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Characteristics of Poisson Cylinder Processes and their Estimation

Dissertation

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Malte Spiess
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Amtierender Dekan:	Prof. Dr. Paul Wentges
1. Gutachter:	Prof. Dr. Evgeny Spodarev
2. Gutachter:	Prof. Dr. Ulrich Stadtmüller
3. Gutachter:	Prof. Dr. Daniel Hug
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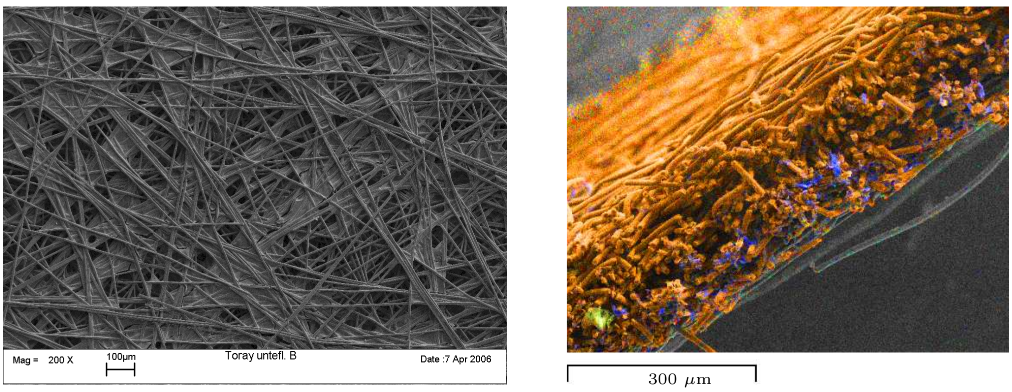
1. Introduction

1.1. Motivation

Many porous materials can be modeled with *random dilated fiber processes*, where a fiber is the image of a (piecewise) C^1-smooth curve.[1] For this, various examples can be found in literature, see e.g., the recent books [OS09] or [Tor02], which discuss the modeling and analysis of such media.

Consider for example polymer electrolyte membrane (PEM) fuel cells, which are widely used in stationary as well as portable devices for power generation. One structural part of a PEM fuel cell which has a high influence on the performance is the so-called gas diffusion layer (GDL). Its main purpose is to assist the gas and water transport. In Figure 1.1 microscopic images of the fiber systems of two GDLs are shown.



(a) Microscopic image of a GDL, top view

(b) Microscopic image of a GDL, side view

Figure 1.1.: Images of gas diffusion layers used in PEM fuel cells (courtesy of the Centre for Solar Energy and Hydrogen Research, Ulm)

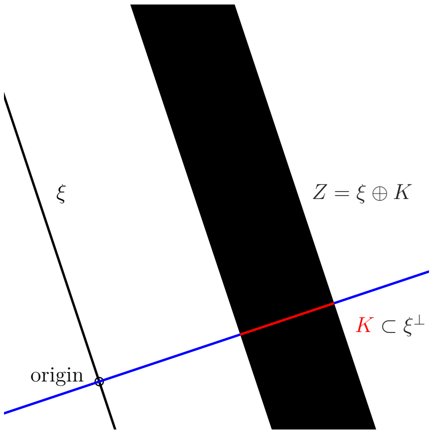
The GDL in the example consists of very long fibers with almost no curvature. Hence, it can be modeled by so-called *cylinder processes*, where a cylinder is defined

[1] In literature, dilated curves are sometimes also called fibers. Here, (for mathematical objects) we always distinguish between fibers and dilated fibers for clarity.

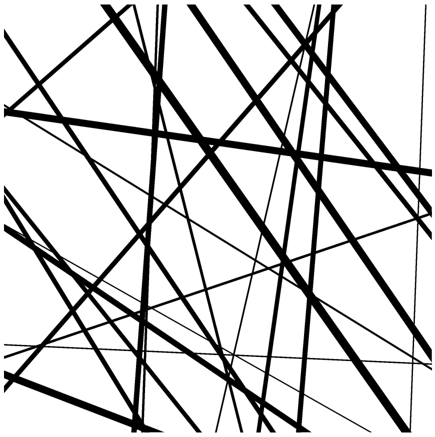
as an infinitely long line dilated with a polyconvex set from the orthogonal space, see Figure 1.2a for a sketch.

An example of a similarly structured material can be found in [SPRB+06], where the influence of the properties of fibrous media used to control the acoustics in cars is analyzed.

These and other interesting applications were our motivation to consider as a basic model throughout this thesis so-called *stationary cylinder processes*, i.e., a random collection of cylinders. In this context "stationarity" means that the distribution of the process is invariant with respect to translations. A pictorial description could be that the process looks the same no matter at which point you view it.



(a) Example of a cylinder as the sum of a line ξ and a set $K \subset \xi^\perp$



(b) Realization of a cylinder process

Figure 1.2.: Construction of a 2D cylinder process

We shall restrict to *Poisson* cylinder processes (abbreviated by PCP), i.e., Poisson point processes of cylinders. Poisson point processes are a common model in stochastic geometry, see, e.g., [DVJ08] or [SKM95] and seem to be a natural choice for material modeling. The property of being Poisson has the effect (among others) that there is no interaction between the cylinders. A rigorous definition of a PCP is given in Section 2.2.

The morphology of a material has a crucial influence on its performance, e.g., on the stability or on the gas transport capabilities. Since the production of new materials is often expensive and time-consuming, it is desirable to find suitable stochastic geometric models which can be optimized on the computer. Thus, for fitting such models to real data, formulas and estimators for the morphological properties of the theoretical models are needed.

One important property of general stationary fiber processes is the *intensity*, i.e., the expected total length of the fibers in the unit cube. For this, there already exist suitable estimators in literature, see, e.g., [SKM95] or [Sch00].

Further fundamental functionals of stationary PCPs are the *specific intrinsic volumes*, which include the *specific surface area*. Among these, the most important is the *volume fraction*, which can easily be accessed for many materials. For the use in asymptotic statistical tests (for growing observation windows), explicit formulas for the asymptotic variance and Berry-Esseen bounds are needed.

Another often crucial characteristic, which has a big influence on the stability and the mechanical properties of the material, is the *directional distribution* of the fibers or cylinders. This is the distribution of the direction of the tangent at a typical point of the fiber process. In material science, sometimes the only information available about a material is pictures from confocal microscopy or polished planar cuts (see [KP05]), i.e., one can observe merely a finite number of thin sections of the material. Yet, even with this two-dimensional information, the directional distribution can be reconstructed with a stereological approach, based on the numbers of intersections of the fibers with the observed planes. For the statistical evaluation of the resulting estimator, consistency and convergence rate are of interest.

In summary, for the model of a stationary PCP the main aims of this thesis are to

- calculate formulas for basic characteristics, namely the Choquet functional, the covariance function, and the specific surface area,
- derive a central limit theorem for the volume fraction, including explicit formulae for the asymptotic variance and Berry-Esseen bounds,
- find a suitable estimator for the directional distribution and analyze its asymptotic statistical properties, where we assume that only the intersection counts of the process with a finite number of test hyperplanes can be observed,
- illustrate the efficiency of the estimator for the directional distribution with simulation studies.

1.2. Cylinder processes

Before we give an outline of this thesis, we want to be a little more precise about the notion of a cylinder process. For this, we need to define first what we call a *cylinder*. Here, a cylinder is the Minkowski sum of a k-flat ξ in the d-dimensional Euclidean space and a polyconvex set K in the orthogonal space $\xi^\perp$, see Figure 1.2a. ξ is called the *direction space*, whereas K denotes the *base* of the cylinder. This generalizes the common notion of a cylinder which has a compact and convex base. A *cylinder process* is – loosely speaking – a measurable random collection of cylinders.

Here, different values of k lead to very different processes. In the three-dimensional case, besides $k = 0$, which yields a usual germ-grain model, there are two possibilities, $k = 1$ and $k = 2$. In Figure 1.3, examples of the resulting processes are shown.

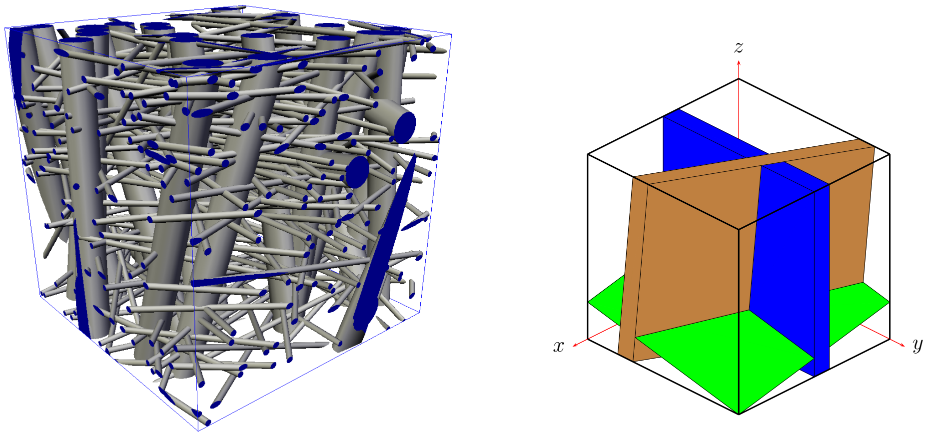


Figure 1.3.: 3D cylinder processes with 1- and 2-dimensional direction space, respectively

Processes of convex cylinders have been considered frequently in literature, for an overview see e.g., [Sch87, Section 3]. They were first mentioned in [Mat75], where they appear as a side product and are used to characterize a certain class of random closed sets. Further formulas for this model have been derived in [Dav78] and [Ser84]. The first rigorous definition of the more general model with a polyconvex base has been given in [Wei87], where the case of cylinder processes has been analyzed without the restriction of being Poisson. Recently, Hoffmann calculated the mixed volumes for not necessarily stationary Poisson cylinder processes with convex bases, see [Hof09b].

1.3. Outline

Chapter 2 starts with the introduction of some basics from integral geometry and stochastic processes, which are needed for the treatment of PCPs. First, in Section 2.1, we introduce some basic sets and notions from convex geometry which form the basis for the rigorous definition of cylinders and also cylinder processes. In Section 2.2, we define Poisson processes on a rather general space. Since it is convenient, we give two definitions of Poisson cylinder processes, which are equivalent up to a set of probability zero as shown in Appendix A.1. After presenting some basic formulas for marked Poisson processes, we briefly turn to the notion of cumulants in Sections 2.3 and 2.4, respectively. We conclude by a short introduction to a principle for solving an inverse linear problem called the *method of the approximate inverse* (short: AI) in Section 2.5. In addition, we define the spherical Radon transform and the cosine transform.

In Chapter 3, we consider some of the most important basic characteristics of stationary Poisson cylinder processes. In Section 3.1, we begin with the so-called *capacity functional* (also known as the *Choquet functional*), which is the probability that the union set of the process hits a compact test set. The functional is of great importance in the theory of random closed sets since it characterizes their distribution. From this basic characteristic, formulas for the covariance function and the contact distribution function of the union set follow easily. Another interesting property of the process is the *specific surface area*, i.e., the expected surface area per unit volume, which is the topic of Section 3.2. To conclude this chapter, we give an example of how the formulas from this chapter can be used in practice. Section 3.3 deals with a question from fuel cell technology which can be solved using the newly derived formulas.

In Chapter 4, we prove a central limit theorem (CLT) for the volume fraction (denoted by $V_\rho^{(d,k)}$) of a stationary PCP within an observation window ρW for $\rho \to \infty$, where W is a compact set which is star-shaped with respect to the origin. At first the preliminaries are discussed. Then, in Section 4.1, the main theorems are presented including the CLT, Berry-Esseen bounds, and large deviation results. We also state formulae for the asymptotic variance for discrete and continuous directional distributions. In Section 4.2 we calculate the order of the variance of $V_\rho^{(d,k)}$ which will be important in the following sections. Sections 4.3 and 4.4 contain rather technical proofs, which are based on a recursive estimation method for the cumulants of $V_\rho^{(d,k)}$ developed by Heinrich in [Hei05]. The application of this rather complex technique becomes necessary because of the long-range dependences caused by the infinitely long cylinders. We conclude by deriving formulae for the asymptotic variance in Section 4.5.

Chapter 5 addresses a problem which originates from stereology: We consider (non-dilated) fiber processes (e.g., line processes) and assume that we can observe the numbers of intersections of the process with some test hyperplanes in a bounded observation window. Our aim is to estimate the directional distribution. Under certain conditions, the resulting point processes (of intersection points) on the test hyperplanes are stationary Poisson processes. In this case the intensity of the latter processes is called the *rose of intersections* and is the cosine transform of the directional distribution of the fiber process. Thus, for the estimation of the directional distribution density from this data, the crucial part is to invert the cosine transform in a numerically stable way. We use the AI method for this purpose, which is shortly described in Section 2.5. The most important part for the use of this method is to derive a reconstruction kernel for a suitable mollifier, which is done for dimension $d = 2$ in Section 5.1.1 and for $d \geq 3$ in Section 5.1.2. This enables us to define the

AI estimator for directional distributions in Section 5.2. We analyze the stochastic properties of this estimator in Section 5.3, including almost sure convergence in the supremum norm (Section 5.3.1), Berry-Esseen bounds (Section 5.3.2), and large deviation properties (Section 5.3.3). Then, in Section 5.4, we demonstrate the performance of our estimator with some numerical experiments, and finally, in Section 5.5, we apply our method to real microscopic image data from GDLs.

In each of the Chapters 3 to 5 we begin with a short introduction to the treated problem and end with a section containing some remarks and commenting on open questions and possible further research topics.

In the Appendix A.1, we show how the two models of the PCP introduced in Section 2.2 are related. Then, in Section A.2, we present the inversion formula for the spherical Radon transform proposed by Rubin in [Rub02], which is related to our approach. We conclude the appendix by the calculation of the intersection area of two specific ellipses, which is needed in Section 5.3.2 for the determination of the variance of the estimator in the 3D case.

All software for the simulation and analysis of the cylinder processes (with exception of Martin Riplinger's work, which is indicated accordingly when applied) used in this thesis is based on and has been integrated into the GeoStoch Java library of the Institute of Stochastics at Ulm University. See `www.geostoch.de` for further details.

2. Poisson cylinder processes and related basic notions

In this chapter, we introduce the basic notation and conventions required for the definition of Poisson line and cylinder processes and present some related basic concepts. It is loosely based on the introductions in [HS09], [HS12], [LRSS11], [RS11], and [SS11].

We begin with some elementary sets and measures from convex geometry in the next Section 2.1. Then we recall the definition of Poisson point processes in a rather general setting, which is used to introduce stationary Poisson cylinder processes in Section 2.2. We give two definitions of PCPs, which are equivalent up to a set of probability zero, see Appendix A.1. We continue by defining the probability generating functional and the n-th order Campbell formula in Section 2.3. We briefly introduce the concept of cumulants in Section 2.4. Finally, a short overview of the basics of the method of the approximate inverse and the functions to which we want to apply it is given in Section 2.5.

2.1. Some basics in convex geometry

We introduce some basic geometric sets, where we mostly follow [SW08]. Throughout this thesis our basic space is the d-dimensional Euclidean space $\mathbb{R}^d$. In this space we denote the set of all compact sets (including the empty set $\emptyset$) by $\mathcal{C}$ and the subset of $\mathcal{C}$ containing all convex sets by $\mathcal{K}$. The set of all polyconvex sets, i.e., all sets which are a finite union of sets from $\mathcal{K}$, is denoted by $\mathcal{R}$. Finally, we write $\mathcal{S}$ for the set of all locally polyconvex sets, i.e., all sets S with $S \cap K \in \mathcal{R}$ for all $K \in \mathcal{K}$.

Furthermore, we denote by $\mathcal{C}'$ the set of all non-empty compact sets (compact bodies) and by $\mathcal{C}^o$ the set of all non-empty compact sets with circumcenter in the origin. For the other sets ($\mathcal{K}$, $\mathcal{R}$, and $\mathcal{S}$) we use the analogous notation. We sometimes add an index to emphasize the dimension, e.g., $\mathcal{C}_s$ for $\mathcal{C}$ in $\mathbb{R}^s$.

We equip the sets $\mathcal{C}$, $\mathcal{K}$, and $\mathcal{R}$ with the topology induced by the (extended) Hausdorff metric, which is defined for sets $A, B \in \mathcal{C}'$ as

$$\delta(A,B) = \max\Big\{ \max_{x\in A} \min_{y\in B} \|x-y\|, \max_{x\in B} \min_{y\in A} \|x-y\| \Big\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm. Further, we let $\delta(\emptyset,\emptyset)=0$, and for $A\in\mathcal{C}'$ we define $\delta(\emptyset,A)=\delta(A,\emptyset)=\infty$.

We denote the set of all closed sets by $\mathcal{F}$, and equip it with the so called *Fell topology*, see [SW08, p. 563]. By $\mathcal{B}(\mathcal{F})$ we denote its Borel sigma-algebra.

By $\mathbb{G}(d,k)$, $k\in\{0,\dots,d\}$ we denote the *Grassmannian manifold*, i.e., the space of all (non-oriented) k-dimensional subspaces of $\mathbb{R}^d$, and by $\mathbb{A}(d,k)$ the set of all affine k-dimensional subspaces. For $B\subset\mathbb{R}^d$ and a linear subspace $\eta\subset\mathbb{R}^d$ we denote by $\pi_\eta(B)$ the orthogonal projection of a B onto η. Often we shall use a projection onto a space $B^\perp$ which contains the vectors which are orthogonal to all vectors of the set $B\subset\mathbb{R}^d$. For the d-dimensional Lebesgue measure of a Borel set $B\subset\mathbb{R}^d$ we write $|B|_d$. Given $B\subset\xi$ for some $\xi\in\mathbb{G}(d,k)$, we shall write $|B|_k^\xi$ for its k-dimensional volume in the subspace ξ.

For the closed ball with radius r centered in a point $x\in\mathbb{R}^d$ we write $B_r^d(x)$, where the dimension is mostly omitted when it equals d. For the volume and surface area of the d-dimensional unit ball $B_1(\mathbf{o})$ we write κ_d and ω_d, respectively, where $\mathbf{o}$ denotes the origin.

For a convex set $K\in\mathcal{K}'$ and $x\in\mathbb{R}^d$, let $p(K,x)$ be the unique point in K which is the closest to x. Then there exist measures $\Phi_k(K,\cdot)$ on $\mathcal{B}(\mathbb{R}^d)$, for $k=0,\dots,d$, with

$$\Big|\{x\in K\oplus B_r(\mathbf{o}): p(K,x)\in B\}\Big|_d=\sum_{k=0}^{d} r^{d-k}\,\kappa_{d-k}\,\Phi_k(K,B),$$

where $K_1\oplus K_2=\{k_1+k_2 : k_1\in K_1, k_2\in K_2\}$ is the *Minkowski sum* of K_1 and K_2. One should remark that this formula is an extension of the so-called *Steiner formula*, see [SW08, Section 14.2]. Furthermore, we define $\Phi_k(\emptyset,B)=0$ for all $B\in\mathcal{B}(\mathbb{R}^d)$. These measures are called *curvature measures*. Since they are locally determined (cf. [SW08, Th. 14.2.3]), they can be extended to functionals with locally polyconvex sets as first argument in such a way that they remain additive. Note that these generalized curvature measures are not necessarily positive, but signed measures. For a detailed introduction, see [SW08]. The *intrinsic volumes* of K can be defined as total curvature measures $V_k^d(K)=\Phi_k(K,\mathbb{R}^d)$ for $k=0,\dots,d$. By $S(K)=2V_{d-1}^d(K)$ we denote the surface area of a set K.

2.2. Poisson cylinder processes

In this section, we introduce the most important model used in this thesis, the Poisson cylinder process (abbreviation: PCP). First, we define Poisson point processes on locally compact Hausdorff spaces. Then we introduce the notion of a cylinder and some associated notations. Finally, we turn to the very definition of a PCP. Since we introduce some random elements, we want to point out that throughout this thesis we assume that all random elements are defined on the common probability space

$(\Omega, \mathcal{A}, \mathbb{P})$. Further, we use the symbol $\mathbb{E}$ (Var) for the expectation value (variance) with respect to $\mathbb{P}$.

Poisson processes For defining a Poisson process on a locally compact Hausdorff space E with countable base, we closely follow [SW08, Section 3.2]. Poisson processes can also be defined on more general spaces, however, the notion presented here is sufficient for our purposes.

Definition 2.1 (point processes)**.** *Let E be a locally compact Hausdorff space with countable base and $N(E)$ be the set of all counting measures on E. Then we denote by $\mathcal{N}(E)$ the smallest σ-algebra which satisfies that for all $B \in \mathcal{B}(E)$ the mapping $N(E) \to \mathbb{N}_0 \cup \{\infty\}$, $\varphi \mapsto \varphi(B)$ is $(\mathcal{N}(E), \mathcal{B}(\mathbb{R}))$-measurable.*

*A **point process** X is a measurable mapping from $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(N(E), \mathcal{N}(E))$.*

*The measure Λ defined by $\Lambda(B) = \mathbb{E}\, X(B)$ is called the **intensity measure** of a point process X.*

A point process is called *locally finite* if the intensity measure Λ is locally finite, i.e., $\Lambda(C) < \infty$ for all compact sets $C \subset E$.

Definition 2.2 (Poisson processes)**.** *Let X be a point process on E with intensity measure Λ. Then X is called **Poisson**, if it satisfies the following two properties:*

(a) for any $B \in \mathcal{B}(E)$ with $\Lambda(B) < \infty$ we have $X(B) \sim \mathrm{Poi}(\Lambda(B))$,

(b) for any pairwise disjoint $B_1, \ldots, B_n \in \mathcal{B}(E)$ the induced random variables $X(B_1), \ldots, X(B_n)$ are independent.

A point process is called *simple*, if it has no multiple points. For Poisson processes this is the case if and only if Λ is *diffuse*, i.e., $\Lambda(\{e\}) = 0$ for all $e \in E$, see [SW08, Lemma 3.2.1].

Cylinders Following the approach introduced in [Wei87], we define a *cylinder* as the Minkowski sum of a flat $\xi \in \mathbb{G}(d, k)$ and a set $K \subset \xi^\perp$ with $K \in \mathcal{R}'$. Note that K is not limited to sets with an associated point in the origin. The flat ξ is also called the *direction space* of $\xi \oplus K$, and K is called the *cross section* or *base*. For a cylinder $Z = K \oplus \xi$ we define the functions $L(Z) = \xi$ and $K(Z) = K$. Furthermore, define $\mathcal{Z}_k$ as the set of all cylinders which have a k-dimensional direction space and base in $\mathcal{R}'$. Let $\mathcal{Z}_k^o$ be the set of all cylinders $Z \in \mathcal{Z}_k$ for which the midpoint of the circumsphere of $K(Z)$ lies in the origin. For the volume of the cross section of the cylinder we introduce the notation $A(Z) = |K(Z)|_{d-k}^{L(Z)}$. With a slight abuse of notation, we shall denote by $S(K)$ the surface area of $K = K(Z)$ in the space $L(Z)^\perp$ for $Z \in \mathcal{Z}_k$.

We consider the space $E = \mathcal{Z}_k$ as a subspace of $\mathcal{F}$ and equip it with the resulting trace topology. Thus, a measure φ on $\mathcal{Z}_k$ is *locally finite* if and only if for all $C \in \mathcal{C}$ we have $\varphi(\{Z \in \mathcal{Z}_k : Z \cap C \neq \emptyset\}) < \infty$, cf. [SW08, Lemma 2.3.1].

2.2.1. Cylinder processes as particle processes

Let $N(\mathcal{F})$ be the set of all locally finite counting measures on $\mathcal{F}$ endowed with the σ-algebra $\mathcal{N}(\mathcal{F})$ (cf. Definition 2.1). A point process Ξ on $\mathcal{F}$ which is a measurable mapping from the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(N(\mathcal{F}), \mathcal{N}(\mathcal{F}))$ with $\Xi \in N(\mathcal{Z}_k)$ almost surely is called a *cylinder process* (in the sense of particle processes). Its distribution is given by the probability measure $\mathbb{P}_\Xi : \mathcal{N}(\mathcal{F}) \to [0,1]$, $\mathbb{P}_\Xi(\cdot) = \mathbb{P}(\Xi \in \cdot)$.

In case of Ξ being a locally finite Poisson process, the union $U_\Xi = \cup_{Z \in \Xi} Z$ is a random closed set, see [SW08, p. 96], where we denote by $Z \in \Xi$ the cylinders Z in the support set of Ξ. The cylinder process Ξ is called *stationary* if its distribution is invariant with respect to translations in $\mathbb{R}^d$ and *isotropic* if it is invariant with respect to rotations about the origin.

Following [Wei87], we consider the following decomposition of Λ for locally finite stationary PCPs Ξ.

Proposition 2.1. *If Ξ is a locally finite stationary* PCP*, then there exist a number $\lambda < \infty$ and a probability measure θ on $\mathcal{Z}_k^o$, such that*

$$\Lambda(A) = \lambda \int_{\mathcal{Z}_k^o} \int_{L(Z)^\perp} \mathbb{1}_A(Z + x)\,\mathrm{d}x\,\theta(\mathrm{d}Z) \tag{2.1}$$

for all Borel sets $A \subset \mathcal{Z}_k$. λ is uniquely determined, and θ is unique for $\lambda > 0$.

Proof. The claim can be shown by generalizing the proof of [SW08, Theorem 4.4.1]. For a fixed linear subspace $U \in \mathbb{G}(d, d-k)$ we define

$$\begin{aligned} G_U &:= \{\xi \in \mathbb{G}(d,k) : \dim(\xi \cap U) = 0\}, \\ Z_U &:= \{Z \in \mathcal{Z}_k^o : L(Z) \in G_U\}, \quad \text{and} \\ A_U &:= \{Z + x : Z \in Z_U, x \in U\}. \end{aligned}$$

Furthermore, we consider the mapping $i\colon (x, Z) \mapsto x + Z$ for $x \in U$ and $Z \in Z_U$, which is a homeomorphism. For Borel sets $A \subset Z_U$, we introduce the measure

$$\eta_A(B) := \Lambda(i(B \times A)), \quad B \in \mathcal{B}(U).$$

Since Ξ is locally finite and stationary, η_A is a locally finite translation invariant measure, i.e., a multiple of the Lebesgue measure in U. We denote the factor by $\rho(A)$, which leads to $\Lambda(i(B \times A)) = |B|_{d-k}^U\,\rho(A)$. As ρ is a finite measure on Z_U, we get

$$i^{-1}(\Lambda)(B \times A) = (|\cdot|_{d-k}^U \otimes \rho)(B \times A).$$

Thus, for every non-negative measurable function f on $\mathcal{Z}_k$, we have

$$\int_{\mathcal{Z}_k} f(Z)\,\Lambda(\mathrm{d}Z) = \int_{Z_U}\int_U f(Z+x)\,\mathrm{d}x\,\rho(\mathrm{d}Z).$$

For $Z \in Z_U$, one can easily see that the projection $\pi_{L(Z)^\perp} : U \to L(Z)^\perp$ is a bijection. Thus, $|\cdot|_{d-k}^U = a(L(Z))\,|\pi_{L(Z)^\perp}(\cdot)|_{d-k}^{L(Z)^\perp}$ for some $a(L(Z)) > 0$ depending only on $L(Z)$. Since $f(Z+x) = f(Z+\pi_{L(Z)^\perp}(x))$, we have

$$\int_U f(Z+x)\,\mathrm{d}x = a(L(Z))\int_{L(Z)^\perp} f(Z+y)\,\mathrm{d}y.$$

We define a measure Λ_U on Z_U by $\Lambda_U(\mathrm{d}Z) = a(L(Z))\,\rho(\mathrm{d}Z)$, which leads to

$$\int_{A_U} f(Z)\,\Lambda(\mathrm{d}Z) = \int_{Z_U}\int_{L(Z)^\perp} f(Z+x)\,\mathrm{d}x\,\Lambda_U(\mathrm{d}Z).$$

We can interpret Λ_U as a measure on all $\mathcal{Z}_k^o$, with $\Lambda_U(\mathcal{Z}_k^o \setminus Z_U) = 0$, which leads to

$$\int_{A_U} f(Z)\,\Lambda(\mathrm{d}Z) = \int_{\mathcal{Z}_k^o}\int_{L(Z)^\perp} f(Z+x)\,\mathrm{d}x\,\Lambda_U(\mathrm{d}Z).$$

Since each set G_U, $U \in \mathbb{G}(d,d-k)$, is open in $\mathbb{G}(d,k)$, there are finitely many subspaces $U_1,\dots,U_m \in \mathbb{G}(d,d-k)$ with $\mathbb{G}(d,k) = \cup_{i=1}^m G_{U_i}$, resulting in $\mathcal{Z}_k^o = \cup_{i=1}^m Z_{U_i}$. Hence, the sets A_{U_i}, $i = 1,\dots,m$, cover $\mathcal{Z}_k$ and are invariant under translation. The translation invariant Borel sets $A_j := A_{U_j} \setminus (A_1 \cup \cdots \cup A_{j-1})$, $j = 1,\dots,m$, form a disjoint covering of $\mathcal{Z}_k$. Introducing the symbol $\llcorner$ for the restriction of a measure, the measure $\Lambda \llcorner A_i$ is translation invariant, and the measure $\Lambda_i := (\Lambda \llcorner A_i)_{U_i}$, defined as above, satisfies

$$\int_{A_i} f(Z)\,\Lambda(\mathrm{d}Z) = \int_{\mathcal{Z}_k^o}\int_{L(Z)^\perp} f(Z+x)\,\mathrm{d}x\,\Lambda_i(\mathrm{d}Z).$$

If we write the measure $\Lambda_0 := \Lambda_1 + \cdots + \Lambda_m$ as $\Lambda_0 = \lambda\,\theta$ for some probability measure θ and $\lambda \geq 0$, (2.1) is satisfied.

From (2.1) we obtain that, for $A \in \mathcal{B}(\mathcal{Z}_k^o)$,

$$\lambda\,\theta(A) = \frac{1}{\kappa_{d-k}}\Lambda\Big(\{Z = Z' + x : Z' \in A, x \in B_1(\mathbf{o})\}\Big).$$

This makes it obvious that λ is finite and uniquely determined, and θ is unique if $\lambda > 0$. $\square$

λ is called the *intensity* and θ the *shape distribution* of Ξ. We sometimes also write $\Xi_{\lambda,\theta}$ for Ξ to emphasize this connection. We shall often use the notation Z_0 for a cylinder with distribution θ. This can be regarded as a typical cylinder of Ξ with circumcenter of the base $K(Z_0)$ in the origin.

By a theorem on disintegration, see e.g. [LB95, Th. A2.2], θ can be decomposed further in the following way. There exist a probability measure α on $\mathcal{B}(\mathbb{G}(d,k))$ (directional distribution of Ξ) and a probability kernel $\beta\colon \mathcal{B}(\mathcal{R}^o) \times \mathbb{G}(d,k) \to [0,1]$ for which $\beta(\cdot,\xi)$ is concentrated on subsets of $\xi^\perp$ such that for Borel sets $A \subset \mathcal{Z}_k^o$ the equation

$$\theta(A) = \int_{\mathbb{G}(d,k)} \int_{\mathcal{R}^o \cap \xi^\perp} \mathbb{1}\{K \oplus \xi \in A\}\, \beta(\mathrm{d}K,\xi)\, \alpha(\mathrm{d}\xi) \tag{2.2}$$

holds.

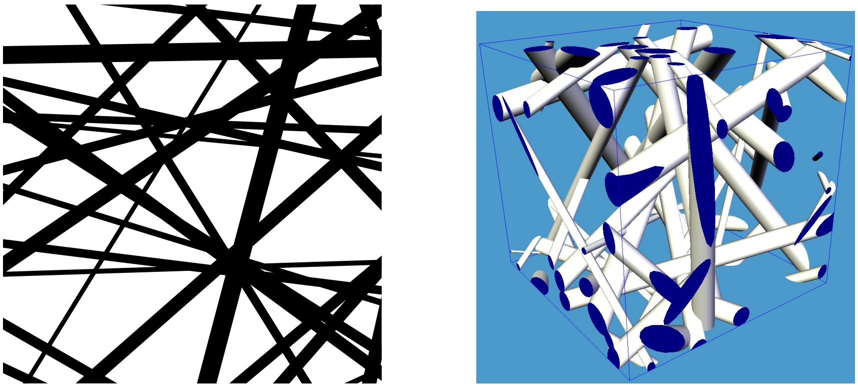


Figure 2.1.: Planar anisotropic and spatial isotropic PCP

Remark 2.1. *In the degenerate case $k = 0$ the union set U_Ξ coincides with the well-studied Boolean (or Poisson grain, Poisson blob, Swiss cheese) model in $\mathbb{R}^d$ with typical grain Z_0, see [Hal88], [Mat75], or [SKM95].*

Remark 2.2. *Another important special case is that of K being almost surely a point. Then the model coincides with a k-flat process.*

2.2.2. Cylinder processes induced by marked processes on $\mathbb{R}^{d-k}$

We present an alternative definition of a cylinder process. In Appendix A.1 we show that the definition from Section 2.2.1 and the one given here are equivalent up to a set of probability zero.

We begin with some notation. Let $\{e_1,\dots,e_d\}$ be the canonical unit vector base of $\mathbb{R}^d$. Furthermore, denote by $E_k = \operatorname{span}\{e_{d-k+1},\dots,e_d\}$ the subspace of $\mathbb{R}^d$ spanned by the last k base vectors. For each $\xi \in \mathbb{G}(d,k)$ there exists an equivalence class of orthogonal matrices defined as $\mathcal{O}_\xi = \{O \in \mathbb{SO}_d : O\,E_k = \xi\}$. We choose from each class the unique representative $O_\xi \in \mathcal{O}_\xi$ which is the lexicographically smallest. For the set of all such matrices we use the symbol $\mathbb{SO}_k^d = \{O_\xi : \xi \in \mathbb{G}(d,k)\}$ and equip it with the usual topology in the set of matrices. This is obviously a measurable set because of the continuity of the mapping onto the lexicographically smallest matrix. Further, let (Θ_0, Ξ_0) be a measurable mapping from $(\Omega, \mathcal{A}, \mathbb{P})$ into the product space $\Gamma_{d,k} = \mathbb{SO}_k^d \times \mathcal{R}_{d-k}^o$, where $\mathcal{R}_{d-k}^o$ is the subset of $\mathcal{R}'_{d-k}$ of all sets with circumcenter $\mathbf{o}$. The image measure $Q := \mathbb{P} \circ (\Theta_0, \Xi_0)^{-1}$ acting on the corresponding Borel product σ-field $\mathcal{B}(\Gamma_{d,k})$ determines the joint distribution of the (not necessarily independent) random elements Θ_0 and Ξ_0.

Remark 2.3. *Since* $\dim \mathbb{G}(d,k) = k(d-k)$*, the set* $\mathbb{SO}_k^d$ *can also be parametrized with a subset of* $\mathbb{R}^{k(d-k)}$*. In the special cases* $d=2, k=1$ *and* $d=3, k=1$*, alternative parameterizations are of the space* $\mathbb{SO}_k^d$ *are*

$$O(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad O(\theta_1,\theta_2) = \begin{pmatrix} \sin\theta_1 & \cos\theta_1\,\cos\theta_2 & \cos\theta_1\,\sin\theta_2 \\ -\cos\theta_1 & \sin\theta_1\,\cos\theta_2 & \sin\theta_1\,\sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix},$$

for $\theta \in [0,\pi)$ *resp.* $(\theta_1,\theta_2) \in [0,2\pi) \times [0, \pi/2]$*. In the dual case* $d=3, k=2$*, the first and the third column of* $O(\theta_1,\theta_2)$ *must be interchanged and the last column multiplied by* -1*.*

Now, we are in a position to introduce a stationary independently marked Poisson process $\Pi_{\lambda',Q} = \sum_{i\ge1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ with intensity λ' and mark distribution $Q(\cdot)$. $\Pi_{\lambda',Q}(\cdot)$ is a random locally finite counting measure (shift-invariant in the first component) on the Borel subsets of $\mathbb{R}^{d-k} \times \Gamma_{d,k}$ such that the numbers $\Pi_{\lambda',Q}(B \times L)$ are Poisson distributed with mean $\lambda'\,|B|_{d-k}\,Q(L)$ for any bounded $B \in \mathcal{B}(\mathbb{R}^{d-k})$ and $L \in \mathcal{B}(\Gamma_{d,k})$. This definition implies that the numbers of atoms of the unmarked Poisson process $\Pi_{\lambda'} = \sum_{i\ge1} \delta_{P_i}$ located in disjoint subsets of $\mathbb{R}^{d-k}$ are independent and the marks (Θ_i, Ξ_i) associated with the atoms P_i are i.i.d. (independent and identically distributed) copies of $(\Theta_0, \Xi_0) \sim Q$ and independent of $\Pi_{\lambda'}$.

Definition 2.3. *Given the Poisson point process* $\Pi_{\lambda',Q} = \sum_{i\ge1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ *with independent marks, satisfying the above assumptions, we introduce the notation*

$$\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda', Q) = \{\Theta_i((\Xi_i + P_i) \times \mathbb{R}^k), i \ge 1\} = \{Z_i, i \ge 1\} \tag{2.3}$$

with $Z_i = \Theta_i((\Xi_i + P_i) \times \mathbb{R}^k) \in \mathcal{Z}_k$ *for* $i \ge 1$*.*

It is shown in Appendix A.1, that $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda', Q)$ is a (measurable) simple stationary Poisson cylinder process as introduced in Section 2.2.1 if $\mathbb{E}\,|\Xi_0 \oplus B_\varepsilon^{d-k}(\mathbf{o})|_{d-k} < \infty$

for some $\varepsilon > 0$. In particular, it has a locally finite intensity measure. In this case, we can write $\Xi_{\lambda,\theta} \stackrel{d}{=} \Pi_{\mathrm{cyl}}^{(d,k)}(\lambda', Q)$ in the notation of the previous section.

It remains to show how λ and θ can be expressed by λ' and Q. Let $B \in \mathcal{B}(\mathcal{R}_{d-k}^o)$ and $S \in \mathcal{B}(\mathbb{SO}_k^d)$. Then the Theorems 12.3.5, 13.1.1, 13.2.1, and 13.2.2 in [SW08] yield that $M = \{\theta\,(R \times \mathbb{R}^k) : \theta \in S, R \in B\} \in \mathcal{B}(\mathcal{Z}_k^o)$ and also the related set $M' = \{Z + x : Z \in M, x \in B_1^d(\mathbf{o})\} \in \mathcal{B}(\mathcal{Z}_k)$ are Borel sets of cylinders with k-dimensional direction space in $\mathbb{R}^d$. With the notation from this section, we calculate

$$\begin{aligned}\Lambda(M') &= \mathbb{E}\,\#\{i \geq 1 : \Theta_i((\Xi_i + P_i) \times \mathbb{R}^k) \in M'\}\\ &= \mathbb{E}\,\#\{i \geq 1 : \Theta_i\,(\Xi_i \times \mathbb{R}^k) \in M, P_i \in B_1^{d-k}(\mathbf{o})\}\\ &= \lambda'\,|B_1^{d-k}(\mathbf{o})|_{d-k}\,\mathbb{P}(\Theta_0 \in S, \Xi_0 \in B)\\ &= \lambda'\,\kappa_{d-k}\,Q(S \times B),\end{aligned}$$

and on the other hand with the notation from the previous Section 2.2.1, Proposition 2.1 yields

$$\Lambda(M') = \lambda \int_{\mathcal{Z}_k^o} \int_{L(Z)^\perp} \mathbb{1}_{M'}(Z + x)\,\mathrm{d}x\,\theta(\mathrm{d}Z) = \lambda\,\kappa_{d-k}\,\theta(M).$$

For $B = \mathcal{R}_{d-k}^o$ and $S = \mathbb{SO}_k^d$ this gives us $\lambda = \lambda'$ (we write λ from now on). For arbitrary $B \in \mathcal{B}(\mathcal{R}_{d-k}^o)$ and $S \in \mathcal{B}(\mathbb{SO}_k^d)$ we have $\theta(\{OR : O \in S, R \in B\}) = Q(S \times B)$.

For the typical cylinder, this means that $Z_0 \stackrel{d}{=} \Theta_0(\Xi_0 \times \mathbb{R}^k)$. Further, with this notation we can write the random union set as

$$U_\Xi = \bigcup_{i \geq 1} \Theta_i((\Xi_i + P_i) \times \mathbb{R}^k). \tag{2.4}$$

2.3. Formulas for marked Poisson point processes

We need two important formulae for marked Poisson processes. We state them here in a form suitable for the process $\Pi_{\lambda,Q}$ introduced in the previous section. Each of them characterizes the distribution of $\Pi_{\lambda,Q}$ uniquely.

The *probability generating functional* $G_{\lambda,Q}(v) = \mathbb{E} \prod_{i \geq 1} v(P_i, \Theta_i, \Xi_i)$ of $\Pi_{\lambda,Q}$ takes the form

$$G_{\lambda,Q}(v) = \exp\Big\{ -\lambda \int_{\mathbb{R}^{d-k}} \int_{\Gamma_{d,k}} (1 - v(x, O, K))\,Q(\mathrm{d}(O, K))\,\mathrm{d}x \Big\} \tag{2.5}$$

for any measurable function $v : \mathbb{R}^{d-k} \times \Gamma_{d,k} \mapsto [0, 1]$ such that $1 - v(\cdot, O, K)$ has bounded support for $(O, K) \in \Gamma_{d,k}$.

The following formula is sometimes called the n-th order *Campbell formula* for marked Poisson processes. It is an immediate consequence of the Slivnyak-Mecke formula ([SW08, Cor. 3.2.3]). For any $n \in \mathbb{N}$ it reads

$$\mathbb{E} \sum_{i_1,\dots,i_n \geq 1}^{*} \prod_{j=1}^{n} f_j(P_{i_j}, \Theta_{i_j}, \Xi_{i_j}) = \lambda^n \prod_{j=1}^{n} \int_{\mathbb{R}^{d-k}} \int_{\Gamma_{d,k}} f_j(x, O, K)\, Q(\mathrm{d}(O,K))\, \mathrm{d}x \quad (2.6)$$

for non-negative measurable functions $f_1, \dots, f_n : \mathbb{R}^{d-k} \times \Gamma_{d,k} \mapsto \mathbb{R}^1$, where the sum $\sum^*$ on the left-hand side of (2.6) runs over all n-tuples of pairwise distinct indices $i_1, \dots, i_n \geq 1$, see also [DVJ08] or [SKM95].

2.4. Some basic facts about cumulants

Let us begin with the definition of the *mixed cumulant* $\mathrm{Cum}(X_1, \dots, X_n)$ (also called *semi-invariant*) of n random variables $X_1, \dots, X_n$ (all having a finite n-th moment). Following [LS59], we define

$$\mathrm{Cum}(X_1, \dots, X_n) = \mathrm{i}^{-n} \frac{\partial^n}{\partial s_1 \dots \partial s_n} \log \mathbb{E} \exp\left\{ \mathrm{i} \sum_{j=1}^{n} s_j X_j \right\}\Bigg|_{s_1=\dots=s_n=0}, \quad (2.7)$$

and $\mathrm{Cum}_n(X) = \mathrm{Cum}(X, \dots, X)$ (by setting $X = X_1 = \dots = X_n$ in (2.7)) denotes the usual n-th *cumulant* of X.

A direct calculation of the derivatives leads to the formula

$$\mathrm{Cum}(X_1, \dots, X_n) = \sum_{j=1}^{n} (-1)^{j-1} (j-1)! \sum_{N_1 \cup \dots \cup N_j = N} \prod_{i=1}^{j} \mathbb{E} \prod_{n_i \in N_i} X_{n_i}, \quad (2.8)$$

where the sum reaches over all disjoint subsets $N_1, \dots, N_j$ of $N = \{1, \dots, n\}$, see, e.g., [LS59], [SS91, p. 13], or [Hei07]. From this formula, it can easily be seen that $\mathrm{Cum}(X_1, \dots, X_n)$ is invariant under permutation of the indices $\{1, \dots, n\}$ and

$$\mathrm{Cum}(\dots, a\,X + b\,Y + c, \dots) = a\,\mathrm{Cum}(\dots, X, \dots) + b\,\mathrm{Cum}(\dots, Y, \dots) \quad (2.9)$$

in each component for any $a, b, c \in \mathbb{R}$, $n \geq 2$.

A random variable whose cumulants of order 3 and higher are 0 is necessarily normally distributed. Additionally, any sequence of random variables whose cumulants of order 3 and higher tend to 0 converges in distribution to a Gaussian random variable. Thus, in statistics and probability theory, cumulant estimates are mainly used to prove asymptotic Gaussianity of functionals of random processes (or fields) over expanding domains. For obtaining even rates of convergence for these limit theorems and exact large deviations probabilities based on cumulant estimates the reader is referred to the monograph [SS91]. Note that in finding optimal rates the corresponding estimation procedures are partly rather lengthy and sophisticated, see [Hei07] for an example.

In this thesis, we are mostly interested in the cumulants of the volume of U_Ξ, i.e., we consider the measurable $\{0,1\}$-valued random field, $\{\mathbb{1}_{U_\Xi}(x),\, x \in \mathbb{R}^d\}$. For $n \geq 2$, using (2.9) in each component of the cumulants, the mixed cumulant function $c_{U_\Xi}(x_1, \ldots, x_n) = \mathrm{Cum}(\mathbb{1}_{U_\Xi}(x_1), \ldots, \mathbb{1}_{U_\Xi}(x_n))$ can be written as

$$\begin{aligned} c_{U_\Xi}(x_1, \ldots, x_n) &= (-1)\ \mathrm{Cum}\left(1 - \mathbb{1}_{U_\Xi}(x_1), \mathbb{1}_{U_\Xi}(x_2), \ldots, \mathbb{1}_{U_\Xi}(x_n)\right) = \ldots \\ &= (-1)^n\ \mathrm{Cum}\left(1 - \mathbb{1}_{U_\Xi}(x_1), \ldots, 1 - \mathbb{1}_{U_\Xi}(x_n)\right) \\ &= (-1)^n\ \mathrm{Cum}(\mathbb{1}_{U_\Xi{}^c}(x_1), \ldots, \mathbb{1}_{U_\Xi{}^c}(x_n)). \end{aligned}$$

By combining the identities $|U_\Xi \cap B_i|_d = \int_{B_i} \mathbb{1}_{U_\Xi}(x_i)\, \mathrm{d}x_i$ for $i = 1, \ldots, n$ with the linearity of (2.7) in each component, we get

$$\begin{aligned} &\mathrm{Cum}\left(|U_\Xi \cap B_1|_d, \ldots, |U_\Xi \cap B_n|_d\right) \\ &= (-1)^n \int_{B_1} \cdots \int_{B_n} c_{U_\Xi{}^c}(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \end{aligned} \tag{2.10}$$

for any bounded $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$.

2.5. The method of the approximate inverse (AI)

The method of the approximate inverse is an approach to derive solutions for inverse (or ill-posed) linear problems. It was first mentioned in [LM90], see also [Sch07] for an introduction. We present some basics notions.

While the method is quite general, we only introduce it for two special operators. The first one is the *cosine transform* which can be defined as

$$(C\, f)(\eta) = \int_{S^{d-1}} |\langle \eta, \xi \rangle|\, f(\xi)\, \mathrm{d}\xi, \quad \eta \in S^{d-1}, \tag{2.11}$$

where f denotes an even measurable function on the unit sphere S^{d-1}, and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$.

The second is the so-called *spherical Radon transform*. For even measurable functions f on the unit sphere the latter is defined as

$$(R\, f)(\eta) = \frac{1}{\omega_{d-1}} \int_{S^{d-1} \cap \eta^\perp} f(\xi)\, \mathrm{d}\xi, \quad \eta \in S^{d-1},$$

where $\mathrm{d}\xi$ is the spherical surface area measure. The spherical Radon transform is frequently considered in tomography, for instance for the reconstruction of convex bodies from the area of their projections onto 2-dimensional subspaces, or in connection with intersection bodies, see also [Gar06].

An overview of the properties of R and C can be found in [Gar06, Appendix C] and [Gro96, Chapter 3].

Both transforms are closely related by $\Box C = R$ (see [GW92]). Here, $\Box$ is the so-called *block operator*, defined as

$$\Box = \frac{\Delta_{d-1} + d - 1}{2\omega_d}, \tag{2.12}$$

where Δ_{d-1} denotes the Beltrami-Laplace operator on the sphere.

Both transforms are self-adjoint. For the cosine transform, this follows directly from Fubini's theorem, for the Radon transform see [Gro96, p. 12].

Now, consider the following inverse problem: Given $T\,f$ for some even measurable function f on the unit sphere, where T is either R or C, we want to approximate f in a numerically stable way. The idea of the method of the approximate inverse is to calculate a "smoothed version" of f, denoted by f_γ for some $\gamma > 0$, which is defined as

$$f_\gamma(\eta) = \int_{S^{d-1}} f(\xi)\, e_\gamma(\eta, \xi)\, \mathrm{d}\xi, \qquad \eta \in S^{d-1},$$

where we assume that

$$f_\gamma \to f \quad \text{as} \quad \gamma \to 0 \qquad \text{(cf. Remark 2.5)}. \tag{2.13}$$

For the so-called *mollifier* $e_\gamma \in L^2_{\mathrm{e}}(S^{d-1} \times S^{d-1})$ it is demanded that

$$\int_{S^{d-1}} e_\gamma(\eta, \xi)\, \mathrm{d}\xi = 1, \qquad \eta \in S^{d-1}.$$

Here and in the following, even functions are denoted by the index "e", in this case $L^2_{\mathrm{e}}(S^{d-1} \times S^{d-1})$ is the space of all square integrable functions which are even in both components.

For a given mollifier, we define the *reconstruction kernel* ψ_γ as the solution of $e_\gamma = T\,\psi_\gamma$. Since both transforms C and R are self-adjoint, this allows us to write the smoothed density for $\eta \in S^{d-1}$ as

$$f_\gamma(\eta) = \langle f, e_\gamma(\eta, \cdot)\rangle_{L^2(S^{d-1})} = \langle f, T\,\psi_\gamma(\eta, \cdot)\rangle_{L^2(S^{d-1})} = \langle T\,f, \psi_\gamma(\eta, \cdot)\rangle_{L^2(S^{d-1})}, \tag{2.14}$$

where $\langle \cdot, \cdot \rangle_{L^2(S^{d-1})}$ denotes the scalar product in $L^2(S^{d-1})$. The last term is the inner product of two known functions, namely the given data $T\,f$ and the reconstruction kernel.

Remark 2.4. *We want to point out that this method also has computational advantages. The reconstruction kernel can be calculated in advance (independent of the given data Tf), and thus the approximate inversion for a function only requires the calculation of an inner product.*

Remark 2.5. *One can easily see that convergence in* (2.13) *holds pointwise and in the L^2-sense if f is continuous, and the support of the mollifier is contained in a ball whose radius tends to zero as γ tends to zero. For a more precise characterization of the convergence see [Rub02].*

3. Characteristics of Poisson cylinder processes

This chapter is based on the results in [SS11]. Here, we assume that we have a simple stationary Poisson cylinder process Ξ with locally finite intensity measure Λ and the measure θ as introduced in Section 2.2. The local finiteness of Λ guarantees that the union set U_Ξ is closed, see [SW08, Theorem 3.6.2]. Furthermore, we use the notation $Z_0 : \Omega \to \mathcal{Z}_k^o$ for the typical cylinder with distribution θ.

This chapter is organized as follows. In the next section, we derive the capacity functional of U_Ξ and some consequential formulae, namely the covariance function and the contact distribution function. In Section 3.2 we find an expression for the specific surface area, and in the final Section 3.2, we introduce and solve a practical optimization problem for cylinder processes.

3.1. Capacity functional and related characteristics

In this section, we calculate the capacity functional (cf. [SKM95, p. 195]) for the union set U_Ξ of the stationary Poisson process Ξ of cylinders with k-dimensional direction space. As a corollary, explicit formulae for the volume fraction, the covariance function, and the contact distribution function of U_Ξ follow easily. It is worth mentioning that the resulting formula (3.1) for the capacity functional generalizes the formula in [Ser84, pp. 572–573], given for Poisson slices in $\mathbb{R}^3$, and a model with this capacity functional has already been proposed in [Mat75, p. 148] for a process with convex cylinder bases.

3.1.1. Capacity functional

For any random closed set X, the *capacity functional* $T_X(B) = \mathbb{P}(X \cap B \neq \emptyset)$, $B \in \mathcal{C}$, determines uniquely the distribution of X, see [Mol05] for a proof.

Lemma 3.1. *The capacity functional of the union set U_Ξ of the cylinder process Ξ is given by*

$$T_{U_\Xi}(B) = 1 - \exp\Big\{ -\lambda\, \mathbb{E}\,\big| - K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)\big|_{d-k}^{L(Z_0)^\perp} \Big\}. \tag{3.1}$$

Proof. Let B be a compact set in $\mathbb{R}^d$. Then by Fubini's theorem and [SW08, p. 96] we get

$$\begin{aligned}
1 - T_{U_\Xi}(B) &= \exp\big\{ -\Lambda(\{Z \in \mathcal{Z}_k : Z \cap B \neq \emptyset\})\big\} \\
&= \exp\Big\{ -\int_{\mathcal{Z}_k} \mathbb{1}\{\tilde{Z} \cap B \neq \emptyset\}\, \Lambda(\mathrm{d}\tilde{Z})\Big\} \\
&= \exp\Big\{ -\lambda \int_{\mathcal{Z}_k^o} \int_{L(Z)^\perp} \mathbb{1}\Big\{(Z + x) \cap B \neq \emptyset\Big\}\, \mathrm{d}x\, \theta(\mathrm{d}Z)\Big\} \\
&= \exp\Big\{ -\lambda\, \mathbb{E} \int_{L(Z_0)^\perp} \mathbb{1}\Big\{(K(Z_0) + x) \cap \pi_{L(Z_0)^\perp}(B) \neq \emptyset\Big\}\, \mathrm{d}x\Big\}.
\end{aligned}$$

One can easily see that $K(Z_0) + x$ hits $\pi_{L(Z_0)^\perp}(B)$ if and only if x belongs to the Minkowski sum of $-K(Z_0)$ and $\pi_{L(Z_0)^\perp}(B)$.

Thus, we have

$$\begin{aligned}
1 - T_{U_\Xi}(B) &= \exp\Big\{ -\lambda\, \mathbb{E} \int_{L(Z_0)^\perp} \mathbb{1}\Big\{(K(Z_0) + x) \cap \pi_{L(Z_0)^\perp}(B) \neq \emptyset\Big\}\, \mathrm{d}x\Big\} \\
&= \exp\Big\{ -\lambda\, \mathbb{E} \int_{L(Z_0)^\perp} \mathbb{1}\Big\{x \in -K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)\Big\}\, \mathrm{d}x\Big\} \\
&= \exp\Big\{ -\lambda\, \mathbb{E}\, \big| -K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)\big|_{d-k}^{L(Z_0)^\perp}\Big\}.
\end{aligned}$$

□

Remark 3.1. *A few remarks are in order.*

(a) *Another interesting consequence of the proof of Lemma 3.1 is that*

$$\mathbb{E}\, \big| -K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)\big|_{d-k}^{L(Z_0)^\perp} = \Lambda(\{Z \in \mathcal{Z}_k : Z \cap B \neq \emptyset\}) < \infty, \qquad (3.2)$$

where the last inequality follows from the local finiteness of Λ.

(b) *As mentioned in Remark 2.1, the choice of* $k = 0$ *leads to the stationary Boolean Model* Ξ' *with the primary grain* K *and intensity* λ. *Here, our formula coincides with the well-known formula for the capacity functional, cf. [Mat75, p. 62]:*

$$T_{\Xi'}(B) = 1 - \exp\big\{ -\lambda\, \mathbb{E}\, | -K \oplus B|_d\big\}.$$

(c) *In case of* K *being almost surely a point, we have a* k*-flat process* Ξ'' *(cf. Remark 2.2). Here, again our formula for the capacity functional coincides with the well-known formula, cf. [Mat75, p. 67], namely*

$$T_{\Xi''}(B) = 1 - \exp\Big\{ -\lambda\, \mathbb{E}\, |\pi_{L(Z_0)^\perp}(B)|_{d-k}^{L(Z_0)^\perp}\Big\}.$$

(d) *For $d = 3, k = 1$ we get*

$$\mathbb{E}\,|-K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)|_2^{L(Z_0)^\perp} = \mathbb{E}\,A(Z_0) + \frac{1}{2\pi}\,\mathbb{E}\Big[S(K(Z_0))\,S(\pi_{L(Z_0)^\perp}(B)) + |\pi_{L(Z_0)^\perp}(B)|_2^{L(Z_0)^\perp}\Big],$$

where $A(Z) = |K(Z)|_2^{L(Z)^\perp}$, and $S(K)$ denotes the boundary length (or the surface area) of K.

(e) *For all $d \geq 2$ and $k = d-1$ we have*

$$\mathbb{E}\,|-K(Z_0) \oplus \pi_{L(Z_0)^\perp}(B)|_1^{L(Z_0)^\perp} = \mathbb{E}\Big[A(Z_0) + |\pi_{L(Z_0)^\perp}(B)|_1^{L(Z_0)^\perp}\Big].$$

(f) *The case of $B = \{\mathbf{o}\}$ yields the volume fraction $p = \mathbb{P}(\mathbf{o} \in U_\Xi) = \mathbb{E}\,|U_\Xi \cap [0,1]^d|_d$ of U_Ξ:*

$$p = T_{U_\Xi}(\{\mathbf{o}\}) = 1 - \exp\{-\lambda\,\mathbb{E}\,A(Z_0)\}\,. \tag{3.3}$$

A variant of this formula can also be found in [Hof09b] in the non-stationary setting.

In this chapter, we assume that $p > 0$, i.e., $\mathbb{E}\,A(Z_0) > 0$. Thus, we have $p \in (0,1)$, cf. inequality (3.2).

3.1.2. Covariance function

In the following we investigate the covariance function of U_Ξ. It is defined as $C_{U_\Xi}(h) = \mathbb{P}(\mathbf{o}, h \in U_\Xi)$, $h \in \mathbb{R}^d$, cf. [SKM95, p. 68].

Because of the relation $C_{U_\Xi}(h) = \mathbb{P}(\mathbf{o}, h \in U_\Xi) = 2p - T_{U_\Xi}(\{\mathbf{o}, h\})$ it is closely connected with the capacity functional of the set $B = \{\mathbf{o}, h\}$, which is

$$T_{U_\Xi}(\{\mathbf{o}, h\}) = 1 - \exp\Big\{-\lambda\,\mathbb{E}\,|\{\mathbf{o}, \pi_{L(Z_0)^\perp}(h)\} \oplus -K(Z_0)|_{d-k}^{L(Z_0)^\perp}\Big\}. \tag{3.4}$$

Let γ_A denote the *covariogram* of a measurable set $A \subset L(Z)^\perp$, $Z \in \mathcal{Z}_k$, defined by

$$\gamma_A(x) = |A \cap (A - x)|_{d-k}^{L(Z)^\perp}$$

for $x \in L(Z)^\perp$.

Lemma 3.2. *For $h \in \mathbb{R}^d$ we have*

$$C_{U_\Xi}(h) = 1 - 2\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)} + \exp\Big\{-2\lambda\,\mathbb{E}\,A(Z_0) + \lambda\,\mathbb{E}\,\gamma_{K(Z_0)}(\pi_{L(Z_0)^\perp}(h))\Big\}. \tag{3.5}$$

Proof. Consider the term $\{\mathbf{o}, \pi_{L(Z)^\perp}(h)\} \oplus -K(Z) = -K(Z) \cup [\pi_{L(Z)^\perp}(h) - K(Z)]$ for some $Z \in \mathcal{Z}_k$. Its volume is equal to

$$\begin{aligned}
&\big| - K(Z) \oplus \{\mathbf{o}, \pi_{L(Z)^\perp}(h)\}\big|_{d-k}^{L(Z)^\perp} \\
&= A(Z) + \big| - \pi_{L(Z)^\perp}(h) \oplus K(Z)\big|_{d-k}^{L(Z)^\perp} - \Big|K(Z) \cap \big[- \pi_{L(Z)^\perp}(h) \oplus K(Z)\big]\Big|_{d-k}^{L(Z)^\perp} \\
&= 2A(Z) - \Big|K(Z) \cap \big[K(Z) - \pi_{L(Z)^\perp}(h)\big]\Big|_{d-k}^{L(Z)^\perp} \\
&= 2A(Z) - \gamma_{K(Z)}(\pi_{L(Z)^\perp}(h)).
\end{aligned}$$

Using equations (3.3) and (3.4), the covariance $C_{U_\Xi}(h)$ rewrites

$$\begin{aligned}
C_{U_\Xi}(h) &= 2p - T_{U_\Xi}(\{\mathbf{o}, h\}) \\
&= 1 - 2\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)} + \exp\Big\{ - \lambda\,\mathbb{E}\,\big| - K(Z_0) \oplus \{\mathbf{o}, \pi_{L(Z_0)^\perp}(h)\}\big|_{d-k}^{L(Z_0)^\perp}\Big\} \\
&= 1 - 2\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)} + \exp\big\{ - 2\lambda\,\mathbb{E}\,A(Z_0) + \lambda\,\mathbb{E}\,\gamma_{K(Z_0)}(\pi_{L(Z_0)^\perp}(h))\big\}. \qquad \square
\end{aligned}$$

Example. In the following, we give an example of a cylinder process in two dimensions with cylinders of constant thickness $2a$ where the expectations in (3.5) can be calculated explicitly.

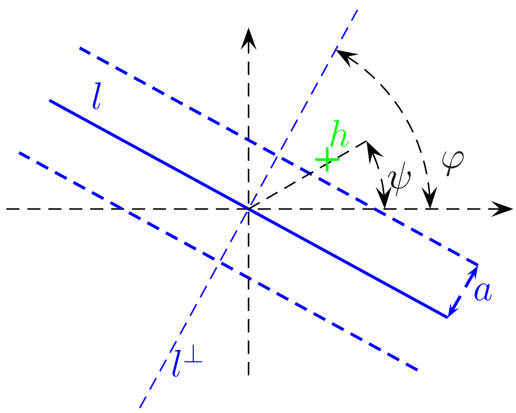


Figure 3.1.: Calculation of the covariance in 2D - sketch of the points and lines

Let $l \in \mathbb{G}(2,1)$ be an arbitrary line through the origin, φ the angle between the x-axis and $l^\perp$, and $h = (r, \psi)$ a vector in polar coordinates (see Figure 3.1). We use the notation $B^1_a(\mathbf{o}) \times \varphi$ with $\varphi \in [0, \pi)$ for a cylinder with radius a and direction space l. Since $|\pi_{l^\perp}(h)| = r|\cos(\varphi - \psi)|$, formula (3.5) rewrites

$$C_{U_\Xi}(h) = 1 - 2\mathrm{e}^{-2\lambda a} + \mathrm{e}^{-4\lambda a + \lambda I},$$

where

$$\begin{aligned} I &= \int_0^\pi (2a - |\pi_{l^\perp}(h)|)\mathbb{1}\{|\pi_{l^\perp}(h)| \le 2a\}\, \theta(B^1_a(\mathbf{o}) \times \mathrm{d}\varphi) \\ &= \int_{\varphi \in [0,\pi] : |\cos(\varphi - \psi)| \le \frac{2a}{r}} (2a - r|\cos(\varphi - \psi)|)\, \theta(B^1_a(\mathbf{o}) \times \mathrm{d}\varphi). \end{aligned}$$

In the isotropic case ($\theta(B^1_a(\mathbf{o}) \times \mathrm{d}\varphi) = {}^{\mathrm{d}\varphi}/_{\pi}$) we can choose ψ arbitrarily, for example $\psi = {}^{\pi}/_{2}$. This yields

$$I = \int_{\varphi \in [0,\pi] : \sin\varphi \le \frac{2a}{r}} (2a - r\sin\varphi) \frac{\mathrm{d}\varphi}{\pi}.$$

In case $r \le 2a$ this simplifies to $I = 2a - r\int_0^\pi \sin\varphi \frac{\mathrm{d}\varphi}{\pi} = 2a - r\frac{2}{\pi}$. For $r > 2a$ we get

$$\begin{aligned} I &= \frac{2a}{\pi}\bigg(\int_0^{\arcsin\frac{2a}{r}} \mathrm{d}\varphi + \int_{\pi - \arcsin\frac{2a}{r}}^{\pi} \mathrm{d}\varphi \bigg) + \\ &\quad + \frac{r}{\pi}\bigg(\int_0^{\arcsin\frac{2a}{r}} (-\sin\varphi)\, \mathrm{d}\varphi + \int_{\pi - \arcsin\frac{2a}{r}}^{\pi} (-\sin\varphi)\, \mathrm{d}\varphi \bigg) \\ &= \frac{4a}{\pi} \arcsin\left(\frac{2a}{r}\right) + \frac{2r}{\pi}\left(\cos\left(\arcsin\frac{2a}{r}\right) - 1\right) \\ &= 2a - \frac{4a}{\pi}\arccos\left(\frac{2a}{r}\right) - \frac{2r}{\pi}\left(1 - \sqrt{1 - \left(\tfrac{2a}{r}\right)^2}\right), \end{aligned}$$

where we have applied that $\cos(\arcsin x) = \sqrt{1 - \sin^2 \arcsin x} = \sqrt{1 - x}$ and $\arcsin x = \frac{\pi}{2} - \arccos x$. This gives us the final formula

$$C_{U_\Xi}(h) = \begin{cases} 1 - 2\mathrm{e}^{-2\lambda a} + \mathrm{e}^{-2\lambda a - \frac{2\lambda r}{\pi}}, & \text{if } r \le 2a, \\ 1 - 2\mathrm{e}^{-2\lambda a} + \exp\bigg\{ -2\lambda a - \frac{\lambda}{\pi}\bigg(4a\arccos\left(\frac{2a}{r}\right) + & \\ \quad +2r\left(1 - \sqrt{1 - \frac{4a^2}{r^2}}\right)\bigg)\bigg\}, & \text{if } r > 2a. \end{cases}$$

The first derivative of $C_{U_\Xi}(h)$ will be needed later for the calculation of the intensity $\overline{S}_\Xi$ of the surface area measure of U_Ξ. For linear subspaces ξ and η, which span a subspace of dimension m, we use the notation $[\xi, \eta]$ for the m-volume of the parallelepiped spanned over the orthonormal bases of ξ and η. $[\xi, \eta]$ is called the *subspace determinant* of ξ and η, see [SW08, Ch. 14.1]. We shall also write $[x, \eta]$ for $[\xi, \eta]$ if ξ is the line spanned by x.

We need some further notation. Denote by $\partial_{\mathrm{e}} A$ the *essential boundary* of a measurable set $A \subset \mathbb{R}^d$, i.e., the set of all points which are neither Lebesgue density points of A nor of A^c. With the *projection measure*, which is defined for $\xi \in \mathbb{R}^d \setminus \{\mathbf{o}\}$ and measurable $A \subset \mathbb{R}^d$ as

$$\mu_\xi(A) = \int_{\xi^\perp} \mathcal{H}^0\big(A \cap (x + \mathbb{R}\,\xi)\big)\,\mathrm{d}x,$$

we can introduce the function $V_\xi(A) = \mu_\xi(\partial_{\mathrm{e}} A)$. For the special case of $A \in \mathcal{K}$, we have $V_\xi(A) = 2\pi_{\xi^\perp}(A)$.

Proposition 3.1. *Suppose that Ξ is a simple stationary Poisson cylinder process. Then the derivative of the covariance function in direction h at the origin is given by*

$$C'_{U_\Xi}(\mathbf{o}, h) = \tfrac{\lambda}{2}\,\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)}\,\mathbb{E}\,V_{\pi_{L(Z_0)^\perp}(h)}(K(Z_0))\,[h, L(Z_0)],$$

where $\gamma'_A(\mathbf{o}, \eta)$ denotes the derivative of γ_A at the origin in direction η.

Proof. For $x \in \mathbb{R}$ we calculate

$$\begin{aligned}
&\frac{\partial}{\partial x} C_{U_\Xi}(x\,h)\Big|_{x=0} \\
&= \frac{\partial}{\partial x}\Big(1 - 2\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)} + \exp\Big\{-2\lambda\,\mathbb{E}\,A(Z_0) + \lambda\,\mathbb{E}\,\gamma_{K(Z_0)}\big(\pi_{L(Z_0)^\perp}(x\,h)\big)\Big\}\Big)\Big|_{x=0} \\
&= \mathrm{e}^{-2\lambda\,\mathbb{E}\,A(Z_0)}\,\frac{\partial}{\partial x}\exp\Big\{\lambda\,\mathbb{E}\,\gamma_{K(Z_0)}\big(x\,\pi_{L(Z_0)^\perp}(h)\big)\Big\}\Big|_{x=0} \\
&= \mathrm{e}^{-2\lambda\,\mathbb{E}\,A(Z_0)}\,\exp\Big\{\lambda\,\mathbb{E}\,\gamma_{K(Z_0)}\big(0\,\pi_{L(Z_0)^\perp}(h)\big)\Big\}\,\frac{\partial}{\partial x}\lambda\,\mathbb{E}\,\gamma_{K(Z_0)}\big(x\,\pi_{L(Z_0)^\perp}(h)\big)\Big|_{x=0} \\
&= \mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)}\,\frac{\partial}{\partial(x/\|\pi_{L(Z_0)^\perp}(h)\|)}\lambda\,\mathbb{E}\,\gamma_{K(Z_0)}\Big(\frac{x\,\pi_{L(Z_0)^\perp}(h)}{\|\pi_{L(Z_0)^\perp}(h)\|}\Big)\Big|_{x=0} \\
&= \lambda\,\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)}\,\|\pi_{L(Z_0)^\perp}(h)\|\,\mathbb{E}\,\gamma'_{K(Z_0)}\big(\mathbf{o}, \pi_{L(Z_0)^\perp}(h)\big),
\end{aligned}$$

where $\|\pi_{L(Z_0)^\perp}(h)\| = [h, L(Z_0)]$. The claim follows from [Gal11, Th. 13], where it is shown that $\gamma'_A(\mathbf{o}, x) = \frac{1}{2}V_x(A)$ for any measurable $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$. $\square$

3.1.3. Contact distribution function

Let B be an arbitrary compact set with $\mathbf{o} \in B$ (called the structuring element), and let $r > 0$. Then the contact distribution function with structuring element B of the union set of the stationary Poisson cylinder process Ξ with volume fraction

$p \in (0,1)$ is defined as $H_B(r) = \mathbb{P}(U_\Xi \cap r\,B \neq \emptyset \mid \mathbf{o} \notin U_\Xi)$, cf. [SKM95, p. 71]. It can be calculated as follows:

$$\begin{aligned}
H_B(r) &= 1 - \frac{\mathbb{P}(U_\Xi \cap r\,B = \emptyset)}{1-p} = 1 - \frac{1 - T_{U_\Xi}(rB)}{1-p} \\
&= 1 - \frac{\exp\Big\{ -\lambda\,\mathbb{E}\,|-K(Z_0) \oplus \pi_{L(Z_0)^\perp}(r\,B)|_{d-k}^{L(Z_0)^\perp} \Big\}}{\exp\{-\lambda\,\mathbb{E}\,A(Z_0)\}} \\
&= 1 - \exp\Big\{ -\lambda\,\mathbb{E}\Big[|-K(Z_0) \oplus \pi_{L(Z_0)^\perp}(r\,B)|_{d-k}^{L(Z_0)^\perp} - A(Z_0)\Big]\Big\}.
\end{aligned} \tag{3.6}$$

Further simplification of this formula is possible in some special cases.

Consider the contact distribution function H_B with B being a line segment between the origin and a unit vector η. In this special case the contact distribution function is called *linear*. With a slight abuse of notation we shall use a vector to represent the line segment between the origin and the endpoint of the vector. It will be clear from the context whether the vector or the line segment is meant.

Lemma 3.3. *If the probability kernel $\beta(\cdot,\xi)$ (cf. (2.2)) is concentrated on convex bodies and isotropic in the first argument for all $\xi \in \mathbb{G}(d,k)$, then for a unit vector η the linear contact distribution function of U_Ξ is given by*

$$H_\eta(r) = 1 - \mathrm{e}^{-\lambda\,r\,C_o(\eta)} \tag{3.7}$$

with

$$C_o(\eta) = c_{d,k} \int_{\mathbb{G}(d,k)} \int_{\mathcal{K}^o \cap \xi^\perp} S(K)\,\beta(\mathrm{d}K,\xi)\,[\xi,\eta]\,\alpha(\mathrm{d}\xi),$$

$c_{d,k} = \frac{\omega_{d-k+1}}{2\pi\,\omega_{d-k}}$, and $\mathcal{K}^o \cap \xi^\perp$ denotes the family of all convex bodies in $\xi^\perp$ with circumcenter in the origin.

Proof. [1] It follows from (3.6) that (3.7) holds if and only if

$$r\,C_o(\eta) = \mathbb{E}\Big[|-K(Z) \oplus \pi_{L(Z)^\perp}(r\,\eta)|_{d-k}^{L(Z)^\perp} - A(Z)\Big].$$

Using the notation introduced in [Sch93, p. 275–279] for mixed volumes (here all mixed volumes and surface measures are with respect to $L(Z)^\perp$) we calculate with [Sch93, Theorem 5.1.6] and [Sch93, Theorem 5.1.7]

$$\begin{aligned}
&|-K(Z) \oplus \pi_{L(Z)^\perp}(r\,\eta)|_{d-k}^{L(Z)^\perp} - A(Z) \\
&= (d-k)V(\pi_{L(Z)^\perp}(r\,\eta), K(Z), \dots, K(Z)) \\
&= \frac{r}{2} \int_{S^{d-1} \cap L(Z)^\perp} |\langle u, \pi_{L(Z)^\perp}(\eta)\rangle|\, S_{d-k-1}(K(Z), \mathrm{d}u),
\end{aligned}$$

[1] The idea of this proof goes back to an anonymous referee.

where $\langle\cdot,\cdot\rangle$ denotes the scalar product, and $S_{d-k-1}(K(Z),\cdot)$ is the surface area measure of $K(Z)$ in $L(Z)^\perp$.

Thus,

$$\begin{aligned} C_o(\eta) &= \frac{1}{2}\int_{\mathcal{Z}_k^o}\int_{S^{d-1}\cap L(Z)^\perp} |\langle u, \pi_{L(Z)^\perp}(\eta)\rangle|\, S_{d-k-1}(K(Z),\mathrm{d}u)\,\theta(\mathrm{d}Z) \\ &= \frac{1}{2}\int_{\mathbb{G}(d,k)}\int_{\mathcal{K}^o\cap\xi^\perp}\int_{S^{d-1}\cap\xi^\perp} |\langle u, \pi_\xi(\eta)\rangle|\, S_{d-k-1}(K,\mathrm{d}u)\,\beta(\mathrm{d}K,\xi)\,\alpha(\mathrm{d}\xi). \end{aligned}$$

Because of the rotation invariance of $\beta(\cdot,\xi)$, the value of the integral does not change if we replace K with ϑK for an arbitrary rotation ϑ in $\xi^\perp$. Furthermore, we get the following equation since the surface area measure is invariant with respect to rotations when they are applied to both arguments.

$$\int_{S^{d-1}\cap\xi^\perp} |\langle u, \pi_\xi(\eta)\rangle|\, S_{d-k-1}(\vartheta K,\mathrm{d}u) = \int_{S^{d-1}\cap\xi^\perp} |\langle \vartheta u, \pi_\xi(\eta)\rangle|\, S_{d-k-1}(K,\mathrm{d}u).$$

Thus, integration over the group $\mathbb{SO}(\xi^\perp)$ of rotations in $\xi^\perp$ equipped with the Haar probability measure leads to

$$\begin{aligned} &\int_{S^{d-1}\cap\xi^\perp} |\langle u, \pi_\xi(\eta)\rangle|\, S_{d-k-1}(K,\mathrm{d}u) \\ &= \int_{\mathbb{SO}(\xi^\perp)}\int_{S^{d-1}\cap\xi^\perp} |\langle u, \pi_\xi(\eta)\rangle|\, S_{d-k-1}(K,\mathrm{d}u)\,\mathrm{d}\vartheta \\ &= \int_{S^{d-1}\cap\xi^\perp}\int_{\mathbb{SO}(\xi^\perp)} |\langle \vartheta u, \pi_\xi(\eta)\rangle|\,\mathrm{d}\vartheta\, S_{d-k-1}(K,\mathrm{d}u) \\ &= 2c_{d,k}\, S(K)\,[\xi,\eta], \end{aligned}$$

where $c_{d,k}$ is the constant from the claim, and we used [Spo02, Corollary 5.2] for the last equality.

This leads to

$$C_o(\eta) = c_{d,k}\int_{\mathbb{G}(d,k)}\int_{\mathcal{K}\cap\xi^\perp} S(K)\,\beta(\mathrm{d}K,\xi)\,[\xi,\eta]\,\alpha(\mathrm{d}\xi).$$

□

Now let the structuring element B be the ball $B_1(\mathbf{o})$. In this case the contact distribution function is called *spherical*. It is obvious that $\pi_{L(Z)^\perp}(B_r(\mathbf{o}))$ is a ball of radius r in the $(d-k)$-dimensional subspace $L(Z)^\perp$. If $K(Z)$ is almost surely convex, then the use of the classical Steiner formula leads to

$$\mathbb{E}\,|-K(Z_0)\oplus\pi_{L(Z_0)^\perp}(B_r(\mathbf{o}))|_{d-k}^{L(Z_0)^\perp} \;=\; \mathbb{E}\,A(Z_0) + \sum_{i=1}^{d-k}\kappa_i\,\mathbb{E}\,V_{d-k-i}^{d-k}(K(Z_0))\,r^i,$$

which yields

$$H_{B_1(\mathbf{o})}(r) = 1 - \exp\Big\{ -\lambda \sum_{i=1}^{d-k} \kappa_i\, r^i\, \mathbb{E}\, V_{d-k-i}^{d-k}(K(Z_0))\Big\}.$$

Example. In what follows, the case of dimensions two and three is considered in detail. It is assumed that the conditions of Lemma 3.3 hold.

(a) For $d = 2, k = 1$ Lemma 3.3 yields

$$C_o(\eta) = c_{2,1} \int_{\mathbb{G}(2,1)} \int_{\mathcal{K}^o \cap \xi^\perp} S(K)\, \beta(\mathrm{d}K, \xi)\, [\xi, \eta]\, \alpha(\mathrm{d}\xi) = \int_{\mathbb{G}(2,1)} 2[\xi, \eta]\, \alpha(\mathrm{d}\xi).$$

Hence, it holds $H_\eta(r) = 1 - \exp\left\{-2\lambda\, r \int_{\mathbb{G}(2,1)} [\xi, \eta]\, \alpha(\mathrm{d}\xi)\right\}$, and so $H_\eta(r)$ does not depend on $K(Z)$.

And for the structuring element being $B = B_1^2(\mathbf{o})$ one gets

$$H_{B_1^2(\mathbf{o})}(r) = 1 - \exp\{ -2\lambda\, r\, \mathbb{E}\, V_0^1(K(Z_0))\} = 1 - \mathrm{e}^{-2\lambda\, r}.$$

Interestingly the result does not depend on the distribution of the base.

(b) For $d = 3, k = 1$ we get

$$C_o(\eta) = \frac{2}{\pi} \int_{\mathbb{G}(3,1)} \int_{\mathcal{K}^o \cap \xi^\perp} S(K)\, \beta(\mathrm{d}K, \xi)\, [\xi, \eta]\, \alpha(\mathrm{d}\xi)$$

which yields

$$H_\eta(r) = 1 - \exp\Big\{ -\frac{2\lambda\, r}{\pi} \int_{\mathbb{G}(3,1)} \int_{\mathcal{K}^o \cap \xi^\perp} S(K)\, \beta(\mathrm{d}K, \xi)\, [\xi, \eta]\, \alpha(\mathrm{d}\xi)\Big\}.$$

For $K(Z) = B_a^2(\mathbf{o})$ we have

$$C_o(\eta) = \frac{2\pi\, a}{\pi} \int_{\mathbb{G}(3,1)} [\xi, \eta]\, \alpha(\mathrm{d}\xi).$$

Thus,

$$H_\eta(r) = 1 - \exp\Big\{ -2\lambda\, r\, a \int_{\mathbb{G}(3,1)} [\xi, \eta]\, \alpha(\mathrm{d}\xi)\Big\}.$$

And if the structuring element is the unit ball ($B = B_1^3(\mathbf{o})$), then

$$\begin{aligned} H_{B_1^3(\mathbf{o})}(r) &= 1 - \exp\Big\{ -\lambda\Big(2r\, \mathbb{E}\, V_1^2(K(Z_0)) + r^2 \int_{\mathcal{Z}_1^o} \kappa_2\, \theta(\mathrm{d}Z)\Big)\Big\} \\ &= 1 - \exp\Big\{ -\lambda\big(r\ \mathbb{E}\, S(K(Z_0)) + r^2\, \pi\big)\Big\}, \end{aligned}$$

where $S(K(Z))$ is the perimeter of $K(Z)$.

If additionally $K(Z)$ is a ball of constant radius a, then

$$H_{B_1(\mathbf{o})}(r) = 1 - \exp\{ -2\pi\, a\, \lambda\, r - \pi\, \lambda\, r^2\}.$$

3.2. Specific surface area

In the recent paper [Hof09b], the specific intrinsic volumes of a rather general non-stationary cylinder process are given. In the stationary anisotropic case, some of these formulae can be simplified. In this section, we give an alternative proof for the specific surface area of the union set U_Ξ of a simple stationary anisotropic Poisson cylinder process Ξ leading to a simpler formula than that of [Hof09b] which can be immediately used in applications.

The specific surface area $\overline{S}_\Xi$ is defined as the mean surface area of U_Ξ per unit volume. More formally, consider the measure $S_{U_\Xi}(B) = \mathbb{E}\,\mathcal{H}^{d-1}(\partial U_\Xi \cap B)$ for all Borel sets $B \subset \mathbb{R}^d$, where $\mathcal{H}^j(\cdot)$ denotes the j-dimensional Hausdorff measure. We assume that this measure is locally finite, i.e., $S_{U_\Xi}(B) < \infty$ for all compact B. Sufficient conditions for this can be found in Lemma 3.4. Due to the stationarity of Ξ, the measure S_{U_Ξ} is translation invariant. By Haar's lemma, there exists a constant $\overline{S}_\Xi \geq 0$ such that $S_{U_\Xi}(B) = \overline{S}_\Xi\,|B|_d$ for all Borel sets B, cf. [Amb90]. The factor $\overline{S}_\Xi$ is called the *specific surface area* of U_Ξ.

Lemma 3.4. *The specific surface area $\overline{S}_\Xi$ of the union set U_Ξ of a stationary an-isotropic cylinder process Ξ is finite if $\mathbb{E}\,S(K(Z_0)) < \infty$.*

Proof. Let $B := B_1(\mathbf{o})$ be the unit ball centered in the origin. Then we calculate using the abbreviation $L_0 = L(Z_0)$ and Campbell's theorem

$$\begin{aligned}
S_{U_\Xi}(B) &= \mathbb{E}\,\mathcal{H}^{d-1}(\partial U_\Xi \cap B) \leq \mathbb{E}\sum_{Z\in\Xi}\mathcal{H}^{d-1}(\partial Z \cap B) = \int_{\mathcal{Z}_k}\mathcal{H}^{d-1}(\partial Z\cap B)\,\Lambda(\mathrm{d}Z)\\
&= \lambda\,\mathbb{E}\int_{L_0^\perp}\mathcal{H}^{d-1}((\partial Z_0 + x)\cap B)\,|\mathrm{d}x|_{d-k}^{L_0^\perp}\\
&= \lambda\,\mathbb{E}\int_{L_0^\perp}\int_{\partial Z_0 + x}\mathbb{1}_B(y)\,\mathcal{H}^{d-1}(\mathrm{d}y)\,|\mathrm{d}x|_{d-k}^{L_0^\perp}\\
&= \lambda\,\mathbb{E}\int_{L_0^\perp}\int_{\partial Z_0}\mathbb{1}_B(y+x)\,\mathcal{H}^{d-1}(\mathrm{d}y)\,|\mathrm{d}x|_{d-k}^{L_0^\perp}\\
&\leq \lambda\,\mathbb{E}\int_{\partial Z_0}\int_{L_0^\perp}\mathbb{1}_{\pi_{L_0}(B)}(\pi_{L_0}(y))\mathbb{1}_{\pi_{L_0^\perp}(B)}(\pi_{L_0^\perp}(y)+x)\,|\mathrm{d}x|_{d-k}^{L_0^\perp}\;\mathcal{H}^{d-1}(\mathrm{d}y)\\
&= \lambda\,\mathbb{E}\int_{\partial Z_0}\mathbb{1}_{\pi_{L_0}(B)}(\pi_{L_0}(y))\,|\pi_{L_0^\perp}(B)|_{d-k}^{L_0^\perp}\,\mathcal{H}^{d-1}(\mathrm{d}y)\\
&= \lambda|\pi_{L_0^\perp}(B)|_{d-k}^{L_0^\perp}\,\mathbb{E}\,\mathcal{H}^{d-1}(\partial Z_0 \cap (\pi_{L_0}(B)\times L_0^\perp))\\
&= \lambda\kappa_{d-k}\,\mathbb{E}\,|\pi_{L_0^\perp}(B)|_{d-k}^{L_0^\perp}\,\mathcal{H}^{d-k-1}(\partial K(Z_0))\\
&= \lambda\kappa_k\kappa_{d-k}\,\mathbb{E}\,S(K(Z_0)).
\end{aligned}$$

This yields $\overline{S}_\Xi = S_{U_\Xi}(B)/|B|_d < \infty$. $\square$

The following results hold for any random closed set X with realizations almost surely from the extended convex ring $\mathcal{S}$ and regular. A closed set is called *regular* if it coincides with the closure of its interior.

Lemma 3.5. *Let X be an almost surely regular, stationary random closed set with realizations in $\mathcal{S}$, which has a finite specific surface area. Then the specific surface area of X is given by*

$$\overline{S}_X = \frac{\omega_d}{\kappa_{d-1}} \int_{\mathbb{G}(d,1)} \lambda(\xi)\,\mathrm{d}\xi,$$

where $\mathrm{d}\xi$ is the Haar probability measure on $\mathbb{G}(d,1)$, $\lambda(\xi) = \frac{1}{2}\,\mathbb{E}\,\Phi_0(X \cap \xi, B_1(\mathbf{o}) \cap \xi)$ is the intensity of the number of connected components of $X \cap \xi$ on a line $\xi \in \mathbb{G}(d,1)$.

Proof. By Crofton's formula for polyconvex sets (cf. [SW08, Th. 6.4.3]) and Fubini's theorem, we have

$$\begin{aligned}\overline{S}_X &= \frac{1}{\kappa_d}\,\mathbb{E}\,\mathcal{H}^{d-1}(\partial X \cap B_1(\mathbf{o})) = \frac{2}{\kappa_d}\,\mathbb{E}\,\Phi_{d-1}(X, B_1(\mathbf{o})) \\ &= \frac{2\Gamma(\frac{d+1}{2})\sqrt{\pi}}{\kappa_d\,\Gamma(d/2)}\,\mathbb{E}\int_{\mathbb{G}(d,1)}\int_{\xi^\perp} \Phi_0(X \cap (\xi + x), B_1(\mathbf{o}) \cap (\xi + x))\,\mathrm{d}x\,\mathrm{d}\xi.\end{aligned}$$

Since X is stationary, $\Phi_0(X \cap (\xi + x), B_1(\mathbf{o}) \cap (\xi + x)) = \Phi_0(X \cap \xi, B_1(-x) \cap \xi)$. The expectation of this term depends only on the length of $B_1(x) \cap \xi$ and the orientation of ξ. Thus, with $\int_{\xi^\perp} |B_1(x) \cap \xi|_1^\xi\,\mathrm{d}x = \kappa_d$ and $|B_1(\mathbf{o}) \cap \xi|_1^\xi = 2$, this leads to

$$\begin{aligned}\overline{S}_X &= \frac{\Gamma(\frac{d+1}{2})\sqrt{\pi}}{\Gamma(d/2)}\,\mathbb{E}\int_{\mathbb{G}(d,1)} \Phi_0(X \cap \xi, B_1(\mathbf{o}) \cap \xi)\,\mathrm{d}\xi \\ &= \frac{\omega_d}{\kappa_{d-1}}\int_{\mathbb{G}(d,1)} \frac{1}{2}\,\mathbb{E}\,\Phi_0(X \cap \xi, B_1(\mathbf{o}) \cap \xi)\,\mathrm{d}\xi.\end{aligned}$$ □

The following theorem has already been stated in dimensions $d = 2, 3$ by Matheron in [Mat67, Paragraph 5] without a rigorous proof. It is a generalization of the formula

$$\overline{S}_X = -\frac{\omega_d}{\kappa_{d-1}} C'_X(0) \tag{3.8}$$

(see, e.g., [SKM95, p. 204]) for stationary, isotropic, and almost surely regular random closed sets $X \in \mathcal{S}$ to the anisotropic case. Note that, since in the isotropic case $C_X(h)$ depends only on the length of $h \in \mathbb{R}^d$ and not on h itself, in this formula C_X is a function of a real variable, namely the length of h.

Theorem 3.1. *Let X be an almost surely regular stationary random closed set with realizations from $\mathcal{S}$ and finite specific surface area. If $C_X(h)$ is its covariance function, then the specific surface area of X is given by the formula*

$$\overline{S}_X = -\frac{\omega_d}{\kappa_{d-1}} \int_{\mathbb{G}(d,1)} C'_X(\mathbf{o}, r_\xi)\,\mathrm{d}\xi, \tag{3.9}$$

where $C'_X(h, v)$ is the derivative of $C_X(h)$ at h in direction of unit vector v, and r_ξ is a direction unit vector of a line $\xi \in \mathbb{G}(d, 1)$.

Proof. For a stationary random closed set $U \subset \mathbb{R}$ from the extended convex ring denote by $-U$ the set reflected at the origin. Define a random variable V which is uniformly distributed on $\{-1, 1\}$ and independent of U. The random closed set UV is obviously isotropic, and thus formula (3.8) yields $\overline{S}_{UV} = -2C'_{UV}(0)$. Since $\overline{S}_U = \overline{S}_{UV}$ and $C'_U(0) = C'_{UV}(0)$ because $\mathbb{P}(\{\mathbf{o}, h\} \in U) = \mathbb{P}(\{\mathbf{o}, -h\} \in U)$, we obtain $\overline{S}_U = -2C'_U(0)$.

For $U = X \cap \xi$, $\xi \in \mathbb{G}(d, 1)$, we get $\lambda(\xi) = \frac{1}{2}\overline{S}_{X\cap\xi} = -C'_{X\cap\xi}(0) = -C'_X(\mathbf{o}, r_\xi)$. Lemma 3.5 completes the proof. □

If X is an almost surely regular two-dimensional stationary random closed set with realizations in $\mathcal{S}$, formula (3.9) simplifies to

$$\overline{S}_X = -\pi \int_0^\pi C'_X(\mathbf{o}, \varphi)\frac{\mathrm{d}\varphi}{\pi} = -\int_0^\pi C'_X(\mathbf{o}, \varphi)\,\mathrm{d}\varphi.$$

Remark 3.2. *Matheron's formula* (3.9) *was independently shown by Galerne for the closely related specific variation, see [Gal11, Th. 17]. The different definition of the specific variation (in comparison to the specific surface area) allows him to state the formula for arbitrary random closed sets, without any restriction on the image space. Under the preliminaries of Theorem 3.1, both formulas coincide.*

The following result is a direct corollary of Proposition 3.1, Theorem 3.1, and Fubini's theorem.

Corollary 3.1. *Let Ξ be a stationary Poisson cylinder process with intensity λ, cylinders with regular cross-section $K(Z_0) \in \mathcal{R}'$ almost surely and finite specific surface area. Then the specific surface area of U_Ξ is given by the formula*

$$\overline{S}_\Xi = -\frac{\lambda\,\omega_d}{2\kappa_{d-1}}\mathrm{e}^{-\lambda\,\mathbb{E}\,A(Z_0)}\,\mathbb{E}\int_{\mathbb{G}(d,1)} V_{\pi_{L(Z_0)^\perp(r_\xi)}}(K(Z_0))\,[\xi, L(Z_0)]\,\mathrm{d}\xi.$$

Example. Assume that $K(Z_0)$ is convex and regular almost surely.

(a) For arbitrary d and $k = d-1$, it holds for $\mathrm{d}\xi$-almost every line $\xi \in \mathbb{G}(d,1)$ that

$$\gamma'_{K(Z)}(\mathbf{o}, \pi_{L(Z)^\perp}(r_\xi)) = -1$$

and

$$\int_{\mathbb{G}(d,1)} [\xi, L(Z)]\,\mathrm{d}\xi = \int_{\mathbb{G}(d,d-1)} [\xi^\perp, L(Z)]\,\mathrm{d}\xi = \frac{\omega_{d+1}\,\kappa_1}{\omega_d\,2\kappa_2} = \frac{\omega_{d+1}}{\omega_d\,\pi},$$

see [Spo02, Corollary 5.2].

This yields

$$\overline{S}_\Xi = \lambda \frac{\omega_{d+1}}{\pi\kappa_{d-1}} \exp\Big\{-\lambda\,\mathbb{E}\,|K(Z_0)|_1^{L(Z_0)^\perp}\Big\} = 2\lambda \exp\Big\{-\lambda\,\mathbb{E}\,|K(Z_0)|_1^{L(Z_0)^\perp}\Big\}.$$

(b) For $d = 3$, $k = 1$, $K = B_a^3(\mathbf{o})$ it can be derived that $\gamma'_{K(Z)}(\mathbf{o}, \pi_{L(Z)^\perp}(\xi)) = -\pi a$, $\int_{\mathbb{G}(3,1)} [\xi, L(Z)]\,\mathrm{d}\xi = 1/2$ (see also [SKM95, p. 298], or [Spo02, Corollary 5.2]), and thus we have

$$\overline{S}_\Xi = 4\lambda \frac{1}{2} \pi\, a\, \mathrm{e}^{-\lambda\,\pi\,a^2} = 2\pi\, a\, \lambda\, \mathrm{e}^{-\lambda\,\pi\,a^2},$$

which coincides with the case of isotropic cylinders, compare [OM00, p. 64].

3.3. An optimization example for PCPs originating from fuel cell research

In this section, we show how the formulae from Sections 3.1 and 3.2 can be applied to solve an optimization problem for cylinder processes. Note that the fibers in this section are not mathematical objects but consist of real polymer material.

The following problem originates from fuel cell research. The gas diffusion layer of a polymer electrolyte membrane fuel cell is a porous material made of polymer fibers (see Figure 1.1) which can be modeled well by an anisotropic Poisson process of cylinders in $\mathbb{R}^3$. In a gas diffusion layer, the volume fraction of the polymer material lies between 70 and 80 percent, and the directional distribution of fibers is concentrated on a small neighborhood of a great circle of a unit sphere S^2, i.e., all fibers are almost horizontal. In order to optimize the water and gas transport properties, it is desirable to have a relatively small variation of the size of pores in the medium, where we define a pore at a point x in the complement of U_Ξ as the maximal ball with center in x which does not hit U_Ξ.

We investigate the following mathematical simplification of this problem, which can be solved analytically in some particular cases.

For a fixed intensity λ of the Poisson cylinder process Ξ, find a shape distribution of cylinders θ which maximizes the volume fraction p of U_Ξ provided that the variance of the typical pore radius H is small. In other words, solve the optimization problem

$$\begin{cases} p \to \max_\theta, \\ \operatorname{Var} H < \varepsilon, \end{cases} \tag{3.10}$$

where H is a random variable with distribution function $H_{B_1(\mathbf{o})}(r)$.

As it will be clear later, the condition on the directional distribution α of fibers that all fibers are almost horizontal can be neglected since the directional component of the shape distribution θ has no influence on the solution.

To simplify the notation, let $c_s = \mathbb{E}\, S(K(Z_0))$ and $\Phi(x)$ be the distribution function of a standard normally distributed random variable.

First we take a look at the moments of the pore radius H (for $r \geq 0$), remembering that $H_{B_1(\mathbf{o})}(r) = 1 - \exp\{-\lambda(r\, c_s + r^2\, \pi)\}$ (as shown in an example in Section 3.1.3), and thus the density of H equals $\frac{\mathrm{d}}{\mathrm{d}r} H_{B_1(\mathbf{o})}(r) = \lambda(c_s + 2\pi\, r) \exp\{-\lambda(r\, c_s + r^2\, \pi)\}$. It holds

$$\begin{aligned} \mathbb{E}\, H &= \int_0^\infty r\, \lambda(c_s + 2\pi\, r) \exp\left\{ -\pi\, \lambda\Big(r + \frac{c_s}{2\pi}\Big)^2 \right\} \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\} \mathrm{d}r \\ &= \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\} \lambda \int_{\frac{c_s}{2\pi}}^\infty \left(r - \frac{c_s}{2\pi} \right) 2\pi\, r\, \mathrm{e}^{-\pi\, \lambda\, r^2}\, \mathrm{d}r \\ &= \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\} \frac{1}{\sqrt{\lambda}} \left(1 - \Phi\left(c_s \sqrt{\tfrac{\lambda}{2\pi}} \right) \right). \end{aligned}$$

Furthermore, it can be calculated that

$$\begin{aligned} \mathbb{E}\, H^2 &= \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\} \lambda \int_0^\infty r^2 (c_s + 2\pi\, r) \exp\left(-\pi\, \lambda \left(r + \frac{c_s}{2\pi} \right)^2 \right) \mathrm{d}r \\ &= \frac{1}{\pi\, \lambda} - \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\} \frac{c_s}{\pi \sqrt{\lambda}} \left(1 - \Phi\left(c_s \sqrt{\tfrac{\lambda}{2\pi}} \right) \right). \end{aligned}$$

Defining $c_e = \exp\left\{ \frac{c_s^2\, \lambda}{4\pi} \right\}$ and $c_\Phi = (1 - \Phi(c_s \sqrt{\lambda/2\pi}))$, this leads to

$$\mathbb{E}\, H^2 - (EH)^2 = \frac{1}{\pi\, \lambda} - \frac{c_e\, c_\Phi\, c_s}{\pi \sqrt{\lambda}} - \frac{c_e^2\, c_\Phi^2}{\lambda} \leq \varepsilon,$$

multiplication with $\pi\lambda$ yields the equivalent condition $1 - \sqrt{\lambda}\, c_e\, c_\Phi\, c_s - \pi\, c_e^2\, c_\Phi^2 \leq \varepsilon\, \pi\, \lambda$, which holds if and only if

$$\left(c_e\, c_\Phi + \frac{\sqrt{\lambda}\, c_s}{2\pi} \right)^2 - \frac{\lambda\, c_s^2}{4\pi^2} + (\varepsilon\, \lambda - 1/\pi) \geq 0.$$

This is always fulfilled if $\varepsilon \geq 1/\pi\lambda$ and $\frac{\lambda c_s^2}{4\pi^2} - (\varepsilon\lambda - 1/\pi) \leq 0$ or, equivalently, $c_s \leq 2\pi\sqrt{\varepsilon - 1/\pi\lambda}$.

In the following, we assume that $\varepsilon \geq 1/\pi\lambda$ and replace the condition $\operatorname{Var} H < \varepsilon$ by a stronger sufficient condition

$$c_s = \mathbb{E}\, S(K(Z_0)) \leq 2\pi\sqrt{\varepsilon - \frac{1}{\pi\lambda}}. \tag{3.11}$$

Hence, (3.10) is reduced to the optimization problem

$$\begin{cases} \mathbb{E}\, A(Z_0) \to \max_\theta, \\ \mathbb{E}\, S(K(Z_0)) \leq 2\pi\sqrt{\varepsilon - \frac{1}{\pi\lambda}}. \end{cases} \tag{3.12}$$

The solution of the optimization problem (3.12) yields cylinders with θ-almost surely circular base. Notice that this solution does not depend on the directional distribution component α of θ. Indeed, cylinders Z_0 can be replaced by cylinders Z' which have the same direction space and surface area ($S(K(Z_0)) = S(K(Z')))$ but are circular. Then the isoperimetric inequality yields $A(Z') \geq A(Z_0)$. Thus, it holds that

$$\mathbb{E}\, S(K(Z_0)) = \mathbb{E}\, S(K(Z'))$$

and

$$\mathbb{E}\, A(Z_0) \leq \mathbb{E}\, A(Z'),$$

which means that the circular version is at least not worse than the original version.

Thus, we assume that the cylinders are θ-almost surely circular and denote the radius of a cylinder Z by $R(Z)$. It follows from condition (3.11) that

$$\mathbb{E}\, S(K(Z_0)) = 2\pi\, \mathbb{E}\, R(Z_0) \leq 2\pi\sqrt{\varepsilon - \frac{1}{\pi\lambda}},$$

i.e., the new condition is that the expectation of the radius of a typical cylinder is less or equal than $\sqrt{\varepsilon - \frac{1}{\pi\lambda}}$.

Furthermore, it follows from (3.12) that maximizing p is equivalent to maximizing $\mathbb{E}\, R(Z_0)^2$.

The above calculation shows that the volume fraction of $70\%-80\%$ in the optimized gas diffusion layer of a fuel cell can be achieved best by taking fibers with circular cross sections, relatively small mean radius and high variance of this radius. Figure 1.1b shows that cross sections of fibers of gas diffusion layers are almost circular. There are also gas diffusion layers with a little variance in the fiber radii, although they are mostly nearly constant. Anyhow the variance of the fiber radii is of course limited, since it is impossible to produce fibers with an arbitrarily large radius.

We have to remark that from a practical point of view the optimization problem (3.10) is not well posed. For the construction of gas diffusion layers, mainly the intensity of the fibers λ can be varied. Hence a practically relevant optimization should involve maximizing the volume fraction p with respect to λ as well. Since the latter problem is much more involved than the one discussed here, it would go beyond the scope of this chapter.

3.4. Concluding remarks and open questions

- In this chapter, we have considered a model for stationary Poisson cylinder processes which seems to be sufficiently general for many applications in the modeling of homogeneous material. For this, we have derived formulae for the most important characteristics, which are usable for practical problems, as demonstrated in Section 3.3.

- Further work on this topic can be done by calculating formulae for other specific intrinsic volumes, like the specific Euler number. Note that for the closely related model of Poisson cylinder processes with convex base (as opposed to the polyconvex base considered in this chapter), these characteristics are already known, see, e.g., [Hof09b]. Furthermore, some special cases for the shape of the cylinder base (e.g., lower-dimensional bodies) or the directional distribution can be analyzed for the formulae derived in this chapter, when it is suitable for a certain application.

- Another possible generalization is to consider further models, possibly non-Poisson ones.

4. Asymptotic behavior of the empirical volume fraction of a PCP

This chapter is based on [HS09] and [HS12]. We analyze the asymptotic behavior of the random d-volume $V_\rho^{(d,k)} = |U_\Xi \cap \rho W|_d$ of the union set of Ξ within the observation window ρW for $\rho \to \infty$. Here, $W \in \mathcal{K}$ is chosen star-shaped (with respect to the origin $\mathbf{o}$)[1] such that $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W > 0$.

We use the notation introduced in Section 2.2.2, i.e., in particular we have

$$U_\Xi = \bigcup_{i \geq 1} \Theta_i\big((\Xi_i + P_i) \times \mathbb{R}^k\big),$$

where $(\Theta_1, \Xi_1), (\Theta_2, \Xi_2), \ldots$ are i.i.d. copies of $(\Theta_0, \Xi_0) : \Omega \to \mathbb{SO}_k^d \times \mathcal{R}_{d-k}^o$, and the process $\Pi_{\lambda,Q} = \sum_{i \geq 1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ is a stationary independently marked Poisson process in $\mathbb{R}^{d-k}$. Further, we shall often make use of the abbreviation π_{d-k} for the projection onto the first $d-k$ components of a vector. Note that in contrast to the very similar projection $\pi_{E_k^\perp}$ with image space $E_k^\perp \subset \mathbb{R}^d$, the function π_{d-k} maps to $\mathbb{R}^{d-k}$ for notational ease.

The first main result of this chapter is a central limit theorem for the estimator $V_\rho^{(d,k)}/|\rho W|_d$ for the volume fraction p in a growing observation window under the condition that $0 < M_2 < \infty$, where $M_s = \mathbb{E}\,|\Xi_0|_{d-k}^s$ is the s-th moment of the volume of the typical base. We also give explicit formulae for the variance of the estimator in the case of discrete and continuous directional distribution. For distributions of mixed type, we show how the formula can be obtained by combining the two calculations. Under the additional assumption that

$$m_a = \mathbb{E}\,\mathrm{e}^{a\,|\Xi_0|_{d-k}} < \infty \quad \text{for some} \quad a > 0, \tag{4.1}$$

we also give Berry-Esseen bounds and derive Cramér-type large deviation results as our second main result.

Because of the long range dependence resulting from the infinitely long cylinders, it is impossible to apply standard techniques to obtain a central limit theorem for $V_\rho^{(d,k)}$ based on M-dependence or mixing conditions. Thus, we generalize the method for the empirical volume fraction of the Boolean model in [Hei05] to cylinder processes.

[1] Note that in Chapter 5 we restrict to the special case $W = B_1(\mathbf{o})$.

This method is based on the analysis of the cumulants of $V_\rho^{(d,k)}$, see Section 2.4. They are closely related to the *n-point probabilities*, which can be introduced as follows. We first recall the fact that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which the marked Poisson process $\Pi_{\lambda,Q} = \sum_{i\geq 1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ is defined can be chosen in such a way that the mapping $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto \mathbb{1}_{U_\Xi(\omega)}(x) \in \{0,1\}$ is measurable with respect to the product-σ-field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}$, see appendix in [Hei05]. This enables us to apply Fubini's theorem to the $\{0,1\}$-valued random field $\{\mathbb{1}_{U_\Xi}(x), x \in \mathbb{R}^d\}$ and implies among others that its n-th order mixed moments (also called n-point probabilities of U_Ξ)

$$p_{U_\Xi}(x_1,\ldots,x_n) := \mathbb{E}\prod_{i=1}^n \mathbb{1}_{U_\Xi}(x_i) = \mathbb{P}(x_1 \in U_\Xi, \ldots, x_n \in U_\Xi)$$

are $\mathcal{B}(\mathbb{R}^{dn})$-measurable for any $n \in \mathbb{N}$ and the void probabilities $p_{U_\Xi{}^c}(x_1,\ldots,x_n)$ take on the following explicit form

$$\begin{aligned} p_{U_\Xi{}^c}(x_1,\ldots,x_n) &:= \mathbb{E}\prod_{i=1}^n (1-\mathbb{1}_{U_\Xi}(x_i)) = \mathbb{P}(x_1 \notin U_\Xi, \ldots, x_n \notin U_\Xi) \\ &= 1 - T_{U_\Xi}(\{x_1,\ldots,x_n\}) \qquad (4.2) \\ &= \exp\Big\{ -\lambda\, \mathbb{E}\Big| \bigcup_{i=1}^n (\Xi_0 - \pi_{d-k}(\Theta_0^T x_i))\Big|_{d-k}\Big\}, \end{aligned}$$

see Lemma 3.1 in Chapter 3. Thus, the n-th order mixed cumulants $c_{U_\Xi{}^c}(x_1,\ldots,x_n)$ of $\{1-\mathbb{1}_{U_\Xi}(x), x \in \mathbb{R}^d\}$ are Borel measurable functions leading to the following integral representation of the n-th order cumulant of $V_\rho^{(d,k)}$, see (2.10),

$$\mathrm{Cum}_n(V_\rho^{(d,k)}) = (-1)^n \int_{(\rho W)^n} c_{U_\Xi{}^c}(x_1,\ldots,x_n)\,\mathrm{d}(x_1,\ldots,x_n) \quad \text{for} \quad n \geq 2. \qquad (4.3)$$

With (2.8), the cumulant function $c_{U_\Xi{}^c}(x_1,\ldots,x_n)$ can be expressed by the l-point probabilities $p_{U_\Xi{}^c}(x_{i_1},\ldots,x_{i_l})$ for $1 \leq i_1 < \cdots < i_l \leq n$ and $l = 1,\ldots,n$.

This chapter is organized as follows. Since here, we concentrate on a few main results, we employ a slightly different format, namely in the next section, we state the main results separately. Then in Section 4.2, we calculate the asymptotic order of the variance with respect to ρ as ρ goes to infinity. Section 4.3 contains the recursive estimation technique for the variance of $V_\rho^{(d,k)}$ adapted from [Hei05]. In Section 4.4, we apply a truncation technique to prove a central limit theorem which only requires that the second moment M_2 of the base exists. In Section 4.5, the asymptotic variance of $V_\rho^{(d,k)}$ is calculated by treating the diffuse and discrete directional distribution separately. Alternative formulae for the asymptotic variance in some special cases are also derived. We conclude the chapter with some remarks in the final section.

4.1. Main results

In this section, we present the main theorems of this chapter.

4.1.1. A CLT for $V_\rho^{(d,k)}$ with explicit asymptotic variance

We begin with a central limit theorem (CLT) for $V_\rho^{(d,k)}$ under the prerequisite that the second moment of $|\Xi_0|_{d-k}$ exists.

Theorem 4.1. *Let U_Ξ be the union set of the stationary* PCP $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ *with typical cylinder base $\Xi_0 \in \mathcal{R}_{d-k}^o$ satisfying $0 < M_2 < \infty$. Further, let $W \subset \mathbb{R}^d$ be compact and star-shaped with respect to $\mathbf{o}$ satisfying $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W \in (0,1]$. Then*

$$\frac{|U_\Xi \cap \rho\, W|_d - p\,\rho^d\,|W|_d}{\sqrt{\mathrm{Var}\big(|U_\Xi \cap \rho\, W|_d\big)}} \quad \underset{\rho\to\infty}{\overset{d}{\longrightarrow}} \quad \mathrm{N}(0,1). \tag{4.4}$$

Note that $p = \mathbb{P}(\mathbf{o} \in U_\Xi) = \mathbb{E}\,|U_\Xi \cap [0,1]^d|_d = 1 - \mathrm{e}^{-\lambda\, M_1}$ is just the *volume fraction* of the stationary random set U_Ξ which coincides with intensity of the random volume measure $|U_\Xi \cap (\cdot)|$, cf. (3.3).

As our second main result the following Theorem 4.2 provides exact asymptotic growth rates of the variances of the d-volume $|U_\Xi \cap \rho\, W|_d$, i.e., of

$$\sigma^2_{\lambda,Q}(W) = \lim_{\rho\to\infty} \frac{\mathrm{Var}\big(|U_\Xi \cap \rho\, W|_d\big)}{\rho^{d+k}} \tag{4.5}$$

in dependence of k, d and W in the cases of purely atomic and diffuse directional distribution $\mathbb{P}_0(\cdot) = Q((\cdot) \times \mathcal{R}_{d-k}^o)$.

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. If the marginal distribution $\mathbb{P}_0(\cdot)$ is discrete, i.e., it is concentrated on $\{\theta_i \in \mathbb{SO}_k^d,\ i \in I\}$ for some at most countable index set I, then*

$$\sigma^2_{\lambda,Q}(W) = \mathrm{e}^{-2\lambda\, M_1} \sum_{i\in I} \int\limits_{\mathbb{R}^k} \left| W \cap \left(W - \theta_i \begin{pmatrix} \mathbf{o}_{d-k} \\ x \end{pmatrix} \right) \right|_d \mathrm{d}x \int\limits_{\mathbb{R}^{d-k}} \left(\mathrm{e}^{\lambda\, f(y,\theta_i)} - 1 \right) \mathrm{d}y, \tag{4.6}$$

where $\mathbf{o}_{d-k}$ denotes the origin in $\mathbb{R}^{d-k}$ and

$$\begin{aligned} f(y,\theta_i) &= \mathbb{E}\big[|\Xi_0 \cap (\Xi_0 + y)|_{d-k}\, \mathbb{1}\{\Theta_0 = \theta_i\}\big] \\ &= \mathbb{E}\big[|\Xi_0 \cap (\Xi_0 + y)|_{d-k} \mid \Theta_0 = \theta_i\big]\, \mathbb{P}_0(\{\theta_i\}). \end{aligned}$$

On the other hand, if $\mathbb{P}_0(\cdot)$ is diffuse, i.e., $\mathbb{P}_0(\{\theta\}) = 0$ for any $\theta \in \mathbb{SO}_k^d$, we have

$$\sigma^2_{\lambda,Q}(W) = \lambda\,\mathrm{e}^{-2\lambda\,M_1} \int_{\mathbb{SO}_k^d} M_2(\theta) \int_{\mathbb{R}^k} \left| W \cap \left(W - \theta \begin{pmatrix} \mathbf{o}_{d-k} \\ x \end{pmatrix} \right) \right|_d \mathrm{d}x\,\mathbb{P}_0(\mathrm{d}\theta), \tag{4.7}$$

where $M_2(\theta) = \mathbb{E}\big[|\Xi_0|^2_{d-k} \mid \Theta_0 = \theta\big]$.

The general case of arbitrary directional distribution is covered by the following result.

Corollary 4.1. *Let the conditions of Theorem 4.1 hold. Consider the unique decomposition of $\mathbb{P}_0(\cdot)$, i.e., $\mathbb{P}_0 = \alpha\,\mathbb{P}_0^{\mathrm{a}} + (1-\alpha)\,\mathbb{P}_0^{\mathrm{c}}$, where we have an atomic distribution $\mathbb{P}_0^{\mathrm{a}}$ and a diffuse distribution $\mathbb{P}_0^{\mathrm{c}}$ on $\mathbb{SO}_k^d$ (implying a decomposition of the mark distribution $Q = \alpha\,Q^{\mathrm{a}} + (1-\alpha)\,Q^{\mathrm{c}}$ on $\mathbb{M}_{d,k}$). Then the limit (4.5) exists and admits the decomposition*

$$\sigma^2_{\lambda,Q}(W) = \sigma^2_{\lambda,Q^{\mathrm{a}},\alpha}(W) + (1-\alpha)\,\sigma^2_{\lambda,Q^{\mathrm{c}}}(W), \tag{4.8}$$

where $\sigma^2_{\lambda,Q^{\mathrm{a}},\alpha}(W)$ resp. $\sigma^2_{\lambda,Q^{\mathrm{c}}}(W)$ is defined as in (4.6) resp. (4.7) with $\mathbb{P}_0$ replaced by $\alpha\,\mathbb{P}_0^{\mathrm{a}}$ (in $f(y,\theta_i)$) resp. by $\mathbb{P}_0^{\mathrm{c}}$.

4.1.2. Berry-Esseen bounds and Cramér-type large deviations for $V_\rho^{(d,k)}$

Now, we provide some results on the convergence speed in Theorem 4.1 above under the preliminary that the exponential moments of $|\Xi_0|_{d-k}$ exists. We begin with estimates of the higher-order cumulants $\mathrm{Cum}_n(V_\rho^{(d,k)})$ of the d-volume $V_\rho^{(d,k)}$.

Theorem 4.3. *Let U_Ξ be the union set* (2.4) *of the stationary* PCP $\Pi^{(d,k)}_{\mathrm{cyl}}(\lambda, Q)$ *with typical cylinder base $\Xi_0 \in \mathcal{R}^o_{d-k}$ satisfying* (4.1) *and $M_1 = \mathbb{E}\,|\Xi_0|_{d-k} > 0$. Further, let $W \subset \mathbb{R}^d$ be compact and star-shaped with respect to the origin, satisfying $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W \in (0,1]$. Then*

$$\left| \mathrm{Cum}_n\left(V_\rho^{(d,k)}\right) \right| \le \rho^{d+(n-1)k}\,(n-1)!\,H_a\,\Delta_a^{n-2} \quad \textit{for} \quad n \ge 2,\ \rho > 0, \tag{4.9}$$

where $H_a = 2^{2k+1}\,|W|_d\,\lambda\,m_a\,(1+\mathrm{e}^{\lambda\,M_1})/a^2$ and $\Delta_a = 2^{2k+3}\,(a+\lambda\,m_a)(1+\mathrm{e}^{\lambda\,M_1})/a^2$.

The next Theorem 4.4 states Cramér's large deviations relations for $V_\rho^{(d,k)}$ and a Berry-Esseen bound of the distance between the distribution function

$$F_\rho(x) = \mathbb{P}\left(\frac{V_\rho^{(d,k)} - \rho^d\,|W|_d\,(1-\mathrm{e}^{-\lambda\,\mathbb{E}\,|\Xi_0|_{d-k}})}{\sigma_\rho\,\rho^{(d+k)/2}} \le x \right)$$

and the distribution function Φ of a standard normally distributed variable.

As shown in (3.3), the volume fraction p of U_Ξ is $p = \mathbb{P}(\mathbf{o} \in U_\Xi) = 1 - \mathrm{e}^{-\lambda\, \mathbb{E}\, |\Xi_0|_{d-k}}$, and the normalized variance σ_ρ^2 of $V_\rho^{(d,k)}$ satisfies the estimate

$$0 < c_1 \le \sigma_\rho^2 \le c_2 < \infty \quad \text{for all } \ \rho \ge 1 \qquad \text{with} \qquad \sigma_\rho^2 = \operatorname{Var}(V_\rho^{(d,k)})/\rho^{d+k}, \tag{4.10}$$

and c_1, c_2 are constants not depending on $\rho \ge 1$, see Lemma 4.1 below.

Theorem 4.4. *Let the assumptions of Theorem 4.3 be satisfied. Then the following asymptotic relations hold in the interval* $0 \le x \le \sigma_\rho\, \rho^{(d-k)/2}/2\,\Delta_a\,(1 + 4\,H_{a,\rho})$ *with* $H_{a,\rho} = H_a/2\,\sigma_\rho^2$*:*

$$\frac{1 - F_\rho(x)}{1 - \Phi(x)} = \exp\left\{ \frac{x^3}{\sigma_\rho\, \rho^{(d+k)/2}} \sum_{s=0}^{\infty} \mu_s^{(\rho)} \Big(\frac{x}{\sigma_\rho\, \rho^{(d+k)/2}} \Big)^s \right\} \left(1 + \mathcal{O}\left(\frac{1+x}{\rho^{(d-k)/2}} \right) \right) \tag{4.11}$$

and

$$\frac{F_\rho(-x)}{\Phi(-x)} = \exp\left\{ \frac{-x^3}{\sigma_\rho\, \rho^{(d+k)/2}} \sum_{s=0}^{\infty} \mu_s^{(\rho)} \Big(\frac{-x}{\sigma_\rho\, \rho^{(d+k)/2}} \Big)^s \right\} \left(1 + \mathcal{O}\left(\frac{1+x}{\rho^{(d-k)/2}} \right) \right) \tag{4.12}$$

as $\rho \to \infty$*, where the coefficients* $\mu_s^{(\rho)}$ *are defined by*

$$\mu_s^{(\rho)} = \frac{1}{(s+2)(s+3)} \sum_{j=1}^{s+1} (-1)^{j-1} \binom{s+j+1}{j} \sum_{\substack{s_1+\cdots+s_j = s+1 \\ s_1,\ldots,s_j \ge 1}} \prod_{i=1}^{j} \frac{\operatorname{Cum}_{s_i+2}(V_\rho^{(d,k)})}{\operatorname{Var}(V_\rho^{(d,k)})\,(s_i+1)!}.$$

In the formulae (4.11) *and* (4.12) *above, the series converge absolutely due to the estimate*

$$\big|\mu_s^{(\rho)}\big| \le 4H_{a,\rho}\,\Delta_a\,\rho^{k(s+1)}\,\big(2\Delta_a\,(1 + 4H_{a,\rho})\big)^s/(s+2)(s+3) \tag{4.13}$$

for all $s \ge 0$.

Further, there exists some constant $c_3 > 0$ *(depending on* a*,* λ*,* m_a*, and* c_1*,* c_2*) such that*

$$\sup_{x \in \mathbb{R}} \big|F_\rho(x) - \Phi(x)\big| \le c_3\, \rho^{-(d-k)/2} \quad \textit{for all} \quad \rho \ge 1. \tag{4.14}$$

Theorem 4.4 is derived from (4.9) combined with a general lemma on large deviations for a single random variable with mean 0 and variance 1, see [Sta66] or Lemma 2.3 in the monograph [SS91]. The relations (4.11) and (4.12) are of particular interest at $x = \varepsilon\,|W|_d\,\rho^{(d-k)/2}/\sigma_\rho$ for small $\varepsilon > 0$.

An immediate consequence of Theorem 4.4 is the following result.

Corollary 4.2. *Let the assumptions of Theorem 4.4 hold. Then we have*

$$\frac{1-F_\rho(x)}{1-\Phi(x)} \le \exp\left\{\frac{b_{1,\rho}\, x^3}{\rho^{(d-k)/2}} \sum_{s=0}^{\infty} \frac{1}{(s+2)(s+3)} \left(\frac{b_{2,\rho}\, x}{\rho^{(d-k)/2}}\right)^s\right\} \left(1+\mathcal{O}\left(\frac{1+x}{\rho^{(d-k)/2}}\right)\right)$$

and

$$\frac{F_\rho(-x)}{\Phi(-x)} \le \exp\left\{\frac{-b_{1,\rho}\, x^3}{\rho^{(d-k)/2}} \sum_{s=0}^{\infty} \frac{1}{(s+2)(s+3)} \left(\frac{-b_{2,\rho}\, x}{\rho^{(d-k)/2}}\right)^s\right\} \left(1+\mathcal{O}\left(\frac{1+x}{\rho^{(d-k)/2}}\right)\right)$$

as $\rho \to \infty$, *where* $b_{1,\rho} = 4H_{a,\rho}\,\Delta_a/\sigma_\rho = 2H_a\,\Delta_a/\sigma_\rho^3$ *and* $b_{2,\rho} = 2\Delta_a\,(1+4H_{a,\rho})/\sigma_\rho$.

Further, the constant c_3 *in Theorem 4.4 is bounded by* $\sqrt{2}\,\Delta_a\,H_a/\sigma_\rho^3$, *which leads to*

$$\sup_{x\in\mathbb{R}} \left|F_\rho(x)-\Phi(x)\right| \le \frac{\sqrt{2}\,\Delta_a\,H_a}{\sigma_\rho^3\,\rho^{(d-k)/2}} \quad \textit{for all} \quad \rho \ge 1.$$

Remark 4.1. *The Cramér-type large deviations relations in Theorem 4.4 imply that the large deviation principle, as introduced, e.g., in [DZ10], holds for the random element* $V_\rho^{(d,k)}$. *The rate function can be calculated by setting* $x = \varepsilon\,|W|_d\,\rho^{(d-k)/2}/\sigma_\rho$ *and using bounds for Mill's ratio, see [Gor41].*

Conjecture 4.1. *Because of the sharp inequalities used in Section 4.3 we suppose that the order of* ρ *in* (4.9), *and thus also in* (4.11), (4.12), *and* (4.14) *is optimal.*

4.2. Order of the asymptotic variance

In this section, we derive a lower and an upper bound for the variance of the random d-volume $V_\rho^{(d,k)} = |U_\Xi \cap \rho\, W|_d$ provided $M_2 < \infty$. For this end, we first derive a closed-term expression of the variance $\mathrm{Var}(|U_\Xi \cap B|_d)$ for any bounded Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ using the formulae for $p_{U_\Xi{}^c}(\mathbf{o},x)$ and $p_{U_\Xi{}^c}(\mathbf{o})$ from (4.2).

By using the very definition of the one- and two-point probabilities $p_{U_\Xi}(\cdot)$ and $p_{U_\Xi}(\cdot,\cdot)$ and the shift-invariance and additivity of the Lebesgue measure $|\cdot|_{d-k}$, we deduce from (4.2) that for $x_1, x_2 \in \mathbb{R}^d$ we have

$$\begin{aligned}
&p_{U_\Xi}(x_1,x_2) - p_{U_\Xi}(x_1)\,p_{U_\Xi}(x_2)\\
&= \mathbb{P}(x_1,x_2 \in U_\Xi) - \mathbb{P}(x_1 \in U_\Xi)\,\mathbb{P}(x_2 \in U_\Xi)\\
&= p_{U_\Xi{}^c}(\mathbf{o}, x_2-x_1) - p_{U_\Xi{}^c}(\mathbf{o})\,p_{U_\Xi{}^c}(\mathbf{o})\\
&= \exp\Big\{-\lambda\,\mathbb{E}\,\big|\Xi_0 \cup (\Xi_0 - \pi_{d-k}(\Theta_0^T\,(x_2-x_1))\big|_{d-k}\Big\} - \exp\Big\{-2\lambda\,\mathbb{E}\,\big|\Xi_0\big|_{d-k}\Big\}\\
&= \mathrm{e}^{-2\lambda\,M_1}\,\Big(\exp\Big\{\lambda\,\mathbb{E}\,\big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\,(x_2-x_1)))\big|_{d-k}\Big\} - 1\Big).
\end{aligned}$$

Hence, by multiple application of Fubini's theorem we get for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$ that

$$\begin{aligned}\operatorname{Var}(|U_\Xi \cap B|_d) &= \mathbb{E} \int_B \int_B \mathbb{1}_{U_\Xi}(x_1)\, \mathbb{1}_{U_\Xi}(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 - \Big(\mathbb{E} \int_B \mathbb{1}_{U_\Xi}(x)\, \mathrm{d}x \Big)^2 \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_B(x_1)\, \mathbb{1}_B(x_2) \Big(p_{U_\Xi}(x_1, x_2) - p_{U_\Xi}(x_1)\, p_{U_\Xi}(x_2) \Big)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \mathrm{e}^{-2\lambda\, M_1} \int_{\mathbb{R}^d} |B \cap (B - x)|_d \\ &\qquad \Big(\exp\Big\{ \lambda\, \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\, x))\big|_{d-k} \Big\} - 1 \Big)\, \mathrm{d}x. \end{aligned} \tag{4.15}$$

Now we replace B by the star-shaped set $\rho\, W$ which increases when ρ does. In view of the relation $\{x \in \mathbb{R}^d : \rho\, W \cap (\rho\, W - x) \neq \emptyset\} = \rho\, (W \oplus (-W)) \subset B_{2\rho}(\mathbf{o})$ and the inequality $\mathrm{e}^y - 1 \leq y\, \mathrm{e}^y$ for $y \geq 0$, we may write

$$\begin{aligned}\operatorname{Var}(V_\rho^{(d,k)}) &\leq \lambda\, \mathrm{e}^{-\lambda\, M_1}\, |\rho\, W|_d \int_{\rho\, (W \oplus (-W))} \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(\Theta_0^T\, x))\big|_{d-k}\, \mathrm{d}x \\ &\leq \lambda\, |W|_d\, \mathrm{e}^{-\lambda\, M_1}\, \rho^d\, \mathbb{E} \int_{B_{2\rho}(\mathbf{o})} \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\big|_{d-k}\, \mathrm{d}x \\ &\leq \lambda\, |W|_d\, \mathrm{e}^{-\lambda\, M_1}\, \rho^d\, \mathbb{E} \int_{[-2\rho, 2\rho]^k} \int_{\mathbb{R}^{d-k}} \big|\Xi_0 \cap (\Xi_0 + y_1)\big|_{d-k}\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &= \lambda\, |W|_d\, \mathrm{e}^{-\lambda\, M_1}\, 4^k\, \mathbb{E}\, |\Xi_0|_{d-k}^2\, \rho^{d+k} \quad \text{for any} \quad \rho > 0. \end{aligned} \tag{4.16}$$

To find a positive lower bound of the ratio σ_ρ^2 we make use of $B_{\delta_W}(\mathbf{o}) \subset W$ which implies $\rho\, W \cap (\rho\, W - x) \supset B_{\delta_W\, \rho}(\mathbf{o}) \cap B_{\delta_W\, \rho}(-x)$ and $\rho\, (W \oplus (-W)) \supset B_{2\delta_W\, \rho}(\mathbf{o})$. This combined with $\mathrm{e}^y - 1 \geq y$ for $y \geq 0$ implies

$$\begin{aligned}&\operatorname{Var}(V_\rho^{(d,k)}) \\ &\geq \lambda\, \mathrm{e}^{-2\lambda\, M_1} \int_{B_{2\delta_W\, \rho}(\mathbf{o})} |B_{\delta_W\, \rho}(\mathbf{o}) \cap B_{\delta_W\, \rho}(x)|_d\, \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(\Theta_0^T\, x))\big|_{d-k}\, \mathrm{d}x \\ &\geq \lambda\, \mathrm{e}^{-2\lambda\, M_1} \int_{B_{\delta_W\, \rho}(\mathbf{o})} |B_{\delta_W\, \rho}(\mathbf{o}) \cap B_{\delta_W\, \rho}(x)|_d\, \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\big|_{d-k}\, \mathrm{d}x \\ &\geq \lambda\, \mathrm{e}^{-2\lambda\, M_1}\, c(d)\, (\rho\, \delta_W)^d \int_{B_{\delta_W\, \rho}(\mathbf{o})} \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\big|_{d-k}\, \mathrm{d}x \\ &\geq \lambda\, \mathrm{e}^{-2\lambda\, M_1}\, c(d)\, (\rho\, \delta_W)^d \int_{[-\rho\, \delta_W/\sqrt{d}, \rho\, \delta_W/\sqrt{d}]^d} \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\big|_{d-k}\, \mathrm{d}x \\ &= \lambda\, 2^k\, d^{-k/2}\, \mathrm{e}^{-2\lambda\, M_1}\, (\rho\, \delta_W)^{d+k}\, c(d)\, I_{d,k}(\rho) \end{aligned}$$

with $c(d) = |B_1(\mathbf{o}) \cap B_1(e_1)|_d > 0$ and

$$I_{d,k}(\rho) = \int_{[-\rho\, \delta_W/\sqrt{d}, \rho\, \delta_W/\sqrt{d}]^{d-k}} \mathbb{E}\, \big|\Xi_0 \cap (\Xi_0 + y)\big|_{d-k}\, \mathrm{d}y.$$

Making use of $\mathbb{P}(|\Xi_0|_{d-k} > 0) > 0$ and standard measure-theoretic arguments it follows that $I_{d,k}(\rho) > 0$ for any $\rho > 0$, and $I_{d,k}(\rho)$ increases with $\rho \to \infty$ to the limit $\mathbb{E}\,|\Xi_0|^2_{d-k}$. In this way we confirm the estimate (4.10) with constants

$$c_1 = \lambda\, 2^k\, d^{-k/2}\, \mathrm{e}^{-2\lambda\, M_1}\, \delta_W^{d+k}\, c(d)\; I_{d,k}(1) \quad \text{and} \quad c_2 = \lambda\, |W|_d\, 4^k\, \mathrm{e}^{-\lambda\, M_1}\, M_2.$$

Another consequence of the above estimates is stated in

Lemma 4.1. *Let U_Ξ be the union set (2.4) of the stationary PCP $\Pi^{(d,k)}_{\mathrm{cyl}}(\lambda, Q)$ with cylinder base $\Xi_0 \in \mathcal{R}^o_{d-k}$ satisfying $0 < M_2 = \mathbb{E}\,|\Xi_0|^2_{d-k} < \infty$. Further, let $W \subset \mathbb{R}^d$ be a compact set satisfying $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W > 0$. Then we have*

$$c_1 \le \liminf_{\rho\to\infty} \frac{\mathrm{Var}(|U_\Xi \cap \rho W|_d)}{\rho^{d+k}} \le \limsup_{\rho\to\infty} \frac{\mathrm{Var}(|U_\Xi \cap \rho W|_d)}{\rho^{d+k}} \le c_2. \tag{4.17}$$

Proof. The above calculations confirm the estimates with constants

$$c_1 = \lambda\, 2^k\, d^{-k/2}\, \mathrm{e}^{-2\lambda\, M_1}\, \delta_W^{d+k}\, c(d)\, M_2 \quad \text{and} \quad c_2 = \lambda\, |W|_d\, 4^k\, \mathrm{e}^{-\lambda\, M_1}\, M_2.$$

$\square$

Remark 4.2. *(4.17) reveals that the variance of $|U_\Xi \cap \rho\, W|_d$ grows with the power $|\rho\, W\,|_d^{1+k/d}$ of the window volume which expresses long-range dependences within the random set (2.4). A similar effect could be observed in studying the asymptotic behavior of the total $(d-k)$-volume of intersection $(d-k)$-flats generated by Poisson hyperplane processes in $B_\rho(\mathbf{o})$ resp. $\rho\, W$ (for convex W) as $\rho \to \infty$, see [HSS06] resp. [Hei09].*

4.3. A recursive estimation method for the cumulants of $V^{(d,k)}_\rho$; proofs of Theorem 4.3 and 4.4

Here, we assume that the preliminaries of Theorem 4.3 hold. This means that U_Ξ is the union set (2.4) of the stationary PCP $\Pi^{(d,k)}_{\mathrm{cyl}}(\lambda, Q)$ with typical cylinder base $\Xi_0 \in \mathcal{R}^o_{d-k}$ satisfying (4.1) and $M_1 > 0$. Further, let $W \subset \mathbb{R}^d$ be compact and star-shaped with respect to $\mathbf{o}$ satisfying $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W \in (0, 1]$.

The main part of this section consists of a combination of recursive estimation procedures carried out in several steps which finally result in the estimate (4.9). This proving idea was developed in [Hei05] to obtain a similar estimate for Boolean models. However, the techniques used there had to be extended to unbounded cylinders which cause long-range dependences in contrast to the classical Boolean model.

To begin with, using the shift-invariance $c_{U_\Xi{}^c}(x_1,\dots,x_n) = c_{U_\Xi{}^c}(\mathbf{o},y_1,\dots,y_{n-1})$ for $y_i = x_{i+1}-x_1$, $i=1,\dots,n-1$, we rewrite (4.3) as follows

$$\begin{aligned}&\mathrm{Cum}_n(V_\rho^{(d,k)})\\ &= (-1)^n \int\limits_{(\rho\,(W\oplus(-W)))^{n-1}} \Big| \bigcap_{y\in Y_{n-1}\cup\{\mathbf{o}\}} (\rho\, W + y)\Big|_d\, c_{U_\Xi{}^c}(Y_{n-1}\cup\{\mathbf{o}\})\,\mathrm{d}Y_{n-1}\end{aligned} \tag{4.18}$$

for any integer $n\ge 2$. Here and in what follows, we denote by $X_m=\{x_1,\dots,x_m\}$ and $Y_n=\{y_1,\dots,y_n\}$ (unordered) sets of distinct points $x_1,\dots,x_m\in\mathbb{R}^d$ and $y_1,\dots,y_n\in\mathbb{R}^d$, respectively. Here, we use the notation $|Y|$ for the number of elements of any finite set $Y\subset\mathbb{R}^d$. For notational simplicity, put $p(Y)=p_{U_\Xi{}^c}(Y)$ and $c(Y)=c_{U_\Xi{}^c}(Y)$ so that, in view of (4.2), we may write

$$p(Y) = \mathrm{e}^{-\lambda\,\mathbb{E}\,|\Xi_0(Y)|_{d-k}} \quad \text{with} \quad \Xi_0(Y) := \bigcup_{y\in Y}\big(\Xi_0 - \pi_{d-k}(\Theta_0^T\,y)\big). \tag{4.19}$$

Further, write $\Xi_0^c(Y)$ for the complement of $\Xi_0(Y)$ in $\mathbb{R}^{d-k}$ and put $\Xi_0(\emptyset)=\emptyset$, $p(\emptyset)=1$, and $c(\emptyset)=0$. Note that $c(\{y\}) = 1-c_{U_\Xi}(y) = p(\{y\}) = \mathrm{e}^{-\lambda\,M_1}$ for any $y\in\mathbb{R}^d$. Since $W\oplus(-W)\subset B_2(\mathbf{o})$ as consequence of $W\subset B_1(\mathbf{o})$, it follows from (4.18) that

$$\Big|\mathrm{Cum}_{n+1}(|U_\Xi\cap\rho\,W|_d)\Big| \le \rho^d\,|W|_d \int_{(B_{2\rho}(\mathbf{o}))^n} \big|c(\{\mathbf{o}\}\cup Y_n)\big|\,\mathrm{d}Y_n. \tag{4.20}$$

The (mixed) cumulant functions $c(Y)$ are connected with the (mixed) moment functions $p(U)$, $\emptyset\ne U\subset Y$, of the random field $\{\mathbb{1}_{U_\Xi{}^c}(x), x\in\mathbb{R}^d\}$ by

$$c(Y) = \sum_{j=1}^{|Y|}(-1)^{j-1}\,(j-1)! \sum_{U_1\cup\dots\cup U_j=Y} p(U_1)\cdots p(U_j) \quad \text{for any finite} \quad Y\subset\mathbb{R}^d,$$

where the inner sum runs over all decompositions of Y into pairwise disjoint, nonempty subsets $U_1,\dots,U_j$. Note the similarity of this formula to (2.8). The equivalent relationships $c(Y)=p(Y)-\sum_{\emptyset\subsetneq X\subsetneq Y} c(X)\,p(Y\setminus X)$ or

$$c(\{x\}\cup Y_n) = p(\{x\}\cup Y_n) - \sum_{\emptyset\subset Y\subsetneq Y_n} c(\{x\}\cup Y)\,p(Y_n\setminus Y) \quad \text{for} \quad x\in\mathbb{R}^d\setminus Y_n$$

do not really help to establish upper bounds of the integral on the right hand side of (4.20). Rather than this, we introduce more general functions $X_m\times Y_n\mapsto c(X_m,Y_n)$ for arbitrary $m\ge1$ and $n\ge1$ (with $X_m\cap Y_n=\emptyset$) by using the recursive relation

$$p(X_m\cup Y_n) = \sum_{\emptyset\subset Y\subset Y_n} c(X_m,Y)\,p(Y_n\setminus Y) \quad \text{with} \quad c(X_m,\emptyset)=p(X_m). \tag{4.21}$$

Obviously, $c(X_m, Y_n)$ is symmetric in $x_1, \dots, x_m$ as well as in $y_1, \dots, y_n$, but the x_i's and the y_j's cannot be interchanged. Furthermore, we have $c(\{x\}, Y_n) = c(\{x\} \cup Y_n)$ for $x \notin Y_n$ and $n \geq 0$.

As an immediate consequence of (4.21) the recursive relation

$$c(X_m, Y_n) = p(X_m \cup Y_n) - \sum_{\emptyset \subset Y \subsetneq Y_n} c(X_m, Y)\, p(Y_n \setminus Y)$$

reveals that $c(X_m, Y_n)$ coincides with the $(n{+}1)$st order mixed cumulant of the $\{0,1\}$-valued random variables $\prod_{i=1}^m \mathbb{1}_{U_\Xi{}^c}(x_i)$ and $\mathbb{1}_{U_\Xi{}^c}(y_j)$, $j = 1, \dots, n$, that means, formally written, that $c(X_m, Y_n) = \mathrm{Cum}_{n+1}(\mathbb{1}\{U_\Xi \cap X_m = \emptyset\}, \mathbb{1}_{U_\Xi{}^c}(y_1), \dots, \mathbb{1}_{U_\Xi{}^c}(y_n))$.

The relation

$$c(X_m, Y_n) = \sum_{\emptyset \subset Y \subset Y_n} (-1)^{|Y|}\, K(X_m, Y)\, c(X_{m-1} \cup Y, Y_n \setminus Y) \quad \text{for } m + n \geq 1, \tag{4.22}$$

where $K(\emptyset, Y) = 0$ for $Y \neq \emptyset$, and

$$K(X_m, Y) = \sum_{\emptyset \subset V \subset Y} (-1)^{|V|}\, \frac{p(X_m \cup V)}{p(X_{m-1} \cup V)} \quad \text{for} \quad m, n \geq 1, \emptyset \subset Y \subset Y_n$$

has been shown in [Hei05] by direct computation applying Möbius' inversion formula. Setting

$$p(V \mid U) := \frac{p(U \cup V)}{p(U)} = \mathbb{P}(U_\Xi \cap V = \emptyset \mid U_\Xi \cap U = \emptyset),$$

we can rewrite (4.22) in following way:

$$c(X_m, Y_n) = \frac{p(X_m)}{p(X_{m-1})} \sum_{\emptyset \subset Y \subset Y_n} (-1)^{|Y|}\, S(X_m, Y)\, c(X_{m-1} \cup Y, Y_n \setminus Y), \tag{4.23}$$

where $S(\emptyset, Y) = 0$ for $Y \neq \emptyset$ and

$$S(X_m, Y) := \sum_{\emptyset \subset V \subset Y} (-1)^{|V|} \frac{p(V \mid X_m)}{p(V \mid X_{m-1})} \quad \text{for} \quad \emptyset \subset Y \subset Y_n \text{ and } m,\, n \geq 1.$$

For our random set model (2.4), we get with (4.19) that

$$\begin{aligned}
&\frac{p(X_m \cup V)}{p(X_{m-1} \cup V)} \\
&= \exp\Big\{ -\lambda\, \mathbb{E}\, |\Xi_0|_{d-k} + \lambda\, \mathbb{E}\, \big|(\Xi_0 - \pi_{d-k}(\Theta_0^T x_m)) \cap \Xi_0(V \cup X_{m-1})\big|_{d-k} \Big\} \\
&= \exp\Big\{ -\lambda\, \mathbb{E}\, \big|(\Xi_0 - \pi_{d-k}(\Theta_0^T x_m)) \cap \Xi_0^c(X_{m-1})\big|_{d-k} \Big\} \exp\{E(X_m, V)\},
\end{aligned}$$

where

$$E(X_m, V) := \lambda\, \mathbb{E}\left|(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x_m)) \cap \Xi_0^c(X_{m-1}) \cap \Xi_0(V)\right|_{d-k} \quad \text{for} \quad \emptyset \subsetneq V \subset Y_n$$

and $E(X_m, \emptyset) = 0$.

This leads to $p(V \mid X_m)/p(V \mid X_{m-1}) = \exp\{E(X_m, V)\}$, and thus

$$S(X_m, Y) = \sum_{\emptyset \subset V \subset Y} (-1)^{|V|} \exp\{E(X_m, V)\} \quad \text{for} \quad Y \subset Y_n$$

and $S(X_m, \emptyset) = 1$ since $E(X_m, \emptyset) = 0$.

As a simple consequence of (4.23) and $c(X_m, \emptyset) = p(X_m) \le p(X_{m-1}) \le 1$, we get the inequality

$$\begin{aligned}
&\int_{(B_{2\rho}(\mathbf{o}))^n} \left|c(X_m, Y_n)\right| \mathrm{d}Y_n \\
&\le \int_{(B_{2\rho}(\mathbf{o}))^n} \left|c(X_{m-1}, Y_n)\right| \mathrm{d}Y_n + \int_{(B_{2\rho}(\mathbf{o}))^n} \left|S(X_m, Y_n)\right| \mathrm{d}Y_n \\
&\quad + \sum_{\emptyset \subsetneq Y \subsetneq Y_n} \int_{(B_{2\rho}(\mathbf{o}))^{|Y|}} \left|S(X_m, Y)\right| \mathrm{d}Y \\
&\qquad\qquad \times \sup_Y \int_{(B_{2\rho}(\mathbf{o}))^{n-|Y|}} \left|c(X_{m-1} \cup Y, Y_n \setminus Y)\right| \mathrm{d}(Y_n \setminus Y).
\end{aligned} \tag{4.24}$$

For any $m \ge 1$ we have $c(X_m, \{y\}) = p(X_m \cup \{y\}) - p(X_m)\, p(\{y\})$ (≥ 0), and thus, by (4.19),

$$\begin{aligned}
&c(X_m, \{y\}) \\
&= \exp\{-\lambda\, \mathbb{E}\,|\Xi_0(X_m \cup \{y\})|_{d-k}\} - \exp\{-\lambda\, \mathbb{E}\,|\Xi_0(X_m)|_{d-k} - \lambda\, \mathbb{E}\,|\Xi_0|_{d-k}\} \\
&= \exp\{-\lambda\, \mathbb{E}\,|\Xi_0(X_m \cup \{y\})|_{d-k}\} \\
&\qquad \times (1 - \exp\{-\lambda\, \mathbb{E}\,|\Xi_0(X_m) \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\, y))|_{d-k}\}) \\
&\le \lambda \exp\{-\lambda\, \mathbb{E}\,|\Xi_0(X_m)|_{d-k}\} \sum_{i=1}^m \mathbb{E}\left|(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x_i)) \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\, y))\right|_{d-k}.
\end{aligned}$$

Therefore, since $M_1 = \mathbb{E}\,|\Xi_0|_{d-k} \le \mathbb{E}\,|\Xi_0(X_m)|_{d-k}$, we get

$$\int_{B_{2\rho}(\mathbf{o})} c(X_m, \{y\})\, \mathrm{d}y \le \lambda\, \mathrm{e}^{-\lambda M_1} \sum_{i=1}^m \int_{B_{2\rho}(\mathbf{o})} \mathbb{E}\left|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\, (y - x_i)))\right|_{d-k} \mathrm{d}y.$$

The integrals on the right hand side can be bounded from above uniformly in the x_i's. Multiple application of Fubini's theorem combined with the shift-invariance of the Lebesgue measure in $\mathbb{R}^{d-k}$ yields

$$\begin{aligned}
&\int\limits_{B_{2\rho}(\mathbf{o})} \Big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\,(y-x)))\Big|_{d-k}\,\mathrm{d}y \\
&= \int\limits_{B_{2\rho}(\mathbf{o})} \Big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y) + \pi_{d-k}(\Theta_0^T\,x))\Big|_{d-k}\,\mathrm{d}y \\
&\le \int\limits_{[-2\rho,2\rho]^k}\int\limits_{\mathbb{R}^{d-k}} \Big|\Xi_0 \cap (\Xi_0 - z_1 + \pi_{d-k}(\Theta_0^T\,x))\Big|_{d-k}\,\mathrm{d}z_1\,\mathrm{d}z_2 = (4\rho)^k\,|\Xi_0|_{d-k}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} \int\limits_{B_{2\rho}(\mathbf{o})} \mathbb{E}\,\Big|(\Xi_0 - \pi_{d-k}(\Theta_0^T\,x)) \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\,y))\Big|_{d-k}\,\mathrm{d}y \\
&\le (4\rho)^k\;\mathbb{E}\,|\Xi_0|_{d-k}^2
\end{aligned} \tag{4.25}$$

so that we arrive at the uniform estimate

$$\sup_{X_m}\int\limits_{B_{2\rho}(\mathbf{o})} c(X_m,\{y\})\,\mathrm{d}y \le C_{m,1}\,\rho^k \quad \text{with} \quad C_{m,1} = 4^k\,m\,\lambda\,\mathrm{e}^{-\lambda\,M_1}\,M_2. \tag{4.26}$$

Let us introduce a further non-negative function $T(y_n;X_m,Y)$ by

$$T(y_n;X_m,Y) := \sum_{\emptyset\subset V\subset Y} (-1)^{|V|}\,\exp\big\{-E(y_n;X_m,V)\big\} \quad \text{for} \quad Y\subset Y_{n-1},\ n\ge 2,$$

where, for $\emptyset \subset V \subset Y_{n-1}$,

$$\begin{aligned}
&E(y_n;X_m,V) \\
&:= \lambda\;\mathbb{E}\,\Big|(\Xi_0 - \pi_{d-k}(\Theta_0^T\,x_m)) \cap \Xi_0^c(X_{m-1}) \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\,y_n)) \cap \Xi_0(V)\Big|_{d-k}.
\end{aligned}$$

In the next step of our estimation procedure, we determine constants A_n and B_n, only depending on n,λ, and the first $n+1$ moments $M_1,\ldots,M_{n+1}$ of $|\Xi_0|_{d-k}$, such that the uniform estimates

$$\int\limits_{(B_{2\rho}(\mathbf{o}))^n} \big|S(X_m,Y_n)\big|\,\mathrm{d}Y_n \le A_n\,\rho^{k\,n}, \quad \int\limits_{(B_{2\rho}(\mathbf{o}))^n} T(y_n;X_m,Y_{n-1})\,\mathrm{d}Y_n \le B_n\,\rho^{k\,n} \tag{4.27}$$

hold. The following relations between S- and T-functions can be shown in quite analogy to the proof of a corresponding [Hei05, Lemma 4]:

Lemma 4.2. *For any m, $n \geq 1$, we have*

$$S(X_m, Y_n) = S(X_m, Y_{n-1})\Big(1 - \exp\{E(X_m, \{y_n\})\}\Big) - \exp\{E(X_m, \{y_n\})\} \\ \times \sum_{\emptyset \subsetneq Y \subset Y_{n-1}} T(y_n; X_m, Y) \exp\{E(X_m, Y)\}\, S(X_m \cup Y, Y_{n-1} \setminus Y).$$

Combining $E(X_m, Y) \leq \lambda\, \mathbb{E}\,\big|(\Xi_0 - \pi_{d-k}(\Theta_0^T x_m)) \cap \Xi_0(Y)\big|_{d-k} \leq \lambda\, M_1$ and (4.25) leads to

$$\begin{aligned} &\int_{B_{2\rho}(\mathbf{o})} \big|S(X_m, \{y\})\big|\,\mathrm{d}y \\ &= \int_{B_{2\rho}(\mathbf{o})} \Big(\exp\{E(X_m, \{y\})\} - 1\Big)\,\mathrm{d}y \leq 4^k\,\lambda\,\mathrm{e}^{\lambda\, M_1}\, M_2\,\rho^k. \end{aligned} \tag{4.28}$$

Thus, from Lemma 4.2 and $S(X_m, \emptyset) = 1$, it follows after obvious arrangements that

$$\begin{aligned} &\int_{(B_{2\rho}(\mathbf{o}))^n} \big|S(X_m, Y_n)\big|\,\mathrm{d}Y_n \\ &\leq 4^k\,\lambda\,\mathrm{e}^{\lambda\, M_1}\, M_2\,\rho^k \int_{(B_{2\rho}(\mathbf{o}))^{n-1}} \big|S(X_m, Y_{n-1})\big|\,\mathrm{d}Y_{n-1} \\ &\quad + \mathrm{e}^{2\lambda\, M_1} \sum_{j=1}^{n-2} \binom{n-1}{j} \int_{(B_{2\rho}(\mathbf{o}))^{j+1}} T(y_{j+1}; X_m, Y_j)\,\mathrm{d}Y_{j+1} \\ &\qquad\qquad \times \sup_{Y_i} \int_{(B_{2\rho}(\mathbf{o}))^{n-j-1}} \big|S(X_m \cup Y_j, Y_{n-1} \setminus Y_j)\big|\,\mathrm{d}(Y_{n-1} \setminus Y_j) \\ &\quad + \mathrm{e}^{2\lambda\, M_1} \int_{(B_{2\rho}(\mathbf{o}))^n} T(y_n; X_m, Y_{n-1})\,\mathrm{d}Y_n. \end{aligned} \tag{4.29}$$

To make the previous estimate explicit, we need upper bounds for the integrals over $T(y_n; X_m, Y_{n-1})$ with respect to the variables $Y_n = \{y_1, \ldots, y_{n-1}, y_n\}$ for each $n \geq 2$.

Lemma 4.3. *For fixed $n \geq 2$, assume that $M_{n+1} < \infty$. Then, for any $m \geq 1$, both estimates in* (4.27) *hold with*

$$B_n = 4^{k\,n}\,(n-1)! \sum_{j=1}^{n-1} \frac{\lambda^j}{j!} \sum_{\substack{n_1 + \cdots + n_j = n-1 \\ n_1, \ldots, n_j \geq 1}} \frac{M_{n_1+2}}{n_1!} \prod_{i=2}^{j} \frac{M_{n_i+1}}{n_i!} \tag{4.30}$$

and

$$A_n = A_{n-1}\,A_1 + \mathrm{e}^{2\lambda\, M_1} \sum_{j=0}^{n-2} \binom{n-1}{j} A_j\,B_{n-j}, \quad A_0 = 1, \quad A_1 = 4^k\,\lambda\,\mathrm{e}^{\lambda\, M_1}\, M_2. \tag{4.31}$$

Proof. Let the finite point sets $X_m,\ Y \subset Y_{n-1} = \{y_1,\dots,y_{n-1}\}$ and $y_n \in \mathbb{R}^d$ be fixed. Using the independently marked Poisson process $\Pi_{\lambda,Q}$ with typical mark $(\Theta_0,\Xi_0) \sim Q$, we introduce, in accordance with (2.3) and (2.4), a new stationary PCP and the corresponding stationary random union set $U_\Xi(y_n;X_m,Y)$ with typical cylinder base $\Xi_0(y_n;X_m,Y) = (\Xi_0 - \pi_{d-k}(\Theta_0^T\, x_m)) \cap \Xi_0^c(X_{m-1}) \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T\, y_n)) \cap \Xi_0(Y)$ as follows:

$$U_\Xi(y_n;X_m,Y) = \bigcup_{i\ge 1} \Theta_i\Big((\Xi_i(y_n;X_m,Y)+P_i)\times\mathbb{R}^k\Big) = \bigcup_{y\in Y} U_\Xi(y_n;X_m,\{y\}), \quad (4.32)$$

where

$$\Xi_i(y_n;X_m,Y) = (\Xi_i - \pi_{d-k}(\Theta_i^T\, x_m)) \cap \Xi_i^c(X_{m-1}) \cap (\Xi_i - \pi_{d-k}(\Theta_i^T\, y_n)) \cap \Xi_i(Y),$$

$i \ge 1$, are i.i.d. random compact sets in $\mathbb{R}^{d-k}$ with $\Xi_i(Y) = \bigcup_{y\in Y}(\Xi_i - \pi_{d-k}(\Theta_i^T\, y))$, see also (4.19). Note that here we allow for the base of the PCP that $\Xi_0 = \emptyset$ for notational ease.

We first show that $T(y_n;X_m,Y_{n-1})$ gives just the probability that the origin $\mathbf{o}$ lies in each of the union set $U_\Xi(y_n;X_m,\{y_j\})$, $j = 1,\dots,n-1$. With the above-introduced notation it is rapidly seen that

$$\mathbb{P}(\mathbf{o} \notin U_\Xi(y_n;X_m,Y)) = \exp\big\{ -\lambda\,\mathbb{E}\,|\Xi_0(y_n;X_m,Y)|_{d-k}\big\} = \exp\big\{ -E(y_n;X_m,Y)\big\}.$$

Taking into account the relations $\sum_{\emptyset\subset Y\subset Y_{n-1}}(-1)^{|Y|} = 0$ and $U_\Xi(y_n;X_m,\emptyset) = \emptyset$ combined with the second part of (4.32), we find by applying the inclusion-exclusion principle that

$$\begin{aligned} T(y_n;X_m,Y_{n-1}) &= \sum_{\emptyset\subset Y\subset Y_{n-1}} (-1)^{|Y|}\,\mathbb{P}(\mathbf{o}\notin U_\Xi(y_n;X_m,Y)) \\ &= \sum_{\emptyset\subsetneq Y\subset Y_{n-1}} (-1)^{|Y|-1}\,\mathbb{P}\Bigg(\bigcup_{y\in Y}\Big\{\mathbf{o}\in U_\Xi(y_n;X_m,\{y\})\Big\}\Bigg) \\ &= \mathbb{P}\Bigg(\bigcap_{j=1}^{n-1}\Big\{\mathbf{o}\in U_\Xi(y_n;X_m,\{y_j\})\Big\}\Bigg) = \mathbb{E}\prod_{j=1}^{n-1}\mathbb{1}_{U_\Xi(y_n;X_m,\{y_j\})}(\mathbf{o}), \end{aligned}$$

whence, again by Fubini's theorem, it follows that

$$\int\limits_{(B_{2\rho}(\mathbf{o}))^{n-1}} T(y_n;X_m,Y_{n-1})\,\mathrm{d}Y_{n-1} = \mathbb{E}\Bigg(\int\limits_{B_{2\rho}(\mathbf{o})} \mathbb{1}_{U_\Xi(y_n;X_m,\{y\})}(\mathbf{o})\,\mathrm{d}y\Bigg)^{n-1}.$$

Furthermore, the subadditivity of the indicator function $\mathbb{1}_{(\cdot)}(\mathbf{o})$ in combination with

$$\Xi_i(y_n;X_m,\{y\}) \subset (\Xi_i - \pi_{d-k}(\Theta_i^T\, x_m)) \cap (\Xi_i - \pi_{d-k}(\Theta_i^T\, y_n)) \cap (\Xi_i - \pi_{d-k}(\Theta_i^T\, y))$$

leads to

$$\int\limits_{B_{2\rho}(\mathbf{o})} \mathbb{1}_{U_\Xi(y_n;X_m,\{y\})}(\mathbf{o})\,\mathrm{d}y \le \sum_{i\ge 1}\int\limits_{B_{2\rho}(\mathbf{o})} \mathbb{1}_{(\Xi_i(y_n;X_m,\{y\})+P_i)\times\mathbb{R}^k}(\mathbf{o})\,\mathrm{d}y$$
$$\le (4\rho)^k \sum_{i\ge 1} \mathbb{1}_{(\Xi_i-\pi_{d-k}(\Theta_i^T x_m))\cap(\Xi_i-\pi_{d-k}(\Theta_i^T y_n))}(-P_i)\,|\Xi_i|_{d-k}.$$

In the last line, we have replaced the integral of $\mathbb{1}_{\Xi_i+P_i}(\pi_{d-k}(\Theta_i^T y))$ over the ball $B_{2\rho}(\mathbf{o})$ by the larger term $(4\rho)^k\,|\Xi_i|_{d-k}$. Some elementary algebraic rearrangements and the application of the higher-order Campbell formula (2.6) together with the reflection invariance of stationary Poisson processes enable us to rewrite the $(n-1)$st moment of the random sum

$$Z(x_m,y_n) = \sum_{i\ge 1} \mathbb{1}_{(\Xi_i-\pi_{d-k}(\Theta_i^T x_m))\cap(\Xi_i-\pi_{d-k}(\Theta_i^T y_n))}(-P_i)\,|\Xi_i|_{d-k}$$

in the following way:

$$\sum_{j=1}^{n-1} \sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\dots,n_j\ge 1}} \frac{(n-1)!}{j!\,n_1!\cdots n_j!}$$
$$\mathbb{E}\sum_{i_1,\dots,i_j\ge 1}^{*} \prod_{q=1}^{j}\Big(\mathbb{1}_{(\Xi_{i_q}-\pi_{d-k}(\Theta_{i_q}^T x_m))\cap(\Xi_{i_q}-\pi_{d-k}(\Theta_{i_q}^T y_n))}(P_{i_q})\,|\Xi_{i_q}|_{d-k}^{n_q}\Big)$$
$$= \sum_{j=1}^{n-1} \sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\dots,n_j\ge 1}} \frac{\lambda^j\,(n-1)!}{j!\,n_1!\cdots n_j!}$$
$$\prod_{q=1}^{j} \mathbb{E}\Big[\Big|(\Xi_0-\pi_{d-k}(\Theta_0^T x_m))\cap(\Xi_0-\pi_{d-k}(\Theta_0^T y_n))\Big|_{d-k}\,|\Xi_0|_{d-k}^{n_q}\Big],$$

where the sum $\sum^*$ runs over all n-tuples of pairwise distinct indices $i_1,\dots,i_n\ge 1$. Together with

$$\int\limits_{B_{2\rho}(\mathbf{o})} \mathbb{E}\Big[\Big|(\Xi_0-\pi_{d-k}(\Theta_0^T x_m))\cap(\Xi_0-\pi_{d-k}(\Theta_0^T y_n))\Big|_{d-k}\,|\Xi_0|_{d-k}^{n_1}\Big]\,\mathrm{d}y_n$$
$$\le (4\rho)^k\,\mathbb{E}\,|\Xi_0|_{d-k}^{n_1+2}$$

we arrive at

$$\int\limits_{(B_{2\rho}(\mathbf{o}))^n} T(y_n;X_m,Y_{n-1})\,\mathrm{d}Y_n \le (4\rho)^{k(n-1)} \int\limits_{(B_{2\rho}(\mathbf{o}))} \mathbb{E}\,(Z(x_m,y_n))^{n-1}\,\mathrm{d}y_n \le B_n\,\rho^{kn}$$

with B_n as given in (4.30). Hence, the second estimate in (4.27) is proved.

From (4.28) and (4.29) we obtain the first estimate of (4.27) with a recursive relation for the constants A_n with $A_1 = 4^k \, \lambda \, \mathrm{e}^{\lambda M_1} M_2$ and $A_0 = 1$. More precisely,

$$\int_{(B_{2\rho}(\mathbf{o}))^n} |S(X_m, Y_n)| \, \mathrm{d}Y_n$$
$$\leq A_1 \, \rho^k A_{n-1} \, \rho^{k(n-1)} + \mathrm{e}^{2\lambda M_1} \, \rho^{k\, n} \sum_{j=1}^{n-1} \binom{n-1}{j} B_{j+1} \, A_{n-j-1} = A_n \, \rho^{kn},$$

which gives (4.31). Thus, the proof of Lemma 4.3 is completed. □

We are now in a position to prove the estimate

$$\sup_{X_m} \int_{(B_{2\rho}(\mathbf{o}))^n} |c(X_m, Y_n)| \, \mathrm{d}Y_n \leq C_{m,n} \, \rho^{k\, n} \quad \text{for any} \quad m, n \geq 1, \tag{4.33}$$

where $C_{m,n}$ depends on m, n, λ, and $M_1, \ldots, M_{n+1}$. From (4.26) we already know that (4.33) is true for $n = 1$ and any $m \geq 1$. Inserting the first estimate of (4.27) with constants (4.31) on the right hand side of (4.24) we get

$$\int_{(B_{2\rho}(\mathbf{o}))^n} |c(X_m, Y_n)| \, \mathrm{d}Y_n$$
$$\leq C_{m-1,n} \, \rho^{k\, n} + A_n \, \rho^{k\, n} + \sum_{j=1}^{n-1} \binom{n}{j} A_j \, \rho^{k\, j} \, C_{m-1+j,n-j} \, \rho^{k\,(n-j)},$$

which immediately implies the estimate (4.33) and the double-index recursion formula

$$C_{m,n} = A_n + \sum_{j=1}^{n} \binom{n}{j} A_{n-j} \, C_{m-1+n-j,j} \quad \text{with} \quad C_{0,n} = 0 \quad \text{for} \quad m,\, n \geq 1. \tag{4.34}$$

This equation allows to determine successively all constants $C_{m,n}$ starting with $C_{m,2}$ depending on A_1, A_2 for all $m \geq 1$ and afterwards $C_{m,3}$ depending on A_1, A_2, A_3 for all $m \geq 1$ etc. For example, we have $C_{m,2} = A_2 + 2A_1 \, C_{m,1} + C_{m-1,2}$, leading to $C_{m,2} = m \, A_2 + m\,(m+1) \, A_1 \, C_{1,1}$ for $m \geq 1$.

Having in mind the identity $c(\{\mathbf{o}\} \cup Y_n) = c(\{\mathbf{o}\}, Y_n)$, we deduce from (4.20) and (4.33) that

$$\Big| \mathrm{Cum}_n \left(|U_\Xi \cap \rho \, W|_d \right) \Big| \leq |W|_d \, C_{1,n-1} \, \rho^{d+k\,(n-1)}, \tag{4.35}$$

where $C_{1,n-1}$ depends on λ and $M_1, \ldots, M_n$. In the final step, we determine the growth of the constants $C_{1,n-1}$ in dependence on $n \geq 2$ under the assumption (4.1).

In this case, we have $M_n \le n!\, a^{-n}\, m_a$ for $n \in \mathbb{N}$, so that formula (4.30) yields

$$B_n \le 4^{k\,n}\,(n-1)! \sum_{j=1}^{n-1} \frac{\lambda^j}{j!} \frac{m_a^j}{a^{j+1}} \sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\ldots,n_j\ge 1}} (n_1+2) \prod_{i=1}^{j} \frac{(n_i+1)}{a^{n_i}}.$$

Since $n+1 \le 2^n$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\ldots,n_j\ge 1}} (n_1+2) \prod_{i=1}^{j} \frac{(n_i+1)}{a^{n_i}} &\le \frac{n}{a^{n-1}} \sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\ldots,n_j\ge 1}} 2^{n_1+1} \prod_{i=2}^{j} 2^{n_i}\\ &= \frac{n\,2^n}{a^{n-1}} \binom{n-2}{j-1},\end{aligned}$$

which in turn gives

$$B_n \le \frac{2^n\,4^{k\,n}}{a^n}\, n! \sum_{j=1}^{n-1} \frac{\lambda^j\, m_a^j}{a^j} \binom{n-2}{j-1} = \frac{\lambda\, m_a}{a} \left(\frac{2\cdot 4^k}{a}\right)^n \left(1+\frac{\lambda\, m_a}{a}\right)^{n-2} n! \quad \text{for} \quad n \ge 2.$$

In summary, using the abbreviations

$$A = \frac{2^{2k+1}}{a}\left(1+\mathrm{e}^{\lambda\,\mathbb{E}\,|\Xi_0|_{d-k}}\right) \quad \text{and} \quad B = \frac{\lambda\,\mathbb{E}\,\mathrm{e}^{a\,|\Xi_0|_{d-k}}}{a},$$

the positive constants A_n and B_n in (4.27) satisfy the estimates $A_1 \le A\,B$ and

$$A_n \le A^n\,B\,(1+B)^{n-1}\,n!, \quad B_n \le B\left(\frac{2^{2k+1}}{a}\right)^n (1+B)^{n-2}\,n! \quad \text{for } n \ge 2. \tag{4.36}$$

The first relation follows from (4.31) by induction on n. In fact, by $M_2 \le 2m_a/a^2$, we have

$$A_1 = 4^k\,\lambda\,\mathrm{e}^{\lambda\,M_1}\,M_2 \le 4^k\,\lambda\,\mathrm{e}^{\lambda\,M_1}\frac{2\,m_a}{a^2} = \frac{2^{2k+1}}{a}\,\mathrm{e}^{\lambda\,M_1}\,B \le A\,B,$$

and, for $n \ge 2$, we combine the recursive relation (4.31) with $A_0 = 1$ and the second (already proved) estimate in (4.36):

$$\begin{aligned} A_n &= A_1\,A_{n-1} + \mathrm{e}^{2\lambda\,M_1} \sum_{j=0}^{n-2} \binom{n-1}{j} A_j\,B_{n-j}\\ &\le A_1\,A_{n-1} + \mathrm{e}^{2\lambda\,M_1}\,B \sum_{j=0}^{n-2} \binom{n-1}{j} A_j \left(\frac{2^{2k+1}}{a}\right)^{n-j} (1+B)^{n-j-2}\,(n-j)! \end{aligned}$$

Replacing A_j by $B\,(1+B)^{j-1}\,j!$ for $j=1,\dots,n-1$, we find after some elementary calculations the asserted first estimate in (4.36).

In the same way, the recursive relation (4.34) suggests an inductive proof of the estimate

$$C_{m,n} \le 2^{m-1}\,4^{n-1}\,A^n\,B\,(1+B)^{n-1}\,n! \quad \text{for} \quad n,\,m\ge 1,$$

whence with (4.35) it follows the desired estimate (4.9) completing the proof of Theorem 4.3.

Now, we apply the general lemma on large deviations, including a Berry-Esseen bound proved by V. Statulevičius in [Sta66], see also [SS91, Lemma 2.3]. This result is formulated for a single random variable ξ satisfying $\mathbb{E}\,\xi = 0$, $\mathrm{Var}(\xi) = 1$, and $|\,\mathrm{Cum}_n(\xi)| \le n!\,H/\Delta^{n-2}$ for $n \ge 2$ and some $H \ge 1/2$ and $\Delta > 0$. In our specific situation, ξ is chosen to be the standardized d-volume $V_\rho^{(d,k)}$, i.e.,

$$\xi = \frac{V_\rho^{(d,k)} - \mathbb{E}\,V_\rho^{(d,k)}}{\sqrt{\mathrm{Var}(V_\rho^{(d,k)})}} = \frac{V_\rho^{(d,k)} - \rho^d\,|W|_d\,\big(1-\mathrm{e}^{-\lambda\,\mathbb{E}\,|\Xi_0|_{d-k}}\big)}{\sigma_\rho\,\rho^{(d+k)/2}}$$

with distribution function $F_\rho(x) = \mathbb{P}(\xi \le x)$. Using (4.9) and the notation introduced in Section 4.1.2, we obtain that

$$|\,\mathrm{Cum}_n(\xi)| \le (n-1)!\,\frac{H_a\,\Delta_a^{n-2}\,\rho^{d+k(n-1)}}{\big(\mathrm{Var}(V_\rho^{(d,k)})\big)^{n/2}} \le n!\;H_{a,\rho}/\Delta_{a,\rho}^{n-2}, \tag{4.37}$$

where $H_{a,\rho} = H_a/2\sigma_\rho^2$ ($\ge 1/2$ by (4.9) for $n=2$) and $\Delta_{a,\rho} = \rho^{(d-k)/2}\,\sigma_\rho/\Delta_a$.

These estimates and the lemma in [Sta66, p. 133] imply the asymptotic relations (4.11) and (4.12) as well as the Berry-Esseen bound (4.14) stated in Theorem 4.4. It should be noted that, according to the general result in [Sta66] or [SS91], the relations (4.11) and (4.12) hold in a smaller interval $0 \le x \le \delta^*\,\Delta_{a,\rho}$ for $\delta^* < \delta(1+\delta)/2$, where $\delta \in (0,1)$ is uniquely determined by the equation $(1-\delta)^3 = 6\,H_{a,\rho}\,\delta$ giving $\delta\,(1+\delta)/2 \le 1/2(1+4\,H_{a,\rho})$. However, a careful check of the original proof reveals that (4.11) and (4.12) remain valid in the desired interval $0 \le x \le \Delta_{a,\rho}/2(1+4H_{a,\rho})$, see [Hei05], which completes the proof of Theorem 4.4.

Proof of Corollary 4.2. The first two inequalities in Corollary 4.2 are an immediate consequence of (4.11), (4.12), and (4.13).

The upper bound for c_3 can be derived as follows. As shown in (4.37), for $n \ge 3$ we have

$$|\,\mathrm{Cum}_n(\xi)| \le n!\,H_{a,\rho}/\Delta_{a,\rho}^{n-2} \le \frac{n!}{(\Delta_{a,\rho}/2H_{a,\rho})^{n-2}} = \frac{n!}{\big(\rho^{(d-k)/2}\,\sigma_\rho^3/(\Delta_a\,H_a)\big)^{n-2}}$$

as $H_{a,\rho} \ge 1/2$. The application of Corollary 2.1 in[SS91] yields the stated bound. $\square$

4.4. A truncation technique for $V_\rho^{(d,k)}$; proof of Theorem 4.1

In this section, we assume that the preliminaries of Theorem 4.1 hold, i.e., we let U_Ξ be the union set of the stationary PCP $\Pi_{\text{cyl}}^{(d,k)}(\lambda, Q)$ with typical cylinder base $\Xi_0 \in \mathcal{R}_{d-k}^o$ satisfying $0 < M_2 < \infty$ and drop the assumption on the exponential moment of Ξ_0. Further, we suppose that $W \subset \mathbb{R}^d$ is compact and star-shaped with respect to $\mathbf{o}$ satisfying $B_{\delta_W}(\mathbf{o}) \subset W \subset B_1(\mathbf{o})$ for some $\delta_W \in (0,1]$. Our aim is to prove (4.4).

We begin by introducing a truncated version $U_\Xi{}^{(\tau)}$ of the union set of the PCP (2.4)

$$U_\Xi{}^{(\tau)} = \bigcup_{i \geq 1} \Theta_i\Big((\Xi_i^{(\tau)} + P_i) \times \mathbb{R}^k \Big), \tag{4.38}$$

where the second component of the typical mark (Θ_0, Ξ_0) in (2.4) is replaced by the truncated typical grain

$$\Xi_0^{(\tau)} = \begin{cases} \Xi_0, & \text{if} \quad |\Xi_0|_{d-k} \leq \tau, \\ \emptyset, & \text{if} \quad |\Xi_0|_{d-k} > \tau, \end{cases} \qquad \text{with} \qquad \tau = \varepsilon\, \rho^{(d-k)/2}. \tag{4.39}$$

for arbitrarily small $\varepsilon > 0$ and large enough $\rho > 0$ such that $\tau \geq 1$ just for convenience. Note that for this cylinder process we allow the base to be empty for notational ease, although in the definition in Section 2.2 this is excluded.

Obviously, by (2.3) and (2.4), we have $U_\Xi{}^{(\tau)} \subset U_\Xi$ as well as the inclusion

$$U_\Xi \setminus U_\Xi{}^{(\tau)} \subset \bigcup_{i \geq 1} \Theta_i \left((\Xi_i \setminus \Xi_i^{(\tau)} + P_i) \times \mathbb{R}^k \right) =: \widetilde{U_\Xi}^{(\tau)},$$

where $\widetilde{U_\Xi}^{(\tau)}$ can be regarded as a PCP with typical mark $(\Xi_0 \setminus \Xi_0^{(\tau)}, \Theta_0)$. The latter relation yields

$$\begin{aligned} \mathbb{E}\big|(U_\Xi \setminus U_\Xi{}^{(\tau)}) \cap \rho\, W\big|_d^2 &\leq \mathbb{E}\big|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W\big|_d^2 \\ &= \operatorname{Var}\big(\big|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W\big|_d\big) + \big(\mathbb{E}\big|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W\big|_d\big)^2. \end{aligned}$$

Replacing Ξ_0 in (4.16) by $\Xi_0 \setminus \Xi_0^{(\tau)}$ we obtain that

$$\begin{aligned} \operatorname{Var}\big(\big|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W\big|_d\big) &\leq \lambda\, |W|_d \exp\big\{ -\lambda\, \mathbb{E}\, |\Xi_0 \setminus \Xi_0^{(\tau)}|_{d-k} \big\}\, 4^k\, \mathbb{E}|\Xi_0 \setminus \Xi_0^{(\tau)}\big|_{d-k}^2\, \rho^{d+k} \\ &\leq \lambda\, |W|_d\, 4^k\, \mathbb{E}|\Xi_0\big|_{d-k}^2\, \mathbb{1}\big(|\Xi_0|_{d-k} > \tau\big)\, \rho^{d+k}, \end{aligned}$$

and, by $\mathbb{E}\,|U_\Xi \cap B|_d = \big(1 - \mathrm{e}^{-\lambda M_1}\big)\,|B|_d \le \lambda\, M_1\, |B|_d$ for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$, we get the inequality

$$\begin{aligned}\big(\mathbb{E}\,|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W|_d\big)^2 &\le \lambda^2\, |W|_d^2\, \rho^{2d}\, \big(\mathbb{E}\,|\Xi_0 \setminus \Xi_0^{(\tau)}|_{d-k}\big)^2 \\ &\le \lambda^2\, |W|_d^2\, \rho^{d+k}\, \varepsilon^{-2} \Big(\mathbb{E}\,|\Xi_0|_{d-k}^2\, \mathbb{1}\big(|\Xi_0|_{d-k} > \tau\big)\Big)^2.\end{aligned}$$

Setting

$$M_2(\varepsilon,\tau) = \varepsilon^{-2}\ \mathbb{E}\,|\Xi_0|_{d-k}^2\, \mathbb{1}(|\Xi_0|_{d-k} > \tau)$$

we arrive together with Chebyshev's inequality at

$$\begin{aligned}\mathbb{P}\big(\rho^{-(d+k)/2}\, \big|(U_\Xi \setminus U_\Xi{}^{(\tau)}) \cap \rho\, W)\big|_d \ge \varepsilon\big) &\le \varepsilon^{-2}\, \rho^{-(d+k)}\ \mathbb{E}\big|(U_\Xi \setminus U_\Xi{}^{(\tau)}) \cap \rho\, W\big|_d^2 \\ &\le \lambda\, |W|_d\, (4^k + \lambda\, |W|_d\, M_2(\varepsilon,\tau))\, M_2(\varepsilon,\tau) \underset{\rho\to\infty}{\longrightarrow} 0\end{aligned}$$

for any $\varepsilon > 0$. By the same arguments,

$$\rho^{-(d+k)/2}\ \mathbb{E}\,|(U_\Xi \setminus U_\Xi{}^{(\tau)}) \cap \rho\, W)|_d \le \Big(\rho^{-(d+k)}\ \mathbb{E}\,|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W)|_d^2\Big)^{1/2} \underset{\rho\to\infty}{\longrightarrow} 0,$$

and, together with $U_\Xi{}^{(\tau)} \subset U_\Xi$ and Minkowski's inequality, we get that

$$\begin{aligned}&\rho^{-(d+k)}\, \big|\operatorname{Var}\big(|U_\Xi \cap \rho\, W|_d\big) - \operatorname{Var}\big(|U_\Xi{}^{(\tau)} \cap \rho\, W|_d\big)\big| \\ &\le \rho^{-(d+k)} \Big|\big(\operatorname{Var}\big(|U_\Xi \cap \rho\, W|_d\big)\big)^{1/2} - \big(\operatorname{Var}\big(|U_\Xi{}^{(\tau)} \cap \rho\, W|_d\big)\big)^{1/2}\Big| \\ &\qquad\times \Big(\big(\operatorname{Var}\big(|U_\Xi \cap \rho\, W|_d\big)\big)^{1/2} + \big(\operatorname{Var}\big(|U_\Xi{}^{(\tau)} \cap \rho\, W|_d\big)\big)^{1/2}\Big) \\ &\le \rho^{-(d+k)}\, \big(\mathbb{E}\,|\widetilde{U_\Xi}^{(\tau)} \cap \rho\, W|_d^2\big)^{1/2} \Big(\big(\operatorname{Var}(|U_\Xi{}^{(\tau)} \cap \rho\, W|_d)\big)^{1/2} + \big(\operatorname{Var}(|U_\Xi \cap \rho\, W|_d)\big)^{1/2}\Big) \\ &\underset{\rho\to\infty}{\longrightarrow} 0.\end{aligned}$$

In summary, by applying Slutzky's theorem, to prove the limit (4.4) in Theorem 4.1 it suffices to verify the CLT

$$\frac{|U_\Xi{}^{(\tau)} \cap \rho\, W|_d - \mathbb{E}\,|U_\Xi{}^{(\tau)} \cap \rho\, W|_d}{\sqrt{\operatorname{Var}\big(|U_\Xi{}^{(\tau)} \cap \rho\, W|_d\big)}} \quad \underset{\rho\to\infty}{\overset{d}{\longrightarrow}} \quad \mathrm{N}(0,1) \tag{4.40}$$

for the truncated model $U_\Xi{}^{(\tau)}$ instead of U_Ξ. Notice that, by standard arguments, $\varepsilon > 0$ can be chosen as null sequence $\varepsilon(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$ such that $\tau(\rho) = \varepsilon(\rho)\, \rho^{(d-k)/2} \underset{\rho\to\infty}{\longrightarrow} \infty$ and $M_2(\varepsilon(\rho), \tau(\rho)) \underset{\rho\to\infty}{\longrightarrow} 0$.

The following lemma yields the proof of (4.40).

Lemma 4.4. *Provided that $M_2 < \infty$, the truncated* PCP *(4.38) with $\tau = \varepsilon\,\rho^{(d-k)/2}$ allows the estimates*

$$\rho^{-(d+k)\,n/2}\,\Big|\operatorname{Cum}_n\big(|U_\Xi{}^{(\tau)} \cap \rho\,W|_d\big)\Big| \le \varepsilon^{n-2}\,c_n(\lambda)\,|W|_d \quad \textit{for} \quad n \ge 3,$$

where the constants $c_n(\lambda)$ depend only on λ, n, and on the moments M_1 and M_2.

Proof. We replace in the calculations in Section 4.3 the typical cylinder base Ξ_0 by the truncated cylinder base (4.39) of the union set of the PCP, $U_\Xi{}^{(\tau)}$. Hence, in Lemma 4.3 the moments M_j in (4.30) are replaced by the truncated moments $M_j^{(\tau)} = \mathbb{E}\,|\Xi_0^{(\tau)}|_{d-k}^j$ for $j = 2, \ldots, n+1$. The inequality $M_j^{(\tau)} \le \tau^{j-2}\,M_2$ leads to

$$B_n \le 4^{k\,n}\sum_{j=1}^{n-1}\frac{(\lambda\,M_2)^j}{j!}\,\tau^{n-j}\sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\ldots,n_j\ge 1}}\frac{(n-1)!}{n_1!\cdots n_j!} \le \tau^{n-1}\,b_n(\lambda),$$

since $\tau \ge 1$, where

$$b_n(\lambda) = 4^{k\,n}\sum_{j=1}^{n-1}\frac{(\lambda\,M_2)^j}{j!}\sum_{\substack{n_1+\cdots+n_j=n-1\\ n_1,\ldots,n_j\ge 1}}\frac{(n-1)!}{n_1!\cdots n_j!}.$$

A simple inductive argument shows that for A_n from Lemma 4.3

$$A_n \le \tau^{n-1}\,a_n(\lambda) \quad \text{for} \quad n \ge 1,$$

for $a_0(\lambda) = 1$, $a_1(\lambda) = A_1$ and $a_n(\lambda) = a_{n-1}(\lambda)\,a_1(\lambda) + \mathrm{e}^{2\lambda\,M_1}\sum_{j=0}^{n-2}\binom{n-1}{j}a_j(\lambda)\,b_{n-j}(\lambda)$ for $n \ge 2$. Finally, we put $c_{m,1}(\lambda) = C_{m,1}$ for $m \ge 1$, where $C_{m,n}$ is defined as in (4.34). In view of $C_{m,2} - C_{m-1,2} = A_2 + 2A_1\,C_{m,1}$ and $C_{0,n} = 0$, it is easy to see that $C_{m,2} = m\,A_2 + 2A_1\,(C_{m,1} + \cdots + C_{1,1}) \le c_{m,2}\,\tau$ with the constants $c_{m,2} = m\,a_2(\lambda) + 2a_1(\lambda)\,(c_{m,1}(\lambda) + \cdots + c_{1,1}(\lambda))$ for any $m \ge 1$. In this way we may proceed for $n = 3, 4, \ldots$ and arrive at $C_{m,n} \le c_{m,n}(\lambda)\,\tau^{n-1}$ for all $n \ge 3$ and $m \ge 1$, where the numbers $c_{m,n}(\lambda)$ are defined recursively by the relation $c_{m,n}(\lambda) = a_n(\lambda) + \sum_{j=0}^{n-1}\binom{n}{j}a_j(\lambda)\,c_{m-1+j,n-j}(\lambda)$. Thus, after inserting $\tau = \varepsilon\,\rho^{(d-k)/2}$, we find that

$$C_{1,n}\,\rho^{k\,n} \le \varepsilon^{n-1}\,c_{1,n}(\lambda)\,\rho^{-d+(d+k)(n+1)/2} \quad \text{for} \quad n \ge 2.$$

This estimate combined with (4.35) and the choice of $\varepsilon(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$ completes the proof. □

4.5. The asymptotic variance of $V_\rho^{(d,k)}$; proof of Theorem 4.2

The aim of this section is to prove that both of the limits in (4.17) coincide. For this, we consider the cases of atomic and diffuse (marginal) distribution of Θ_0 separately.

4.5.1. Diffuse directional distributions

We first prove the second result of Theorem 4.2 with diffuse distribution $\mathbb{P}_0$ of Θ_0, i.e., $\mathbb{P}(\Theta_0 = \theta) = 0$ for all $\theta \in \mathbb{SO}_k^d$. The inequality $0 \le \mathrm{e}^x - 1 - x \le x^2\,\mathrm{e}^x/2$ for $x \ge 0$ leads to

$$\Big| \mathrm{Var}\big(|U_\Xi \cap \rho W|_d\big) - \lambda \mathrm{e}^{-2\lambda M_1} \int\limits_{\mathbb{R}^d} |\rho W \cap (\rho W - x)\big|_d \, \mathbb{E}\,\big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(\Theta_0^T x)\big)\big|_{d-k} \,\mathrm{d}x \Big|$$
$$\le \frac{\lambda^2}{2}\,\mathrm{e}^{-\lambda M_1}\, |\rho\, W|_d \int_{\rho\,(W \oplus (-W))} \Big(\mathbb{E}\,\big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x)\big)\big|_{d-k} \Big)^2 \,\mathrm{d}x$$
$$\le \frac{\lambda^2}{2}\,\mathrm{e}^{-\lambda M_1}\, \rho^d\, |W|_d \int_{B_{2\rho}(\mathbf{o})} \Big(\mathbb{E}\,\big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x)\big)\big|_{d-k} \Big)^2 \,\mathrm{d}x.$$

We divide both sides of the previous inequality by ρ^{d+k} and show in the next step that

$$J_\rho = \rho^{-k} \int_{B_\rho(\mathbf{o})} \Big(\mathbb{E}\,\big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x)\big)\big|_{d-k} \Big)^2 \,\mathrm{d}x \underset{\rho\to\infty}{\longrightarrow} 0. \tag{4.41}$$

Taking an independent copy $(\widetilde{\Theta}_0, \widetilde{\Xi}_0)$ of the mark $(\Theta_0, \Xi_0) \sim Q$, applying Fubini's theorem and substituting $x = \Theta_0\, y$, we may rewrite J_ρ with the total expectation formula in the following way:

$$\begin{aligned}
J_\rho &= \rho^{-k}\,\mathbb{E}\Big[\int_{B_\rho(\mathbf{o})} \big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(\Theta_0^T\, x)\big)\big|_{d-k} \big|\widetilde{\Xi}_0 \cap \big(\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\Theta}_0^T\, x)\big)\big|_{d-k} \,\mathrm{d}x \Big] \\
&= \rho^{-k}\,\mathbb{E}\Big[\int_{B_\rho(\mathbf{o})} \big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(y)\big)\big|_{d-k} \big|\widetilde{\Xi}_0 \cap \big(\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\Theta}_0^T \Theta_0\, y)\big)\big|_{d-k} \,\mathrm{d}y \Big] \\
&= \rho^{-k} \int_{\mathbb{SO}_k^d} \int_{\mathbb{SO}_k^d} \mathbb{E}\Big[\int_{B_\rho(\mathbf{o})} \big|\Xi_0 \cap \big(\Xi_0 - \pi_{d-k}(y)\big)\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap \big(\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, y)\big)\big|_{d-k} \,\mathrm{d}y \;\Big|\; \Theta_0 = \theta, \widetilde{\Theta}_0 = \widetilde{\theta} \Big] \mathbb{P}_0(\mathrm{d}\widetilde{\theta})\,\mathbb{P}_0(\mathrm{d}\theta).
\end{aligned}$$

Since $\mathbb{P}_0$ is diffuse and Θ_0 and $\widetilde{\Theta}_0$ are stochastically independent, it follows that $\mathbb{P}(\Theta_0 = \widetilde{\Theta}_0) = 0$. Thus, it suffices to show that the inner integral disappears as $\rho \to \infty$ for any pair $(\theta, \widetilde{\theta}) \in \mathbb{SO}_k^d \times \mathbb{SO}_k^d$ with $\theta \ne \widetilde{\theta}$. For this purpose, we consider the subspace $E = (\theta^T\, \widetilde{\theta}\, E_k) \cap E_k$ with dimension $\dim E =: l \in \{0, \ldots, k-1\}$ depending on the choice of the distinct orthogonal matrices θ and $\widetilde{\theta}$. We note that $\dim E = k$ would imply $\theta^T \widetilde{\theta} E_k = E_k$, and this gives $\theta = \widetilde{\theta}$ by the very definition of $\mathbb{SO}_k^d$. Furthermore, let $\vartheta \in \mathbb{SO}_d$ be chosen such that $E = \vartheta E_l$ and $\vartheta E_k = E_k$ (such ϑ always exists). Now, setting $y = (y_1, y_2)^T$ with $y_1 \in \mathbb{R}^{d-l}$ and $y_2 \in \mathbb{R}^l$, we can continue to estimate the above inner integral over $B_\rho(\mathbf{o})$ as follows:

$$
\begin{aligned}
&\rho^{-k} \int_{B_\rho^d(\mathbf{o})} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y))\big|_{d-k} \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, y))\big|_{d-k} \,\mathrm{d}y \\
&\le \rho^{-k} \int_{B_\rho^l(\mathbf{o})} \int_{B_\rho^{d-l}(\mathbf{o})} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(y_1, y_2)^T))\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, y_2)^T))\big|_{d-k} \,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&\le \rho^{-k} \int_{B_\rho^l(\mathbf{o})} \int_{B_\rho^{d-l}(\mathbf{o})} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(y_1, \mathbf{o}_l)^T))\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, \mathbf{o}_l)^T))\big|_{d-k} \,\mathrm{d}y_1\,\mathrm{d}y_2,
\end{aligned}
\tag{4.42}
$$

using that $\widetilde{\theta}^T \theta\, \vartheta\, E_l$ and $\vartheta\, E_l$ are subspaces of E_k with dimension less than k. It follows that $\pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\, y) = \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, \mathbf{o}_l))$ and $\pi_{d-k}(\vartheta\, y) = \pi_{d-k}(\vartheta\,(y_1, \mathbf{o}_l))$, i.e., the integrand does not depend on y_2, and we can take $y_2 = \mathbf{o}_l$ and evaluate the integral over $y_2 \in B_\rho^l(\mathbf{o})$.

Further, by setting $y_1 = (z_1, z_2)^T$ with $z_1 \in \mathbb{R}^{d-k}$ and $z_2 \in \mathbb{R}^{k-l}$, we get the following upper bound of term (4.42):

$$
\begin{aligned}
&\rho^{-(k-l)}\, \kappa_l \int_{B_\rho^{k-l}(\mathbf{o})} \int_{\mathbb{R}^{d-k}} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(z_1, z_2, \mathbf{o}_l)^T))\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(z_1, z_2, \mathbf{o}_l)^T))\big|_{d-k} \,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \kappa_l \int_{B_1^{k-l}(\mathbf{o})} \int_{\mathbb{R}^{d-k}} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta(z_1, \mathbf{o}_k)^T))\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta(z_1, \rho\, z_2, \mathbf{o}_l)^T))\big|_{d-k} \,\mathrm{d}z_1\,\mathrm{d}z_2 \underset{\rho\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

where we have used the relations $\pi_{d-k}(\vartheta\,(z_1, z_2, \mathbf{o}_l)^T) = \pi_{d-k}(\vartheta\,(z_1, \mathbf{o}_k)^T)$ and $\|\pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta(z_1, \rho\, z_2, \mathbf{o}_l)^T)\| \underset{\rho\to\infty}{\longrightarrow} \infty$ for $z_2 \neq \mathbf{o}_{k-l}$ and any $z_1 \in \mathbb{R}^{d-k}$. Finally, applying the dominated convergence theorem completes the proof of (4.41).

Turning back at the beginning of Subsection 4.5.1 we see that in case of diffuse $\mathbb{P}_0$

the limit (4.5) is obtained in the following way

$$\begin{aligned}
&\frac{\lambda\,\mathrm{e}^{-2\lambda\,M_1}}{\rho^{d+k}}\int_{\rho(W\oplus(-W))}\big|\rho\,W\cap(\rho\,W-x)\big|_d\;\mathbb{E}\,\big|\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T\,x)\big|_{d-k}\,\mathrm{d}x\\
&=\frac{\lambda\,\mathrm{e}^{-2\lambda\,M_1}}{\rho^{d+k}}\;\mathbb{E}\Big[\int_{\mathbb{R}^{d-k}}\int_{\mathbb{R}^k}\big|\rho\,W\cap(\rho\,W-\Theta_0(x_1,x_2)^T)\big|_d\,\big|\Xi_0\cap(\Xi_0-x_1)\big|_{d-k}\,\mathrm{d}x_2\,\mathrm{d}x_1\Big]\\
&=\lambda\,\mathrm{e}^{-2\lambda\,M_1}\;\mathbb{E}\Big[\int_{\mathbb{R}^{d-k}}\int_{\mathbb{R}^k}\big|W\cap(W-\Theta_0(\tfrac{x_1}{\rho},x_2)^T)\big|_d\,\big|\Xi_0\cap(\Xi_0-x_1)\big|_{d-k}\,\mathrm{d}x_2\,\mathrm{d}x_1\Big]\\
&\underset{\rho\to\infty}{\longrightarrow}\;\lambda\,\mathrm{e}^{-2\lambda\,M_1}\int_{\mathbb{SO}_k^d}\mathbb{E}\Big[|\Xi_0|_{d-k}^2\;\Big|\;\Theta_0=\theta\Big]\int_{\mathbb{R}^k}\big|W\cap(W-\theta(\mathbf{o}_{d-k},x)^T)\big|_d\,\mathrm{d}x\,\mathbb{P}_0(\mathrm{d}\theta).
\end{aligned}$$

This finishes the proof of (4.7).

4.5.2. Discrete directional distributions

Let $\mathbb{P}_0$ be an atomic distribution, i.e., its support is some finite or countably infinite set $\{\theta_i\in\mathbb{SO}_k^d,\ i\in I\}$ of distinct matrices in $\mathbb{SO}_k^d$; for convenience let $I=\mathbb{N}$. With the notation of Theorem 4.2 we have $f(y,\theta_i)=\mathbb{E}\big[|\Xi_0\cap(\Xi_0-y)|_{d-k}\;\big|\;\Theta_0=\theta_i\big]\,\mathbb{P}_0(\{\theta_i\})$ for $i\in\mathbb{N}$ and $y\in\mathbb{R}^{d-k}$.

To begin with we state the elementary inequality

$$\mathrm{e}^{x_1+\cdots+x_n}-1-\sum_{i=1}^{n}(\mathrm{e}^{x_i}-1)\le\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mathrm{e}^{x_i}-1)(\mathrm{e}^{x_j}-1)\mathrm{e}^{x_1+\cdots+x_n}$$

for $x_1,\ldots,x_n\ge0$, which can be verified by induction on $n\in\mathbb{N}$ and remains valid also in the limit $n\to\infty$.

Applying the previous inequality to the points $x_i=\lambda\,f(\pi_{d-k}(\theta_i^T x),\theta_i)$ for $i\in\mathbb{N}$ and $x\in\mathbb{R}^d$, we are led to the estimate

$$\begin{aligned}
&\Big|\mathrm{Var}(|U_\Xi\cap\rho\,W|_d)\\
&\qquad-\mathrm{e}^{-2\,\lambda\,M_1}\int_{\mathbb{R}^d}\big|\rho\,W\cap(\rho\,W-x)\big|_d\sum_{i=1}^{\infty}\Big(\exp\{\lambda\,f(\pi_{d-k}(\theta_i^T x),\theta_i)\}-1\Big)\,\mathrm{d}x\Big|\\
&\le\lambda^2\,|W|_d\,\rho^d\int_{B_{2\rho}(\mathbf{o})}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}f(\pi_{d-k}(\theta_i^T x),\theta_i)\,f(\pi_{d-k}(\theta_j^T x),\theta_j)\,\mathrm{d}x,
\end{aligned}$$

where the simple relations $x_i+x_j+\sum_{k=1}^{\infty}x_k\le2\lambda\,M_1$ for all $i<j$ and $\mathrm{e}^{x_i}-1\le x_i\,\mathrm{e}^{x_i}$ have been used.

In analogy to (4.41) we divide both sides of the previous inequality by ρ^{d+k} and prove that

$$I_\rho=\rho^{-k}\int_{B_\rho(\mathbf{o})}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}f(\pi_{d-k}(\theta_i^T x),\theta_i)\,f(\pi_{d-k}(\theta_j^T x),\theta_j)\,\mathrm{d}x\underset{\rho\to\infty}{\longrightarrow}0.$$

For any $\varepsilon > 0$ there exists an integer $n = n(\varepsilon)$ such that $\sum_{i=n+1}^{\infty} f(\mathbf{o}_{d-k}, \theta_i) \leq \varepsilon$, and this yields the estimate

$$\begin{aligned} I_\rho \leq \varepsilon\, \rho^{-k} \sum_{i=1}^{\infty} \int_{B_\rho(\mathbf{o})} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, \mathrm{d}x \\ + \rho^{-k} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_{B_\rho(\mathbf{o})} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, f(\pi_{d-k}(\theta_j^T x), \theta_j)\, \mathrm{d}x. \end{aligned} \tag{4.43}$$

By setting $x = (x_1, x_2)^T$ with $x_1 \in \mathbb{R}^{d-k}$ and $x_2 \in \mathbb{R}^k$, it is easily seen that the first summand in (4.43) is equal to

$$\begin{aligned} &\varepsilon\, \rho^{-k} \sum_{i=1}^{\infty} \int_{B_\rho^d(\mathbf{o})} \mathbb{E}\left[\left|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\theta_i^T (x_1, x_2)^T))\right|_{d-k} \mathbb{1}\{\Theta_0 = \theta_i\} \right] \mathrm{d}(x_1, x_2) \\ &= \varepsilon\, \rho^{-k} \int_{B_\rho^d(\mathbf{o})} \mathbb{E} \left|\Xi_0 \cap (\Xi_0 - \pi_{d-k}((x_1, x_2)^T))\right|_{d-k} \mathrm{d}(x_1, x_2) \\ &\leq \varepsilon\, \rho^{-k} \int_{B_\rho^k(\mathbf{o})} \int_{\mathbb{R}^{d-k}} \mathbb{E} \left|\Xi_0 \cap (\Xi_0 - x_1)\right|_{d-k} \mathrm{d}x_1\, \mathrm{d}x_2 = \varepsilon\, \kappa_k\, M_2. \end{aligned}$$

In order to treat the finite double sum in (4.43), it suffices to consider the integral

$$\begin{aligned} &\rho^{-k} \int_{B_\rho(\mathbf{o})} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, f(\pi_{d-k}(\theta_j^T x), \theta_j)\, \mathrm{d}x \\ &= \rho^{-k} \int_{B_\rho(\mathbf{o})} f(\pi_{d-k}(\theta_i^T \theta_j y), \theta_i)\, f(\pi_{d-k}(y), \theta_j)\, \mathrm{d}y \\ &= \rho^{-k} \int_{B_\rho(\mathbf{o})} \mathbb{E}\Big[|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\theta_i^T \theta_j y))|_{d-k} \;\Big|\; \Theta_0 = \theta_i \Big]\, \mathbb{P}_0(\{\theta_i\}) \\ &\qquad\qquad \times \mathbb{E}\Big[|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y))|_{d-k} \;\Big|\; \Theta_0 = \theta_j \Big]\, \mathbb{P}_0(\{\theta_j\})\, \mathrm{d}y. \end{aligned}$$

for a single pair $i < j$. This integral can be shown to converge to 0 as $\rho \to \infty$ by repeating quite the same steps carried out to show that the integral (4.42) disappears as $\rho \to \infty$. Thus, the total sum in (4.43) can be made arbitrarily small. This means that the existence and the explicit form of the limit (4.5) in case of atomic $\mathbb{P}_0$ is proved by finding the limit (as $\rho \to \infty$) of

$$\frac{\mathrm{e}^{-2\lambda M_1}}{\rho^{d+k}} \int_{\rho\,(W \oplus (-W))} \left|\rho\, W \cap (\rho\, W - x)\right|_d \sum_{i=1}^{\infty} \Big(\exp\{\lambda\, f(\pi_{d-k}(\theta_i^T x), \theta_i)\} - 1 \Big)\, \mathrm{d}x.$$

Making use of the monotone convergence theorem, we first interchange integration and summation, and then we pass to the limit for each term of the above sum:

$$\begin{aligned}
&\frac{1}{\rho^{d+k}} \int_{\rho\,(W\oplus(-W))} |\rho\, W\cap(\rho\, W-x)|_d \Big(\exp\{\lambda\, f(\pi_{d-k}(\theta_i^T x),\theta_i)\}-1\Big)\,\mathrm{d}x \\
&= \frac{1}{\rho^k}\int_{\mathbb{R}^{d-k}}\int_{\mathbb{R}^k}\Big|W\cap\Big(W-\theta_i\Big(\frac{x_1}{\rho},\frac{x_2}{\rho}\Big)^T\Big)\Big|_d\,\mathrm{d}x_2\,(\mathrm{e}^{\lambda\, f(x_1,\theta_i)}-1)\,\mathrm{d}x_1 \\
&= \int_{\mathbb{R}^{d-k}}\int_{\mathbb{R}^k}\Big|W\cap\Big(W-\theta_i\Big(\frac{x_1}{\rho},x_2\Big)^T\Big)\Big|_d\,\mathrm{d}x_2\,(\mathrm{e}^{\lambda\, f(x_1,\theta_i)}-1)\,\mathrm{d}x_1 \\
&\underset{\rho\to\infty}{\longrightarrow} \int_{\mathbb{R}^k}\Big|W\cap\Big(W-\theta_i(\mathbf{o}_{d-k},x_2)^T\Big)\Big|_d\,\mathrm{d}x_2\int_{\mathbb{R}^{d-k}}(\mathrm{e}^{\lambda\, f(x_1,\theta_i)}-1)\,\mathrm{d}x_1.
\end{aligned}$$

The last step is justified by the dominated convergence theorem. Thus, the proof of (4.6) is finished, and Theorem 4.2 is completely proved.

4.5.3. Mixed directional distributions

Here, we prove Corollary 4.1. Recall the formula (4.15) for the variance of $V_\rho^{(d,k)}$, where $B\subset\mathbb{R}^d$ is a Borel set:

$$\begin{aligned}
&\mathrm{Var}(|U_\Xi\cap B|_d) \\
&= \mathrm{e}^{-2\lambda\, M_1}\int_{\mathbb{R}^d}|B\cap(B-x)|_d\Big(\exp\Big\{\lambda\,\mathbb{E}\,\big|\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T x))\big|_{d-k}\Big\}-1\Big)\,\mathrm{d}x.
\end{aligned}$$

Splitting the exponential term gives

$$\begin{aligned}
&\exp\Big\{\lambda\,\mathbb{E}\,\big|\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T x))\big|_{d-k}\Big\}-1 \\
&= \mathrm{e}^{\lambda\, T^{\mathrm{a}}(x)}-1+\mathrm{e}^{\lambda\, T^{\mathrm{c}}(x)}-1+(\mathrm{e}^{\lambda\, T^{\mathrm{a}}(x)}-1)\,(\mathrm{e}^{\lambda\, T^{\mathrm{c}}(x)}-1),
\end{aligned}$$

where $T^{\mathrm{a}}(x)$ (resp. $T^{\mathrm{c}}(x)$) denotes the atomic (resp. diffuse) part of the expectation term $T(x)=\mathbb{E}\,\big|\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T x))\big|_{d-k}$. For the variance of $V_\rho^{(d,k)}$, this means that

$$\begin{aligned}
&\mathrm{Var}(|U_\Xi\cap B|_d) \\
&= \mathrm{e}^{-2\lambda\, M_1}\int_{\mathbb{R}^d}|B\cap(B-x)|_d\Big(\mathrm{e}^{\lambda\, T^{\mathrm{a}}(x)}-1\Big)\,\mathrm{d}x \\
&\quad+\mathrm{e}^{-2\lambda\, M_1}\int_{\mathbb{R}^d}|B\cap(B-x)|_d\Big(\mathrm{e}^{\lambda\, T^{\mathrm{c}}(x)}-1\Big)\,\mathrm{d}x \\
&\quad+\mathrm{e}^{-2\lambda\, M_1}\int_{\mathbb{R}^d}|B\cap(B-x)|_d\Big(\mathrm{e}^{\lambda\, T^{\mathrm{a}}(x)}-1\Big)\Big(\mathrm{e}^{\lambda\, T^{\mathrm{c}}(x)}-1\Big)\,\mathrm{d}x.
\end{aligned}$$

For the first two summands, we have to repeat the procedures of Subsections 4.5.1 and 4.5.2 with $T(x)$ replaced by $T^{\mathrm{a}}(x)$ and $T^{\mathrm{c}}(x)$, respectively. The additional third term can be shown to disappear as $\rho\to\infty$ using (4.42) and the inequality

$$(\mathrm{e}^{\lambda\, T^{\mathrm{a}}(x)}-1)\,(\mathrm{e}^{\lambda\, T^{\mathrm{c}}(x)}-1)\le\lambda^2\,\mathrm{e}^{\lambda\, M_1}\,T^{\mathrm{a}}(x)\,T^{\mathrm{c}}(x).$$

4.5.4. Some special cases

Spherical sampling window

For $W = B_1(\mathbf{o})$ the formulae (4.6) and (4.7) can be substantially simplified. This relies on the formula

$$\int_0^2 |B_1(\mathbf{o}) \cap (B_1(\mathbf{o}) + s\, e_1)\,|_d\, s^{k-1}\,\mathrm{d}s = 2\kappa_{d-1} \int_0^2 \int_0^{s/2} \left(\sqrt{1-y^2}\right)^{d-1} \mathrm{d}y\, s^{k-1}\,\mathrm{d}s$$
$$= \frac{2^k \kappa_{d-1}}{k} \int_0^1 z^{\frac{k+1}{2}-1} (1-z)^{\frac{d+1}{2}-1}\,\mathrm{d}z = \frac{2^k\, \kappa_{d+k}}{\pi\, k\, \kappa_{k-1}},$$

which, together with $2\pi\,\kappa_{k-1} = (k+1)\,\kappa_{k+1} = \omega_{k+1}$, yields

$$\int_{\mathbb{R}^k} \left|W \cap (W - \theta_i(\mathbf{o}_{d-k}, x)^T)\right|_d \mathrm{d}x = \int_{\mathbb{R}^k} \left|B_1(\mathbf{o}) \cap (B_1(\mathbf{o}) - (\mathbf{o}_{d-k}, x)^T)\right|_d \mathrm{d}x$$
$$= \frac{2^{k+1}\,\kappa_k\,\kappa_{d+k}}{\omega_{k+1}}.$$

Thus, we obtain in the discrete case

$$\sigma^2_{\lambda,Q}(B_1(\mathbf{o})) = \mathrm{e}^{-2\lambda\, M_1}\, \frac{2^{k+1}\,\kappa_k\,\kappa_{d+k}}{\omega_{k+1}} \sum_{i\in I} \int_{\mathbb{R}^{d-k}} (\mathrm{e}^{\lambda\, f(x,\theta_i)} - 1)\,\mathrm{d}x,$$

and analogously in the diffuse case

$$\sigma^2_{\lambda,Q}(B_1(\mathbf{o})) = \lambda\, \mathrm{e}^{-2\lambda\, M_1}\, \frac{2^{k+1}\,\kappa_k\,\kappa_{d+k}}{\omega_{k+1}}\, M_2.$$

The case of motion-invariant union sets U_Ξ

Another important special case arises when the stationary random set (2.4) is additionally isotropic, i.e., $\mathbb{P}_0$ is the uniform distribution on $\mathbb{SO}_k^d$ induced by the normalized Haar measure on the Grassmannian $\mathbb{G}(d,k)$. If the conditional second moment $M_2(\theta)$ does not depend on $\theta \in \mathbb{SO}_k^d$ (e.g., Θ_0 and Ξ_0 are independent), we obtain

$$\int_{\mathbb{SO}_k^d} M_2(\theta) \int_{\mathbb{R}^k} \left|W \cap (W - \theta\,(\mathbf{o}_{d-k}, x)^T)\right|_d \mathrm{d}x\, \mathbb{P}_0(\mathrm{d}\theta)$$
$$= M_2 \int_{\partial B_1^k(\mathbf{o})} \int_0^\infty \int_{\mathbb{SO}_k^d} \left|W \cap (W - r\,\theta\,(\mathbf{o}_{d-k}, u)^T)\right|_d \mathbb{P}_0(\mathrm{d}\theta)\, r^{k-1}\,\mathrm{d}r\, \mathcal{H}_{k-1}(\mathrm{d}u)$$
$$= \frac{\omega_k}{\omega_d}\, M_2 \int_{\partial B_1^d(\mathbf{o})} \int_0^\infty |W \cap (W - r\,v)\,|_d\, r^{k-1}\,\mathrm{d}r\, \mathcal{H}_{d-1}(\mathrm{d}v)$$
$$= \frac{\omega_k}{\omega_d}\, M_2 \int_{\mathbb{R}^d} \frac{|W \cap (W-x)|_d}{\|x\|^{d-k}}\,\mathrm{d}x = M_2\, I_{k+1}(W),$$

where $\mathcal{H}_k(\cdot)$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^d$ and the functional

$$I_{k+1}(W) = \frac{\omega_k}{\omega_d} \int_W \int_W \frac{\mathrm{d}y \, \mathrm{d}x}{\|y-x\|^{d-k}}$$

is known as $(k+1)$st order *chord power integral* of W (up to occasionally other multiplicative constants).

Other expressions for the asymptotic variance in case of isotropy

The application of Blaschke-Petkantschin-type formulae for convex bodies W in $\mathbb{R}^d$ leads to the identities, see [SW08, pp. 362–364],

$$I_{k+1}(W) = \frac{\kappa_k}{k+1} \int_{\mathbb{A}(d,1)} \big(V_1(W \cap E)\big)^{k+1} \mu_1(\mathrm{d}E) = \int_{\mathbb{A}(d,k)} \big(V_k(W \cap E)\big)^2 \mu_k(\mathrm{d}E),$$

where $\mathbb{A}(d,k)$ is the space of affine k-flats in $\mathbb{R}^d$ equipped with the motion-invariant k-flat measure μ_k satisfying $\mu_k(\{E \in \mathbb{A}(d,k) : E \cap B_1^d(\mathbf{o}) \neq \emptyset\}) = \kappa_{d-k}$, see [SW08] for precise definitions and more details. By virtue of Carleman's inequality we get the estimate

$$I_{k+1}(W) \leq \frac{2^{k+1} \kappa_k \, \kappa_{d+k}}{d \, \omega_{k+1}} \left(\frac{|W|_d}{\kappa_d} \right)^{(d+k)/d} \quad , \quad k = 1, \ldots, d-1,$$

for convex W in $\mathbb{R}^d$ with "=" iff $W = B_r^d(\mathbf{o})$. Hence, for given volume of W, the variance of the volume of the motion-invariant set (2.4) is maximal in case of a spherical window.

4.6. Concluding remarks and open questions

- One should remark that although throughout this chapter we demanded that $\Xi_0 \in \mathcal{R}^o$ for consistency with Chapter 3, it is not necessary in this context when we are only interested in the volume fraction. Here, it suffices to assume that $\Xi_0 \in \mathcal{C}$, see [HS09] and [HS12].

  Under these conditions it is not guaranteed that the union set is closed with probability 1. Indeed, there exist simple examples so that $\mathbb{P}(U_\Xi$ is closed$) = 0$. E.g., if the typical cylinder base is defined by $\Xi_0 = \bigcup_{1 \leq i_1, \ldots, i_{d-k} \leq N} \bigtimes_{j=1}^{d-k} \left[i_j, i_j + \frac{1}{N} \right]$ for some positive random integer N satisfying $\mathbb{E}\, N = \infty$, then the union set is closed with probability zero, no matter which distribution Θ_0 has.

  One can even show that under the assumption $\mathbb{E}\, |\Xi_0|_{d-k} < \infty$ the additional condition $\mathbb{E}\, |\Xi_0 \oplus \pi_{d-k}(B_\varepsilon(\mathbf{o}))|_{d-k} < \infty$ for some $\varepsilon > 0$ is not only sufficient, but even necessary for the closedness of the stationary random union set (2.4):

Lemma 4.5. *Let* Ξ_0 *be a compact typical cylinder base of the* PCP (2.3) *satisfying* $\mathbb{E}\,|\Xi_0|_{d-k} < \infty$ *and* $\mathbb{E}\,|\Xi_0 \oplus \pi_{d-k}(B_\varepsilon(\mathbf{o}))|_{d-k} = \infty$ *for any* $\varepsilon > 0$*. Then for the union set we have* $\mathbb{P}(U_\Xi \text{ is closed in } \mathbb{R}^d) = 0$*.*

The proof of Lemma 4.5 can be done quite similarly to that in [Hei05] for Boolean models.

- We mention further that the above theorems can be extended to analogous results for estimators of the *covariance* $C_{U_\Xi{}^c}(u)$ of the random set $U_\Xi{}^c$ defined by the two-point probability $p_{U_\Xi{}^c}(\mathbf{o}_d, u)$ for any $u \in \mathbb{R}^d$, see Section 3.1.2. This is seen from the obvious relation $C_{U_\Xi{}^c}(u) = 1 - \mathbb{P}(\mathbf{o}_d \in U_\Xi \cup (U_\Xi - u))$ and the fact that $U_\Xi \cup (U_\Xi - u)$ forms the union set of a PCP with typical base $\Xi_0 \cup (\Xi_0 - \pi_{d-k}(\Theta_0^T u))$.
- It is an open question whether the Berry-Esseen estimate (4.14) can be obtained under weaker conditions on the cylinder base. Perhaps it suffices to require $\mathbb{E}\,|\Xi_0|_{d-k}^3 < \infty$ as one would expect from the CLT for independent random variables.
- A rigorous proof of Conjecture 4.1 is still missing.
- Open problems in this context are to derive central limit theorems for the other specific intrinsic volumes, including the specific surface area, see Section 3.2.

5. The approximate inverse estimator for directional distributions

This chapter is based on the articles [LRSS11] and [RS11]. The images in the figures with the unit spheres have been generated by Martin Riplinger who gave us the kind permission of reproduction. He has also implemented the AI inversion method for the 3D cosine transform and the Kiderlen-Pfrang algorithm which is used for the images.

In this chapter, we present and analyze a method for the estimation of the *directional distribution* (also called the *rose of directions*) of stationary fiber processes, where a fiber is the image of a C^1-smooth curve. This provides important information about the structure of the fibers.

We assume that Ξ is a stationary random set of fibers with *intensity* λ (the expected total length of fibers per unit volume) and directional distribution density φ (with respect to the spherical Lebesgue measure), i.e., φ is a function on the unit sphere S^{d-1} which we define to be symmetric for notational ease. In case of Ξ being a Poisson line process, φ can be introduced with the probability measure α from (2.2) as a symmetric function for which

$$\alpha(G) = \int_{S^{d-1}} \mathbb{1}\Big\{x \in \bigcup_{l \in G} l\Big\}\, \varphi(x)\, \mathrm{d}x \tag{5.1}$$

holds for any $G \in \mathcal{B}(\mathbb{G}(d,1))$. In (5.1) the integration is over all points on the unit sphere which lie on a line from G. For general fiber processes, the directional distribution is the distribution of the tangent direction at a typical point of the fiber process. See [SKM95, Section 9.3 and 9.4] for a rigorous definition of fiber processes and their directional distribution.

For the estimation of φ, we take a stereological approach (cf. Remark 5.1). Since Ξ is stationary, the intersection $\Xi \cap \xi^\perp$, $\xi \in S^{d-1}$, is a stationary point process on the hyperplane $\xi^\perp$. The function g which maps ξ onto the intensity (expected number of points / intersections per unit area) of this process $\Xi \cap \xi^\perp$ is called the *rose of intersections*. It is well-known that g is the cosine transform (see Section 2.5) of the directional distribution of Ξ (up to a multiplicative constant), which can easily be seen for discrete directional distributions (e.g., the Manhattan model), and then can be generalized to arbitrary measures, see also [SW08, Theorem 4.4.6]. This can be written as

$$g(\xi) = \lambda\,(C\,\varphi)(\xi), \tag{5.2}$$

where C is the cosine transform (2.11). Thus, if we have an estimation of g at some points $\xi_1, \dots, \xi_n$, an even measure (directional distribution of Ξ) on the unit sphere has to be reconstructed from finitely many (approximate) values of its cosine transform. For this purpose, stable numerical inversion algorithms are needed.

Remark 5.1. *Stereology aims to retrieve higher-dimensional information about geometric objects from lower-dimensional samples. One important example is 3D porous media that cannot be observed entirely, but only within sections with finitely many flats, lines or points. This includes the estimation of the volume fraction by determining the ratio of grid points which hit the media and the estimation of the surface area by counting how many times test lines intersect the surface of the media within a bounded observation window. See [BJ05] for an introduction.*

In 2005, Kiderlen and Pfrang ([KP05]) presented three non-parametric algorithms to estimate the rose of directions of a spatial fiber system. They are based on least square or other optimization problems. To be able to determine the rose of directions numerically, they restrict their considerations to atomic measures. Hoffmann ([Hof09a]) used in 2007 a least square estimator to invert the sine transform for a similar estimation approach. There exist convergence results and a proof of consistency ([GKM06]) for these algorithms. But all these algorithms only lead to discrete reconstructions, which are concentrated only on a finite number of points and look often artificial. Continuous reconstructions, which provide the chance of better model fits, are missing so far in literature.

We use the method of the approximate inverse as introduced in Section 2.5 to invert the cosine transform in a numerically stable way and get a continuous function as a result.

Remark 5.2. *For both the cosine and the spherical Radon transform, there exist analytic inversion formulas even in more general settings. Helgason obtained in [Hel08] an analytic inversion formula for the spherical Radon transform which was modified in [Spo01, Lemma 5.1] to a more compact form:*

For $f \in C_e(S^{d-1})$ and $d \geq 3$ holds

$$f(\xi) = \frac{(-1)^{d-2}\,2^{d-3}}{(d-3)!\,\omega_{d-1}} \left(\frac{\partial}{\partial(\mu^2)}\right)^{d-2} \left[\,\int\limits_{\langle \xi,\eta\rangle^2 > \mu^2} \frac{Rf(\eta)\,|\langle \xi,\eta\rangle|}{(\langle \xi,\eta\rangle^2 - \mu^2)^{\frac{4-d}{2}}}\,\mathrm{d}\eta\right]_{\mu=0}, \qquad \xi \in S^{d-1}.$$

While this formula forms a proper basis for theoretical considerations, it cannot be used for numerical calculations because of the numerical instability of some operations such as the differentiation.

This chapter is organized as follows. In the next section, we derive reconstruction kernels for the cosine transform for arbitrary dimensions $d \geq 2$. For this, we make use of the closely related spherical Radon transform for $d \geq 3$, for which we also derive a reconstruction kernel as a side product. In Section 5.2, we give a rigorous definition of our estimator for directional distributions. Its asymptotical properties are analyzed in the following Section 5.3. We begin with analyzing strong convergence of the estimator $\hat{\varphi}_\gamma$ to the mollified density φ_γ. We show that the supremum of the difference between the two functions converges almost surely to zero under mild assumptions on the directions of the test hyperplanes and the growth rate of the observation window radius. In Section 5.3.2, we derive Berry-Esseen bounds for the estimator and show how a central limit theorem can be used to construct asymptotic tests on the directional distribution. Section 5.3.3 contains an analysis of the large deviation behavior of the estimator. We conclude this chapter by presenting results of simulation studies in Section 5.4 and examining examples of real data in Section 5.5. Here, we apply our estimation method to microscopic images of gas diffusion layers of fuel cells, see Figure 1.1.

Note that while the estimator presented in Section 5.2 can also be introduced for general stationary fiber processes, we focus on Poisson line processes (PLPs).

5.1. Reconstruction kernels for the cosine transform

For the method of the approximate inverse, the crucial part is to derive a reconstruction kernel for a suitable mollifier. Then (2.14) can be used to calculate the smoothed reconstructed function or density. We derive reconstruction kernels for the cosine transform in all dimensions $d \geq 2$. Since the calculation techniques for the reconstruction kernels differ in 2D, we consider the case $d = 2$ separately in the next section. In Section 5.1.2, we derive a reconstruction kernel for all higher dimensions for the cosine and – as a side product – the spherical Radon transform.

5.1.1. Two-dimensional case

The even functions on S^1 correspond one-to-one to the π-periodic functions in $\mathbb{R}$. Thus, to simplify the notation, we consider densities with respect to the Lebesgue measure on the interval $[0, \pi]$ in this section. Here, a point $x \in [0, \pi]$ corresponds to the point $(\cos x, \sin x)^T$ on the unit sphere. Notice the difference from the higher-dimensional case, where we consider densities on the unit sphere with respect to the spherical surface area measure. In the two-dimensional case, the cosine transform can be written in the form

$$(C\,\varphi)(x) = \int_0^\pi |\cos(x-t)|\,\varphi(t)\,\mathrm{d}t.$$

Furthermore, in 2D it is closely related to the *sine transform*

$$(S\,\varphi)(x) = \int_0^\pi |\sin(x-t)|\,\varphi(t)\,\mathrm{d}t$$

by $(C\,\varphi)(x) = (S\,\varphi)(x + \pi/2)$. Thus, it suffices to consider the sine transform in the following, which seems to be more common in the 2D case. The following proposition (see [Hil62, Mec81, SKM95]) enables us to calculate the reconstruction kernel.

Proposition 5.1. *Let $g \in C^2(\mathbb{R})$ be an arbitrary π-periodic function. Then we have*

$$Sf = g \qquad \textit{with} \qquad f = \tfrac{1}{2}\left(g + g''\right).$$

Example. Let us consider the mollifier

$$e_\gamma(x,y) = \gamma^{-1} c_\nu \left(1 - \frac{(x-y)^2}{\gamma^2}\right)^\nu \mathbb{1}\{|x-y| \le \gamma\}$$

as a π-periodic function for $\nu \in \mathbb{N}$ and $\gamma \le \pi/2$ (see for example [Sch07]), where

$$c_\nu^{-1} = \int_{-1}^1 (1-x^2)^\nu\,\mathrm{d}x = \frac{\Gamma(\nu+1)\sqrt{\pi}}{\Gamma(\nu+\frac{3}{2})} \qquad (c_3^{-1} = 32/35,\ c_4^{-1} = 256/315).$$

For $t = |x-y| < \gamma$ and $\nu \ge 3$, the reconstruction kernel is then calculated using Proposition 5.1:

$$\psi_\gamma(t) = \frac{c_\nu}{2\gamma}\left[\left(1-\frac{t^2}{\gamma^2}\right)^\nu + \frac{4t^2\,\nu\,(\nu-1)}{\gamma^4}\left(1-\frac{t^2}{\gamma^2}\right)^{\nu-2} - \frac{2\nu}{\gamma^2}\left(1-\frac{t^2}{\gamma^2}\right)^{\nu-1}\right].$$

5.1.2. Higher-dimensional case

In this section, we derive reconstruction kernels for the cosine transform in arbitrary dimensions $d \ge 3$. For $d = 3$, a closed expression can also be found in [LRSS11], see Remark 5.3 at the end of this section. We first calculate a reconstruction kernel for the spherical Radon transform, which is of interest of its own. Then we use the relatedness of the spherical Radon and the cosine transform to arrive at the desired reconstruction kernel. Basically, we generalize the procedure by Martin Riplinger from [LRSS11] for the three-dimensional case to arbitrary dimensions $d \ge 3$. We concentrate on the Gaussian mollifier. One should also mentioned that Rubin suggested a method for the inversion of the spherical Radon transform which is related to ours, cf. [Rub02].

An explicit calculation for one example of the functions he suggested can be found in Appendix A.2.

We begin with some facts about the spherical Radon transform. It commutes with rotations (see [Gar06, Lemma C.2.7]), i.e.,

$$R(T_A\, f(\eta)) = T_A\,(R\, f)(\eta), \quad A \in \mathbb{SO}_d,\ \eta \in S^{d-1},$$

with $T_A f(\xi) := f(A^{-1}\xi)$. Therefore, it is sufficient to construct a kernel for only one fixed point $\xi_0 \in S^{d-1}$. To simplify the notation, we choose ξ_0 to be e_d, the d-th vector of the standard basis of $\mathbb{R}^d$.

If the mollifier depends only on the geodesic distance $d_{\mathrm{geo}}(\xi, \eta)$ between $\xi, \eta \in S^{d-1}$ and not on ξ and η themselves, then for $\eta \in S^{d-1}$ the mollifier e_γ depends only on the polar angle of η, denoted by θ. We introduce the notation $e_\gamma(\theta) := e_\gamma(\xi_0, \eta)$.

For the construction of the reconstruction kernel, we shall use the following result.

Theorem 5.1 ([Gar06], p. 432–434)**.** *Let $f, g \in C^1_{\mathrm{e}}(S^{d-1})$ be rotationally symmetric functions with $R\, f = g$. Then for $0 < t \le 1$,*

$$f(\arccos t) = \frac{1}{(d-3)!}\, t \left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{d-2} \int_0^t g(\arcsin x)\, x^{d-2}\, (t^2 - x^2)^{(d-4)/2}\, \mathrm{d}x. \tag{5.3}$$

To calculate the normalizing constant, we recall that we require $\int_{S^{d-1}} e_\gamma(\xi) \mathrm{d}\xi = 1$ for all $\gamma > 0$, which can be rewritten for rotationally symmetric mollifiers $e_\gamma(\theta)$ as

$$2\omega_{d-1} \int_0^{\pi/2} e_\gamma(\theta)\, (\sin\theta)^{d-2}\, \mathrm{d}\theta = 1.$$

For the Gaussian mollifier,

$$e_\gamma(\theta) = \frac{1}{c(\gamma)} \exp\left\{ -\frac{\sin^2\theta}{\gamma^2} \right\}, \quad \theta \in [0, \pi/2],$$

the substitution $x = \cos\theta$ leads to

$$\begin{aligned} c(\gamma) &= 2\omega_{d-1} \int_0^{\pi/2} \exp\left\{ -\frac{\sin^2\theta}{\gamma^2} \right\} (\sin\theta)^{d-2}\, \mathrm{d}\theta \\ &= 2\omega_{d-1} \int_0^1 \exp\left\{ -\frac{1 - x^2}{\gamma^2} \right\} (1 - x^2)^{(d-3)/2}\, \mathrm{d}x. \end{aligned} \tag{5.4}$$

We need the following lemma for our main result.

Lemma 5.1. *For $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and $t > 0$, it holds that*

$$\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n t^k = c_{k,n}\, t^{k-2n},$$

where $c_{k,n} = \prod_{j=0}^{n-1} (k - 2j)$.

Proof. $n = 0$ is obvious. For the $(n+1)$st derivative, we calculate

$$\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+1} t^k = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t} c_{k,n} t^{k-2n} = (k-2n) c_{k,n} t^{k-2n-2} = c_{k,n+1} t^{k-2(n+1)}.$$

□

Theorem 5.2. *In dimensions $d \geq 3$, for the Gaussian mollifier the reconstruction kernel for the spherical Radon transform is given by*

$$\psi_\gamma^R(\theta) = \frac{B\left(\frac{d-1}{2}, \frac{d-2}{2}\right)}{2(d-3)!\, c(\gamma)} \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{(d-1)/2 + r}{d - 3/2 + r}\right) \frac{(-1)^k\, c_{2d+2k-5,d-2}}{k!\, \gamma^{2k}} (\cos\theta)^{2k},$$

for $\theta \in [0, \pi/2]$, where the $c_{k,n}$ are as in Lemma 5.1, $c(\gamma)$ is as calculated in (5.4)*, and $B(\cdot,\cdot)$ denotes the beta function.*

Proof. According to (5.3), for $t \in [0,1]$, we have

$$\psi_\gamma^R(\arccos t) = \frac{1}{(d-3)!\, c(\gamma)} t \left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{d-2} \int_0^t \mathrm{e}^{-x^2/\gamma^2} x^{d-2} (t^2 - x^2)^{(d-4)/2} \,\mathrm{d}x.$$

We begin with

$$\begin{aligned}\int_0^t \mathrm{e}^{-x^2/\gamma^2} x^{d-2} (t^2 - x^2)^{(d-4)/2} \,\mathrm{d}x &\overset{x=st}{=} t \int_0^1 \mathrm{e}^{-(st)^2/\gamma^2} (st)^{d-2} (t^2 - (st)^2)^{(d-4)/2} \,\mathrm{d}s \\ &\overset{u=s^2}{=} \frac{1}{2} t^{2d-5} \int_0^1 \mathrm{e}^{-u\,t^2/\gamma^2} u^{(d-3)/2} (1-u)^{(d-4)/2} \,\mathrm{d}u,\end{aligned}$$

where the integral is $B((d-1)/2, (d-2)/2)$ times the moment generating function of a Beta$((d-1)/2, (d-2)/2)$-distributed random variable evaluated at $-t^2/2\gamma^2$, thus we get (see [JKB95]) for $X \sim$ Beta$((d-1)/2, (d-2)/2)$

$$\begin{aligned}&\frac{1}{2} t^{2d-5} B\left(\frac{d-1}{2}, \frac{d-2}{2}\right) m_X(-t^2/\gamma^2) \\ &= \frac{1}{2} B\left(\frac{d-1}{2}, \frac{d-2}{2}\right) \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{(d-1)/2 + r}{d - 3/2 + r}\right) \frac{(-1)^k\, t^{2d+2k-5}}{k!\, \gamma^{2k}}.\end{aligned}$$

With Lemma 5.1 this leads to

$$\begin{aligned}\psi_\gamma^R(\arccos t) &= \frac{B\left(\frac{d-1}{2}, \frac{d-2}{2}\right)}{2(d-3)!\, c(\gamma)} t \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{(d-1)/2 + r}{d - 3/2 + r}\right) \frac{(-1)^k \left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{d-2} t^{2d+2k-5}}{k!\, \gamma^{2k}} \\ &= \frac{B\left(\frac{d-1}{2}, \frac{d-2}{2}\right)}{2(d-3)!\, c(\gamma)} \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{(d-1)/2 + r}{d - 3/2 + r}\right) \frac{(-1)^k\, c_{2d+2k-5,d-2}\, t^{2k}}{k!\, \gamma^{2k}}.\end{aligned}$$

□

Corollary 5.1. *For the Gaussian mollifier, the reconstruction kernel for the cosine transform is given by*

$$\psi_\gamma(\theta) = \frac{1}{2\omega_{d-1}} \left(\frac{1}{\sin^{d-2}\theta} \frac{\partial}{\partial\theta} \left(\sin^{d-2}\theta \frac{\partial\, \psi_\gamma^R(\theta)}{\partial\theta} \right) + (d-1)\, \psi_\gamma^R(\theta) \right), \quad \theta \in [0, \pi/2],$$

where $\psi_\gamma^R(\theta)$ is the reconstruction kernel for the Radon transform from Theorem 5.2.

Proof. For the block operator (see (2.12)), it holds that $C^{-1} = \Box R^{-1}$ (see [Spo01]). Since the Beltrami-Laplace operator on the sphere can be written using spherical coordinates $\Delta_{d-1} = \frac{1}{\sin^{d-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{d-2}\theta\frac{\partial}{\partial\theta}\right)$ for rotationally symmetric functions (cf. [EMOT81, p. 235]), this yields the result. □

Remark 5.3. *For $d = 3$ formula (5.3) can be highly simplified which leads to the compact formula for the reconstruction kernel for the Gaussian mollifier derived by Martin Riplinger in [LRSS11]. For the Radon transform this leads to*

$$\psi_\gamma^R(\theta) = \frac{1}{c(\gamma)} \left[1 - \frac{\sqrt{\pi}\cos\theta}{\gamma} \exp\left\{ - \frac{\cos^2\theta}{\gamma^2} \right\} \operatorname{erfi}\left(\frac{\cos\theta}{\gamma} \right) \right], \quad \theta \in [0, \pi/2],$$

where erfi *is defined as* $\operatorname{erfi}(x) := -\mathrm{i}\operatorname{erf}(\mathrm{i}\,x)$*, with the well known error function* $\operatorname{erf}(x) = 2/\sqrt{\pi}\int_0^x \exp\{-t^2\}\,\mathrm{d}t$*. For the cosine transform we get*

$$\psi_\gamma(\theta) = \frac{1}{4\pi} \left(\frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\theta} \psi_\gamma^R(\theta) + \frac{\partial^2}{\partial\theta^2} \psi_\gamma^R(\theta) + 2\psi_\gamma^R(\theta) \right), \quad \theta \in [0, \pi/2].$$

5.2. Definition of the AI estimator for directional distributions

We begin with some notation, where we concentrate on the case that Ξ is a stationary Poisson line process (PLP). We assume that the numbers of intersections of Ξ with hyperplanes orthogonal to the unit vectors $\xi_1, \dots, \xi_n \in S^{d-1}$ in a window $\rho\, W = B_\rho(\mathbf{o})$ can be observed. These intersection counts are denoted by $Y_1, \dots, Y_n$, i.e., $Y_i = \#\{\Xi \cap \xi_i^\perp \cap \rho\, W\}$.

Since Ξ is a stationary PLP, the process $\Xi \cap \xi_i^\perp$ is also stationary. It has the intensity $\lambda\,(C\,\varphi)(\xi_i)$ (see [SW08, Theorem 4.4.6]), and thus $Y_i \sim \mathrm{Poi}(\kappa_{d-1}\,\rho^{d-1}\,\lambda\,(C\,\varphi)(\xi_i))$. It is reasonable to introduce the notation $\tilde{Y}_i = \frac{Y_i}{\kappa_{d-1}\rho^{d-1}\lambda}$, which leads to $\mathbb{E}\,\tilde{Y}_i = (C\,\varphi)(\xi_i)$ and $\operatorname{Var}\tilde{Y}_i = \frac{(C\,\varphi)(\xi_i)}{\kappa_{d-1}\,\rho^{d-1}\,\lambda}$ for all ρ, n, i.

Now we are in a position to introduce the estimator

$$\hat{\varphi}_\gamma(\eta) := \sum_{i=1}^{n} \tilde{Y}_i\, \psi_\gamma(\eta, \xi_i)\, \Delta_i \tag{5.5}$$

for the density φ at the point $\eta \in S^{d-1}$, which can be seen as a discretized version of formula (2.14). Here, the weight Δ_i is defined as follows. Introduce the spherical Voronoi cell of ξ_i as

$$\Omega_i = \Big\{\nu \in S^{d-1} : d_{\text{geo}}(\nu, \xi_i) \leq \min\{d_{\text{geo}}(\nu, \xi_j), d_{\text{geo}}(\nu, -\xi_j)\} \quad \text{for all} \quad j \neq i\Big\},$$

where $d_{\text{geo}}(\cdot,\cdot)$ denotes the geodesic distance, i.e., the length of the shortest path in S^{d-1} between two unit vectors. Then we set the weight Δ_i of the estimator (5.5) to be two times the area of the Voronoi cell Ω_i of ξ_i which results in $\sum_{i=1}^n \Delta_i = \omega_d$.

Remark 5.4. *Note that the non-negativity of the resulting estimated density is not guaranteed, but in the experiments conducted in [LRSS11], the problem could be solved by setting the negative values which appeared only in small regions to zero.*

5.3. Asymptotic properties of the AI estimator

We analyze some of the most interesting stochastic properties of the estimator presented in Section 5.2, again making the assumption on the mollifier $e_\gamma(\eta, \xi)$ (and thus also on the reconstruction kernel $\psi_\gamma(\eta, \xi)$) that it depends only on the geodesic distance between ξ and η. Since we are interested in convergence and also consider a growing number of measurement points, we need a slightly different notation, i.e., we introduce double indices for ξ and Y. This means that instead of $\xi_1, \dots, \xi_n$ we assume that we can observe the number of intersections of Ξ with hyperplanes orthogonal to the unit vectors $\xi_{1n}, \dots, \xi_{nn} \in S^{d-1}$ within ρW. Similarly, we denote the intersections by $Y_{1n}, \dots, Y_{nn}$, i.e., $Y_{in} = \#\{\Xi \cap \xi_{in}^\perp \cap \rho W\}$. To simplify the notation, we shall write ξ_i instead of ξ_{in} when the value of n is clear, and analogously Y_i for Y_{in}.

One should remark that the supremum of the absolute value of the reconstruction kernel ψ_γ tends to infinity for $\gamma \to 0$. Therefore, $\hat{\varphi}_\gamma$ does not converge to φ pointwise or in the $L^2(S^{d-1})$-sense. To overcome this, we fix $\gamma > 0$ and analyze the properties of the estimator $\hat{\varphi}_\gamma$, especially convergence to the mollified density φ_γ as $\rho, n \to \infty$. For suitable mollifiers, φ_γ approximates φ as $\gamma \to 0$, cf. [LRSS11].

Remark 5.5. *Throughout this chapter, we assume that λ is known. However, the information given by $Y_1, \dots, Y_n$ can also be used to estimate λ. More precisely, in Proposition 5.2 below, we show that, under certain conditions on ρ and $\xi_{1n}, \dots, \xi_{nn}$, the random variable*

$$\hat{\lambda} := \frac{1}{2\kappa_{d-1}} \sum_{i=1}^n \frac{Y_i}{\kappa_{d-1}\, \rho^{d-1}}\, \Delta_i. \tag{5.6}$$

is a strongly consistent estimator for λ.

5.3.1. Almost sure convergence

In this section, we derive sufficient conditions for almost sure convergence of the estimator $\hat{\varphi}_\gamma$ in the supremum norm, i.e., for some $\rho = \rho(n) \to \infty$ $(n \to \infty)$ we get

$$\sup_{\eta \in S^{d-1}} \left|\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)\right| \underset{n\to\infty}{\longrightarrow} 0.$$

For this, we also derive some results on the convergence speed of the estimator (in Lemma 5.4) which are of interest of their own. Yet, we focus on almost sure convergence, so no attempt has been made to achieve optimality in the convergence rates.

In this section, we often assume that the reconstruction kernel is Lipschitz continuous. By this, we mean that it is Lipschitz continuous in both components, i.e., for $\eta, \xi_1, \xi_2 \in S^{d-1}$ we have

$$\left|\psi_\gamma(\eta, \xi_1) - \psi_\gamma(\eta, \xi_2)\right| \le L_{\psi_\gamma}\, d_{\mathrm{geo}}(\xi_1, \xi_2)$$

for some $L_{\psi_\gamma} < \infty$. Since S^{d-1} is compact, this also means that ψ_γ is bounded.

We begin with some notation. Similarly to [GKM06], we define the *symmetrized mesh norm* of a set of points $\{\xi_1, \dots, \xi_n\}$ as the mesh norm of the set including the reflected vectors, $\{\xi_1, \dots, \xi_n, -\xi_1, \dots, -\xi_n\}$, i.e.,

$$h^*(\xi_1, \dots, \xi_n) = h(\xi_1, \dots, \xi_n, -\xi_1, \dots, -\xi_n) = \max_{\nu \in S^{d-1}} \min_{1 \le i \le n} \{d_{\mathrm{geo}}(\nu, \xi_i), d_{\mathrm{geo}}(\nu, -\xi_i)\}.$$

Note that the weights Δ_i, $i = 1, \dots, n$ of the estimator (5.5) are thus bounded by $2\kappa_{d-1}\, h^*(\xi_1, \dots \xi_n)^{d-1}$, as the maximal geodesic radius of one cell is $h^*(\xi_1, \dots \xi_n)$.

Throughout this section, we use the following auxiliary result on the cosine transform.

Lemma 5.2. *For arbitrary densities φ on S^{d-1} the cosine transform $C\,\varphi$ is Lipschitz continuous with respect to the geodesic distance. The Lipschitz constant is at most 1. Furthermore, $C\,\varphi$ is bounded by 1.*

Proof. For $\zeta, \eta \in S^{d-1}$ we have

$$\begin{aligned}\left|(C\,\varphi)(\zeta) - (C\,\varphi)(\eta)\right| &= \left|\int_{S^{d-1}} \left(|\langle \zeta, \nu\rangle| - |\langle \eta, \nu\rangle|\right)\varphi(\nu)\,\mathrm{d}\nu\right| \\ &\le \sup_{\nu \in S^{d-1}} \left||\langle \zeta, \nu\rangle| - |\langle \eta, \nu\rangle|\right| \le d_{\mathrm{geo}}(\zeta, \eta).\end{aligned}$$

For the second claim, we calculate $(C\,\varphi)(\zeta) = \int_{S^{d-1}} |\langle \zeta, \nu\rangle|\, \varphi(\nu)\,\mathrm{d}\nu \le 1$. □

Deterministic measurement directions

At first, we assume that the series of measurement directions is deterministic. See, e.g., [HS96] or [SW04] for an approach to choose a suitable point configuration. In the next section, we give one example of a class of random vectors which covers a case which often appears in applications.

Let us begin with the analysis of the bias of the estimator (5.5). Note that $\mathbb{E}\,\tilde{Y}_{in} = (C\,\varphi)(\xi_{in})$ does not depend on the observation window radius ρ.

Theorem 5.3. *Assume that the reconstruction kernel ψ_γ is Lipschitz continuous. Denote by $L_{C\,\varphi}$ and L_{ψ_γ} the Lipschitz constants of $C\,\varphi$ and ψ_γ respectively. Then*

$$\sup_{\eta\in S^{d-1}} \left|\,\mathbb{E}\,\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)\right| \le \omega_{d-1}\,(L_{C\,\varphi}\,\overline{\psi}_\gamma + L_{\psi_\gamma})\,h^*(\xi_1,\dots,\xi_n)$$

uniformly for all $\rho > 0$, where $\overline{\psi}_\gamma$ denotes the supremum of $|\psi_\gamma|$.

Proof. Let $\eta \in S^{d-1}$ be an arbitrary unit vector. Then, for $\nu \in \Omega_i$, $i = 1,\dots,n$,

$$\begin{aligned}
&\left|(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i) - (C\,\varphi)(\nu)\,\psi_\gamma(\eta,\nu)\right| \\
&\le \left|(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i) - (C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\nu)\right| + \left|(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\nu) - (C\,\varphi)(\nu)\,\psi_\gamma(\eta,\nu)\right| \\
&= \left|(C\,\varphi)(\xi_i)\right| \cdot \left|\psi_\gamma(\eta,\xi_i) - \psi_\gamma(\eta,\nu)\right| + \left|\psi_\gamma(\eta,\nu)\right| \cdot \left|(C\,\varphi)(\xi_i) - (C\,\varphi)(\nu)\right| \\
&\le L_{\psi_\gamma} h^*(\xi_1,\dots,\xi_n) + \overline{\psi}_\gamma\, L_{C\,\varphi} h^*(\xi_1,\dots,\xi_n),
\end{aligned}$$

since $C\,\varphi$ is bounded by 1 (see Lemma 5.2). As $C\,\varphi$ and ψ are symmetric functions, finally we obtain

$$\begin{aligned}
\left|\,\mathbb{E}\,\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)\right| &= \left|\sum_{i=1}^{n} (C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i - \int_{S^{d-1}} (C\,\varphi)(\nu)\,\psi_\gamma(\eta,\nu)\,\mathrm{d}\nu\right| \\
&\le 2\sum_{i=1}^{n} \int_{\Omega_i} \left|(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i) - (C\,\varphi)(\nu)\,\psi_\gamma(\eta,\nu)\right| \mathrm{d}\nu \\
&\le \omega_{d-1}\,(L_{C\,\varphi}\,\overline{\psi}_\gamma + L_{\psi_\gamma})\,h^*(\xi_1,\dots,\xi_n). \qquad \square
\end{aligned}$$

To analyze weak convergence of $\hat{\varphi}_\gamma$, we need the following lemma.

Lemma 5.3. *If $|\psi_\gamma|$ is bounded by $\overline{\psi}_\gamma < \infty$, then, for $\varepsilon > 0$,*

$$\mathbb{P}\Bigg(\sup_{\eta\in S^{d-1}} \sum_{i=1}^{n} \left|\Big(\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big)\,\psi_\gamma(\eta,\xi_i)\right| \Delta_i > \varepsilon\Bigg) \le \frac{\omega_{d-1}^2\,\overline{\psi}_\gamma^{\,2}}{\kappa_{d-1}\,\rho^{d-1}\,\lambda\,\varepsilon^2}$$

uniformly for all $n \in \mathbb{N}$.

Proof. Using the notation $Y(\xi) = \#\{\Xi \cap \xi^\perp \cap \rho\, W\}$ and $\tilde{Y}(\xi) = \frac{1}{\kappa_{d-1}\,\rho^{d-1}\,\lambda} Y(\xi)$, we get $Y(\xi) \sim \mathrm{Poi}\big(\kappa_{d-1}\,\rho^{d-1}\,\lambda\,(C\,\varphi)(\xi)\big)$ and $\operatorname{Var}\tilde{Y}(\xi) = \frac{(C\,\varphi)(\xi)}{\kappa_{d-1}\,\rho^{d-1}\,\lambda}$. Thus, Lemma 5.2 and Chebyshev's inequality yield

$$\begin{aligned}
\mathbb{P}\Big(\sup_{\eta\in S^{d-1}} \sum_{i=1}^{n} \Big|(\tilde{Y}_i - (C\,\varphi)(\xi_i))\psi_\gamma(\eta,\xi_i)\Big|\,\Delta_i > \varepsilon\Big) &\le \mathbb{P}\Big(\sum_{i=1}^{n} \Big|\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big|\,\Delta_i > \varepsilon/\overline{\psi}_\gamma\Big) \\
\le \frac{\overline{\psi}_\gamma^2}{\varepsilon^2} \operatorname{Var}\Big(\sum_{i=1}^{n}\Big|\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big|\,\Delta_i\Big) &\le \frac{\omega_{d-1}^2\,\overline{\psi}_\gamma^2}{\varepsilon^2} \sup_{\nu\in S^{d-1}} \operatorname{Var}\Big(\big|\tilde{Y}(\nu) - (C\,\varphi)(\nu)\big|\Big) \\
\le \frac{\omega_{d-1}^2\,\overline{\psi}_\gamma^2}{\varepsilon^2} \sup_{\nu\in S^{d-1}} \frac{(C\,\varphi)(\nu)}{\kappa_{d-1}\,\rho^{d-1}\,\lambda} &\le \frac{\omega_{d-1}^2\,\overline{\psi}_\gamma^2}{\kappa_{d-1}\,\rho^{d-1}\,\lambda\,\varepsilon^2}.
\end{aligned}$$

□

Lemma 5.4. *Let ψ_γ be Lipschitz continuous. If $h^*(\xi_1,\dots,\xi_n) < c_h^{-1}\,\varepsilon$, we have*

$$\mathbb{P}\Big(\sup_{\eta\in S^{d-1}} |\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)| > \varepsilon\Big) \le \frac{\omega_{d-1}^2\overline{\psi}_\gamma^2}{\kappa_{d-1}\,\rho^{d-1}\,\lambda\,(\varepsilon - \omega_{d-1}\,c_h\,h^*(\xi_1,\dots,\xi_n))^2},$$

where the constant $c_h = \omega_{d-1}(L_{C\,\varphi}\,\overline{\psi}_\gamma + L_{\psi_\gamma})$ does not depend on $\xi_1,\dots,\xi_n$.

Proof. We consider the split

$$\hat{\varphi}_\gamma(\eta) = \sum_{i=1}^{n}\Big(\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i + \sum_{i=1}^{n}(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i.$$

For the specified h^*, Lemma 5.3 and the proof of Theorem 5.3 lead to

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\eta\in S^{d-1}} |\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)| > \varepsilon\Big) \\
&\le \mathbb{P}\Big(\sup_{\eta\in S^{d-1}} \Big(\sum_{i=1}^{n}\Big|\Big(\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big)\,\psi_\gamma(\eta,\xi_i)\Big|\,\Delta_i + \\
&\qquad + \Big|\sum_{i=1}^{n}(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i - \varphi_\gamma(\eta)\Big|\Big) > \varepsilon\Big) \\
&\le \mathbb{P}\Big(\sup_{\eta\in S^{d-1}} \sum_{i=1}^{n}\Big|\Big(\tilde{Y}_i - (C\,\varphi)(\xi_i)\Big)\,\psi_\gamma(\eta,\xi_i)\Big|\,\Delta_i \\
&\qquad > \varepsilon - \omega_{d-1}(L_{C\,\varphi}\,\overline{\psi}_\gamma + L_{\psi_\gamma})\,h^*(\xi_1,\dots,\xi_n)\Big) \\
&\le \frac{\omega_{d-1}^2\overline{\psi}_\gamma^2}{\kappa_{d-1}\,\rho^{d-1}\,\lambda\,(\varepsilon - c_h\,h^*(\xi_1,\dots,\xi_n))^2}.
\end{aligned}$$

□

This immediately leads to the following result.

Theorem 5.4. *Suppose that ψ_γ is Lipschitz continuous, the symmetrized mesh norm $h^*(\xi_{1n},\dots,\xi_{nn})$ tends to zero and $\rho = \rho(n) \geq c_1 \cdot n^{(1+c_2)/(d-1)}$ for some constants $c_1, c_2 > 0$. Then, almost surely,*

$$\sup_{\eta\in S^{d-1}} \left|\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. The result follows from the Borel-Cantelli lemma. Since $h^*(\xi_{1n},\dots,\xi_{nn})$ tends to zero, there exists an integer N such that $h^*(\xi_{1n},\dots,\xi_{nn}) \leq \frac{1}{2}\, c_h^{-1}\, \varepsilon$ for $n \geq N$. Thus, with Lemma 5.4 we get for arbitrary $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\sup_{\eta\in S^{d-1}} \left|\hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta)\right| > \varepsilon \Big) \leq N + \sum_{n=N+1}^{\infty} \frac{4\,\omega_{d-1}^2 \overline{\psi}_\gamma^2}{\kappa_{d-1}(c_1\, n^{(1+c_2)/(d-1)})^{d-1} \lambda\, \varepsilon^2} < \infty.$$

□

Random measurement directions

In this section, instead of deterministic measurement directions, we consider the following setting which should be a suitable model for applications in which the measurement directions cannot be chosen but are given in a random way. We consider as measurement directions the series of i.i.d. unit vectors $\xi_1, \xi_2, \dots$ which are assumed to be independent of Ξ. For the estimator (5.5) we use the same weights as in the deterministic setting.

Lemma 5.5. *Let $\xi_1, \xi_2, \dots$ be a sequence of i.i.d. random unit vectors with positive density with respect to the spherical Lebesgue measure. Then $h^*(\xi_1,\dots,\xi_n))$ converges to zero almost surely.*

Proof. To simplify the notation, we show the stronger claim that the non-symmetrized mesh norm $h(\xi_1,\dots,\xi_n)$ tends to zero almost surely.

Let $\varepsilon > 0$. Obviously, there exists some integer $m(\varepsilon) < \infty$ and a sequence of unit vectors $\nu_1,\dots,\nu_m \in S^{d-1}$ with $h(\nu_1,\dots,\nu_m) < \varepsilon/2$. As the density of ξ_1 is positive, it holds for every cell $\Omega(\nu_i)$ of the tessellation induced by $\nu_1,\dots,\nu_m$ that $\mathbb{P}(\xi_1 \in \Omega(\nu_i)) > 0$. Thus, there is a constant p with $0 < p \leq \mathbb{P}(\xi_1 \in \Omega(\nu_i))$, $i = 1,\dots,m$. With this, the probability that there is a cell which contains none of the vectors $\xi_1,\dots,\xi_n$ can be estimated from above by

$$\mathbb{P}\big(\exists i : \{\xi_1,\dots,\xi_n\} \cap \Omega(\nu_i) = \emptyset\big) \leq \sum_{i=1}^{m} \mathbb{P}\big(\{\xi_1,\dots,\xi_n\} \cap \Omega(\nu_i) = \emptyset\big) \leq m(1-p)^n.$$

Since $\sum_{n=1}^{\infty} m(1-p)^n < \infty$, it follows from the Borel-Cantelli lemma that every cell contains at least one point almost surely as n tends to infinity. This means that the mesh norm of the sequence $\xi_1,\dots,\xi_n$ can at most be ε. □

Theorem 5.5. *Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random unit vectors with non-negative density with respect to the spherical surface measure. Then*

$$\sup_{\eta \in S^{d-1}} \big| \mathbb{E}\, \hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta) \big| \underset{n\to\infty}{\longrightarrow} 0.$$

If additionally ψ_γ is Lipschitz continuous, and the observation window radii fulfill $\rho(n) \geq c_1 \cdot n^{(1+c_2)/(d-1)}$ for some $c_1, c_2 > 0$,

$$\sup_{\eta \in S^{d-1}} \big| \hat{\varphi}_\gamma(\eta) - \varphi_\gamma(\eta) \big| \underset{n\to\infty}{\longrightarrow} 0$$

holds almost surely.

Proof. This is a direct consequence of Lemma 5.5 in combination with Theorem 5.3 and Theorem 5.4, respectively. □

We conclude this section with a result on the estimator $\hat{\lambda}$ introduced in (5.6).

Proposition 5.2. *If the symmetrized mesh norm $h^*(\xi_{1n}, \ldots, \xi_{nn})$ of the measurement directions tends to 0, and $\rho(n) \geq c_1\, n^{(1+c_2)/(d-1)}$ for some constants $c_1, c_2 > 0$, then the estimator $\hat{\lambda}$ for the intensity of Ξ is strongly consistent for $n \to \infty$.*

Proof. For $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(|\hat{\lambda} - \lambda| > \varepsilon) &= \mathbb{P}\biggl(\biggl|\frac{1}{2\kappa_{d-1}} \sum_{i=1}^n \frac{Y_i}{\kappa_{d-1}\, \rho^{d-1}}\, \Delta_i - \lambda\biggr| > \varepsilon\biggr) \\ &\leq \mathbb{P}\biggl(\biggl|\frac{1}{2\kappa_{d-1}} \sum_{i=1}^n \frac{Y_i - \mathbb{E}\, Y_i}{\kappa_{d-1}\, \rho^{d-1}} \Delta_i\biggr| > \frac{\varepsilon}{2}\biggr) + \mathbb{P}\biggl(\biggl|\frac{1}{2\kappa_{d-1}} \sum_{i=1}^n \frac{\mathbb{E}\, Y_i}{\kappa_{d-1}\, \rho^{d-1}} \Delta_i - \lambda\biggr| > \frac{\varepsilon}{2}\biggr). \quad (5.7)\end{aligned}$$

The first probability in (5.7) can be estimated from above by repeating the steps in the proof of Lemma 5.3,

$$\mathbb{P}\biggl(\biggl|\frac{1}{2\kappa_{d-1}} \sum_{i=1}^n \frac{Y_i - \mathbb{E}\, Y_i}{\kappa_{d-1}\, \rho^{d-1}}\, \Delta_i\biggr| > \varepsilon/2\biggr) \leq \frac{\omega_{d-1}^2\, \lambda}{\kappa_{d-1}^3\, \rho^{d-1}\, \varepsilon^2}.$$

For the second summand in (5.7), it follows from [Gar06, p. 428] that

$$\begin{aligned}\int_{S^{d-1}} (C\,\varphi)(\xi)\, \mathrm{d}\xi &= \int_{S^{d-1}} \int_{S^{d-1}} |\langle \xi, \nu \rangle|\, \varphi(\nu)\, \mathrm{d}\nu\, \mathrm{d}\xi = \int_{S^{d-1}} \int_{S^{d-1}} |\langle \xi, \nu \rangle|\, \mathrm{d}\xi\, \varphi(\nu)\, \mathrm{d}\nu \\ &= 2\kappa_{d-1}.\end{aligned}$$

Combined with the Lipschitz continuity of $C\,\varphi$, this leads to

$$\biggl|\sum_{i=1}^n (C\,\varphi)(\xi_i)\, \Delta_i - 2\kappa_{d-1}\biggr| \leq 2 \sum_{i=1}^n \int_{\Omega_i} \Bigl|(C\,\varphi)(\xi_i) - (C\,\varphi)(\xi)\Bigr|\, \mathrm{d}\xi \leq \omega_d\, h^*(\xi_{1n}, \ldots, \xi_{nn}).$$

With $\mathbb{E}\,Y_i = \lambda\,(C\,\varphi)(\xi_i)$, this shows that the second term in (5.7) is 0 for n large enough since $h^*(\xi_{1n},\dots,\xi_{nn})$ tends to 0.

In combination, this means that the sum $\sum_{n=1}^{\infty}\mathbb{P}(|\hat{\lambda}(n)-\lambda|>\varepsilon)$ is finite if $\sum_{n=1}^{\infty} 1/\rho^{(d-1)}$ is. This is clearly the case under the condition on $\rho(n)$ above. The claim follows from the Borel-Cantelli lemma. $\square$

5.3.2. Berry-Esseen bounds

In this section, we show how for some fixed $\eta \in S^{d-1}$ the estimator $\hat{\varphi}_\gamma(\eta)$ can be written as a compound Poisson process and use some well-known results to derive Berry-Esseen bounds, see [KS10]. Since we are interested in the distribution of $\hat{\varphi}_\gamma$, we can simplify the notation in the following way. The line process is stationary, so instead of increasing the radius of the observation window, we can also increase the intensity, which will have the same effect on the estimator. Thus, it suffices to let λ tend to infinity and restrict to $\rho = 1$, i.e., $\rho\,W = W$.

We begin with some notation. Denote by N_W the number of the observed lines, i.e., lines which hit W, thus $N_W \sim \mathrm{Poi}(\kappa_{d-1}\lambda)$. Introducing the symbol Q_W for the distribution of a typical line of Ξ hitting W, we obtain that Ξ is distributed as the set of the i.i.d. lines $L_1,\dots,L_{N_W}$ with distribution Q_W, which are independent of N_W.

The definition of the estimator

$$\hat{\varphi}_\gamma(\eta) = \sum_{i=1}^{n} \tilde{Y}_i\,\psi_\gamma(\eta,\xi_i)\,\Delta_i.$$

can be rewritten as follows. For a line l, let $\hat{\varphi}_\gamma(l,\eta)$ be the estimator for the process when $\Xi \cap W = \{l\} \cap W$. Analogously, define $\tilde{Y}_i(l) = \frac{1}{\kappa_{d-1}\,\lambda}\mathbb{1}\{l \cap \xi_i \cap W \neq \emptyset\}$. This leads to

$$\hat{\varphi}_\gamma(\eta) = \sum_{l\in\Xi}\hat{\varphi}_\gamma(l,\eta) = \sum_{l\in\Xi}\sum_{i=1}^{n}\tilde{Y}_i(l)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i \stackrel{d}{=} \sum_{j=1}^{N_W}\sum_{i=1}^{n}\tilde{Y}_i(L_j)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i.$$

As the terms $\sum_{i=1}^{n}\tilde{Y}_i(L_j)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i$, $j = 1,2,\dots$ are i.i.d. random variables, this means that $\hat{\varphi}_\gamma(\eta)$ has a compound Poisson distribution with size N_W and summands $\sum_{i=1}^{n}\tilde{Y}_i(L_0)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i$, where $L_0 \sim Q_W$ is a typical line of Ξ hitting W. Thus, we can apply the following well-known result, where we use the constant for the upper bound from [KS10].

Lemma 5.6. *Let $S_N = X_1 + \dots + X_N$ be a compound Poisson process, the random variables $X_1, X_2, \dots$ being uniformly distributed with $\mathbb{E}\,|X_1|^3 < \infty$, $N \sim \mathrm{Poi}(\mu)$, and*

$N, X_1, X_2, \ldots$ *are independent, then*

$$\sup_x \left| \mathbb{P}\Big(\frac{S_N - \mu\, \mathbb{E}\, X_1}{\sqrt{\mu\, \mathbb{E}\, |X_1|^2}} \leq x \Big) - \Phi(x) \right| \leq 0.3041 \frac{\mathbb{E}\, |X_1|^3}{\Big(\mathbb{E}\, |X_1|^2\Big)^{3/2} \sqrt{\mu}}.$$

Together with the considerations above, this can be used to arrive at the main result of this section.

Theorem 5.6. *Let $\eta \in S^{d-1}$, L_0 be a typical line of Ξ hitting W, and $\hat{\varphi}_\gamma(L_0, \eta)$ as above. If $|\psi_\gamma|$ is bounded,*

$$\sup_x |F_\lambda(x) - \Phi(x)| \leq 0.3041 \frac{\mathbb{E}\, (\kappa_{d-1}\, \lambda\, |\hat{\varphi}_\gamma(L_0, \eta)|)^3}{\Big(\mathbb{E}\, (\kappa_{d-1}\, \lambda\, \hat{\varphi}_\gamma(L_0, \eta))^2\Big)^{3/2} \sqrt{\kappa_{d-1}\, \lambda}}, \tag{5.8}$$

where $F_\lambda(x) = \mathbb{P}\left(\frac{\hat{\varphi}_\gamma(\eta) - \kappa_{d-1}\, \lambda\, \mathbb{E}\, \hat{\varphi}_\gamma(L_0, \eta)}{\sqrt{\kappa_{d-1}\, \lambda\, \mathbb{E}\, \hat{\varphi}_\gamma(L_0, \eta)^2}} \leq x \right)$.

Proof. Since the reconstruction kernel ψ_γ is bounded, for the third moment we have $\mathbb{E}(\kappa_{d-1}\, \lambda\, |\hat{\varphi}_\gamma(L_0, \eta)|)^3 \leq \omega_{d-1}^3\, \overline{\psi}_\gamma^3 < \infty$, see Theorem 5.7. Thus, Lemma 5.6 can be applied with $\mu = \lambda\, \kappa_{d-1}$. □

One should remark that on the right hand side of (5.8) we have expanded the fraction by $\kappa_{d-1}^3\, \lambda^3$ because then the expectation values do not depend on λ (and κ_{d-1}).

With Theorem 5.6, it remains to derive (bounds for) the second and third absolute moment of $\sum_{i=1}^n \tilde{Y}_i(L_0)\, \psi_\gamma(\eta, \xi_i)\, \Delta_i$. This makes it in particular necessary to determine the mixed moments of $Y_1, \ldots, Y_n$, i.e., the probability that the typical line hits a set of test hyperplanes within the observation window. For notational ease we assume that these are the first m of our test hyperplanes, i.e., $\xi_1^\perp, \ldots, \xi_m^\perp$ for $m = 1, 2$. To avoid longish formulas, we use an upper bound for the third moment. For this calculation, we introduce the notation

$$H_{1,\ldots,m} = H_{\xi_1,\ldots,\xi_m} = \big\{ l \in \mathbb{A}(d,1) : l \cap \xi_1^\perp \cap W \neq \emptyset, \ldots, l \cap \xi_m^\perp \cap W \neq \emptyset \big\},$$

where $\mathbb{A}(d,1)$ is the set of all lines in $\mathbb{R}^d$. $H_{1,\ldots,m}$ is the set of all lines which hit the hyperplanes $\xi_1^\perp, \ldots, \xi_m^\perp$ within W, which means that

$$Q_W(H_{1,\ldots,m}) = \mathbb{P}(L_0 \cap \xi_1^\perp \cap W \neq \emptyset, \ldots, L_0 \cap \xi_m^\perp \cap W \neq \emptyset) = \mathbb{E}\, Y_1(L_0) \cdots Y_m(L_0).$$

Theorem 5.7. *Let the conditions of Theorem 5.6 hold.*

1) If the measurement directions $\xi_1, \ldots, \xi_n$ *are deterministic,*

$$\mathbb{E}\left[\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)\right] = \sum_{i=1}^{n} (C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i,$$
$$\mathbb{E}\left(\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_\gamma(\eta,\xi_i)\,\psi_\gamma(\eta,\xi_j)\,\Delta_i\,\Delta_j\,Q_W(H_{i,j})$$

do not depend on λ*.*

2) If the measurement directions $\xi_1, \ldots, \xi_n$ *are i.i.d. with common density* ζ *and independent of* Ξ*, then*

$$\begin{aligned}
&\mathbb{E}\left[\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)\right]\\
&\quad= 2n\int_{(S^{d-1})^n} \psi_\gamma(\eta,\nu_1)\,\omega_d(\Omega_1(\nu_1,\ldots,\nu_n))\,Q_W(H_{\nu_1})\,\zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n),\\
&\mathbb{E}(\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta))^2\\
&\quad= n\,\mathbb{E}\left[\psi_\gamma^2(\eta,\xi_1)\,\Delta_1^2\,Y_1(L_0)\right] + n(n-1)\,\mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_2)\,\Delta_1\,\Delta_2\,Y_1(L_0)\,Y_2(L_0)\right],
\end{aligned}$$

where $\omega_d(\cdot)$ *denotes the spherical Lebesgue measure,* $\Omega_1(\nu_1,\ldots,\nu_n)$ *is the Voronoi cell at* ν_1 *induced by the points* $\nu_1,\ldots,\nu_n,-\nu_1,\ldots,-\nu_n$*, and*

$$\begin{aligned}
&\mathbb{E}\left[\psi_\gamma^2(\eta,\xi_1)\,\Delta_1^2\,Y_1(L_0)\right]\\
&\quad= 4\int_{(S^{d-1})^n} \psi_\gamma(\eta,\nu_1)^2\,\omega_d(\Omega_1(\nu_1,\ldots,\nu_n))^2\,Q_W(H_{\nu_1})\zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n),\\
&\mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_2)\,\Delta_1\,\Delta_2\,Y_1(L_0)\,Y_2(L_0)\right]\\
&\quad= 4\int_{(S^{d-1})^n} \psi_\gamma(\eta,\nu_1)\,\psi_\gamma(\eta,\nu_2)\,\omega_d(\Omega_1(\nu_1,\ldots,\nu_n))\,\omega_d(\Omega_2(\nu_1,\ldots,\nu_n))\,Q_W(H_{\nu_1,\nu_2})\\
&\qquad\zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n).
\end{aligned}$$

3) For both the deterministic and the i.i.d. case,

$$\mathbb{E}\left(\kappa_{d-1}\,\lambda\,|\hat{\varphi}_\gamma(L_0,\eta)|\right)^3 \le \omega_{d-1}^3\,\overline{\psi}_\gamma^3.$$

Proof. At first, let $\xi_1,\ldots,\xi_n$ be deterministic. Since $\mathbb{E}\,N_W = \kappa_{d-1}\,\lambda$ we get with Wald's identity

$$\mathbb{E}\left[\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)\right] = \mathbb{E}\sum_{j=1}^{N_W}\hat{\varphi}_\gamma(L_0,\eta) = \mathbb{E}\,\hat{\varphi}_\gamma(\eta) = \sum_{i=1}^{n}(C\,\varphi)(\xi_i)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i,$$

which yields the first claim. With

$$\mathbb{E}\, Y_i(L_0)\, Y_j(L_0) = \mathbb{P}(L_0 \cap \xi_i^\perp \cap W \neq \emptyset \text{ and } L_0 \cap \xi_j^\perp \cap W \neq \emptyset) = Q_W(H_{i,j}),$$

we get

$$\begin{aligned}\mathbb{E}(\hat{\varphi}_\gamma(L_0,\eta))^2 &= \mathbb{E}\left(\sum_{i=1}^n \tilde{Y}_i(L_0)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i\right)^2 \\ &= \frac{1}{\kappa_{d-1}^2\,\lambda^2}\sum_{i=1}^n\sum_{j=1}^n \psi_\gamma(\eta,\xi_i)\,\psi_\gamma(\eta,\xi_j)\,\Delta_i\,\Delta_j\,Q_W(H_{i,j}).\end{aligned}$$

To prove the second statement, we calculate

$$\begin{aligned}&\mathbb{E}\left[\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)\right] = \kappa_{d-1}\,\lambda\;\mathbb{E}\left[\sum_{i=1}^n \tilde{Y}_i(L_0)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i\right] = n\,\mathbb{E}\left[Y_1(L_0)\,\psi_\gamma(\eta,\xi_1)\,\Delta_1\right] \\ &= n\int\limits_{(S^{d-1})^n} \mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\Delta_1\,Y_1(L_0)\;\middle|\;\xi_1=\nu_1,\ldots,\xi_n=\nu_n\right]\zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n) \\ &= 2n\int_{(S^{d-1})^n}\psi_\gamma(\eta,\nu_1)\,\omega_d(\Omega_1(\nu_1,\ldots,\nu_n))\,Q_W(H_{\nu_1})\,\zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n),\end{aligned}$$

and for the second moment, we get

$$\begin{aligned}&\mathbb{E}(\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta))^2 \\ &= n\sum_{j=1}^n \mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_j)\,\Delta_1\,\Delta_j\,Y_1(L_0)\,Y_j(L_0)\right] \\ &= n\,\mathbb{E}\left[\psi_\gamma^2(\eta,\xi_1)\,\Delta_1^2\,Y_1(L_0)\right] + n(n-1)\,\mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_2)\,\Delta_1\,\Delta_2\,Y_1(L_0)\,Y_2(L_0)\right],\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_2)\,\Delta_1\,\Delta_2\,Y_1(L_0)\,Y_2(L_0)\right] \\ &= \int_{(S^{d-1})^n}\mathbb{E}\left[\psi_\gamma(\eta,\xi_1)\,\psi_\gamma(\eta,\xi_2)\,\Delta_1\,\Delta_2\,Y_1(L_0)\,Y_2(L_0)\;\middle|\;\xi_1=\nu_1,\ldots,\xi_n=\nu_n\right] \\ &\qquad \zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n) \\ &= 4\int_{(S^{d-1})^n}\psi_\gamma(\eta,\nu_1)\,\psi_\gamma(\eta,\nu_2)\,\omega_d(\Omega_1(\nu_1,\ldots,\nu_n))\,\omega_d(\Omega_2(\nu_1,\ldots,\nu_n))\,Q_W(H_{\nu_1,\nu_2}) \\ &\qquad \zeta(\nu_1)\cdots\zeta(\nu_n)\,\mathrm{d}(\nu_1,\ldots,\nu_n).\end{aligned}$$

The calculation for $\mathbb{E}\left[\psi_\gamma^2(\eta,\xi_1)\,\Delta_1^2\,Y_1(L_0)\right]$ can be done analogously.

For the third moment we get

$$\begin{aligned}\mathbb{E}\,|\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta)|^3 &= \mathbb{E}\,\bigg|\sum_{i=1}^{n} Y_i(L_0)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i\bigg|^3 \\ &\le \overline{\psi}_\gamma^3 \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \Delta_i\,\Delta_j\,\Delta_k = \omega_{d-1}^3\,\overline{\psi}_\gamma^3. \qquad \square\end{aligned}$$

Remark 5.6. *Equation* (5.2) *is equivalent to* $Q_W(H_1) = (C\,\varphi)(\xi_1)$*, so* $Q_W(H_1)$ *can be replaced accordingly in the theorem above.*

In the following, we derive specific formulas for the term $Q_W(H_{1,2})$, which appears in Theorem 5.7 for the most interesting dimensions $d = 2, 3$. It is convenient to define the set $H_{1,2}(\nu) = H_{1,2}(-\nu)$ as the subset of $H_{1,2}$ containing the lines with direction $\nu \in S^{d-1}$. This can be regarded as a subset of $\nu^\perp$, and thus we get

$$\begin{aligned}Q_W(H_{1,2}) &= \int\limits_{S^{d-1}} \mathbb{P}(L_0 \cap \xi_1 \cap W \neq \emptyset, L_0 \cap \xi_2 \cap W \neq \emptyset \mid L_0 \in \mathbb{A}(d,\nu))\,\varphi(\nu)\,\mathrm{d}\nu \\ &= \int_{S^{d-1}} \frac{|H_{1,2}(\nu)|_{d-1}}{\kappa_{d-1}}\,\varphi(\nu)\,\mathrm{d}\nu,\end{aligned} \tag{5.9}$$

where $\mathbb{A}(d,\nu)$ is the set of all lines in $\mathbb{R}^d$ with direction ν.

Two-dimensional case

For the two-dimensional case, we introduce some special notation which seems to be more common, see Section 5.1.1. Instead of even functions on the sphere, we consider densities on the interval $[0,\pi]$, where each value corresponds to an angle between a vector and the x-axis. For simplicity, all functions, in particular the density φ, are defined to be π-periodic. For unit vectors we use the notation $\vec{\nu} := (\cos\nu, \sin\nu)^T$, where $\nu \in [0, 2\pi]$, and for the line through $\vec{\nu}$ (and $\mathbf{o}$) we write $\overline{\nu}$.

Furthermore, in this section ξ_{ij} denotes the angle between the x-axis and the test lines instead of the vectors orthogonal to the test hyperplanes, i.e., instead of counting the intersections of Ξ and $\vec{\nu}^\perp$, we define $Y_{in} := \#(\Xi \cap \overline{\xi}_{in} \cap W)$. Thus, the rose of intersections is the sine transform of φ:

$$g(x) = \lambda\,S\varphi(x) = \lambda \int_0^\pi |\sin(x-t)|\,\varphi(t)\,\mathrm{d}t.$$

Because of (5.9), we need to calculate $|H_{1,2}(\nu)|_1$. As it can be seen in Figure 5.1, for one line, the measure of this set is $|H_1(\nu)|_1 = 2\sin|\nu - \xi_1|$. Similarly, if we have two

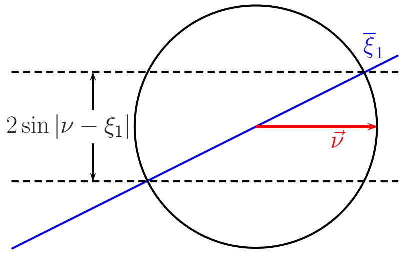


Figure 5.1.: Example of lines with direction $\vec{\nu}$ hitting one line $\bar{\xi}_1$

lines $\bar{\xi}_1$ and $\bar{\xi}_2$, we get $|H_{1,2}(\nu)|_1 = |H_1(\nu) \cap H_2(\nu)|_1 = 2\sin(\min\{|\nu - \xi_1|, |\nu - \xi_2|\})$. Assuming that $\xi_1 \leq \xi_2$, this leads to the formula

$$Q_W(H_{1,2}) = \frac{1}{2}\int_0^\pi \left|H_{1,2}(\nu)\right|_1 \varphi(\nu)\,\mathrm{d}\nu = \int_0^\pi \sin\big(\min\{|\nu - \xi_1|, |\nu - \xi_2|\}\big)\,\varphi(\nu)\,\mathrm{d}\nu$$
$$= \int\limits_0^{\frac{\xi_2-\xi_1}{2}} \sin\nu\,(\varphi(\xi_1+\nu) + \varphi(\xi_2-\nu))\mathrm{d}\nu + \int\limits_0^{\frac{\pi}{2}-\frac{\xi_2-\xi_1}{2}} \sin\nu\,(\varphi(\xi_2+\nu) + \varphi(\xi_1-\nu))\mathrm{d}\nu.$$

Example. In the case of uniformly distributed directions ($\varphi(x) = 1/\pi$), the formulae for $Q_W(H_{1,\dots,m})$ can be simplified to

$$Q_W(H_1) = S\varphi(\xi_1) = \frac{2}{\pi}\int_0^{\pi/2} \sin\nu\,\mathrm{d}\nu = \frac{2}{\pi}$$

and

$$\begin{aligned} Q_W(H_{1,2}) &= \frac{2}{\pi}\left(\int\limits_0^{\frac{\xi_2-\xi_1}{2}} \sin\nu\,\mathrm{d}\nu + \int\limits_0^{\frac{\pi}{2}-\frac{\xi_2-\xi_1}{2}} \sin\nu\,\mathrm{d}\nu\right) \\ &= \frac{2}{\pi}\left[2 - \sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\xi_2-\xi_1}{2}\right)\right], \end{aligned} \tag{5.10}$$

where we used an addition rule for the cosine.

Three-dimensional case

Here, $H_{1,2}(\nu)$ is $\xi_1^\perp \cap B_1(\mathbf{o})$ projected onto $\nu^\perp$ intersected with $\xi_2^\perp \cap B_1(\mathbf{o})$ projected onto $\nu^\perp$, i.e., $H_{1,2}(\nu) = \pi_{\nu^\perp}(\xi_1^\perp \cap B_1(\mathbf{o})) \cap \pi_{\nu^\perp}(\xi_2^\perp \cap B_1(\mathbf{o}))$, where $\pi_{\nu^\perp}$ denotes the projection onto the plane $\nu^\perp$, see Figure 5.2.

This means that $H_{1,2}(\nu)$ is the intersection of two ellipses with major axis length 1 and minor axis lengths $|\langle\nu, \xi_1\rangle|$ and $|\langle\nu, \xi_2\rangle|$, respectively. The angle between the

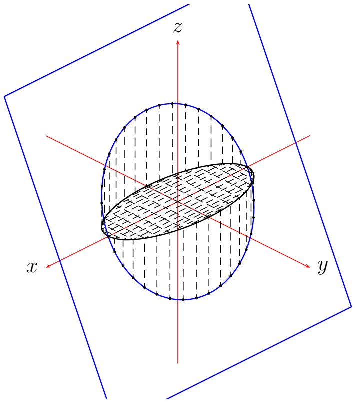


Figure 5.2.: Sketch of the projection of $\xi_1^\perp \cap B_1^3(\mathbf{o})$ onto $\nu^\perp$. Here, we have $\nu = (0, 0, 1)^T$ and $\xi_1 = \big(0, 1/\sqrt{5}, 2/\sqrt{5}\big)^T$.

major axes is the same as the angle between $\pi_{\nu^\perp}\xi_1$ and $\pi_{\nu^\perp}\xi_2$ and can be determined easily. For the calculation of the intersection area see Appendix A.3. Then

$$Q_W(H_{1,2}) = \int_{S^2} \frac{|H_{1,2}(\nu)|_2}{\pi} \varphi(\nu)\,\mathrm{d}\nu. \tag{5.11}$$

Application: testing the directional distribution

In this section, we present some tests of the directional distribution of a stationary PLP Ξ, i.e., we want to test the hypothesis $H_0 : \varphi = \varphi_0$ vs. $H_1 : \varphi \neq \varphi_0$ for some symmetric density φ_0 on the unit sphere, where the uniform distribution (i.e., isotropy) is of course the most interesting. We denote by $\mathbb{P}_0$ the probability measure under the null hypothesis, and analogously we write $\mathbb{E}_0$, Cov_0, and $Q_{W,0}$. For simplicity, we restrict to the case when the measurement directions $\xi_1, \dots, \xi_n$ are deterministic.

As shown in Theorem 5.6, an asymptotic test on the density in one point η can be constructed with the asymptotically standard Gaussian distributed random variable

$$\frac{\hat{\varphi}_\gamma(\eta) - \kappa_{d-1}\,\lambda\,\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta)}{\sqrt{\kappa_{d-1}\,\lambda\,\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta)^2}} \xrightarrow[\rho\to\infty]{\mathrm{d}} \mathrm{N}(0,1). \tag{5.12}$$

This test is of course only applicable for large ρ. Note that alternative expressions for the expectation values have been derived in Theorem 5.7.

One problem of this test is the choice of η. If several independent realizations of the PLP are available, an asymptotic χ^2-test can be constructed using (5.12) with a different η for each realization. If only one realization can be observed, several directions (denoted by $\eta_1, \dots, \eta_m$) can be taken into account with the following approach.

The vector of the random variables

$$X_i := \frac{\hat{\varphi}_\gamma(\eta_i) - \kappa_{d-1}\,\lambda\,\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta_i)}{\sqrt{\kappa_{d-1}\,\lambda\,\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta_i)^2}}, \quad i = 1,\dots,m \tag{5.13}$$

has an asymptotic multivariate normal distribution, which can easily be seen with Lemma 5.6, since for any linear combination a central limit theorem holds. Furthermore, their expectation is zero, and the covariance is independent of λ, since

$$\begin{aligned}\mathrm{Cov}_0(X_i,X_j) &= \frac{\mathrm{Cov}_0(\hat{\varphi}_\gamma(\eta_i),\hat{\varphi}_\gamma(\eta_j))}{\kappa_{d-1}\,\lambda\sqrt{\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta_i)^2}\sqrt{\mathbb{E}_0\,\hat{\varphi}_\gamma(L_0,\eta_j)^2}}\\ &= \frac{\sum_{k=1}^n\sum_{l=1}^n \psi_\gamma(\eta_i,\xi_k)\,\psi_\gamma(\eta_j,\xi_l)\,\Delta_k\,\Delta_l\,Q_{W,0}(H_{k,l})}{\sqrt{\mathbb{E}_0(\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta_i))^2}\sqrt{\mathbb{E}_0(\kappa_{d-1}\,\lambda\,\hat{\varphi}_\gamma(L_0,\eta_j))^2}},\end{aligned}$$

where we used that

$$\begin{aligned}\mathrm{Cov}(\hat{\varphi}_\gamma(\eta_i),\hat{\varphi}_\gamma(\eta_j)) &= \mathrm{Cov}\left(\sum_{k=1}^n \frac{Y_k}{\kappa_{d-1}\lambda}\,\psi_\gamma(\eta_i,\xi_k)\,\Delta_k, \sum_{l=1}^n \frac{Y_l}{\kappa_{d-1}\lambda}\,\psi_\gamma(\eta_j,\xi_l)\,\Delta_l\right)\\ &= \frac{1}{\kappa_{d-1}^2\,\lambda^2}\sum_{k=1}^n\sum_{l=1}^n \psi_\gamma(\eta_i,\xi_k)\,\psi_\gamma(\eta_j,\xi_l)\,\Delta_k\,\Delta_l\,\mathrm{Cov}(Y_k,Y_l)\\ &= \frac{1}{\kappa_{d-1}\,\lambda}\sum_{k=1}^n\sum_{l=1}^n \psi_\gamma(\eta_i,\xi_k)\,\psi_\gamma(\eta_j,\xi_l)\,\Delta_k\,\Delta_l\,Q_W(H_{k,l}).\end{aligned}$$

It follows from the Cramér-Wold theorem that the random variable $X^T K_X^{-1} X$ is asymptotically χ^2_m-distributed, where $X = (X_1,\dots,X_m)^T$, and K_X is the covariance matrix of X for some $\lambda > 0$. The resulting rule for a level α test is to reject the null hypothesis if $X^T K_X^{-1} X > \chi^2_{m,1-\alpha}$.

This is most interesting when the null hypothesis is that the process is isotropic. Then, for $Q_W(H_{k,l})$, one can use formula (5.10) in the two-dimensional case and formula (5.11) in the three-dimensional case with $\varphi \equiv 1/4\pi$.

5.3.3. Large deviations

Another interesting asymptotic property of the estimator $\hat{\varphi}_\gamma$ is the large deviation behavior. The following result is based on Cramér's theorem, see [dH00, Theorem I.4].

Theorem 5.8. *Let $\rho = 1$. It holds that for $a > \mathbb{E}\,\hat{\varphi}_\gamma(\eta)$*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(\hat{\varphi}_\gamma(\eta) \geq a) = -I(a),$$

and similarly, for $a < \mathbb{E}\,\hat{\varphi}_\gamma(\eta)$, we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}(\hat{\varphi}_\gamma(\eta) \le a) = -I(a).$$

The rate function is given by

$$I(z) = \sup_{t\in\mathbb{R}} \Big[zt - \kappa_{d-1} \Big(m_{\widetilde{\varphi}_\gamma(L_0)}(t) - 1 \Big) \Big]$$

for $\widetilde{\varphi}_\gamma(L_0) = \frac{1}{\kappa_{d-1}} \sum_{k=1}^n Y_k(L_0)\, \psi_\gamma(\eta, \xi_k)\, \Delta_k$, where $L_0 \sim Q_W$ is a typical line and

$$\begin{aligned} &m_{\widetilde{\varphi}_\gamma(L_0)}(t) \\ &= \sum_{j=1}^n \sum_{1\le i_1<\cdots<i_j\le n} \exp\left\{ \frac{t}{\kappa_{d-1}} \sum_{k=1}^j \psi_\gamma(\eta, \xi_{i_k})\, \Delta_{i_k} \right\} \mathbb{P}(Y^*_{i_1,\ldots,i_j}(L_0) = 1). \end{aligned} \tag{5.14}$$

Here, we use the notation $m_X(t) = \mathbb{E}\,\mathrm{e}^{tX}$ for the moment generating function of a random variable X and introduce

$$Y^*_{i_1,\ldots,i_j}(\,\cdot\,) = \prod_{k\in\{i_1,\ldots,i_j\}} Y_k(\,\cdot\,) \prod_{k\in\{i_1,\ldots,i_j\}^c} (1 - Y_k(\,\cdot\,)),$$

i.e., for $l \in \mathbb{A}(d,1)$ we have $Y^*_{i_1,\ldots,i_j}(l) = 1$ if and only if l hits the test hyperplanes $\xi_{i_1}, \ldots, \xi_{i_j}$ (and no others) within W.

Proof. We use the idea with the compound Poisson distribution from Section 5.3.2 and additionally note that for integer λ and i.i.d. random variables $N_1, \ldots, N_\lambda$ with $N_1 \sim \mathrm{Poi}(\kappa_{d-1})$ we have $\sum_{i=1}^\lambda N_i \overset{d}{=} N_W$, as $N_W \sim \mathrm{Poi}(\kappa_{d-1}\,\lambda)$. Assuming that $N_1, \ldots, N_\lambda$ are independent of the i.i.d. random lines $L_{ij} \sim Q_W$ for $i, j \ge 1$, we get

$$\hat{\varphi}_\gamma(\eta) \overset{d}{=} \sum_{i=1}^{\lambda} \sum_{j=1}^{N_i} \hat{\varphi}_\gamma(L_{ij}, \eta) = \frac{1}{\lambda} \sum_{i=1}^{\lambda} \sum_{j=1}^{N_i} \frac{1}{\kappa_{d-1}} \sum_{k=1}^n Y_k(L_{ij})\, \psi_\gamma(\eta, \xi_k)\, \Delta_k.$$

Thus, $\hat{\varphi}_\gamma(\eta)$ can be written as the average of λ i.i.d. random variables.

To apply Cramér's theorem, it remains to calculate the moment generating function of $\sum_{j=1}^{N_1} X_j$, where $X_j = \frac{1}{\kappa_{d-1}} \sum_{k=1}^n Y_k(L_{1j})\, \psi_\gamma(\eta, \xi_k)\, \Delta_k = \widetilde{\varphi}_\gamma(L_{1j})$ which can be done with [Mit97, Lemma 3.1]:

$$\begin{aligned} m_{\sum_{j=1}^{N_1} X_j}(t) &= m_{N_1}[\log m_{X_1}(t)] = \exp\left\{\kappa_{d-1}(\exp[\log m_{X_1}(t)] - 1)\right\} \\ &= \exp\left\{\kappa_{d-1}\,(m_{X_1}(t) - 1)\right\}. \end{aligned}$$

Thus, we arrive at (5.14) by calculating

$$\begin{aligned}
m_{X_1}(t) = \mathbb{E}\,\mathrm{e}^{t\,X_1} &= \mathbb{E}\exp\left\{\frac{t}{\kappa_{d-1}}\sum_{i=1}^{n} Y_i(L_0)\,\psi_\gamma(\eta,\xi_i)\,\Delta_i\right\} \\
&= \mathbb{E}\sum_{j=1}^{n}\sum_{1\le i_1<\cdots<i_j\le n}\exp\left\{\frac{t}{\kappa_{d-1}}\sum_{k=1}^{j}\psi_\gamma(\eta,\xi_{i_k})\,\Delta_{i_k}\right\}\mathbb{1}\left\{Y^*_{i_1,\dots,i_j}(L_0)=1\right\} \\
&= \sum_{j=1}^{n}\sum_{1\le i_1<\cdots<i_j\le n}\exp\left\{\frac{t}{\kappa_{d-1}}\sum_{k=1}^{j}\psi_\gamma(\eta,\xi_{i_k})\,\Delta_{i_k}\right\}\mathbb{P}\left(Y^*_{i_1,\dots,i_j}(L_0)=1\right). \qquad \square
\end{aligned}$$

Note that some summands in (5.14) are zero because it is not possible that one line hits an arbitrary set of test hyperplanes within the observation window, only sequences of neighboring hyperplanes can be intersected.

5.4. Numerical experiments with simulated data

In this section, we present the results of some numerical experiments. As already mentioned, it is a well-known approach to estimate the directional distribution of fiber processes by counting intersections with test planes and applying an inversion of the cosine transform to retrieve the directional distribution. In the following, we perform numerical simulations and apply the method introduced in this chapter to the data, i.e., we calculate the values of the estimator $\hat{\varphi}_\gamma$. We also apply other approaches and compare the results.

5.4.1. Two-dimensional case

For the 2D case, we compare our approach to other methods for the estimation of the directional distribution, namely a Fourier method as described in [MN80] and a method suggested by Digabel (see [Dig75]). An overview of such estimators can be found in [RS92]. For our tests we assume that we can access the data, i.e., the estimation of the rose of intersections, at the angles $\xi_i = \frac{(i-1)\,\pi}{100}$, $i = 1,\dots,100$ and use the same points to evaluate the results. For all reconstructions, we use the same parameters. Depending on the degree of distortion resulting from the estimation of the cosine transform of the density, the smoothing is sometimes a little too much or too less. For our approach we use the polynomial kernel with parameters $\nu = 5$ and $\gamma = 0.4$. The parameter for Digabel's method has been chosen such that the results are similar to the ones of our approach.

A first analysis (see Figure 5.3) shows the results of the methods applied to a theoretical cosine transform of a density, in this case a mixture of a von Mises and a beta distribution. The von Mises density on the interval $[0, 2\pi)$ is defined as

$f_{\mathrm{vM}}(x) = \frac{\exp\{\kappa \cos(x-\mu)\}}{2\pi\, I_0(\kappa)}$, for $\mu \in [0, 2\pi)$, $\kappa > 0$, where I_0 is the modified Bessel function of order zero. Thus, the function $\tilde{f}_{\mathrm{vM}}(x) = f_{\mathrm{vM}}(x) + f_{\mathrm{vM}}(x+\pi)$ is a probability density on $[0, \pi)$. The beta density on $[0, 1]$ is defined as $f_\beta(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, where $\alpha, \beta > 0$, and $B(\cdot, \cdot)$ is the beta function. To get a density function on $[0, \pi)$ we have simply rescaled it accordingly. For the density presented here, we chose the parameters $\kappa = 10$, $\mu = 1$ for the von Mises distribution and $\alpha = 2$, $\beta = 10$ for the beta distribution, respectively.

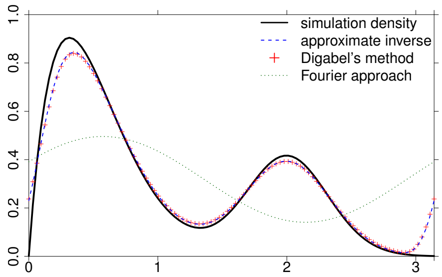


Figure 5.3.: Comparison of the three reconstruction methods on theoretical data

In our first simulation study we simulate a PLP in the unit ball with intensity 10 000 and count the intersections with the set of test lines analytically. For each direction we consider only one test line, namely the one through the origin, which produces highly distorted data. The results can be found in Figure 5.4.

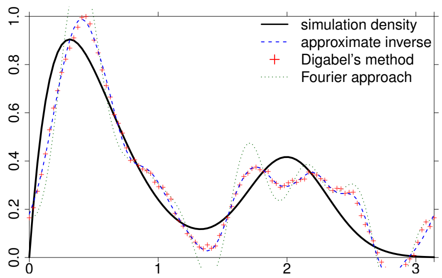


Figure 5.4.: Comparison of the methods on simulated data in the unit ball, intersections counted analytically

For our second simulation study we have considered PCPs with radius 3 and intensity 25 in the unit square. These processes have been voxelized with a resolution of 1000 x 1000 pixels, i.e., a cylinder is 6 pixels thick. Then the images are skeletonized with the software Avizo which produces a set of line segments as result. With this set we have estimated the rose of intersections taking an average over 10 simulations. Since the intensity is always underestimated considerably we renormalized the graphs to get a valid density function. The results are depicted in Figure 5.5.

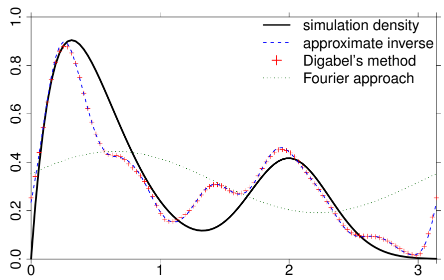


Figure 5.5.: Comparison of the methods on simulated voxel data in the unit square, intersections counting based on skeletonization

Remark 5.7. *The apparent similarity of the curves resulting from the* AI *method and Digabel's approach is not surprising but to be expected, because both methods base on the notion presented in Proposition 5.1. In other words, both are based on the second derivative of the rose of intersections.*

5.4.2. Three-dimensional case

We compare our results with a method introduced in [KP05] (called Kiderlen and Pfrang's method or "KP method" in the following for briefness). We use the least square ansatz, which is given in their paper: Assuming that the value of the cosine transform in n directions is available, the proposed loss-free discretization becomes a least squares problem with $\frac{n(n-1)}{2}$ unknowns, so the number of unknown grows quadratically in the number of measurement points. Here, we consider 900 directions which leads to 404 550 unknowns. The solution is an even discrete measure on the sphere which is concentrated on the directions which are orthogonal to two of the directions $\xi_1, \ldots, \xi_n$. Of course the solution of this optimization problem is much more expensive than the reconstruction with our method, and furthermore the calculating time depends on the input.

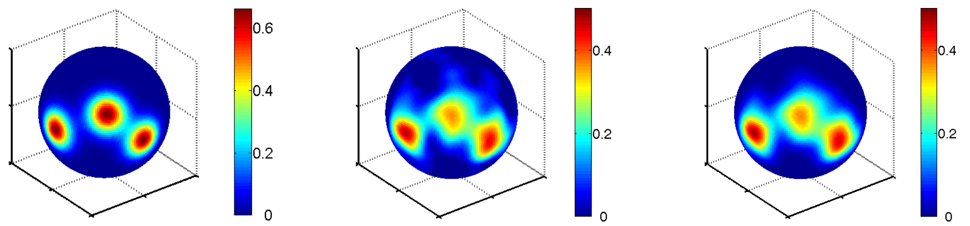
To show the effectiveness of the estimator introduced in Section 5.2, we carry out two simulation studies with stationary PLPs and PCPs, respectively, and compare it to the KP method.

In the first experiment we simulate stationary PLPs in the observation window $B_1^3(\mathbf{o})$. For each measuring direction ξ_i, $i = 1, \ldots, n$, we intersect the process with the orthogonal plane $\xi_i^\perp$ and analytically determine the number of intersections. Finally, we use the method described in Section 5.1.2 and (5.2) to estimate the directional distribution of the PLP. For all reconstructions, we use the regularization parameter $\gamma = 0.22$.

First, we consider PLPs with a mixed von Mises-Fisher directional distribution. The density of a von Mises-Fisher distributed random vector can be written as

$f_{\mu,\kappa}(x) = c(\kappa)\exp\{\kappa\langle\mu, x\rangle\}$, where $c(\kappa) = \frac{\sqrt{\kappa}}{(2\pi)^{3/2} I_{1/2}(\kappa)}$, and I_r denotes the modified Bessel function of first kind and order r. We consider the following symmetric mixed distribution with three peaks: $f(\eta) = \frac{1}{6}\sum_{i=1}^{3}(f_{\mu_i,25}(\eta) + f_{-\mu_i,25}(\eta))$, where $\mu_1 = 1/\sqrt{1.02}(1, 0.1, -0.1)^T$, $\mu_2 = (0, 1, 0)^T$, and $\mu_3 = \sqrt{4/17}(-1, -1.5, 1)^T$.

Figure 5.6 shows the estimates. It should be mentioned that these data sets are generated by only one simulation of the process, so they can be seen as strongly perturbed data.



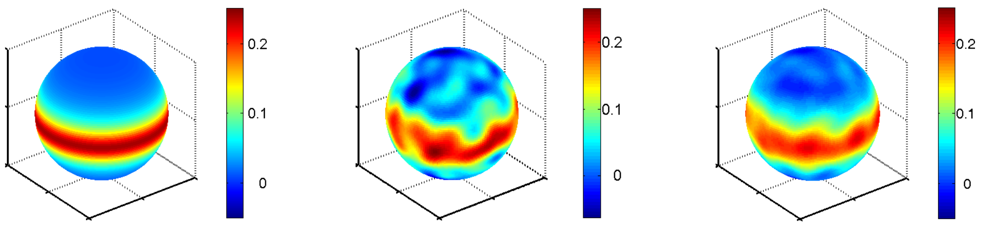
(a) Density function (b) AI method (c) KP method + smoothing

Figure 5.6.: Mixed von Mises-Fisher distribution (process intensity = 1000)

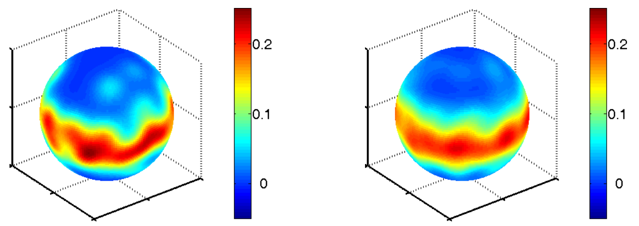
As a second example we consider the directional fiber distribution introduced in [SPRB+06], which is used in modeling foams or granular porous media. The density of the directional distribution, which is independent of the azimuth angle ϕ, is given by

$$p(\phi, \theta) = \frac{\beta}{4\pi\left[1 + (\beta^2 - 1)\cos^2\theta\right]^{3/2}}, \quad \theta \in [0, \pi]. \tag{5.15}$$

The parameter β is called *anisotropy parameter*. In the case $\beta = 1$ this is the density of the directional distribution of an isotropic fiber process. For increasing β, the fibers tend to be more and more parallel to the xy-plane (the material plane). We choose $\beta = 3$ in our experiment. Figure 5.7 shows the density function and its reconstruction based on one or 5 realizations respectively of the corresponding PLPs.



(a) Density function (b) AI method (c) AI method



(d) KP method + smoothing (e) KP method + smoothing

Figure 5.7.: Reconstruction of the distribution defined in (5.15) (reconstructions from one realization (middle), reconstructions from 5 realizations (right))

Again the KP method followed by additional kernel smoothing leads to similar results, where a suitable choice of the smoothing parameter is crucial. For these two distributions the L^2-error of the smoothed KP method is only a little bit smaller than the error of our method since in the large regions with zero density our method shows some small values in contrast to the KP method (cf. Figure 5.7).

In a further experiment, we have simulated stationary PCPs with radius 0.005, different directional distributions, and intensity 500 in the unit cube. The union sets of these PCPs have been voxelized with a resolution of 500 voxels per unit length to generate a setting similar to the analysis of microscopic images, see Figure 5.8.

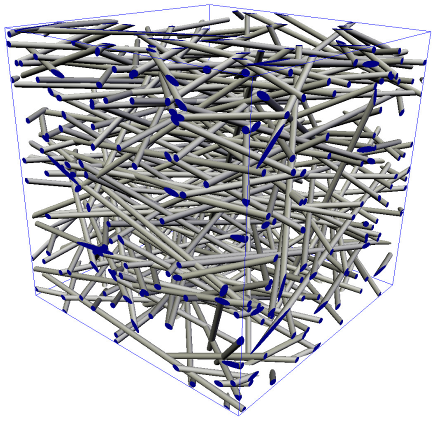
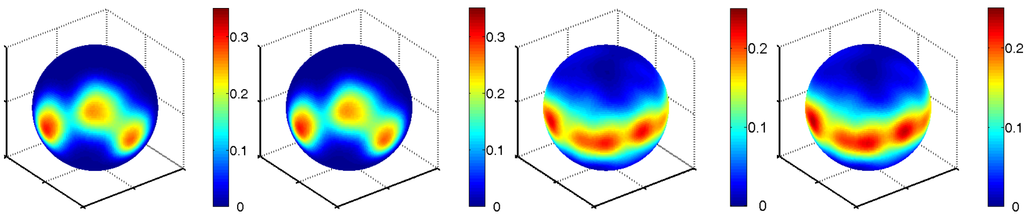


Figure 5.8.: Cylinder process in the unit cube with directional distribution as defined in (5.15) ($\beta = 3$), intensity 200, and cylinder radius 0.01

Counting the intersection points of a plane with the voxelized image is a rather difficult task, since cylinders which are (almost) parallel to the plane may be counted multiple times, while overlapping cylinders may be counted only once. Unfortunately, this effect also depends on the direction of the plane, so with the approach to discretize the plane and count the intersections in the resulting images, it seems impossible to generate estimates without heavy systematical bias. To overcome this problem, we have skeletonized the data with the 3D image analysis software Avizo and estimated the intersection intensities with the resulting skeleton. Here we have also computed the values for 900 directions. Figure 5.9 shows the reconstructions from voxelized data.



(a) AI method (b) KP method and smoothing (c) AI method (d) KP method and smoothing

Figure 5.9.: Reconstruction from voxelized data: mixed von Mises distribution (left), distribution as defined in (5.15) (right)

5.4.3. Asymptotic behavior for growing observation window radius

For applications, it is important to know how good the normal approximation of the test statistics in Section 5.3.2 is at a certain radius and intensity. Thus, in a further experiment we analyze simulation data with different observation window radii for the most important case, the uniform distribution (i.e., isotropy). For this, we have simulated 100 000 independent copies of a stationary isotropic PLP with intensity 1 in balls with different radii. For different observation window radii, we have analyzed the deviation of the distribution of the X_i as defined in (5.13) from the standard Gaussian distribution, where the expectation values in the definition have been computed numerically by Martin Riplinger with Theorem 5.7 and the results from Section 5.3.2. To evaluate the results, we have split our simulation data into 100 groups of size 1000. For each group, we have conducted a level 0.05 Kolmogorov-Smirnoff and Shapiro-Wilk test. In Table 5.1, the rejection rates are listed for different radii at the exemplary point $(-0.40382, -0.23972, 0.88287)^T$, which we have selected from 20 random points because the values seem representative. We consider three of the point sets in [SW04] as measurement points. At a radius between 15 and 20, the approximation seems to be reasonably good with $\gamma = 0.2$. For differing values of γ, convergence gets slower.

	49 points		100 points		225 points	
	KS test	SW test	KS test	SW test	KS test	SW test
radius 5	0.19	0.05	0.46	0.15	0.28	0.02
radius 10	0.13	0.1	0.13	0.07	0.11	0.08
radius 15	0.08	0.07	0.09	0.04	0.08	0.05
radius 20	0.07	0.07	0.06	0.08	0.08	0.06

Table 5.1.: Rejection rates for level 0.05 Kolmogorov-Smirnoff and Shapiro-Wilk tests at different radii and measurement point sets. The hypothesis of isotropy is being tested for an isotropic sample. $\gamma = 0.2$ for all reconstructions.

5.5. Application to real data

In the following, we present the results of our algorithm applied to real microscopic data. For this purpose, we examine images of the gas diffusion layer of a polymer electrolyte membrane fuel cell (with kind permission of the Centre for Solar Energy and Hydrogen Research, Ulm).

For both the two- and the three-dimensional data we have first applied the skeletonization algorithm of Avizo, then estimated the rose of intersections analytically with the resulting set of line segments, and finally applied our method to approximately invert the cosine transform.

5.5.1. Two-dimensional microscopic images

In this section, we analyze 10 electron microscopic images of 10 different gas diffusion layers of the same kind. One of them can be seen in Figure 1.1a. Each has a resolution of 1024 x 696 pixels and shows different layers of the fiber tissue. The fibers are approximately 6 pixels thick.

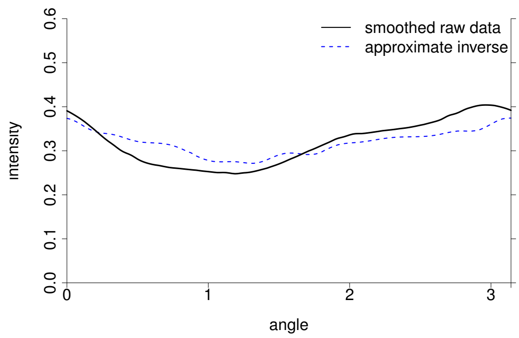


Figure 5.10.: Smoothed data from the line segments of the skeleton and the result of our algorithm

In Figure 5.10 we present the result of our reconstruction compared to a kernel density estimation based on the directions of the lines and weighted with the length of the lines of the skeleton. This shows that our method works well to reconstruct the directional distribution of two-dimensional real data, as the result of our method is very close to the original data from the skeletonization.

5.5.2. Three-dimensional synchrotron images

Here, we reconstruct the directional distribution of one synchrotron image with a resolution of approximately 1000 x 1000 x 200 voxels, see also Figure 5.11, where a cut-out of the skeleton generated by Avizo can be seen.

Because of the production process the directional distribution of the fibers should be approximately isotropic with respect to the x-y plane, which is also shown in our reconstruction. As it can be seen in the smoothed raw data (i.e., the directions and lengths of the segments) from the Avizo skeleton in Figure 5.12a, again there is an artifact in the reconstruction from the skeletonization, the values at the axis directions are too low, whereas at the bisector they are too high. Of course, this can be seen in our reconstruction (cf. Figure 5.12b) as well, although this is not a problem of our method but of the input data. Thus, this shows that our method works well on real three-dimensional data.

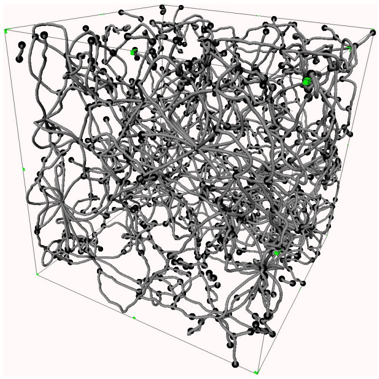
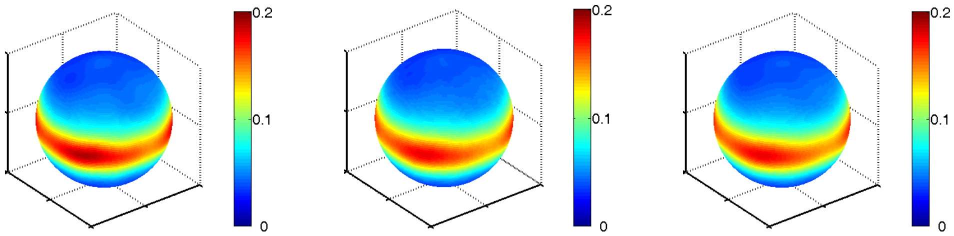


Figure 5.11.: Skeleton of a 3D image of a GDL, generated with Avizo



(a) Kernel density estimation, based on the raw data

(b) Reconstruction with AI method

(c) Reconstruction with the KP method + smoothing

Figure 5.12.: Reconstruction from real data, both reconstructions are based on the skeleton generated by Avizo

5.6. Concluding remarks and open questions

- In [LRSS11], a regularization method for the AI-estimator in the 2D and 3D case can be found along with a more in-depth analysis of the numerical properties. Furthermore, the numerical stability of the integration on the sphere S^2 is discussed.

- The skeleton generated by Avizo can also be used to directly estimate the directional distribution of a fiber system, see Section 5.5. However, in our setting, we assume that only the intersection counts of the process with some test hyperplanes can be observed and not the full information about the process. Hence, this estimator does not seem to be appropriate for comparison to the AI estimator and has not been used in the simulation studies.

- For the analysis of PLPs in $\mathbb{R}^d$, there are still some open problems, including the regularization in arbitrary dimension. Functional limit theorems for the supremum considered in Section 5.3.1 are missing as well. One could also consider certain assumptions on the directional distribution, for example in case $d = 3$ isotropy in the xy-plane, which is also interesting for applications.
- Further ideas for future work are extended simulation studies, most interesting of course in the 3-dimensional case. These could involve the implementation of other (dilated) fiber processes and also the comparison of different mollifiers.
- One should also mention the ongoing project in which different estimators for the directional distribution in $\mathbb{R}^3$ are compared, see [ARR+11]. In this paper, a detailed comparison of the best-known estimators is given, where various simulated data with different process types and parameters and also some real data is taken into account.

A. Appendix

A.1. The relatedness of the two definitions of a PCP in Section 2.2

In this section, we show that the process $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ defined in Section 2.2.2 is a PCP as introduced in Section 2.2.1 and give conditions under which the process is stationary and has a locally finite intensity measure.

Proposition A.1. *Let $\Pi_{\lambda,Q}$ be an independently marked Poisson process with intensity λ and mark distribution Q acting on $\mathcal{B}(\Gamma_{d,k})$, where $\Gamma_{d,k} = \mathbb{SO}_k^d \times \mathcal{R}_{d-k}^o$ (see Section 2.2.2). Additionally, we demand that $\mathbb{E}\,|\Xi_0 \oplus B_\varepsilon^{d-k}(\mathbf{o})|_{d-k} < \infty$ for some $\varepsilon > 0$. Then the following assertions hold:*

(a) There is a version of $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ which is a PCP *in the sense of the particle process in Section 2.2.1.*

(b) $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ is stationary.

(c) The process has a locally finite intensity measure.

Proof. (a) It follows from the Theorems 12.3.5, 13.1.1, 13.2.1, and 13.2.2 in [SW08] that the mapping

$$\mathbb{R}^{d-k} \times \mathbb{SO}_k^d \times \mathcal{R}_{d-k}^o \to \mathcal{Z}_k, \quad (p, \theta, C) \mapsto \theta((C + p) \times \mathbb{R}^k)$$

is continuous. Thus, the process $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ is a (measurable) Poisson process in the space of cylinders. It remains to show the local finiteness, which can only be achieved in an almost sure sense.

Let $C \in \mathcal{C}$ be an arbitrary compact set and r the radius of the smallest centered ball containing C. Then

$$\begin{aligned}
&\mathbb{P}\Big(\#\{\Theta_i((\Xi_i + P_i) \times \mathbb{R}^k) \cap C \neq \emptyset,\, i \geq 1\} < \infty\Big) \\
&\geq \mathbb{P}\Big(\#\{\Theta_i((\Xi_i + P_i) \times \mathbb{R}^k) \cap B_r^d(\mathbf{o}) \neq \emptyset,\, i \geq 1\} < \infty\Big) \\
&= \mathbb{P}\Big(\#\{(\Xi_i + P_i) \cap B_r^{d-k}(\mathbf{o}) \neq \emptyset,\, i \geq 1\} < \infty\Big) = 1,
\end{aligned}$$

where in the last formula we have the usual Boolean model $\{(P_i + \Xi_i),\, i \geq 1\}$ in $\mathbb{R}^{d-k}$. For this, it is known that the process is almost surely locally finite if $\mathbb{E}\,|\Xi_0 \oplus B_\varepsilon(\mathbf{o})|_{d-k} < \infty$ for some $\varepsilon > 0$, see [Hei05].

(b) Let $B \in \mathcal{B}(\mathcal{Z}_k)$ and $x \in \mathbb{R}^d$. Then the distribution of $\Xi(B)$ does not change under translation of B by x, which can be seen by calculating

$$\begin{aligned}
&\#\Big\{\bigcup_{i=1}^{\infty} \Theta_i\big((P_i + \Xi_i) \times \mathbb{R}^k\big) \cap (B + x)\Big\} \\
&= \#\Big\{\bigcup_{i=1}^{\infty} \Theta_i\big((P_i - \Theta_i^T x + \Xi_i) \times \mathbb{R}^k\big) \cap B\Big\} \\
&\stackrel{d}{=} \#\Big\{\bigcup_{i=1}^{\infty} \Theta_i\big((P_i + \Xi_i) \times \mathbb{R}^k\big) \cap B\Big\},
\end{aligned}$$

where we used that the process $\{P_i - \Theta_i^T x,\, i \geq 1\}$ is also a stationary Poisson process with intensity λ and independent of $\{(\Theta_i, \Xi_i),\, i \geq 1\}$.

(c) As in (a), let $C \in \mathcal{C}$ be an arbitrary compact set and r the radius of the smallest centered ball containing C. Then analogously to (a) we calculate

$$\begin{aligned}
\Lambda(\{Z \in \mathcal{Z}_k : Z \cap C \neq \emptyset\}) &\leq \Lambda(\{Z \in \mathcal{Z}_k : Z \cap B_r^d(\mathbf{o}) \neq \emptyset\}) \\
&= \mathbb{E}\,\#\{(\Xi_i + P_i) \cap B_r^{d-k}(\mathbf{o}) \neq \emptyset,\, i \geq 1\} \\
&= \mathbb{E}\,\#\{P_i \in -\Xi_0 \oplus B_r^{d-k}(\mathbf{o}),\, i \geq 1\} \\
&= \lambda\, \mathbb{E}\,|\Xi_0 \oplus B_r^{d-k}(\mathbf{o})|_d < \infty.
\end{aligned}$$

The last inequality holds as $\mathbb{E}\,|\Xi_0 \oplus B_\varepsilon^{d-k}(\mathbf{o})|_{d-k} < \infty$, see [SW08, Theorem 4.1.2].

□

A.2. Rubin's inversion formula for the spherical Radon transform

We explicitly calculate a formula for the inversion of the spherical Radon transform introduced by Rubin in [Rub02]. It is based on a convolution-backprojection method, thus, it is related with the method from Section 5.1.2. The method can be applied for $d \geq 3$, but we restrict ourselves to the most interesting case $d = 3$. Formally, our aim is to reconstruct f from its spherical Radon transform Rf, where $f \in L^1_{\mathrm{e}}(S^2)$.

As described in [Rub02, Sect. 5], for this, we need to find a function $a : (0, \infty) \to \mathbb{C}$ which is locally integrable and for which

$$\lambda(x) := \left(I_{0+}^{1/2}\left[\frac{a(\sqrt{s})}{\sqrt{s}}\right]\right)(x) = \frac{1}{\Gamma(1/2)} \int_0^x \frac{a(\sqrt{s})}{\sqrt{s\,(x-s)}}\,\mathrm{d}s \quad \in L^1(0,\infty), \tag{A.1}$$

i.e.,

$$\int_0^\infty \left| \int_0^x \frac{a(\sqrt{s})}{\sqrt{s\,(x-s)}}\,\mathrm{d}s \right| \mathrm{d}x < \infty$$

with the Riemann-Liouville fractional integral

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(x)(t-x)^{\alpha-1}\,\mathrm{d}x, \quad t \geq 0.$$

To avoid trivial (and useless) cases we assume that

$$\gamma = \sqrt{\pi} \int_0^\infty \lambda(x)\,\mathrm{d}x \neq 0. \tag{A.2}$$

Then with [Rub02, Theorem 5.1] we get

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\eta^\perp} a\left(\frac{\sin[d_{\mathrm{geo}}(x,\xi)]}{\varepsilon}\right)(Rf)(\xi)\,\mathrm{d}\xi = \gamma\, f(\eta), \qquad \eta \in S^2.$$

Thus, it remains to find a locally integrable solution to (A.1) and the γ in (A.2). Rubin suggests the following in [Rub02, Example 5.2]. For $x \geq 0$ let

$$\chi_{\alpha,m}(x) = \left(\frac{d}{d\,x}\right)^m \frac{x^m}{(x+\mathrm{i})^{1+\alpha}}, \quad m \in \mathbb{N}, \alpha > 0,$$

and

$$\lambda_{\alpha,m}(x) = \left(I_{0+}^{\alpha}\chi_{\alpha,m}\right)(x) = \frac{\mathrm{i}^{m-\alpha} m!}{\Gamma(1+\alpha)} \frac{x^a}{(x+\mathrm{i})^{m+1}}.$$

Now he suggests to choose $\alpha = 1/2$ and $\lambda(x) = \lambda_{1/2,m}(x)$ for any $m > 1/2$, e.g., $m = 1$. Further, he sets $a(x) = x\,\chi_{1/2,1}(x^2)$.

We calculate

$$\chi_{1/2,1}(s) = \frac{d}{d\,s}\frac{s}{(s+\mathrm{i})^{3/2}} = \frac{(s+\mathrm{i})^{3/2} - \frac{3}{2}s(s+\mathrm{i})^{1/2}}{(s+\mathrm{i})^3} = \frac{(s+\mathrm{i}) - \frac{3}{2}s}{(s+\mathrm{i})^{5/2}} = \frac{-s+2\mathrm{i}}{2(s+\mathrm{i})^{5/2}}$$

which leads to

$$a(x) = \frac{-x^3 + 2\mathrm{i}\,x}{2(x^2+\mathrm{i})^{5/2}} \qquad \text{and} \qquad \lambda(x) = \frac{\sqrt{\mathrm{i}\,x}}{\Gamma(3/2)\,(x+\mathrm{i})^2}.$$

For the constant γ we calculate $\gamma = \pi$ with the formula given in [Rub02, Example 5.2].

Since the real (or imaginary) part of $a(\cdot)$ is also locally integrable and fulfills equations (A.1) and (A.2), this may be considered as well:

$$a^*(x) := \text{Re } a(x) = \frac{-x^3 \cos\left(\frac{5}{2}\arctan\frac{1}{x^2}\right) + 2x \sin\left(\frac{5}{2}\arctan\frac{1}{x^2}\right)}{2(x^4+1)^{5/4}},$$

where we used

$$\begin{aligned}(s+\mathrm{i})^{-5/2} &= \left(\sqrt{s^2+1}\right)^{-5/2} \exp\left\{-\tfrac{5}{2}\,\mathrm{i}\,\arctan\tfrac{1}{s}\right\} \\ &= \frac{\cos\left(\frac{5}{2}\arctan\frac{1}{s}\right) - \mathrm{i}\sin\left(\frac{5}{2}\arctan\frac{1}{s}\right)}{(s^2+1)^{5/4}}.\end{aligned}$$

This leads to the following inversion formula for the complex case:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\eta^\perp} a\left(\frac{\sin[d_{\text{geo}}(x,\xi)]}{\varepsilon}\right)(Rf)(\xi)\,\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\eta^\perp}\frac{-\left(\sin[d_{\text{geo}}(x,\xi)]/\varepsilon\right)^3 + 2\mathrm{i}\left(\sin[d_{\text{geo}}(x,\xi)]/\varepsilon\right)}{2\left[\left(\frac{\sin[d_{\text{geo}}(x,\xi)]}{\varepsilon}\right)^2 + \mathrm{i}\right]^{5/2}}(Rf)(\xi)\,\mathrm{d}\xi \\ &= \gamma f(\eta) = \pi f(\eta).\end{aligned}$$

And in the real-valued case we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\eta^\perp} a\left(\frac{\sin[d_{\text{geo}}(x,\xi)]}{\varepsilon}\right)(Rf)(\xi)\,\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_{\eta^\perp}\Bigg[-\frac{(\sin[d_{\text{geo}}(x,\xi)]/\varepsilon)^3\cos\left(\frac{5}{2}\arctan\left(\varepsilon/\sin[d_{\text{geo}}(x,\xi)]\right)^2\right)}{2\left[\left(\sin[d_{\text{geo}}(x,\xi)]/\varepsilon\right)^4+1\right]^{5/4}} \\ &\qquad + \frac{\sin[d_{\text{geo}}(x,\xi)]\,\sin\left(\frac{5}{2}\arctan\left(\varepsilon/\sin[d_{\text{geo}}(x,\xi)]\right)^2\right)}{\varepsilon\left[\left(\sin[d_{\text{geo}}(x,\xi)]/\varepsilon\right)^4+1\right]^{5/4}}\Bigg](Rf)(\xi)\,\mathrm{d}\xi \\ &= \gamma^*\, f(\eta),\end{aligned}$$

where

$$\gamma^* = \sqrt{\pi}\int_0^\infty \lambda^*(x)\,\mathrm{d}x \qquad \text{with} \qquad \lambda^*(x) = \frac{1}{\Gamma(1/2)}\int_0^x \frac{a^*(\sqrt{s})}{\sqrt{s\,(x-s)}}\,\mathrm{d}s.$$

One should remark that since no smoothing has been applied, this method leads to numerically very unstable results, as analyzed by Martin Riplinger in [LRSS11].

A.3. Intersection area of two ellipses

We calculate the area of the intersection of two ellipses within the unit disk in $\mathbb{R}^2$ needed in Section 5.3.2. Our results allow the numerical calculation of $Q(H_{1,2})$. It suffices to consider centered ellipses with length 1 and arbitrary widths not exceeding 1. We denote the angle between their major axes by θ.

For the first ellipse E_1, we assume without loss of generality that its major axis is the first coordinate axis, and it has a width a_1 with $0 < a_1 \leq 1$, i.e.,

$$E_1 = \left\{(x, y)^T : x^2 + {}^{y^2}/_{a_1^2} \leq 1\right\}.$$

The other ellipse E_2 also has length 1, and the width is denoted by a_2, assuming again that $0 < a_2 \leq 1$. Its major axis has an angle θ with respect to the first coordinate axis. Formally,

$$E_2 = \{(x, y)^T : (x \cos\theta + y \sin\theta)^2 + (x \sin\theta - y \cos\theta)^2/a_2^2 \leq 1\}, \tag{A.3}$$

see also Figure A.1.

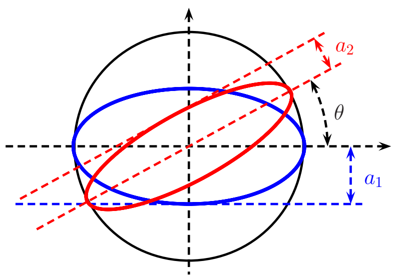


Figure A.1.: Draft of the two ellipses under consideration (E1 in blue, E2 in red)

For the calculation of $|E_1 \cap E_2|_2$, we consider the ellipses stretched by the factor $1/a_1$ in the y-direction. We denote this linear map by Y_{1/a_1}. This operation depicts E_1 onto the unit circle, and E_2 onto the ellipse $\widetilde{E}_2 = Y_{1/a_1}E_2$. Our calculation consists of two parts, namely the determination of the axis lengths of $\widetilde{E}_2$ (Lemma A.1) and the calculation of the intersection area of a centered ellipse and the unit disk (Lemma A.2).

Lemma A.1. *The square lengths of the axes of the ellipse $\widetilde{E}_2 = Y_{1/a_1} E_2$, where E_2 is defined in* (A.3)*, are*

$$\lambda_{1,2} = \frac{1}{2}\left[\left(1+\frac{a_2^2}{a_1^2}\right)\cos^2\theta + \left(\frac{1}{a_1^2}+a_2^2\right)\sin^2\theta\right] \pm \sqrt{\frac{1}{4}\left[\left(1+\frac{a_2^2}{a_1^2}\right)\cos^2\theta + \left(\frac{1}{a_1^2}+a_2^2\right)\sin^2\theta\right]^2 - \frac{a_2^2}{a_1^2}}.$$

Proof. The ellipse $\widetilde{E}_2$ can be written as the result of a linear map applied to the unit disk:

$$\begin{aligned}\widetilde{E}_2 = Y_{1/a_1}\, R_\theta\, Y_{a_2}\, B_1(\mathbf{o}) &= \begin{pmatrix} 1 & 0 \\ 0 & {}^1/_{a_1}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a_2\end{pmatrix} B_1(\mathbf{o}) \\ &= \begin{pmatrix}\cos\theta & -a_2\,\sin\theta \\ \frac{1}{a_1}\sin\theta & \frac{a_2}{a_1}\cos\theta\end{pmatrix} B_1(\mathbf{o}),\end{aligned}$$

where R_θ denotes the rotation about the angle θ.

With a singular value decomposition of the latter matrix one can find the desired axis lengths. □

A few remarks are in order.

- It can be shown that the constants λ_1 and λ_2 in Lemma A.1 are always positive.
- $\lambda_1 = \lambda_2$ holds if and only if $a_1 = a_2 = 1$ or we have $\theta = 0$ and $a_1 = a_2$. In both cases the ellipse $\widetilde{E}_2$ is a circle with radius one which leads to $\lambda_1 = \lambda_2 = 1$.

For the main result of this section, we need one more lemma.

Lemma A.2. *For a centered ellipse E with axis lengths a and b, where $1 \le a$ and $0 < b \le 1$, it holds that*

$$|E \cap B_1(\mathbf{o})|_2 = 2\arccos\sqrt{\frac{a^2\,(1-b^2)}{a^2-b^2}} + 2a\,b\,\arcsin\sqrt{\frac{1-b^2}{a^2-b^2}}.$$

Proof. The claim follows by calculating the intersection points of the boundary of E and the unit sphere and the respective areas. □

Combining these two lemmas, we get

Proposition A.2. *Let E_1 and E_2 be two centered ellipses with major axis lengths 1 and minor axis lengths a_1 and a_2, respectively ($0 < a_1, a_2 \leq 1$), where the angle between the two major axes is denoted by θ (see Figure A.1). Then, for $a_1 = a_2 = 1$ and for $a_1 = a_2$, $\theta = 0$ we have $|E_1 \cap E_2|_2 = a_1\,\pi$. Otherwise,*

$$|E_1 \cap E_2|_2 = 2a_1 \left(\arccos \sqrt{\frac{\lambda_1 (1 - \lambda_2)}{\lambda_1 - \lambda_2}} + \sqrt{\lambda_1 \lambda_2} \arcsin \sqrt{\frac{1 - \lambda_2}{\lambda_1 - \lambda_2}} \right),$$

where

$$\begin{aligned}\lambda_{1,2} = \frac{1}{2} \left[\left(1 + \frac{a_2^2}{a_1^2} \right) \cos^2\theta + \left(\frac{1}{a_1^2} + a_2^2 \right) \sin^2\theta \right] \\ \pm \sqrt{\frac{1}{4} \left[\left(1 + \frac{a_2^2}{a_1^2} \right) \cos^2\theta + \left(\frac{1}{a_1^2} + a_2^2 \right) \sin^2\theta \right]^2 - \frac{a_2^2}{a_1^2}}.\end{aligned}$$

Bibliography

[Amb90] R. V. Ambartzumian. *Factorization calculus and geometric probability*, volume 33 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[ARR+11] H. Altendorf, C. Redenbach, M. Riplinger, K. Sandau, M. Spiess, and O. Wirjadi. Directional analysis of fiber systems using 3d image data. preprint, 2011.

[BJ05] A. Baddeley and E. B. V. Jensen. *Stereology for statisticians*, volume 103 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[Dav78] P. Davy. *Stereology — A Statistical Viewpoint*. PhD thesis, Australian National University, Canberra, 1978.

[dH00] F. den Hollander. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000.

[Dig75] H. Digabel. Détermination pratique de la rose des directions. In *15 Fasicules de morphologie mathématique appliqué*, chapter 6. Fontainebleau, 1975.

[DVJ08] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I / II.* Probability and its Applications (New York). Springer, New York, second edition, 2003 / 2008.

[DZ10] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[EMOT81] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. II.* Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.

[Gal11] B. Galerne. Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. *Image Anal. Stereol.*, 30(1):39–51, 2011.

[Gar06] R. J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2006.

[GKM06] R. J. Gardner, M. Kiderlen, and P. Milanfar. Convergence of algorithms for reconstructing convex bodies and directional measures. *Ann. Statist.*, 34(3):1331–1374, 2006.

[Gor41] R. D. Gordon. Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. *Ann. Math. Statistics*, 12:364–366, 1941.

[Gro96] H. Groemer. *Geometric applications of Fourier series and spherical harmonics*, volume 61 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[GW92] P. Goodey and W. Weil. Centrally symmetric convex bodies and the spherical Radon transform. *J. Differential Geom.*, 35(3):675–688, 1992.

[Hal88] P. Hall. *Introduction to the theory of coverage processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1988.

[Hei05] L. Heinrich. Large deviations of the empirical volume fraction for stationary Poisson grain models. *Ann. Appl. Probab.*, 15(1A):392–420, 2005.

[Hei07] L. Heinrich. An almost-Markov-type mixing condition and large deviations for Boolean models in the line. *Acta Appl. Math.*, 96(1-3):247–262, 2007.

[Hei09] L. Heinrich. Central limit theorems for motion-invariant Poisson hyperplanes in expanding convex bodies. *Rend. Circ. Mat. Palermo (2) Suppl.*, 81:187–212, 2009.

[Hel08] S. Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2008.

[Hil62] J. E. Hilliard. Specification and measurement of microstructural anisotropy. *Trans. Metall. Soc. AIME*, 224:1201–1211, 1962.

[Hof09a] L. M. Hoffmann. Estimating an even spherical measure from its sine transform. *Appl. Math.*, 54(1):67–78, 2009.

[Hof09b] L. M. Hoffmann. Mixed measures of convex cylinders and quermass densities of Boolean models. *Acta Appl. Math.*, 105(2):141–156, 2009.

[HS96] R. H. Hardin and N. J. A. Sloane. McLaren's improved snub cube and other new spherical designs in three dimensions. *Discrete Comput. Geom.*, 15(4):429–441, 1996.

[HS09] L. Heinrich and M. Spiess. Berry-Esseen bounds and Cramér-type large deviations for the volume distribution of Poisson cylinder processes. *Lith. Math. J.*, 49(4):381–398, 2009.

[HS12] L. Heinrich and M. Spiess. Central Limit Theorem for the Volume of Stationary Poisson Cylinder Processes in Expanding Domains. submitted, 2012.

[HSS06] L. Heinrich, H. Schmidt, and V. Schmidt. Central limit theorems for Poisson hyperplane tessellations. *Ann. Appl. Probab.*, 16(2):919–950, 2006.

[JKB95] N. L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous univariate distributions. Vol. 2.* Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1995.

[KP05] M. Kiderlen and A. Pfrang. Algorithms to estimate the rose of directions of a spatial fibre system. *J. Microsc.*, 219(2):50–60, 2005.

[KS10] V. Yu. Korolev and I. G. Shevtsova. An improvement of the Berry-Esseen inequality with applications to Poisson and mixed Poisson random sums. *Scand. Actuar. J.*, 2010. to appear.

[LB95] G. Last and A. Brandt. *Marked point processes on the real line.* Probability and its Applications (New York). Springer-Verlag, New York, 1995. The dynamic approach.

[LM90] A. K. Louis and P. Maass. A mollifier method for linear operator equations of the first kind. *Inverse Problems*, 6(3):427–440, 1990.

[LRSS11] A. K. Louis, M. Riplinger, M. Spiess, and E. Spodarev. Inversion algorithms for the spherical Radon and cosine transform. *Inverse Problems*, 27(3):035015, 25, 2011.

[LS59] V. P. Leonov and A. N. Shiryaev. On a method of semi-invariants. *Theor. Probability Appl.*, 4:319–329, 1959.

[Mat67] G. Matheron. *Eléments pour une théorie des milieux poreux.* Masson, Paris, 1967.

[Mat75] G. Matheron. *Random Sets and Integral Geometry.* J. Wiley & Sons, New York, 1975.

[Mec81] J. Mecke. Formulas for stationary planar fibre processes. III. Intersections with fibre systems. *Math. Operationsforsch. Statist. Ser. Statist.*, 12(2):201–210, 1981.

[Mit97] H. Mita. Probabilities of large deviations for sums of random number of i.i.d. random variables and its application to a compound Poisson process. *Tokyo J. Math.*, 20(2):353–364, 1997.

[MN80] J. Mecke and W. Nagel. Stationäre räumliche Faserprozesse und ihre Schnittzahlrosen. *Elektron. Informationsverarb. Kybernet.*, 16(8-9):475–483, 1980.

[Mol05] I. Molchanov. *Theory of random sets.* Probability and its Applications (New York). Springer London Ltd., London, 2005.

[OM00] J. Ohser and F. Mücklich. *Statistical Analysis of Microstructures in Materials Science.* J. Wiley & Sons, Chichester, 2000.

[OS09] J. Ohser and K. Schladitz. *3D Images of Materials Structures: Processing and Analysis.* Wiley-VCH, Weinheim, 2009.

[RS92] J. Rataj and I. Saxl. Estimation of direction distribution of a planar fibre system. *Acta Stereologica*, 11:631–636, 1992. Suppl. I.

[RS11] M. Riplinger and M. Spiess. Asymptotic properties of the approximate inverse estimator for directional distributions. submitted, 2011.

[Rub02] B. Rubin. Inversion formulas for the spherical Radon transform and the generalized cosine transform. *Adv. in Appl. Math.*, 29(3):471–497, 2002.

[Sch87] R. Schneider. Geometric inequalities for Poisson processes of convex bodies and cylinders. *Results Math.*, 11(1-2):165–185, 1987.

[Sch93] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1993.

[Sch00] K. Schladitz. Estimation of the intensity of stationary flat processes. *Adv. in Appl. Probab.*, 32(1):114–139, 2000.

[Sch07] T. Schuster. *The method of approximate inverse: theory and applications*, volume 1906 of *Lecture Notes in Mathematics.* Springer, Berlin, 2007.

[Ser84] J. Serra. *Image analysis and mathematical morphology.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1984.

[SKM95] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic Geometry and its Applications.* J. Wiley & Sons, Chichester, second edition, 1995.

[Spo01] E. Spodarev. On the rose of intersections of stationary flat processes. *Adv. in Appl. Probab.*, 33(3):584–599, 2001.

[Spo02] E. Spodarev. Cauchy-Kubota-type integral formulae for the generalized cosine transforms. *Izv. Nats. Akad. Nauk Armenii Mat.*, 37(1):52–69 (2003), 2002.

[SPRB+06] K. Schladitz, S. Peters, D. Reinel-Bitzer, A. Wiegmann, and J. Ohser. Design of acoustic trim based on geometric modelling and flow simulation for non-woven. *Computational Materials Science*, 38(1):56–66, 2006.

[SS91] L. Saulis and V. A. Statulevičius. *Limit theorems for large deviations*, volume 73 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1991. Translated and revised from the 1989 Russian original.

[SS11] M. Spiess and E. Spodarev. Anisotropic Poisson Processes of Cylinders. *Methodology and Computing in Applied Probability*, 13(4):801–819, 2011.

[Sta66] V. A. Statulevičius. On large deviations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 6:133–144, 1966.

[SW04] I. H. Sloan and R. S. Womersley. Extremal systems of points and numerical integration on the sphere. *Adv. Comput. Math.*, 21(1-2):107–125, 2004.

[SW08] R. Schneider and W. Weil. *Stochastic and integral geometry.* Probability and its Applications (New York). Springer, Berlin, 2008.

[Tor02] S. Torquato. *Random heterogeneous materials*, volume 16 of *Interdisciplinary Applied Mathematics.* Springer, New York, 2002.

[Wei87] W. Weil. Point processes of cylinders, particles and flats. *Acta Appl. Math.*, 9(1-2):103–136, 1987.

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Declaration

I hereby declare that this thesis was performed and written on my own and that references and resources used within this work have been explicitly indicated.

I am aware that making a false declaration may have serious consequences.

Ulm, July 5, 2012

(Signature)

Curriculum vitae

Date of birth:	May 6, 1980
Marital status:	Married
Nationality:	German
Birthplace:	Waiblingen, Germany

Scholar education

1990 – 1999	Remstalgymnasium Weinstadt **Degree:** Abitur

Academic studies

Oct. 2000 – Dec. 2006	Studies in mathematics at Ulm University **Degree:** Diplom **Thesis:** Simulation Methods for Spatial Gibbs Point Processes **Supervisor:** Prof. Dr. Volker Schmidt

PhD studies

since Jan. 2007	PhD student in the Institute of Stochastics at Ulm University **Supervisor:** Prof. Dr. Evgeny Spodarev sponsored by the DFG since January 2009

Research visits

April 2009	Research stay at the Moscow State University for two weeks
April 2010	Invitation to a conference in Oberwolfach

List of publications

- Lothar Heinrich and M.S.,
  Berry-Esseen bounds and Cramér-type large deviations for the volume distribution of Poisson cylinder processes,
  Lithuanian Mathematical Journal, 49(4):381–398, 2009,

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  Anisotropic Poisson Processes of Cylinders,
  Methodology and Computing in Applied Probability, 13(4):801–819, 2011,

- Alfred K. Louis, Martin Riplinger, M.S., and Evgeny Spodarev,
  Inversion algorithms for the spherical Radon and cosine transform,
  Inverse Problems, 27(3):035015, 25, 2011,

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  Asymptotic Properties of the Approximate Inverse Estimator for Directional Distributions,
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- Lothar Heinrich and M.S.,
  Central Limit Theorem for the Volume of Stationary Poisson Cylinder Processes in Expanding Domains,
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  Directional analysis of fiber systems using 3d image data,
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