcal{Z}_t^x(\theta)\, \mathcal{Z}_t^y(\theta) & \text{if } \ell \in \mathcal{S}^{xy} \\ \eta_t^{(\ell)}(\theta) = L_{\ell,t}^x(\theta)\, \mathcal{Z}_t^y(\theta) & \text{if } \ell \in \mathcal{L}^x \\ \eta_t^{(13)}(\theta) = \Phi_t^x(\theta)\, \mathcal{Z}_t^y(\theta), & \end{cases} \tag{2.20}$$

where $\Phi_t^i(\theta)$ and $\Phi_t^{xy}(\theta)$ respectively, represent the marginal and bivariate transition densities belonging to the normal distribution each conditioned on the last price x_{t-1}', y_{t-1}' or z_{t-1}', respectively, observed at date $t-1$. The explicit expressions are stated in (2.11) and (2.16), respectively. Then, $\mathcal{Z}_t^i(\theta)$ represents the marginal transition density corresponding to those spike regime states $\tilde{s}_t \in \mathcal{S}^i$ suggesting $s_t^i = 2$. The corresponding transition density is introduced in (2.13) (or in (2.12) by means of *Markov model II*). Most importantly (with regard to the current section), for all

$\ell \in \mathcal{L}^i$ the terms $L^i_{\ell,t}(\theta)$ and for all $\ell \in \mathcal{L}^{xy}$ the terms $L^{xy}_{\ell,t}(\theta)$ respectively represent the marginal and bivariate tansition densities involving latent prices.

The next section is then dedicated to provide the method we propose for deriving those likelihood functions involving latent prices and thus to answer the preceeding question. We will call them marginal and bivariate **(r -stage) transition densities**.

2.3.2 Derivation of (r -stage) Transition Densities

This work involves several different approaches to model the comovement of electricity and gas prices adequately as proposed throughout Section 2.1.3. In order to make the subsequent exposition fully understandable it would be desirable to specify the model framework chosen to work with: It is reasonable to restrict the issue of latent prices to those regime states inferring mean reverting dynamics. Hence, the actual form of the dynamics goverened by the spike regime states does not matter . According to Section 2.1.3 these observation process dynamics for all $u \in [t-r, t]$ and $t \in \mathbb{T}$ are given by

$$\begin{cases} dx_u = -\alpha^x \left(x_u - f^x \right) du + \sigma^x \, dM^x_u, & \text{if } s^x_t = 1 \\ \\ dy_u = -\alpha^y \left(y_u - f^y \right) du + \sigma^y \, dM^y_u, & \text{if } s^y_t = 1 \end{cases} \tag{2.21}$$

satisfying all assumptions posed before.

To begin with, we concentrate on the bivariate case dealing with latent prices. Exemplarily, think of one specific scenario when $\tilde{s}_t = 7$ or equivalently when $s_t = 1$ and $s_{t-1} = 4$. (That complies with Example 2.1 chosen in the introductionary part of the present Section 2.3.) Then the task is to calculate the density $L^{xy}_{7,t}(\theta)$ of observing the data $z_t = z'_t$ at time t given the process dynamics are modulated by $\tilde{s}_t = 7$, given the observations obtained through date $t-1$, i.e. Ω_{t-1} , and based on the knowledge of the parameter vector θ .

Now, we proceed in three different steps to derive explicit expressions for the *(r -stage) transition densities* for all $\ell \in \mathcal{L}$: First, to picture the explicit scenario and more importantly to picture the possible past evolution of the Markov chain inferred by regime states $\tilde{s}_t$ assuming values in the set $\mathcal{L}$ the corresponding scenarios are illustrated. Second, we apply Bayes theory to derive an analytical expression for the transition densities $L^{xy}_{\ell,t}(\theta) = f(z'_t \,|\, \Omega_{t-1}, \tilde{s}_t = \ell)$ for all $\ell \in \mathcal{L}^{xy}$. Third, the analogue result is stated for the marginal cases, i.e. for all $\ell \in \mathcal{L}^i$ the expressions $L^i_{\ell,t}$ are provided.

Markov Chain Scenarios. Just like Example 2.1 suggests, the realization $\tilde{s}_t = 7$ of the Markov chain yields the following scenario: Today's observations x'_t and y'_t are assumed to stem both from the *stable dynamics*, i.e. $s_t = 1$ (or equivalently $s^x_t = 1$ and $s^y_t = 1$). The previously observed prices at date $t-1$ are assumed to stem both from the spike regime, i.e. $s_{t-1} = 4$ (or equivalently $s^x_{t-1} = 2$ and $s^y_{t-1} = 2$). As explained before we face a scenario involving latent prices. Figure 2.1 now illustrates the possible evolution of the Markov chain $\tilde{s}$ in the case, when at time t the Markov chain $\tilde{s}$ assumes the regime state $\tilde{s}_t = 7$. Thereafter, the analogous illustrations are provided for the remaining states inferring bivariate mean reverting dynamics involving the issue of latent prices. In particular Figure 2.2 corresponds to the state $\tilde{s}_t = 3$ and Figure 2.3 corresponds to the state $\tilde{s}_t = 5$.

Key Idea. Having gained an understanding of how the Markov chain $\tilde{s}$ possibly evolved in the past, when at time t one of the bivariate latent states $\tilde{s}_t = \ell \in \mathcal{L}^{xy}$ is predominant, we turn to the task of calculating an analytical expression for the corresponding bivariate *(r-stage) transition densities* $L^{xy}_{\ell,t}(\theta)$. The key idea to address this issue is to find an integer $r = 2, 3, \ldots, K$ at time t with $r < t$, such that the most recent observation z'_{t-r}, again modulated by *stable dynamics*, has occured at date $t-r$. Due to the unobservability of the regime states the exact point of time cannot be fixed. Hence, the *(r-stage) transition density* will be derived as the sum of transition densities conditioned on the event, that at date $t-r$ the dynamics stem again from the *stable regime* (i.e. $s^i_{t-r} = 1$) weighted by the corresponding probabilities, where summation is taken over the dates $r = 2, 3, \ldots, K$. To account for the requirement of z'_{t-r} (x'_{t-r} or y'_{t-r} respectively) being the **most recent** prices observed again in the *stable regime* (such that e.g. all prices $z'_{t-\nu}$ for $0 < \nu < r$ are assumed to stem from spike regime states) we need to derive the conditional probability, that is e.g. for all $\ell \in \mathcal{L}^{xy}$ given by

$$p^{\ell}_{t-r,t} := \mathbb{P}(s_{t-r} = 1,\, s_{t-\nu} \neq 1,\, 1 < \nu < r \,|\, \tilde{s}_t = \ell, \Omega_t; \theta)$$

for all $t \in \mathbb{T}$. In general each latent state $\ell \in \mathcal{L}$ implies certain subsets $\mathcal{J}^*_\ell$ and $\mathcal{J}_\ell$ of the state space $\mathcal{B}$. These sets are given by (compare Figures 2.1, 2.2 and 2.3)

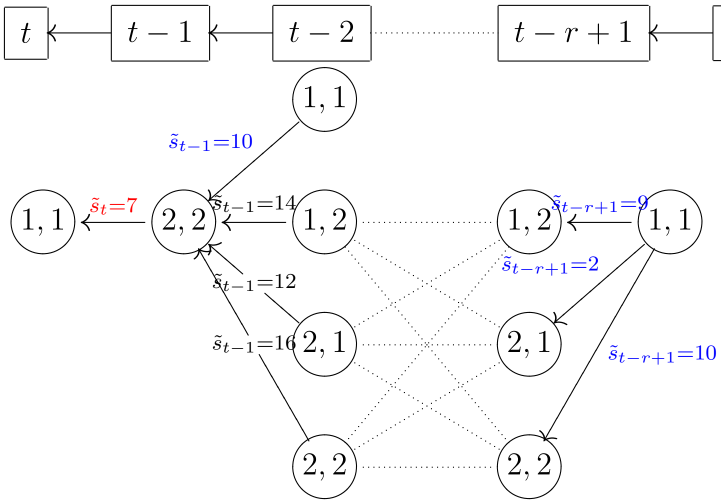


Figure 2.1: Possible state sequence inferred by regime state $\tilde{s}_t = 7$ at time t. The circle entries (i, j) for $i, j \in \{1, 2\}$ refer to the regime state $s^x_{t-r} = i$ and $s^y_{t-r} = j$ at the specific date $t-r$ with $r = 1, 2, \ldots, K$ and $r < t$. The rectangular boxes depict the course of time.

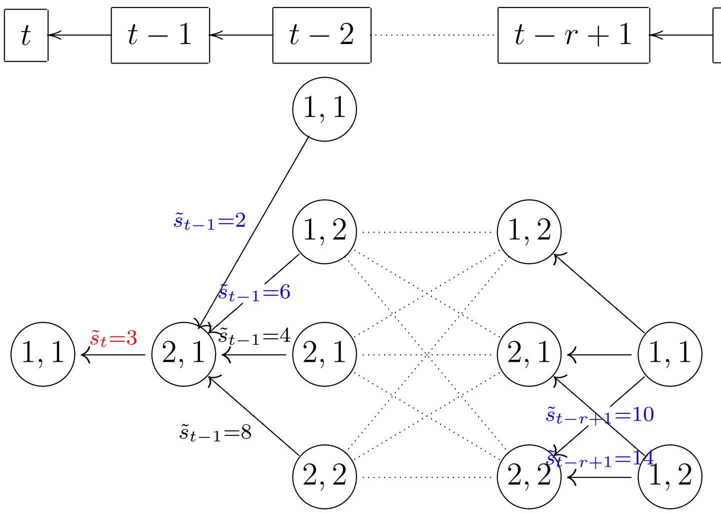


Figure 2.2: Possible state sequence inferred by the regime state $\tilde{s}_t = 3$ at time t. The circle entries (i, j) for $i, j \in \{1, 2\}$ refer to the regime state $s^x_{t-r} = i$ and $s^y_{t-r} = j$ at the specific date $t-r$ with $r = 1, 2, \ldots, K$ and $r < t$. The rectangular boxes depict the course of time.

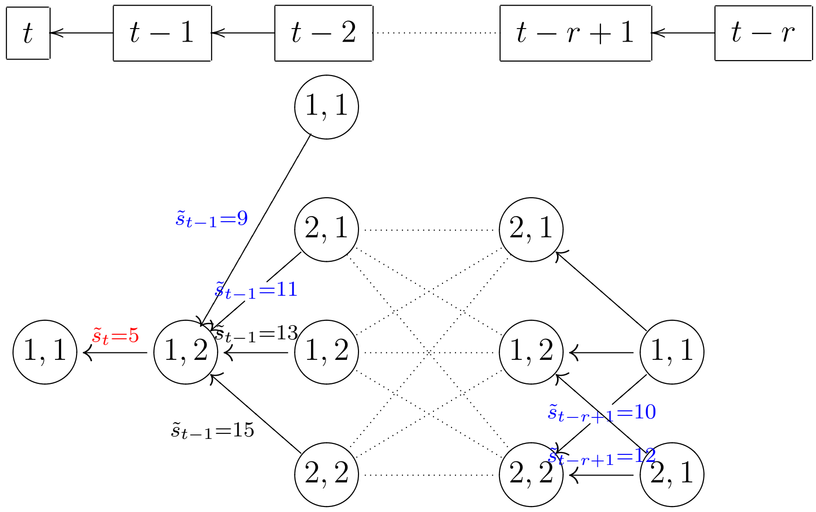


Figure 2.3: Possible state sequence inferred by the regime state $\tilde{s}_t = 5$ at time t. The circle entries (i, j) for $i, j \in \{1, 2\}$ refer to the regime state $s^x_{t-r} = i$ and $s^y_{t-r} = j$ at the specific date $t-r$ with $r = 1, 2, \ldots, K$ and $r < t$. The rectangular boxes depict the course of time.

$$\mathcal{J}_\ell = \begin{cases} \{2,9,10\} & \text{if } \ell \in \mathcal{L}^{xy}, \\ \{2,6,10,14\} & \text{if } \ell \in \mathcal{L}^{x}, \\ \{9,10,11,12\} & \text{if } \ell \in \mathcal{L}^{y} \end{cases} \quad \text{and} \quad \mathcal{J}^*_\ell = \begin{cases} \{10\} & \text{if } \ell = 7, \\ \{2\} & \text{if } \ell = 3, \\ \{9\} & \text{if } \ell = 5, \\ \{2,6\} & \text{if } \ell = 11, \\ \{10,14\} & \text{if } \ell = 15, \\ \{9,11\} & \text{if } \ell = 6, \\ \{10,12\} & \text{if } \ell = 8. \end{cases}$$

Then with the specific choice of these sets implied by regime state $\ell \in \mathcal{L}$ at time t the conditional probability $p^\ell_{t-r,t}$ is given by the next lemma.

Lemma 2.1. *For any $t \in \mathbb{T}$ and $\ell \in \mathcal{L}$ the probability $p^\ell_{t-r,t}$ is given by*

$$p^\ell_{t-r,t} = \begin{cases} \sum_{j^* \in \mathcal{J}^*_\ell} \mathbb{P}(\tilde{s}_{t-1} = j^* \,|\, \tilde{s}_t = \ell, \Omega_t; \theta) & r = 2 \\ \sum_{j \in \mathcal{J}_\ell} \mathbb{P}(\tilde{s}_{t-r+1} = j \,|\, \tilde{s}_t = \ell, \Omega_t; \theta) \cdot \tilde{p}(j, \ell) & r \in \{3, 4, \ldots, t\} \end{cases} \tag{2.22}$$

where

$$\tilde{p}(j, \ell) = \sum_{\substack{k_\nu \in \mathcal{B} \setminus \mathcal{J}_\ell \\ \nu = 1, \ldots, r-2}} \tilde{p}_{k_1,k_2} \cdot \tilde{p}_{k_2,k_3} \cdot \cdots \cdot \tilde{p}_{k_{r-2},j}$$

with $\tilde{p}_{k_\nu,k_{\nu+1}} := \mathbb{P}(\tilde{s}_{t-\nu} = k_\nu | \tilde{s}_{t-(\nu+1)} = k_{\nu+1}, \tilde{s}_t = \ell, \Omega_t; \theta)$.

Proof. The proof is given in Appendix A. □

With $\mathbb{P}(s_0 = 1)$ (compare Assumption 2.2) we immediately have

$$\sum_{r=2}^{t} p^{\ell}_{t-r,t} = 1$$

for all $t \in \mathbb{T}$ and $\ell \in \mathcal{L}$.

Remark 2.4. With spikes occuring rather rarely in the historical time series of electricity and gas spot prices, it makes sense to restrict r by an integer K. Kosater & Mosler [KM05] even suggested to fix K at the value of 5. Throughout our numerical analysis the employed data supports the reasonability of such a choice. Hence we use $K = 5$. Moreover, we have seen that using time-dependent values of $K = K(t)$ or different values for the marginal and bivariate likelihoods does not significantly improve the results.

Now an explicit expression for the bivariate *(r-stage) transition densities* $L^{xy}_{\ell,t}(\theta) = f(z'_t \,|\, \Omega_{t-1}, \tilde{s}_t = \ell\,; \theta)$ involving latent prices for all $\ell \in \mathcal{L}^{xy}$ can be derived by applying the subsequent Theorem 2.2 with the specific sets $\mathcal{J}_\ell$ and $\mathcal{J}^*_\ell$.

Theorem 2.2 (Bivariate (r-stage) Transition Density)**.** *For all $\ell \in \mathcal{L}^{xy}$, $t \in \mathbb{T}$ the transition density $L^{xy}_{\ell,t}(\theta)$ of observing $z_t = z'_t$ at date t inferred by regime state $\tilde{s}_t = \ell$ given Ω_{t-1} is given by*

$$L^{xy}_{\ell,t}(\theta) = \sum_{j^* \in \mathcal{J}^*_\ell} \mathbb{P}(\tilde{s}_{t-1} = j^* \,|\, \tilde{s}_t = \ell, \Omega_{t-1}; \theta) \cdot \Phi^{xy}_t(\theta | s_{t-2} = 1)$$
$$+ \sum_{r=3}^{t} \sum_{j \in \mathcal{J}_\ell} \mathbb{P}(\tilde{s}_{t-r+1} = j \,|\, \tilde{s}_t = \ell, \Omega_{t-1}; \theta) \cdot \tilde{p}(j, \ell) \cdot \Phi^{xy}_t(\theta | s_{t-r} = 1),$$

where $\Phi^{xy}_t(\theta | s_{t-r} = 1)$ is given by (2.18) *for all $r \in \{2, 3, \ldots, t\}$ and $\tilde{p}(j, \ell)$ is given according to Lemma 2.1 for all j, ℓ.*

Proof. Define the events $A^j_r := \{\tilde{s}_{t-r+1} = j\}$, $B := \{z_t = z'_t\}$ and $C := \{\Omega_{t-1}, \tilde{s}_t = \ell\}$ for all $j \in \mathcal{J}_\ell$, $t \in \mathbb{T}$ and $r \in \{2, 3, \ldots, t\}$. By a specific application of Bayes Theorem, i.e.

$$\mathbb{P}(A | B \cap C; \theta) = \frac{\mathbb{P}(A | C; \theta)\, \mathbb{P}(B | A \cap C; \theta)}{\mathbb{P}(B | C; \theta)},$$

for all $j \in \mathcal{J}_\ell$, $t \in \mathbb{T}$ and $r \in \{2, 3, \ldots, t\}$ we obtain

$$\mathbb{P}(\tilde{s}_{t-r+1} = j | \Omega_t, \tilde{s}_t = \ell;\, \theta) = \frac{\mathbb{P}(\tilde{s}_{t-r+1} = j | \Omega_{t-1}, \tilde{s}_t = \ell;\, \theta) \cdot \mathbb{P}(z_t = z'_t | \tilde{s}_{t-r+1} = j, \Omega_{t-1}, \tilde{s}_t = \ell;\, \theta)}{\mathbb{P}(z_t = z'_t | \Omega_{t-1}, \tilde{s}_t = \ell;\, \theta)},$$

where the denominator is assumed to be positive. Otherwise, we have $L_{\ell,t}^{xy}(\theta) = 0$. Using the notation $\mathbb{P}_{\tilde{s}_t=\ell}(A) := \mathbb{P}(A|\tilde{s}_t = \ell)$ together with Lemma 2.1 and

$$\sum_{r=2}^{t} p_{t-r,t}^{\ell} = 1$$

for all $\ell \in \mathcal{L}^{xy}$ we obtain

$$\sum_{r=3}^{t} \sum_{j \in \mathcal{J}_\ell} \mathbb{P}_{\tilde{s}_t=\ell}(\tilde{s}_{t-r+1} = j|\Omega_t;\, \theta) \cdot \tilde{p}(j,\ell) + \sum_{j^* \in \mathcal{J}_\ell^*} \mathbb{P}_{\tilde{s}_t=\ell}(\tilde{s}_{t-1} = j^*|\Omega_t;\, \theta) = 1.$$

This finally implies

$$\begin{aligned}
&\mathbb{P}_{\tilde{s}_t=\ell}(z_t = z_t'|\Omega_{t-1};\, \theta) = \\
&\quad \sum_{r=3}^{t} \sum_{j \in \mathcal{J}_\ell} \mathbb{P}_{\tilde{s}_t=\ell}(\tilde{s}_{t-r+1} = j|\Omega_{t-1};\, \theta) \cdot \tilde{p}(j,\ell) \cdot \mathbb{P}_{\tilde{s}_t=\ell}(z_t = z_t'|\tilde{s}_{t-r+1} = j, \Omega_{t-1};\, \theta) \\
&\quad + \sum_{j^* \in \mathcal{J}_\ell^*} \mathbb{P}_{\tilde{s}_t=\ell}(\tilde{s}_{t-1} = j^*|\Omega_{t-1};\, \theta) \cdot \mathbb{P}_{\tilde{s}_t=\ell}(z_t = z_t'|\tilde{s}_{t-1} = j, \Omega_{t-1};\, \theta).
\end{aligned}$$

Since $L_{\ell,t}^{xy}(\theta) = \mathbb{P}_{\tilde{s}_t=\ell}(z_t = z_t'|\Omega_{t-1};\, \theta)$ and $\Phi_t^{xy}(\theta|s_{t-r} = 1) = \mathbb{P}_{\tilde{s}_t=\ell}(z_t = z_t'|\tilde{s}_{t-r+1} = j, \Omega_{t-1}, ;\, \theta)$ for all $j \in \mathcal{J}_\ell$ the statement is proved. □

Analogously for all $\ell \in \mathcal{L}^i$ the marginal *(r-stage) transition densities* $L_{\ell,t}^i(\theta)$ can be derived in form of the subsequent Corollary 2.2.

Corollary 2.2 (Marginal (r-stage) Transition Density)**.** *For all $\ell \in \mathcal{L}^i$, $t \in \mathbb{T}$ the transition density $L_{\ell,t}^i(\theta)$ of observing $x_t = x_t'$ ($y_t = y_t'$ respectively) at date t inferred by regime state $\tilde{s}_t = \ell$ given Ω_{t-1} is given by*

$$\begin{aligned}
L_{\ell,t}^i(\theta) = &\sum_{j^* \in \mathcal{J}_\ell^*} \mathbb{P}(\tilde{s}_{t-1} = j^* \,|\, \tilde{s}_t = \ell, \Omega_{t-1}; \theta) \cdot \Phi_t^i(\theta|s_{t-2}^i = 1) \\
+ &\sum_{r=3}^{t} \sum_{j \in \mathcal{J}_\ell} \mathbb{P}(\tilde{s}_{t-r+1} = j \,|\, \tilde{s}_t = \ell, \Omega_{t-1}; \theta) \cdot \tilde{p}(j,\ell) \cdot \Phi_t^i(\theta|s_{t-r}^i = 1),
\end{aligned}$$

where $\Phi_t^i(\theta|s_{t-r}^i = 1)$ is given by (2.11) *for all $r \in \{2, 3, \ldots, t\}$ and $\tilde{p}(j,\ell)$ is given according to Lemma 2.1 for all j, ℓ.*

Remark 2.5. The previously stated algorithms yield the *posterior* and *prior probabilities.* Moreover, values for the *smoothed inferences*, i.e. $\mathbb{P}(\tilde{s}_{t-r} = j \,|\, \Omega_t; \theta)$ have been calculated (compare Section 2.2.1). Similar to the techniques involved in the derivation of the smoothed probabilities it is possible to calculate values for the probabilities $\mathbb{P}(\tilde{s}_{t-r+1} = j \,|\, \tilde{s}_t = \ell, \Omega_{t-1}; \theta)$. We omit the calculations at this point and refer to Appendix A.

Conclusion. To summarize, we note that we have found explicit expressions for those transition density functions involving latent prices. Hence, we have answered the preceeding question of how to determine that number of periods r one needs to look back into the past, such that at date $t-r$ the process is (most recently) again driven by *stable* (mean reverting) *dynamics*. The key to solving the issue of latent prices is to derive the specific (r-stage) probabilities $p^{\ell}_{t-r,t}$ provided in Lemma 2.1. Thereafter, the *(r -stage) transition densities* can be derived as the sum of densities conditioned on the event, that at date $t-r$ the dynamics stem again from regime $\{1\}$, weighted by the corresponding probability, where summation is taken over the dates $r = 2, 3, \ldots, t$.

2.4 Spike Regime Transition Densities

After the derivation of the so called *(r -stage) transition densities*, it is left to propose a way to calculate transition densities corresponding to spike regime states. These states suggest, that either the (electricity price) process x or the (gas price) process y or both processes are modeled by jump(-diffusion) dynamics. We will call them **spike regime transition densities**.

Now, the explicit form of the process dynamics governed by regime state variables $\tilde{s}_t$ assuming values in the set $\mathcal{S}$ matters. Hence, we must distinguish between the two possible models for the spike regime dynamics. On the one hand, the parameter choice valid within the spike regime of *Markov model II* yields mean reverting jump diffusion dynamics. On the other hand, the respective parameter choice valid within *Markov model I* yields pure jump dynamics having a constant long term level but no diffusive term.

Independence of Jumps. Based on economic considerations within the here presented bivariate regime-switching model, we assume the dependence relationship to stem purely from the bivariate mean reverting regime states $\mathcal{L}^{xy} \cup \{1\}$. After that, we do not expect to observe a direct link between electricity and gas prices when observed in an "abnormal" state, i.e. a spike regime state. The comovement of electricity and gas prices is due to gas being a possible fuel for power stations. However, shortage of gas supply must not necessarily lead to increasing electricity prices. The electricity price is made during an auction balancing supply and demand including all kinds of power stations, i.e. all kinds of fuels. On the other hand, a direct impact of e.g. an electricity generation outage on the gas price is neither obvious nor compulsory. Relying further on the study of historical data these considerations can also be transferred to other fuels. Hence, in terms of our regime-switching model it is reasonable to pose the next assumption.

Assumption 2.3 (Independence of Jumps). *The jump times* τ_m^i *and the jump sizes* J_m^i *introduced in* (2.3) *corresponding to the observation processes* x *and* y*, respectively, are independent of each other.*

Then immediately for all $\tilde{s}_t = j \in \mathcal{S}^{xy} := \{10, 12, 14, 16\} \subset \mathcal{S}$ (compare (2.20)) the transition density of z_t given Ω_{t-1} is given by

$$f(z_t' | \, \Omega_{t-1}, \tilde{s}_t = j; \, \theta) = \mathcal{Z}_t^x(\theta) \cdot \mathcal{Z}_t^y(\theta).$$

Thus, explicit expressions for the transition densities $\mathcal{Z}^i(\theta)$ belonging to the spike regime states $\tilde{s}_t \in \mathcal{S}$ must be proposed in such a way that they can be incorporated into the calibration routine suggested in Section 2.2. Due to the independence assumption it is sufficient to deal with the marginal cases.

Absence of Latent Prices in the Spike Regime. With the discussion of latent prices we have formed the basis for the study of the suggested different bivariate regime-switching models. The intuition of switching between different dynamics has been elaborated. At this point we want to make one true simplification: We assume the absence of latent prices in the spike regime. The reasoning for such an assumption is that spikes are modeled by processes where especially the jump size is represented by a random variable. Hence although the Markov chain indicates today's price to stem from the price regime the jump size might be close to zero. That possibility supports our idea to condition the spike transition densities always on the last observed price no matter if it has been suggested to stem from the spike regime or the normal mean reverting regime. If the latter is the case then we pretend the last price to stem from the spike regime where the jump size has been close to zero and the price has been somewhere around the long term level of the mean reverting dynamics. The other way round such a reasoning would not work. If one conditions the normal mean reverting price on a spike price the mean reversion speed parameter might become arbitrarily large to pull prices back to the long term level. The same is true for the volatility parameter. Our intention is to estimate the mean reverting parameter exclusively from "normal" prices. To accomplish this task we have provided the so-called *(r-stage) transition densitiess* as introduced in Section 2.3.

2.4.1 Estimating the Spike Parameters of Markov model II

As mentioned earlier, we must propose a way to include the *spike regime transition densities* and the estimation of the corresponding parameters $\theta_J \subset \theta$ into the calibration routine introduced in Section 2.2. For now, we deal with the marginal mean reverting jump diffusion dynamics present within the spike regime of *Markov model II.* More specifically, the underlying observation process dynamics inferred by the spike regime states $\mathcal{S}$ are according to Section 2.1.3 given by

$$\begin{cases} dx_u = -\alpha^x \left(x_u - f^x\right) du + \sigma^x \, dS_u^x + J^x \, dq_u^x, & \text{if } s_t^x = 2 \\ \\ dy_u = -\alpha^y \left(y_u - f^y\right) du + \sigma^y \, dS_u^y + J^y \, dq_u^y, & \text{if } s_t^y = 2. \end{cases} \tag{2.23}$$

The results we have gained about the probabilistic features of the model in Section 2.1.4.3 yield the probability of observing x_t inferred by any fixed regime state $\tilde{s}_t \in \mathcal{S}^x$ at date t given Ω_{t-1}, i.e.

$$\mathcal{Z}_t^x(\theta) = \sum_{k=0}^{\infty} \frac{e^{-\lambda^x} (\lambda^x)^k}{k!} \, \phi\left(\cdot;\, e_{t,1}^x + \mu^{Jx} \,,\, \sqrt{v_1^x} + \sigma^{Jx}\right) \tag{2.24}$$

with $e_{t,1}^x$ and v_1^x given according to (2.11) and $\mu^{Jx} := \mu_J^x \, e^{-\alpha^x} k$, $\sigma^{Jx} := \sigma_J^x \, e^{-\alpha^x} \sqrt{k}$. Clearly, the analogue form models the transition density $\mathcal{Z}_t^y(\theta)$ corresponding to the (gas price) process y.

2.4.1.1 Simulated Maximum Likelihood

Earlier studies on the estimation of such jump-diffusion models using maximum likelihood techniques have been accomplished by Beckers [Bec81] and Ball & Torous [BT99]. Obviously the density of a jump-diffusion model (as given in (2.24)) is a discrete mixture of N normally distributed variables, where N tends to infinity. As argued within the work of Kiefer [Kie78] and Honoré [Hon98] for such models there exist parameter specifications such that the likelihood function is unbounded. At this point care must be taken. In the literature several methods are provided to deal with the spoken to pitfalls in the simultaneous estimation of jump-diffusion models. For example Ait-Sahalia [AS04] suggests using analytical expansions to approximate transition densities. Due to the limited transparency and applicability of that approach, we choose to apply another tool applicable to virtually any jump-diffusion model.

That tool is the so called **simulated maximum likelihood** (SML) method first introduced independently by Pedersen [Ped95] and Santa Clara [BSC02]. Following the paper of Santa Clara & Brandt [BSC02] the method to estimate the parameters of the jump-diffusion dynamics from discretly sampled data and further to approximate the transition densities can be summarized in terms of the following list:

1. **Approximating the transition densities.** First construct consistent approximations to the transition densities of the diffusion and use these approximations to evaluate the likelihood function: For that, one applies an Euler discretization to the diffusion such that the time interval between any two consecutive observations is split into smaller intervals of length $\delta := \frac{1}{N}$. Then a high-frequency discrete time process with Gaussian transitions is constructed that converges to the diffusion as the discretization becomes finer, i.e. as $N \to \infty$. However, the Gaussian transitions are still unknown in closed-form. Therefore, an intuitive and computationally efficient simulation scheme is applied to numerically evaluate the transition densities of the Euler discretization. Note, both the Euler discretization and the simulation scheme are consistent such that the resulting approximations to the transition densities are consistent as well.

2. **Maximum likelihood estimation.** Next, the approximated transition densities are maximized with respect to the unknown parameter vector $\theta_J = \theta_J^x \cup \theta_J^y$. With the transition densities being consistent, so is the approximated likelihood function. Hence, asymptotically the SML estimator behaves just like the exact maximum likelihood estimator. It is shown that as long as the maximum likelihood estimator converges to the true paramater vector $\theta_J^{(0)}$, so does the simulated maximum likelihood estimator.

For clearness of exposition we want to omit stating all of the explicit formulas and details. These can be found in the paper of Santa Clara and Brandt. From their paper we have taken Figure 2.4 that illustrates the procedure. Moreover, a convenient summary along with algorithms (we apply in Section 2.5.2) is provided by Fusai & Roncoroni in their book [FR08].

The results of an empirical analysis implementing the estimation procedure for both the mean reverting parameter θ_{MR} and the spike regime parameter θ_J are then provided by Section 2.5.2.

2.4.2 Estimating the Spike Parameters of Markov model I

Next, we deal with the other case, when the specifications of *Markov model I* are valid. Then according to Section 2.1.3 we have the following observation process

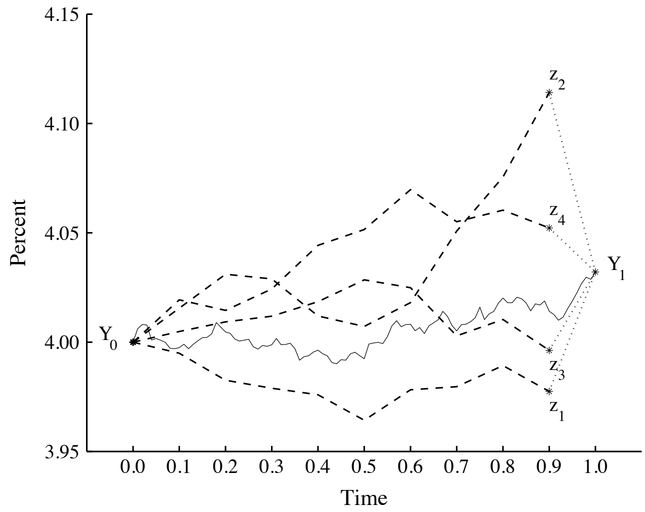


Figure 2.4: This figure is taken from the paper [BSC02] and illustrates the approximation of the transition densities of a diffusion via SML. The solid line represents the unobserved continuous-time sample path of a univariate diffusion. The four dashed lines represent incomplete ten-step Euler discretizations.

dynamics inferred by the spike regime states $\mathcal{S}$ in the framework of *Markov model I*

$$\begin{cases} dx_u = f_J^x\, du + J^x\, dq_u^x, & \text{if } s_t^x = 2 \\ \\ dy_u = f_J^y\, du + J^y\, dq_u^y, & \text{if } s_t^y = 2 \end{cases} \tag{2.25}$$

satisfying all assumptions posed before. Again referring to the already discussed probabilistic features of these model dynamics, the jump transition density of x_t given Ω_{t-1} when the process is governed by one of the spike regime states $s_t \in \{2, 4\}$ at date t is given by

$$\mathcal{Z}_t^x(\theta_J^x) := \sum_{k=0}^{\infty} \frac{e^{-\lambda^x}(\lambda^x)^k}{k!}\, \phi(\cdot;\, \mu_t^{cx} + \mu^{Jx}, \sigma^{Jx}), \tag{2.26}$$

where $\mu_t^{cx} = x_{t-1} + f_J^x$, $\mu^{Jx} := \mu_J^x\, k$ and $\sigma^{Jx} := \sigma_J^x \sqrt{k}$. Moreover, the analogue form models the transition density $\mathcal{Z}_t^y(\theta_J^y)$ corresponding to the (gas price) process y.

2.4.2.1 Maximum Likelihood Estimates

For this approach lacking a diffusion term in the spike regime state, we want to use another more simple tool than SML to estimate the spike regime parameter θ_J. For that we rely on another assumption supported by the subsequent considerations made with respect to the marginal process x. Thereafter the analgoue tool can be applied to the process y.

The set $\Omega_t^x = \{x_0', x_1', \ldots, x_t'\}$ is assumed to consist of observations obtained through date t for all $t \in \mathbb{T}$. Here the set is supposed to represent daily observed electricity prices. As mentioned earlier, in the framework of the regime-switching model the concrete parameter specification is determined through the value $j \in \mathcal{B}$ the Markov chain $\tilde{s}$ assumes at date t. In turn the concrete parameter specification determines, whether we follow the dynamics of a jump process or the dynamics of a mean reverting process, i.e. we say the Markov chain is supposed to determine whether the price process has a jump at date t or not. Thus, in the light of *Markov model I* we require the occurence of exactly one jump for every $\tilde{s}_t = s$ satisfying $s \in \mathcal{S}^x$. Such a requirement is summarized by the spoken to assumption.

Assumption 2.4. *If the driving Markov chain $\tilde{s}$ assumes a value in the spike regime set $\mathcal{S}^x$, then with probability one we observe a jump at date t, i.e. if $s_t^x = 2$ the corresponding process dynamics are given by*

$$x_t = x_{t-1} + f_J^x + J_t^x,$$

where J_t^x for all $t \in \mathbb{T}$ are independent identically normal distributed random variables with expected value μ_J^x and standard deviation σ_J^x.

Hence, for this approach we assume the process x to exhibit exactly one jump per unit of time if the Markov chain $\tilde{s}$ indicates the (marginal) process dynamics to stem from a spike regime state $s \in \mathcal{S}^x$. The main advantage of such an assumption is then the possibility to calculate explicit expressions for the maximum likelihood estimates of the spike regime parameter θ_J. These can be derived from the resulting form of the spike regime transition density approximating $\mathcal{Z}_t^x(\theta_J^x)$ stated within the next corollary.

Corollary 2.3. *Under Assumption 2.4 the* spike regime transition density *approximating $\mathcal{Z}_t^x(\theta_J^x)$ on a time interval $[t-1, t]$ for any $\tilde{s}_t = j \in \mathcal{S}^x$ given Ω_{t-1} is given by*

$$f(x_t' | \Omega_{t-1}, \tilde{s}_t = j; \theta_J) = \phi(\cdot;\, \mu_t^{cx} + \mu_J^x,\, \sigma_J^x).$$

Clearly, analogue considerations hold for the approximation of $\mathcal{Z}_t^y(\theta_J)$ for all $\tilde{s}_t \in \mathcal{S}^y$.

The results of an empirical analysis implementing all these steps together with the results of De Jong & Huisman [DJH02] indicate that it is even reasonable to model the spike regime state as truly independent event. That is the spikes are independent even of recent price levels. Such a behavior is reflected by the next even more restrictive assumption based on the model suggested by De Jong & Huisman [DJH02].

Assumption 2.5. *If* $s_t^x = 2$ *the corresponding process dynamics are given by*

$$x_t = f_J^x + J_t^x,$$

where J_t^x *for all* $t \in \mathbb{T}$ *are independent identically normal distributed random variables with expected value* μ_J^x *and standard deviation* σ_J^x.

Clearly, the transition densities need to be adopted as well. Throughout the empirical study *Markov model I* based on the preceeding two assumptions performed best (compare Section 2.5.1).

2.5 Empirical Results

The available data is edited to obtain two consecutive time series. The electricity prices are auctioned at the EEX (European Energy Exchange [eex]) in Germany relative to the hour 9:00-10:00 during the time period 01/01/2004 to 10/31/2008. The prices are day-ahead prices, i.e. they are auctioned the day before delivery.

The gas prices are traded at the virtual trading point TTF (Title Transfer Facility) in the Netherlands during the time period 01/09/2004 to 10/31/2008. Again we have day-ahead prices. Here the prices are the arithmetic average of bid and offer quotes of the different brokers Argus, Heren and Spectron [1]. The gas prices are only traded from Monday to Friday. For the weekend the prices are traded separately through Monday to Friday. In order to get a complete time series, we used the "weekend" price traded on Friday of the corresponding week as the price for the weekend and for holidays during this week (starting at Friday). For example, Wednesday, the 5th of March 2005, is a holiday in the Netherlands. On this day no prices were traded. We take the last traded "weekend" price available. In this

[1] Argus and Heren Energy are two provider of price assessments, business intelligence and market data on the global gas industries (amongst others). The Spectron platform is an electronic trading system operated by Spectron, a broker in gas products (amongst others).

case, it is the "weekend" price from Friday, 29th of April 2005. In case there is no data available on a Friday, we take the "weekend" price from Thursday of this week and so on. In total we replace 42 missing data values including holidays, excluding weekends. Thereafter, Figure 2.5 illustrates the complete consecutive time series.

The empirical study is executed on basis of deseasonalized historical electricity and gas prices as being observed in the time period 09/01/2004 until 10/31/2008 at the EEX in Leipzig and the TTF in Amsterdam. Both series consist of $T = 1521$ price quotes. Using the seasonality function as of Section 2.1.4 the data is extracted from seasonality. Moreover, extreme outliers (outside the three times standard deviation range) have been cut off. The resulting time series are plotted in Figure 2.6.

2.5.1 Calibration Routine Markov model I

To generate the desired empirical results we must fix initial parameter values to start the calibration routine. For this we rely on the results of [DJH02] and choose the according values to constitute a reasonable starting point. The parameter estimates calculated by running Algorithm 2.4 based on real world data are listed in Table 2.2.

$s_t^i = 1$	α^x	α^y	f^x	f^y	σ^x	σ^y
Real World	0.569	0.124	-0.011	0.0009	0.284	0.112
$s_t^i = 2$	μ_J^x	μ_J^y	σ_J^x	σ_J^y	f_J^x	f_J^y
Real World	-0.071	-0.147	0.718	0.838	0.597	0.242
$s_t^i = 1$	ρ					
Real World	0.121					

Table 2.2: Estimates $\hat{\theta}_{MR}$ and $\hat{\theta}_J$ modulating *Markov model I.*

On basis of the calculated parameter estimates it is possible to simulate paths representing the specific bivariate model dynamics under consideration. We rely on classical tools as e.g. summarized in Seydel [Sey06] to implement the discrete version of the process dynamics as stated in (2.11). Additionally, we must generate a Markov chain scenario for the hidden latent regime state variable $\tilde{s}_t$. At any stage within the simulation we proceed in two steps: First generate the (current) state of the Markov chain $\tilde{s}$ (given the last state $\tilde{s}_{t-1} = j$) by adopting multinomial random variables distributed according to the transition probability parameter estimates $\hat{p}_{jk}$

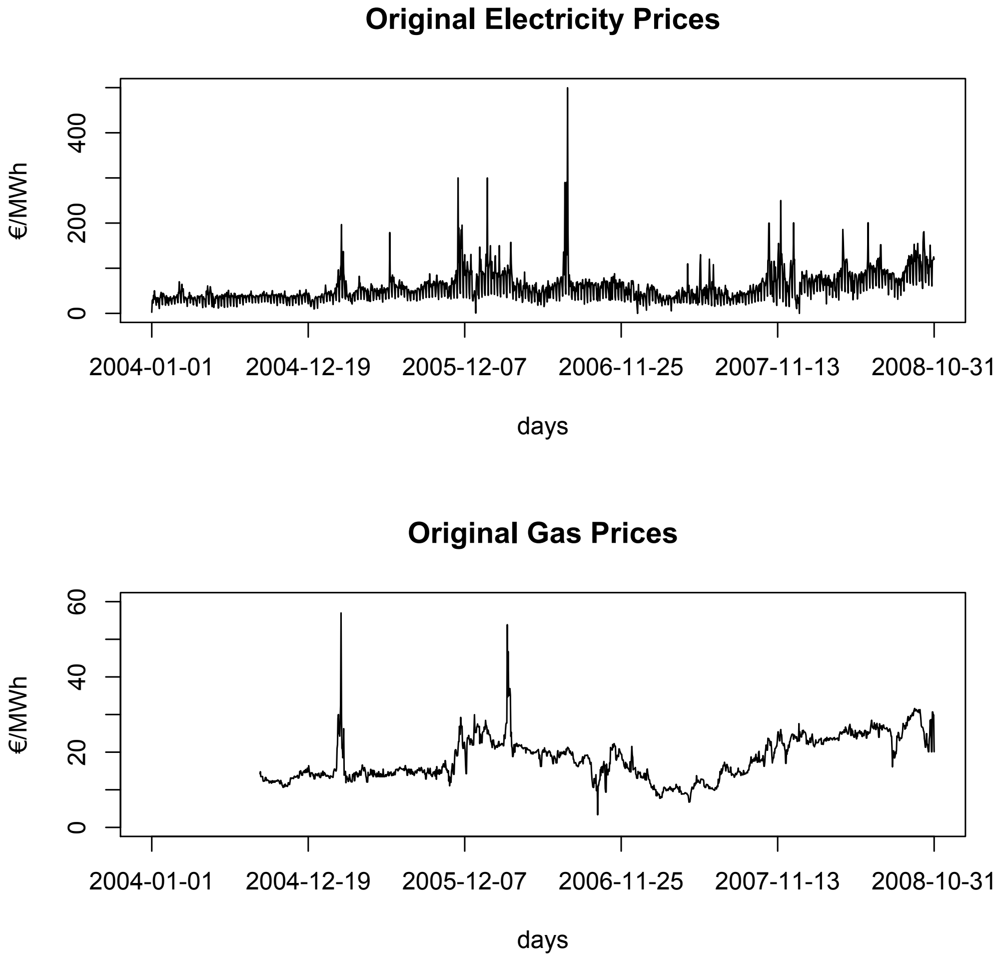


Figure 2.5: Top: Day-ahead power prices from the EEX Germany relatively to the hour 9:00-10:00 in €/MWh from 01/01/2004 to 10/31/2008. Bottom: Completed day-ahead gas prices from the TTF Netherlands in €/MWh from 09/01/2004 to 10/31/2008.

for $k = 1, \ldots, 4$. If $\tilde{s}_t = l \in \mathcal{L}$ suggests a latent state, then we determine the most recent point of time $t-r$ such that $\tilde{s}_{t-r+1} \in \mathcal{J}_\ell$. Next, based on the simulated current regime state $\tilde{s}_t$ we use the corresponding respective choice of

$$\begin{cases} x_t = x_{t-r}e^{-\alpha^x r} + f^x(1 - e^{-\alpha^x r}) + \sigma_r^x R_r^x, & \text{if } s_t^x = 1 \\ y_t = y_{t-r}e^{-\alpha^y r} + f^y(1 - e^{-\alpha^y r}) + \sigma_r^y R_r^y, & \text{if } s_t^y = 1 \\ x_t = f_J^x + J_t^x, & \text{if } s_t^x = 2 \\ y_t = f_J^y + J_t^y, & \text{if } s_t^y = 2 \end{cases}$$

to calculate the current price. All specifications hold as before. Note, special care must be taken for the different choices of r according to the different regime states. Further, the random variables R_r^i are given according to (2.10). Implementing the simulation procedure along with the calculated parameter estimates yields the exemplary graph pictured in Figure 2.7.

2.5.2 Calibration Routine Markov model II

What is left to point out is how the *SML method* can be incorporated into the calibration routine provided in Section 2.2. More precisely, we need to answer the following questions: How does the method work in our specific framework? What do we gain from applying the method? How can we incorporate the gains into the calibration routine, i.e. Algorithm 2.4? We omit the details at this point and refer to Appendix A.

Then the procedure and the calibration routine are combined. For that it is naturally to split the estimation procedure into two parts: One using the *SML method* to estimate the spike regime parameter θ_J and one to estimate the mean reversion regime parameter θ_{MR} just following the outlined steps of Algorithm 2.4.

Algorithm 2.5 (Adapted Calibration Routine).

1. Fix initial values for the spike regime parameter θ_J and for the parameter $\xi_{t|T}$ and P determining the distribution of the latent variables $\tilde{s}_t$.

2. Iterate through the different steps of Algorithm 2.4 to obtain optimal estimates $\hat{\theta}_{MR}$ for the mean reverting parameter θ_{MR} along with an estimate of the distribution of the latent variables $\tilde{s}_t$, i.e. $\hat{\xi}_{t|T}$ and $\hat{P}$.

3. Apply the proposed simulated maximum likelihood technique to obtain optimal parameter estimates $\hat{\theta}_J$ based on the input values $\hat{\theta}_{MR}$, $\hat{\xi}_{t|T}$ and $\hat{P}$.

4. Rerun Algorithm 2.4 (as in Step 2) now using $\hat{\theta}_J$, $\hat{\xi}_{t|T}$ and $\hat{P}$ as updated input parameter values to obtain an updated estimate $\hat{\theta}_{MR}$ (which should be close to the previous one).

To generate the desired empirical results we choose the according initial values as for *Markov model I* to constitute a reasonable starting point and to keep things comparable. Running through all the steps of Algorithm 2.5 then yields the parameter estimates based on real world data listed in Table 2.3. Analogously to the simulation

$s_t^i = 1$	α^x	α^y	f^x	f^y	σ^x	σ^y
Real World	0.611	0.156	−0.0004	0.0008	0.290	0.075
$s_t^i = 2$	μ_J^x	μ_J^y	σ_J^x	σ_J^y	λ^x	λ^y
Real World	0.171	0.038	0.927	0.482	0.980	0.998
$s_t^i = 1$	ρ					
Real World	0.049					

Table 2.3: Estimates $\hat{\theta}_{MR}$ and $\hat{\theta}_J$ modulating *Markov model II.*

procedure explained within *Markov model I* we use the respective choice of

$$\begin{cases} x_t = x_{t-r}e^{-\alpha^x r} + f^x(1-e^{-\alpha^x r}) + \sigma_r^x R_r^x, & \text{if } s_t^x = 1 \\ y_t = y_{t-r}e^{-\alpha^y r} + f^y(1-e^{-\alpha^y r}) + \sigma_r^y R_r^y, & \text{if } s_t^y = 1 \\ x_t = x_{t-1}e^{-\alpha^x} + f^x(1-e^{-\alpha^x}) + \sigma_\epsilon^x R_1^x + \sum_{m=1}^{q_1^x} J_{\tau_m}^x, & \text{if } s_t^x = 2 \\ y_t = y_{t-1}e^{-\alpha^y} + f^y(1-e^{-\alpha^y}) + \sigma_\epsilon^y R_1^y + \sum_{m=1}^{q_1^y} J_{\tau_m}^y, & \text{if } s_t^y = 2 \end{cases}$$

inferred by the simulated regime state $\tilde{s}_t$ to calculate the current prices based on the obtained parameter estimates listed in Table 2.3. All specifications hold as before. Implementing all stages of the simulation procedure then yields the for the fourth approach exemplary graph pictured in Figure 2.8.

2.5.3 Goodness of Fit

The goodness of fit is tested by different measures and methods. First of all, we have chosen as the *Benchmark model* the most simple case, when both price dynamics follow mean reverting Ohrnstein Uhlenbeck processes not involving any jump term or regime-switching behaviour. Such a model has been studied in detail in [Jen09] and thus constitutes a profound tool to accurately assess and judge the parameter values. All values are listed in Table 2.5.3. Next, we plot a path for each model to evaluate the visual performance and to judge whether the typical properties of electricity

Parameter	Benchmark	Markov model I	Markov model II
α^x	0.613	0.569	0.611
α^y	0.192	0.124	0.156
f^x	0.0001	-0.011	-0.0004
f^y	-0.001	0.0009	0.0008
σ^x	0.219	0.284	0.290
σ^y	0.079	0.112	0.075
ρ	0.024	0.121	0.049
μ_J^x	–	-0.071	0.171
μ_J^y	–	-0.147	0.038
σ_J^x	–	0.718	0.927
σ_J^y	–	0.838	0.482
f_J^x (λ^x)	–	0.597	0.980
f_J^y (λ^y)	–	0.242	0.998
log-Likelihood	1849.3	1911.2	2147.5

Table 2.4: Parameter estimates $\hat{\theta}_{MR}$ and $\hat{\theta}_J$ corresponding to different model approaches.

and gas prices are reflected appropriately (to be found in Figure 2.7 and Figure 2.8). Thereafter, lacking an overall theoretical distribution we apply bootstrapping methods for assigning measures of accuracy to the gained sample estimates. For that we proceed by simulating $R = 10000$ process paths (each containing $T = 1521$ observations) with the real world parameter estimates. Similar to Bierbrauer et al. [BTW05] the performance of the models is then assessed by comparing the first moments, the price distribution quantiles and the extreme events. The comparison is carried out in between the respective values calculated on the one hand based on the real world observations, on the other hand based on the bootstrap samples generated by the *Benchmark model*, *Markov model I* and *Markov model II*, respectively. Table 2.5 lists the spoken to statistics of interest.

2.5.4 Implications, Comments

- To judge the overall magnitude of estimates it is worth looking at the empiric mean reverting parameter estimates obtained by regressing the observations $x(t+1)$ against $x(t)$. Following the outline of Blanco & Soronow [BS01] the mean reversion speed is found at $\alpha^x = 0.613$ and $\alpha^y = 0.188$ respectively. The values are significant at the 1 % level. The long run mean values f^i

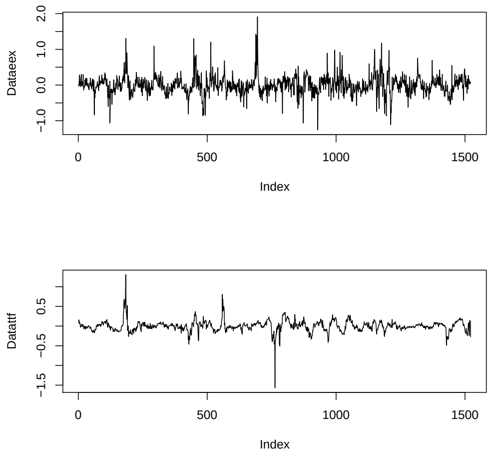


Figure 2.6: Real World Data

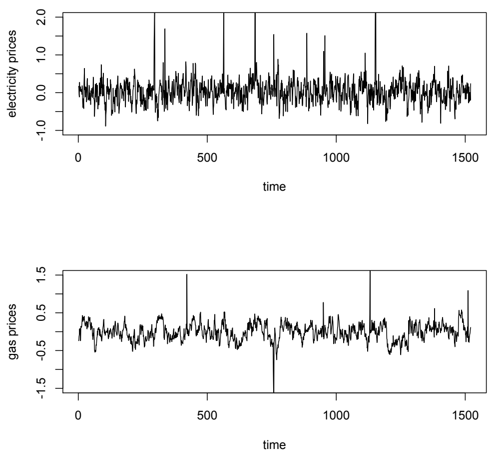


Figure 2.7: Simulated path corresponding to *Markov model I*.

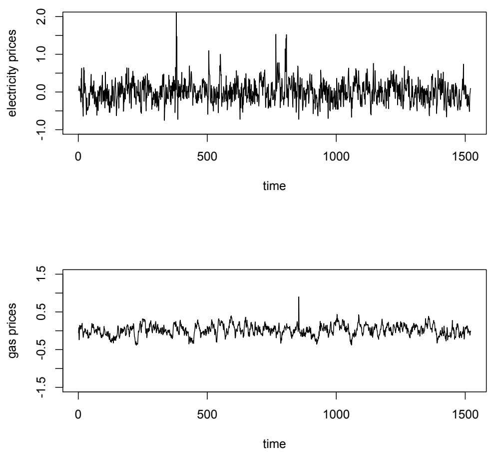


Figure 2.8: Simulated path corresponding to *Markov model II*.

Electricity Moments	Real World	Benchmark	MM I	MM II
mean	≈ 0	-0.0006 (1%)	-0.0007 (1%)	0.001 (1%)
standard deviation	0.261	0.197 (1%)	0.289 (1%)	0.272 (1%)
skewness	0.720	-0.003 (7%)	0.612 (34%)	0.210 (54%)
kurtosis	9.05	2.99 (13%)	6.349 (214%)	5.879 (549%)
first quartile	-0.136	-0.13 (1%)	-0.188 (1%)	-0.177 (1%)
third quartile	0.132	0.13 (1%)	0.175 (1%)	0.177 (1%)
maximum	1.91	0.65 (7%)	1.91 (39%)	1.52 (74%)
minimum	-1.25	-0.65 (7%)	-1.03 (24%)	-1.15 (47%)
Gas Moments	Real World	Benchmark	MM I	MM II
mean	≈ 0	-0.001 (1%)	0.001 (2%)	0.001 (1%)
standard deviation	0.140	0.127 (1%)	0.230 (1%)	0.136 (1%)
skewness	0.092	0.002 (12%)	0.077 (38%)	0.061 (52%)
kurtosis	21.19	2.98 (20%)	5.501 (330%)	5.179 (621%)
first quartile	-0.062	-0.08 (1%)	-0.149 (2%)	-0.089 (1%)
third quartile	0.062	0.08 (1%)	0.151 (2%)	0.091 (1%)
maximum	1.31	0.41 (5%)	1.26 (49%)	0.631 (34%)
minimum	-1.57	-0.41 (5%)	-1.13 (45%)	-0.57 (28%)
correlation	0.21	0.022 (4%)	0.082 (4%)	0.037 (4%)

Table 2.5: Summary statistics for real world data compared to the bootstrap replicates (i.e. the mean value with corresponding standard deviation) calculated from $R = 10000$ samples each of length $T = 1521$.

are almost zero. The volatility parameter then calculate to $\sigma^x = 0.219$ and $\sigma^y = 0.078$. The R^2 value is at $68\,\%$. The model used for regression is simply the uncorrelated discrete version of the *Benchmark model*. As desired the empiric estimates gained by regression are very close to the benchmark estimates. Hence, the overall magnitude of the estimates is validated.

- We expect the mean reversion speed parameter α^i to assume the largest values in the benchmark model, that does not include a spike regime. In order to pull prices back to the long term level these values should be relatively large. Moreover, not respecting the spikes in the model the correlation parameter is supposed to be quite low.

- The lowest values of α^i and the largest value of ρ are found in *Markov model I*. In that cases the parameter represent solely the mean reverting dynamics, i.e. the parameter are not involved in the specifications of the spike regime dynamics. The estimates are supposed to stem from the mean reverting regime state data. Hence, a comparable low mean reversion speed and high correlation is reasonable.

- Note, the mean reverting parameter θ_{MR} stemming from the model of *Markov model II* are again estimated solely on basis of the data that is supposed to stem from the mean reverting regime states according to the distribution of the latent state variable. However, the parameter θ_{MR} are involved in both the "normal" and spike regime state dynamics.

- Looking at the simulated paths of Figure 2.7 and Figure 2.8 generated on basis of the different approaches indicates overestimated volatility parameter σ^i belonging to the "normal" regime states. The values are close to the overall standard deviation of the data series. Especially the gas price fluctuation in *Markov model I* seems to be overestimated. However, that is a known phenomenon in the literature of fitting electricity spot price model to market data.

- Another known phenomenon is the difficulty to match the kurtosis of the real world data with the bootstrap replicates. Across the Markov modulated model approaches that moment could only be estimated with a huge standard deviation. That is due to the low probability of spike states in the distribution of the Markov chain, such that one simulated path might involve a certain number of jumps, but another does not. Looking at different periods of time in historical data, that seems to be reasonable.

- The *Benchmark model*, *Markov model I* or *Markov model II*, which one should be preferred? The testing results clearly point out that the Markov modulated models outperform the *Benchmark model*. *Markov model I* is computationally less expansive (with spikes modelled as truly independent events). However, the log-likelihood values of *Markov model I* indicate that *Markov model II* incorporates the main features of electricity markets more accurately. Separating the estimation of the jump parameter θ_J and the mean reversion parameter θ_{MR} (as in *Markov model II*) might be more reliable since it reduces the number of parameters that have to be estimated simultaneously. Applying two different methods to estimate the parameter separately on the one hand increases the reliability. On the other hand it reduces the transparency of the tool. After all, from a practical point of view we believe *Markov model I* is preferable, since it is not only computationally less expansive but also generates the best match of moments.

Conclusion. Our intention for the actual chapter has not been in the first place to address the elusive task of finding the "perfect" model. We believe looking at a regime-switching model is a huge step in the right direction of fitting a model to electricity market data. However, the calibration procedure becomes quite complex when it involves a latent state variable. In addition we have studied a bivariate regime-switching model such that the calibration procedure becomes even more complex. Thus, one of the main issues we wanted to address was to state all the details of the proceeding and all tools we have applied in the course of fitting the model to market data. Clearly, the specific procedure we have suggested has been chosen to deal with the features of electricity markets in the first place. However, we believe the primal proceeding can be carried over to any kind of bivariate regime-switching model. By looking at the different approaches we have shown the flexibility of the general model dynamics involving different states assigning different dynamics. Since the perfect model to reflect the features of electricity and/or gas prices still does not exist, looking at a flexible approach seems to be the right ansatz.

2.6 Managing and Valuing the Plant

The starting point of this chapter has been the question of how to manage the price risk a power plant faces in the liberalized energy markets. For that, we have taken the view of a power plant owner. In order to hedge against price risk, the real options approach has been chosen to determine the value of the power plant. That is, we model the plant value by the pay-off of a series of spark spread options. Note, for reasons of simplicity and to keep things comparable to the stochastic dynamic programming approach of Chapter 3 we do not include any discounting terms. The main objective is hence to answer the question of how likely the electricity price is beyond the marginal costs of the plant at time of maturity. That is how likely the owner exercises the right to transform fuel into power.

Supposing the plant manger intends to maximize the expected value generated from that right, the valuation formula for that specific scenario is given by the difference of the value of two call options on the spark spread. The expected plant pay-off per hour at maturity day is given by

$$\mathbb{E}^*\left[\left((x_T - HR\,y_T - \alpha)\overline{K}\right)^+ |\mathcal{F}_0\right] - \mathbb{E}^*\left[\left((x_T - HR\,y_T - \alpha)\underline{K}\right)^- |\mathcal{F}_0\right]$$

where $\overline{K}$, $\underline{K}$ denote the maximum, minimum capacity of the plant, α are the fixed (non-fuel) production costs and HR is the conversion efficiency of the plant, denoted heat rate. The expectation is calculated with respect to the pricing measure $\mathbb{P}^*$, assuming a zero market price of risk (similar e.g. to Burger et al. [BKMS04]). If the risk manager wants to charge a risk premium then this can be included with no further restrictions. Thus the expected value refers to the complete probability distribution of cash-flows. The specific form of these cash-flows at time of maturity naturally depends on the characteristics of the plant and the ideas of the plant manager.

Virtual Power Plant. The gas-fired plant we want to consider in this scenario has

- a netto maximum electric capacity of $\overline{K} := 250$ MW,
- a netto minimum electric capacity of $\underline{K} := 10$ MW,
- a netto conversion efficiency rate of $HR := 50\,\%$,
- generates fixed production costs of $\alpha := 20$ euros per MWh,
- purchases gas and sells electricity solely on the spot market,
- has full flexibility in between the capacity boundaries,
- has a baseload generation character, i.e. is operated 24 h a day and
- covers all other costs (like stand-on costs, switching costs ect.) with the fixed production costs.

The plant is switched on before the four month time horizon, will not be turned off and can either be operated at minimum capacity or up to full capacity. The plant manager can decide on a day to day basis about the generation level under full knowledge of the current spark spread.

The Valuation. The expected value is then calculated by applying the traditional Monte Carlo method. Here the idea is to generate a large number of sample paths of the process $z = (z_u)_{u\in[0,T]} = ((x_u, y_u)^{\mathsf{T}})_{u\in[0,T]}$ over the interval $[0, T]$ for each of the sample paths to compute the spark spread function whose expectation the risk manger wants to evaluate and then average those values over the sample paths. Since the terminal distribution of z_T is not known in closed-form, one has to generate samples from the entire path z_u for $0 \le u \le T$. This requires the choice of a discretization time step Δu and the generation of discrete time samples $z(0 + j\Delta u)$ for $j = 0, \dots, \frac{T}{\Delta u}$. As argued by Carmona & Durleman [CD03] these steps should be taken with great care to make sure that the numerical scheme used to generate these discrete samples produces reasonable approximations. In the course of Section 2.1.4 we have provided a discrete version for all $t \in \mathbb{T}$ of the process dynamics corresponding to the different model approaches. Note, of course one deals with the known difficulties in quantifying and controlling the embedded error connected to the MC method. It is out of the scope of this work to study this in more detail. We choose a large enough sample of $N = 10000$ to get a profound result. The plant value can then be calculated according to

$$\sum_{n=1}^{N}\sum_{t=0}^{T}\Big((x_t(n,\hat\theta) - HR\, y_t(n,\hat\theta) - \alpha)\overline{K}\Big)^{+} - \Big((x_t(n,\hat\theta) - HR\, y_t(n,\hat\theta) - \alpha)\underline{K}\Big)^{-},$$

where $T = 120$, $\Delta u = 1$, $N = 10000$ and $\hat\theta$ are the parameter estimates generated based on the different model approaches.

Results of the Valuation. On basis of such an intuitive valuation the different spot price models should then be tested with respect to their ability to reflect the comovement of power and fuel prices. That is, applying Monte Carlo simulation techniques allows us to arrive at the desired risk management tools. First of all the plant value is found by the spoken to spark spread valuation, then the optimal operation schedule (i.e. the decision whether to produce at minimum or maximum capacity) depending on todays expectations about the future spark spread evolution can be determined. Last but not least, having set up a portfolio of spark spread options the corresponding performance and risk numbers can be calculated. That is the return, volatility, profit at risk (here defined as the difference between the mean value of cash flows and the 5 % quantile of the cash flow distribution) and the 5 % Value of Risk. Plants differ significantly in their production flexibility. “Plants with daily operating flexibility can potentially generate considerably more money by treating every delivery period as a series of individual options to cash in on the spark spread”, according to De Jong in [DJW07]. De Jong finds that such a calculation easily overestimates the true value since on the one hand exercise decisions are being made optimally (i.e. spot prices are known before decisions are being made). On the other hand the plant is always in the right state to exploit the spark spread even though this may be associated with high switching or stand-on costs. Thus by looking at our results these findings should be kept in mind. Hence, for our study the relative difference in value in between the different underlying spot price models is of greater interest than the absolute value itself. The effect of the model on the difference in plant value and risk numbers is what we want to point out.

Spot Model	**Benchmark**	**Markov model II**	**Markov model I**
Plant Value	6.35	7.52	7.82
Max. Cap. Fraction	50.6%	49.2%	48.2%
5% **VaR**	5.77	6.62	6.71
Profit at Risk	0.58	0.89	1.09
Minimum	5.14	5.69	6.09
Maximum	7.49	12.05	10.41
Volatility	35%	63%	68%
Sample Correlation	2.1%	3.4%	8.2%

Table 2.6: Spark spread valuation risk analysis, where all numbers are given in percentages or milion euros.

We find, that the plant value increases significantly about 18.4 % and 23.1 % when transforming the spot price model from the *Benchmark model* to *Markoc model II* and *Markov model I* respectively, i.e. by incorporating specific alternations between stable and unstable regime states in the underlying price dynamics. As indicated by Carmona et al. [CD03] the correlation of the underlying indexes is one of the most influencing parameter on the value of the spark spread, since it has a main impact on the bivariate probability distribution. Similarily, we find an obvious impact of the correlation on the risk numbers. The more correlated the price series are, i.e the more the electricity and gas prices move together, the more revenues can be generated even in the worst price scenarios (here measured by the positive profit at risk and value at risk numbers). That is exactly what a risk manager would expect. Here the regime-switching models produce a revenues portfolio, that has a $5\,\%$ chance of making more than 6.62 milion euros (6.71 milion euros) over the four month time horizon. Again the numbers are significantly improved with regard to the *Benchmark model* VaR (that generates a $5\,\%$ chance of making 5.77 milion euros over the four month).

A more profound idea of the impact of the different underlying price models on the plant value yields a look at the complete probability distributions of the revenues the plant generates over the four month time horizon with respect to the different spot price dynamics. *Markov model II* yields a maximum plant value of 12 milion euros, whereas *Markov model I* yields a maximum value of "only" 10 milion euros and the *Benchmark model* just 7.49 milion euros that is not even the mean of the regime-switching based approaches. The increase in value goes together with a significant increase in volatility and less operation time at maximum capacity on average. However, the probability distributions of cash flows are negatively skewed with more probability mass on extreme high cash flows (in the Markov modulated models).

With these numbers at hand the risk manager is in the position to limit the risk influencing the operative strategy of the plant. Of course, a risk manager can further determine a marginal spark spread value that makes production profitable or not (under knowledge of all other costs). In our specific plant scenario the marginal value is exactly given by the fixed production costs of 20 euros per MWh. Such boundaries are useful for the plant management in the daily operational decisions.

Conclusion. After all an adequate price model allows risk mangers to better understand the impact of price behaviour and risks on values and hedges. Reducing the valuation problem to spark spread options is hence a good way to start the risk management of the plant. However, as mentioned before such a simple valuation assumes away certain operational constraints and management possibilities. To include such constraints and the possibility to sell electricity also through forward contracts in the valuation and operation problem we choose a stochastic dynamic programming approach presented in the next chapter.

Chapter 3

A Stochastic Dynamic Model for the Optimal Valuation and Operation of a Power Plant

3.1 Economic Motivation

The real option based spark spread valuation discussed in the second chapter only works under conditions that assume away several operational or market constraints. These constraints however have a significant impact on the plant value and the optimal operating strategy as e.g. argued by Gardner & Zhuang [GZ00] or Deng & Oren [DO03]. For example Deng et al. state that “ignoring operating characteristics in the valuation of a real asset would almost certainly lead to overvaluation”. A way to introduce such constraints is to consider a stochastic dynamic programming (SDP) representation of the problem. Bjorgan et al. [BSLD00] described SDP as an optimal procedure characterized by the states, time stages and decision options pertinent to the process. In terms of SDP the value of a unit’s generation capacity over a future period of time is determined by summing up the expected revenues the plant accumulates on every stage. Clearly, the scheduling of the plant, the uncertain energy prices and the operating constraints affect significantly the reward the plant manager can expect by selling electricity to the market. With respecting operating constraints the problem of finding an optimal operating strategy maximizing the plant value becomes path dependent.

Literature dealing with such models adapted to energy markets has been provided e.g. by Tseng & Barz [TB02] who focused on the short-term generation asset valuation problem. Deng & Oren [DO03] find a significant affect of physical operating constraints on the valuation of a power plant for different underlying price models. The paper of Gardner & Zhuang [GZ00] describes how SDP can be used to calculate plant values and optimal operating policies while considering plant operating constraints. Their numerical results imply that constraints such as minimum/maximum up/down times, ramp rates or capacity constraints may have a significant influence on the power plant's value.

After the deregulation of energy markets actors can adopt a variety of trading relations for the purchase and sale of electricity. Full term supply contracts are about to be abandoned. The peculiar feature of electricity spot prices of being highly volatile induces power generators to sell electricity through forward contracts. By selling forward it is possible for them to lock in a certain reward. On the other hand the extreme price risk induces electricity retail companies and large-scale consumers to purchase electricity through bilateral contracts to lock in a fixed purchase price. Looking e.g. at the forward selling strategy of *RWE Power AG* in the german market

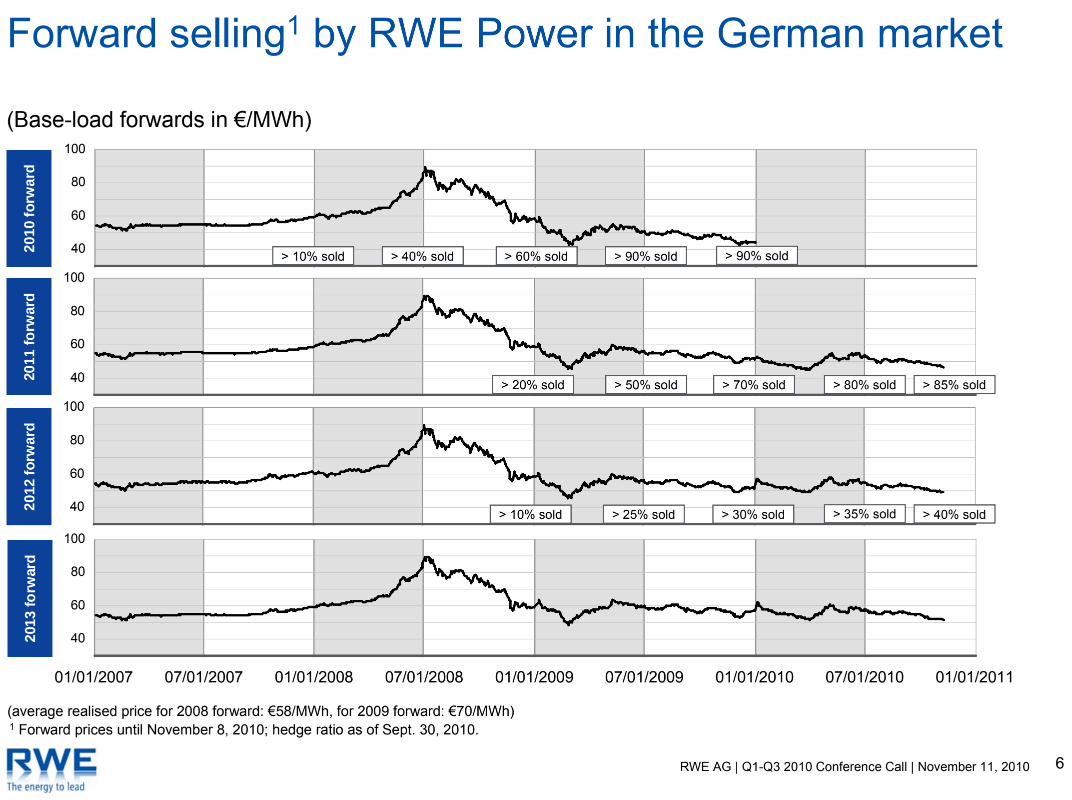


Figure 3.1: The graph is taken from *RWE Power AG*.

as indicated by Figure 3.1 supports this development. Approximately three years before maturity they hedge against spot price risks by selling a certain percentage of a year's available baseload capacity through forward contracts. The chart then records semi-annually the capacity fraction already locked in through forwards maturing e.g. in the year 2010. By the end of 2009 more than 90 % of the capacity has been sold through 2010 forward contracts. On such a basis the RWE mangers can decide on a day-to-day basis whether they produce the contracted power at the locked in price or buy the power in the market instead. In times of recession, i.e. decreasing prices, that might lead to an additional margin.

With this in mind our intention is to focus on the problem of finding the optimal hedging strategy between forward and spot markets. Moreover, we want to include trading rules practiced at the energy exchanges (e.g. EEX in Leipzig) such that spot bidding is done under uncertainty. Hence, the exposure of the plant to spot price risks is even more pronounced. Before the spot price is set on the day before delivery the plant intending to sell electricity on the spot must place their (volume) bids at the market. The auction mechanism then determines the so-called *market clearing price*. Thus the plant manager decides on a day-to-day basis how much capacity to devote to the day-ahead spot market. If the risk managers (such as those from *RWE Power*) already have signed forward contracts over a certain fraction of the available capacity with delivery e.g. in the first quarter of the year 2010, then only the remaining production capacity can be bid on the spot in the respective quarter.

In such a way the risk manager face an allocation problem at the signment date of the forward contracts: According to their future price expectations they must decide *how much capacity to devote to forward contracts and how much should be kept for bidding on the spot market*? The coordination of the bidding on the spot market, the hedging through forward selling and the scheduling of the plant is at the heart of our study. Naturally, to address the allocation problem the capacity constraints have significant influence. Besides that we do not want to interfuse the impact of the spot and capacity constraints by introducing a bulk of different operational constraints. Hence we focus on the uncertain spot bidding procedure and the capacity constraints of the plant. Thus the plant under consideration is characterized by the following assumptions:

Assumption 3.1 (Plant Characteristics). *(i) The facilities'* ***maintanence and operation costs*** *per unit of production are constant and denoted by* α *. (According to Deng [Den00] it is reasonable for a typical gas turbine combined cycle cogeneration plant, that these costs are stable over time.)*

(ii) Over the considered period of time the power station is not turned off and operated in between the capacity boundaries, where $\underline{K}$ *and* $\overline{K}$ *denote the* ***minimum and maximum capacity****, respectively.*

(iii) *To ensure a certain reward the plant locks in at least the minimum capacity through forward selling for all maturities. The contracted units then must be produced at maturity (such that the plant is not turned off). To keep the chance of participating in a spot price rally the plant managers must keep a so-called* ***spot reserve*** *(denoted by δ) for bidding on the spot market. Hence, the maximum possible capacity to be locked in through forwards is $\overline{K} - \delta$.*

(iv) *Ramp ups and downs in between the capacity constraints of the plant can be done with day-ahead notice without generating any further costs.*

(v) *Thinking e.g. of pipeline gas or long-term purchase contracts we assume the necessary fuel to produce the scheduled electricity to be constant over time. The price is then incorporated in the fixed costs α per unit of production.*

(vi) *The* ***contract lead time****, i.e. the minimum time between the time of scheduling the electricity and the time of the actual delivery, will be one day for spot contracts and a quarter (e.g. one month or a quarter of a year) for forward contracts.*

The problem is formulated as a stochastic dynamic program based on the continuous-time stochastic price process reflecting the uncertain electricity spot price. On the day of decision the spot price of the contract being deliverd today along with the price and units devoted to forward contracts with signment up to the current date are known and part of the state vector of the dynamic system. Within such an SDP approach it is straightforward to introduce a utility function representing the company-specific risk preferences. These preferences then build the fundament for selecting the optimal operation schedule and sales strategy. We start by choosing the exponential utility and generalize it to the class of strictly concave, continuously differentiable utility functions. Of course, it is not clear which utility function to choose. However, being a well-studied tool and keeping it in a desirable general form makes its application meaningful.

The solution of finding an optimal value function and associated operational strategy to address the allocation problem is approached by posing the first order Bellman equation for the value function. With regard to the path dependence evoked by the capacity constraints the underlying price process is approximated by the discrete time observations modelled by a familiy of random variables $(S_t)_{t\in\mathbb{T}}$. For a numerical study we then approximate the price process by different recombining lattice structures.

We state existence and uniqueness of a solution to the SDP representation of the allocation problem. Solving a full-blown SDP problem and its approximation is helpful in identifying the interaction of the operational constraints. Hence, it provides an insight into the structure of the problem. Solution techniques to achieve such an insight might be splitted into three parts:

1. First backward induction techniques in binomial or trinomial lattice structures are applied e.g. by Gardner & Zhuang [GZ00] or Guan et al. [GWGS08]. The paper of Wegner et al. [ER06] constructs optimal value functions by a stability based scenario tree construction that rests upon pertubation theory for multistage stochastic programs.

2. Second an alternative approach to implement backward induction is the MC simulation for Bermudan options, that has been applied to the valuation of swing options by Ghuieva et al. [GLS01]. Applications to power generation can be found in Tseng & Barz [TB02].

3. Third the Langrangian relaxation method is a basic technique in integer programming. It consists in relaxing complicating constraints by adjusting the objective function. Literature with application to power plant operation is provided by Takriti, Spugati & Wu [TSW01], Takriti, Birge & Long [TBL96] or Guan et al. [GWGS08].

We choose the backward dynamic programming technique and apply it to binomial and trinomial market approximations.

The present chapter starts in Section 3.2 by introducing the necessary mathematical notations and assumptions that lead to the introduction of the dynamic system in Section 3.3 being the building block for the multistage problem representation. Section 3.4 provides the structure of the optimal value function and the corresponding operating strategy calculated in terms of the DP technique. Thereafter, Section 3.5 states existence and uniqueness of a solution. The remaining sections then outline the numerical results gained by applying the theoretical model framework to different underlying market structures. First, in Section 3.6 the Cox Ross Rubinstein market is chosen. Then in Section 3.7 the optimal generation capacity allocation strategy is determined based on a one factor model representation approximated by a trinomial tree structure. We close in Section 3.8 with an economic interpretation and some concluding remarks in Section 3.9.

3.2 Conventions

The subsequent notations and assumptions are used throughout the theoretical representation of the allocation problem.

Time Conventions. The problem is based on a finite four quarter time horizon $[0,T]$ with $T < \infty$ and $T := 4n$ for any $n \in \mathbb{N}$. At the discrete dates $0, 1, \ldots, T-1$ a decision is made by the decision maker. These dates will be referred to as the **trading dates** and will be collected in the set $\mathbb{T} := \{0, 1, \ldots, T-1\}$.

Process Conventions. Let P be a probability measure defined on the measurable space $(\Omega, \mathcal{F})$. Let $S = (S_u)_{u \in \mathbb{R}_+}$ be a Markov process that evolves continuously in time and is observed at the discrete trading dates $t \in \mathbb{T}$. At the discrete dates t let the observations be given by the random variables $S_t(\omega)$ taking on the values s_t such that $s_t \in \mathbb{R}$ for all $t \in \mathbb{T}$. The process will be referred to as the **spot price process**. Let $s_0 \in \mathbb{R}$ be some given initial value. The units of capacity bid on the spot market at date t are denoted by b_t and referred to as **spot control**.
Let $D := T + n$ and let $F : \mathbb{T} \times \{n, n+1, \ldots, D\} \times \mathbb{R} \to \mathbb{R}$ be a function mapping the current value of the spot price process to a real number F_t^{t+n}, i.e. $F_t^{t+n} = F(t, t+n, s_t)$. The value F_t^{t+n} will be referred to as the **forward price** observed at date $t \in \mathbb{T}$ for contracts maturing at some future date $t+n$. The units of capacity devoted to forward contracts at date t maturing at date $t+n$ are denoted by f_{t+n} and referred to as **forward control**.

State Variables. For all $t \in \mathbb{T}$ the triple $X_t = (\mathbf{H}_t, \mathbf{h}_t, s_t) \in \mathbb{X}$ denotes the **state of the system** at date t composed of the historic price vector $\mathbf{H}_t$, the historic unit vector $\mathbf{h}_t$ and the currently observed spot price s_t. The **initial (historic) forward price and unit vector**, $\mathbf{H}_0 \in \mathbb{R}^D$ and $h_0 \in \mathbb{R}^D$ respectively, are given by

$$\mathbf{H}_0 = (\underbrace{H, \ldots, H}_{n}, 0, \ldots, 0) \in \mathbb{R}^D$$

and

$$\mathbf{h}_0 = (\underbrace{h, \ldots, h}_{n}, 0, \ldots, 0) \in \mathbb{R}^D,$$

where $H \in \mathbb{R}$ is some constant and $h \in [\underline{K}, \overline{K}]$ is some constant real number satisfying the capacity constraints, i.e. the minimum capacity $\underline{K}$ and the maximum capacity $\overline{K}$ such that $0 < \underline{K} < \overline{K} < \infty$.

The **historic forward price vector**, $\mathbf{H}_t \in \mathbb{R}^D$, collects all forward prices observed until date $t \in \mathbb{T}$ such that

$$\mathbf{H}_t = (\underbrace{H, \ldots, H}_{n}, \underbrace{F_0^n, F_1^{1+n}, \ldots, F_t^{t+n}}_{t+1}, \underbrace{0, \ldots, 0}_{T-(t+1)}) \in \mathbb{R}^D$$

and the **historic forward unit vector**, $\mathbf{h}_t \in \mathbb{R}^D$, collects the corresponding contracted forward units until date t such that

$$\mathbf{h}_t = (\underbrace{h, \ldots, h}_{n}, \underbrace{f_n, f_{1+n}, \ldots, f_{t+n}}_{t+1}, \underbrace{0, \ldots, 0}_{T-(t+1)}) \in \mathbb{R}^D,$$

where $h \in [\underline{K}, \overline{K}]$ and $H \in \mathbb{R}$ are given by the initial vectors $\mathbf{h}_0$ and $\mathbf{H}_0$.

Space Conventions. For all $t \in \mathbb{T}$ the **state space** $\mathbb{X}$ consists of all state values $X_t = (\mathbf{H}_t, \mathbf{h}_t, s_t)$ the system can attain at time t. Here, we assume

$$\mathbb{X} = \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$$

for all $t \in \mathbb{T}$. Suppose the set $\mathbb{X}$ is endowed with a σ-algebra Σ.
For all $t \in \mathbb{T}$ the **control space** C is a given set of all control variables $c_t = (f_{t+n}, b_t)$. We assume

$$C = \mathbb{R}^2$$

for all $t \in \mathbb{T}$.
For all $t \in \mathbb{T}$ we denote by F_t the set of all extended real-valued functions $J_{T,\pi_t} : \mathbb{X} \to \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ corresponding to some fixed (truncated) policy $\pi_t = (\mu_t, \ldots, \mu_{T-1})$.
For all $t \in \mathbb{T}$ the measurable space $(\Omega, \mathcal{F})$ is referred to as the **disturbance space**.
For each $X_t = (\mathbf{H}_t, \mathbf{h}_t, s_t)$ and each $t \in \mathbb{T}$ there exists a non-empty subset $K(X_t)$ of the **control space** C referred to as the **control constraint set at** X_t.
For all $t \in \mathbb{T}$ let the control constraint set at X_t be decomposable into the cross product of two compact intervals $I^{(j)}(X_t) \subset \mathbb{R}$ for $j = 1, 2$, such that

$$K(X_t) := I^{(1)}(X_t) \times I^{(2)}(X_t) \subset C.$$

For all $t \in \mathbb{T}$ we denote by $\mathbb{K}_t$ the set of all measurable control functions $\mu_t : \mathbb{X} \to C$, such that $\mu_t(X_t) \in K(X_t)$ for all $X_t \in \mathbb{X}$. The set $\mathbb{K}_t$ is referred to as the **constrained control function set** at time t. We denote by Π the set of all sequences $\pi = (\mu_0, \ldots, \mu_{T-1})$ such that $\mu_t \in \mathbb{K}_t$ for all t. Elements of Π are referred to as **policies**.

Decision Operator Conventions. The sequence $\pi = (\mu_0, \ldots, \mu_{T-1}) \in \Pi$ is the **policy** the decision maker applies during the trading period. The policy π specifies via the measurable **control function**

$$\mu_t : \mathbb{X} \longrightarrow C$$

the **control variable**

$$c_t = \mu_t(X_t) = (f_{t+n}, b_t)$$

to be chosen at date t for every state $X_t \in \mathbb{X}$. For all $t \in \mathbb{T}$ let the control function μ_t be given by the tuple of functions $\mu_t = (\mu_t^{(1)}, \mu_t^{(2)})$ with $\mu_t^{(1)} : \mathbb{X} \to \mathbb{R}$ and $\mu_t^{(2)} : \mathbb{X} \to \mathbb{R}$.

3.3 Dynamic System

This section provides all necessary conventions referred to the stochastic program. It translates the assumptions characterizing the plant into the theoretical framework. Additionally, a preliminary problem formulation is given.

System Equation. Given a policy π the system equation

$$\begin{cases} X_{t+1} = \Gamma(X_t, \mu_t(X_t), \omega) \\ X_0 = x = (\mathbf{H}_0, \mathbf{h}_0, s_0) \end{cases}$$

defines for all $t \in \mathbb{T}$ a controlled stochastic process in discrete time

$$(X^{x,\pi}(t))_{t \in \{0,1,\ldots,T\}}$$

where the system dynamics are given by the tuple of functions $\Gamma(X_t, \mu_t(X_t), \omega) = (\eta_1(X_t), \eta_2(X_t, \mu_t^{(1)}(X_t)), S_{t+1}(\omega))$ with $\eta_1 : \mathbb{X} \to \mathbb{R}^D$, $\eta_2 : \mathbb{X} \times \mathbb{R} \to \mathbb{R}^D$ and the random variable $S_{t+1} : \Omega \to \mathbb{R}$. Hence, the consecutive state is given by

$$\begin{cases} \mathbf{H}_{t+1} = \eta_1(X_t) := \mathbf{H}_t + F(t, t+n, s_t) \cdot \mathbf{e}_{t+n+1} \\ \mathbf{h}_{t+1} = \eta_2(X_t, \mu_t^{(1)}(X_t)) := \mathbf{h}_t + f_{t+n} \cdot \mathbf{e}_{t+n+1} \\ s_{t+1} = S_{t+1}(\omega_{t+1}) \end{cases},$$

where $\mathbf{e}_t$ denotes the t-th unit vector.

Stochastic Transition Kernel. For each $A \in \mathcal{F}$ and $t \in \mathbb{T}$ the **disturbance kernel** is a stochastic transition kernel on $(\Omega, \mathcal{F})$ given s_t, denoted by

$$P(A|\, s_t).$$

The stochastic **state transition kernel** on $(\mathbb{X}, \Sigma)$ given X_t and c_t is then denoted by

$$\mathcal{T}(B;\, X_t, c_t)$$

for all $B \in \Sigma$ and $t \in \mathbb{T}$. Note, the disturbance kernel can be expressed in terms of the state transition kernel and vice versa. For all $B \in \Sigma$ we have,

$$\begin{aligned} \mathcal{T}(B; X_t, c_t) &= P(\{\omega \in \Omega \,|\, \Gamma(X_t, c_t, \omega) \in B\}|\, s_t) \\ &= P(\Gamma^{-1}(B)_{(X_t, c_t)}|\, s_t). \end{aligned}$$

Thus $\mathcal{T}(B;\, X_t, c_t)$ is the probability that the $(t+1)$-th state is in B given that the t-th state is X_t and the system is controlled by c_t.

When the process dynamics are used successively to express the uncertain spot prices $S_{t+1}, S_{t+2}, \ldots, S_T$ exclusively in terms of $\omega_{t+1}, \ldots, \omega_T$ and s_t, one can see that for each fixed $s_t \in \mathbb{R}$ the probability measures $P(\cdot\,|\, s_t)$, $P(\cdot\,|\, s_{t+1})$, $\ldots$, $P(\cdot\,|\, s_{T-1})$ together with the system dynamics define a unique **measure**

$$P(d(\omega_{t+1}, \omega_{t+2}, \ldots, \omega_T)|\, s_t)$$

on the cartesian product Ω^{T-t} of $T-t$ copies of Ω.

Well-Defined Expectation. Let $G : \mathbb{X} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be an integrable random variable for every $X_t \in \mathbb{X}$ and $y \in \mathbb{R}^d$. We say that the (conditional) expectation function $g(X_t, y) := \mathbb{E}[G(X_t, y, \omega)|s_t]$ is **well-defined**, if it is measurable in both arguments. We adopt the notation of Bertsekas & Shreve [BS80], such that $g(X_t, y) = \mathbb{E}[G(X_t, y, \omega)|s_t] = \mathbb{E}[G(X_t, y, \omega)|S_t = s_t]$. The designation $g(X_t, y)$ is integrable means the random variable $\mathbb{E}[G(X_t, y, \omega)|S_t]$ is integrable.

Costs and Revenues. For all $t \in \mathbb{T}$ the function

$$R_t(X_t, c_t, \omega) = (H(t+1) - \alpha)h(t+1) + (S_{t+1}(\omega) - \alpha)b_t$$

is the uncertain reward incurred at the $(t+1)$-th stage, i.e. within $[t+1, t+2)$, where

- $H(t+1) = \mathbf{H}_t^\intercal \cdot \mathbf{e}_{t+1}$ is the $(t+1)$-th entry of the vector $\mathbf{H}_t$,

- $h(t+1) = \mathbf{h}_t^{\intercal} \cdot \mathbf{e}_{t+1}$ is the $(t+1)$-th entry of the vector $\mathbf{h}_t$,
- $S_{t+1}(\omega)$ is the uncertain spot price at time $t+1$ and
- α is any positive real number representing the fixed costs per unit of production.

Utility Function. For all $t \in \mathbb{T}$ the random variable R_t may assume negative values. In that regard we assume the utility function $U : \mathbb{R} \to \mathbb{R}$ to be a member of the following class:

Assumption 3.2 (Utility Class)**.** *The utility function* $U : \mathbb{R} \to \mathbb{R}$ *is increasing on* $\mathbb{R}$*, continuous, differentiable and strictly concave, and satisfies*

$$U'(-\infty) = \lim_{x \downarrow -\infty} = +\infty \quad \textit{and} \quad U'(+\infty) = \lim_{x \to +\infty} = 0.$$

Moreover, U *must be chosen such that* $UR_{t+1}(X_t, c_t) := \mathbb{E}[U(R_t(X_t, c_t, \omega) \,|s_t]$ *is well-defined.*

To start with we choose one specific member of that class, namely the **exponential utility** function given by

$$U(R_t) = -\exp\{-\gamma R_t\}, \tag{3.1}$$

where γ is a positive constant that represents the degree of risk aversion.

Control Constraint Set. Let $\underline{K}$, $\overline{K}$ and δ be some positive constants such that $\underline{K} < \overline{K}$ and $\delta < \overline{K} - \underline{K}$. For all $t \in \{0, n, 2n, 3n\}$ we have

$$\begin{aligned} K(X_t) := I^{(1)}(X_t) \times I^{(2)}(X_t) = \{ c_t = (f_{t+n}, b_t) \in C \,| \\ \underline{K} \le f_{t+n} \le \overline{K} - \delta \text{ for } t \neq 3n, \quad f_{t+n} = 0 \text{ else}, \\ 0 \le b_t \le \overline{K} - h(t+1) \big\} \,. \end{aligned}$$

For all $t \in \mathbb{T} \setminus \{0, n, 2n, 3n\}$ we have

$$\begin{aligned} K(X_t) := I^{(1)}(X_t) \times I^{(2)}(X_t) = \{ c_t = (f_{t+n}, b_t) \in C \,| \\ f_{t+n} = h((t+n) - t \pmod{n} + 1), \\ 0 \le b_t \le \overline{K} - h(t+1) \big\} \,. \end{aligned}$$

Preliminary Problem Formulation. Choose an optimal policy $\pi^* \in \Pi$ such that for all X_0

$$J_{T,\pi^*}(X_0) := \sup_{\pi \in \Pi} J_{T,\pi} = \sup_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{T-1} U[R_t(X_t, \mu_t(X_t), \omega)]\right]$$

where expectation is taken with respect to the measure $P(d(\omega_1, \omega_2, \ldots, \omega_T)|\, s_0)$ subject to

- $X_{t+1} = (\mathbf{H}_{t+1}, \mathbf{h}_{t+1}, s_{t+1})$ is generated according to the system equation

$$X_{t+1} = \Gamma(X_t, \mu_t(X_t), \omega)$$

  for all $t \in \mathbb{T}$ and

- $\mu_t(X_t) \in K(X_t)$ for all $t \in \mathbb{T}$, i.e. the control variable is admissible.

Remark 3.1. The plant manager intends to maximize the "reward-to-go" given by the accumulated costs and revenues evaluated according to the risk preferences (the plant managers have agreed on) by choosing an optimal operating schedule (i.e. policy π^*). Applying the risk aversion on every stage is a more conservative strategy than maximizing the total expected wealth. It will be shown, that it makes the operation more flexible. Moreover, the whole problem can be split into several subproblems each affecting a certain subperiod (e.g. a quarter) of the considered time horizon.

That is the value of the plant at initial time $t = 0$ can be decomposed mainly into three parts.

- The reward generated within the **"present to go quarter"** based on the initial forward units h, given through the inital state X_0, and the optimal spot decision b_0 to be chosen in the admissible set $I^{(2)}(X_0)$ such that the *"day-ahead reward"* is maximized.

- The reward generated within the **"next to go quarter"** based on the current forward decision f_n to be chosen in the admissible set $I^{(1)}(X_0)$ such that the *"next to go quarter reward"* is maximized. Moreover, it is based on the future optimal spot decisions $b^*_{n+k} \in I^{(2)}(X_{n+k})$ for all $k \in \{0, 1, \ldots, n-1\}$. Here, due to the specific form of the spot control constraint set (that is determined by the optimal forward control f^*_n), i.e. $I^{(2)}(X_{n+k}) = [0, \overline{K} - f^*_n]$, allocation of the maximum capacity might be necessary.

- The reward generated within the **"remaining to go quarters"** based on future optimal forward and spot decisions f^*_i and b^*_{i+k} for all $i \in \{2n, 3n\}$.

All steps of the dynamic programming technique applied to obtain that specific structure of the value function (in terms of well-defined expectations) are then provided within Section 3.4.

3.4 Bellman Equation & Dynamic Programming

Due to the formulated assumptions and conventions, the sequence of system states $\{X_0, X_1, \ldots, X_T\}$ forms a finite Markov sequence, which is completely described by the state transition kernel $\mathcal{T}(dX_{t+1};\, X_t, c_t)$ and the initial state X_0. In order to obtain a solution to the optimization problem the dynamic programming algorithm will be applied and discussed. According to Bertsekas [Ber76], "the **Markov property** is at the heart of the dynamic programming technique." Such a technique decomposes the problem into a sequence of simpler maximization problems, that are carried out over the constrained control set at X_t, i.e. $K(X_t)$, rather than over the constraint control function set $\mathbb{K}_t$.

The DP algorithm is based on the so-called **principle of optimality** due to Bellman, "who contributed a great deal to the popularization of DP and to its transformation into a systematic tool" (Bertsekas [Ber76]). In our setting the principle can be stated as follows:

- Suppose $\pi^* = (\mu_0^*, \ldots, \mu_{T-1}^*)$ is an optimal policy for the T-stage optimization problem.
- Considering the subproblem of starting at state X_t at time $t > 0$ and maximizing the reward-to-go from time t to time T, i.e. $J_{T,\pi_t}(X_t)$. Then the (truncated) policy
$$\pi_t^* = (\mu_t^*, \mu_{t+1}^*, \ldots, \mu_{T-1}^*)$$
is also optimal for the subproblem.

3.4.1 Mappings Underlying the DP Model

Here Ω is not countable, thus matters are complicated since the function

$$g^*(X_t) := \sup_y g(X_t, y) = \sup_y \mathbb{E}[G(X_t, y, \omega)|s_t]$$

is not necessarily measurable, even if the (conditional) expectation function $g(X_t, y)$ is. For a thorough treatment of that problem we refer to the book of Bertsekas &

Shreve [BS80]. They suggest two approaches to overcome this difficulty. The first possibility is to define the expected value as an outer integral, as we do until stated differently. (All details on the outer integral formulation are discussed in [BS80], thus we only state the basic definition concerning our model framework in Appendix B.1.) The other approach is to impose an appropriate measurable space structure on $\mathbb{X}$, C and Ω and require that the functions $\mu_t \in \mathbb{K}_t$ are measurable as we do with effect from Section 3.5.

Definition 3.1. *Let a mapping* $H_t : \mathbb{X} \times C \times F_{t+1} \to \overline{\mathbb{R}}$ *be given by*

$$H_t(X_t, c_t, J) \;=\; \mathbb{E}\left[\, U[R_t(X_t, c_t, \omega)] + J(\Gamma(X_t, c_t, \omega)) \,|\, s_t \,\right]$$

for all $t \in \mathbb{T}$, *where the following are assumed:*

(1) ω *takes values in the measurable space* $(\Omega, \mathcal{F})$. *For each fixed* $s_t \in \mathbb{R}$ *a probability measure* $P(d\omega|\, s_t)$ *on* $(\Omega, \mathcal{F})$ *is given and* $\mathbb{E}[\,.\,|\, s_t]$ *denotes the outer integral (compare Appendix B) with respect to that measure such that no further measurability assumptions are needed.*

(2) R_t *and* Γ *map* $\mathbb{X} \times C \times \Omega$ *into the domain of the utility function* U *and* $\mathbb{X}$ *respectively.*

(3) $J \in F_{t+1}$ *maps* $\mathbb{X}$ *into* $\overline{\mathbb{R}}$.

Hence, we are given the function H_t which maps the state X_t, control c_t and the function J into $\overline{\mathbb{R}}$. We then define the function $J : \mathbb{X} \to \overline{\mathbb{R}}$.

Definition 3.2. *(a) At date* T *we define*

$$J_T(X_T) \equiv 0$$

for all $X_T \in \mathbb{X}$.

(b) For each $\mu_t \in \mathbb{K}_t$ *the* ***reward-to-go operator*** *is defined as the mapping* $T_{\mu_t} : F_{t+1} \to F_t$ *such that for every* $X_t \in \mathbb{X}$, $J \in F_{t+1}$ *it holds*

$$T_{\mu_t}(J)(X_t) := H_t(X_t, \mu_t(X_t), J)$$

for all $t \in \mathbb{T}$.

(c) The ***optimal reward-to-go operator*** $T_t : F_{t+1} \to F_t$ *is then defined as*

$$T_t(J)(X_t) := \sup_{\mu_t \in \mathbb{K}_t} T_{\mu_t}(J)(X_t)$$

for all $X_t \in \mathbb{X}$, $J \in F_{t+1}$ *and all* $t \in \mathbb{T}$.

(d) *The* ***optimal reward-to-go function*** $J_t : \mathbb{X} \to \overline{\mathbb{R}}$ *is then defined as*

$$J_t(X_t) := J_{T,\pi_t^*}(X_t) := \sup_{\pi_t \in \Pi_t} J_{T,\pi_t}(X_t)$$

for all $X_t \in \mathbb{X}$ *and all* $t \in \mathbb{T}$ *.*

Remark 3.2. The term "reward-to-go" involved e.g. in the denotation of J_t for $t \in \mathbb{T}$ has been chosen to account for $J_t(X_t)$ denoting the maximum reward the plant accumulates from the current date $t \in \mathbb{T}$ until the end of the time horizon T.

3.4.2 Reward-to-Go Iteration

Now, the value of the reward-to-go function J_{T,π_t} at date t can be iteratively expressed in terms of the operator T_{μ_t}, $T_{\mu_{t+1}}$, ..., $T_{\mu_{T-1}}$. Hence, at any date $t \in \mathbb{T}$ the operator can be iteratively applied to compute the reward-to-go for a given initial state X_t and corresponding to a given policy π_t.

Theorem 3.1 (Reward Iteration). *Let* $\pi_t = (\mu_t, \ldots, \mu_{T-1})$ *be a truncated policy. For all* $t \in \mathbb{T}$ *it holds*

$$J_{T,\pi_t}(X_t) = (T_{\mu_t} \cdot T_{\mu_{t+1}} \cdots T_{\mu_{T-1}})(J_T)(X_t) \text{ for all } X_t \in \mathbb{X},$$

where $(T_{\mu_t} \cdot T_{\mu_{t+1}} \cdots T_{\mu_{T-1}})$ *denotes the composition of the mappings* T_{μ_t} *,* $T_{\mu_{t+1}}$ *, ...,* $T_{\mu_{T-1}}$ *and* J_{T,π_t} *denotes the "reward-to-go" function* $J_{T,\pi_t} : \mathbb{X} \to \overline{\mathbb{R}}$ *corresponding to* π_t *.*

Proof. 3.2(b) yields $J_{T,\pi_t}(X_t) = T_{\mu_t}(J)(X_t)$ for all $X_t \in \mathbb{X}$, all $J \in F_{t+1}$ and all $t \in \mathbb{T}$. Especially, for a fixed $\mu_{T-1} \in \mathbb{K}_{T-1}$ we have $J_{T,\pi_{T-1}}(X_{T-1}) = T_{\mu_{T-1}}(J_T)(X_{T-1})$ for all $X_{T-1} \in \mathbb{X}$. For some fixed t and $\pi_t = (\mu_t, \ldots, \mu_{T-1})$ let the induction hypothesis $J_{T,\pi_t}(X) = (T_{\mu_t} \cdot T_{\mu_{t+1}} \cdots T_{\mu_{T-1}})(J_T)(X)$ be satisfied for all $X_t \in \mathbb{X}$. Then, the desired result follows by backward induction. □

3.4.3 Optimal Reward-to-Go Iteration

We have introduced an expression for the expected reward J_{T,π_t} generated over the subsequent $T - t$ stages. The value can be calculated iteratively by successive application of the reward-to-go operator T_{μ_k} to the reward-to-go functions J_{T,π_k} starting with $k = T - 1$ proceeeding backwards in time until $k = t$. The result is stated in Theorem 3.1. In view of the formulated optimization problem the following related questions arise:

1. Does the relation $J_0 = J_{T,\pi^*} = (T_0 \cdot T_1 \cdot \dots \cdot T_{T-1})(J_T)$ hold, i.e. is it possible to decompose the computation of the optimal expected reward into a sequence of maximization problems in a similar way as for the expected reward? Or equivalently is the dynamic programming technique applicable?

2. Does an optimal control function $\mu_t \in \mathbb{K}_t$ for every $t \in \mathbb{T}$ exist, such that a (uniformly) optimal policy $\pi^* = (\mu_0^*, \mu_1^* \dots, \mu_{T-1}^*) \in \Pi$ exists? We call a policy π^* uniformly optimal, if for all $t \in \mathbb{T}$ the truncated policy π_t^* is ($T-t$ stage) optimal.

Let us assume for now, that these questions are affirmative. For later reference we summarize these assumptions on the optimization problem:

Assumption 3.3. *Let $T = 4n$ for some $n \in \mathbb{N}$. Then the following statements hold:*

(I) For all $X_t \in \mathbb{X}$, $\mu_t \in \mathbb{K}_t$ and all $J \in F_{t+1}$ the optimal operator T_t satisfying $T_t(J)(X_t) = \sup_{\mu_t} H(X_t, \mu_t(X_t), J)$ can be iteratively applied such that

$$J_{T,\pi^*}(X_0) = \sup_{\pi \in \Pi} J_{T,\pi} = (T_0 \cdot T_1 \cdots T_{T-1})(J_T)(X_0)$$

is the optimal (T-stage) value function.

(E) A uniformly optimal policy $\pi^* = (\mu_0^*, \dots, \mu_{T-1}^*)$ exists with $\mu_t^* = (\mu_t^{(1)*}, \mu_t^{(2)*}) \in \mathbb{K}_t$ for all $t \in \mathbb{T}$.

Now, let (I) and (E) be valid such that the dynamic programming technique due to Bellman can be applied. It decomposes the problem into a sequence of simpler maximization problems, that are carried out over the constraint control set at X_t, i.e. $K(X_t)$, for all $t \in \mathbb{T}$. The problem can then be re-formulated.

Problem Formulation. Find an optimal policy $\pi^* = (\mu_0^*, \dots, \mu_{T-1}^*) \in \Pi$ such that for all $X_0 \in \mathbb{X}$ the optimal value function is given by

$$\begin{aligned} J_0(X_0) &= J_{T,\pi^*}(X_0) \\ &= (T_0 \cdot T_1 \cdot \dots \cdot T_{T-1})(J_T)(X_0) \\ &= \sup_{\pi \in \Pi} J_{T,\pi}(X_0) \\ &= \sup_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{T-1} U[R_t(X_t, \mu_t(X_t), \omega)]\right] \end{aligned}$$

for all $t \in \mathbb{T}$ subject to

- $X_{t+1} = (\mathbf{H}_{t+1}, \mathbf{h}_{t+1}, s_{t+1})$ is generated according to $X_{t+1} = \Gamma(X_t, \mu_t(X_t), \omega)$,
- $\mu_t(X_t) \in K(X_t)$,

where $K(X_t)$ is the non-empty control constraint set at X_t for all $t \in \mathbb{T}$.

Remark 3.3. (i) Again expectation denotes the outer integral with respect to the measure $P(d(\omega_1, \omega_2, \ldots, \omega_T)|\, s_0)$ such that no further measurability assumptions are necessary (compare Definition 3.1). Note, when calling the problem well-defined we refer to the problem rewritten in terms of ordinary integration such that all necessary measurability assumptions are satisfied.

(i) It is not guaranteed a priori that a maximizing control policy exists. One must in any case require, that

$$J_{T,\pi}(X_0) < \infty.$$

for all $X_0 \in \mathbb{X}$.

(ii) We denote the smallest upper bound on the set of real numbers $\{J_{T,\pi}(X_0)|\, \pi \in \Pi\}$ by

$$J_0(X_0) = \sup_{\pi \in \Pi} J_{T,\pi}(X_0).$$

(iii) The optimal value of the problem depends on the initial state $X_0 \in \mathbb{X}$. We refer to the function J_0 that assigns to each initial state X_0 the corresponding optimal value $J_0(X_0)$ as the **optimal value function**.

3.4.4 Dynamic Programming

The following notations are introduced with regard to the specific choice of an exponential utility function U, i.e. $U(R) = -\exp\{-\gamma\, R\}$ for all $R \in \mathbb{R}$ with $\gamma > 0$, being a member of the utility class specified by Assumption 3.2. From now on and especially throughout the recursive DP equations, resulting in the structural result stated in Theorem 3.2, we use that notation and the property of the exponential utility that $U(x+y) = -U(x) \cdot U(y)$ for all $x, y \in \mathbb{R}$. At the end of our theoretical analysis, however, we argue that all results generalize to the class of utility functions specified by Assumption 3.2.

Spot Reward. For all $t \in \mathbb{T}$ the mapping $SR_{t+1} : I^{(2)}(X_t) \times \Omega \to \mathbb{R}$ is a real valued function given by

$$SR_{t+1}(b, \omega) := U((S_{t+1}(\omega) - \alpha)\, b)$$

with $\alpha > 0$. The function value SR_{t+1} will be referred to as the **(uncertain) spot reward** incurred at the next stage after choosing the spot control b, i.e. the second entry of the control variable c, in the corresponding control constraint set at X_t, $I^{(2)}(X_t)$. The **optimal spot reward value function** is then denoted by SR^*_{t+1}. Particularly, for all $t \in \mathbb{T}$ and any $X_t \in \mathbb{X}$ we define

$$ESR(s_t, b) := \mathbb{E}[SR_{t+1}(b, \omega) \mid s_t], \tag{3.2}$$

$$SR^*_{t+1}(X_t) := \sup_{b \in I^{(2)}(X_t)} ESR(s_t, b), \tag{3.3}$$

$$SR^*_{t+1}(X_k) := \mathbb{E}[\sup_{b \in I^{(2)}(X_t)} ESR(S_t, b) \mid s_k] \tag{3.4}$$

for all $k \in \{0, 1, \ldots, t-1\}$, Note, expectation in Definition 3.4 denotes the outer integral with respect to the measure $P^{t-k}|_{s_k} := P(d(\omega_{k+1}, \omega_{k+2}, \ldots, \omega_t)| \, s_k)$.

Forward Reward. For all $t \in \mathbb{T}$ the mapping $FR_t : I^{(1)}(X_t) \times \mathbb{R} \to \mathbb{R}$ is a real valued function given by

$$FR_t(f_t^{t+n}, F_t^{t+n}) := U((F_t^{t+n} - \alpha)\, f_t^{t+n})$$

with $\alpha > 0$. By convention F_t^{t+n} is a deterministic real valued function of the current spot price s_t, i.e. $F_t^{t+n} = F(t, t+n, s_t)$. The function value FR_t will be referred to as the **forward reward** incurred at one stage within the next quarter (per unit of production) after choosing the forward control f, i.e. the first entry of the control variable c, in the corresponding control constraint set $I^{(1)}(X_t)$. The **optimal quarter value function** for all $i \in \{n, 2n, 3n\}$ and any $X_{i-n+j} \in \mathbb{X}$ for all $j \in \{0, 1, \ldots, n-1\}$ is given by

$$Q^*_i(X_{i-n+j}) := \sup_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n+j}), \tag{3.5}$$

$$Q_i(f_i, X_{i-n+j}) := -FR(f_i, F^i_{i-n}) \cdot \mathbb{E}\left[\sum_{k=0}^{n} SR^*_{i+k+1}(X_{i+k}) | s_{i-n+j}\right], \tag{3.6}$$

$$FR^*_i := \sup_{f \in I^{(1)}(X_{i-n})} FR(f_i, F^i_{i-n}) \tag{3.7}$$

Note, expectation in Definition 3.6 denotes the outer integral with respect to the measure $P^{n-j}|_{s_{i-n+j}} := P(d(\omega_{i-n+j+1}, \ldots, \omega_i)| \, s_{i-n+j})$.

Dynamic Programming. To derive a structural result about the optimal value function J_0 we use the DP technique. For now, let (I) and (E) given in Assumption

3.3 hold true. Then for all $X_t \in \mathbb{X}$ the optimal reward-to-go function J_t is given via the Bellman equation

$$\begin{aligned} J_t(X_t) &:= \sup_{\mu_t \in \mathbb{K}_t} \mathbb{E}\left[\underbrace{U[R_t(X_t, \mu_t(X_t), \omega)] + J_{t+1}(\Gamma(X_t, \mu_t(X_t), \omega))}_{:=G_t(J_{t+1}, \omega)} \,\middle|\, s_t \right] \\ \mu_t^*(X_t) &:= \arg\max_{\mu_t \in \mathbb{K}_t} \mathbb{E}\left[U[R_t(X_t, \mu_t(X_t), \omega)] + J_{t+1}(\Gamma(X_t, \mu_t(X_t), \omega)) \,|\, s_t\right] \end{aligned}$$

for all $t \in \mathbb{T}$.

Thereafter the structural result is stated in the next theorem.

Theorem 3.2. *Let (I) and (E) given according to Assumption 3.3 hold true. For any $X_0 \in \mathbb{X}$ the optimal value function $J_0(X_0)$ has the structure*

$$J_0(X_0) = Q_0^*(X_0) + Q_n^*(X_0) + \mathbb{E}[Q_{2n}^*(X_n) \,|\, s_0] + \mathbb{E}[Q_{3n}^*(X_{2n})|s_0],$$

where for all $i \in \{n, 2n, 3n\}$ the optimal quarter value function $Q_i^*(X_{i-n})$ is given by

$$Q_i^*(X_{i-n}) = -FR_{i-n}^* \cdot \mathbb{E}\left[\sum_{j=i}^{i+n-1} SR_{j+1}^*(X_i)|s_{i-n}\right]$$

and expectation denotes the outer integral with respect to the measure $P^n|_{s_{i-n}}$. If $i = 0$ the optimal quarter value function is given by

$$Q_0^*(X_0) = -FR(h, H) \cdot \sum_{j=0}^{n-1} SR_{j+1}^*(X_0)$$

with $f_0^* = h$.

For all $i \in \{0, n, 2n, 3n\}$ the optimal controls are given by

$$\begin{aligned} f_i^* &:= \mu_{i-n}^{(1)}(X_{i-n}) = \arg\max_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n}) \qquad \text{if } i \neq 0 \qquad \text{and} \\ b_{i+k}^* &:= \mu_{i+k}^{(2)}(X_{i+k}) = \arg\max_{b \in I^{(2)}(X_{i+k})} ESR(s_{i+k}, b) \end{aligned}$$

for all $k \in \{0, 1, \ldots, n-1\}$. For all $t \in \mathbb{T}$ and $z := t - t \pmod{n}$ it holds

$$\begin{aligned} I^{(1)}(X_t) &= \begin{cases} \{h(z+n+1)\} & \text{for all } t \in \mathbb{T} \setminus \{0, n, 2n, 3n\}, \\ [\underline{K}, \overline{K} - \delta] & \text{for all } t \in \{0, n, 2n\}, \\ \{0\} & \text{for } t = 3n \qquad \text{and} \end{cases} \\ I^{(2)}(X_t) &= [0, \overline{K} - h(t)] = [0, \overline{K} - f_{z+n}^*]. \end{aligned}$$

Proof. The imposed conventions in the first part along with the definition of the dynamic system in the second part and the introduced notation yield the specific form of the dynamic programming equations. The structural result is determined on basis of the outer integral formulation. We will frequently make use of the measurability assumption that $UR_{t+1}(X_t, c_t) := \mathbb{E}[U(R_t(X_t, c_t, \omega)\,|s_t]$ is well-defined. Then the property of the outer integral, that $\int^* (f+h)\,dP = \int^* f\,dP + \int^* h\,dP$ holds true for all such f and h (compare Appendix B.1).

For all $X_{3n+k} \in \mathbb{X}$ with $k \in \{0, 1, \ldots, n\}$ we have

$$\begin{aligned}
J_T(X_T) :=& 0 \\
J_{T-1}(X_{T-1}) =& \sup_{\mu_{T-1} \in \mathbb{K}_{T-1}} \mathbb{E}\left[U[(H(T)-\alpha)h(T) + (S_T(\omega)-\alpha)b_{T-1}] \,|\, s_{T-1}\right] \\
=& \underbrace{-U[(H(T)-\alpha)h(T)]}_{=-FR(h(T),H(T))} \cdot \sup_{\mu_{T-1} \in \mathbb{K}_{T-1}} \mathbb{E}\left[U[(S_T(\omega)-\alpha)b_{T-1}] \,|\, s_{T-1}\right] \\
=& -FR^*_{2n} \cdot SR^*_T(X_{T-1}) \\
c^*_{T-1} = \mu^*_{T-1}(X_{T-1}) :=& -FR^*_{2n} \cdot \arg \max_{c_{T-1} \in K(X_{T-1})} \mathbb{E}\left[SR_T(b_{T-1}, \omega) \,|\, s_{T-1}\right]
\end{aligned}$$

where the optimal control c^*_{T-1} is to be found in $K(X_{T-1})$ with $I^{(1)}(X_{T-1}) = \{h(T+1)\} = \{f^*_{4n}\}$ and $I^{(2)}(X_{T-1}) = [0, \overline{K} - h(T-1)] = [0, \overline{K} - f^*_{3n}]$, such that $f^{T-1+n*}_{T-1} = f^*_{4n}$ and $b^*_{T-1} = \arg \max_{b \in I^{(2)}(X_{T-1})} ESR(s_{T-1}, b)$.

Backward induction yields

$$\begin{aligned}
J_{T-2}(X_{T-2}) =& \sup_{\mu_{T-2} \in \mathbb{K}_{T-2}} \mathbb{E}\left[U[(H(T-1)-\alpha)h(T-1) + (S_{T-1}(\omega)-\alpha)b_{T-2}]\right. \\
& \qquad\qquad \left. -FR^*_{2n} \cdot SR^*_T(X_{T-1}) \,|\, s_{T-2}\right] \\
=& -FR^*_{2n} \cdot [SR^*_{T-1}(X_{T-2}) + SR^*_T(X_{T-2})] \\
J_{T-3}(X_{T-3}) =& -FR^*_{2n} \cdot [SR^*_{T-2}(X_{T-3}) + SR^*_{T-1}(X_{T-3}) + SR^*_T(X_{T-3})] \\
& \vdots \\
J_{3n}(X_{3n}) =& -FR^*_{2n} \cdot \sum_{j=3n}^{T-1} SR^*_{j+1}(X_{3n}) \\
c^*_{3n} = \mu^*_{3n}(X_{3n}) :=& -FR^*_{2n} \cdot \arg \max_{c_{3n} \in K(X_{3n})} [ESR(s_{3n}, b_{3n}) + \sum_{j=3n+1}^{T-1} SR^*_{j+1}(X_{3n})]
\end{aligned}$$

where

$$\begin{cases} I^{(1)}(X_{3n}) = \{0\} \\ I^{(2)}(X_{3n}) = [0, \overline{K} - h(3n)] = [0, \overline{K} - f^*_{3n}]. \end{cases}$$

and

$$\begin{cases} f_{4n}^* = 0 \\ b_{3n}^* = \arg \max\limits_{b \in I^{(2)}(X_{3n})} ESR(s_{3n}, b) \end{cases}$$

Note, e.g. expectation in $SR_T^*(X_{T-3})$ is the outer integral with respect to the measure $P^2|_{s_{T-3}}$ (compare page 79). Turning to the third quarter for all $X_{2n+k} \in \mathbb{X}$ such that $k \in \{0, 1, \dots, n-1\}$ we obtain recursively

$$\begin{aligned} J_{3n-1}(X_{3n-1}) &= \sup_{\mu_{3n-1} \in \mathbb{K}_{3n-1}} \mathbb{E}\Big[U[(H(3n) - \alpha)h(3n) + (S_{3n}(\omega) - \alpha)b_{3n-1}] \\ &\qquad - FR_{2n}^* \cdot \sum_{j=3n}^{T-1} SR_{j+1}^*(X_{3n}) \mid s_{3n-1}\Big] \\ &= \underbrace{-FR(f_{2n}^*, F_n^{2n})}_{=-FR_n^*} \cdot SR_{3n}^*(X_{3n-1}) \underbrace{-FR_{2n}^* \cdot \mathbb{E}[\sum_{j=3n}^{T-1} SR_{j+1}^*(X_{3n})|s_{3n-1}]}_{=Q_{3n}^*(X_{3n-1})} \\ &\;\;\vdots \end{aligned}$$

$$J_{2n}(X_{2n}) = \underbrace{-FR_n^* \cdot \sum_{j=2n}^{3n-1} SR_{j+1}^*(X_{2n})}_{=Q_{2n}^*(X_{2n})} + Q_{3n}^*(X_{2n})$$

$$\begin{aligned} c_{2n}^* = \mu_{2n}^*(X_{2n}) := \arg \max_{c_{2n} \in K(X_{2n})} \Bigg\{ &-FR_n^* \cdot \Bigg[ESR(s_{2n}, b_{2n}) + \sum_{j=2n+1}^{3n-1} SR_{j+1}^*(X_{2n})\Bigg] \\ &\underbrace{-FR(f_{3n}, F_{2n}^{3n}) \cdot \mathbb{E}[\sum_{j=3n}^{T-1} SR_{j+1}^*(X_{3n})|s_{2n}]}_{=Q_{3n}(f_{3n}, X_{2n})} \Bigg\} \end{aligned}$$

where

$$\begin{cases} I^{(1)}(X_{2n}) = [\underline{K}, \overline{K} - \delta] \\ I^{(2)}(X_{2n}) = [0, \overline{K} - h(2n)] = [0, \overline{K} - f_{2n}^*] \end{cases}$$

and

$$\begin{cases} f_{3n}^* = \arg \max\limits_{f \in I^{(1)}(X_{2n})} Q_{3n}(f, X_{2n}) \\ b_{2n}^* = \arg \max\limits_{b \in I^{(2)}(X_{2n})} ESR(s_{2n}, b). \end{cases}$$

Proceeding analogously for the remaining two quarters by backward induction arguments for any given initial state $X_0 \in \mathbb{X}$ we obtain the specific structure of $J_0(X_0)$ along with the corresponding optimal operating policy. $\square$

3.4.5 Structure of the Optimal Value Function

Supposing (I) and (E) given according to Assumption 3.3 hold true the DP technique could be applied and we have derived a specific structure of the optimal value function J_0 of the SDP problem, that is given by

$$\begin{aligned}
J_0(X_0) &= Q_0^*(X_0) + Q_n^*(X_0) + \mathbb{E}[Q_{2n}^*(X_n) \mid s_0] + \mathbb{E}[Q_{3n}^*(X_{2n}) | s_0] \\
&= FR(h,H) \cdot \sup_{b \in I^{(2)}(X_0)} \left[ESR(s_0, b) + \sum_{j=1}^{n-1} SR_{j+1}^*(X_0) \right] \\
&+ \sup_{f \in I^{(1)}(X_0)} Q_n(f, X_0) \\
&+ \mathbb{E}[Q_{2n}^*(X_n) \mid s_0] + \mathbb{E}[Q_{3n}^*(X_{2n}) | s_0],
\end{aligned}$$

where expectation denotes the outer integral and for all $i \in \{n, 2n, 3n\}$ the quarter value function is given by

$$Q_i(f_i, X_{i-n}) = -FR(f_i, F_{i-n}^i) \cdot \mathbb{E}\left[\sum_{k=0}^{n} SR_{i+k+1}^*(X_i) | s_{i-n} \right].$$

The current decisions to be optimized do only affect the current and the next quarter. Due to the additive character of the "reward-to-go" function J_t for all $t \in \mathbb{T}$ the optimization problem to be solved on every stage of the DP algorithm can be split into two parts with respect to the current decision variables f_{t+n} and b_t. Hence, throughout the trading period $[0, T]$ for all $i \in \{0, n, 2n, 3n\}$ and $k \in \{0, 1, \ldots, n-1\}$ the following subproblems need to be solved:

1. **Optimal Spot Production:** Find $b_{i+k}^* \in I^{(2)}(X_{i+k})$ such that the expected *"day-ahead" spot reward*

$$ESR(s_{i+k}, b_{i+k})$$

   is maximized.

2. **Optimal Future Production:** Find $f_i^* \in I^{(1)}(X_{i-n})$ such that the *"next to go quarter" reward*

$$Q_i(f_i, X_{i-n})$$

   is maximized.

Allocating Character. To what extend do we face an allocation problem? Looking at the derived structure and the decomposition of the problem into a sequence of subproblems, where does this structure contribute to answer the previously posed main economic question of the problem: *How much capacity should be contracted in the forward market and how much capacity should be kept for bidding in the spot market?* The answer is then simple: The second subproblem exactly reflects the allocating character that needs to be respected when evaluating future production.

In mathematical terms the allocating character can be expressed by reformulating the *"optimal future production"* problem: Using for all $i \in \{0, n, 2n, 3n\}$ the notation

$$J_n^*(X_i) := \sum_{j=i}^{i+n-1} SR_{j+1}^*(X_i)$$

we have (for $i \in \{n, 2n, 3n\}$)

$$Q_i(f_i, X_{i-n}) = -FR(f_i, F_{i-n}^i) \cdot \mathbb{E}[J_n^*(X_i) \,|\, s_{i-n}],$$

where $J_n^*(X_i)$ depends indirectly on f_i^* in terms of the state vector X_i. The state X_i contains f_i^* through the "historic unit vector" h_i for all i. More specifically, the recursive mapping $J_n^*(X_i)$ is the sum of the expected optimal spot reward functions $SR_{i+k}^*(X_i)$ that are attained at $b_{i+k}^* = \mu_{i+k}^{(2)}(X_{i+k}) \in I^{(2)}(X_{i+k})$. Moreover, the spot control set for all $k \in \{0, 1, \ldots, n-1\}$ is given by $I^{(2)}(X_{i+k}) = [0, \overline{K} - f_i^*]$ due to the system dynamics and the definition of the forward control constraint set $I^{(2)}(X_{i+k})$. Hence, when optimizing the forward control f_i^* the spot control sets $I^{(2)}(X_{i+k})$ for all $k \in \{0, 1, \ldots, n-1\}$ are directly influenced by the value of f_i^* and we face the allocation problem.

Let us rearrange the quarter value function $Q_i(f_i, X_{i-n})$ such that we obtain as a corollary to Theorem 3.2 an explicit form of the *"optimal future production"* problem reflecting the allocating character:

Corollary 3.1. *Let $J_0(X_0)$ be given according to Theorem 3.2 then for any $i \in \{n, 2n, 3n\}$ and $X_{i-n} \in \mathbb{X}$ the quarter value function $Q_i(f_i, X_{i-n})$ is equivalent to*

$$\begin{aligned} Q_i(f_i, X_{i-n}) \;\; = \;\; \mathbb{E}\Bigg[& \sum_{k=0}^{n-1} U\left((F_{i-n}^i - S_{i+k+1}) f_i + (S_{i+k+1} - \alpha)\overline{K}\right) \mathbb{1}_{A_k}(\omega^{n+k}) \\ & + U\left((F_{i-n}^i - \alpha) f_i + (S_{i+k+1} - \alpha) b_{i+k}^*\right) \mathbb{1}_{B_k}(\omega^{n+k}) \,|\, s_{i-n} \Bigg], \end{aligned}$$

where expectation denotes the outer integral with respect to the measure $P^{2n-1}|_{s_{i-n}}$ and the spot control sets A_k and B_k, respectively, are given by

$$A_k = \left\{ \omega^{n+k} \;\middle|\; \arg\max_{b \in [0, \overline{K} - f_i]} \mathbb{E}[SR(b, \omega_{i+k}) | s_{i-n}] = \overline{K} - f_i \right\}$$

$$B_k = \left\{ \omega^{n+k} \,\middle|\, \arg \max_{b \in [0, \overline{K} - f_i]} \mathbb{E}[SR(b, \omega_{i+k}) | s_{i-n}] < \overline{K} - f_i \right\}$$

with $\omega^{n+k} := (\omega_{i-n+1}, \ldots, \omega_{i+k})$ *for all* $k \in \{0, 1, \ldots, n-1\}$.

Remark 3.4. (i) The spot price dynamics are used successively to express the spot prices $S_{i-n+1}, S_{i-n+2}, \ldots S_{i+n}$ exclusively in terms of $\omega_{i-n+1}, \omega_{i-n+2}, \ldots, \omega_{i+n}$ and s_{i-n} .

(ii) From an economic point of view intuitively one would assume (denoting any F^i_{i-n} with F , b^*_{i+k} with b^* and f^*_i with f^*):

- If $F \geq \alpha$ then forward production is profitable. Not accounting for the possibility of future spot bidding (above the spot reserve) the optimal forward control is at the maximum level, i.e. $\overline{K} - \delta$. If any future spot price scenario (within the delivery quarter) suggests an optimal spot control b^* above the spot reserve δ , spot price production is supposed to be profitable as well. In that scenario production is profitable and hence the plant is supposed to be operated at full capacity such that allocation of total capacity is necessary (compare Assumption 3.4 in Section 3.5.2).
- If $F < \alpha$ then forward production is not profitable. Hence, the optimal forward control is at the minimum level, i.e. $f^* = \underline{K}$. Such a conclusion will be verified in Section 3.5.2 within Lemma 3.3.

(iii) Let $F^i_{i-n} \geq \alpha$ and assume that an optimal solution exists. Then if for all future scenarios the spot prices S_{i+k+1} exceed the forward price F^i_{i-n} currently traded at the market, i.e. for all k

$$P^{n+k+1}(S_{i+k+1} > F^i_{i-n} \,|\, s_{i-n}) = 1,$$

then $b^*_{i+k} = \overline{K} - f_i$ for all k and immediately Corollary 3.1 implies $f^*_i = \underline{K}$. Contrary, if for all k

$$P^{n+k+1}(S_{i+k+1} < F^i_{i-n} \,|\, s_{i-n}) = 1,$$

then $f^*_i = \overline{K} - \delta$.

Hence, the optimal forward control $f^*_i \in I^{(1)}(X_{i-n})$ for all $X_{i-n} \in \mathbb{X}$ and $i \in \{n, 2n, 3n\}$ must be determined such that it maximizes the expected quarter value function given in Corollary 3.1 to reflect the allocating character of the problem.

Of course both subproblems can only be solved, if the maxima exist and the corresponding optimal value functions are finite. We will deal with these questions in the next section.

3.5 Existence and Uniqueness of a Solution

In view of the formulated optimization problem we now want to answer the preceeding questions:

1. Does the relation $J_0 = J_{T,\pi^*} = (T_0 \cdot T_1 \cdots T_{T-1})(J_T)$ hold, i.e. is it possible to decompose the computation of the optimal expected reward into a sequence of maximization problems in a similar way as for the expected reward? Or equivalently is the dynamic programming technique applicable?

2. Does an optimal control function $\mu_t \in \mathbb{K}_t$ for every $t \in \mathbb{T}$ exist, such that a (uniformly) optimal policy $\pi^* = (\mu_0^*, \mu_1^* \ldots, \mu_{T-1}^*) \in \Pi$ exists? We call a policy π^* uniformly optimal, if for all $t \in \mathbb{T}$ the truncated policy π_t^* is ($T-t$ stage) optimal.

That is we want to show, that the statements (I) and (E) of Assumption 3.3 hold true within the SDP problem formulation: By assuming (I) and (E) we have been able to derive a specific structure of the optimal value function J_0 and the operating policy stated in Theorem 3.2. With this structure at hand the problem has been decomposed into two kinds of subproblems with respect to the different control variables b and f. Proving existence and uniqueness for these subproblems (on every stage) then proves the validity of Assumption 3.3. In addition we show, that by posing appropriate measurability assumptions, i.e. $UR_{t+1}(X_t, c_t) := \mathbb{E}[U(R_t(X_t, c_t, \omega)|s_t]$ to be well-defined and μ_t to be measurable for every $t \in \mathbb{T}$, the whole problem can be understood and solved in terms of ordinary integration. Thereafter, the main result of the SDP problem is stated in Section 3.5.3.

3.5.1 Optimal Control - First Subproblem

To begin with, we derive statements concerning existence and uniqueness of a solution to the first subproblem, i.e. the problem of determining $b^* \in I^{(2)}(X_t)$ for all $t \in \mathbb{T}$ such that the expected *"day-ahead" spot reward* given according to (3.2)

$$ESR(s_t, b) = \mathbb{E}[SR_{t+1}(b, \omega) \mid s_t]$$

is maximized, where by convention we have $I^{(2)}(X_t) = [0, a_t]$ with $a_t := \overline{K} - h(t+1) > 0$ with $h(t) = f_{z+n}$ for all $t \in \mathbb{T}$ and $z = t - t \pmod{n}$ and $\mu_t^{(2)} : \mathbb{X} \to C$ is appropriately measurable.

Due to the properties of the utility function U (summarized by Assumption 3.2) for any $t \in \mathbb{T}$ the function $SR_{t+1}(\cdot, \omega)$ is continuous, concave, and differentiable for P-almost every $\omega \in \Omega$ and $SR_{t+1}(b, \cdot)$ is measurable and integrable for every $b \in I^{(2)}(X_t)$.

It is then our aim to prove that (given $s_t \in \mathbb{R}$) the expected value function $ESR(s_t, b)$ inherits certain properties of the integrand $SR_{t+1}(b, \omega)$. The next lemmata state these properties following the results of Ruszczyński & Shapiro stated in Chapter 2 of [RS03].

Lemma 3.1. *Given $s_t \in \mathbb{R}$ for any $t \in \mathbb{T}$ the expected value function $ESR(s_t, b)$ is continuous at $b \in I^{(2)}(X_t)$ and integrable.*

Proof. For any $t \in \mathbb{T}$ the integrand function $SR_{t+1}(b, \cdot)$ is measurable for every $b \in I^{(2)}(X_t)$ and there exists a P-integrable function $Z(\omega)$ such that $|SR_{t+1}(b, \omega)| \leq Z(\omega)$ for P-almost every $\omega \in \Omega$ and all $b \in I^{(2)}(X_t)$. Then also $ESR(s_t, b)$ is integrable. Moreover, by the Lebesgue Dominated Convergence Theorem we can take the limit inside the integral. Together with the continuity of $SR_{t+1}(\cdot, \omega)$ for P-almost every $\omega \in \Omega$ that implies

$$\begin{aligned} \lim_{b \to b^*} \int_\Omega SR_{t+1}(b, \omega)\, P(d\omega; s_t) &= \int_\Omega \lim_{b \to b^*} SR_{t+1}(b, \omega)\, P(d\omega; s_t) \\ &= \int_\Omega SR_{t+1}(b^*, \omega)\, P(d\omega; s_t). \end{aligned}$$

This shows continuity of $ESR(s_t, b)$ at $b^* \in I^{(2)}(X_t)$ for any $t \in \mathbb{T}$ and $s_t \in \mathbb{R}$. □

Lemma 3.2. *Given $s_t \in \mathbb{R}$ for any $t \in \mathbb{T}$ the expected value function $ESR(s_t, \cdot)$ is concave and differentiable.*

Proof. For any $t \in \mathbb{T}$ the concavity of $ESR(s_t, b)$ for given s_t follows immediately from the concavity of $SR_{t+1}(\cdot, \omega)$ for P-almost every $\omega \in \Omega$. For all $b_t \in I^{(2)}(X_t)$ and any given s_t the expected value function $ESR(s_t, b_t)$ is finite. Hence, $ESR(s_t, b^*)$ and $ESR(s_t, b^*+h_0)$ are finite for some $h_0 > 0$ and b^*, $b^*+h_0 \in I^{(2)}(X_t)$. It follows from the concavity and differentiability of $SR_{t+1}(\cdot, \omega)$ that the ratio

$$g_h(\omega) := h^{-1} \left[SR_{t+1}(b^* + h, \omega) - SR_{t+1}(b^*, \omega) \right]$$

is monotonically increasing to $SR'_{t+1}(b^*, \omega)$ as $h \downarrow 0$. Also we have that

$$\mathbb{E}[|g_{h_0}(\omega)|] \leq h_0^{-1} \left(\mathbb{E}[|SR_{t+1}(b^* + h_0, \omega)|] + \mathbb{E}[|SR_{t+1}(b^*, \omega)|] \right) < +\infty.$$

Then it follows by Lemma 3.1 and the Monotone Convergence Theorem that

$$\lim_{h \downarrow 0} \mathbb{E}[g_h(\omega) \mid s_t] = \mathbb{E}[\lim_{h \downarrow 0} g_h(\omega) \mid s_t] = \mathbb{E}[SR'_{t+1}(b^*, \omega) \mid s_t].$$

Thus $ESR(s_t, \cdot)$ is differentiable at b^*, i.e.

$$ESR'(s_t, b^*) = \mathbb{E}[SR'_{t+1}(b^*, \omega)|s_t].$$

□

Theorem 3.3. *The set* $\underset{b \in I^{(2)}(X_t)}{\arg\max} ESR(s_t, b)$ *is non-empty. If additionally* b^* *exists with* $ESR'(s_t, b^*) = 0$, *then* b^* *is a global maximizer of* $ESR(s_t, \cdot)$.

Proof. Due to Lemma 3.1 the expected value function $ESR(s_t, \cdot)$ is continuous for any given $s_t \in \mathbb{R}$ with bounded domain $[0, a_t]$. That implies the first statement of the theorem, i.e. that the set $\underset{b \in I^{(2)}(X_t)}{\arg\max} ESR(s_t, b)$ is non-empty. Due to Lemma 3.2 the derivative of the expected value function exists. If there exists a b^* with $ESR'(s_t, b^*) = 0$, then the concavity of $ESR(s_t, \cdot)$ for any given $s_t \in \mathbb{R}$ immediately implies that b^* is a global maximum of $ESR(s_t, \cdot)$ and $b^* \in [0, a_t]$. □

We have been able to show, that an optimal solution to the first subproblem exists on every stage of the DP algorithm, i.e. for all $t \in \mathbb{T}$ the supremum in the relation $SR^*_{t+1}(X_t) = \sup_{b \in I^{(2)}(X_t)} ESR(s_t, b)$ is attained. We have shown, that $ESR(s_t, b)$ is continuous, concave and integrable. Moreover, the supremum is to be found in the compact set $I^{(2)}(X_t)$. Consequently, also $SR^*_{t+1}(X_t)$ is integrable for all $t \in \mathbb{T}$.

After that, we can turn to the *second subproblem.* That is the task of finding an optimal (forward) control variable f, such that the *next to go quarter reward* is maximized. Due to the structure of the optimal value function V^*_T and the corresponding optimal policy stated in Theorem 3.2, it is sufficient to derive the desired properties of the optimal quarter value functions $Q^*_i(X_{i-n})$ and the corresponding optimal control variables f^*_i, to be found in the compact set $I^{(1)}(X_{i-n})$, for all $X_{i-n} \in \mathbb{X}$ and $i \in \{n, 2n, 3n\}$. Moreover, f^*_i fixes the left end point of the interval $I^{(2)}(X_{i+k}) = [0, a_{i+k}]$ for all $i \in \{n, 2n, 3n\}$ and $k \in \{0, 1, \ldots, n-1\}$. Thus, we are presented with the allocation problem. Existence, uniqueness and well-posedness of the so-called "optimal future production" problem will be stated within the next section.

3.5.2 Optimal Control - Second Subproblem

The *"optimal future production"* problem has been specified by the task of finding (at time $i - n$) an optimal (forward) control $f^*_i \in I^{(1)}(X_{i-n})$ for all $i \in \{n, 2n, 3n\}$

such that the *"next to go quarter reward"* given according to (3.6)

$$Q_i(f_i, X_{i-n}) = -FR(f_i, F_{i-n}^i) \cdot \mathbb{E}[\sum_{j=i}^{i+n-1} SR_{j+1}^*(X_i)|s_{i-n}] \tag{3.8}$$

is maximized.

Remark 3.5. For convenience let us summarize the conventions affecting the *optimal future production problem* by the following list.

(i) The mapping $J_n^* : \mathbb{R} \to \mathbb{R}$ is given by

$$J_n^*(X_i) := \sum_{j=i}^{i+n-1} SR_{j+1}^*(X_i)$$

for all $i \in \{0, n, 2n, 3n\}$.

(ii) For all $i \in \{n, 2n, 3n\}$

$$Q_i^*(X_{i-n}) = \sup_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n})$$

denotes the **optimal quarter value function** and

$$f_i^* = \mu_{i-n}^{(1)*}(X_{i-n}) := \arg \max_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n})$$

the corresponding **optimal control variable** with respect to the current state X_{i-n}, where $\mu_t^{(1)}$ is appropriately measurable for all $t \in \mathbb{T}$.

(iii) The **(forward) control constraint set** at X_t is given by

$$I^{(1)}(X_t) = \begin{cases} \{h(z+n+1)\} & \text{for all } z = t - t \pmod{n} \text{ and } t \in \mathbb{T} \setminus \{0, n, 2n, 3n\}, \\ [\underline{K}, \overline{K} - \delta] & \text{for all } t \in \{0, n, 2n\}, \\ \{0\} & \text{for } t = 3n. \end{cases}$$

Obviously, for all $i \in \{n, 2n, 3n\}$ the **optimal quarter value function** is considerably influenced by the optimal spot reward accumulated at the future stages $\{i+1, i+2, \ldots, i+n\}$, given by $J_n^*(X_i)$. The sum is affected by the optimal (spot) control variables b_{i+k}^* for all $k \in \{0, 1, \ldots, n-1\}$. Hence, the next theorem states existence and uniqueness of $SR_{j+1}^*(X_i)$ for all j and any given $s_i \in \mathbb{R}$. Moreover, it states integrability of the *optimal spot value function* $J_n^*(X_i)$.

Theorem 3.4. *For $i \in \{0, n, 2n, 3n\}$, $k \in \{0, 1, \ldots, n-1\}$ and $n \in \mathbb{N}$ the optimal spot value function has the specific form*

$$J^*_{n-k}(X_{i+k}) = \sum_{v=k}^{n-1} SR^*_{i+v+1}(X_{i+k})$$

and $J^*_{n-k}(X_{i+k})$ is integrable. For all $v \in \{k, k+1, \ldots, n-1\}$ the suprema in the relation

$$SR^*_{i+v+1}(X_{i+v}) = \sup_{b \in I^{(2)}(X_{i+v})} ESR(s_{i+v}, b_{i+v})$$

are attained at $b^*_{i+v} = \mu^{(2)*}_{i+v}(X_{i+v}) = \arg \max_{b_{i+v} \in I^{(2)}(X_{i+v})} ESR(s_{i+v}, b_{i+v})$.

Proof. Without loss of generality let $i = 0$. The proof will be carried out by using backward induction techniques.

- For $t = n-1$ the above lemmata imply that for any given $s_{n-1} \in \mathbb{R}$ the conditional expectation $ESR(s_{n-1}, b_{n-1})$ is continuous, concave and differentiable with respect to b_{n-1}, where $b_{n-1} \in I^{(2)}(X_{n-1}) = [0, \overline{K} - h(n)]$. Applying Theorem 3.3 yields the existence of $J^*_1(X_{n-1}) = SR^*_n(X_{n-1}) = \sup_{b_{n-1} \in I^{(2)}(X_{n-1})} ESR(s_{n-1}, b_{n-1})$. The optimal control variable $b^*_{n-1} = \mu^{(2)*}_{n-1}(X_{n-1}) = \arg \max_{b_{n-1} \in I^{(2)}(X_{n-1})} ESR(s_{n-1}, b_{n-1})$ is uniquely found in the compact interval $[0, a_{n-1}]$. With the properties of $ESR(s_{n-1}, b_{n-1})$ the integrability of $J^*_1(X_{n-1})$ holds true.

(IH) For any fixed $k \in \{0, 1, \ldots, n-1\}$ the value function is given by $J^*_{n-k}(X_k) = \sum_{v=k}^{n-1} SR^*_{v+1}(X_k)$ and $J^*_{n-k}(X_k)$ is integrable. The suprema in the relation

$$SR^*_{v+1}(X_v) = \sup_{b \in I^{(2)}(X_v)} ESR(s_v, b)$$

are attained at $b^*_v = \mu^{(2)*}_v(X_v) = \arg \max_{b \in I^{(2)}(X_v)} ESR(s_v, b)$ for all $v \in \{k, k+1, \ldots, n-1\}$.

- The induction step is carried out by going backwards from k to $k-1$. We have

$$\begin{aligned}
&\mathbb{E}[SR_k(b_{k-1}, \omega) + J^*_{n-k}(X_k) | s_{k-1}] \\
&\overset{(A)}{=} ESR(s_{k-1}, b_{k-1}) + \mathbb{E}[J^*_{n-k}(X_k) | s_{k-1}] \\
&\overset{(IH)}{=} ESR(s_{k-1}, b_{k-1}) + \mathbb{E}[\sum_{j=k}^{n-1} SR^*_{j+1}(X_k) | s_{k-1}] \\
&\overset{(B)}{=} ESR(s_{k-1}, b_{k-1}) + \sum_{j=k}^{n-1} SR^*_{j+1}(X_{k-1}),
\end{aligned}$$

where (A) is true since by induction hypothesis $J^*_{n-k}(X_k)$ is integrable. Assertion (B) uses the notation introduced in (3.4), that is applicable due to the linearity of the expectation. Applying the line of arguments stated in the case $k = n-1$ we can conclude that the optimal value

$$SR^*_k(X_{k-1}) = \sup_{b_{k-1}\in I^{(2)}(X_{k-1})} ESR(s_{k-1}, b_{k-1})$$

is attained at $\mu^{(2)*}_{k-1}(X_{k-1}) = \arg\max_{b\in I^{(2)}(X_{k-1})} ESR(s_{k-1}, b)$ for all $X_{k-1} \in \mathbb{X}$ and $SR^*_k(X_{k-1})$ is integrable. Using the induction hypothesis we conclude, that the optimal value function is given by

$$\begin{aligned} J^*_{n-(k-1)}(X_{k-1}) &= \sup_{b\in I^{(2)}(X_{k-1})} \{ESR(s_{k-1}, b) + \sum_{j=k}^{n-1} SR^*_{j+1}(X_{k-1})\} \\ &= \sum_{j=k-1}^{n-1} SR^*_{j+1}(X_{k-1}) \end{aligned}$$

with $b^*_v = \arg\max_{b\in I^{(2)}(X_v)} ESR(s_v, b)$ for all $v \in \{k-1, k, \ldots, n-1\}$ and is integrable.

Thus, the statement holds for all $k \in \{0, 1, \ldots, n-1\} \subset \mathbb{N}$. □

Then immediately the next result about the first quarter value function follows.

Corollary 3.2. *The first quarter value function is given by*

$$Q_0(h, X_0) = -FR(h, H) \cdot J^*_n(X_0)$$

with $f^*_0 = h$ and $Q_0(h, X_0)$ is optimal, well-defined and all involved suprema are attained.

It is then left to state the corresponding result of existence and uniqueness with respect to the (forward) control variable f_i for all $i \in \{n, 2n, 3n\}$. First we consider the case when $F^i_{i-n} < \alpha$. Our intuition, that the optimal (forward) control should be attained at the minimum (forward) production level, i.e. $f^*_i = \underline{K}$, can now be verified.

Lemma 3.3. *If $F^i_{i-n} < \alpha$ then $f^*_i = \underline{K}$ for all $i \in \{n, 2n, 3n\}$.*

Proof. Let i be fixed (we omit it in the notation as before) and assume $F < \alpha$ and $s \in \mathbb{R}$ given. Then we need to find $f^* \in [\underline{K}, \overline{K} - \delta]$ such that

$$Q(f, F) = -FR(f, F) \cdot \mathbb{E}\left[\sum_{k=0}^{n-1} \sup_{b_k\in[0,\overline{K}-f]} ESR(s_k, b_k) | \, s\right]$$

is maximized with respect to f. Clearly, $-FR(f,F)$ is concave, continuous and strictly increasing with respect to f (since $F<\alpha$). On the other hand $ESR(s_k,b_k)$ for all b_k is negative and with Theorem 3.4 for all k the suprema b_k^* are attained. Hence, the unique solution exists and is attained at $f^*=\underline{K}$. □

Next, in the case $F\geq\alpha$ the structure of the quarter value function $Q(f,F)$ as given in Corollary 3.1 must be further specified by introducing the subsequent assumption.

Assumption 3.4. *If $F_{i-n}^i\geq\alpha$ and $b_{i+k}^*=\mu_{i+k}^{(2)}(X_{i+k})>\delta$ for any $X_{i+k}\in\mathbb{X}$ with $k\in\{0,1,\dots,n-1\}$ and $i\in\{n,2n,3n\}$ then we suppose $f_i^*+\mu_{i+k}^{(2)}(X_{i+k})=\overline{K}$.*

The forward reward function $FR(f,F)$ is concave, continuous and strictly increasing with respect to f (if $F\geq\alpha$). In contrast, if f increases, then the supremum b^* of the expected *"day-ahead" spot reward* $ESR(s,b)$ is searched within an interval, where the left endpoint of that interval decreases as soon as f increases. Once more, this reflects the allocating character of the problem. To deal with that *allocation problem* we need the preceeding assumption that $b+f=\overline{K}$ for all scenarios implying $b>\delta$ in order to calculate an optimal solution. Such an assumption can also be economically motivated.

Remark 3.6. Let the forward price observed by the plant manager be above the fixed costs per unit of production. Not accounting for the possibility of spot bidding in the delivery quarter, the plant manager would devote all available capacity to forward contracts. In contrast, not accounting for bilateral agreements the plant manager has the possibility to determine based on future price scenarios (generated according to the underlying factor model) the optimal spot units to bid on the spot market on a certain day within that delivery quarter. Altogether, he decides (by assumption) to run the power plant at maximum capacity for all future scenarios that suggest to bid more than the spot reserve at the spot market. Hence, in that situation the spot reserve somehow depicts a boundary that suggests full capacity production to be profitable. Next he transfers such a consideration to all future scenarios within the delivery quarter. Finally, based on today's forward price and expected future spot price scenarios, he is in the position to calculate the optimal forward units he agrees to deliver within the next quarter.

Back to the mathematical formulation, that is Corollary 3.1 can be applied with the specific choice of

$$A_k := \left\{\omega^{n+k}\;\middle|\; \arg\max_{b\in[0,\overline{K}-f_i]}\mathbb{E}[SR(b,\omega_{i+k})|s_{i-n}]>\delta\right\} \tag{3.9}$$
$$B_k := \left\{\omega^{n+k}\;\middle|\; \arg\max_{b\in[0,\overline{K}-f_i]}\mathbb{E}[SR(b,\omega_{i+k})|s_{i-n}]\leq\delta\right\}$$

for all $k \in \{0, \ldots, n-1\}$, where $\omega^{n+k} := (\omega_{i-n+1}, \ldots, \omega_{i+k})$.

Thereafter, in the general case existence and uniqueness of an optimal solution is stated by the next Theorem.

Theorem 3.5. *For* $k \in \{0, \ldots, n-1\}$, $n \in \mathbb{N}$ *and* $i \in \{n, 2n, 3n\}$ *the optimal quarter value function* $Q_i^*(X_{i-n})$ *is given by* (3.8) *and* $Q_i^*(X_{i-n})$ *is integrable. The suprema in the relations*

$$Q_i^*(X_{i-n}) = \sup_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n})$$

and

$$SR_{i+k+1}^*(X_{i+k}) = \sup_{b \in I^{(2)}(X_{i+k})} ESR(s_{i+k}, b)$$

are attained at

$$f_i^* = \mu_{i-n}^{(1)*}(X_{i-n}) = \arg \max_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n})$$

and

$$b_{i+k}^* = \mu_{i+k}^{(2)*}(X_{i+k}) = \arg \max_{b \in I^{(2)}(X_{i+k})} ESR(s_{i+k}, b),$$

respectively.

Proof. Due to Theorem 3.4 $J_n^*(X_i)$ is integrable and the optimal spot controls are attained at b_{i+k}^* for all k and all $i \in \{n, 2n, 3n\}$. From Theorem 3.2 we have

$$Q_i(f, X_{i-n}) = -FR(f, F_{i-n}^i) \cdot \mathbb{E}[J_n^*(X_i) | s_{i-n}]$$

and thus $Q_i(f, X_{i-n})$ is integrable. The involved expectation can now be understood in terms of ordinary integration with respect to the measure $P^n|_{s_{i-n}}$, where the integral has the following representation

$$\begin{aligned}\int_\Omega \int_\Omega \cdots \int_\Omega J_n^*(X_i)\, P(d\omega_i | s_{i-1}) \cdots P(d\omega_{i-n+2} | s_{i-n+1}) P(d\omega_{i-n+1} | s_{i-n}) \\ \overset{\text{Fubini}}{=} \int_{\Omega^n} J_n^*(X_i) P(d(\omega_{i-n+1}, \ldots, \omega_i) | s_{i-n}).\end{aligned}$$

It is then left to show existence and uniqueness of f_i^*.

In the case $F_{i-n}^i < \alpha$ Lemma 3.3 states that $Q_i(f, X_{i-n})$ attains its supremum at $f_i^* = \underline{K}$. If $F_{i-n}^i \geq \alpha$ the expectations involved in the quarter value function Q_i, given according to Corollary 3.1 by

$$\begin{aligned}Q_i(f, X_{i-n}) = \sum_{k=0}^{n-1} \mathbb{E}\left[\underbrace{U\left((F_{i-n}^i - S_{i+k+1}) f + (S_{i+k+1} - \alpha) \overline{K} \right) \mathbb{1}_{A_k}(\omega^{n+k})}_{\text{integrand (1)}} \Big| s_{i-n} \right] \\ + \sum_{k=0}^{n-1} \mathbb{E}\left[\underbrace{U\left((F_{i-n}^i - \alpha) f + (S_{i+k+1} - \alpha) b_{i+k}^* \right) \mathbb{1}_{B_k}(\omega^{n+k})}_{\text{integrand (2)}} \Big| s_{i-n} \right],\end{aligned}$$

can now be understood in terms of ordinary integration with respect to the respective measure $P^{n+k}|_{s_{i-n}}$ for all k. Together with Assumption 3.4 the optimal spot controls b^*_{i+k} for all k imply the sets A_k and B_k defined by (3.9). Due to the properties of U (given according to Assumption 3.2) for almost every $\omega^{n+k} = (\omega_{i-n+1}, \ldots, \omega_{i+k}) \in A_k$ *integrand (1)* and for almost every $\omega^{n+k} \in B_k$ *integrand (2)* is concave, continuous and differentiable with respect to $f \in I^{(1)}(X_{i-n})$. Hence, as a consequence of Lemmata 3.1 and 3.2 (that apply analogously) the corresponding expected value functions inherit these properties. Then again the sum of expected value functions inherits these properties. Finally the compactness of $I^{(1)}(X_{i-n})$ guarantees that the unique maximum is attained at $f^*_i = \mu^{(1)*}_{i-n}(X_{i-n}) = \arg \max_{f \in I^{(1)}(X_{i-n})} Q_i(f, X_{i-n})$. Thus also $Q^*_i(X_{i-n})$ is integrable. □

3.5.3 Main Result of the SDP Problem

To summarize the above derived results, we state the main theorem, concerning the stochastic dynamic programming problem, within the given model framework.

Theorem 3.6. *Let $T = 4n$ for some $n \in \mathbb{N}$. Let Assumption 3.4 be satisfied and let $U : \mathbb{R} \to \mathbb{R}$ denote the exponential utility function given by (3.1). Then the following statements hold:*

(E) A uniformly optimal policy $\pi^* = (\mu^*_0, \ldots, \mu^*_{T-1})$ exists with $\mu^*_t = (\mu^{(1)*}_t, \mu^{(2)*}_t) \in \mathbb{K}_t$ for all $t \in \mathbb{T}$.

(I) For all $X_t \in \mathbb{X}$, $\mu_t \in \mathbb{K}_t$ and all $J \in F_{t+1}$ the optimal operator T_t satisfying $T_t(J)(X_t) = \sup_{\mu_t} H(X_t, \mu_t(X_t), J)$ can be iteratively applied such that

$$J_{T,\pi^*}(X_0) = \sup_{\pi \in \Pi} J_{T,\pi} = (T_0 \cdot T_1 \cdots T_{T-1})(J_T)(X_0)$$

is the optimal (T-stage) value function.

(S) The optimal value function $J_{T,\pi^*}(X_0)$ has the specific form

$$J_{T,\pi^*}(X_0) = Q^*_0(X_0) + Q^*_n(X_0) + \mathbb{E}[Q^*_{2n}(X_n) \,|\, s_0] + \mathbb{E}[Q^*_{3n}(X_{2n})|s_0]$$

and $J_{T,\pi^*}(X_0)$ is well-defined.

Proof. Corollary 3.2 and Theorem 3.5 imply that $Q^*_0(X_0)$ and $Q^*_i(X_{i-n})$ for all $i \in \{n, 2n, 3n\}$ are integrable and that for all well-defined $J \in F_{t+1}$ the function

$$H(X_t, \mu_t(X_t), J) = \mathbb{E}\left[\, U[R_t(X_t, \mu_t(X_t), \omega)] + J(\Gamma(X_t, \mu_t(X_t), \omega)) \,|\, s_t \,\right]$$

is well-defined. Note, the proof is given within Theorem 3.4 and Theorem 3.5.

Additionally, Corollary 3.2 and Theorem 3.5 imply that all suprema in the relation

$$T_t(J)(X_t) = \sup_{\mu_t} T_{\mu_t}(J)(X_t) = \sup_{\mu_t} H(X_t, \mu_t(X_t), J)$$

are attained at $c_t^* = (f_{t+n}^*, b_t^*) = \mu_t^*(X_t)$ for all $t \in \mathbb{T}$ and all $X_t \in \mathbb{X}$. This is exactly equivalent to the existence of an uniformly optimal policy π^* such that the existence statement (E) is proved. Due to Bertsekas & Shreve [BS80] (compare also Appendix B) the existence of an uniformly optimal policy π^* immediately implies statement (I), i.e. the problem can be decomposed and the DP technique applies. Then Theorem 3.2 applies and yields the structure of the optimal value function $J_{T,\pi^*}(X_0)$, that is also well-defined. Such that (S) holds true and the statement is proved. □

Obviously, Theorem 3.6 extends to a finite time horizon that is an arbitrary number M of segments of length n.

Corollary 3.3. *Let $T = (M+1) \cdot n$ for some $n, M \in \mathbb{N}$ such that $T < \infty$. Then the optimal value function is given by*

$$J_{T,\pi^*}(X_0) = Q_0^*(X_0) + Q_n^*(X_0) + \sum_{m=2}^{M} \mathbb{E}[Q_{m\cdot n}^*(X_{(m-1)n}) |\, X_0].$$

Extension to General Utility Class. Due to the structure of the problem all main results are valid for any utility function U in the class of concave, continuously differentiable utilities specified by Assumption 3.2. Factoring the negative of the forward reward function $-FR_{i-n}(f, F_{i-n}^i)$ back inside the spot reward functions $SR_{i+k+1}(b,\omega)$ for all $k = 0, 1, \ldots, n-1$ and $i \in \{n, 2n, 3n\}$ yields the general form. To put it differently, rearranging the notation such that $FR(f_i, F_{i-n}^i) := -1$ and $SR_{i+k+1}(b,\omega)$ is replaced by $UR_{i+k+1}(c,\omega) := U(R_{i+k}(X_{i+k}, c, \omega))$ for all i and k all results apply analogously.

3.6 Cox Ross Rubinstein Market

3.6.1 Motivation

To get an insight into the structure of the problem, to identify the interaction of the operational constraints and the implications of different market evolutions on an optimal policy, we choose a basic model for the underlying stochastic process S observed at the discrete dates $t \in \mathbb{T}$, i.e. $(S_t)_{t\in\mathbb{T}}$, to generate such specific market situations. We study the well-known binomial model first proposed by Cox, Ross and Rubinstein in 1979 [CRR79] with different choices of the involved market parameter u and d. Of course, one could calibrate the CRR model to market data and use these parameters to reflect the market through the model dynamics. However, we feel that the CRR market is only in a narrower sense a possible candidate for reflecting the special properties of the electricity spot market. Hence, we consider the underlying market structure as a tool to understand the implications and interactions of the SDP model. Thus, we examine the somehow extreme cases of such a simple market structure and try to point out a relationship between a certain basic market evolution and the basic structure of a corresponding optimal policy. These ideas should then be tested and verified by using another, "more realistic" factor model. This task will be accomplished in Section 3.7.

3.6.2 Spot Market

According to the plant characteristics posed in Assumption 3.1 the price of the chosen input product per unit of production is assumed to be constant. One could think of a long-term purchase contract e.g. with a gas importing company supplying pipeline gas. The CRR model hence represents the electricity price dynamics. We assume a quarterly time structure with $n = 30$ (i.e. a four month time horizon).

Then the electricity spot market is given as a T-period Cox Ross Rubinstein model under no arbitrage assumptions with $t \in \mathbb{T}$ and $T = 120$. Assuming a constant risk-free interest rate of $r = 0$ it consists of

- one bond $B_{t+1} = 1 \cdot B_t$ with $B_0 = \alpha$ (representing the fixed costs) and
- one risky asset $S_{t+1} = R_{t+1}\, S_t$ with $S_0 = s_0$ (representing the uncertain electricity spot price),

that evolves according to the identically, independently distributed sequence of random variables $(R_t)_{t\in\mathbb{T}}$ in discrete time assuming the values

$$R_{t+1} = \begin{cases} u & \text{with probability } p_u \text{ and} \\ d & \text{with probability } p_d. \end{cases}$$

All random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The random variables S_t are adapted to the filtration. The so generated spot market is free of arbitrage if and only if $d < 1 < u$. Hence, we now deal with a specific choice of the process conventions posed in general in Section 3.2.

3.6.3 Forward Market

The forward contracts are assumed to deliver at continuous flow at each day within the next month (so called *swap contracts*). The settlement of the forward contracts should be at the beginning of each day within the next quarter. Thus, the forward price of a contract signed at date t written on electricity, delivering in $[T^S, T^S+1)$ is given by

$$F_t^{T^S} = F(t, T^S, s_t) = \mathbb{E}_{\mathbb{Q}}[S_{T^S} \mid \mathcal{F}_t]$$

for $T^S = t+30$. Here, the equivalent martingale measure $\mathbb{Q}$ represents the risk-adjusted probability measure such that the scenario tree under $\mathbb{Q}$ matches the volatility and term structure of forward and future prices observable at the market. Thereafter, the forward price is explicitly given by

$$F_0^{T^S} = F(0, T^S, s_0) = \sum_{j=0}^{T^S} \binom{T^S}{j} q_u^j \, q_d^{T^S-j} \, (s_0 \, u^j \, d^{T^S-j}), \tag{3.10}$$

where the martingale probabilities q_u and q_d are given by the risk-neutral probabilities

$$\begin{cases} q_u = \frac{1-d}{u-d} \\ q_d = \frac{u-1}{u-d}. \end{cases}$$

Note, such a choice can only be verified with regard to our motivation to look at the most simple case in order to learn about the model and develop expectations on the form of an optimal control.

3.6.4 First Subproblem

Let us once more point out, that througout the numerical analysis all results are calculated for the exponential utility function introduced in (3.1). According to

Section 3.4.5 the optimization problem can be splitted into two subproblems. Hence, the first task is to find $b^*_{i+k} \in I^{(2)}(X_{i+k})$ such that the expected "day-ahead" spot reward

$$ESR(s_{i+k}, b_{i+k}) = \mathbb{E}[SR(b_{i+k}, \omega) \,|\, s_{i+k}]$$

is maximized for all $i \in \{0, n, 2n, 3n\}$ and $k \in \{0, 1, \ldots, n-1\}$. Given the currently observable spot price $s_t \in \mathbb{R}$ it is possible to calculate an explicit analytical solution in case of the now underlying market structure.

Lemma 3.4 (Optimal Spot Control)**.** *Given $s_t \in \mathbb{R}$ for all $t \in \mathbb{T}$ it holds*

$$b^*_t = \begin{cases} a_t & \textit{if } s_t d > \alpha \textit{ or } \tilde{b}^*_t > a_t \\ 0 & \textit{if } s_t u \leq \alpha \textit{ or } \tilde{b}^*_t \leq 0. \\ \tilde{b}^*_t & \textit{else} \end{cases} \tag{3.11}$$

with

$$\tilde{b}^*_t = \frac{\ln(\frac{p_u}{p_d}) + \ln(\frac{s_t u - \alpha}{\alpha - s_t d})}{\gamma\, s_t(u-d)} = \arg\max_{b_t \in \mathbb{R}} ESR(b_t, s_t)$$

and $a_t = \overline{K} - f_{z+n}$ such that $z = t - t \pmod{n}$.

Proof. Let $s = s_t$ be given then

$$\begin{aligned} ESR(s, b) =& \mathbb{E}[-\exp\{-\gamma\,(S_{t+1} - \alpha)\, b\} \,|\, s] \\ =& p_u\,(-e^{-\gamma\,(su-\alpha)\,b}) + p_d\,(-e^{-\gamma\,(sd-\alpha)\,b}). \end{aligned}$$

Obviously, $ESR(s, \cdot)$ is concave with respect to b for given s (compare Lemma 3.2). Hence, setting the first derivative with respect to b equal to zero yields

$$\begin{aligned} & p_u\,(su-\alpha)\cdot(e^{-\gamma\,(su-\alpha)\,b}) + p_d\,(sd-\alpha)\cdot(e^{-\gamma\,(sd-\alpha)\,b}) = 0 \\ & p_u\,(su-\alpha)\cdot(e^{-\gamma\,(su-\alpha)\,b}) = p_d\,(\alpha-sd)\cdot(e^{-\gamma\,(sd-\alpha)\,b}) \\ & \log(p_u) + \log(su-\alpha) - \gamma\,(su-\alpha)b = \log(p_d) + \log(\alpha-sd) - \gamma\,(sd-\alpha)b \\ & b^* = \frac{\log(\frac{p_u}{p_d}) + \log(\frac{su-\alpha}{\alpha-sd})}{\gamma s(u-d)}. \end{aligned}$$

Then the stationary point b^* is a global maximum. With respect to the constraint control set at X_t, i.e. $I^{(2)}(X_t)$, this yields the stated result. Moreover, the optimal value function $SR^*_{t+1}(X_t)$ is finite and integrable (compare Theorem 3.6). $\square$

3.6.5 Second Subproblem

As explained in Section 3.4.5 the second subproblem exactly reflects the allocating character of the optimization problem and thus tackles the question: *How much capacity should be devoted to the forward market and how much capacity should be kept for bidding in the spot market?* Here the task is to find $f_i^* \in I^{(1)}(X_{i-n})$ such that the "next to go quarter" reward

$$Q_i(f_i, X_{i-n}) = -FR(f_i, F^i_{i-n}) \cdot \mathbb{E}[J^*_n(X_i)|s_{i-n}]$$

is maximized for all $i \in \{n, 2n, 3n\}$. For the underlying market structure the *optimal future production* problem is equivalent to maximizing the weighted sum of exponential functions with respect to the forward control variable f. Contrary to the *optimal spot production* problem no obvious analytical solution exists. One could approximate the sum by an appropriate function. However, it is not clear - or even arguable - that such a complex task yields an explicit analytical expression. Thus, we decide to study the optimization problem numerically.

Our proceeding is based on the following idea: *"Walk on every price path from the signment of the forward contract at date $(i-n)$ up to date i, i.e. one stage before the first delivery date. For every path consider the optimal spot decision b_i^* based on the corresponding spot price s_i. Walk one step ahead on every path and consider the next optimal spot decision b_{i+1}^* based on the next corresponding spot price S_{i+1}. Repeat that procedure until one step before the end of the delivery period (that is the last spot decision date within that quarter, i.e. date $i+n-1$). Thereafter, consider all dates that imply a bid on the spot market b_{i+k}^*, that infers a production exceeding the spot reserve δ for delivery at the next date. Collect those price paths, that lead to such a decision in a certain set A_z, where z refers to the date of decision."*

For the present market structure the spot price dynamics and the preceeding idea can be illustrated by means of a tree. Note, that the tree is recombining in the sense that an "up"-move followed by a "down"-move gives the same result as a "down"-move followed by an "up"-move. However, in terms of finding a control f^* that is optimal for our second subproblem the specific path, i.e. the exact sequence of "up" and "down"-moves must be taken into account.

Let a tree structure $S_{t,\ell}$ be given such that $t \in \{0, \ldots, 2n-1\}$ refers to the time and $\ell \in \{0, 1, \ldots, t\}$ refers to the level. Hence, for our specific model we consider a tree representing the spot price scenarios within the two next quarters. That period encloses one quarter until delivery starts and the corresponding delivery quarter.

Keeping the preceeding idea in mind, we derive an algorithm tackling the allocation problem. It collects for all $z \in \{n, 1, \ldots, 2n-1\}$ and $\ell \in \{0, 1, \ldots, z\}$ specific future spot price scenarios $S_{z,\ell}$ in the set A_z, that is generally defined by (3.9). Then, according to Assumption 3.4 the set A_z contains all (spot price) scenarios $S_{z,\ell}$ that imply an optimal spot control $b^*_{z,\ell}$, which makes production at maximum capacity $\overline{K}$ and its allocation to bilateral and spot contracts necessary. Additionally, the two possible spot prices $S_{z+1,\ell,k}$ (for $k = 0, 1$) are collected that can be realized after applying the optimal spot control $b^*_{z,\ell}$. For clearness of exposition we denote F^i_{i-n} with F, f_i with f, X_{i-n} with X and s_{i-n} with s. Hence, the results below and especially the following algorithm holds for all $i \in \{n, 2n, 3n\}$.

Algorithm 3.1. Let X imply $F \geq \alpha$. Let $z = n$, $\ell = 0$ and proceed as follows:

1. Calculate the *"spot level prices"* $S_{z,\ell} = s\, u^\ell\, d^{z-\ell}$ and the corresponding *"spot level controls"* $b^*_{z,\ell}$ according to (3.11). Then set

$$\begin{cases} A_z = A_z \cup \{\ell\} & \text{if } b^*_{z,\ell} > \delta \\ B_z = B_z \cup \{\ell\} & \text{if } b^*_{z,\ell} \leq \delta. \end{cases}$$

   Set $\ell = \ell + 1$ and repeat step 1. until $\ell = z + 1$.

2. Set $z = z + 1$ and repeat step 1. until $z = 2n$.

3. For $k = 0, 1$ calculate the possible "realized spot level prices" $S_{z,\ell,k} = S_{z,\ell}\, u^k\, d^{1-k}$ that occur with probability $p_u^{\ell+k}\, p_d^{(z+1)-(\ell+k)}$ after control $b^*_{z,\ell}$ has been applied.

Thus, the above algorithm yields the necessary input to reformulate the second optimization problem with regard to the underlying CRR market structure: The optimal forward control $f^* \in I^{(1)}(X_{i-n})$ for any given $X \in \mathbb{X}$ is the maximizer of the expected quarter value function

$$\begin{aligned} Q_i(f, X) = \sum_{k=0}^{1} \sum_{z=n}^{2n-1} \sum_{l=0}^{z} & pr(l, z, k)\; U\left[(F - S_{z+1,l,k}) f \;+\; (S_{z+1,l,k} - \alpha)\, \overline{K}\right] \mathbb{1}_{A_z}(l) \\ & + pr(l, z, k)\; U\left[(F - \alpha) f \;+\; (S_{z+1,l,k} - \alpha)\, b^*_{z,l}\right] \mathbb{1}_{B_z}(l). \end{aligned}$$

At this, we deal with a specific form of Corollary 3.1. Here, for all z, l, k we have

(i) $pr(l, z, k) := \binom{z+1}{l+k} p_u^{l+k}\, p_d^{(z+1)-(l+k)}$

(ii) the sets A_z and B_z can be obtained via Algorithm 3.1,

(iii) the *"realized spot level prices"* $S_{z+1,l,k}$ can be obtained via Algorithm 3.1,

(iv) the *"spot level controls"* $b^*_{z,l}$ can be obtained via Algorithm 3.1;

3.6.6 Numerical Analysis - CRR Model

Now, the numerical analysis of the presented problem is based on **three specific market situations** generated according to the Cox Ross Rubinstein model given by

1. an **upward moving market** generated with $u = 1.9$, $d = 0.9$ that imply a negative risk premium of -0.8,
2. a **downward moving market** generated with $u = 1.1$, $d = 0.1$ that imply a positive risk premium of 0.8 and
3. a **sideways moving market** generated with $u = 1.1$, $d = 0.9$ that imply a risk premium of 0.

In this setting the risk premium is simply the expected return per unit of risk, i.e. $RP = \frac{\mathbb{E}[R]}{\sigma}$ with $\mathbb{E}[R] = p_u u + p_d d - 1$, $\sigma^2 = p_u p_d (u-d)^2$ and $p_u = p_d = 0.5$.

In order to implement the desired market structures represented by the CRR model without loss of generality all other initial parameter are fixed: They include the initial spot price s_0, the initial forward price vector H_0, the initial forward unit vector h_0 (both affecting the first quarter by convention (compare Section 3.2)), the capacity constraints $\underline{K}$, $\overline{K}$ and δ (such that $I^{(1)}_{i-n} = [\underline{K}, \overline{K} - \delta]$ and $I^{(2)}_{i+k} = [0, \overline{K} - f]$ for all $k = 0, 1, \ldots, n-1$), the utility function parameter γ and the time horizon T, comprised of the number of segments M and the length of one segment n (compare Corollary 3.3). The parameter α modeling the fixed production costs is chosen as the anchor of the binomial grid.

3.6.6.1 Upward Moving Market

Choosing the initial parameter values to represent an upward moving market as listed in Table 3.1, the implemented binomial model yields a spot tree that exhibits the following features: In the upper and mid sceanarios of the tree the spot prices explode, whereas the prices stay close to zero in the lower scenarios. The price range of historical prices can be rediscovered in the lower mid scenarios. A certain number of spot price scenarios are illustrated in Figure 3.2.

Having generated an underlying tree structure we are able to determine the optimal control. Looking at the derived values for the spot control indicated in the middle

Parameter	Value	Parameter	Value
$s_0(=H)$	38.8	M	3
p_u	0.5	n	30
p_d	0.5	$\overline{K}$	250
u	1.9	$\underline{K}(=h)$	10
d	0.9	δ	24
α	121294.3	γ	1

Table 3.1: Initial parameter used to implement the CRR model to represent an upward moving market. The parameter α is chosen such that the binomial grid is anchored at it.

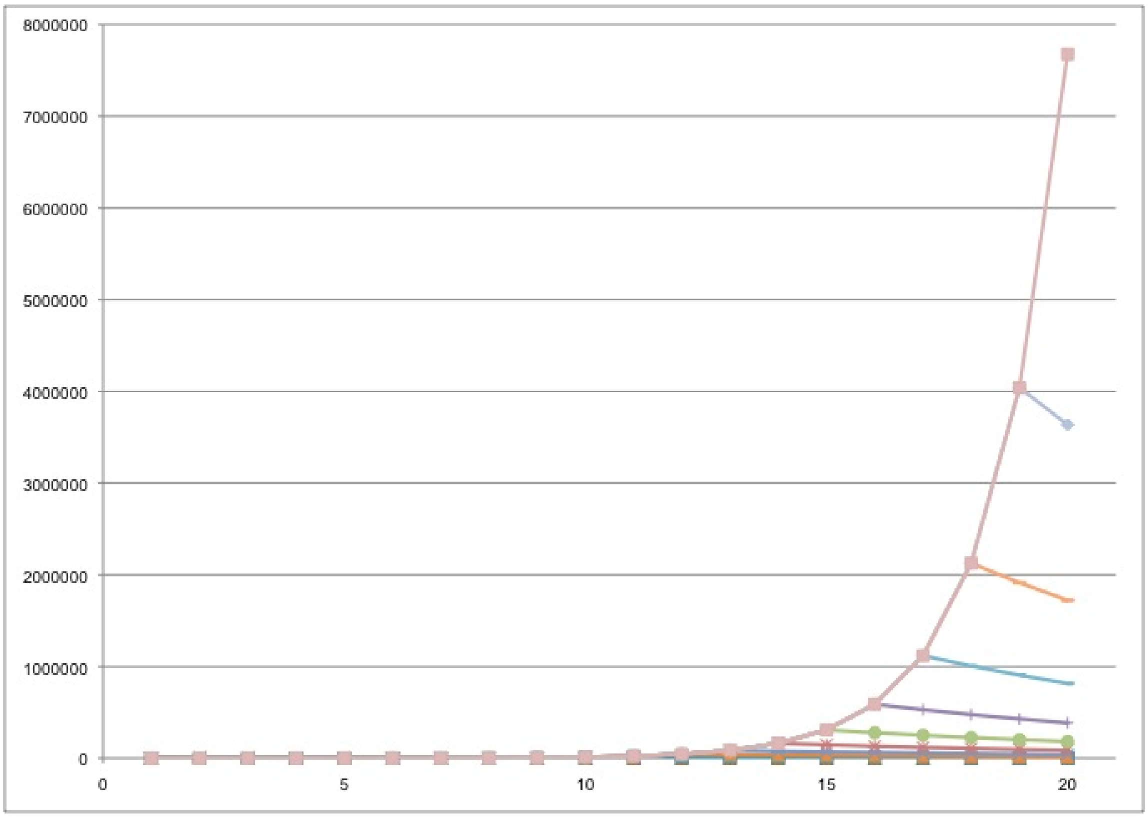


Figure 3.2: Spot price scenarios generated to reflect the underlying CRR upwards moving market, where $n = 5$ has been used for illustration. The pink line represents the upmost scenario path.

frame of Figure 3.3 we observe the significant impact of the fixed production costs α. Due to the choice of α at the beginning of the generated tree the optimal spot control is at the lower boundary for all scenarios. Starting from day 13 on the further in time in the tree the more "up scenarios" yield an optimal spot control at the upper boundary and the less "low scenarios" an optimal control at the lower boundary. In between the trivial up and down scenarios calculation according to Lemma 3.4 yields b^*. The resulting values are in case of an upward moving market

m	0	$\overline{K} - \underline{K}$	$[0, \delta]$
0	69.6%	28.2%	2.2%
1	37.5%	61.2%	1.3%
2	28.2%	70.9%	0.9%
3	24.2%	75.3%	0.5%

Table 3.2: Percentage of optimal spot control values splitted to different categories within each quarter calculated on an upward moving market structure. At this point not respecting the forward control affecting the same delivery period.

m	$\underline{K}$	$(\underline{K}, \overline{K} - \delta)$	$\overline{K} - \delta$
1	100%	0%	0%
2	48.4%	3.2%	48.4%
3	32.8%	1.6%	65.6%

Table 3.3: Percentage of optimal forward control values splitted to different categories within each quarter calculated on an upward moving market structure. For the calculations 1365 spot price scenarios and 1, 31 or 61 forward contracts with delivery in quarter $m = \frac{i}{n} = 1, 2, 3$ respectively have been used. Here, the optimal controls taking on values in between the boundaries are $f_{2n} = 136.7$ and $f_{3n} = 144.9$.

close to zero. Compare these findings also with the values listed in Table 3.2. The effect of the upward moving market becomes obvious e.g. in the last quarter (i.e. $m = 3$). Here 75.3 % of the scenarios suggest to bid all available capacity for the next day on the spot market. (Note, due to the structure of the tree the calculations corresponding to $m = 3$ are based on the largest number of spot price scenarios.)

Then Table 3.3 states the analogue percentages for the optimal forward control on an upward moving market structure. Here the according characteristic is obvious. The further in the tree the more scenarios imply a forward control at the upper boundary. At this point we must remark, that the forward control affecting the first delivery quarter ($m = 0$) is given as initial parameter h_0 and the one affecting the second delivery quarter ($m = 1$), i.e. f_n is calculated based on the initial node. Hence, in that case one underlying scenario leads to only one optimal forward control value f_n^* .

3.6.6.2 Downward and Sideways Moving Market

Parameter	Value	Parameter	Value
$s_0(=H)$	38.8	M	3
p_u	0.5	n	30
p_d	0.5	$\overline{K}$	250
u	1.1	$\underline{K}(=h)$	10
d	0.1	δ	24
α	$1.62e^{-13}$	γ	1

Table 3.4: Initial parameter used to implement the CRR model to represent a downward moving market. The parameter α is chosen such that the binomial grid is anchored at it.

Parameter	Value	Parameter	Value
$s_0(=H)$	38.8	M	3
p_u	0.5	n	30
p_d	0.5	$\overline{K}$	250
u	1.1	$\underline{K}(=h)$	10
d	0.9	δ	24
α	33.37	γ	1

Table 3.5: Initial parameter used to implement the CRR model to represent a sideways moving market. The parameter α is chosen such that the binomial grid is anchored at it.

The analogue proceeding for the downward and sideways moving market representations yields the associated optimal control values. Since the findings have the according characteristics (that have been stated for the upward moving market) now adapted to the respective market structures, we restrict ourselves to list the derived percentages in Tables 3.6 and 3.7, present the illustrations in Figures 3.4 and 3.5 and to compare the percentages for $m = 3$ with those of the upward moving market. Obviously, the market structure has a huge imapct on these numbers. In case of a downward moving market only $17.4\,\%$ of the spot price scenarios suggest to bid all available capacity for the next day on the spot. The according number based on the

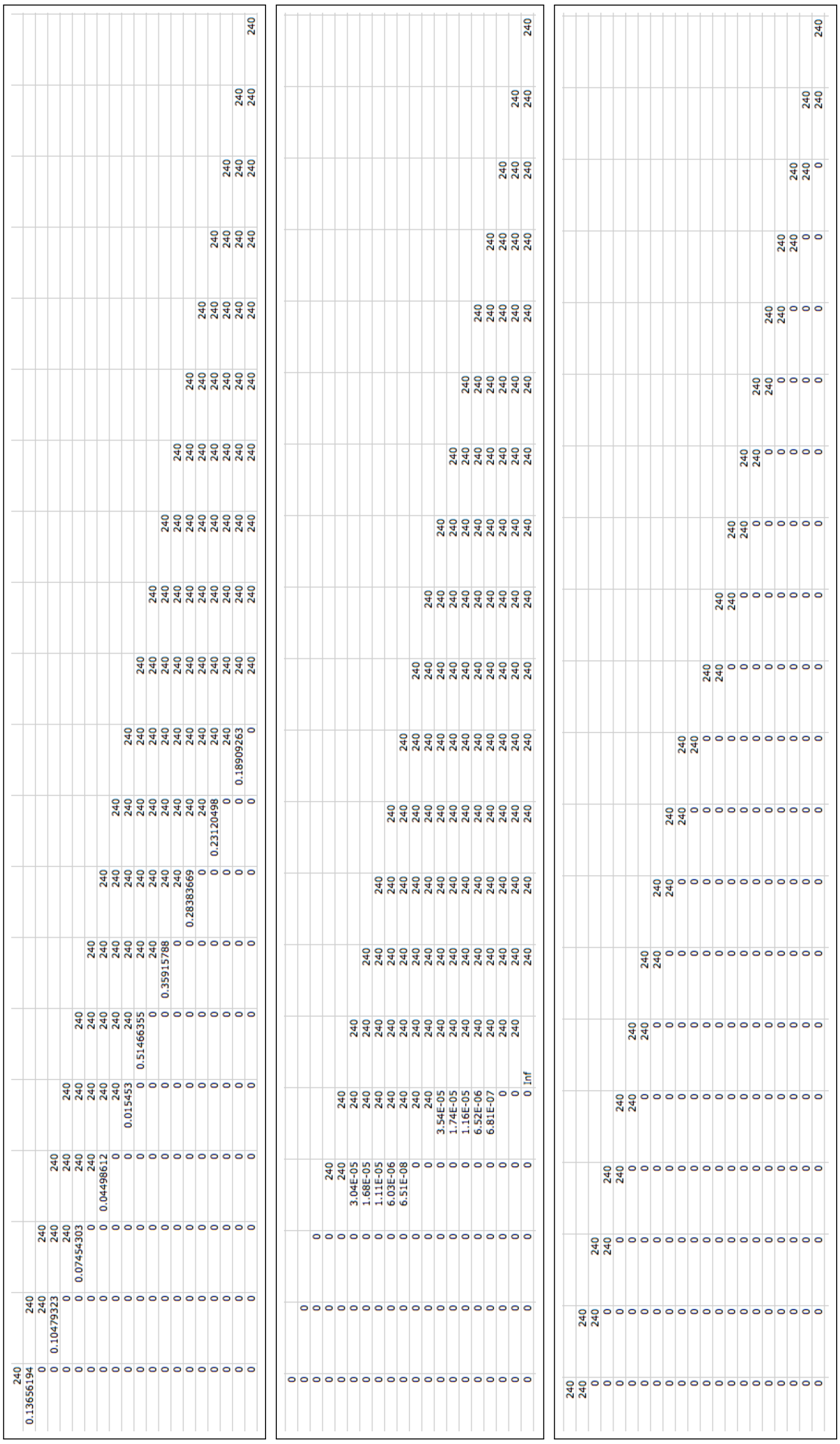


Figure 3.3: Value of optimal spot controls corresponding to different spot price scenarios within a sideways, upwards and downwards (from left to right) moving CRR tree structure.The initial node is situated in the bottom left corner. The vertical axis captures the level, the horizontal axis captures the time in the tree. The lower part of the tree indicates to produce at the lower boundary $s^* = 0$, the upper part to produce at the upper boundary $s^* = \overline{K} - \underline{K} = 240$. In between the optimal value is calculated according to Lemma 3.4.

m	0	$\overline{K} - \underline{K}$	$[0, \delta]$
0	25.8%	74.2%	0%
1	65.3%	34.7%	0%
2	77.3%	22.7%	0%
3	82.6%	17.4%	0%

m	0	$\overline{K} - \underline{K}$	$[0, \delta]$
0	44.3%	49.5%	6.2%
1	50.8%	48%	1.2%
2	51.4%	47.9%	0.7%
3	51.8%	47.7%	0.5%

Table 3.6: Percentage of optimal spot control values splitted to different categories within each quarter calculated on a downward (left frame) or sideways (right frame) moving market structure.

m	$\underline{K}$	$(\underline{K}, \overline{K} - \delta)$	$\overline{K} - \delta$
1	0%	0%	100%
2	48.4%	3.2%	48.4%
3	72.1%	0%	27.9%

m	$\underline{K}$	$(\underline{K}, \overline{K} - \delta)$	$\overline{K} - \delta$
1	0%	100%	0%
2	51.6%	0%	48.4%
3	50.8%	0%	49.2%

Table 3.7: Percentage of optimal forward control values splitted to different categories within each quarter calculated on a downward (left frame) or sideways (right frame) moving market structure. For the calculations 1365 spot price scenarios and 1, 31 or 61 forward contracts with delivery in quarter $m = 1, 2, 3$ respectively have been used. Here, the optimal controls taking on values in between the boundaries are $f_{2n} = 158.3$ on the downward market and $f_n = 141.2$ on the sideways market.

sideways moving market structure calculates to $47.7\,\%$. An economic interpretation can then be found in Section 3.8.

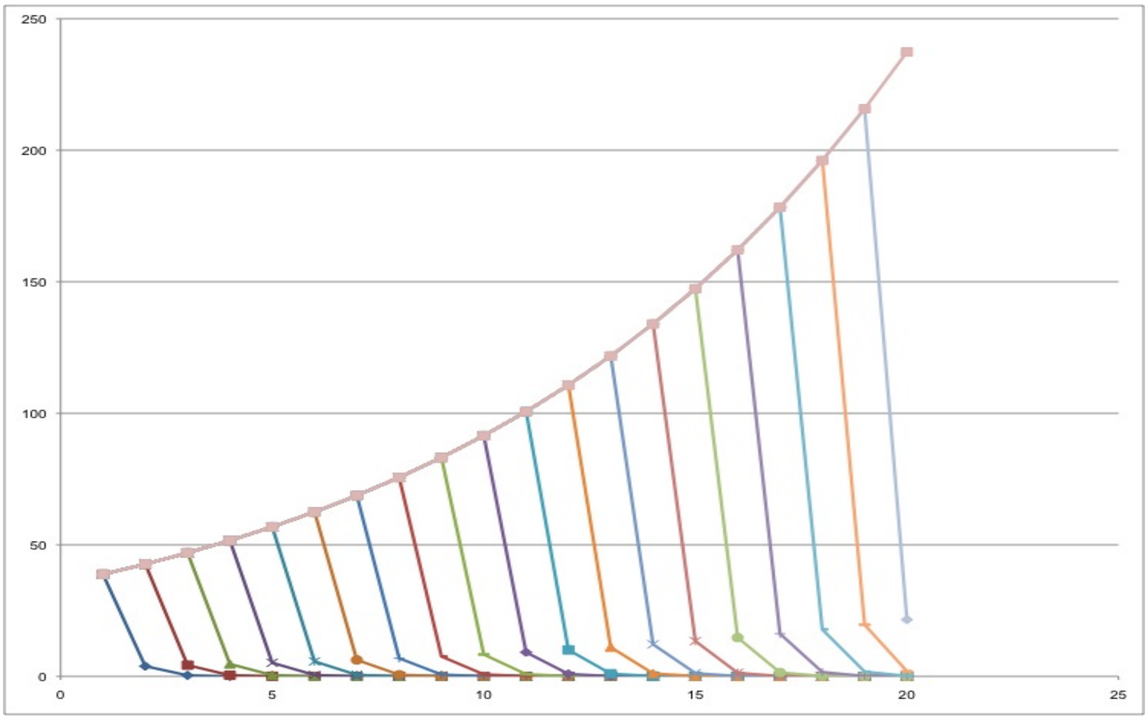


Figure 3.4: Spot price scenarios generated to reflect the underlying CRR downward moving market, where $n = 5$ has been used for illustration.

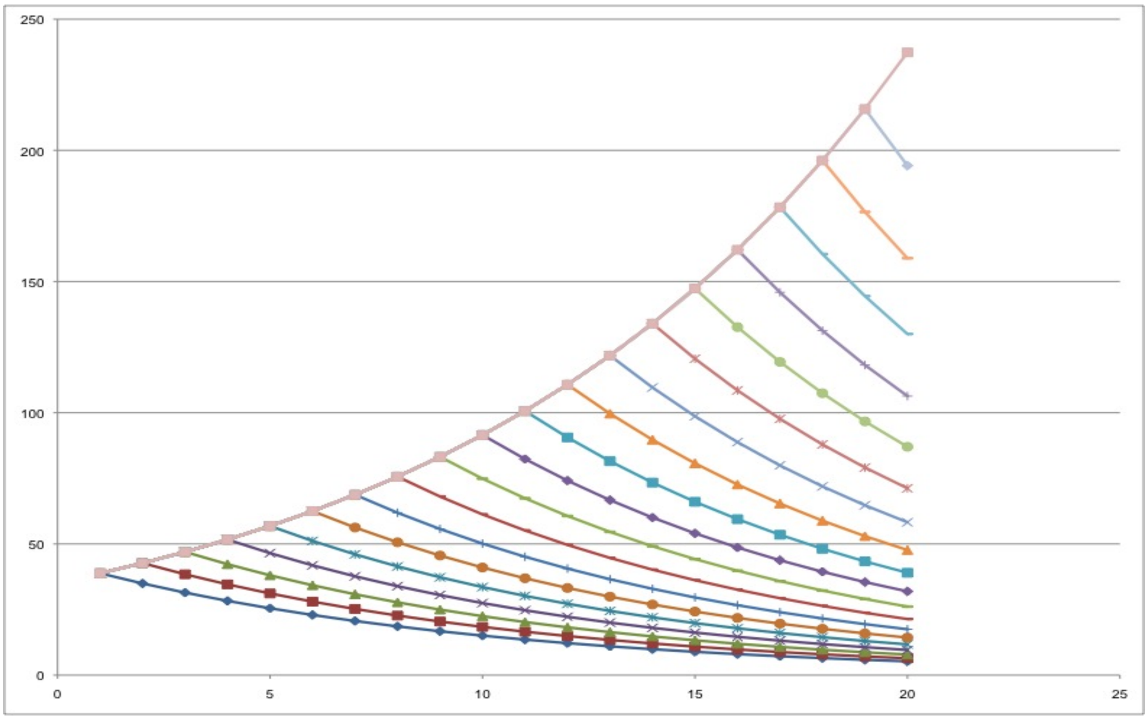


Figure 3.5: Spot price scenarios generated to reflect the underlying CRR sideways moving market, where $n = 5$ has been used for illustration.

3.7 One Factor Model consistent with Market Data

As indicated within the motivation of the Cox Ross Rubinstein model as an underlying framework to solve the present optimization problem, we now want to test and verify our findings by using another somehow “more realistic” factor model. Especially, we now want to fit the suggested model to market observable data. The overall framework, however, is required to be consistent with Section 3.6 and naturally to be consistent with the conventions posed throughout the theoretical part in Section 3.2.

For that once more we refer to the plant characteristics posed in Assumption 3.1 and especially recall the fuel price to be constant. Thus, the now presented one factor model represents the electricity price dynamics. Again the forward contracts are assumed to deliver at continuous flow at each day within the next month (i.e. so called swap contracts) and the settlement of the forward contracts should be at the beginning of each day within the next quarter.

3.7.1 Fixed Delivery Dynamics

We basically follow the ansatz of Clewlow & Strickland as in [CS99], where they develop a single-factor modeling framework which is consistent with market observable forward prices and volatilities. It is based on the trinomial tree building procedure developed by Hull & White in 1994 (compare [HW94a], [HW94b]). Moreover, the model is an extension of the one factor model suggested by Schwartz in 1997 [Sch97]. The starting point is the stochastic evolution of the fixed delivery electricity forward curve $F(t, T, s_t)$. To obtain a Markovian sequence of spot prices the volatility of the forward prices must have a negative exponential form. As similarily argued by Börger, Kiesel & Schindlmayr [KSB09] we remark, that the model of Clewlow & Strickland tries to fit the term structure of volatility to market data, but does not account for the absence of fixed delivery forward products in electricity markets. Our ansatz to overcome this issue will be explained when outlining the steps to fit the trinomial tree structure to market data. We consider the fixed delivery forward dynamics as building block for modeling the market observable swap contracts.

Thus, the starting point for solving our optimization problem - based on a one factor market structure calibrated to market data - is to define the **fixed delivery forward curve dynamics** by

$$\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\kappa(T-t)}\, dW^Q(t),$$

where the volatility function is comprised of the constant level of spot and forward price volatility σ and the constant rate at which the volatility of increasing maturity forward prices declines (that is also the speed of mean reversion of the spot prices) κ. These parameter will be fitted to historical volatilities of observable electricity swap prices. The **associated spot price dynamics** are implicity given by

$$\frac{dS(t)}{S(t)} = \kappa(\mu(t) - \ln S(t))dt + \sigma\, dW^Q(t),$$

where the long term risk adjusted drift is given by

$$\mu(t) = \frac{1}{\kappa}\frac{\partial \ln F(0,t)}{\partial t} + \ln F(0,t) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t}).$$

Thus, after some calculations the **explicit forward curve** at date t

$$\begin{aligned} F(t,T,s_t) &= F(0,T)\cdot\left(\frac{s_t}{F(0,T)}\right)^{\exp\{-\kappa(T-t)\}} \\ &\quad \cdot \exp\{-\frac{\sigma^2}{4\kappa}e^{-\kappa T}(e^{2\kappa t}-1)(e^{-\kappa T} - e^{-\kappa t})\} \end{aligned} \tag{3.12}$$

is obtained. That is the forward curve at date t is a function of the current spot price s_t, the initial forward curve $F(0,T)$ and the volatility parameters σ and κ. Having an explicit form for the forward price is not only computationally but also with regard to our optimization problem extremly useful. For more details on the derivation of the spot price process and the explicit form of the forward curve we refer the interested reader again to Clewlow & Strickland [CS99].

3.7.2 Time and Space Discretization

We use the trinomial tree building procedure introduced 1994 by Hull and White with an application to term structure models to build a tree stucture to approximate logarithmic spot price dynamics that are consistent with the initial forward curve observed on the market. Here we follow the approach of Clewlow and Strickland. Let $x(t) := \ln S(t)$ and $\theta :\equiv \kappa \cdot \mu$. Then the tree building procedure mainly consists of three steps:

1. Calculation of the spot prices $\overline{x}_{ij}$ to approximate

   $$d\overline{x}(t) = -\kappa\overline{x}(t)dt + \sigma dW^Q(t)$$

   resulting in a preliminary tree structure for x assuming $\theta(t) = 0$ for all t and $x(0) = 0$.

2. Calculation of the level shift parameter a_i such that $x_{ij} = \overline{x}_{ij} + a_i$ approximates
$$dx(t) = (\theta(t) - \kappa x(t))dt + \sigma dW^Q(t).$$
3. Add market price of risk $\lambda(t,T)$ to obtain also a tree structure under the real world measure $\mathbb{P}$.

For that of course the question is how to choose the involved parameter appropriately?

3.7.3 Fitting Trinomial Tree to Market Data

To answer the above question one must take the already spoken to issue of missing fixed delivery forward contracts into account. On the electricity market those contracts are not traded. Hence, market data is only available for the so called swap contracts having a delivery period. How can we use those contracts to fit the above presented model framework to market data? With regard to the present setting the idea is to decompose the fitting of the volatility parameter and the long term level shift parameter into two separate steps. The approach will be outlined in the following and uses the ideas of Benth, Koekebakker & Ollmar [BKO07] as well as the tree fitting procedure suggested by Clewlow & Strickland [CS99]. In order to build a trinomial tree structure as outlined above and fitting the tree to available swap contracts in our approach the following relations must be assumed:

- We model the atomic swap contracts by
$$dG(t, T_m^S, T_m^E) = \Sigma(t, T_m^S, T_m^E)\, dW^{\mathbb{Q}}(t)$$
with $t < T_m^S$ for all $m = 0, \ldots, M$ (corresponding to the month of delivery), where $T_0^S < T_0^E < T_1^S < T_1^E < \cdots < T_M^S < T_M^E$ is the ascending sequence of start and end days of each month respectively. Then due to Benth et al. [BKO07] for given $t < T_m^S$ it holds
$$\Sigma(t, T_m^S, T_m^E) = \frac{1}{T_m^E - T_m^S} \int_{T_m^S}^{T_m^E} \sigma e^{-\kappa(u-t)} du.$$
- The (fixed delivery) **forward prices** settled at date i, i.e. $F(t, i, S_t)$, are calculated based on
$$F(t, i, S_t) = G(t, T_m^S, T_m^E)$$
for all $t < T_m^S$ and i such that $i \in [T_m^S, T_m^E]$. The market data used for $G(t, T_m^S, T_m^E)$ is given by Phelix baseload month futures quoted at the EEX in Leipzig.

- The **bond prices** are calculated based on

$$L(t,i) = L(t, T_m^E)$$

for all $i \in [T_m^S, T_m^E]$. For the left hand side we use the market quoted LIBOR rates maturing at the end of month m, i.e. at T_m^E.

Under these considerations we now have all ingredients available for proceeding in three steps to build the spoken to trinomial tree structure that is consistent with the initial forward curve. The steps are outlined below. (For more details compare with Appendix B.)

1. Fitting the volatility of the swap contracts to quoted swap data and extracting the implied fixed delivery volatility function in the least square sense yields

$$\hat{\sigma}_{\text{imp}}(t,T) = \hat{\sigma} e^{-\hat{\kappa}(T-t)},$$

where $\hat{\sigma}$ and $\hat{\kappa}$ are the resulting parameter estimators. Thereafter, the estimates are used to calculate the values $\overline{x}_{ij}$, p_{uij}, p_{mij} and p_{dij} to approximate $d\overline{x}(t) = -\kappa\overline{x}(t)dt + \sigma dW^{\mathbb{Q}}(t)$.

2. We choose the level shifts in a way to reflect the available swap market data and the settlement structure of the SDP problem, i.e.

$$a_i = \ln\left(\frac{p(0,i)\, F(0,i,s_0)}{\sum_j Q_{ij} e^{\overline{x}_{ij}}}\right),$$

where Q_{ij} are the so called state prices, that is the time zero price of a security that pays one unit of cash if node (i,j) in the tree is reached and zero otherwise, $p(0,i) = p(0,T_m^E)$ and $F(0,i,s_0) = G(0,T_m^S,T_m^E)$ for all $T_m^S \le i \le T_m^E$. Thus, $x_{ij} = \overline{x}_{ij} + a_i$ approximates $dx(t) = (\theta(t) - \kappa x(t))dt + \sigma dW^{\mathbb{Q}}(t)$.

3. Based on the Girsanov Theorem a spot tree under the real world measure $\mathbb{P}$ is derived, i.e. we use Girsanov to a add market price of risk term $\lambda(t)$ such that

$$dX(t) = \left(\theta(t) + \kappa(\lambda(t) - X(t))\right) dt + \sigma dW^{\mathbb{P}}(t)$$

is approximated by $x_{ij}^{\mathbb{P}} = \overline{x}_{ij} + a_i + \kappa\lambda_i$ where $\hat{\kappa}$, $\hat{\sigma}$ are given through the implied volatility function, a_i are the calculated level shifts to add proper drift, i.e. $\theta(t)$, at the discrete dates i, and $\lambda(t)$ is a (piecewise) constant function sth. $\int_i^{i+1} \lambda(t)dt = \lambda_i$ for all i referred to as the market price of risk.

After all based on the trinomial spot tree under $\mathbb{Q}$ that is consistent with the initial forward curve one can calculate electricity forward prices having an explicit formula. Additionally, the spot tree under the real world measure $\mathbb{P}$ provides spot price scenarios such that we have constructed the framework to solve the discussed optimization problem based on the underlying one factor model.

3.7.4 Solution calculated on a Trinomial Tree Structure

Having generated the necessary framework for solving the optimization problem where the underlying one factor model is approximated by a trinomial tree structure, we can outline the different steps to find the optimal control values. For clearness of exposition we focus on those nodes that are in the middle of the tree. (At the boundaries of the tree the procedure can be formulated analogously with adequate movements and their corresponding probabilities.) The proceeding derived with regard to the two subproblems explained in detail in Sections 3.6.4 and 3.6.5 is again valid at this point. However, we can not give an explicit solution for the optimal spot control and therefore use numerical optimization techniques (as used before throughout the calculation of the optimal forward control).

1. **First Subproblem:** The aim is to find b^* such that

$$\mathbb{E}[U(S(\omega)-\alpha)b|s]$$

   is maximized with respect to b. That is, given $x^{\mathbb{P}}_{ij}$ and p_{uij}, p_{mij}, p_{dij} for all (i,j) solve

$$p_{uij}\,U((S_{i+1,j+1}-\alpha)b_{ij})+p_{mij}\,U((S_{i+1,j}-\alpha)b_{ij})+p_{dij}\,U((S_{i+1,j-1}-\alpha)b_{ij}) \xrightarrow{b_{ij}} \max,$$

   where e.g. p_{uij} is the probability of moving upwards, i.e. starting from node (i,j) moving to node $(i+1,j+1)$.

2. **Second Subproblem:** Naturally, the proceeding is based on the results of Section 3.4.5. Given the forward price $F^i_{i-n} = F_{i-n,j}$ (denoted by F) valid for contracts delivering within the quarter $[i,i+n]$ and given the current spot price $s_{i-n,j}$ (denoted by s) for some fixed node $(i-n,j)$ we need to solve the analogoue to the problem stated in Corollary 3.1: Find f^* such that

$$\sum_{k=0}^{n-1}\sum_{j=1}^{J}\sum_{\ell=j-1}^{j+1} \qquad p_{i+k+1,\ell}\,U((F-S_{i+k+1,\ell})f+(S_{i+k+1,\ell}-\alpha)\overline{K})\mathbb{1}_{A_k}(\ell)$$
$$+p_{i+k+1,\ell}\,U((F-\alpha)f+(S_{i+k+1,\ell}-\alpha)b^*_{i+k,j})\mathbb{1}_{B_k}(\ell)$$

is maximized with respect to the forward control f. For that the following steps must be carried out:

(a) Check whether one of the trivial cases implying the optimal control to be attained at the boundaries is present (that is all scenarios within the delivery month indicate to produce at the upper respectively lower boundary or we have $F < \alpha$).

(b) Calculate the probability tree starting from node $(i-n, j)$ reaching until the end of the delivery quarter, i.e. reaching two quarters into the future. That is, calculate $p_{i+k+1,\ell}|_s$ for all $k = 0, \ldots, n-1$ and $\ell = 1, \ldots, J$ i.e. the probability of moving to node $(i+k+1, \ell)$ when starting from node $(i-n, j)$.

(c) Collect all scenarios (or level ℓ in the tree at time $i+k$) implying $b^*_{i+k,\ell} > \delta$ in the set A_k all other scenarios in the set B_k for all $k = 0, \ldots, n-1$ and all $\ell = 1, \ldots, J$ (i.e. proceed similar to Algorithm 3.1).

Following the outlined steps to calculate an optimal control vector maximizing the value function when the underlying system evolves according to the principles of the specific one factor model yields the numerical results, that are presented within the next section.

3.7.5 Numerical Analysis - One Factor Model

Parameter	Value	Parameter	Value
κ	0.001	M	3
σ	0.02	n	30
h_0	$\underline{K}$	$\overline{K}$	250
$G(0, T^S, T^E)$	as of 01.03.2011	$\underline{K}$	10
$L(0, T^E)$	as of 01.03.2011	δ	24
α	35	γ	1

Table 3.8: Initial Parameter used to calibrate the model to market data. The resulting volatility estimators are used to implement the trinomial tree structure representing the introduced one factor model.

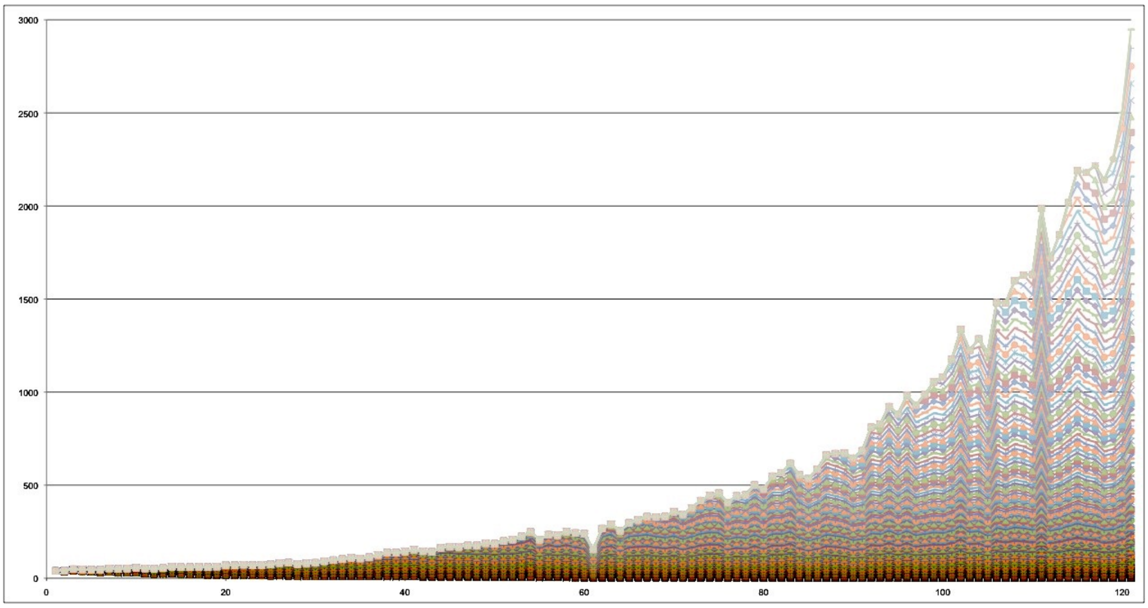


Figure 3.6: Spot price scenarios generated to reflect the underlying one factor model.

After implementing the previously outlined steps, we have generated a trinomial tree structure under the real world measure $\mathbb{P}$ representing the spot price scenarios underlying the considered optimization problem. Moreover, forward prices have been calculated based on the according spot price tree under the risk neutral measure $\mathbb{Q}$ (compare Section 3.7.1). For the scenario generation the in Table 3.8 stated initial parameter have been used, where the vector $G(0, T^S, T^E) = (50.53, 49.3, 49.08, 47.26)$ is the initial swap curve as observed at 01.03.2011 at the EEX for delivery in the next four months (i.e. in march, april, may and june) and $L(0, T^E) = (0.58, 0.82, 0.91, 1.05, 1.13)$ are the according monthly LIBOR rates. All values are listed in euros. The fixed production cost parameter α is chosen close to the mean of the spot price scenario tree in order to create a scenario tree comparable with that created by the basic market structures. Moreover, we want options priced based on that scenario tree to be in the money.

Next, the resulting spot price scenarios are illustrated in Figure 3.6. Their obvious features are collected with the following list: In the middle of the tree the scenarios stay around the initial price s_0. The price scenarios grow exponentially in the upper scenarios and tend to zero smoothly in the lower scenarios. The level shifts in the tree exactly reflect the shape of historical spot prices, that have been used throughout the calibration procedure explained in Section 3.7.3.

m	0	$\overline{K}-\underline{K}$	$[0,\delta]$
0	46.3%	47.9%	5.8%
1	49.5%	48.3%	2.2%
2	50.9%	47.8%	1.4%
3	48.5%	50.5%	1%

Table 3.9: Percentage of optimal spot control values splitted to different categories within each quarter. The optimal forward control affecting the according delivery quarter have not been respected so far.

m	$\underline{K}$	$(\underline{K},\overline{K}-\delta)$	$\overline{K}-\delta$
1	0%	0%	100%
2	31.1%	4.9%	64%
3	40.5%	3.3%	56.2%

Table 3.10: Percentage of optimal forward control values splitted to different categories within each quarter. For the calculations 2640 spot price scenarios and 1, 61 or 121 forward contracts with delivery in quarter $m = \frac{i}{n} = 1, 2, 3$ respectively have been used.

Based on such a scenario tree the optimal control values are calculated according to Section 3.7.4. Generally, Table 3.9 indicates that the percentages of scenarios implying the optimal spot control to be at the lower or upper boundary, respectivly, are nearly equalized. A more detailed view yields Figure 3.7, where in between the trivial up and down scenarios numerical optimization yields values that are around zero. (Note, those values are given without respecting the optimal constraint control

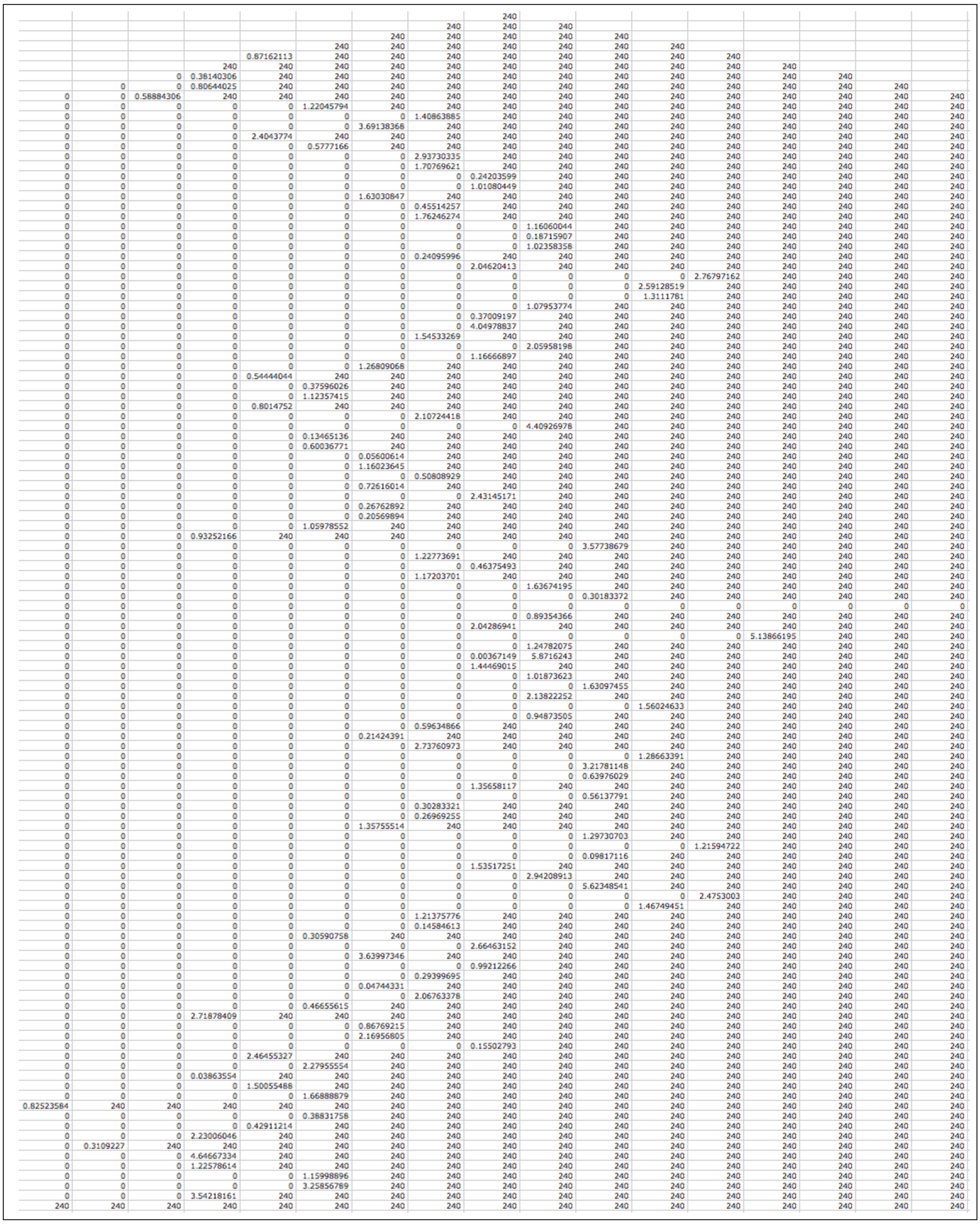


Figure 3.7: Section of the optimal spot control corresponding to different spot price scenarios within a trinominal tree structure approximating a one factor model. The left part of the tree indicates to produce at the lower boundary $b^* = 0$, the right part to produce at the upper boundary $b^* = \overline{K} - \underline{K} = 240$. In between the optimal value is found by numerical optimization. The initial node of the tree is at the top of the tree. The tree grows in time in the vertical direction and in level in the horizontal direction.

sets.) The overall position in the tree, where these values must be calculated is significantly influenced by the choice of the value α.

Finally, Table 3.10 lists the percentages of scenarios leading to an optimal forward control that is at the lower or upper boundary, respectively. As before the forward control affecting the first delivery quarter ($m = 0$) is given as initial parameter h_0 and the one affecting the second delivery quarter ($m = 1$), i.e. f_n, is calculated based on the initial node. Hence, clearly one underlying scenario leads to only one optimal forward control value. Thereafter, the calculations concerning the third ($m = 2$) and fourth ($m = 3$) quarter are based on 61 and 121 forward contracts respectively. For each forward contract the allocation problem is based on 2640 spot price scenarios within the corresponding delivery quarter.

3.8 Economic Interpretation and Analysis

How much capacity should be devoted to the forward market and how much capacity should be kept for bidding in the spot market? To further address the originating issue of our work, we have studied different model structures to generate a scenario for the price evolution on the corresponding markets. We are now in the position to state the insight we have gained.

CRR Basic Market Structures. We can identify an obvious interaction between the specific market structure and the optimal control that has been calculated on such a market. The impact of the basic market structure e.g. on the corresponding optimal spot control values is obvious: Within a four month time horizon the optimal spot control is attained at the upper boundary in approximately 48 %, 59 % or 37 % of the future price scenarios calculated on a sideways, upwards or downwards moving market repectively. These values are then chosen as the benchmark to judge the results calculated based on the one factor model.

Trend Cycles in the One Factor Model. Comparing the trend of historic electricity spot prices as illustrated in Figure 3.8 with the calculated optimal control values identifies a true relationship. If there is a downward trend we can identify the same behaviour in the optimal control as we have for the basic downward moving market structure. The same holds for an upward or sideways trend. The four quarter time horizon could be interpreted as a sequence of four trend cycles: A downward cycle is followed by two months of sideway cycles with a downward spike in between and finally followed by an upward cycle. Having learned from the basic market structures, where we can identify an obvious interaction between the specific market structure and the optimal control that has been calculated on such a market, the associated behavior can be recovered in the present framework. We support this hypothesis when comparing our ideas with Table 3.9 and Figure 3.7: Within the first downward trend quarter it is optimal to bid only in 47.9 % of the future scenarios all available capacity at the spot market. That number increases to 48.3 % within the second quarter and is thus close to the benchmark value expected for a sideways moving market (of 48 %). The percentage decreases again to 47.8 % when calculated based on third quarter scenarios, but is still close to the sideways benchmark. Finally, the highest percentage of 50.5 % is calculated based on fourth quarter scenarios. Historic spot prices suggest an upward moving trend in june prices. Thus, the increased percentage is what we expect based on the benchmark value of 59 %. After all, within the one factor model the total magnitude of percentages indicating full spot production is close to the benchmark value calculated on a sideways moving market.

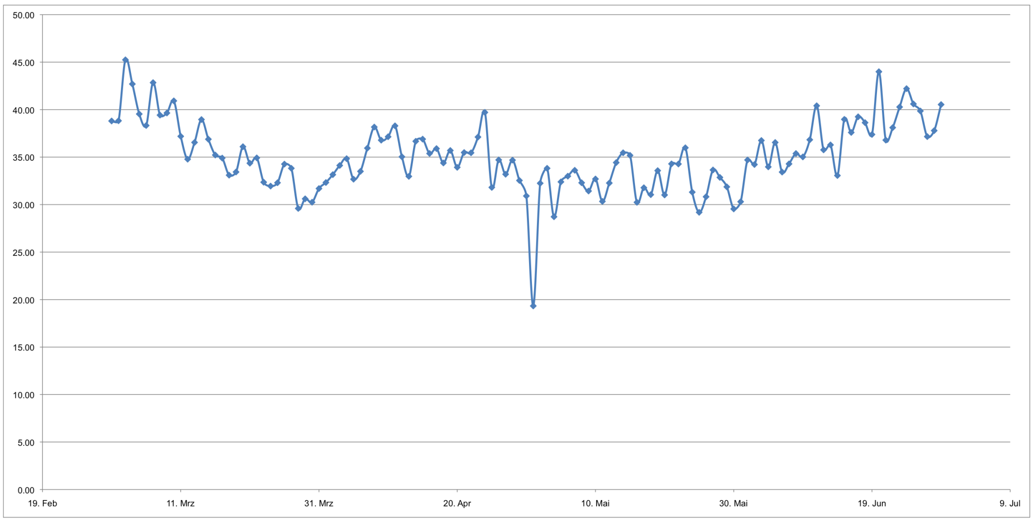


Figure 3.8: Average of historical daily base load electricity spot prices as observed in the months march, april, may and june (throughout the years 2001 - 2010).

Fixed Production Costs. The value of the fixed production costs somehow marks the boundary between full or no production. Within the benchmark models on the boundary the probability of spot prices to move above the fixed costs is the same as the probability of moving below the fixed costs. Such a boundary is situated in the middle, lower or upper part of the tree depending whether a sideways, upwards or downwards moving market structure is underlying (compare Figure 3.3).

Then based on the one factor model the shape of that boundary apparent in Figure 3.7 exactly resembles the historic trend cycles.

After all, we identify the optimal spot control calculated based on the trinomial tree structure fitted to historical prices to be a composition of the optimal control values calculated for the specific market structures that reflect an upwards, sideways or downwards moving market. Having learned from the basic market structures chosen to depict the benchmark models, we find that the optimal control calculated within a one factor model matches exactly the knowledge gained. Based on that we identify the tree structure generated to reflect the one factor model to represent an overall sideways moving market.

Realized Risk Premium & Optimal Forward Control. The structural result stated in Corollary 3.1 suggests the "realized risk premium" (RRP), i.e. the difference between the forward price F and the spot price at maturity "realized" in a certain scenario $S(\omega)$, to play a significant role in the solution to the allocation problem. Such a finding is now supported by the empirical results we have gained within the one factor model. The positive RRP fraction (being the number of positive RRP divided by the total number of scenarios within the delivery quarter) is almost identical to the scenario fraction suggesting to sell all available capacity through forward contracts. The according numbers are listed in Table 3.11. Naturally, there

m	$f^* = \overline{K} - \delta$	$RRP > 0$	$ØRP$	$ØRP_{\text{Side}}$	$ØRP_{\text{Down}}$	$ØRP_{\text{Up}}$
1	100%	61.4%	-14.22	$-1e^{-13}$	38.8	$-1e^{+10}$
2	64%	64.1%	-19.82	$-4e^{-13}$	24.03	$-2e^{+16}$
3	56.2%	58.5%	-20.41	$-4e^{-12}$	213.07	$-3e^{+24}$

Table 3.11: The fraction of positive realized risk premium scenarios compared to the fraction of full forward production scenarios and the risk premium. For the calculations based on the one factor model (left part) 2640 spot price scenarios and 1 , 61 or 121 forward contracts with delivery in quarter $m = 1, 2, 3$ respectively have been used. For the benchmark models (right part) 1365 spot price scenarios and 1 , 31 or 61 forward contracts with delivery in quarter $m = 1, 2, 3$ respectively have been used.

is a direct link of the RRP to the risk premium defined in the classical sense. With respect to one specific forward contract (having price F^i_{i-n} delivering in quarter

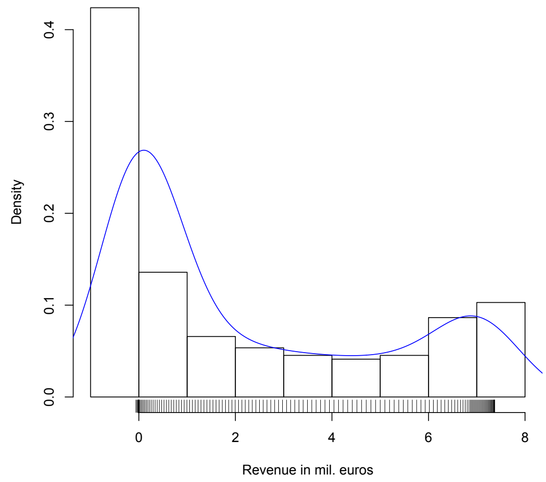
$m = \frac{i}{n}$) the relationship is given by

$$RP^i_{i-n} := F^i_{i-n} - \frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}[S_{i+k+1} \mid s_{i-n}] = \frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}[RRP^i_{i-n} \mid s_{i-n}].$$

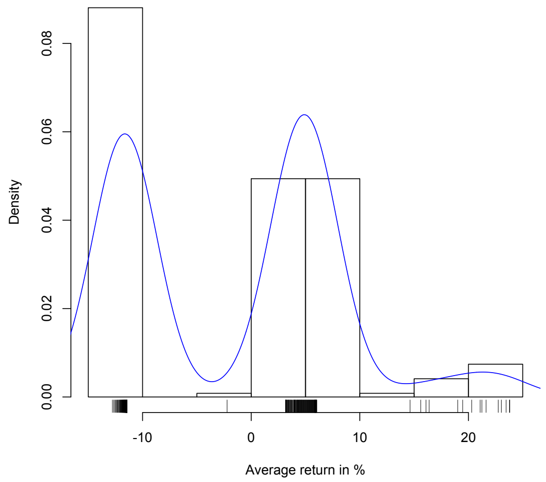
Thus, the answer to the preceeding question of how much capacity to devote to forward contracts and how much to keep for spot bidding is closely related to the risk premium, that "measures the difference between the risk-neutral and the market predictions" (Benth et al. [BŠBK08]). Once more, we find the one factor model risk premium values to be closest to the according sideways benchmark. Based on the calculated numbers along with the theoretical findings we conclude, that the (realized) risk premium has a significant influence on the optimal sales strategy and thus the value of the considered power plant.

Total Revenues and Daily Return accumulated by the Plant. The evaluation of the return distribution of the plant is closely related to the introductionary question of determining the plant value along with an optimal operating strategy. The calulations here are based on a representative choice of (243) scenario paths in the tree. We have chosen the trivial paths along the tree such that the total number (among all scenarios) of neutral steps equals the number of down steps as well as the number of up steps. Then we observe, that the nature of the spot tree is reflected by the revenue distribution: Even in the far down scenarios of the tree we do not loose that much as we earn in the far up scenarios. However, we loose much more frequently such that the mean value of the average return per day is -1.48 %. Looking at the daily average returns we calculate a 5 % Value at Risk of -12.05 %. That is with a probability of 5 % on average the plant value decreases by more than 12 % per day. Besides the daily average return, we calculate the total revenues of the plant over the four month time horizon. On average the plant accumulates 2.181 million euros based on the different scenarios. The 5 % Value at Risk is determined to be -0.009 million euros. That is, with a probability of 5 % the plant accumulates in total a loss of 9000 euros or more within the four month time horizon. Note, the VaR numbers have been calculated by ordering the simulated performance values and determining the 5 % worst numbers. From that, we can easily calculate the VaR applying interpolation techniques.

The obvious two humps in the return distribution (plotted in Figure 3.9) can be explained with the capacity boundaries. In the far up scenarios it is optimal to produce at full capacity, whereas in the far down scenarios it is optimal to produce at



(a) Total Revenues



(b) Daily Return

Figure 3.9: Histogram and density plot of the total revenues and the daily return generated by applying the optimal control strategy to a representative choice of scenario paths in the trinomial tree.

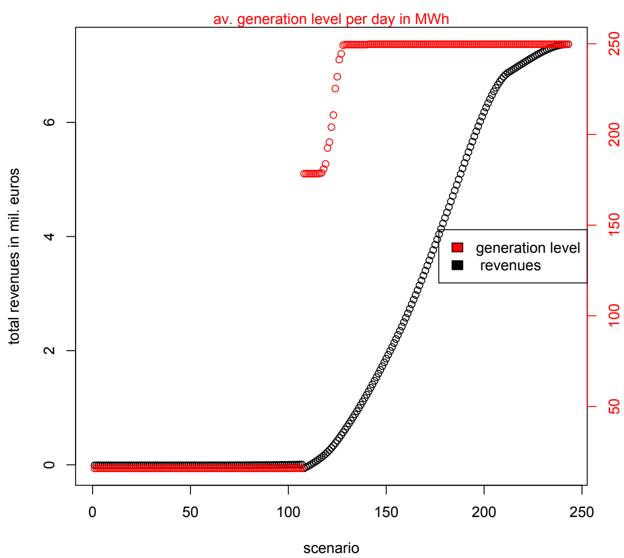


Figure 3.10: Total revenues generated by running the power station with the optimal generation level based on a representative choice of scenarios in the trinomial tree.

lower capacity. Thus, the daily return is either negative (due to minimum load restriction), around 4 % or for some scenarios even around 20 % (due to the maximum capacity constraint and the exploding character of prices in only a certain number of scenarios). Compare also with the summary statistics of the distribution given in Table 3.12

One Factor Model	**Total Revenues**	**Daily Return**
plant value (or mean)	2.181	−1.48%
5% **VaR**	−0.009	−12.05%
minimum	−0.062	−12.78%
maximum	7.369	23.81%
1st quartile	−0.009	−11.47%
3rd quartile	4.566	5.44%

Table 3.12: SDP valuation risk analysis, where all numbers are given in milion euros or percentages.

3.9 Concluding Remarks

Maximizing the Revenues on each stage in contrast to Maximizing the Terminal Wealth. The reader familiar with optimization theory might have noticed, that in our work we have not maximized the utility of the total wealth accumulated by the plant over the whole considered time horizon. For our application we have evaluated the costs and revenues the plant accumulates optimally on every stage. Hence, the plant manager is supposed to be more interested in evaluating the reward he gathers on each stage with regard to his risk preferences in contrast to applying his risk view over the entire time horizon. Such a short-term view is even more conservative due to the concavity of the utility function. Moreover, such an approach is very flexible in the sense that the planning is done with regard to each stage separately. If an unexpected event as e.g. an generation outage occurs during the period of consideration and if it is possible to sell the already signed forward contracts the reward so far is not affected. Last but not least the flexibility also applies to the technical side. Splitting the optimization problem with respect to the different control variables is only possible if the decisions are made independently from each other, i.e. the spot bidding of today does not affect the spot bidding of tomorrow. In that sense we have been able to derive a structure of the optimal value function, that is basically the sum of the reward gathered within each quarter. The reward gathered in each future quarterly period under consideration can then be maximized separately by allocating the capacity optimally to the forward contract (delivering within that quarter) and the spot market based on the information available at the current stage.

In favor of maximizing the total wealth speaks that normally the risk manager is interested in maximizing the generated wealth over a fixed time horizon and not what happens in between. As he can choose the period of consideration arbitrary short also the short term view is possible. However, in our problem we deal with quarterly forward contracts such that a one quarter view is the minimum period that is possible. Moreover, the problem can not be split with respect to the different control variables. Now, each control generating wealth on a specific stage also affects the total expected wealth from now until the end of the period. After all, the formulation we have chosen can be applied for every strictly concave, continuously differentiable utility function. Once more that makes the approach quite flexible. But which utility to choose?

Does the Utility Function reflect the Risk Preferences of a Power Plant? At the beginning of this chapter we have chosen the exponential utility function within the class of strictly concave, continuously differentiable utility functions in order to reflect the risk preferences of the power plant. Such a choice of constant absolute risk aversion might be arguable as stated e.g. by Rabin [Rab01] from a behavioral economics point of view. However, the exponential utility is a popular choice derived from its separability. The outlined procedure for calculating an optimal sales strategy in view of a power plant can be carried over to any other utility function of the class specified by Assumption 3.2 representing the desired risk preference. Of course in practice it is not clear which utility function exactly reflects the risk preferences of the power plant. It is even not clear if only one utility is the right choice or if several utility functions corresponding to several individuals representing the plant must be applied. Another issue about the choice of one specific utility function is the fact that a power plant can be owned (at least partially) by the government. In such a case it is even arguable if the utility function can be applied solely to financial input quantities. To find an answer to these questions is a delicate task and it shows once more that bringing theory into harmony with practice has its limitations. Thus, we are aware of the concerns about our theoretical work. However we believe, that having set up and solved a problem that can be solved for a specific class of utility functions and can be further adopted in many ways, depicts a conceivable tool in the daily decisions of a power plant manager.

Chapter 4

Summary & Contribution

The present thesis is partitioned in two main chapters. **Part I** (i.e. the second chapter) presents and discusses an adequate model for the comovement of electricity and gas prices. The model is thereafter used to calculate a gas fired plant's value by a series of spark spread options and for risk management purposes. Considering a complex two-factor regime-switching model generating reliable forecasts makes it reasonable to apply a more simple valuation scheme such as the spark spread valuation. In contrast **Part II** (i.e. the third chapter) is based on a more simple factor model for representing the price evolution, however it accounts for a complex valuation scheme. Part II focuses on setting up and solving a full blown stochastic dynamic programming representation of the problem of optimally scheduling and valuing a power plant when plant characteristics are respected and spot price risks are hedged by selling generation capacity through forward contracts. The main steps and findings of both chapters are now summarized.

Part I

In the presented model the peculiar characteristics of historical electricity spot price series and most importantly the sudden increases in value (i.e. the electricity spikes) are reflected by systematic alternations between stable and unstable states, referred to as the normal and spike states. These states are generated by a first-order Markov chain in discrete-time. It governs the model parameters of the stochastic processes that are chosen to reflect the comovement of electricity and gas prices. Here we contribute to the existing literature by considering a two-factor version of a regime-switching model. In addition we present a flexible approach, such that numerous alternative process specifications can be introduced. All components of the regime-switching model are presented separately: Firstly, the Markov chain underlying the process dynamics and generating the alternations between different states is introduced. Secondly, the system of stochastic drivers reflecting the small variations in prices (when normal trading takes place) are suggested. Last but not least the jump component reflecting the sudden and extreme price changes is specified. Accounting additionally for the mean reverting property of prices we argue that these different model components (when observed in different regime states) can be adjusted in various ways e.g. with regard to the needs and expectations of a plant's manager.

Model Proposals. We restrict our analysis to two different approaches. Both models exhibit the same mean reverting dynamics when inferred by a normal regime state. They differ in the jump specifications. One approach (referred to as *Markov model I*) reflects the spikes by a compound Poisson process. The other approach (referred to as *Markov model II*) additionally accounts for the mean reverting property when the dynamics are inferred by spike states. The dynamics are then given by a mean reverting process plus a compound Poisson process. These Markov modulated models are then further compared with regard to their ability to reflect the peculiar characteristics of both electricity and gas spot prices to a *Benchmark model*, that is given by correlated mean reverting stochastic processes not including a jump term. A discrete version of the process dynamics is derived by stating the unique strong solution to the system of SDEs. Thereafter the probabilistic features of the processes are analyzed to form the basis for the subsequently proposed calibration routine. We point out, that due to the mean reverting property of the model care has to be taken in the derivation of transition densities needed to execute the calibration routine. Here we are dealing with the *issue of latent prices* provoked by the unobservability of the Markov chain.

Our main contribution is to provide analytic expressions for the multistage transition densities that need to be used when latent prices are inferred by the Markov chain. Our proceeding is outlined from illustrating the issue in terms of different Markov chain scenarios, stating the key idea to overcome the problem, providing an explicit expression for the conditional probability that the most recent price observation stemming again from a stable regime has occured at a certain point of time in the past until proposing such multistage transition densities for the bivariate and marginal case. Thereafter we propose a way of how to deal with the pitfalls in estimating jump diffusion models. All these results (especially the derived transition densities) are then used to implement the suggested calibration routine for the different model approaches.

Empirical Results. The goodness of fit is tested by various measures and methods. The *Benchmark model* is found to constitute a profound tool to accurately assess and judge the model parameter values. Thereafter the Markov modulated model clearly outperform the benchmark: They not only provide increased log-likelihood values, but also a better match of moments (based on 10000 sample paths each of length 1521). Comparing the two regime-switching model proposals we find that the model alternating between mean reverting dynamics and a pure jump component simply modeled by a level parameter and a normally distributed random variable is the preferable approach. When including also the mean reverting property to the spike regime dynamic specifications the model provides the largest log-likelihood value and a good match of moments. However, the other approach is computationally less expansive and provides a similar or even better match of moments.

Plant Valuation & Management. Part I closes by accounting for the motivating application of the complex two factor regime-switching model to calculate a power plant's value by a series of spark spread options, where decisions are being made optimally. That is the price forecasts (generated by the different model approaches) determine whether to produce at maximum or minimum capacity at a certain future time (or day). In that sense they determine the plant's operation schedule. We calculate the plant value and quantify the risk and performance numbers for all proposed forecasting models with regard to a specific gas fired power station. Once more the Markov modulated models outperform the benchmark by generating increased plant values, i.e. more profit on average. The risk numbers support that finding. Moreover, a main impact of the correlation of the underlying indexes on the risk numbers is found. That is exactly what a risk manager would expect.

After all an adequate price model allows risk managers to better understand the impact of price behaviour and risks on values and hedges. Reducing the valuation problem to spark spread options is hence a good way to start the risk management of the plant. However, as mentioned before such a simple valuation assumes away certain operational constraints and management possibilities. To include such constraints and the possibility to sell electricity also through forward contracts in the valuation and operation problem we have chosen a stochastic dynamic programming approach presented in **Part II** (i.e. the third chapter).

Part II

The third chapter presents a path dependent sequential decision making process to asses the plant value of a power generating unit by respecting certain plant characteristics, operational constraints and management possibilities.

Theoretical Model Framework. After setting up the SDP representation of the problem our main contribution is to state existence and uniqueness of a solution along with a specific structure of the optimal value function. The SDP model is presented in such a way that also the incorporation of a model for the spread between electricity and any fuel required to produce that electricity is possible. However, our empirical analysis is carried out and further suggests to use an one factor model due to the path dependence of the problem.

Throughout the third chapter several challenges a power plant faces in the uncertain energy market environment are addressed:

1. The uncertain spot bidding procedure at the energy exchanges is explicitly incorporated in the model set up. Spot bids are optimized based on "day-ahead" price expectations. This extremely adds up to the complexity of the problem.

2. The risks caused by the extreme and sudden price changes in electricity spot prices are addressed by incorporating forward contracts to the sales portfolio of the power plant. The choice of forward contracts is supported by considering e.g. the sales portfolio of *RWE Power AG* in the German market. Obviously, the RWE managers sell a huge fraction of their capacity on a forward basis.

3. We include capacity constraints to the problem setting, since they have a significant influence on the plant's daily operation.

4. Last but not least the risk preferences of the plant are incorporated by the choice of a specific utility function. We propose a general class of utility functions. All results are then derived for a specific member of that class, namely the expected utility function. Thereafter we argue, that all results extend to the specified general class.

Focusing on these challenges the coordination of the bidding on the spot market, the hedging through forward selling and the scheduling of the plant are at the heart of the third chapter.

Empirical Findings. After describing the theoretical model framework, stating existence and uniqueness of a solution and presenting the structure of the optimal value function the interaction of the optimal controls and the impact of the underlying price model is addressed by an empirical analysis based on different specific factor models.

At first an optimal policy and value function is calculated based on the Cox Ross Rubinstein market, where different market movements are analyzed, namely upwards, downwards and sideways movements are elaborated. The calculated values then form the benchmark for judging the values calculated on basis of a specific one factor model for the electricity price dynamics. That model is fitted to market observable swap prices and LIBOR rates and is approximated by a trinomial tree structure. We identify a true relationship between the market trend cycles and the optimal operation schedule of the plant. The value of the fixed production costs can be identified to depict some kind of boundary between full or no production. Again a true relationship between the optimal operating schedule and the fixed costs is obvious. The magnitude of optimal control values with regard to the benchmark indicates an overall sideways moving market implied by the one factor model fitted to historical swap prices.

Our theoretical results as well as the empirical study imply an interaction between the ("realized") risk premium and the fraction of capacity devoted to forward contracts. This makes good sense. Hence, the risk premium depicts the decisive factor to answer the central question of this chapter: *How much capacity should be devoted to the forward market and how much capacity should be kept for bidding in the spot market?*

Using the one factor model for calculating plant values on basis of the SDP approach yields considerably low values in comparison to those calculated based on the spark spread valuation executed in the second chapter. Such a result is reasonable, since the spark spread valuation purely addresses the price risks induced by the spikes, but assumes full production flexibility in between the capacity boundaries. That is exercise decisions of the options are being made optimally. Such a valuation easily overestimates the true value. Additionally incorporating hedging possibilities through forward contracts, accounting for uncertain spot bidding and including the risk preferences clearly has its price, but makes good sense with regard to the market practice.

Outlook. After all one could ask why not solving the presented optimal operation and valuation problem including uncertain spot bids and hedging through forward contracts based on a forecasting model such as the proposed regime-switching models. The main issue of connecting them is the path dependence of the sequential decision making problem presented in the third chapter. Hence it requires a lattice structure to calculate optimal solutions which easily blows up by using complex model specifications. Additionally, it is necessary to have explicit formulas to calculate forward prices. Since both problems and models on their own are rather complex, we believe it is an extremely challenging task to connect the two. We leave that exciting challenge for future research.

Zusammenfassung

Die vorliegende Arbeit gliedert sich in zwei Hauptteile. Der **erste Teil** (d.h. Kapitel 2) beschreibt und analysiert ein Modell, welches das Ziel hat die gleichzeitige Entwicklung von Elektrizitäts- und Gaspreisen möglichst adequat abzubilden. Dieses Modell gehört zur Klasse der sogenannten *Regime-Switching* Modelle. Verschiedene Arbeiten haben gezeigt, dass diese gut geeignet sind die Wechsel zwischen stabilen Preiszuständen und den für Elektrizitätspreise charakteristischen extremen Preisausschlägen zu modellieren.

Wir stellen ein solches flexibles, bivariates Modell dar und beschreiben den Prozess wie die Modellparameter an historische Marktdaten angepasst werden können. Hierbei gehen wir insbesondere auf die besonderen Herausforderungen ein, die von der Markov-Kette ausgehen, welche die unbeobachtbaren Regimezustände abbildet.

Eine Routine zur Kalibrierung des Modells an Marktdaten wird dargestellt und durchgeführt. Auf Basis der berechneten Modellparameter wird anschliessend der Wert eines Gaskraftwerkes über sogenannte *Spark Spread Optionen* bestimmt. Das vorgestellte Modell bildet somit die Grundlage für das Portfolio- und Risikomanagement eines Gaskraftwerkes.

Das Hauptanliegen des **zweiten Teils** (d.h. des dritten Kapitels) besteht darin das Problem der Bewertung und optimalen Auslastung eines Kraftwerkes durch ein stochastisches dynamisches Programm abzubilden und dieses zu lösen.

Die für den Elektrizitätsmarkt spezifischen Spotpreisrisiken können hierbei durch das Eingehen von Forwardkontrakten abgesichert werden. Desweiteren werden die Kapazitätsgrenzen und die Risikoaversion des Kraftwerkes bzw. deren Manager explizit berücksichtigt.

Wir zeigen Existenz und Eindeutigkeit einer Lösung und leiten eine spezielle Struktur der optimalen Wertfunktion her. Für verschiedene Faktormodelle wird das Problem anschliessend numerisch betrachtet und die entsprechenden Risiko- und Performancegrößen berechnet.

Appendix A

Regime-Switching Model

Itô Calculus

The Itô formula for jump-diffusion processes, also called the *Itô-Doeblin* formula with respect to our application is given by the next proposition. For more details we refer the interested reader to Cont & Tankov [CT04] (in particular to Chapter 8).

Proposition A.1 (Itô formula for jump-diffusion processes)**.** *Let x be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:*

$$x_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma dW_s + \sum_{m=1}^{q_t} J_m$$

where $b_s = -\alpha(x_s - f)$ and all specifications made in Definition 2.2 are valid. Then for any twice continuously differentiable function $h : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process $Y_t = h(t, x_t)$ can be represented as:

$$\begin{aligned} h(t, x_t) =& h(0, x_0) + \int_0^t \left[\frac{\partial h}{\partial s}(s, x_s) + \frac{\partial h}{\partial x}(s, x_s) b_s \right] ds \\ &+ \frac{1}{2} \int_0^t \sigma^2 \frac{\partial^2 h}{\partial x^2}(s, x_s) ds + \int_0^t \frac{\partial h}{\partial x}(s, x_s) \sigma dW_s \\ &+ \sum_{0<s\leq t} [f(s, x_s) - f(s^-, x_{s^-})]. \end{aligned} \tag{A.1}$$

(r-stage) Transition Probabilities

Lemma A.2. *For any $t \in \mathbb{T}$ and $\ell \in \mathcal{L}$ the probability $p^{\ell}_{t-r,t}$ is given by*

$$p^{\ell}_{t-r,t} = \begin{cases} \sum_{j^* \in \mathcal{J}^*_\ell} \mathbb{P}(\tilde{s}_{t-1} = j^* \mid \tilde{s}_t = \ell, \Omega_t; \theta) & r = 2 \\ \sum_{j \in \mathcal{J}_\ell} \mathbb{P}(\tilde{s}_{t-r+1} = j \mid \tilde{s}_t = \ell, \Omega_t; \theta) \cdot \tilde{p}(j, \ell) & r \in \{3, 4, \ldots, t\} \end{cases} \tag{A.2}$$

where

$$\tilde{p}(j, \ell) = \sum_{\substack{k_\nu \in \mathcal{B} \setminus \mathcal{J}_\ell \\ \nu = 1, \ldots, r-2}} \tilde{p}_{k_1,k_2} \cdot \tilde{p}_{k_2,k_3} \cdot \cdots \cdot \tilde{p}_{k_{r-2},j}$$

with $\tilde{p}_{k_\nu,k_{\nu+1}} := \mathbb{P}(\tilde{s}_{t-\nu} = k_\nu | \tilde{s}_{t-(\nu+1)} = k_{\nu+1}, \tilde{s}_t = \ell, \Omega_t; \theta)\,.$

Proof. For $A \in \mathcal{F}$ we define the conditional probability measure $\mathbb{Q}$ such that $\mathbb{Q} := \mathbb{P}(A | \tilde{s}_t = \ell, \Omega_t; \theta)$ for all $\ell \in \mathcal{L}$. Then for $r = 2$ we have

$$p^{\ell}_{t-2,t} = \mathbb{Q}(s_{t-2} = 1) = \sum_{j^* \in \mathcal{J}^*_\ell} \mathbb{Q}(\tilde{s}_{t-1} = j^*).$$

For $r \in \{3, \ldots, t\}$ it holds

$$p^{\ell}_{t-r,t} = \sum_{j \in \mathcal{J}_\ell} \mathbb{Q}(\tilde{s}_{t-r+1} = j, \tilde{s}_{t-\nu} \neq j, 0 < \nu < r).$$

Hence,

$$\begin{aligned}
&p^{\ell}_{t-r,t} = \\
&= \sum_{j \in \mathcal{J}_\ell} \mathbb{Q}(\tilde{s}_{t-r+1} = j, \tilde{s}_{t-r+2} \neq j, \ldots, \tilde{s}_{t-1} \neq j) \\
&= \sum_{j \in \mathcal{J}_\ell} \sum_{\substack{k_\nu \in \mathcal{B} \setminus \mathcal{J}_\ell \\ \nu = 1, \ldots, r-2}} \mathbb{Q}(\tilde{s}_{t-r+1} = j, \tilde{s}_{t-r+2} = k_{r-2}, \ldots, \tilde{s}_{t-1} = k_1) \\
&= \sum_{j \in \mathcal{J}_\ell} \sum_{\substack{k_\nu \in \mathcal{B} \setminus \mathcal{J}_\ell \\ \nu = 1, \ldots, r-2}} \mathbb{Q}(\tilde{s}_{t-1} = k_1 | \tilde{s}_{t-2} = k_2) \cdots \mathbb{Q}(\tilde{s}_{t-r+1} = j | \tilde{s}_{t-r+2} = k_{r-2}) \cdot \mathbb{Q}(\tilde{s}_{t-r+1} = j) \\
&= \sum_{j \in \mathcal{J}_\ell} \sum_{\substack{k_\nu \in \mathcal{B} \setminus \mathcal{J}_\ell \\ \nu = 1, \ldots, r-2}} \tilde{p}_{k_1,k_2} \cdot \tilde{p}_{k_2,k_3} \cdot \cdots \cdot \tilde{p}_{k_{r-2},j} \cdot \mathbb{Q}(\tilde{s}_{t-r+1} = j) \\
&= \sum_{j \in \mathcal{J}_\ell} \mathbb{Q}(\tilde{s}_{t-r+1} = j) \cdot \tilde{p}(j, \ell).
\end{aligned}$$

Hence, Lemma 2.1 is proven. □

By properties of the Markov chain the probabilities $\mathbb{P}(\tilde{s}_{t-r+1}=j|\tilde{s}_t=\ell,\Omega_{t-1})$ can be transformed to

$$\begin{aligned}\mathbb{P}(\tilde{s}_{t-r+1}=j|\tilde{s}_t=\ell,\Omega_{t-1}) &= \frac{\mathbb{P}(\tilde{s}_{t-r+1}=j,\tilde{s}_t=\ell,\Omega_{t-1})}{\mathbb{P}(\tilde{s}_t=\ell|\Omega_{t-1})}\\ &= \frac{\mathbb{P}(\tilde{s}_t=\ell|\tilde{s}_{t-r+1}=j,\Omega_{t-1})\cdot\mathbb{P}(\tilde{s}_{t-r+1}=j|\Omega_{t-1})}{\mathbb{P}(\tilde{s}_t=\ell|\Omega_{t-1})}\\ &= \frac{\mathbb{P}(\tilde{s}_{t-r+1}=j|\Omega_{t-1})}{\mathbb{P}(\tilde{s}_t=\ell|,\Omega_{t-1})}\cdot\sum_{\substack{k_\nu\in\mathcal{B}\\ \nu=1,\ldots,r-2}}\hat{p}_{\ell,k_1}\hat{p}_{k_1,k_2}\cdot\hat{p}_{k_2,k_3}\cdot\ldots\cdot\hat{p}_{k_{r-2},j},\end{aligned}$$

where $\hat{p}_{k_\nu,k_{\nu+1}}:=\mathbb{P}(\tilde{s}_{t-\nu}=k_\nu|\tilde{s}_{t-(\nu+1)}=k_{\nu+1},\Omega_{t-1})$. Thereafter, all involved probabilities are given by the *smoothed inferences*. These can be estimated by applying Algorithm 2.3.

Analogously, the transition probabilities $\tilde{p}_{k_\nu,k_{\nu+1}}$ conditioned on the event $\{\tilde{s}_t=\ell\}$ can be estimated.

Conditional Moments

The conditional moments involved in the conditional transition densities $\Phi^i_t(\theta|s^i_{t-r}=1)$ for all $\ell\in\mathcal{L}^i$ and $\Phi^{xy}_t(\theta|s_{t-r}=1)$ for all $\ell\in\mathcal{L}^{xy}$ can also be iteratively derived, such that they are given for any $r\in\{1,2,\ldots,K\}$.The explicit forms of these conditional transition densities can be found in (2.11) and (2.18), respectively.

Lemma A.3 (Conditional Moments)**.** *For all $t\in\mathbb{T}$ the expected value and variance of x_t conditioned on x_{t-r} for any $r\in\{1,2,\ldots,K\}$ are given by*

$$\begin{aligned}e^x_{t,r} &:= \mathbb{E}[x_t|x_{t-r}] = x_{t-r}\,e^{-\alpha^x r}+f^x(1-e^{-\alpha^x})\sum_{\nu=0}^{r-1}e^{-\alpha^x\nu}\\ &\ = x_{t-r}\,e^{-\alpha^x r}+f^x(1-e^{-\alpha^x r}),\\ v^x_r &:= \mathbb{V}[x_t|x_{t-r}] = \sum_{\nu=0}^{r-1}e^{-2\alpha^x((r-1)-\nu)}(\sigma^x_1)^2=(\sigma^x_r)^2\,r.\end{aligned}$$

The analogue specifications of $e^y_{t,r}$ and v^y_r hold for the expected value and variance of y_t conditioned on y_{t-r}.

Proof. Having the discrete version of the observation price process given in (2.11) the proof of Lemma A.3 can be conducted using induction. Exemplarily, we look at the electricity price dynamics. The values $\tilde{s}_t\in\mathcal{L}^x$ imply $s^x_t=1$. Thus, all parameters modeling the jump part of the process are equal to zero and the discrete time series becomes

$$x_t = x_{t-1}e^{-\alpha^x} + f^x(1 - e^{-\alpha^x}) + \sigma_\epsilon^x R_t^x,$$

where $R_1^x \sim \mathcal{N}(0,1)$, $\sigma_1^x := \sigma^x \sqrt{\frac{1-e^{-2\alpha^x}}{2\alpha^x}}$ and α^x, f^x, σ^x are constant for any fixed $t \in \mathbb{T}$. Now, for $r = 1$ we have

$$\mathbb{E}[x_t|x_{t-1}] = x_{t-1}\, e^{-\alpha^x} + f^x(1 - e^{-\alpha^x}).$$

Assume for some $r \in \{1, 2, \ldots, K\}$ it holds

$$\mathbb{E}[x_t|x_{t-r}] = x_{t-r}\, e^{-r\alpha^x} + f^x(1 - e^{-\alpha^x}) \sum_{\nu=0}^{r-1} e^{-\nu\alpha^x}. \tag{A.3}$$

Then for $r + 1 \in \{2, \ldots, K\}$ we obtain

$$\begin{aligned}
\mathbb{E}[x_t|x_{t-r-1}] \quad &= \quad \mathbb{E}[x_{t-1}|x_{t-r-1}]\, e^{-\alpha^x} + f^x(1 - e^{-\alpha^x}) \\
&\overset{(A.3)}{=} \quad \{x_{t-r-1}\, e^{-\alpha^x r} + f^x(1 - e^{-\alpha^x}) \sum_{\nu=0}^{r-1} e^{-\alpha^x} \nu\}\, e^{-\alpha^x} + f^x(1 - e^{-\alpha^x}) \\
&= \quad x_{t-r-1}\, e^{-\alpha^x(r+1)} + f^x(1 - e^{-\alpha^x}) \sum_{\nu=0}^{r} e^{-\alpha^x \nu} \\
&\overset{\text{telescoping}}{=} \quad x_{t-r-1}\, e^{-\alpha^x(r+1)} + f^x(1 - e^{-\alpha^x(r+1)}).
\end{aligned}$$

Hence, by induction Hypothesis (A.3) is true for all $r \in \{1, 2, \ldots, K\}$ and $e_{t,r}^x$ has the desired form.

Analogue for $r = 1$ it holds

$$\mathbb{V}[x_t|x_{t-1}] = (\sigma_1^x)^2.$$

Assume for some $r \in \{1, 2, \ldots, K\}$ it holds

$$\mathbb{V}[x_t|x_{t-r}] = \sum_{\nu=0}^{r-1} e^{-2\alpha^x((r-1)-\nu)} (\sigma_1^x)^2. \tag{A.4}$$

Then for $r + 1 \in \{2, \ldots, K\}$ we obtain

$$\begin{aligned}
\mathbb{V}[x_t|x_{t-r-1}] \quad &= \quad \mathbb{V}[x_{t-1}|x_{t-r-1}]\, e^{-2\alpha^x} + (\sigma_\epsilon^x)^2 \\
&\overset{(A.4)}{=} \quad \sum_{\nu=0}^{r-1} e^{-2\alpha^x(r-\nu)} (\sigma_1^x)^2 + (\sigma_1^x)^2 \\
&= \quad \sum_{\nu=0}^{r} e^{-2\alpha^x(r-\nu)} (\sigma_1^x)^2 \overset{\text{telescoping}}{=} (\sigma_r^x)^2\, r.
\end{aligned}$$

Again, by induction Hypothesis (A.4) is true for all $r \in \{1, 2, \ldots, K\}$ and v_r^x has the desired form. Analogue calculations lead to expressions $e_{t,r}^y$ and v_r^y. $\square$

SML Method

First of all, we remark that the SML method is based on one main assumption.

Assumption A.1. *On each interval of length δ in the refinement of the axis (i.e. between any two consecutive observations) no more than one jump can occur with probability $p^1 := 1 - e^{-\lambda x \delta}$, where λ^x is the intensity parameter of the Poisson process q^x.*

If one chooses any (discretized) sample path $\omega \in [1, K]$ and walks from date $t-1$ along the refinement of the grid up to date $t-\delta$, one can calculate the time $t-\delta$ realization $\hat{x}_{t-\delta}(\omega)$ (using the discretized version of the observation process dynamics as in Corollary 2.1). Then, for all $\tilde{s}_t = j \in \mathcal{S}^x$ the corresponding transition density for that specific path is given by the weighted sum

$$\hat{f}(\hat{x}_t|\hat{x}_{t-\delta}(\omega),\, \tilde{s}_t = j, \theta) = p^0 \phi(\hat{x}_t;\, \mu^c_\delta, \sigma^c_\delta) + p^1 \phi(\hat{x}_t;\, \mu^d_\delta, \sigma^d_\delta),$$

where $\mu^c_\delta = f^x(1 - e^{-\alpha^x \delta}) + e^{-\alpha^x \delta}\hat{x}_{t-\delta}(\omega)$, $\sigma^c_\delta = \sigma^x \sqrt{\frac{1-e^{-2\alpha^x \delta}}{2\alpha^x}}$, $\mu^d_\delta = \mu^c_\delta + \mu_J$, $\sigma^d_\delta = \sqrt{(\sigma^c_\delta)^2 + \sigma^2_J}$ and $p^0 = 1 - p^1$.

Thereafter, the approximated spike regime transition density resulting from K sample paths is calculated by

$$f(\hat{x}_t|\hat{x}_{t-\delta},\, \tilde{s}_t = j, \theta) = \frac{1}{K} \sum_{\omega=1}^{K} \hat{f}(\hat{x}_t|\hat{x}_{t-\delta}(\omega),\, \tilde{s}_t = j, \theta)$$

and further according to the EM algorithm, we obtain the approximated log-Likelihood function

$$\begin{aligned}\mathbb{E}[\log L(\theta) \,|\, \Omega_T;\, \theta] = \sum_{t=k}^{T} \sum_{j \in \mathcal{S}^x} & \mathbb{P}(\tilde{s}_t = j|\Omega_T;\, \theta) \\ & \cdot \log\left[\mathbb{P}(\tilde{s}_t = j|\Omega_{t-1};\, \theta)\, f(\hat{x}_t|\hat{x}_{t-\delta},\, \tilde{s}_t = j, \theta)\right].\end{aligned}$$

where the transition density $f(\hat{x}_t|\hat{x}_{t-\delta},\, \tilde{s}_t = j, \theta)$ approximates $\mathcal{Z}^x(\theta)$. Analogue considerations hold for the approximation of $\mathcal{Z}^y(\theta)$. Thereafter, maximizing the expected log-Likelihood function for both sets of spike regime states $\mathcal{S}^i \subset \mathcal{S}$ with respect to the corresponding spike regime parameter $\theta_J \subset \theta$ then yields the desired estimates.

Appendix B

Stochastic Dynamic Programming

Theoretical Part

Outer Integral Let P be a probability measure on the measurable space $(\Omega, \mathcal{F})$. Let f, g and h be functions from Ω to $[-\infty, +\infty]$ then in accordance with Bertsekas & Shreve [BS80] we define:

Definition B.1. *If $f \geq 0$ the* outer integral *of f with respect to P is defined by*

$$\int^* f \, dP = \inf \left\{ \int g \, dP \mid f \leq g, \; g \text{ is } \mathcal{F}\text{-measurable} \right\}.$$

If $f \leq 0$ define $\int^* f \, dP := -\int^* (-f) \, dP$ to apply the outer integral formulation analogously.

Lemma B.1. *If $f \geq 0$ and $h \geq 0$, then*

$$\int^* (f + h) \, dP \leq \int^* f \, dP + \int^* h \, dP. \tag{B.1}$$

If either f or h is $\mathcal{F}$-measurable, then equality holds in (B.1).

Proof. Given in Appendix A of [BS80]. □

Again an analogue result holds, if $f \leq 0$ and $h \leq 0$.

Uniformly Optimal Policy As regards the completeness of exposition, we state (without proof) the subsequent results of Bertsekas & Shreve (stated on page 44 f. of their book [BS80]) applied in the proof of Theorem 3.6 in terms of our model framework.

Proposition B.2. *A policy $\pi^* = (\mu_0^*, \mu_1^*, \dots)$ is uniformly T-stage optimal if and only if*

$$(T_{\mu^*_{T-k}} \cdot T^{k-1})(J_T) = T^k(J_T)$$

for all $k = 0, 1, 2, \dots, T-1$, where T^k denotes the composition of T with itself k times.

As a corollary of Proposition B.2 Bertsekas & Shreve obtain the following:

Corollary B.1. *(a) There exists a uniformly T-stage optimal policy if and only if the supremum in the relation*

$$T^{k+1}(J_T)(X_{T-k-1}) = \sup_{c \in K(X_{T-k-1})} H(X_{T-k-1}, c, T^k(J_T))$$

is attained for all $X_{T-k-1} \in \mathbb{X}$ and all $k \in \{0, 1, \dots, T-1\}$.

(b) If there exists a uniformly T-stage optimal policy, then

$$J_{T,\pi^*} = T^T(J_T)$$

is optimal.

Empirical Part

Fitting Procedure The stepwise procedure of fitting the exponential volatility function determining the evolution of fixed delivery forward contracts to market observable swap price data as stated in Section 3.7.3 is accomplished by using the following ideas:

1. The first step of the approach is based on the work of Benth, Koekkebakker and Ollmar [BKO07]. The task is to use sample deviations of historical daily swap contract returns to estimate the exponential volatility parameter of fixed delivery forward price dynamics. At this point bear in mind the difference of fixed delivery dynamics and market observable swap contracts having a monthly delivery period. Futhermore, the stochastic evolution of forward prices is modelled under an equivalent martingale measure $\mathbb{Q}$ whereas the atomic swap contracts $G(t, T_m^S, T_m^E)$ are observed under the real world measure $\mathbb{P}$. Although

there might exist a risk premia that cause forward prices exhibit non-zero drift terms, the diffusion terms under both measures are equal. Hence, Σ and σ can be estimated from real world data. Note, according to Cortazar & Schwartz [CS94] this is only correct if the observations are sampled continuously, i.e. in our application daily samples are used. We have (according to Benth et al. [BKO07] in Section 4.2.2.) for $m = 0, \ldots, M$

$$\begin{aligned}\Sigma(t, T_m^S, T_m^E) &= \frac{1}{T_m^E - T_m^S} \int_{T_m^S}^{T_m^E} \sigma e^{-\kappa(u-t)} du \\ &= -\frac{1}{T_m^E - T_m^S} \frac{\sigma}{\kappa} (e^{-\kappa(T_m^E - t)} - e^{-\kappa(T_m^S - t)}).\end{aligned}$$

Assuming settlement of the forward contracts at K points in a month $m \in \{0, \ldots, M\}$, i.e. $T_{1m} < T_{2m} < \cdots < T_{Km}$ with $T_m^S = T_{1m}$ and $T_m^E = T_{Km}$, we have

$$dG(t, T_m^S, T_m^E) = \frac{1}{T_m^E - T_m^S} \sum_{k=1}^{K} \sigma(t, T_{km}) dW_t.$$

Let $dG(t_j, T_m^S, T_m^E) \approx G(t_j, T_m^S, T_m^E) - G(t_{j-1}, T_m^S, T_m^E) = x_{jm}^G$ for $j = 1, \ldots, N$ and $m = 0, \ldots, M$. With these approximations at hand we use the monthly sets of observations under the real world measure collected in the matrix

$$X_{N\times(M+1)}^G = \begin{bmatrix} x_{10}^G & x_{11}^G & \ldots & x_{1M}^G \\ x_{20}^G & \ldots & \ldots & x_{2M}^G \\ \vdots & \vdots & \vdots & \vdots \\ x_{N0}^G & \ldots & \ldots & x_{NM}^G \end{bmatrix}$$

and obtain $\hat{\Sigma}_m = \sqrt{\frac{1}{N-1}\sum_{j=1}^N (x_{jm}^G - \overline{x}_{jm}^G)^2}$ as proxy for the swap contract volatility, where $\overline{x}_{jm}^G$ denotes the average daily price differences $\overline{x}_{jm}^G = \frac{1}{N}\sum_{j=1}^N x_{jm}^G$ corresponding to the m-th month of observation. Now, minimizing

$$\begin{aligned}&\sum_{m=1}^{M}\sum_{j=1}^{N} (\hat{\Sigma}_m - \frac{1}{T_m^E - T_m^S}\sum_{k=1}^{K} \sigma\, e^{\kappa(T_{km}-t_j)})^2 \\ &= \sum_{m=1}^{M}\sum_{j=1}^{N} (\hat{\Sigma}_m + \frac{1}{T_m^E - T_m^S} \frac{\sigma}{\kappa} (e^{-\kappa(T_m^E - t_j)} - e^{-\kappa(T_m^S - t_j)}))^2\end{aligned}$$

with respect to the volatility parameter σ and κ yields the implied volatility function for the fixed delivery period

$$\hat{\sigma}_{\text{imp}}(t, T_{km}) = \hat{\sigma}_{\text{imp}}\, e^{-\hat{\kappa}_{\text{imp}}(T_{km}-t)}$$

for all $k = 1, \ldots, K$ such that T_{km} satisfies $T_m^S \leq T_{km} \leq T_m^E$ and all $m = 0, \ldots, M$.

2. According to the approach of Clewlow and Strickland [CS99] in the second step the state prices Q_{ij} for all i, j are obtained by forward induction using the recursion
$$Q_{ij} = \sum_{j'} Q_{ij'} p_{j'j} \, p(i, i+1),$$
where $Q_{00} = 1$, $p_{j'j}$ is the probability of moving from node (i, j') to node $(i+1, j)$ and $p(i, i+1)$ is the bond price at date i maturing at date $i+1$. Note, in order to have daily quoted bond prices available, linear interpolation techniques have been used.

3. Fitting the **market price of risk** to historic logarithmic spot price returns is then accomplished by using the relation
$$\overline{S}_i = e^{\overline{X}_i + \kappa\lambda_i}$$
such that $\lambda_i = \frac{1}{\kappa}(\ln \overline{S}_i - \overline{X}_i)$, where we use the average spot prices of the last ten years as a proxy for $\overline{S}_i$, i.e. $\overline{S}_i = \frac{1}{10}\sum_{y=1}^{10} S_i(y)$ with $y = 1, \ldots, 10$, and $\overline{X}_i = \frac{1}{J}\sum_{j=1}^{J} x_{ij}$ where J is the number of scenarios in the tree.

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Ehrenwörtliche Erklärung

Ich versichere hiermit, dass ich die Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Die Arbeit wurde bisher nicht im In- oder Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt.

Ulm, 21. Dezember 2011

(Unterschrift)