Singular limits and maximally continued solutions of moving boundary problems

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Referent:	Prof. Dr. J. Escher
Korrefferent:	Prof. Dr. C. Walker
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Abstract

We prove the converge of abstract dynamical systems to their associated quasistationary approximations and apply these results to a moving boundary problem modeling the growth of avascular tumors.
Moreover, we introduce the notion of maximal continued solutions of moving boundary problems in the sense that we characterize what inhibits global in time existence of solutions. Again, our test object is the tumor model.

Keywords: Moving boundary problem, blow-up, quasistationary approximation

Zusammenfassung

Wir beweisen die Konvergenz abstrakter dynamischer Systeme gegen ihre assoziierten quasistationären Approximationen und geben eine Anwendung auf ein freies Randwertproblem, welches das Wachstum avaskularer Tumoren beschreibt.
Darüberhinaus erklären wir den Begriff der maximal fortgesetzten Lösung eines freien Randwertproblems. Wir charakterisieren, welche Phänomene die globale Existenz von Lösungen verhindern und diskutieren diese am Beispiele des Tumor Modells.

Stichworte: Freies Randwertproblem, blow-up, quasistationäre Approximation

Contents

1 Preface

The theory of partial differential equations as the probably most important mathematically tool to describe various processes in nature, has already been developed to be a very extensive and multifarious theory. Nevertheless, many problems are still very hard to treat, in particular those problems, which involve an unknown domain as a part of the differential equations. One of the simplest examples of these so called 'moving boundary problems' is the well understood mean curvature flow $V = H$: One looks for a family of surfaces whose normal velocity equals their mean curvature. In suitable coordinates this simple law turns out to be a complicated nonlinear parabolic equation, and solutions will have a finite life span in general. In deed, under this law any convex initial geometry shrinks into a round point in finite time as Gerhard Huiskens was able to prove, see [Hu1984]. This also means finite time blow-up of the surfaces curvature.
Much less is known in the case of more complicated systems. Recently, an abstract functional analytic view has been successfully applied to the famous Stefan problem, so called Hele-Shaw flows, the averaged mean curvature flow and models of avascular tumor growth, just to mention a few, c.f. [K2007], [EsSi97a], [EsSi98], [MaSi00], [Es2000]. This semigroup theory based method seems to be the right tool not only to prove existence and smoothness properties of local solutions, but also to understand equilbrium states of these problems.
Nevertheless, many questions have not yet been answered. One major thing is the nonexistence of a general well posedness statement analogous to the ordinary case of fixed domains, stating, that a solution or some derivative must blow up, if it has a finite life span only. In the case of a moving boundary problem, blow-up can obviously also mean the development of a singularity in the moving boundary. In the second part of this work we shall give a first approach to such a theorem, c.f. Theorem 4.2. It should be mentioned, that, although we do only consider one special model, the method of the proof does apply to all the problems numbered above.
Another problem is the precise relation between the so called full problems and their 'quasistationary approximations'. In order to point out qualitative properties, it is still necessary in many cases formally to drop out a disturbing time derivative of a function describing the development of some involved physical quantity. In many cases, a physical justification of this procedure can be given by reasoning, that this quantity evolves along a different time scale than the boundary manifold, and therefore is approximately in a stationary state relative to the development of the boundary. In the first part of this work we shall introduce a small parameter and prove convergence of the corresponding solution to a solution of the quasistationary model, if this parameter tends to zero. This result, although stated in a rather general setting, applies first of all to the special model which we consider in this work. This special model will be explained in the introduction.

At this place, I want to thank professor Joachim Escher for his very helpfull support, but never tutelage, in a various manner, concerning not only this thesis, but also the time at the end of my undergraduate studies.

2 Introduction

Throughout this work we will consider the following moving boundary problem:

$$
(2.1) \qquad \begin{cases} -\triangle p = f(v) & \text{in } \Omega(t) \\ \varepsilon\partial_t v - \triangle v = -h(v) & \text{in } \Omega(t) \\ V = -\partial_\nu p & \text{on } \Gamma(t) \\ p = cH & \text{on } \Gamma(t) \\ v = \psi & \text{on } \Gamma(t), \end{cases}
$$

where we set $\Gamma(t) := \partial\Omega(t)$. The system is completed by the initial conditions $v(0,\cdot) = v_0$, $\Gamma(0) = \Gamma_0$. System (2.1) is a model describing the growth of an avascular tumor: $\Omega(t)$ is the domain occupied by the tumor at time t. By p we denote the cell pressure and v describes the concentration of a nutrient, for example glucose, diffusing through the tumor. The normal velocity of the family $\{\Gamma(t)\}$ and the mean curvature of the surface $\Gamma(t)$ are denoted by V and $H = H(t)$, respectively. The functions h and f are known: the rate of consumption of the nutrient and of cell-proliferation.
The model (2.1) was introduced by Greenspan in [Gr1956] and [Gr1976]. The idea is to study tumor growth from the point of view of fluid dynamics: The tumor cells are considered as particles of an incompressible fluid, while the tumor itself is a moving domain, whose velocity field is commensurate with the pressure gradient. Forces of surface tension counteract the internal cell pressure.
Cell proliferation makes the cellular tissue grow, fed with some nutrient by diffusion. Thus, the cell proliferation rate f can be interpreted as the source term in the balance of mass equation. To illustrate this, let

$$
\mathrm{vol}(t) := \int_{\Omega(t)} 1 \, dx
$$

measure the volume of the tumor. Then it is well known that $d/dt\, \mathrm{vol}(t) = \int_{\Gamma(t)} V \, d\sigma$, c.f. [EsSi97a]. Thus, by the equations in (2.1) and the divergence theorem,

$$
d/dt\, \mathrm{vol}(t) = \int_{\Omega(t)} f(v(t,x)) \, dx,
$$

meaning, that in dependence on the present concentration of the nutrient, f acts as a source or as a sink. A typical choice would be $f(v) = -\mu_0(v - \bar{v})$, where μ_0, $\bar{v}$ are positive constants, but may also be of logistic type, see [ByCh95], [ByCh96].
Next, let us comment on the positive constants c, ε. At first, $c > 0$ is the surface tension coefficient related to cell to cell adhesive forces. In this work we are not interested in

the dependence of the system (2.1) on c and thus normalize it to be 1. This is different in the case of ε, which determines the time scale where the evolution of the nutrient takes place. As already mentioned in the preface, a huge part of this work is devoted to prove the convergence of the solution $(v_\varepsilon, p_\varepsilon, \Gamma_\varepsilon)$ of (2.1) to a solution of the model obtained by setting $\varepsilon = 0$, as ε tends to 0.

Finally, let us discuss some results which are already available. First of all, in the situation of a constant consumption rate $h(v) = \lambda v$, $\lambda > 0$, (2.1) has already been studied in [Es2000], where the existence of smooth local solutions has been proved. A radially symmetric setting for a quasi stationary analogue of (2.1) has been investigated in [MatA08]. In fact, it is shown there, that in case the tumor is a disc and there is a critical rate of cell death, then the tumor domain will vanish. The corollary 4.3 of this work may be viewed as a first approximation of a generalization of this result.

This work is organized as follows: Section 3 is devoted to the study of the problems dependence on the parameter ε. In Section 3.1 we shall define an abstract Banach space setting which allows the treatment of a quite general formulation of a dynamical system in Section 3.2. These results are applied to system (2.1) in Section 3.3.

In Section 4 we develop the notion of maximal continued solutions. The results we obtain are stated in Section 4.1. Section 4.2 will be used to define much of our notation and to discuss some helpful differential geometric material. Section 4.3 will deal with a proof of our main theorem.

Finally, Section 5 is needed to fill some gaps.

3 Singular limits in nonlinear dynamical systems

3.1 The abstract setting and linear equations

Let $\Sigma_\vartheta := \{z \in \mathbb{C};\ |\arg(z)| \leq \vartheta + \pi/2\} \cup \{0\}$. Troughout this section we shall assume

- E_1, E_0 are Banach spaces, $E_1 \overset{d}{\hookrightarrow} E_0$;
- J is a perfect subinterval of $\mathbb{R}^+$ containing 0, $0 < \rho < 1$;
- There are $M, \eta > 0$ as well as $\vartheta \in (0, \pi/2)$ such that

$$(3.1) \qquad \begin{cases} \mathcal{A} \subset C^\rho(J, \mathcal{H}(E_1, E_0)), & \|A\|_{C^\rho(J,\mathcal{L}(E_1,E_0))} \leq \eta, \\ \Sigma_\vartheta \subset \rho(-A(s)) & \\ \|A(s)\|_{\mathcal{L}(E_1,E_0)} + (1+|\lambda|)^{1-j}\|(\lambda + A(s))^{-1}\|_{\mathcal{L}(E_0,E_j)} \leq M, & \end{cases}$$

where $(s, \lambda, A) \in J \times \Sigma_\vartheta \times \mathcal{A}$ and $j = 0, 1$.

Here, $\mathcal{H}(E_1, E_0)$ denotes the set of all bounded linear operators $A \in \mathcal{L}(E_1, E_0)$, such that $-A$, considered as a closed operator in E_0 generates a strongly continuous analytic semigroup of operators on E_0, i.e. in $\mathcal{L}(E_0)$, which we shall denote by e^{-tA}. The symbol $\rho(A)$ denotes the resolvent set of A, and, given metric spaces X, Y, $C^\rho(X, Y)$ is the set of ρ - Hölder continuous functions. (3.1) has the well-known implication

$$(3.2) \qquad 1/M\|x\|_{E_1} \leq \|A(s)x\|_{E_0} \leq M\|x\|_{E_1},$$

$(s, x, A) \in J \times E_1 \times \mathcal{A}$.

It is also well known, that (3.1) guarantees the existence of a parabolic fundamental solution $U_A(t, s)$ possessing E_1 as a regularity subspace for any $A \in \mathcal{A}$. A detailed construction can be found in [LaQPP]. Moreover, estimates are proven, whose importance is revealed in the study of quasilinear problems. In the sequel we shall use these techniques to derive estimates of the fundamental solutions belonging to a family of the form $\frac{1}{a} \cdot A$, $a > 0$, $A \in \mathcal{A}$. In deed, Lemma III 2.2.1 in [LaQPP], (3.1), (3.2) and the fact that semigroup and generator commute, lead to the following statement:

Lemma 3.1 *There exist $C = C(k, M)$, $\sigma = \sigma(\vartheta, M) > 0$ such that*

$$(3.3) \qquad \|[tA(s)]^k e^{-tA(s)}\|_{\mathcal{L}(E_j)} + t\|[tA(s)]^k e^{-tA(s)}\|_{\mathcal{L}(E_0,E_1)} \le C \cdot e^{-\sigma t},$$

$k \in \mathbb{N}$, $(t,s,A) \in \mathbb{R}^{>0} \times J \times \mathcal{A}$, $j = 0, 1$.

Let X, Y, Z be Banach spaces and $J^*_\triangle := \{(t,s) \in J \times J;\ s < t\}$. If $f : J^*_\triangle \to \mathcal{L}(Y,Z)$ and $g : J^*_\triangle \to \mathcal{L}(X,Y)$ are suitable functions, let

$$(f \star g)(t,s) := \int_s^t f(t,\tau) g(\tau,s)\, d\tau$$

denote their convolution. First just formally we define for $(t,s) \in J^*_\triangle$

i) $a_A(t,s) := e^{-(t-s)A(s)}$

ii) $k_A(t,s) := -[A(t) - A(s)] e^{-(t-s)A(s)}$

iii) $w_A(t,s) := \sum_j (\bigstar_{i=1..j}\ k_A(t,s))$

iv) $e_A(t,s) := A(t) e^{-(t-s)A(t)} - A(s) e^{-(t-s)A(s)}$,

$A \in \mathcal{A}$, and notice, that the equality

$$U_A(t,s) = a_A(t,s) + (a_A \star w_A)(t,s)$$

holds true for $A \in \mathcal{A}$. Let $\varepsilon > 0$, $A \in \mathcal{A}$ and $A_\varepsilon := \frac{1}{\varepsilon} A$. The fact

$$e^{-t(A_\varepsilon)(s)} = e^{-\frac{t}{\varepsilon} A(s)}$$

is behind the following lemma:

Lemma 3.2 *There exist $\delta = \delta(M, \vartheta, \rho) > 0$, $\varepsilon_0 = \varepsilon_0(M, \eta, \vartheta, \rho) > 0$, such that*

$$(3.4) \qquad \begin{array}{lcl} \|(a_{A_\varepsilon} \star w_{A_\varepsilon})(t,s)\|_{\mathcal{L}(E_0)} & \le & C \cdot (t-s)^\rho \cdot e^{-\delta(t-s)/\varepsilon} \\ \|(a_{A_\varepsilon} \star w_{A_\varepsilon})(t,s)\|_{\mathcal{L}(E_1,E_0)} & \le & C \cdot \frac{(t-s)^{\rho+1}}{\varepsilon} \cdot e^{-\delta(t-s)/\varepsilon} \\ \|(a_{A_\varepsilon} \star w_{A_\varepsilon})(t,s)\|_{\mathcal{L}(E_0,E_1)} & \le & C \cdot \varepsilon \cdot (t-s)^{\rho-1} \cdot e^{-\delta(t-s)/\varepsilon} \\ \|(a_{A_\varepsilon} \star w_{A_\varepsilon})(t,s)\|_{\mathcal{L}(E_1)} & \le & C \cdot (t-s)^\rho \cdot e^{-\delta(t-s)/\varepsilon} \end{array}$$

hold true for $(t,s) \in J^*_\triangle$, $A \in \mathcal{A}$ and

- $\varepsilon < \varepsilon_0$ *and a constant* $C = C(M, \eta, \rho) > 0$ *in the case* $J = [0, \infty)$
- $\varepsilon > 0$ *and a constant* $C = C(M, \eta, \rho, T) > 0$*, monotone increasing in* $T > 0$ *in the case* $J = [0, T]$.

In order to prove Lemma 3.2 we shall use the following techniquality:

Lemma 3.3 *There are* $C = C(\vartheta) > 0$ *and* $b = b(\rho, \sigma) > 0$ *such that*

$$\|e_{A_\varepsilon}(t, s)\|_{\mathcal{L}(E_0)} \leq C \cdot (t - s)^{\rho^2 - 1} \cdot e^{-b(t-s)/\varepsilon},$$

$(t, s) \in J^*_\triangle$.

The two lemmata will be proved at the very end of the thesis. At the moment, we need some more notation: If X, Y are sets, Y^X is the set of all mappings of X into Y. $\mathcal{P}(X)$ denotes the set of all subsets of X. Given metric spaces $(X, d_X), (Y, d_Y)$, let $C_{ue}(X, Y) \subset \mathcal{P}(Y^X)$ be the set of all uniformly equicontinuous families of maps from X to Y, and let $C^a_{ue}(X, Y) \subset \mathcal{P}(Y^X)$, $a > 0$, be the set of all equi-Hölder continuous families of maps from X to Y, that is the set of families $\{v_b : X \to Y;\ b \in \mathsf{B}\}$ satisfying

$$\sup_{b \in \mathsf{B}} d_Y(v_b(x), v_b(y)) \leq c \cdot d_X(x, y)^a, \qquad x, y \in X.$$

Finally, let E_θ, $\theta \in [0, 1]$, be admissible interpolation spaces between E_1 and E_0. The open ball in the space E_θ with radius R is denoted by $\mathbb{B}^R_\theta$. The next theorem generalizes a result of S. G. Krein (see Theorem 4.4.5 in [Pazy]):

Theorem 3.4 (The Linear Case) *Let* $0 < \delta < T$, $J := [0, T]$, $J_\delta := [\delta, T]$, $\alpha_0 \in (0, 1)$. *Assume* $(B_\varepsilon)_{\varepsilon > 0} \subset \mathcal{A}$ *to be a net in* $\mathcal{A}$ *and* $(F_\varepsilon)_{\varepsilon > 0} \subset (E_0)^J$ *such that for some* $R > 0$ *and* $\mu \in (0, 1)$

i) $\{F_\varepsilon\} \in C_{ue}(J_\delta, \mathbb{B}^R_0) \cap \mathcal{P}(C(J, \mathbb{B}^R_0))$

or

ii) $\{F_\varepsilon\} \in C_{ue}(J_\delta, \mathbb{B}^R_\mu) \cap \mathcal{P}(C(J, \mathbb{B}^R_\mu))$.

Further, there may exist a pair $(B, F) \in (\mathcal{L}_{is}(E_1, E_0))^{J \setminus \{0\}} \times (E_0)^{J \setminus \{0\}}$*, such that* $(B_\varepsilon, F_\varepsilon)(t) \overset{\varepsilon \to 0}{\longrightarrow} (B, F)(t)$ *holds true in* $\mathcal{L}(E_1, E_0) \times E_0$*, uniformly on* J_δ. *If* u_ε *is the mild solution of*

$$\varepsilon u'(t) + B_\varepsilon(t)u(t) = F_\varepsilon(t), \qquad u(0) = u_0 \in E_{\alpha_0}, \qquad\qquad t \in J,$$

then $u_\varepsilon(t) \overset{\varepsilon\to 0}{\longrightarrow} B(t)^{-1}F(t)$ *uniformly on* J_δ *with respect to*

- *the topology on* E_α *in the case i) for all* $\alpha \in [0,1)$,
- *the topology on* E_1 *in the case ii).*

Notice that in the case ii) the mild solution is a classical one.

Proof: For simplicity let $U_\varepsilon := U_{\frac{1}{\varepsilon}B_\varepsilon}$, $a_\varepsilon := a_{\frac{1}{\varepsilon}B_\varepsilon}$ and so on. Then

$$\begin{array}{rcl} u_\varepsilon(t) & = & U_\varepsilon(t,0)u_0 + \frac{1}{\varepsilon}\int_0^t U_\varepsilon(t,s)F_\varepsilon(s)\,ds \\ & = & U_\varepsilon(t,0)u_0 + \frac{1}{\varepsilon}\int_0^t [a_\varepsilon(t,s) + (a_\varepsilon \star w_\varepsilon)(t,s)].F_\varepsilon(s)\,ds. \end{array}$$

Interpolation gives $U_\varepsilon(t,0)u_0 \overset{\varepsilon\to 0}{\longrightarrow} 0$ in $C([a,T],E_\beta)$, $0 < a < T$, $0 \le \beta \le 1$. From Lemma 3.2 we conclude

$$\begin{array}{rcl} \|\frac{1}{\varepsilon}\int_0^t (a_\varepsilon \star w_\varepsilon)(t,s)F_\varepsilon(s)\,ds\|_{E_1} & \le & C \cdot \int_0^t (t-s)^{\rho-1} \cdot e^{-\delta(t-s)/\varepsilon}\,\|F_\varepsilon(s)\|_{E_0}\,ds \\ & \le & CR \cdot \varepsilon^\rho \cdot \int_0^\infty x^{\rho-1} \cdot e^{-\delta x}\,dx \\ & & \overset{\varepsilon\to 0}{\longrightarrow} 0. \end{array}$$

In order to get along with the remaining term we set

$$\begin{array}{rcl} \frac{1}{\varepsilon}\int_0^t a_\varepsilon(t,s)F_\varepsilon(s)\,ds & = & \frac{1}{\varepsilon}\int_0^t a_\varepsilon(t,s)[F_\varepsilon(s) - F_\varepsilon(t)]\,ds + \frac{1}{\varepsilon}\int_0^t a_\varepsilon(t,s)F_\varepsilon(t)\,ds \\ & =: & I_{\varepsilon,1}(t) + I_{\varepsilon,2}(t) \end{array}$$

and treat $I_{\varepsilon,1}$ at first. Assume
$\underline{F_\varepsilon \in C_{ue}(J_\delta, \mathbb{B}_0^R)}$: Then, by interpolation of the morphism $a_\varepsilon(t,s) : (E_0,E_0) \to (E_0,E_1)$ we get

$$\begin{array}{rcl} \|I_{\varepsilon,1}(t)\|_{E_\alpha} & \le & \frac{C}{\varepsilon} \cdot \varepsilon^\alpha \cdot \int_0^t (t-s)^{-\alpha} \cdot e^{-\sigma(t-s)/\varepsilon}\,\|F_\varepsilon(t) - F_\varepsilon(s)\|_{E_0}\,ds \\ & = & \frac{C}{\varepsilon} \cdot \varepsilon^\alpha \cdot \int_0^t x^{-\alpha} \cdot e^{-\sigma x/\varepsilon}\,\|F_\varepsilon(t) - F_\varepsilon(t-x)\|_{E_0}\,dx \\ & \le & \frac{C}{\varepsilon} \cdot \varepsilon^\alpha \cdot \int_0^{r\varepsilon} x^{-\alpha} \cdot e^{-\sigma x/\varepsilon}\,\|F_\varepsilon(t) - F_\varepsilon(t-x)\|_{E_0}\,dx \\ & & +2CR \cdot \int_r^\infty x^{-\alpha} \cdot e^{-\sigma x}\,dx \\ & \le & C \cdot \int_0^r y^{-\alpha} \cdot e^{-\sigma y}\,\|F_\varepsilon(t) - F_\varepsilon(t-\varepsilon y)\|_{E_0}\,dy \\ & & +2CR \cdot \int_r^\infty x^{-\alpha} \cdot e^{-\sigma x}\,dx, \end{array}$$

$r > 0$. If $b > 0$ is given, we choose $r > 0$ big enough, such that the second integral is smaller than $b/2$. Then, by our knowledge about F_ε, we can choose $\varepsilon > 0$ small enough that the first integral is smaller than $b/2$, too. Thus $I_{\varepsilon,1}(\cdot) \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ in $C(J_\delta, E_\alpha)$, $0 \le \alpha < 1$. Assume

$F_\varepsilon \in C_{ue}(J_\delta, \mathbb{B}_\mu^R)$: In this case we find

$$\begin{array}{rcl} \|I_{\varepsilon,1}(t)\|_{E_1} & \le & \frac{1}{\varepsilon}\cdot\int_0^t \|e^{-(t-s)A_\varepsilon(s)}\|_{\mathcal{L}(E_\mu,E_1)}\ \|F_\varepsilon(t)-F_\varepsilon(s)\|_{E_\mu}\ ds \\ & \le & \frac{C}{\varepsilon}\cdot\varepsilon^{1-\mu}\cdot\int_0^t (t-s)^{\mu-1}\cdot e^{-\sigma(t-s)/\varepsilon}\ \|F_\varepsilon(t)-F_\varepsilon(s)\|_{E_\mu}\ ds \end{array}$$

and continue as before. Let us consider $I_{\varepsilon,2}(t)$ to be an element of E_0 auf. Setting

$$I_{\varepsilon,2}^a(t) := \frac{1}{\varepsilon}\cdot\int_0^{t-a} a_\varepsilon(t,s)F_\varepsilon(t)\ ds,$$

we have $I_{\varepsilon,2}^a(t) \in E_1$ as well as

$$\begin{array}{rcl} I_{\varepsilon,2}^a(t) & = & \frac{1}{\varepsilon}\cdot(B_\varepsilon(t))^{-1}\ [\int_0^{t-a}[B_\varepsilon(t)-B_\varepsilon(s)]a_\varepsilon(t,s)F_\varepsilon(t)\ ds] \\ & & +(B_\varepsilon(t))^{-1}\ [\int_0^{t-a}\frac{1}{\varepsilon}B_\varepsilon(s)a_\varepsilon(t,s)F_\varepsilon(t)\ ds] \\ & =: & (B_\varepsilon(t))^{-1}\ J_{\varepsilon,1}^a + (B_\varepsilon(t))^{-1}\ J_{\varepsilon,2}^a. \end{array}$$

We calculate

$$\begin{array}{rcl} J_{\varepsilon,2}^a(t) & = & -\int_0^{t-a} e_\varepsilon(t,s)F_\varepsilon(t)\ ds + \int_0^{t-a}\frac{1}{\varepsilon}B_\varepsilon(t)e^{-(t-s)/\varepsilon\, B_\varepsilon(t)}(t,s)F_\varepsilon(t)\ ds \\ & = & -\int_0^{t-a} e_\varepsilon(t,s)F_\varepsilon(t)\ ds + [e^{-a(\frac{1}{\varepsilon}B_\varepsilon)(t)}F_\varepsilon(t) - e^{-t(\frac{1}{\varepsilon}B_\varepsilon)(t)}F_\varepsilon(t)]. \end{array}$$

Sending $a \to 0$, from the estimates

$$\begin{array}{rcl} \|\int_0^t e_\varepsilon(t,s)F_\varepsilon(t)\ ds\|_{E_0} & \le & CR\cdot\int_0^t (t-s)^{\rho^2-1}\ e^{-c(t-s)/\varepsilon}\ ds \\ & \le & CR\cdot\varepsilon^{\rho^2}\cdot\int_0^\infty x^{\rho^2-1}\ e^{-cx}\ dx \end{array}$$

as well as

$$\begin{array}{rcl} \|\frac{1}{\varepsilon}\cdot\int_0^t [B_\varepsilon(t)-B_\varepsilon(s)]a_\varepsilon(t,s)F_\varepsilon(t)\ ds\|_{E_0} & \le & \frac{CR}{\varepsilon}\cdot\int_0^t (t-s)^{\rho-1}\cdot\varepsilon\cdot e^{-\sigma(t-s)/\varepsilon}\ ds \\ & \le & CR\cdot\varepsilon^\rho\cdot\int_0^\infty x^{\rho-1}\ e^{-\sigma x}\ dx \end{array}$$

we conclude that $u_\varepsilon(t) \stackrel{\varepsilon\to 0}{\longrightarrow} B(t)^{-1}F(t)$. In the last step we have also used the continuity (in fact analyticity) of the inversion map.

■

We close this section by proving a simple inequality of Gronwall-type:

Lemma 3.5 *Let J be a bounded perfect subinterval of $\mathbb{R}^{\geq 0}$ containing 0, and let $a, u \in L_\infty(J, \mathbb{R})$ satisfy*

$$u(t) \leq a(t) + C \cdot \int_0^t (t-\tau)^{-\alpha} \cdot u(\tau)\, d\tau,$$

$C > 0$, $t \in J$, $\alpha \in (0,1)$. Then

$$\|u\|_{L_\infty(J,\mathbb{R})} \leq c \cdot \|a\|_{L_\infty(J,\mathbb{R})},$$

$c = c(C, \alpha, \sup(J))$.

PROOF: Let $k(t,s) := C \cdot (t-s)^{-\alpha}$. By Lemma II 3.2.1 in [LaQPP], $w(t,s) := \sum_{j=1}^{\infty} \bigstar_{i=1}^{j} k(t,s)$ is well defined. Let $b := a + k \star u - u$. Then $b \geq 0$ and $w \geq 0$, since $k \geq 0$. From Theorem II 3.2.2 in [LaQPP] we know that $u = (a-b) + w \star (a-b)$. Moreover, Theorem II 3.2.1 in [LaQPP] implies, that w is integrable. Thus, $u \leq a + w \star a$ and the assertion follows. ∎

3.2 Quasilinear systems

From now on we focus on a system of the form

$$\left\{ \begin{array}{rcl} \varepsilon\dot{u} + A(\rho)u &=& F(u,\rho) \\ \dot{\rho} + B(\rho)\rho &=& G(u,\rho) \\ u(0) &=& u_0 \\ \rho(0) &=& \rho_0. \end{array} \right. \tag{3.5}$$

Our assumptions are as follows:

- E_0, E_1, F_0, F_1 are Banach spaces and and the embeddings $E_1 \overset{d}{\hookrightarrow} E_0$, $F_1 \overset{d}{\hookrightarrow} F_0$ are compact;
- $0 < \gamma \leq \beta < \alpha \leq 1$, Y_β is open in F_β;
- given $\delta \in (\beta, 1]$, $Y_\delta := Y_\beta \cap F_\delta$ carries the topology of F_δ;
- $A \in C^{1-}(Y_\beta, \mathcal{H}(E_1, E_0))$, $B \in C^{1-}(Y_\beta, \mathcal{H}(F_1, F_0))$;

- $F \in C^{1-}(E_\beta \times Y_\beta, E_\gamma)$, $G \in C^{1-}(E_\beta \times Y_\beta, F_\gamma)$;
- $(u_0, \rho_0) \in E_\alpha \times Y_\alpha$.

Here, E_θ, F_θ are interpolation spaces of exponent $\theta \in (0,1)$ with respect to any admissible interpolation functor. We want to call a set $\mathcal{M} \subset \mathcal{H}(E_1, E_0)$ *regularly bounded*, if there exist $M > 0$, $\vartheta \in (0, \pi/2)$, $\omega \in \mathbb{R}$, such that

$$\left\{ \begin{array}{l} \omega + \Sigma_\vartheta \subset \rho(-C) \\ \\ \|\omega + C\|_{\mathcal{L}(E_1,E_0)} + (1+|\lambda|)^{1-j} \|(\lambda + \omega + C)^{-1}\|_{\mathcal{L}(E_0,E_j)} \leq M \end{array} \right. \tag{3.6}$$

holds true, whenever $(\lambda, C, j) \in \Sigma_\vartheta \times \mathcal{M} \times \{0,1\}$. Clearly, finite unions of regularly bounded sets are regularly bounded. It is known, that any $C \in \mathcal{H}(E_1, E_0)$ possesses a regularly bounded neighborhood (c.f Corollary I 1.4.3 in [LaQPP]).

Let $\rho \in C^\nu([0,T], Y_\beta)$, $T > 0$, $\nu \in (0,1)$. Then

$$F_{\rho,T}(t,u) := F(\rho(t), u) \in C^{\nu,1-}([0,T] \times E_\beta, E_\gamma).$$

Thus (cf. [Ama88]), the equation

$$\left\{ \begin{array}{rcl} \varepsilon \dot{u} + A(\rho) u & = & F_{\rho,T}(t,u) \\ u(0) & = & u_0 \end{array} \right. \tag{3.7}$$

possesses a unique maximal continued solution $u_{\varepsilon,T}(\rho) \in C([0,t_\varepsilon^+), E_\alpha) \cap C^\alpha([0,t_\varepsilon^+), E_0)$. Fixing this notation, we can proceed as follows:

Given a regularly bounded neighborhood U of $B(\rho_0)$ in $\mathcal{H}(F_1, F_0)$, we choose a closed, convex and bounded neighborhood X of ρ_0 in Y_β such that $B[X] \subset U$ and set

$$W^\nu_{X(U),T} := \{ r \in C([0,T], X) | \ \|r(t) - r(s)\|_{F_\beta} \leq |t-s|^\nu \},$$

which is not the empty set, because it contains the constant function $t \mapsto \rho_0$. We have

Lemma 3.6 *The set $W^\nu_{X(U),T}$ is a closed, bounded and convex subset of $C([0,T], Y_\beta)$.*

PROOF: $W^{\nu}_{X(U),T}$ is bounded, since X is bounded. It is convex, since X is convex and

$$\|(b \cdot r_0 + (1-b) \cdot r_1)(s) - (b \cdot r_0 + (1-b) \cdot r_1)(t)\|_{F_\beta} \leq [b + (1-b)]|t-s|^{\nu}.$$

To see that it is closed, let $(r_n) \subset W^{\nu}_{X(U),T}$ converge to some $r \in C([0,T], X)$. Then, given $\delta > 0$, there exists $n \in \mathbb{N}$ such that

$$\|r(t) - r(s)\|_{F_\beta} \leq 2\|r - r_n\|_{C([0,T],X)} + \|r_n(t) - r_n(s)\|_{F_\beta} \leq \delta + |t-s|^{\nu}$$

and the assertion follows by sending δ to 0. ■

We now make the cruical assumption:

- Given $\rho \in W^{\nu}_{X(U),T}$, there is a $C = C(X)$, such that

  $$\|u_{\varepsilon,T}(\rho)(t)\|_{E_\alpha} \leq C, \qquad 0 < t < t^+_{\varepsilon}(\rho), \tag{3.8}$$

  where $C(X) \leq C(\tilde{X})$, provided $X \subset \tilde{X}$.

Finally, we use the abbreviations $B_a := \bar{\mathbb{B}}_{F_\beta}(\rho_0, a)$ as well as $W^{\nu}_{a(U),T} := W^{\nu}_{B_a(U),T}$. The symbol '$\hookrightarrow\hookrightarrow$' means compact embedding.

Theorem 3.7 *There is a positive time t^* independent of ε, such that, given $\varepsilon > 0$, system (3.5) possesses a unique classical solution $(u_\varepsilon, \rho_\varepsilon) \in C([0,t^*], E_\alpha \times Y_\alpha) \cap C^{\alpha}([0,t^*], E_0 \times F_0)$.*

Moreover, $\bigcup_{\varepsilon>0} u_\varepsilon[[0,t^*]] \times \rho_\varepsilon[[0,t^*]]$ is a bounded subset of $E_\alpha \times Y_\alpha$. If, given $0 < \delta < t^*$, there is a number $\mu > 0$, such that $u_\varepsilon \in C^{\mu}_{ue}([\delta,t^*], E_0)$, then there is a pair $(u,\rho) \in C((0,t^*], E_1) \times C([0,t^*], Y_\alpha)$ with the following properties:

- *$(u_\varepsilon, \rho_\varepsilon)(t) \to (u,\rho)(t)$ in $E_\alpha \times Y_\alpha$, locally uniformly on $(0,t^*]$, after possibly passing to a subnet;*

- *$\rho(0) = \rho_0$;*

- *(u,ρ) satisfies (3.5) in the case $\varepsilon = 0$ pointwise on $t \in (0,t^*]$.*

Proof: Let regularly bounded neighborhoods U of $B(\rho_0)$ in $\mathcal{H}(F_1, F_0)$ and W of $A(\rho_0)$ in $\mathcal{H}(E_1, E_0)$, $r_0 \in (0, \alpha - \beta)$ as well as $a > 0$ be chosen such that $B[B_a] \subset U$, $A[B_a] \subset W$. Moreover, we may assume A, B to be Lipschitz continuous on B_a. We set

$$W_{a,T} := W^{r_0}_{a(U),T}.$$

(3.8) implies the global existense of the function $u_{\varepsilon,T}(\rho)$, $\rho \in W_{a,T}$, i.e. on $[0,T]$. Moreover, the set

$$M_a := \bigcup_{\substack{\varepsilon > 0 \\ r \in W_{a,T}}} u_{\varepsilon,T}(r)[[0,T]]$$

is bounded in E_α and thus relatively compact in E_β. Thus, we can choose $a > 0$ in a way, that F, G are Lipschitz continuous on $M_a \times B_a$.
Next we prove that, given $\varepsilon > 0$, the mapping

$$\rho \mapsto u_{\varepsilon,T}(\rho) : W_{a,T} \to C([0,T], E_\alpha)$$

is Lipschitz continuous. In deed, from the Variation of Constants (VoC) formula we know that

$$u_{\varepsilon,T}(\rho)(t) = U_{A(\rho)_\varepsilon}(t,0)u_0 + 1/\varepsilon \int_0^t U_{A(\rho)_\varepsilon}(t,s) F_{\rho,T}(s, u(s))\, ds.$$

Thus, Lemma II 5.1.4 in [LaQPP] tells us

$$\begin{aligned}
&\|[u_{\varepsilon,T}(\rho) - u_{\varepsilon,T}(\sigma)](t)\|_{E_\alpha} \\
&\leq C \cdot \big[\|A(\rho)_\varepsilon - A(\sigma)_\varepsilon\|_{L_\infty(0,T,\mathcal{L}(E_1,E_0))} \cdot \{\|u_0\|_{E_\alpha} \\
&+ \textstyle\int_0^t (t-s)^{\gamma-\alpha} \|F_{\omega,\rho,T}(s, u_{\varepsilon,T}(\rho))(s)\|_{E_\gamma}\, ds\Big\} \\
&+ \textstyle\int_0^t \|U_{[A(\sigma)+\omega]_\varepsilon}(t,s)\|_{\mathcal{L}(E_\gamma,E_\alpha)} \|F_{\omega,\rho,T}(s, u_{\varepsilon,T}(\rho))(s) - F_{\omega,\sigma,T}(s, u_{\varepsilon,T}(\sigma))(s)\|_{E_\gamma}\, ds\Big],
\end{aligned}$$

for a suitable $\omega \in \mathbb{R}$ and $F_{\omega,\sigma,T}(s,v) := F_{\sigma,T}(s,v) + \omega v$. Thus,

$$\begin{aligned}
\|[u_{\varepsilon,T}(\rho) - u_{\varepsilon,T}(\sigma)](t)\|_{E_\alpha} \quad \leq \quad & C \cdot [\|\rho - \sigma\|_{C([0,T],Y_\beta)} \\
& + \textstyle\int_0^t (t-s)^{\gamma-\alpha} \|[u_{\varepsilon,T}(\rho) - u_{\varepsilon,T}(\sigma)](s)\|_{E_\alpha}\, ds],
\end{aligned}$$

where $C = C(\varepsilon, T, \gamma, \alpha)$ is independent of $\rho, \sigma \in W_{a,T}$. Lemma 3.5 finally gives

$$\|[u_{\varepsilon,T}(\rho) - u_{\varepsilon,T}(\sigma)](t)\|_{E_\alpha} \le \tilde{C} \cdot \|\rho - \sigma\|_{C([0,T],Y_\beta)}.$$

In general, of course, $\tilde{C}$ will tend to infinity as ε tends to 0. Nevertheless, denoting by $\sigma_{\varepsilon,T}(\rho)$ the unique solution of the equation

$$\sigma_t + B(\rho)\sigma = G(u_{\varepsilon,T}(\rho), \rho), \qquad \sigma(0) = \rho_0,$$

the VoC formula, interpolation and Lemma II 5.1.4 in [LaQPP] imply the Lipschitz continuity of the mapping

$$\Phi_\varepsilon : W_{a,T} \to C([0,T], Y_\beta), \qquad \rho \mapsto \sigma_{\varepsilon,T}(\rho),$$

c.f. Theorem II 5.2.1 in [LaQPP]. Again, the corresponding Lipschitz constants will depend on $\varepsilon > 0$ in general. But, following Theorem II 5.3.1 in [LaQPP], thanks to (3.8) and $0 < r_0 < \alpha - \beta$, we find that

$$\bigcup_{\varepsilon>0} \Phi_\varepsilon[W_{a,T}] \subset W_{a,T} \cap C([0,T], P_\alpha), \tag{3.9}$$

where P_α denotes a bounded subset of Y_α containing ρ_0, provided $a, T > 0$ are small enough. Since $W_{a,T}$ is a convex, closed and bounded subset of $C([0,T], Y_\beta)$ by Lemma 3.6, and since any of the mappings Φ_ε is compact due to (3.9), $F_1 \hookrightarrow\hookrightarrow F_0$ (thus also $F_\alpha \hookrightarrow\hookrightarrow F_\beta$) and Simons compactness theorem (vgl. [Si88]), there exists a family of fixed points ρ_ε inside $W_{a,T}$. Thus, if $\varepsilon > 0$ is given, the pair $(u_\varepsilon, \rho_\varepsilon)$ is a solution of (3.5), where $u_\varepsilon := u_{\varepsilon,T}(\rho_\varepsilon)$. The uniqueness and regularity results from Theorem 12.1 in [Ama93] complete the proof of the theorems first statement.

Let us now fix two sufficiently small numbers a, T and denote them by A, t^*. By construction, it is clear, that the set $\{\rho_\varepsilon |\ \varepsilon > 0\}$ is relatively compact in $C([0,t^*], Y_\beta)$. It follows from $\rho_\varepsilon \in W_{a,T}$,

$$\varepsilon(u_\varepsilon)_t + A(\rho_\varepsilon)u_\varepsilon = F(u_\varepsilon, \rho_\varepsilon)$$

and (3.8), that $\|u_\varepsilon(t)\|_{E_\alpha} \le C(A)$, $t \in [0,t^*]$, $\varepsilon > 0$. Given $0 < \delta < t^*$, let $J_\delta := [\delta, t^*]$ and $J := [0,t^*]$. Assume that $u_\varepsilon \in C^\mu_{ue}(J_\delta, E_0)$. Then the set $\{u_\varepsilon |\ \varepsilon > 0\}$ is relatively compact in $C(J_\delta, E_\beta)$. Therefore we find a subnet $(u_{\varepsilon'}, \rho_{\varepsilon'})_{\varepsilon'>0}$, again denoted by $(u_\varepsilon, \rho_\varepsilon)_{\varepsilon>0}$, satisfying

$$(u_\varepsilon, \rho_\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} (u, \rho) \in C(J_\delta, E_\beta) \times C(J, Y_\beta)$$

in $C(J_\delta, E_\beta) \times C(J, Y_\beta)$. Let $(u_{\varepsilon_k}, \rho_{\varepsilon_k})_{\varepsilon_k > 0}$ be a sequence in $(u_\varepsilon, \rho_\varepsilon)_{\varepsilon > 0}$ such that $\varepsilon_k \to 0$. Denote this sequence again by $(u_\varepsilon, \rho_\varepsilon)$. It is not difficult to see, that the set

$$A[\rho[J]] \times B[\rho[J]] \cup \bigcup_{\varepsilon > 0} A[\rho_\varepsilon[J]] \times B[\rho_\varepsilon[J]] \tag{3.10}$$

is a compact subset of $\mathcal{H}(E_1, E_0) \times \mathcal{H}(F_1, F_0)$. Thus, we find $\omega \in \mathbb{R}$, such that the operators $\omega + A(\rho_\varepsilon)$, $\omega + B(\rho_\varepsilon)$ as well as the limits $\omega + A(\rho)$, $\omega + B(\rho)$ satisfy the assumptions of Theorem 3.4. Moreover, $F(u_\varepsilon, \rho_\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} F(u, \rho)$ in E_γ uniformly on J_δ. Remember that the set $\bigcup_{\varepsilon > 0} u_\varepsilon[J]$ is bounded in E_α. Thus $\{u_\varepsilon |\ \varepsilon > 0\} \in C_{ue}^{\mu\beta/\alpha}(J_\delta, E_\beta)$:

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{E_\beta} \leq \|u_\varepsilon(t) - u_\varepsilon(s)\|_{E_\alpha}^{1-\beta/\alpha} \cdot \|u_\varepsilon(t) - u_\varepsilon(s)\|_{E_0}^{\beta/\alpha} \leq C \cdot |t - s|^{\mu\beta/\alpha},$$

$t, s \in J_\delta$. By construction, the set $\{\rho_\varepsilon|_{J_\delta};\ \varepsilon > 0\}$ is equi - r_0 - Hölder continuous with respect to the topology of Y_β. Thus,

$$F(u_\varepsilon, \rho_\varepsilon) \in C_{ue}^{\min\{r_0, \mu\beta/\alpha\}}(J_\delta, E_\gamma),$$

and Theorem 3.4 implies

$$u(t) \overset{\varepsilon \leftarrow 0}{\longleftarrow} u_\varepsilon(t) \overset{\varepsilon \to 0}{\longrightarrow} (\omega + A(\rho(t)))^{-1}(F(u(t), \rho(t)) + \omega \cdot u(t)),$$

where the right arrow represents convergence in $C(J_\delta, E_1)$. Therefore, the left arrow has to represent convergence in $C(J_\delta, E_1)$, too. Thus, $u \in C(J_\delta, E_1)$, and so (δ has been arbitrary) $u \in C((0, t^*], E_1)$. We are left to show that the pair (u, ρ) solves the second equation, but this is an easy consequence from the VoC-formula and the continuous dependence of the evolution operator on the family of generating operators (cf. Lemma 5.1.4 in [LaQPP]).

■

Remember that, thanks to (3.8), $u_\varepsilon[[0, t^*]]$ is a bounded subset of E_α, uniformly in $\varepsilon > 0$. Thus, $u[(0, t^*]]$ is bounded in E_α, too, and we find a convergent sequence $u(h_k)$, $h_k \to 0$ with respect to the topology in E_β. Denoting its limit by $x := u(0) \in E_\beta$, continuity implies that

$$u(0) = (\omega + A(\rho(0)))^{-1}(F(u(0), \rho(0)) + \omega \cdot u(0)).$$

Thus $x \in E_1$ as well as $A(\rho_0)x = F(x, \rho_0)$. Unfortunately, we cannot guarantee uniform continuity of u. In applications, the uniqueness and continuous dependence of the solution of the problem $Ay = F(y)$ on the data A and F may help us to identify $x = u(0)$ as a real continuation.

The following observation will be usefull in applications:

Corollary 3.8 *Let $\rho \in W^{\nu}_{a(U),T}$ be given, $T > 0$, $\nu \in (0,1)$. We assume, there exists a Banach space V such that $E_0 \overset{d}{\hookrightarrow} V$ and an interpolation functor $\mathcal{F}$ of exponent $\mu \in (0,1)$, such that*

- *there is a $\delta_0 \in [0, \alpha]$ such that $\mathcal{F}(E_{\delta_0}, V) \hookrightarrow E_{\beta'}$ for some $\beta' > \beta$,*
- *there is a Banach space Z, $E_\beta \subset Z \subset V$, such that F can be continued to a function $\tilde{F} : Z \times Y_\beta \to V$. Moreover, given $y \in Y_\beta$, there may exist a neighborhood $U \subset Y_\beta$ of y, such that $\tilde{F}(\cdot, z)$ is bounded on bounded subsets of Z uniformly with respect to $z \in U$,*
- *$A \in C^{1-}(Y_\beta, \mathcal{H}(E_{\delta_0}, V))$.*

Then, if (3.8) is replaced by

$$\|u_{\varepsilon,T}(\rho)(t)\|_Z \le C, \qquad 0 < t < t^+_\varepsilon(\rho), \qquad C = C(a),$$

the statement of Theorem 3.7 still holds true. In fact, in order to establish the convergence result, it suffices to possess the a priori information

$$u_\varepsilon \in C^\mu_{ue}(J_\delta, V), \qquad \delta \in (0, t^*), \qquad \mu \in (0,1).$$

Notice that Z is not needed to be an interpolation space.

Proof: Again, let regularly bounded neighborhoods U of $B(\rho_0)$ and W of $A(\rho_0)$ (and also the corresponding constants M, ω), $r_0 \in (0, \alpha - \beta)$ as well as $a > 0$ be chosen small enough, that $B[B_a] \subset U$, $A[B_a] \subset W$ and

$$\bigcup_{\substack{\varepsilon > 0 \\ r \in W_{a,T}}} (F \circ (u_{\varepsilon,T}(r), r))[[0,T]] + \omega \cdot u_{\varepsilon,T}(r)[[0,T]]$$

is bounded in V. It follows

$$\begin{array}{rcl} \|u_{\varepsilon,T}(r)(t)\|_{E_{\beta'}} & \leq & C(a) + 1/\varepsilon \int_0^t \|U_{(\omega+A(r))_\varepsilon}(t,s)\|_{\mathcal{L}(V,\mathcal{F}(E_{\delta_0},V))} \cdot C(a)\ ds \\ & \leq & C(a) + C(a) \cdot \varepsilon^{\mu-1} \int_0^t [(t-s)^{-\mu} + (t-s)^{(\rho-1)\mu}] e^{-\sigma(t-s)/\varepsilon}\ ds \\ & \leq & C(a) + C(a) \cdot (1+\varepsilon^{\rho\mu}) \cdot [\int_0^\infty x^{-\mu} \cdot e^{-\sigma x}\ dx + \int_0^\infty x^{(\rho-1)\mu} \cdot e^{-\sigma x}\ dx] \\ & \leq & C(a). \end{array}$$

From $E_1 \hookrightarrow\hookrightarrow E_0$ (thus also $E_{\beta'} \hookrightarrow\hookrightarrow E_\beta$) we conclude, that $F \circ (u_{\varepsilon,T}(r), r)$ is actually bounded in E_γ, uniformly with respect to $r \in W_{a,T}$. Thus we find even boundedness of $u_{\varepsilon,T}(r)$ in E_α as before. We can proceed as in the proof of the last theorem. The second statements proof is straight forward. ■

Finally, let us consider an important special case:

Corollary 3.9 *Under the assumptions of Corollary 3.8 we assume the nonlinearity F to possess the the following structure:*

$$F(u,\rho) = \varepsilon \cdot H(u,\rho) + J(u,\rho),$$

where

- $J(\cdot,z) : V \to V$ *is bounded on bounded sets, locally uniformly with respect to* $z \in Y_\beta$;
- $H : Z \times Y_\beta \to V$, *and, given* $z \in Y_\beta$ *there is a neighborhood* Y *of* z *in* Y_β *such that*

  $$\|H(v,y)\|_V \leq C \cdot (\|v\|_Z + 1),$$

  where $C > 0$ *can be chosen independent of* $y \in Y$, *but may depend on* $\|v\|_V$;
- $(H|_{E_\beta \times Y_\beta}, J|_{E_\beta \times Y_\beta}) \in C^{1-}(E_\beta \times Y_\beta, [E_\gamma]^2)$.

Assuming

$$\|u_{\varepsilon,T}(\rho)(t)\|_V \leq C, \qquad 0 < t < t_\varepsilon^+(\rho), \qquad C = C(a)$$

all the statements of Corollary 3.8 remain valid in the sense, that the corresponding limit is a solution of the system

$$
\text{(3.11)} \qquad \left\{ \begin{array}{rcl} A(\rho)u & = & J(u,\rho) \\ \dot{\rho} + B(\rho)\rho & = & G(u,\rho) \\ \rho(0) & = & \rho_0. \end{array} \right.
$$

PROOF: The assumptions imply

$$
\|u_{\varepsilon,T}(\rho)(t)\|_Z \leq C(a) + C(a) \cdot \varepsilon \|u_{\varepsilon,T}(\rho)\|_{L_\infty(0,t,Z)},
$$

which offers an estimate of $\|u_{\varepsilon,T}(\rho)(t)\|_Z$, provided $\varepsilon < \varepsilon_0$. We proceed as before. ∎

Remark 3.10 Statements, similar to those of Theorem 3.7, Corollary 3.8 and 3.9 could be achieved in another way avoiding the usage of the estimates of Lemma 3.2: One could replace the arguments from the proofs of Corollary 3.8, 3.9 by a 'real' scaling argument. Of course, to establish the converge results, stronger a priori estimates, containing at least the information

$$
\sup_{0<\varepsilon<\varepsilon_0} \|\dot{u}_\varepsilon\|_{L_\infty(J_\delta,V)} < \infty
$$

are necessary in this case. Also, the linear case would not be covered by this approach. Moreover, analyzing the above proofs, one recognizes that it is in fact enough to possess the a priori information

$$
\{u_\varepsilon\} \in C_{ue}(J_\delta, V).
$$

From this point of view, our results seem to be nearly optimal.

3.3 Application to the tumor model

In this Section we want to identify system (2.1):

$$
\left\{ \begin{array}{rclll} -\triangle p & = & f(v) & \text{in} & \Omega(t) \\ \varepsilon \partial_t v - \triangle v & = & -h(v) & \text{in} & \Omega(t) \\ V & = & -\partial_\nu p & \text{on} & \Gamma(t) \\ p & = & cH & \text{on} & \Gamma(t) \\ v & = & \psi & \text{on} & \Gamma(t) \end{array} \right.
$$

as a special case of Corollary 3.9. Local existence for the case $\varepsilon = 1$, $h(v) = \lambda v$, $\lambda > 0$, has already been shown in [ES2000]. We make some general assumptions which we keep fixed afterwards:

- $n \geq 2$, $q > n+1$;
- $f \in C^\infty(\mathbb{R})$, $h \in C^\infty(\mathbb{R})$, $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $c = 1$.

Throughout this section we use the same symbol '$\cdot$' to denote the scalar products in all $\mathbb{R}^l$, where $l \in \mathbb{N}$.
Let us introduce some function spaces. Given $q \in (1, \infty)$ and $s \geq 0$, H^s_q denotes the Bessel potential space, built over L_q. Remember that this space coincides with the Sobolev space W^s_q provided s is a natural number. Moreover, if $p \geq 1$ and $s \in \mathbb{R}$, B^s_{pp} is the Besov space as introduced in [Trie1], [Trie2]. Observe that $B^s_{\infty\infty} = BUC^s$, if $s > 0$ is not a natural number. Letting $s = k + s'$, where $k \in \mathbb{N}$ and $s' \in (0,1)$, BUS^s stands for the space of all continously differentiable functions up to the order k, having s'- Hölder continuous derivatives of order k. As usual, function spaces, built over a manifold, are defined by means of a sufficiently smooth atlas.
We want to make a short discussion about the initial state of system (2.1): Given a domain D with smooth boundary Σ and a tubular neighborhood N of Σ with radius a, i.e. $\mathrm{dist}(\partial N, \Sigma) = a$, let $\mathrm{Ad}(\Sigma) := \{\rho \in C^2(\Sigma);\ \|\rho\|_{C^1(\Sigma)} < a/4\}$, $\Gamma_\rho := \theta_\rho[\Sigma]$ and $\Omega_\rho := \theta_\rho[D]$, $\rho \in \mathrm{Ad}$. Here θ_ρ denotes the Hanzawa diffeomorphism:

$$\theta_\rho(y) = \begin{cases} P(y) + \rho(P(y)) \cdot [\Lambda(y) + \varphi(\Lambda(y))] \cdot \mu(P(y)), & y \in N \\ y & y \notin N, \end{cases}$$

where $\varphi \in \mathcal{D}(\mathbb{R})$ satisfies $\varphi|_{[-a,a]} = 1$, $\mathrm{supp}(\varphi) \subset (-3a, 3a)$ and μ is the outward unit normal field of Σ. $P = P_{[\Sigma]}$ and $\Lambda = \Lambda_{[\Sigma]}$ are the metric projection map and the signed distance function with respect to Σ. Especially, $\theta_\rho(y) = y + \rho(y) \cdot \mu(y)$, if $y \in \Sigma$.
We assume a to be small enough, that P and Λ are smooth function in $\bar{N}$ and mention, that the surface Γ_ρ is the zero-levelset of the function

$$\varphi_\rho : N \to \mathbb{R}, \qquad x \mapsto \Lambda(x) - \rho(P(x)),$$

$\rho \in \mathrm{Ad}$. Finally, we set $N_D := N \cap D$. Let Ω_0 be a domain in $\mathbb{R}^n$ of class $B^{4-1/q}_{qq}$. We assume that there is a smooth hypersurface Σ and a function $\rho_0 \in B^{4-1/q}_{qq}(\Sigma) \cap \mathrm{Ad}$ such that $\partial\Omega_0 = \Gamma_0 = \Gamma_{\rho_0}$.
At the moment, this is only an assumption. In Section 4.2 we will discuss in detail how restrictive it is. Also the notions from differential geometry which we use here are explained there in more detail.
Finally, the pressure p at time $t = 0$ is uniquely determined by the solution of the elliptic problem

$$\begin{cases} -\triangle p = f(v_0) & \text{in} \quad \Omega_0 \\ p = H_{\Gamma_0} & \text{on} \quad \Gamma_0. \end{cases}$$

Let us now consider time dependent functions $\rho : J \subset \mathbb{R}^{\geq 0} \to \mathrm{Ad}$ and define

$$\Omega_{\rho,J} := \bigcup_{t\in J}(\{t\} \times \Omega_{\rho(t)}).$$

We give a first notion of a solution of system (2.1). Let $W^2_{q,0}(\Omega_0) := \{u \in W^2_q(\Omega_0);\ u|_{\Gamma_0} = 0\}$.

Definition 3.11 *Let $(v_0 - \psi) \in W^2_{q,0}(\Omega_0)$. A triple $(v_\varepsilon, p_\varepsilon, \Gamma_\varepsilon)$ is called a classical solution of system (2.1), if there is an interval $J_\varepsilon := [0, T_\varepsilon)$ and a function $\rho_\varepsilon \in C^1(J_\varepsilon, C(\Sigma)) \cap C(J_\varepsilon, Ad)$, such that*

- $\Gamma_\varepsilon(t) = \Gamma_{\rho_\varepsilon(t)}$, $t \in J_\varepsilon$,
- $v_\varepsilon \in C^\infty(\Omega_{\rho_\varepsilon, j_\varepsilon})$,
- $v_\varepsilon(t,\cdot), p_\varepsilon(t,\cdot) \in W^2_q(\Omega_{\rho_\varepsilon(t)})$, $t \in J_\varepsilon$ *and*
- $(v_\varepsilon, p_\varepsilon, \Gamma_\varepsilon)$ *satisfy the equations in (2.1) pointwise on* $\bigcup_{t\in J_\varepsilon}(\{t\} \times \overline{\Omega}_{\rho_\varepsilon(t)})$.

Given $\sigma \in \mathrm{Ad}$, let θ^*_σ, θ^σ_* denote the pull-back and push-forward operators induced by θ_σ, i.e. $\theta^*_\sigma f = f \circ \theta_\sigma$, $\theta^\sigma_* g = g \circ \theta^{-1}_\sigma$. If suitable functions b, ρ are time dependent, i.e. $b = b(t,x)$, $\rho = \rho(t,x)$, we define $[\theta^*_\rho b](t,x) := [\theta^*_{\rho(t)} b(t,\cdot)](x)$, analogue for θ^ρ_*. It is already known, that, using the Hanzawa diffeomorphism, system (2.1) can be transformed to the system

$$(3.12)\qquad \left\{ \begin{array}{rcll} A(\rho)r & = & f(w) & \text{in} \quad J \times D \\ \varepsilon\partial_t w + A(\rho)w & = & \varepsilon R(w,r,\rho) - h(w) & \text{in} \quad J \times D \\ \partial_t\rho + B(\rho)r & = & 0 & \text{on} \quad J \times \Sigma \\ r & = & H(\rho) & \text{on} \quad J \times \Sigma \\ w & = & \chi(\rho) & \text{on} \quad J \times \Sigma \\ w(0,\cdot) & = & w_0 & \text{in} \quad D \\ \rho(0,\cdot) & = & \rho_0 & \text{on} \quad \Sigma, \end{array} \right.$$

involving the transformed quantities $w_0 := \theta^*_{\rho_0} v_0$, $w := \theta^*_\rho v$, $r := \theta^*_\rho p$. Here: $A(\rho)u := -\theta^*_\rho(\Delta(\theta^\rho_* u))$, $B(\rho)u := -\theta^*_\rho(\nabla(\theta^\rho_* u)) \cdot \nabla\varphi_\rho)$, $H(\rho) := \theta^*_\rho H_{[\Gamma_\rho]}$ and $\chi(\rho) := \theta^*_\rho \psi$, c.f. [Es2000]. We mention that $A(\rho)$ is just the Laplace-Beltrami operator with respect to the Riemannian metric induced by θ_ρ. The term R arises from the transformation of the time derivative v_t and is determined by

$$R(w,r,\sigma)(y) = r_0(B(\sigma)r, B_\mu(\sigma)w)(y), \qquad y \in D$$

for $r, w \in C^1(\bar{D})$, $\sigma \in \mathrm{Ad}$ and

$$r_0(h,k)(y) := \begin{cases} \varphi(\Lambda(y)) \cdot h(P(y)) \cdot k(y), & \text{if } \ y \in N_D \\ 0, & \text{if } \ y \in D \setminus N_D, \end{cases} \tag{3.13}$$

$$B_\mu(\sigma)v(y) = \theta_\sigma^* \, \nabla(\theta_*^\sigma v)(y) \cdot (\mu \circ P)(y), \qquad y \in N.$$

Further, letting S and T be the solution operators of the elliptic problems

$$\begin{cases} A(\rho)r &= f \quad \text{in} \quad D \\ r &= 0 \quad \text{on} \quad \Sigma, \end{cases} \tag{3.14}$$

$$\begin{cases} A(\rho)r &= 0 \quad \text{in} \quad D \\ r &= h \quad \text{on} \quad \Sigma, \end{cases} \tag{3.15}$$

the function $r(w,\rho) := S(\rho)f(w) + T(\rho)H(\rho)$ is the unique solution of the elliptic problem

$$\begin{cases} A(\rho)r &= f(w) \quad \text{in} \quad D \\ r &= H(\rho) \quad \text{on} \quad \Sigma. \end{cases} \tag{3.16}$$

The precise regularity properties of the operators S and T can be found in Lemma 3.1, [Es2000] and in (2.9) and Lemma 2.3 in [EsSi97a]. Using (3.16) we get

$$\begin{cases} \varepsilon\partial_t w + A(\rho)w &= \varepsilon R(w, r(w,\rho), \rho) - h(w) \quad \text{in} \quad \dot{J} \times D \\ w &= \chi(\rho) \quad \text{on} \quad J \times \Sigma \\ \partial_t \rho + B(\rho)T(\rho)H(\rho) &= B(\rho)S(\rho)f(w) \quad \text{on} \quad \dot{J} \times \Sigma \\ w(0,\cdot) &= w_0 \quad \text{in} \quad D \\ \rho(0,\cdot) &= \rho_0 \quad \text{in} \quad \Sigma. \end{cases} \tag{3.17}$$

Letting $u := w - \chi(\rho)$ we have

$$
(3.18) \qquad \begin{cases} \varepsilon\partial_t u + A(\rho)u &= \varepsilon R(u_\chi, r(u_\chi,\rho),\rho) - h(u_\chi) \\ & \quad -A(\rho)\chi(\rho) - \varepsilon Q(u,\rho) \qquad \text{in} \quad \dot{J}\times D \\ u &= 0 \qquad \text{on} \quad J\times\Sigma \\ \partial_t\rho + B(\rho)T(\rho)H(\rho) &= B(\rho)S(\rho)f(u_\chi) \qquad \text{on} \quad \dot{J}\times\Sigma \\ u(0,\cdot) &= w_0 - \chi(\rho_0) \qquad \text{in} \quad D \\ \rho(0,\cdot) &= \rho_0 \qquad \text{in} \quad \Sigma, \end{cases}
$$

where we used the notation $u_\chi := u_\chi(\rho) := u + \chi(\rho)$. Here Q results from the differentiation of the transformed boundary-data $\chi(\rho)$ with respect to time and is given by

$$
Q(w,\sigma)(y) := [\varphi\circ\Lambda]\cdot[B(\sigma)r(w_{\chi(\sigma)},\sigma)\circ P]\cdot[(\theta^*_\sigma\nabla\psi)\cdot(\mu\circ P)](y), \qquad y\in N_D,
$$

$Q(v,\sigma)(y) = 0$, $y \in D\setminus N_D$. Considering (3.18) as the system

$$
(3.19) \qquad \begin{cases} \varepsilon\partial_t u + A(\rho)u &= \varepsilon R(u_\chi, r(u_\chi,\rho),\rho) - h(u_\chi) \\ & \quad -A(\rho)\chi(\rho) - \varepsilon Q(u,\rho) \qquad \text{in} \quad \dot{J}\times D \\ \partial_t\rho + B(\rho)T(\rho)H(\rho) &= B(\rho)S(\rho)f(u_\chi) \qquad \text{on} \quad \dot{J}\times\Sigma \\ u(0,\cdot) &= w_0 - \chi(\rho_0) \qquad \text{in} \quad D \\ \rho(0,\cdot) &= \rho_0 \qquad \text{in} \quad \Sigma \end{cases}
$$

over a suitable function space with zero-boundary-condition, we arrive at

$$
(3.20) \qquad \begin{cases} \varepsilon\partial_t u + A(\rho)u &= F(u,\rho) \qquad \text{in} \quad \dot{J}\times D \\ \partial_t\rho + B(\rho)T(\rho)P(\rho)\rho &= G(u,\rho) \qquad \text{on} \quad \dot{J}\times\Sigma \\ u(0,\cdot) &= w_0 - \chi(\rho_0) \qquad \text{in} \quad D \\ \rho(0,\cdot) &= \rho_0 \qquad \text{in} \quad \Sigma, \end{cases}
$$

if we make use of the decomposition $H(\rho) = P(\rho)\rho + K(\rho)$ which has been introduced in [EsSi97a], Lemma 3.1, and set

$$
\begin{aligned} F(u,\rho) &:= \varepsilon R(u_\chi, r(u_\chi,\rho),\rho) - h(u_\chi) - A(\rho)\chi(\rho) - \varepsilon Q(u,\rho) \\ G(u,\rho) &:= -B(\rho)[T(\rho)K(\rho) - S(\rho)f(u_\chi)]. \end{aligned}
$$

We define

Definition 3.12 *Let $\rho_0 \in B_{qq}^{4-\frac{1}{q}}(\Sigma) \cap Ad$ und $w_0 \in W_q^2(D)$. A pair $(u_\varepsilon, \rho_\varepsilon)$ is called a classical solution of system (3.20), if there is a nontrivial interval J_ε such that*

- $\rho_\varepsilon \in C(J_\varepsilon, B_{qq}^{4-\frac{1}{q}}(\Sigma) \cap Ad) \cap C^1(J_\varepsilon, B_{qq}^{1-\frac{1}{q}}(\Sigma)) \cap C^\infty(\dot{J}_\varepsilon, B_{qq}^{4+k-\frac{1}{q}}(\Sigma))$ *for any* $k \in \mathbb{N}$
- $u_\varepsilon \in C(J_\varepsilon, H_q^2(D)) \cap C^1(J_\varepsilon, L_q(D)) \cap C^\infty(\dot{J}_\varepsilon, H_q^{2+k}(D))$ *for any* $k \in \mathbb{N}$
- $u_\varepsilon(t)|_\Sigma = 0$ *for* $t \in J_\varepsilon$ *und*
- $(u_\varepsilon, \rho_\varepsilon)$ *satisfy the equations of (3.20) pointwise on* J_ε.

It is obvious how to obtain a classical solution of (2.1) from a classical solution of (3.20). In order to apply Corollary 3.9, we make the additional assumptions

- $v_0 \geq 0, \psi \equiv 1$
- $h(0) = 0$, $h' > 0$.

Notice that these assumptions are biologically reasonable: the nutrient concentration u cannot be negative. Moreover, if there is no nutrient, there cannot be consumption, and the more nutrient is present, the more will actually be consumed. Moreover, it is convenient to assume the nutrient concentration outside the tumor tissue to be constant. The assumptions can be dropped at the end of this Section.
Let us proceed by choosing appropriate numbers $0 < \gamma < \beta < \alpha < 1$, $s > 0$ (cf. [ES2000], section 4) and define

- $E_1 := H_{q,0}^{2+2s}(D)$, $E_0 := H_q^{2s}(D)$
- $F_1 := B_{qq}^{4+3s-1/q}(\Sigma)$, $F_0 := B_{qq}^{1+3s-1/q}(\Sigma)$
- $Y_\beta := F_\beta \cap \mathrm{Ad}$.

Here E_θ, F_θ are, say, complex interpolation spaces of exponent $\theta \in (0,1)$. The setting is made to obtain $E_\alpha \times F_\alpha = H_{q,0}^2(D) \times B_{qq}^{4-1/q}(\Sigma)$, see [Es2000], Lemma 4.1 and the considerations after the proof of Lemma 4.2 for more information. Moreover, Theorem 5.5.3, 1 and 2 in [RuSi] imply

$$h \in C^\infty(E_\beta, E_\gamma),$$

where h denotes now the Nemytskij Operator induced by the function h.
Using the Lemmata 4.4 -4.6 in [Es2000], we find, that the mappings A, BTP, F, G satisfy the assumptions from the beginning of the last Section:

$$(A, BTP) \in C^\infty(Y_\beta, \mathcal{H}(E_1, E_0) \times \mathcal{H}(F_1, F_0)), \qquad (F, G) \in C^\infty(Y_\beta \times E_\beta, E_\gamma \times F_\gamma).$$

Further, letting

- $Z := BUC^1(D)$, $V := L_q(D)$, $\tilde{V} := BUC(D)$,

one easily verifies that

- $E_\beta \overset{d}{\hookrightarrow} Z \overset{d}{\hookrightarrow} \tilde{V} \overset{d}{\hookrightarrow} V$
- $A \in C^\infty(Y_\beta, \mathcal{H}(E_\alpha, V))$
- $[E_\alpha, V]_\mu \hookrightarrow E_{\beta'}$ for $\beta' > \beta$, the complex interpolation functor $[\cdot,\cdot]_\mu$ and μ close enough to 1.

Let $\rho \in W^\nu_{a(U),T}$ be given, and let $u_\varepsilon := u_\varepsilon(\rho) \in C([0, t^+_\varepsilon), E_\alpha) \cap C^1([0, t^+_\varepsilon), V)$ be the unique solution of the first equation in (3.20) with respect to that function ρ. Letting

- $J := [0, S]$, $0 < S < t^+_\varepsilon(\rho)$, and
- $\mathcal{P}[J \times D]$ be the parabolic boundary of the set $J \times D$,

we get

Lemma 3.13

$$\sup_{0<t<t^+_\varepsilon} |u_\varepsilon(t)|_{C(\bar{D})} \leq \max\{\max v_0, 1\} =: M.$$

PROOF: Let $u := u_\varepsilon$, $w := u + 1$, for simplicity. Then w is a solution of the first two equations in (3.17). Considering w as a function on $J \times D$ and writing again w instead of $w|_{J\times D}$, we find $w \geq 0$:
It suffices to show, that w cannot be negative in all points $(t_0, x_0) \in J \times D \setminus \mathcal{P}[J \times D]$. In deed, we have $F_\beta \hookrightarrow BUC^3(\Sigma)$. Since $E_\alpha = H^2_{q,0}(D) \hookrightarrow BUC^{1+\tau}(D)$, the function $x \mapsto R(u(t), r(u(t), \rho(t)), \rho(t))(x) - h(u(t))(x)$ belongs to the space $BUC^\tau(D)$ (cf. the end of this Section). The regularity of the coefficients of $A(\rho)$ and parabolic theory imply, that w possesses a continuous second x - differential in all points $(t, x) \in (0, S] \times D$. If now $\min w = w(t_0, x_0)$ for some $(t_0, x_0) \in J \times D \setminus \mathcal{P}[J \times D]$, it follows that

$$w_t(t_0, x_0) \leq 0, \qquad R(w(t_0), r(w(t_0), \rho(t_0)), \rho(t_0))(x_0) = 0.$$

The assumption $w(t_0, x_0) \le 0$ then implies that $-h(w(t_0, x_0)) \ge 0$, i.e. $A(\rho(t_0))w(t_0, x_0) \ge 0$. But this contradicts the ellipticity of $A(\rho(t_0))$, since it is an operator without zero order terms, i.e.

$$A(\rho(t_0)) = \sum_{i,j=1}^{n} a_{ij}(t_0, x)\partial_i\partial_j + a_j(t_0, x)\partial_j.$$

Analogously we see, that w must achieve its maximal value on $\mathcal{P}[J \times D]$. Since $0 < S < t_\varepsilon^+$ was arbitrary, we find all in all:

$$0 \le w \le \max\{\max w_0, 1\} = \max\{\max v_0, 1\},$$

giving the assertion.

■

In order to verify the remaining assumptions of Corollary 3.9 we need to have a closer look onto the nonlinearities R and Q in (3.20). First observe, that $Q \equiv 0$, since ψ is constant. Remember the definition of the mapping

$$r_0(h, k)(y) := \begin{cases} \varphi(\Lambda(y)) \cdot h(P(y)) \cdot k(y), & \text{if} \quad y \in N_D \\ 0, & \text{if} \quad y \in D \setminus N_D. \end{cases}$$

It is clear that this mapping, considered as a subset of $(C(\Sigma) \times C(\bar{N}_D)) \times C(\bar{D})$, is bilinear and bounded. Moreover, remember

$$R(u, r, \rho)(y) = r_0(B(\rho)r, B_\mu(\rho)u)(y).$$

It is not difficult to verify, that the action of the operators $B(\sigma)$ and $B_\mu(\sigma)$ for $\sigma \in Y_\beta$ is determined by

$$B(\sigma)v(s) = \vec{b}_\sigma(s) \cdot \nabla v(s), \qquad s \in \Sigma$$

and

$$B_\mu(\sigma)v(y) = (B_\sigma(y)\nabla v(y)) \cdot \mu(P(y)), \qquad y \in N_D,$$

respectively, where,

$$\sigma \mapsto (\vec{b}_\sigma, B_\sigma) \in C^\infty(Y_\beta, C^\tau(\Sigma, \mathbb{R}^n) \times C^\tau(\bar{D}, \mathcal{L}(\mathbb{R}^n))).$$

Thus, we find a constant $C = C(\sigma)$ with the property

$$\begin{array}{rcl} \|B(\sigma)v\|_{C(\Sigma)} & \leq & C \cdot \|\nabla v\|_{C(\Sigma)}, \\ \|B_\mu(\sigma)v\|_{C(\bar{N}_D)} & \leq & C \cdot \|\nabla v\|_{C(\bar{D})}, \end{array}$$

$v \in C^1(\bar{D})$, locally uniformly with respect to $\sigma \in Y_\beta$, by continuity. Standard elliptic theory implies, that

$$\|r(v_{\chi\equiv 1}, \sigma)\|_{C^1(\bar{D})} \leq C(\sigma, \|f(v_{\chi\equiv 1})\|_\infty),$$

locally uniformly with respect to $\sigma \in Y_\beta$, (cf. Theorem 8.33 in [GilTru]). Therefore

$$\|R(v_{\chi\equiv 1}, r(v_{\chi\equiv 1}, \sigma), \sigma)\|_{C(\bar{D})} \leq C(\sigma, M) \cdot (\|v\|_{C^1(\bar{D})} + 1),$$

locally uniformly with respect to $\sigma \in Y_\beta$. Finally, we observe $A(\sigma)\chi(\sigma) = A(\sigma)1 = 0$, and we find the mapping

$$J(\cdot, \sigma) := v \mapsto h(v) : C(\bar{D}) \to C(\bar{D}),$$

$\sigma \in Y_\beta$, obviously to be bounded on bounded sets, locally uniformly with respect to $\sigma \in Y_\beta$. Summarizing, we have shown:

Theorem 3.14 *Let $\rho_0 \in B_{qq}^{4-\frac{1}{q}}(\Sigma) \cap Ad$ and $w_0 \in W_q^2(D)$. Then, given $\varepsilon > 0$ there is a unique classical solution of system (3.20) on some nontrivial interval $[0, t^*]$ independent of $\varepsilon > 0$.*

Therefore:

Theorem 3.15 *Let Ω_0 be of class $B_{qq}^{4-1/q}$ and $v_0 - 1 \in W_{q,0}^2(\Omega_0)$. Then, given $\varepsilon > 0$, system (2.1) possesses a unique classical solution $(v_\varepsilon, p_\varepsilon, \Gamma_\varepsilon)$ on some nontrivial interval $[0, t^*]$ independent of $\varepsilon > 0$. Moreover, $v_\varepsilon \geq 0$.*

Finally, we want to prove convergence of the solutions. Assume $(u_\varepsilon, \rho_\varepsilon)$ to solve (3.20). It suffices to derive an $L_\infty([a, T] \times D)$-bound for $(u_\varepsilon)_t$, if $[a, T] \subset (0, t^*]$ is given.
First of all, due to the abstract theory (c.f. Theorem 3.7, Corollary 3.8, 3.9), we know that the set

$$\bigcup_{\varepsilon_0 > \varepsilon > 0} \rho_\varepsilon[[0, t^*]] \times \dot{\rho}_\varepsilon[[0, t^*]] \times u_\varepsilon[[0, t^*]] \times \varepsilon \dot{u}_\varepsilon[[0, t^*]] \times \varepsilon \dot{u}_\varepsilon[[a, t^*]] \tag{3.21}$$

is a bounded subset of $c^{3+\tau}(\Sigma) \times c^{\tau}(\Sigma) \times H^2_{q,0}(D) \times L_q(D) \times BUC(D)$ for some small positive τ. We can bootstrap to find uniform bounds in an arbitrary strong topology, provided, we are away from 0. Letting $w_\varepsilon := u_\varepsilon + 1$ and $v_\varepsilon := w_\varepsilon \circ \theta^{-1}_{\rho_\varepsilon}$ (i.e. v_ε solves the original model (2.1)), we have

$$z_\varepsilon(t,x) := (v_\varepsilon)_t(t,x) = (w_\varepsilon)_t(t, \theta^{-1}_{\rho_\varepsilon(t)}(x)) + R(w_\varepsilon, r_\varepsilon(w_\varepsilon, \rho_\varepsilon), \rho_\varepsilon)(t, \theta^{-1}_{\rho_\varepsilon(t)}(x)), \tag{3.22}$$

(cf. the proof of Lemma 2.1 in [Es2000]), where $r_\varepsilon = r(w_\varepsilon, \rho_\varepsilon)$ is defined by (3.16). Since z_ε obviously solves

$$\varepsilon \dot{z}_\varepsilon - \Delta z_\varepsilon + h'(v_\varepsilon) \cdot z_\varepsilon = 0 \qquad \text{in } \ \Omega_{\rho_\varepsilon,(0,t^*)},$$

since $h' \geq \alpha_0 > 0$, the parabolic maximum principle (c.f Theorem 2.10, 2.11 in [Lieb96]) implies

$$\sup_{\Omega_{\rho_\varepsilon,[\delta,T]}} |z_\varepsilon| \leq \sup_{\mathcal{P}[\Omega_{\rho_\varepsilon,[\delta,T]}]} |z_\varepsilon|,$$

$\delta \geq 0$. But $w_\varepsilon|_{(0,t^*]\times\Sigma} = 1$, so

$$\sup |(w_\varepsilon)_t|\,|_{(0,t^*]\times\Sigma} = 0.$$

.

Thus, it suffices to estimate $\dot{u}_\varepsilon$ near $t = a$, where $a > 0$ is as small as we want it to be. Given $b > 0$, we may assume $\|\rho_0\|_{C^2(\Sigma)} < b$ (c.f. Theorem 4.6). Since $h' \geq \alpha_0 > 0$ and $A(0) = -\triangle$, we therefore have

$$-\alpha_0/2 \in \bigcap_{\varepsilon_0 > \varepsilon > 0,\ t \in [J]} \rho(-[A(\rho_\varepsilon(t)) + h'(u_\varepsilon(t))]),$$

where $J := [0, t_0] \supset [0, a]$ is sufficiently small. Here $A(\rho(t))$ is considered as a closed operator in $L_q(D)$. Moreover, thanks to (3.21), we can find suitable corresponding resolvent estimates. Setting $\tilde{R}(u, \rho) := R(u_{\chi\equiv 1}, r(u_{\chi\equiv 1}, \rho), \rho)$, $\dot{u}_\varepsilon$ solves

$$\begin{aligned} \varepsilon(\dot{u}_\varepsilon)_t + [A(\rho_\varepsilon) + h'(u_\varepsilon + 1)]\dot{u}_\varepsilon \quad &= \quad \varepsilon D\tilde{R}(u_\varepsilon, \rho_\varepsilon)(\dot{u}_\varepsilon, \dot{\rho}_\varepsilon) \\ &\quad\; - DA(\rho_\varepsilon)(\dot{\rho}_\varepsilon)u_\varepsilon \end{aligned}$$

and the abstract theory (analogously to the proofs of corollary 3.8, 3.9) yields that at least

$$\|\dot{u}_\varepsilon(a+h)\|_{BUC(D)} \leq C_1 \cdot \|\dot{u}_\varepsilon(a)\|_{L_q(D)} \cdot e^{-(c(\alpha_0,b)\cdot h)/\varepsilon} + C_2 \leq C_3/\varepsilon \cdot e^{-(c\cdot h)/\varepsilon} + C_2,$$

$a > 0$, $t_0 \geq h > 0$.

Thus, we have shown:

Theorem 3.16 *Given $\varepsilon > 0$, let $(u_\varepsilon, \rho_\varepsilon)(u_0, \rho_0)$ be the unique classical solution of (3.20) from Theorem 3.14. Then, given $\delta \in (0, t^*]$, $\{u_\varepsilon\} \in C^{1-}_{ue}([\delta, t^*], \tilde{V}) \subset C_{ue}([\delta, t^*], V)$.*

Notice that our abstract theory only guarantees $H^2_q \times B^{4-1/q}_{qq}$ - convergence of the $(u_\varepsilon, \rho_\varepsilon)$. But of course, thanks to our considerations behind (3.21), we can bootstrap to find all in all:

Theorem 3.17 *Given $\varepsilon > 0$, let $(v_\varepsilon, p_\varepsilon, \Gamma_\varepsilon)$ be the unique classical solution of (2.1) from Theorem 3.15. Let $J := [0, t^*]$. There exists $\rho \in C(J, B^{4-1/q}_{qq}(\Sigma)) \cap C^1(\dot{J}, C^\infty(\Sigma))$ such that (v, p, Γ_ρ) solve (2.1) with $\varepsilon = 0$, where*

- *$(v(t), p(t)) \in BUC^\infty(\Omega_{\rho(t)})$, $t > 0$;*
- *$v(0)$ and $p(0)$ are determined by the unique solution of the elliptic problem*

$$\begin{cases} -\triangle(p, v) &= (f(v), -h(v)) \quad \text{in} \quad \Omega_0 \\ \quad (p, v) &= (H_{\Gamma_0}, 1) \quad\quad\;\; \text{on} \quad \Gamma_0. \end{cases}$$

- *Letting $(w_\varepsilon, r_\varepsilon)$, as at the beginning of this section, we have*

  $$(w_\varepsilon, r_\varepsilon, \rho_\varepsilon)(t) \to (v \circ \theta_\rho, p \circ \theta_\rho, \rho)(t) \qquad \text{in} \qquad BUC^\infty(D) \times BUC^\infty(D) \times C^\infty(\Sigma),$$

  uniformly on compact subsets of $(0, t^*]$.

4 The blow-up mechanism of moving boundary problems

4.1 The main result

Let us first supplement our general assumptions from the beginning of Section 3.3 by

- $\beta_0 > 1 - 1/q$.

We refine the notion of a classical local solution of system (2.1):

Definition 4.1 *Let Ω_0 be a domain in $\mathbb{R}^n$ of class $c^{3+\beta_0}$ and $u_0 - \psi \in W^2_{q,0}(\Omega_0)$. A classical solution of problem (2.1) is a triple $(v(t,x), p(t,x), \Gamma(t))$ defined on a nontrivial interval $J := [0, S)$ such that*

i) $\bigcup_{t\in j}(\{t\} \times \Gamma(t))$ *is a smooth submanifold of* $\mathbb{R}^{n+1}$

ii) $\{\Gamma(t),\ t \in J\}$ *is of class* $(mb)^{(3,\beta_0)}$

iii) v *is smooth on* $\bigcup_{t\in j}(\{t\} \times \Omega(t))$

iv) $v(t,\cdot), p(t,\cdot) \in W^2_q(\Omega(t))$ *for* $t \in J$ *and*

v) (v, p, Γ) *satisfy the equations (3.1) pointwise on* $\bigcup_{t\in J}(\{t\} \times \overline{\Omega}(t))$.

The class $(mb)^{(k,\alpha)}$ is a convenient tool to measure the spatial regularity of a family of hypersurfaces in $\mathbb{R}^{n+1}$. The precise definition of this class is provided in Section 4.2. Clearly, a classical solution is maximally continued, if there is no proper extension of it. In this case, t^+ denotes the maximal time of existence.

Theorem 4.2 *Let Ω_0 be a domain in $\mathbb{R}^n$ of class $c^{3+\beta_0}$ and $v_0 - \psi \in W^2_{q,0}(\Omega_0)$. Then there exists a unique maximal continued classical solution of problem (2.1). If $t^+ < \infty$, then either*

- $\|v(t)\|_{BUC^1(\Omega(t))} + \|p(t)\|_{BUC^1(\Omega(t))} \longrightarrow \infty$ *as* $t \to t^+$

  or

- $A(\Gamma(t)) \longrightarrow 0$ *as* $t \to t^+$.

The quantity $A(\Gamma)$ measures the maximal possible size of spheres which intersect with Γ only at one point, see Section 4.2. Therefore, the second condition contains the situation where different regions of the tumor grow together, mathematically speaking the occurrence of self-intersection. This has been the case in numerical experiments, see [CrLoNi]. The condition can also describe shrinking to a point or the development of singularities in $\Gamma(t)$:

Corollary 4.3 *Let $n = 2$ and $c > 0$. Assume $v_0 \equiv c$, $\psi \equiv c$, $h(c) = 0$ and $f(c) = -\alpha_0 < 0$. Let (v, p, Γ) be the unique classical solution of system (2.1) corresponding to some initial surface Γ_0. Then*

$$A(\Gamma(t))^{-1} + \|\nabla p\|_{C(\Gamma(t))} \to \infty$$

as $t \to t^+$.

4.2 Notations and helpful material

In the following by a $C^{k+\alpha}$ $[c^{k+\alpha}]$ -domain Ω we mean a bounded connected open subset of $\mathbb{R}^n$ such that its boundary $\Gamma := \partial\Omega$ is a compact embedded hypersurface of regularity $C^{k+\alpha}$ $[c^{k+\alpha}]$. If $U \subset \mathbb{R}^n$ is an open set, $c^{k+\alpha}(U)$ denotes the little Hölder space, that is the closure of $BUC^\infty(U)$ in the usual Hölder norm. If M is a sufficiently smooth manifold, the spaces $c^{k+\alpha}(M)$ are defined by means of a (sufficiently) smooth atlas for M.

If Ω is a domain, we define a *tubular neighborhood* of Γ to be an open set N which is the diffeomorphic image of the map

$$X : \Gamma \times (-\delta, \delta) \to \mathbb{R}^n, \qquad (x, a) \mapsto x + a \cdot \nu(x),$$

where $\nu(x)$ is the outer unit normal vector at $x \in \Gamma$ and $\delta > 0$ is sufficiently small. We decompose the inverse of X into $X^{-1} = (P_{[\Gamma]}, \Lambda_{[\Gamma]})$, where $P_{[\Gamma]}$ is the metric projection of a point x onto Γ and $\Lambda_{[\Gamma]}$ is the signed distance function with respect to Γ. Clearly, if $\mathrm{im}(X)$ is a tubular neighborhood of Γ, so is $\mathrm{im}(X|_{\Gamma\times(-a,a)})$, if $a < \delta$. The set of all tubular neighborhoods of a surface Γ is denoted by $\mathcal{TN}(\Gamma)$.

We say that $\Omega \subset \mathbb{R}^n$ satisfies an interior sphere condition (ISC) if for any $x \in \Gamma$ there is a ball $B_x \subset \bar{\Omega}$ such that $\Gamma \cap B_x = \{x\}$. It satisfies an exterior sphere condition (ESC), if the complement has ISC. For a suitable domain Ω and $x \in \Gamma$, let $R_i(x)$ $[R_e(x)]$ be the supremum over all radii of possible interior [exterior] spheres at x and define

$$A_i(\Gamma) := \inf_{x\in\Gamma} R_i(x), \qquad A_e(\Gamma) := \inf_{x\in\Gamma} R_e(x)$$

as well as

$$A(\Gamma) := \min\{A_i(\Gamma), A_e(\Gamma)\}.$$

Finally, if $N \in \mathcal{TN}(\Gamma)$, $r(N) := \mathrm{dist}(\partial N, \Gamma)$ is the *radius* of N.

Definition 4.4 *We say that a family $\{\Gamma_\alpha,\ \alpha \in \mathsf{A}\}$ of submanifolds of $\mathbb{R}^n$ has* uniformly bounded C^k-geometry, *if there exists numbers $m \in \mathbb{N}$, $K, L > 0$ such that*

- *for each $\alpha \in \mathsf{A}$ there is a C^k-atlas $(U_\alpha^l, \varphi_\alpha^l)$ for Γ_α where $1 \le l \le m$;*
- *if $\varphi_\alpha^l \in \mathit{diff}^k(V_\alpha^l, W_\alpha^l)$ (i.e. $U_\alpha^l = (\varphi_\alpha^l)^{-1}[W_\alpha^l] \cap \Gamma_\alpha$), then $\|\varphi_\alpha^l\|_{C^k(V_\alpha^l)} + \|(\varphi_\alpha^l)^{-1} - x_\alpha^l\|_{C^k(W_\alpha^l)} \le K$ for $(l, \alpha) \in \{1, ..., m\} \times \mathsf{A}$ and some $x_\alpha^l \in V_\alpha^l$;*
- *for each $\alpha \in \mathsf{A}$ there is a partition of the unity $\{\pi_\alpha^l\}$, $1 \le l \le m$, subordinated to the above covering, such that $\|\pi_\alpha^l \circ (\varphi_\alpha^l)^{-1}\|_{C^k(W_\alpha^l)} \le L$ for $(l, \alpha) \in \{1, ..., m\} \times \mathsf{A}$.*

Lemma 4.5 *Let $\mu > 0$ be given and let $\{\Omega_\alpha,\ \alpha \in \mathsf{A}\}$ be a family of C^2-domains in $\mathbb{R}^n$. Assume Γ_α to be the boundary of Ω_α and suppose*

- $\inf\limits_{\alpha \in \mathsf{A}} \min(A_i(\Gamma_\alpha), A_e(\Gamma_\alpha)) \ge \mu$.

Then for each $\alpha \in \mathsf{A}$ we find $N_\alpha \in \mathcal{TN}(\Gamma_\alpha)$ such that $\inf\limits_{\alpha \in \mathsf{A}} r(N_\alpha) \ge \mu$.

Moreover, if $N \in \mathcal{TN}(\Gamma_\alpha)$ satisfies $r(N) \le a < \mu$, the quantity

$$\sup_{\Gamma_\beta \subset N} \|\Lambda_{[\Gamma_\alpha]}\|_{C^2(\Gamma_\beta)}$$

is estimated from above in terms of the numbers a and μ. Finally, if $\sup_{\alpha \in \mathsf{A}} \mathit{area}(\Gamma_\alpha) < \infty$, then $\{\Gamma_\alpha\}$ has uniformly bounded C^2-geometry. Here $\mathit{area}(\Gamma) := \int_\Gamma 1$.

PROOF: The first statement can be easily seen by a careful reading of chapter 14.6 in [GilTru]. For the second one we notice

$$\sup_{\Gamma_\beta \subset N} \|\Lambda_{[\Gamma_\alpha]}\|_{C(\Gamma_\beta)} \le r(N) \le a.$$

Let $\kappa^1(x_\alpha), ..., \kappa^{n-1}(x_\alpha)$ be the principal curvatures of the surface Γ_α at $x_\alpha \in \Gamma_\alpha$. Inside N we have

$$\nabla(\Lambda_{[\Gamma_\alpha]})(x) = \nu(z_\alpha),$$

where $z_\alpha := P_{[\Gamma_\alpha]}(x)$, as well as

$$D^2(\Lambda_{[\Gamma_\alpha]})(x) = \mathrm{diag}[\frac{-\kappa^1(z_\alpha)}{1-\kappa^1(z_\alpha)\Lambda_{[\Gamma_\alpha]}(x)}, ..., \frac{-\kappa^{n-1}(z_\alpha)}{1-\kappa^{n-1}(z_\alpha)\Lambda_{[\Gamma_\alpha]}(x)}, 0]$$

with respect to a principal coordinate system centered at z_α, since $|\kappa^i| \le 1/\mu$. Thus, to prove the second statement, it suffices to construct a suitable C^2-atlas for each of the surfaces Γ_α. We fix $\alpha \in \mathsf{A}$. Because translation (since $\sup_{\alpha\in\mathsf{A}} \mathrm{dist}(\Gamma_\alpha, 0) < \infty$) and rotation will not influence our estimates we may work nearby $0 \in \mathbb{R}^n$, which we assume to be an element of Γ_α.
Let Γ_α be locally the graph of some C^2-function g satisfying $g(0) = 0$, $\nabla g(0) = 0$. For $x \in \Gamma_\alpha$ fix a principal coordinate system of unit length $\{v_1(x), ..., v_{n-1}(x)\}$ centered at x. Then

$$\frac{D^2 g(x)}{\sqrt{1+|\nabla g(x)|^2}} \cdot v_i(x) = \kappa^i(x)(I + \nabla g(x)(\nabla g(x))^t) \cdot v_i(x), \tag{4.1}$$

provided, the evaluations of g and its partial derivatives at x are well defined. Keep in mind that $|\kappa^i| \le 1/\mu$. Let $\mathcal{V}$ be the largest open set where g can be defined. It is not difficult to see, that $|\nabla g(x)| \to \infty$ as $x \to \partial\mathcal{V}$. Let

$$W := \bigcup\{U \in \mathcal{U}(0) \cap \mathcal{V};\ \|\nabla g\|_{C(U)} \le 1\} \cap \mathbb{B}(0,1).$$

Here, $\mathcal{U}(0)$ denotes the neighbourhood filter of $0 \in \mathbb{R}^{n-1}$. Let $y \in \partial W$ be given. Then, then, if $|y| < 1$, it follows from (4.1) and the mean-value theorem, that $|y| \ge \mu/(2\sqrt{2})$. Moreover, $\|g\|_{C^2(W)} \le \max\{1, (2\sqrt{2})/\mu\}$. Since this estimate is independent of $\alpha \in \mathsf{A}$, we get the second assertion by constructing a suitable family π^l_α of partitions of the unity, subordinated to the sets $g[W]$. This is possible because of the lower bound for the size of W. To prove the last statement, we fix an open real interval J containing 0 and define local coordinates by

$$\varphi^{-1} : W \times J \to \mathbb{R}^n, \qquad (x, y) \mapsto (x, y + g(x)).$$

Clearly, $\varphi(x, y) = (x, y - g(x))$, $(x, y) \in \varphi^{-1}[W \times J] \subset \mathbb{R}^n$. Moreover, if $\sup_{\alpha\in\mathsf{A}} \mathrm{area}(\Gamma_\alpha) < \infty$, we can cover each surface Γ_α by, say, $m \in \mathbb{N}$ charts constructed as above, because, if not, there would be a sequence $(\alpha_n) \subset \mathsf{A}$ such that

$$\int_{\Gamma_{\alpha_n}} 1 = \sum_{l=1}^{a_{\alpha_n}(n)} \int_{\mathrm{supp}(\pi^l_{\alpha_n})} \pi^l_{\alpha_n} \ge a_{\alpha_n}(n) \cdot C,$$

where $a_{\alpha_n}(n) \geq n$ and $C = C(\min\{1, \mu/(2\sqrt{2})\})$. This completes the proof.

■

Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Given any manifold M of class C^k, let $\mathbb{B}^k_\varepsilon(M)$ be the open ball in $C^k(M)$ with radius $\varepsilon > 0$. The next theorem ensures that any hypersurface M of class $c^{k+\alpha}$ with $k \geq 2$ can always be represented as a graph of a $c^{k+\alpha}$-function in normal direction of an analytic surface. The proof is based on a level set approximation of M and the implicit function theorem. Because the idea of the proof is due to Matthias Bergner and not to the author of this thesis, a precise proof is omitted here. It will be given in an article containing the natural generalization of the techniques which we use here, see [SEM] for its probable title.

Theorem 4.6 *Let Ω be a $C^{k+\alpha}$ [$c^{k+\alpha}$] -domain in $\mathbb{R}^n$, $k \geq 2$, and let $\varepsilon > 0$ be given. Then there exists*

- *a domain D whose boundary Σ is an analytic embedded hypersurface*
- *a tubular neighborhood N of Σ containing $\partial\Omega$*
- *a function $\rho \in C^{k+\alpha}(\Sigma)$ [$\rho \in c^{k+\alpha}(\Sigma)$]*

such that the map

$$\theta_\rho : \Sigma \to \partial\Omega, \qquad x \mapsto x + \rho(x) \cdot \nu(x)$$

is a $C^{k+\alpha}$ [$c^{k+\alpha}$] -diffeomorphism. After having fixed a norm on $C^k(\Sigma)$ one can choose $\rho \in \mathbb{B}^1_\varepsilon(\Sigma)$.

In view of Theorem 3.16 it should be remarked, that it is in fact possible to assume $\rho \in \mathbb{B}^2_\varepsilon(\Sigma)$, provided, $k \geq 3$. However, we are now ready to introduce the classes $(MB)^{(k,\alpha)}$:

Definition 4.7 *Let J be a real interval. The family $\{\Gamma(t),\ t \in J\}$ is of class $(MB)^{(k,\alpha)}$, if, given $t_0 \in J \setminus \sup J$, there is a smooth manifold $\Sigma = \Sigma(t_0)$, a positive number $\delta = \delta(t_0)$, and a function*

$$\rho \in C([t_0, t_0 + \delta), C^{k+\alpha}(\Sigma)) \cap C^1([t_0, t_0 + \delta), C(\Sigma))$$

such that $\Gamma(t_0 + h) = \theta_{\rho(h)}[\Sigma]$ for $h \in [0, \delta)$. It is of class $(mb)^{(k,\alpha)}$ if this holds true for the little Hölder spaces $c^{k+\alpha}$.

The abbreviation 'MB' should remind on 'moving boundary'.

4.3 Localizations in time

Let the data of the problem be given and fix a smooth manifold Σ, a tubular neighborhood N of Σ containing Γ_0, and a function $\rho_0 \in c^{3+\beta_0}(\Sigma)$ such that θ_{ρ_0} is a $c^{3+\beta_0}$-diffeomorphism from Σ onto Γ_0. If $a := r(N)$, by Theorem 4.6 we may assume that $\|\rho_0\|_{C^1(\Sigma)} < a/5$. Notice that, thanks to Theorem 4.6, the existence of ρ_0 is no longer an assumption (as in section 3.3) but a fact. Unfortunately, it seems to be harder to prove Theorem 4.6 in the regularity scale of Besov spaces. Therefore, we decide to choose our initial domain to be a bit more regular than needed in order to apply the abstract theory.
We remember from section 3.3 that inside N the diffeomorphisms $\theta_{\rho(\cdot)}$ transform problem (2.1) into (3.20). We shall define now a slightly modified setting:

We choose $b := \max\{2\|\rho_0\|_{C^{1+\beta_0}(\Sigma)}, a/5\}$ and let

$$\mathcal{U} := H^2_{q,0}(D) \times \{\rho \in B^{4-1/q}_{qq}(\Sigma);\ \|\rho\|_{c^{1+\beta_0}(\Sigma)} < b;\ \|\rho\|_{c^1(\Sigma)} < a/5\},$$

abbreviated by $\mathcal{U} := H^2_{q,0}(D) \times \mathcal{A}$. Then $\mathcal{U}$ is an open subset of $\mathbb{D} := H^2_{q,0}(D) \times B^{4-1/q}_{qq}(\Sigma)$ and it contains (u_0, ρ_0). Now, as in section 3.3, one shows the existence of a unique solution of (3.20), that means a pair of functions (u, ρ) defined on a nontrivial interval $J := [0, T)$ such that

- $\rho \in C(J, B^{4-\frac{1}{q}}_{qq}(\Sigma) \cap \mathcal{A}) \cap C^1(J, B^{1-\frac{1}{q}}_{qq}(\Sigma)) \cap C^\infty(\dot{J}, B^{4+k-\frac{1}{q}}_{qq}(\Sigma))$ for all $k \in \mathbb{N}$
- $u \in C(J, H^2_q(D)) \cap C^1(J, L_q(D)) \cap C^\infty(\dot{J}, H^{2+k}_q(D))$ for all $k \in \mathbb{N}$
- $u(t)|_\Sigma = 0$ for $t \in J$ and
- (u, ρ) satisfy the equations in (3.20) pointwise in J.

Moreover, modifying Lemma 4.6 in [Es2000] in an obvious way and using the generation properties of the operator $B(\cdot)T(\cdot)P(\cdot)$ in the Hölder space setting which are stated for example in [EsSi97a], one gets the maximal regularity of the distance function ρ at 0 by standard arguments based on interpolation and the variation-of-constants formula: $\rho \in C(J, c^{3+\beta_0}(\Sigma))$. Further, following the abstract theory, T can be chosen maximal in the sense that

$$\|(u(t), \rho(t))\|_{\mathbb{D}} \xrightarrow{t \to T} \infty \qquad \text{or} \qquad (u(t), \rho(t)) \xrightarrow{t \to T} \partial\mathcal{U}, \tag{4.2}$$

provided, $T < \infty$, c.f. [Ama93], Section 12. Notice, that the transformation of problem (2.1) was made in such a way, that, if (u, ρ) is a solution to (3.20) in the sense stated above and $r = r(u, \rho)$ is given by (3.16), then

$$(v(t), p(t), \Gamma(t)) := (u(t) + \chi(\rho(t)) \circ \theta_{\rho(t)}^{-1}, r(t) \circ \theta_{\rho(t)}^{-1}, \theta_{\rho(t)}[\Sigma])$$

solves (2.1). Let us introduce the notion of time-local solutions:

Definition 4.8 *A solution (u, ρ) of (3.20) corresponding to the initial value (u_0, ρ_0) is said to be a time-local solution of (2.1).*

Lemma 4.9 *Let (u, ρ) be a time-local solution of (3.20) and assume $T < \infty$. If $(u, r) \in L_\infty(J, BUC^1(D) \times BUC^1(D))$, then (u, ρ) extends to a continuous function on $\overline{J} := [0, T]$ with values in $H_q^2(D) \times c^{3+\beta_0}(\Sigma)$. Moreover, $\rho(T) \in \partial\mathcal{A}$.*

PROOF: By definition, ρ takes its values in the set $\mathcal{A}$. Thus, ρ is bounded in $c^{1+\beta_0}(\Sigma)$. Therefore, the set $\{\rho(t);\ t \in [0, T)\}$ is relatively compact in $c^{1+\beta'}(\Sigma)$ for $0 < \beta' < \beta_0$. Notice, that the mappings $P(\cdot)$, $K(\cdot)$ and $\sigma \mapsto \theta_\sigma$ can be defined on the closure of $\mathcal{A}$ in the norm of $c^{1+\beta'}(\Sigma)$. Therefore

$$\sup_{\delta \in J} \|\theta_{\rho(\delta)}\|_{C^1(\overline{D}, \mathbb{R}^n)} + \|\theta_{\rho(\delta)}^{-1}\|_{C^1(\overline{\Omega}_{\rho(\delta)}, \mathbb{R}^n)} < \infty.$$

Direct calculations (see [Kn2007]) show, that from this and from our assumption on r we get a bound for $B(\rho)r$ in the norm of $C(\Sigma)$, so that the velocity ρ_t is bounded in the same norm, cf. the third equation in (3.12). Also, from $P(\rho)\rho = r - K(\rho)$ and well-known generation properties of the operator $P(\rho)$, we are allowed to conclude $\|\rho\|_{C(J, c^{2+\mu}(\Sigma))} < \infty$ for some $\mu > 0$. Thus, interpolating $C(\Sigma)$ against $c^{2+\mu}(\Sigma)$ and using the mean value theorem, we find, that $\rho \in BUC^\varepsilon(J, c^{2+\mu'}(\Sigma))$, where $\mu > \mu' > 0$ and ε is a suitable positive number. Thus, our assumption with respect to u implies, that $G(u, \rho)$ is bounded in $c^{\mu'}(\Sigma)$. Since the mapping $B(\cdot)T(\cdot)P(\cdot)$ can be defined on the closure of $\mathcal{A}$ in the norm of $c^{2+\mu'}(\Sigma)$, the second equation in (3.20) and the generation properties of $B(\rho)T(\rho)P(\rho)$ provide a (two times) variation of constants formula based bootstrapping procedure ending up at a bound for ρ in the norm of $c^{3+\beta_0}(\Sigma)$.
Turning to the function u, we find, that $F(u, \rho)$ is bounded in $L_q(D)$. Since $A(\cdot)$ can be defined on the closure of $\mathcal{A}$ in the norm of $c^{2+\mu'}(\Sigma)$, a (two-times) variation of constants formula based bootstrapping procedure leads to a bound for u in the $H_q^2(D)$-norm, so we also get a bound for the time derivative u_t in the norm of $L_q(D)$. Therefore $u \in BUC^{\varepsilon'}(J, H_q^s)$ for $s < 2$ and suitable $\varepsilon' > 0$.
By a bootstrapping argument we conclude, that in fact $(u, \rho) \in BUC^{\varepsilon''(k)}([\gamma, T), H_q^k(D) \times c^k(\Sigma))$ is true for any positive $\gamma < T$, $k \in \mathbb{N}$ and an interpolation exponent $\varepsilon'' > 0$ depending on k. Thus, $\rho(T) \in \partial\mathcal{A}$ by (4.2). ■

Remark 4.10 The a priori estimates for ρ which we used in the proof of the last lemma depend highly on the geometry of Σ. As we will see later, we don't need them to be uniform with respect to that manifold in any sense. What we need to be uniform, is just *the property* of $P(\sigma)$, $B(\sigma)T(\sigma)P(\sigma)$, $\sigma \in \mathcal{A}$, to generate an analytic semigroup. We will comment on this in more detail at the very end of this chapter, when we will have defined the necessary notation, cf. Remark 4.11.

In order to proof our main result, we need to remember the following important relation between the original problem and its transformed version: If (u, ρ) is a time-local solution of (2.1), then the surface $\Gamma(t) = \theta_{\rho(t)}[\Sigma]$ is the zero-levelset of the function

$$\varphi_{\rho(t)} : N \to \mathbb{R}, \qquad x \mapsto \Lambda_{[\Sigma]}(x) - \rho(t)(P_{[\Sigma]}(x)).$$

From this we conclude easily $V(t,x) = \frac{\rho_t(t)(P_{[\Sigma]}(x))}{|\nabla(\varphi_{\rho(t)})(\theta_{\rho(t)}(x))|}$ (see again [Es2000]). We are now prepared for the

Proof [of Theorem 4.2]:

Let the initial data of the problem be given and fix a suitable reference manifold Σ as described in this section. The existence result for time-local solutions ensures, that we find a solution of problem (2.1) either on $[0, \infty)$ or on some finite interval $[0, T)$. Then the maximal interval of existence is

$$[0, t^+) := \bigcup_{T>0} \{[0,T),\ (2.1) \text{ has a solution on } [0,T)\},$$

where the property of being a solution is to be understood in the sense of Definition 4.1. Let us first take care of uniqueness: Since any solution of problem (2.1) is of class $(mb)^{(3,\beta_0)}$, the uniqueness of time-local solutions implies, that on some nontrivial interval $[0, \tau)$ there can only be one solution. Let us assume, that there are two different maximal solutions, say (v_1, p_1, Γ_1), (v_2, p_2, Γ_2). Then continuity implies, that

$$D := \{t;\ (v_1, p_1, \Gamma_1)(t) \neq (v_2, p_2, \Gamma_2)(t)\}$$

is an open subset of $[0, t^+)$. Thus $t_* := \inf D \notin D$. But we can use $(v_1, p_1, \Gamma_1)(t_*) = (v_2, p_2, \Gamma_2)(t_*)$ as initial value for a unique time-local solution, existing on, say, $[t_*, t_* + \tau')$. Thus, we can extend (v_1, p_1, Γ_1) to a unique classical solution on $[0, t_* + \tau')$. But $[t_*, t_* + \tau') \cap D \neq \emptyset$, which is a contradiction.

Now, let $N(t)$ be a tubular neighborhood of $\Gamma(t)$ and suppose that there are constants $C, \varepsilon_0 > 0$ such that

i) $\|v(t)\|_{BUC^1(\Omega(t))} + \|p(t)\|_{BUC^1(\Omega(t))} \le C$

ii) $A(\Gamma(t)) \ge \varepsilon_0$.

Let us assume t^+ to be finite. First of all we observe (c.f. [EsSi97a], Section 1)

$$|\frac{d}{dt}\,\mathrm{area}(\Gamma(t))| = |\frac{d}{dt}\int_{\Gamma(t)} 1| = |\int_{\Gamma(t)} V(t)\cdot H(t)| \le 1/\varepsilon_0 \cdot C \cdot \mathrm{area}(\Gamma(t)),$$

i.e $\sup_{t\in[0,t^+)} \mathrm{area}(\Gamma(t)) < \infty$, since $t^+ < \infty$. Therefore, invoking Lemma 4.5 and (4.4) below, the family $\{\Gamma(t)\}$ has uniformly bounded C^2-geometry. We fix a smooth atlas for each surface $\Gamma(t)$ and a corresponding partition of the unity as they have been constructed in the proof of Lemma 4.5. In particular, we choose appropriate numbers m, K, L according to Lemma 4.5. Keep in mind that the norms of the function spaces built over the $\Gamma(t)$ are fixed from now on. If now $t^* \in [0, t^+)$ is given, then for some $\delta^* = \delta^*(t^*) > 0$ the solution has a time-local representation (u^*, ρ^*) in the form (3.20) on some intervall $J(t^*) := [t^*, t^* + \delta^*)$, where $u_0^* = v(t^*) - \psi$, $\rho_0^* = 0$, since $\Gamma(t^*)$ is smooth. By ii) we may assume that $r(N(t^*)) = \varepsilon_0$. For technical reason (cf. Lemma 4.5) we choose $r(N(t^*)) = a_0$, where $0 < a_0 < \varepsilon_0$. Then (u^*, ρ^*) is arranged to take values in the set

$$\mathcal{U}(t^*) := H^2_{q,0}(\Omega(t^*)) \times \{\rho \in B^{4-1/q}_{qq}(\Gamma(t^*));\ \|\rho\|_{c^{1+\beta}(\Gamma(t^*))} < a_0/5\},$$

but our assumptions imply even stronger a priori bounds: Later on we will see, that for some number $\tilde{\varepsilon}$

$$\sup_{t^*\in[0,t^+)} \sup_{x\in N(t^*), h\in J(t^*)} |\nabla(\varphi_{\rho^*(h)})(x)| \le \tilde{\varepsilon}. \tag{4.3}$$

From this, $V = -p_\nu$ and i) we conclude

$$\sup_{t^*\in[0,t^+)} \sup_{J(t^*)} |\rho_t^*| < \infty, \tag{4.4}$$

meaning, that $\|\rho^*(\cdot)\|_{C(J(t^*), C(\Gamma(t^*)))}$ will vanish as $t^* \to t^+$. We will see at the very end of the proof, that also the quantity

$$\|\rho^*\|_{C(J(t^*), C^{1+\beta_0}(\Gamma(t^*)))} \tag{4.5}$$

will vanish as $t^* \to t^+$.

On the other hand, if $t^* \in [0, t^+)$ is given, as in the proof of Lemma 4.9 the definition of the mapping $\sigma \mapsto \theta_\sigma$ immediately implies

$$\sup_{\delta \in J(t^*)} \|\theta_{\rho^*(\delta)}\|_{C^1(\overline{\Omega(t^*)}, \mathbb{R}^n)} + \|\theta^{-1}_{\rho^*(\delta)}\|_{C^1(\overline{\Omega(t^*+\delta)}, \mathbb{R}^n)} < \infty.$$

This estimate may depend on t^*. Nevertheless, $(r^*, u^*) \in L_\infty(J(t^*), BUC^1(\Omega(t^*)) \times BUC^1(\Omega(t^*)))$, and we can arrange the time-local solution (u^*, ρ^*) to exist until it reaches the boundary of $\mathcal{U}(t^*)$, thanks to Lemma 4.9, contradicting (4.5).

We are left to prove (4.3), (4.5):

Fix $t^* \in (0, t^+)$ and let $x \in N(t^*)$. Let $\{(W_1, \varphi_1^{-1}), ..., (W_{m_x}, \varphi_{m_x}^{-1})\}$ be those charts of $\Gamma(t^*)$ that contain $P_{[\Gamma(t^*)]}(x)$, that is

$$P_{[\Gamma(t^*)]}(x) \in \bigcap_{l=1}^{m_x} W_l \cap \Gamma(t^*).$$

If $\{\pi_j;\ 1 \leq j \leq m\}$ is the corresponding subordinated partition of the unity, and if $\operatorname{supp}(\pi_l) \subset W_l \cap \Gamma(t^*)$ for $1 \leq l \leq m_x$, the assertion follows from the decomposition

$$\begin{aligned} \rho^*(t) \circ P_{[\Gamma(t^*)]} &= (\textstyle\sum_{l=1}^{m_x} \pi_l)\rho^*(t) \circ \varphi_l^{-1} \circ \varphi_l \circ P_{[\Gamma(t^*)]} \\ &= [\textstyle\sum_{l=1}^{m_x} (\pi_l \cdot \rho^*(t)) \circ \varphi_l^{-1}] \circ [\varphi_l \circ P_{[\Gamma(t^*)]}], \end{aligned}$$

due to the bounded geometry of $\{\Gamma(t)\}$, Lemma 4.5 and $\|\rho^*\|_{C(J(t^*), C^1(\Gamma(t^*)))} \leq a_0/5$. An estimate for $DP_{[\Gamma(t^*)]}$ in termes of the principal curvatures of $\Gamma(t^*)$ (dominated by $1/\varepsilon_0$) can be obtained by inverting equation (14.97) in chapter 14.6 in [GilTru].

In order to prove (4.5), let us economize our notation and drop t^*, l out of it, provided, it is not imperative. Let $Y := \varphi[\Gamma \cap \varphi^{-1}[W]] \subset \mathbb{R}^{n-1}$ and write again φ^{-1} instead of $\varphi^{-1}|_Y$. If $\varphi^{-1}(x) = (x, g(x)) \subset T_p\Gamma \times N_p\Gamma$ for some $p \in \Gamma$, a careful inspection of the proof of Lemma 4.5 shows, that, given $\delta_0 > 0$, we may assume

- $\|\nabla g\|_{C(\bar{Y})} \leq \delta_0$;
- $Y = \mathbb{B}(0, R)$ for some $R = R(\delta_0) > 0$;
- $\bigcup_{l=1}^m \varphi^{-1}(l, t^*)[Y_\varepsilon] = \Gamma(t^*)$, where $Y_\varepsilon := \mathbb{B}(0, R - \varepsilon)$, $0 < \varepsilon < R$;
- π is subordinated to to $\varphi^{-1}[Y_\varepsilon]$.

Then, in order to prove (4.5), it suffices to show, that for some constant $\tilde{K}$

$$\|\rho^*(h)\circ\varphi^{-1}(l,t^*)\|_{W^2_{\tilde q}(Y_\varepsilon)} \le \tilde K, \qquad l\in\{1,...,m\},\quad h\in J(t^*),$$

where $\tilde K$ depends on the global constants K, L, a_0 and $\tilde q$ is big enough such that $W^2_{\tilde q} \hookrightarrow c^{1+\gamma}$ where $\gamma > \beta_0$.
Let $G(x,z) := g(x) - z$, $x\in Y$, $z\in\mathbb{R}$, $|z|<R$. Then $\Gamma\cap\varphi^{-1}[Y] = G^{-1}[\{0\}]$. Choose $\delta_0 > 0$ in such a way, that the matrix

$$a_{ij}(\zeta) := \delta_{ij} - (\zeta_i\zeta_j)/(1+|\zeta|^2), \qquad \zeta\in\mathbb{R}^{n-1},$$

is positive definite for $|\zeta|\le\delta_0$ (δ_{ij} being the Kronecker symbol). Then, the operator A defined by

$$\begin{array}{rcl} H(x,g(x)) & = & \operatorname{div}(\frac{\nabla g(x)}{\sqrt{1+|\nabla g(x)|^2}}) \\ & = & 1/|\nabla G|\cdot\sum_{i,j=1}^{n-1} a_{ij}(\nabla g(x))\partial_i\partial_j g(x) \\ & =: & Ag(x), \end{array}$$

is uniformly elliptic in Y. Moreover, $\max_{i,j}\|a_{ij}(\nabla g(x))\|_{C^1(\bar Y)} \le c$, $c = c(K)$, and

$$\nabla Ag = A(\nabla g) + B,$$

where $B = B(D^2 g)$ is also uniformly bounded in terms of K. From the fourth equation in (2.1) and our assumption on p we know, that $\nabla[H\circ(x,g(x))] = [A\nabla g + B](x)$ is a priori bounded in terms of K and C. Thus, elliptic theory implies, that

$$\|\partial_k g\|_{W^2_{\tilde q}(Y_\varepsilon)} \le c, \qquad c = c(K,n,C,\varepsilon,\tilde q),\ \ 1\le k\le n-1.$$

Now, let $\nu = \nu(t^*)$ be the outward unit normal vector field of $\Gamma = \Gamma(t^*)$, let $\hat\nu := \nu\circ\varphi^{-1}$, $\hat\rho := \rho^*(h)\circ\varphi^{-1}$. Let $\hat\mu$ be the outward unit normal vector field of the surface $\theta_{\hat\rho}[\Gamma]$. Then $\|\hat\nu_j\|_{W^2_{\tilde q}(Y_\varepsilon)}$ is estimated from above in terms of $c = c(K,n,C,\varepsilon,\tilde q)$. Since $\|\rho^*(h)\|_{c^{1+\beta_0}} \le a_0$, interpolation yields, that

$$\|\rho^*(h)\circ\varphi^{-1}(t^*)\|_{C^1(\bar Y)}$$

will vanish as t^* reaches t^+, thanks to (4.4). Observe

$$0 = \partial_j 1 = \partial_j(\hat\nu\cdot\hat\nu) = 2\partial_j\hat\nu\cdot\hat\nu$$

and thus

$$(\partial_j(\varphi^{-1} + \hat{\rho}\hat{\nu})) \cdot \hat{\nu} = \partial_j \hat{\rho}.$$

Therefore, since the left factor in the last equation is tangential to $\theta_{\hat{\rho}}[\Gamma]$, we may assume $\hat{\nu} \cdot \hat{\mu}$ to be close to 1. As in the proof of Lemma 4.5 it can be seen, that the coordinate representation of the second fundamental form of the surface $\theta_{\hat{\rho}}[\Gamma]$:

$$[II(\theta_{\hat{\rho}}[\Gamma])]_{ij} := (\partial_i\partial_j\varphi^{-1} + \partial_i\partial_j\hat{\rho}\hat{\nu} + \partial_i\hat{\rho}\partial_j\hat{\nu} + \partial_j\hat{\rho}\partial_i\hat{\nu} + \hat{\rho}\partial_i\partial_j\hat{\nu}) \cdot \hat{\mu}$$

is estimated in terms of ε_0 and K. The desired estimate of $\partial_i\partial_j\hat{\rho}$ follows immediately.

■

Remark 4.11 We use the same notation as at the end of the last proof and let $\rho := \rho^*(h)$. Moreover, $D(\hat{\rho})$ denotes the localized version of the operator D, $D \in \{P, BTP\}$, i.e. $(D(\rho)f) \circ \varphi^{-1} = D(\hat{\rho})(f \circ \varphi^{-1})$, where f denotes a sufficiently regular function on $\Gamma(t^*)$. From [EsSi97a] we know that the operator $P(\hat{\rho})$ is elliptic in $x \in Y$, if $\hat{\rho}$ is a priori bounded in $C^1(\bar{Y})$ and the matrix $[w_{jk}(x)]^{-1}$ with

$$w_{jk}(x) := [\partial_j\varphi^{-1} \cdot \partial_k\varphi^{-1} + \hat{\rho}(h)(\partial_j\hat{\nu} \cdot \partial_k\varphi^{-1} + \partial_k\hat{\nu} \cdot \partial_j\varphi^{-1}) + \hat{\rho}(h)^2(\partial_j\hat{\nu} \cdot \partial_k\hat{\nu})](x)$$

is positive definite, cf. the proof of Lemma 3.2 in [EsSi97a]. Thus, the generation property of $P(\rho)$ depends on a smallness assumption for ρ which can be made uniformly with respect to a uniformly bounded C^2-geometry of the family $\{\Gamma(t)\}$. Basically, three facts guarantee the generation property of $B(\rho)T(\rho)P(\rho)$ for $\rho \in c^{2+\varepsilon}(\Gamma)$:

i) ellipticity of $P(\hat{\rho})$;

ii) ellipticity of $A(\rho)$;

iii) positivity of $\vec{b}(\rho) \cdot \nu$,

where $\vec{b}(\rho)$ defines the action of $B(\rho)$ by $B(\rho)w = \vec{b}(\rho) \cdot \nabla w$. Notice that the requirements ii) and iii) are automatically fullfilled, if θ_ρ is a diffeomorphism - they do not need a further smallness assumption for ρ.
We want to discuss the sufficiency of i)-iii) in some detail: First of all, given $\kappa \in (0, a_0]$, the atlas for $\Gamma := \Gamma(t^*)$ induces an atlas for the set

$$N_\kappa := X[\Gamma \times [-\kappa, 0]]$$

via the local charts

$$\tilde{\varphi}^{-1}(x,r) := \varphi^{-1}(x) + r \cdot \hat{\nu}(x), \qquad x \in Y, \;\; r \in [-\kappa, 0].$$

Here X denotes the map $(y,r) \mapsto y + r \cdot \nu(y)$. This is to be done in order to control the coefficients of $A(\hat{\rho})$ near Γ, where $A(\hat{\rho})$ is the localized version of $A(\rho)$ via $\tilde{\varphi}^{-1}$, i.e. $A(\hat{\rho})(g \circ \tilde{\varphi}^{-1}) = (A(\rho)g) \circ \tilde{\varphi}^{-1}$.
Let us assume $0 := 0^{\mathbb{R}^{n-1}} \in Y$, without loss of generality. Let (a_{jk}, p_{jk}, b_j) be the coefficients of $A(\hat{\rho}), P(\hat{\rho}), B(\hat{\rho})$ and set $(a^0_{jk}, p^0_{jk}, b^0_j) := (a^0_{jk}(0, 0^{\mathbb{R}}), p^0_{jk}(0), b^0_j(0))$. If $T_0 g$ denotes the unique solution of the elliptic problem

$$\begin{cases} (1 - \sum_{j,k=1}^n a^0_{jk} \partial_j \partial_k) u & = & 0 & \text{in} & \mathbb{R}^{n-1} \times \mathbb{R}^{>0} \\ u & = & g & \text{on} & \mathbb{R}^{n-1}, \end{cases}$$

and if B_0 and P_0 are the constant coefficient operators defined by freezing the coefficients of B and P at $0 \in Y$, we can build the nicely behaving operator $B_0 T_0 (1 - P_0)$. In fact, this operator is a Fourier multiplier, and the properties of its symbol guarantees a suitable generation property, cf. Lemma 5.1, Lemma 5.2 in [EsSi97a]. The proof of this fact precisely needs the following set of information:

- ellipticity of P_0;
- ellipticity of A_0;
- positivity of b^0_n.

To achieve now the generation property of $B(\rho)T(\rho)P(\rho)$, Escher and Simonett apply a pertubation argument: Observe that this operator 'can be defined on C^2', roughly speaking. Letting ρ take its values in $c^{2+\varepsilon}$, the difference

$$(\pi B(\sigma) T(\sigma) P(\sigma) h) \circ \varphi^{-1} - B_0 T_0 P_0 ((\pi h) \circ \varphi^{-1}),$$

$h \in c^{3+\tilde{\varepsilon}}(\Gamma)$, can be controlled by the size of Y and the value of κ, cf. the estimates (5.6), (5.7) in [EsSi97a]. More precisely, one makes a subtile partition of the above difference analogous to that one in step b) in the proof of Lemma 5.1 in [EsSi97b]. After that one estimates the difference of the freezed and variable coefficients. Here, the treatment of the difference between $T(\rho)$ and T_0 makes essential use of an obvious generalization to the higher dimensional case of formula a) in Lemma 6.6 in [EsSi95]. The arguments from the proof of Lemma 6.7 in the same paper, used to estimate the difference $a_{jk} - a^0_{jk}$, can be carried over by using continuity aspects of multiplication (even in the little Hölder spaces of negative real exponent) stated in Theorem 2.8.2 in [Trie1]. We emphasize, that shrinking the chart domain and thus changing also the

corresponding partition of the unity leads to equivalent norms built over the special surface we are working on. These things cannot be done uniformly for our hole family $\{\Gamma(t)\}$, but they do require only the function ρ to belong to a certain regularity class built over the special surface and not any further smallness assumption.
Thats why we don't have to worry about i) - iii) when choosing the number a_0 in the proof of the preceeding theorem.

PROOF [of Corollary 4.3]: First observe that $u \equiv c$. We have

$$\begin{aligned} d/dt\,\mathrm{vol}(t) &= \int_{\Gamma(t)} V(t)\,dx \\ &= -\int_{\Gamma(t)} p_\nu(t,x)\,d\sigma(x) \\ &= -\int_{\Omega(t)} \Delta p(t,x)\,dx \\ &= \int_{\Omega(t)} f(v(t,x))\,dx \\ &= -\alpha_0\,\mathrm{vol(t)}. \end{aligned}$$

Thus, if $t^+ = \infty$, then $\mathrm{vol}(\Omega(t)) \to 0$ as $t \to \infty$, thus $A(\Gamma(t)) \to 0$: In deed, if not, there would be a ball $B \subset \bar{\Omega}(t)$ for all $t \in [0, t^+)$ and $\mathrm{vol}(\Omega(t)) \geq \mathrm{vol}(B)$. Let us assume $t^+ < \infty$, but $A(\Gamma(t)) \geq a_0$ and $\|\nabla p(t)\|_{C(\Gamma(t))} \leq C$.
Since

$$\begin{aligned} |d/dt\,\mathrm{area}(t)| &= |\int_{\Gamma(t)} H(t)V(t)\,dx| \\ &= |-\int_{\Gamma(t)} H(t,x)p_\nu(t,x)\,d\sigma(x)| \\ &\leq \mathrm{area}(\Gamma(t)) \cdot C \cdot 1/a_0, \end{aligned}$$

and $t^+ < \infty$, it follows (since $n = 2$)

$$D := \sup_t\ \mathrm{diam}(\Omega(t)) \leq \sup_t \mathrm{area}(\Gamma(t)) < \infty.$$

Thus $|p(t,x)| \leq C_1(\alpha_0, a_0, D)$ (cf. Theorem 3.7 in [GilTru]). Moreover, observe that $\partial_i p(t)$ is harmonic in $\Omega(t)$ for $i = 1, ..., n$, meaning that

$$\sup_t |\nabla p(t,x)| \leq C_2, \qquad x \in \Omega(t),$$

$C_2 = C_2(C, D)$ (cf. again Theorem 3.7 in [GilTru]), contradicting Theorem 4.2. This completes the proof.

■

5 Proofs

The thesis will be closed by giving the missing proofs. It must be emphasized that the following is just a variation of the construction of parabolic fundamental solutions which can be found in [LaQPP]. We shall use the notation from chapter 3 (which is also taken from [LaQPP]).

Proof [of Lemma 3.2]:
We use the notations from (3.1), set $a_\varepsilon := a_{A_\varepsilon}$, $k_\varepsilon := k_{A_\varepsilon}$, etc. First we conclude from (3.1), (3.2)

$$\|k_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} \leq \eta \cdot C \cdot (t-s)^{\rho-1} \cdot e^{-\sigma(t-s)/\varepsilon}$$

as well as

$$\|k_\varepsilon(t,s)\|_{\mathcal{L}(E_1,E_0)} \leq \frac{\eta \cdot C}{\varepsilon} \cdot (t-s)^{\rho} \cdot e^{-\sigma(t-s)/\varepsilon},$$

where $C = C(M)$, $\sigma = \sigma(\vartheta, M)$. Using induction, it is easy to see that

$$\|\bigstar_{j=1}^{n} k_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} \leq \frac{e^{-\sigma(t-s)/\varepsilon}}{t-s} \cdot \frac{[C \cdot (t-s)^\rho]^n}{\Gamma(n \cdot \rho)},$$

where Γ denotes the Eulerian Gamma function. Here, $C = C(M, \eta)$. Because of

$$\sum_{n=1}^{\infty} \frac{x^n}{\Gamma(n \cdot \beta)} \leq C(\beta) \cdot x \cdot e^{2x^{1/\beta}},$$

$x \geq 0$, $0 < \beta < 1$, it follows

(5.1)
$$\begin{aligned}\|w_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} = \|\textstyle\sum_{n=1}^{\infty} \bigstar_{j=1}^{n} k_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} &\leq \tfrac{e^{-\sigma(t-s)/\varepsilon}}{t-s} \cdot C \cdot C(\rho) \cdot (t-s)^\rho \cdot e^{2[C(t-s)^\rho]^{1/\rho}} \\ &= C \cdot C(\rho) \cdot (t-s)^{\rho-1} \cdot e^{(2C^{1/\rho} - \sigma/\varepsilon)\cdot(t-s)}.\end{aligned}$$

This leads to

$$\|w_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} \leq C \cdot C(\rho) \cdot (t-s)^{\rho-1} \cdot e^{\frac{-\sigma}{3\cdot\varepsilon}\cdot(t-s)},$$

$\varepsilon \leq \sigma/3 \cdot C^{-1/\rho}$, $C = C(M, \eta)$, and in the case $J = [0, T]$

$$\|w_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} \le \tilde{C}\cdot(t-s)^{\rho-1}\cdot e^{-\sigma(t-s)/\varepsilon},$$

where $\tilde{C} = C(M,\eta)\cdot e^{2C^{1/\rho}T}$. Denoting $\tilde{C}$ again by C, we find that $C = C(M,\eta,\rho,T)$ is monotone increasing in T. Because of

$$w_\varepsilon = k_\varepsilon + k_\varepsilon \star w_\varepsilon = k_\varepsilon + w_\varepsilon \star k_\varepsilon$$

(cf. chapter 4.3 in [LaQPP]) we find

$$\begin{array}{rcl}
\|w_\varepsilon(t,s)\|_{\mathcal{L}(E_1,E_0)} & \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +\int_s^t \|w_\varepsilon(t,\tau)\|_{\mathcal{L}(E_0)}\cdot\|k_\varepsilon(\tau,s)\|_{\mathcal{L}(E_1,E_0)}\,d\tau \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +C\cdot\int_s^t e^{-\sigma(t-\tau)/(3\varepsilon)}\cdot(t-\tau)^{\rho-1}\cdot\frac{C}{\varepsilon}(\tau-s)^\rho\cdot e^{-\sigma(\tau-s)/(3\varepsilon)}\,d\tau \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +\frac{C}{\varepsilon}\cdot e^{-\sigma(t-s)/(3\varepsilon)}\cdot(t-s)^\rho\cdot\int_s^t(t-\tau)^{\rho-1}\,d\tau \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +\frac{C}{\varepsilon}\cdot e^{-\sigma(t-s)/(3\varepsilon)}\cdot(t-s)^\rho\cdot(t-s)^\rho \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +\frac{C}{\varepsilon}\cdot e^{-\sigma(t-s)/(3\varepsilon)}\cdot(t-s)^\rho\cdot e^{\rho\cdot(t-s)} \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon} \\
& & +\frac{C}{\varepsilon}\cdot e^{-\sigma(t-s)/(12\varepsilon)}\cdot(t-s)^\rho \\
& \le & \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/(12\cdot\varepsilon)},
\end{array}$$

$\varepsilon \le \sigma/(4\cdot\rho)$, $C = C(M,\eta,\rho)$. In the case $J = [0,T]$ we have

$$\|w_\varepsilon(t,s)\|_{\mathcal{L}(E_1,E_0)} \le \frac{C}{\varepsilon}\cdot(t-s)^\rho\cdot e^{-\sigma(t-s)/\varepsilon},$$

$C = C(M,\eta,\rho,T)$ monotone increasing in T. From this we see

$$\begin{array}{rcl}
\|a_\varepsilon \star w_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} & \le & C\cdot\int_s^t e^{-\sigma(t-\tau)/\varepsilon}\cdot(\tau-s)^{\rho-1}\cdot e^{-\sigma(\tau-s)/(3\cdot\varepsilon)}\,d\tau \\
& \le & C\cdot e^{-\sigma(t-s)/(3\cdot\varepsilon)}\cdot(t-s)^\rho
\end{array}$$

as well as

$$\begin{array}{rcl}
\|a_\varepsilon \star w_\varepsilon(t,s)\|_{\mathcal{L}(E_1,E_0)} & \le & C\cdot\int_s^t e^{-\sigma(t-\tau)/\varepsilon}\cdot\frac{(\tau-s)^\rho}{\varepsilon}\cdot e^{-\sigma(\tau-s)/(12\cdot\varepsilon)}\,d\tau \\
& \le & \frac{C}{\varepsilon}\cdot e^{-\sigma(t-s)/(12\cdot\varepsilon)}\cdot(t-s)^{\rho+1}.
\end{array}$$

The estimates for the case $J = [0,T]$ can be obtained in an obvious way.

Let us now estimate $\|a_\varepsilon \star w_\varepsilon\|_{\mathcal{L}(E_j,E_1)}$. We have

$$\begin{array}{rcl} \varepsilon^{-1}A(a_\varepsilon \star w_\varepsilon) & = & \varepsilon^{-1}AU_\varepsilon - \varepsilon^{-1}Aa_\varepsilon \\ & = & -\partial_1 U_\varepsilon - \varepsilon^{-1}Aa_\varepsilon \\ & = & -\partial_1 a_\varepsilon - \varepsilon^{-1}Aa_\varepsilon - \partial_1(a_\varepsilon \star w_\varepsilon) \\ & = & k_\varepsilon - \partial_1(a_\varepsilon \star w_\varepsilon) \end{array}$$

and therefore $A(a_\varepsilon \star w_\varepsilon) = \varepsilon k_\varepsilon - \varepsilon\partial_1(a_\varepsilon \star w_\varepsilon)$, meaning, that

$$\|a_\varepsilon \star w_\varepsilon\|_{\mathcal{L}(E_j,E_1)} \leq \varepsilon \cdot C(M) \cdot \|k_\varepsilon\|_{\mathcal{L}(E_j,E_0)} + \varepsilon \cdot C(M) \cdot \|\partial_1(a_\varepsilon \star w_\varepsilon)\|_{\mathcal{L}(E_j,E_0)}.$$

Observe (cf. (4.3.26), (4.3.30) in chapter II.4 of [LaQPP])

$$\begin{array}{rcl} \varepsilon\partial_1(a_\varepsilon \star w_\varepsilon) & = & \varepsilon \cdot e^{-(t-s)\varepsilon^{-1}A(t)} w_\varepsilon(t,s) \\ & & +\varepsilon \cdot \int_s^t e_\varepsilon(t,\tau) w_\varepsilon(\tau,s)\, d\tau \\ & & +\int_s^t A(t) e^{-(t-\tau)\varepsilon^{-1}A(t)} [w_\varepsilon(t,s) - w_\varepsilon(\tau,s)]\, d\tau. \\ & =: & J_1 + J_2 + J_3. \end{array}$$

It is clear how to estimate J_1 and J_2. In order to estimate J_3 we make use of the decomposition (cf. (4.3.8), (4.3.22) in chapter II.4 in [LaQPP])

$$\begin{array}{rcl} w_\varepsilon(t,s) - w_\varepsilon(\tau,s) & = & k_\varepsilon(t,s) - k_\varepsilon(\tau,s) \\ & & + \int_s^\tau [k_\varepsilon(t,\omega) - k_\varepsilon(\tau,\omega)] w_\varepsilon(\omega,s)\, d\omega \\ & & + \int_\tau^t k_\varepsilon(t,\omega) w_\varepsilon(\omega,s)\, d\omega. \\ & =: & I_1 + I_2 + I_3, \end{array}$$

where $s < \tau < t$. Note that

$$k_\varepsilon(t,s) - k_\varepsilon(\tau,s) = \varepsilon^{-1}[A(\tau) - A(t)] a_\varepsilon(t,s) - \varepsilon^{-1}[A(\tau) - A(s)][a_\varepsilon(t,s) - a_\varepsilon(\tau,s)],$$

where $s < \tau < t$. We have

$$\|\varepsilon^{-1}[A(\tau) - A(t)] a_\varepsilon(t,s)\|_{\mathcal{L}(E_j,E_0)} \leq \eta \cdot C(M) \cdot \varepsilon^{-j} \cdot [(t-s)^{j-1}(t-\tau)^\rho] \cdot e^{-\sigma(t-s)/\varepsilon},$$

$j = 0, 1$. Moreover, since $a_\varepsilon(t,s) - a_\varepsilon(\tau,s) = -\varepsilon^{-1} \int_\tau^t A(s)\, e^{-(\omega-s)\varepsilon^{-1}A(s)}\, d\omega$, Lemma 3.1 gives us

$$\begin{aligned}\|a_\varepsilon(t,s)-a_\varepsilon(\tau,s)\|_{\mathcal{L}(E_j,E_1)} &\le C(M)\cdot\varepsilon^{1-j}\cdot\textstyle\int_\tau^t(\omega-s)^{j-2}\underbrace{e^{-\sigma\varepsilon^{-1}(\omega-s)}}_{\le 1}\,d\omega\\ &\le C(M)\cdot\varepsilon^{1-j}\cdot[(t-\tau)(\tau-s)^{j-2}].\end{aligned}$$

Thus,

$$(5.2)\quad \|\varepsilon^{-1}[A(\tau)-A(s)][a_\varepsilon(t,s)-a_\varepsilon(\tau,s)]\|_{\mathcal{L}(E_j,E_0)}\le\eta\cdot C(M)\cdot\varepsilon^{-j}\cdot[(t-\tau)(\tau-s)^{j-2+\rho}].$$

On the other hand, clearly,

(5.3)

$$\begin{aligned}\|\varepsilon^{-1}[A(\tau)-A(s)][a_\varepsilon(t,s)-a_\varepsilon(\tau,s)]\|_{\mathcal{L}(E_j,E_0)} &\le \eta\cdot C(M)\cdot[(\tau-s)^\rho\,\varepsilon^{-1}]\cdot\\ &\qquad[[(\varepsilon/(t-s))^{1-j}\,e^{-\sigma\varepsilon^{-1}(t-s)}]\\ &\qquad+[(\varepsilon/(\tau-s))^{1-j}\,e^{-\sigma\varepsilon^{-1}(\tau-s)}]]\\ &\le (\tau-s)^{\rho-1+j}\,\varepsilon^{-j}\,e^{-\sigma\varepsilon^{-1}(\tau-s)}.\end{aligned}$$

Therefore, by the multiplication $(5.2)^\rho\cdot(5.3)^{1-\rho}$, we arrive at

$$\|\varepsilon^{-1}[A(\tau)-A(s)][a_\varepsilon(t,s)-a_\varepsilon(\tau,s)]\|_{\mathcal{L}(E_j,E_0)}\le\eta\cdot C(M)\cdot\varepsilon^{-j}\cdot[(t-\tau)^\rho(\tau-s)^{j-1}]\cdot e^{-\sigma(1-\rho)\varepsilon^{-1}(\tau-s)},$$

and thus

$$(5.4)\quad \|k_\varepsilon(t,s)-k_\varepsilon(\tau,s)\|_{\mathcal{L}(E_j,E_0)}\le\eta\cdot C(M)\cdot\varepsilon^{-j}\cdot[(t-\tau)^\rho(\tau-s)^{j-1}]\cdot e^{-\sigma(1-\rho)\varepsilon^{-1}(\tau-s)},$$

where $s<\tau<t$. Since, obviously,

(5.5)

$$\begin{aligned}\|k_\varepsilon(t,s)-k_\varepsilon(\tau,s)\|_{\mathcal{L}(E_0)} &\le \eta\cdot C(M)\cdot[(t-s)^{\rho-1}+(\tau-s)^{\rho-1}]\cdot e^{-\sigma\varepsilon^{-1}(\tau-s)}\\ &\le 2\cdot\eta\cdot C(M)\cdot(\tau-s)^{\rho-1}\cdot e^{-\sigma\varepsilon^{-1}(\tau-s)},\end{aligned}$$

by taking the product $(5.4)^{1/2}\cdot(5.5)^{1/2}$, we also get

$$\|k_\varepsilon(t,s)-k_\varepsilon(\tau,s)\|_{\mathcal{L}(E_0)}\le\eta\cdot C(M)\cdot[(t-\tau)^{\rho/2}(\tau-s)^{\rho/2-1}]\cdot e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)},$$

$s < \tau < t$. In the sequel we shall make frequently use of the following simple facts: If $t > \tau > s > 0$ and $a > \tilde{a} > 0$, then

$$e^{-\sigma a(t-s)} \leq e^{-\sigma a(\tau-s)}, \qquad e^{-\sigma ab} \leq e^{-\sigma \tilde{a} b}, \qquad b > 0.$$

Moreover, we denote the number $\sigma/12$ again by σ. We calculate

$$\|I_1\|_{\mathcal{L}(E_0)} \leq c \cdot [(t-\tau)^{\rho/2}(\tau-s)^{\rho/2-1}] \cdot e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)},$$

where $c = c(M, \eta)$,

$$\begin{array}{rcl} \|I_2\|_{\mathcal{L}(E_0)} & \leq & c \cdot \int_s^\tau (t-\tau)^{\rho/2}(\tau-\omega)^{\rho/2-1} e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-\omega)} \cdot (\omega-s)^{\rho-1} e^{-\sigma\varepsilon^{-1}(\omega-s)}\, d\omega \\ & \leq & c \cdot (t-\tau)^{\rho/2} e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)} \cdot \int_s^\tau (\tau-\omega)^{\rho/2-1} \cdot (\omega-s)^{\rho-1}\, d\omega \\ & = & c \cdot (t-\tau)^{\rho/2}(\tau-s)^{3\rho/2-1} \cdot e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)} \cdot \mathsf{B}(\rho/2, \rho) \\ & \leq & c \cdot (t-\tau)^{\rho/2}(\tau-s)^{\rho/2-1} \cdot e^{\rho(\tau-s)} \cdot e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)} \cdot \mathsf{B}(\rho/2, \rho), \end{array}$$

where $c = c(M, \eta, \rho)$ ($c = c(M, \eta, \rho, T)$ in the case $J = [0, T]$) and B denotes the Eularian Beta function. Finally,

$$\begin{array}{rcl} \|I_3\|_{\mathcal{L}(E_0)} & \leq & c \cdot \int_\tau^t (t-\omega)^{\rho-1}(\omega-s)^{\rho-1} \cdot e^{-\sigma\varepsilon^{-1}(t-s)}\, d\omega \\ & \leq & c \cdot (\tau-s)^{\rho-1} \cdot e^{-\sigma\varepsilon^{-1}(t-s)} \cdot \int_\tau^t (t-\omega)^{\rho-1}\, d\omega \\ & \leq & c \cdot [(t-\tau)^{\rho}(\tau-s)^{\rho-1}] \cdot e^{-\sigma\varepsilon^{-1}(t-s)} \\ & \leq & c \cdot [(t-\tau)^{\rho/2}(\tau-s)^{\rho/2-1}] \cdot e^{\rho/2(t-s)} \cdot e^{-\sigma\varepsilon^{-1}(t-s)}, \end{array}$$

$c = c(M, \eta, \rho)$ ($c = c(M, \eta, \rho, T)$ in the case $J = [0, T]$). Summing up, we find

$$\|w_\varepsilon(t, s) - w_\varepsilon(\tau, s)\|_{\mathcal{L}(E_0)} \leq c \cdot [(t-\tau)^{\rho/2}(\tau-s)^{\rho/2-1}] \cdot e^{-\sigma a(\rho)\varepsilon^{-1}(\tau-s)},$$

where $c = c(M, \eta, \rho)$ and $\varepsilon < \varepsilon_0 = \varepsilon_0(M, \eta, \rho, \vartheta)$ in the case $J = [0, \infty)$, $c = c(M, \eta, \rho, T)$ monotone increasing in $T > 0$ and all $\varepsilon > 0$ in the case $J = [0, T]$, respectively. Moreover,

$$\|I_1\|_{\mathcal{L}(E_1, E_0)} \leq c \cdot \varepsilon^{-1} \cdot (t-\tau)^{\rho} \cdot e^{-\sigma(1-\rho)\varepsilon^{-1}(\tau-s)},$$

$$\begin{array}{rcl} \|I_2\|_{\mathcal{L}(E_1, E_0)} & \leq & c \cdot \varepsilon^{-1} \cdot \int_s^\tau (t-\tau)^{\rho/2}(\tau-\omega)^{\rho/2-1} e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-\omega)} \cdot (\omega-s)^{\rho} e^{-\sigma\varepsilon^{-1}(\omega-s)}\, d\omega \\ & \leq & c \cdot \varepsilon^{-1} \cdot (t-\tau)^{\rho/2} e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)} \cdot \int_s^\tau (\tau-\omega)^{\rho/2-1} \cdot (\omega-s)^{\rho}\, d\omega \\ & = & c \cdot \varepsilon^{-1} \cdot (t-\tau)^{\rho/2}(\tau-s)^{3\rho/2} \cdot e^{-\sigma(1-\rho/2)\varepsilon^{-1}(\tau-s)} \end{array}$$

and

$$
\begin{aligned}
\|I_3\|_{\mathcal{L}(E_1,E_0)} &\le c\cdot\varepsilon^{-1}\cdot\int_\tau^t (t-\omega)^{\rho-1}(\omega-s)^\rho\cdot e^{-\sigma\varepsilon^{-1}(t-s)}\,d\omega\\
&\le c\cdot\varepsilon^{-1}\cdot(t-s)^\rho\cdot e^{-\sigma\varepsilon^{-1}(t-s)}\cdot\int_\tau^t (t-\omega)^{\rho-1}\,d\omega\\
&\le c\cdot\varepsilon^{-1}\cdot[(t-\tau)^\rho(t-s)^\rho]\cdot e^{-\sigma\varepsilon^{-1}(\tau-s)},
\end{aligned}
$$

$c=c(M,\eta,\rho)$ ($c=c(M,\eta,\rho,T)$ in the case $J=[0,T]$). Since

$$
\|w_\varepsilon(t,s)-w_\varepsilon(\tau,s)\|_{\mathcal{L}(E_1,E_0)}\le\|I_1\|_{\mathcal{L}(E_1,E_0)}+\|I_2\|_{\mathcal{L}(E_1,E_0)}+\|I_3\|_{\mathcal{L}(E_1,E_0)},
$$

putting everything together and using the symbol $a(\rho)$ to denote a 'generic constant', we arrive at

$$
\begin{aligned}
\|J_3\|_{\mathcal{L}(E_0)} &\le c\cdot\varepsilon\cdot\int_s^t (t-\tau)^{\rho/2-1}\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(t-\tau)}\cdot(\tau-s)^{\rho/2-1}\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(\tau-s)}\,d\tau\\
&\le c\cdot\varepsilon\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(t-s)}\cdot\int_s^t[(t-\tau)\cdot(\tau-s)]^{\rho/2-1}\,d\tau\\
&= c\cdot\varepsilon\cdot(t-s)^{\rho-1}\cdot\mathsf{B}(\rho/2,\rho/2)\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(t-s)},
\end{aligned}
$$

$$
\begin{aligned}
\|J_3\|_{\mathcal{L}(E_1,E_0)} &\le c\cdot\Big[\int_s^t e^{-\sigma\varepsilon^{-1}(t-\tau)}\cdot(t-\tau)^{\rho-1}\cdot e^{-\sigma\varepsilon^{-1}(1-\rho)(\tau-s)}\,d\tau\\
&\quad+\int_s^t e^{-\sigma\varepsilon^{-1}(t-\tau)}\cdot(t-\tau)^{\rho/2-1}(\tau-s)^{3\rho/2}\cdot e^{-\sigma\varepsilon^{-1}a(\rho)(\tau-s)}\,d\tau\\
&\quad+\int_s^t e^{-\sigma\varepsilon^{-1}(t-\tau)}\cdot(t-\tau)^{\rho-1}(t-s)^\rho\cdot e^{-\sigma\varepsilon^{-1}a(\rho)(\tau-s)}\,d\tau\Big]\\
&\le c\cdot\Big[e^{-\sigma a(\rho)\varepsilon^{-1}(t-s)}\cdot(t-s)^\rho\\
&\quad+(t-s)^\rho\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(t-s)}\cdot\int_s^t(t-\tau)^{\rho/2-1}(\tau-s)^{\rho/2}\,d\tau\\
&\quad+(t-s)^\rho\cdot e^{-\sigma a(\rho)\varepsilon^{-1}(t-s)}\cdot\int_s^t(t-\tau)^{\rho-1}\,d\tau\Big]\\
&\le c\cdot\big[(t-s)^\rho\cdot e^{-\sigma\tilde a(\rho)\varepsilon^{-1}(t-s)}\big],
\end{aligned}
$$

where $c=c(M,\eta,\rho)$ and $\varepsilon<\varepsilon_0=\varepsilon_0(M,\eta,\rho,\vartheta)$ in the case $J=[0,\infty)$, $c=c(M,\eta,\rho,T)$ monotone increasing in $T>0$ and all $\varepsilon>0$ in the case $J=[0,T]$, respectively.

■

PROOF [of Lemma 3.3]:
We shall use again the notation from (3.1) and denote by C a generic constant. First of all, given $\varepsilon>0$ and $\vartheta\in(0,\pi/2)$, we find that $\varepsilon\cdot\Sigma_\vartheta=\Sigma_\vartheta$ and because of

$$
\lambda+A_\varepsilon(t)=\lambda+\frac{1}{\varepsilon}A(t)=\frac{1}{\varepsilon}\cdot(\varepsilon\lambda+A(t))
$$

also

$$
\|(\lambda+A_\varepsilon(t))^{-1}\|_{\mathcal{L}(E_0,E_j)}\le M\cdot\varepsilon\cdot(1+\varepsilon|\lambda|)^{j-1},
$$

where $j \in \{0,1\}$, $\lambda \in \Sigma_\vartheta$, $t \in J$. This implies

$$\begin{array}{rl} & \|(\lambda + A_\varepsilon(t))^{-1} - (\lambda + A_\varepsilon(s))^{-1}\|_{\mathcal{L}(E_0)} \\ \leq & \|(\lambda + A_\varepsilon(t))^{-1}\|_{\mathcal{L}(E_0)} \cdot \|A_\varepsilon(t) - A_\varepsilon(s)\|_{\mathcal{L}(E_1,E_0)} \cdot \|(\lambda + A_\varepsilon(s))^{-1}\|_{\mathcal{L}(E_0,E_1)} \\ \leq & M^2 \cdot \varepsilon \cdot \eta \cdot (t-s)^\rho \cdot \frac{1}{1+\varepsilon|\lambda|}. \end{array}$$

From

$$\begin{array}{rl} & \|A_\varepsilon(t)e^{-(t-s)A_\varepsilon(t)} - A_\varepsilon(s)e^{-(t-s)A_\varepsilon(s)}\|_{\mathcal{L}(E_0)} \\ = & \|\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda(t-s)}[(\lambda + A_\varepsilon(t))^{-1} - (\lambda + A_\varepsilon(s))^{-1}]\, d\lambda\|_{\mathcal{L}(E_0)} \end{array}$$

as well as $\frac{\varepsilon|\lambda|}{1+\varepsilon|\lambda|} \leq 1$ and $|e^{\lambda(t-s)}| \leq C \cdot e^{\Re(\lambda)(t-s)}$, $\lambda \in \Gamma$, it follows (theorem 4.1.1 in [LaQPP]) that

$$\|e_\varepsilon\|_{\mathcal{L}(E_0)} \leq C \cdot M^2 \cdot (t-s)^{\rho-1}.$$

Here, $\Gamma \subset \Sigma_\vartheta \setminus \{0\}$ runs from $\infty e^{-i(\vartheta+\pi/2)}$ to $\infty e^{i(\vartheta+\pi/2)}$. On the other hand, we clearly have (cf. Lemma 3.1)

$$\|e_\varepsilon\|_{\mathcal{L}(E_0)} \leq C \cdot (t-s)^{-1} \cdot e^{-\sigma(t-s)/\varepsilon}.$$

All in all

$$\begin{array}{rl} & \|e_\varepsilon(t,s)\|_{\mathcal{L}(E_0)} \leq \|e_\varepsilon(t,s)\|^\rho_{\mathcal{L}(E_0)} \cdot \|e_\varepsilon(t,s)\|^{1-\rho}_{\mathcal{L}(E_0)} \\ \leq & C \cdot (t-s)^{\rho^2-\rho} \cdot (t-s)^{\rho-1} \cdot e^{-\sigma\cdot(1-\rho)\cdot(t-s)/\varepsilon} \\ = & C \cdot (t-s)^{\rho^2-1} \cdot e^{-b(t-s)/\varepsilon}, \end{array}$$

$b = \sigma \cdot (1-\rho)$.

■

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Curriculum Vitae

Friedrich Lippoth

Name:	**Lippoth**, Friedrich-Matthias
Date of birth:	26th November 1979

Education

2006-2009	Stipendiary in the DFG Research Training Group 615
2006	Diploma in mathematics, University of Hanover
2000-2006	Study of mathematics and theoretic informatics, University of Hanover

Participation in Conferences

Dec. 2009	Elliptic and Parabolic Equations, WIAS Berlin.
Feb. 2009	Wave Motion, Oberwolfach.
Sept. 2008	Mathematical Fluid Dynamics, Darmstadt.
Sept. 2008	Summer School on Functional Analytic Methods for PDEs, Hanover.
Dec. 2007	Evolution Equations: The state of the art, Reisensburg.
Sept. 2007	Analysis and Geometry, Hanover.