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COHOMOLOGICAL ASPECTS OF MODULI OF CURVES AND ABELIAN VARIETIES

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Contents

Introduction

Moduli spaces of curves and abelian varieties

A fundamental problem in algebraic geometry is the classification of algebraic varieties. Moduli spaces represent a solution for this kind of problem. Let us suppose that we want to classify all geometric objects (algebraic varieties) is a certain class $\mathcal{C}$ up to a fixed equivalence relation. Then the associated moduli space (if it exists) would be a geometric object M whose points are in one-to-one correspondence with the equivalence classes of varieties in the class $\mathcal{C}$. Furthermore, the local geometry of M should describe the geometry of deformations of the varieties in $\mathcal{C}$. When this correspondence is complete, we speak of *fine moduli spaces*. By relaxing these requirements, one can get what is called a *coarse moduli space*.

If one wants to classify algebraic varieties of fixed dimension, the first interesting case is the classification of algebraic varieties of dimension 1, i.e., of algebraic curves. In this case, there is a fundamental invariant to be taken into account: namely, the *genus* of the curve. For a non-singular curve C over the fields of complex numbers, the genus can be defined geometrically as the dimension of the space of regular differentials on C, or topologically interpreting C as a real orientable surface and taking its topological genus (i.e. the number of "holes" of the surface). The importance of the genus lies in the fact that two non-singular curves can be deformed into each other if and only if they have the same genus. In particular, to obtain a connected moduli space, we need to fix the genus of the curves we consider. It is a fundamental result that the moduli space of smooth curves of genus g exists for every $g \geq 2$. The coarse moduli space M_g is a quasi-projective variety that can be constructed using geometric invariant theory [M65, Thm 5.11].

Furthermore, the costruction of the moduli space can be extended to the case of smooth genus g curves with n distinct marked points. In this case the objects that are classified are $n+1$-tuples $(C, p_1, \ldots, p_n)$ where C is a smooth genus g curve and the p_i are distinct points on C. Two n-pointed curves $(C, p_1, \ldots, p_n)$ and $(C', p'_1, \ldots, p'_n)$ are isomorphic if there is an isomorphism $C \to C'$ that maps p_i to p'_i for every $i = 1, \ldots, n$. For every pair (g, n) with $2g - 2 + n > 0$, the coarse moduli space $M_{g,n}$ of n-pointed smooth genus g curves exists and is a quasi-projective variety with locally quotient singularities. A fine moduli space $\mathcal{M}_{g,n}$ exists as well, but in general not in the category of quasi-projective varieties. Instead, to define $\mathcal{M}_{g,n}$ one needs to endow it with a more flexible structure and work in the category of algebraic stacks [DM69, Kn83]. Then $\mathcal{M}_{g,n}$ exists and is a smooth irreducible Deligne–Mumford stack of dimension $3g-3+n$.

Moduli spaces of smooth curves are almost never complete (the only exception is $\mathcal{M}_{0,3}$, which is a point). On the other hand, they admit a natural compactification, the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, the moduli space of n-pointed stable curves of genus g, i.e. nodal n-pointed curves of arithmetic genus g with finite automorphism group.

A different class of projective varieties whose moduli spaces have been intensively studied is the moduli space of abelian varieties. An abelian variety is a projective variety A, together with morphisms $+:\ A\times A\to A$ and $\iota:\ A\to A$ defining a group structure on the set of points of A. A polarized abelian variety is a pair $(A,[L])$ where A is an abelian variety and $[L]:=c_1(L)\in H^2(A;\mathbf{Z})$ is the first Chern class of an ample line bundle L on A; if the space of sections of L is 1-dimensional, $(A,[L])$ is called a principally polarized abelian variety.

Over the field of complex numbers, principally polarized abelian varieties of dimension g are isomorphic to complex tori. In particular, they can always be written as $\mathbf{C}^g/\Lambda$ for a lattice $\Lambda=\mathbf{Z}^g+\mathbf{Z}\tau_1+\cdots+\mathbf{Z}\tau_g$, where $\tau_1,\ldots,\tau_g$ denote the rows of a complex $g\times g$-matrix τ of maximal rank. On the other hand, not every complex torus admits a polarization, i.e., not all of them are projective varieties. More specifically, by the Riemann bilinear relations, a complex torus of the form $\mathbf{C}^g/\Lambda$ admits a principal polarization $[L]$ if and only if the matrix τ is symmetric with positive definite imaginary part, i.e. if τ is an element of *Siegel space*

$$\mathbf{H}_g:=\{\tau\in M(g\times g,\mathbf{V}):\ \tau={}^t\tau,\Im(\tau)>0\}.$$

In this case, the theory of theta functions provides a canonical choice of the principal polarization on $\mathbf{C}^g/\Lambda$.

It is also possible to describe explicitly when two principally polarized abelian varieties $\mathbf{C}^g/(\mathbf{Z}^g+\tau\mathbf{Z})$ and $\mathbf{C}^g/\tau'\mathbf{Z}$ are isomorphic. Let us consider the symplectic group

$$\mathrm{Sp}(2g,\mathbf{Z})=\left\{\gamma\in\mathrm{GL}(2g,\mathbf{Z}):\ \gamma\left(\begin{array}{c|c}0&\mathbf{1}_g\\\hline -\mathbf{1}_g&0\end{array}\right){}^t\gamma=\left(\begin{array}{c|c}0&\mathbf{1}_g\\\hline -\mathbf{1}_g&0\end{array}\right)\right\}.$$

There is a natural action of the symplectic group on Siegel space, defined by

$$\begin{array}{ccc}\mathrm{Sp}(2g,\mathbf{Z})\times\mathbf{H}_g&\longrightarrow&\mathbf{H}_g\\ \left(\begin{pmatrix}A&B\\C&D\end{pmatrix},\tau\right)&\longmapsto&(A\tau+B)(C\tau+D)^{-1}.\end{array}$$

It is possible to prove that $\tau,\tau'\in\mathbf{H}_g$ give rise to isomorphic abelian varieties if and only if they lie in the same orbit for the action of $\mathrm{Sp}(2g,\mathbf{Z})$. Hence, the arithmetic quotient $\mathcal{A}_g = \mathbf{H}_g / \operatorname{Sp}(2g, \mathbf{Z})$ is the moduli space of g-dimensional ppavs. Whether the moduli space so constructed is a fine or a coarse one depends on the category in which we construct the quotient. The quotient stack yields a fine moduli space, which is again a smooth Deligne–Mumford stack. If we consider the quotient as an algebraic variety, we get the coarse moduli space of $\mathcal{A}_g$, which is a quasi-projective variety with locally quotient singularities.

Differently from the situation with the moduli space of curves and its Deligne–Mumford compactification, for the moduli space $\mathcal{A}_g$ one can consider several different compactifications, which are "natural" from different points of view. For instance, the theory of moduli forms allows to construct the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$, which is in some sense the minimal compactification of $\mathcal{A}_g$. However, this compactification does not have a modular interpretation and moreover it is also very singular. To solve this problem, Mumford et al. [AMRT75] introduced the theory of *toroidal compactifications* of $\mathcal{A}_g$. Their approach allows to construct a compactification of $\mathcal{A}_g$ starting from a certain combinatorial object: namely, an *admissible fan* in the cone of positive semidefinite quadratic forms in g variables. In this thesis we shall consider two of the most notable toroidal compactifications of $\mathcal{A}_g$: the perfect cone compactification $\mathcal{A}_g^{\mathrm{perf}}$ and the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$. The former is the toroidal compactification which is most natural from the point of view of the birational geometry [SB06]. The latter compactification is more natural from the point of moduli theory, because it is the normalization of the main component of the moduli stack of certain generalizations of principally polarized abelian varieties [Al02, Ol08].

Cohomological investigations

In this thesis, we deal with moduli spaces from the point of view of a basic topological invariant: the cohomology with rational coefficients. This is an important invariant, which can be used for instance to get information on the Chow groups of the spaces considered and on their compactifications. Furthermore, in the specific case of moduli spaces of curves and abelian varieties, the knowledge of the rational cohomology has applications in number theory (mainly in the study of modular forms) and in mathematical physics (string theory). We will concentrate on specific moduli spaces and describe the rational cohomology groups as graded vector spaces with mixed Hodge structures.

Curves of genus 3 with extra structure

In the first chapter, we calculate the cohomology of the moduli space $\mathcal{M}_{3,2}$ of curves of genus 3 with 2 distinct marked points. The starting point is the well known fact that a non-hyperelliptic curve of genus 2 can be canonically embedded as a smooth cruve of degree 4 in the projective plane $\mathbf{P}^2$. If we denote by $\mathcal{H}_{3,2}$ the locus of hyperelliptic curves in $\mathcal{M}_{3,2}$, we can interpret $\mathcal{M}_{3,2}\setminus\mathcal{H}_{3,2}$ as parametrizing equivalence classes of triples (φ, p, q) where φ is a non-singular homogeneous quartic polynomial in three variables and p, q are two distinct points in $\mathbf{P}^2$ lying in the vanishing locus of φ, up to the natural action of GL(3) on the coordinates. On the one hand, the quotient map from the space $\mathcal{I}_2$ parametrizing the triples (φ, p, q) to $\mathcal{M}_{3,2} \setminus \mathcal{H}_{3,2}$ allows to deduce the cohomology of $\mathcal{M}_{3,2}$ from that of $\mathcal{I}_2$ in a straightforward way via a Leray spectral sequence. On the other hand, the cohomology of $\mathcal{I}_2$ can be computed explicitly. This part of the computation is based on a development of Vassiliev–Gorinov's method for the computation of complements of discriminants (see [Vas99, Gor05, Tom05]). This requires to study the Borel–Moore homology of the space $\mathcal{D}$ of triples (φ, p, q) where the polynomial φ is *singular*. The space $\mathcal{D}$ possesses a structure given by the singular locus of its elements. This allows to construct a cubical space whose geometric realization is properly homotopy equivalent to that of $\mathcal{D}$, so that the Borel–Moore homology of both spaces coincides. Furthermore, this geometric realisation admits a natural filtration that defines a spectral sequence converging to the Borel–Moore homology of $\mathcal{D}$. From the Borel–Moore homology of $\mathcal{D}$ we recover the cohomology of $\mathcal{I}_2$ by combining a long exact sequence and the duality between Borel–Moore homology and cohomology.

It is important to observe that this whole construction respects mixed Hodge structures and the natural action of the symmetric group $\mathfrak{S}_2$ by interchanging the two marked points. Therefore, in our result we get a description of $H^\bullet(\mathcal{M}_{3,2}; \mathbf{Q})$ including its structure as $\mathfrak{S}_2$-representation and its mixed Hodge structures.

In the second chapter, we use similar techniques to approach a different task: determining the cohomology of the moduli space of smooth spin curves of genus 3. Smooth spin curves of genus g are pairs (C, L) where C is a smooth genus g curve and L is a *theta characteristic* on C, i.e. a line bundle satisfying $L^{\otimes 2} \cong \omega_C$. A spin structure is called odd (respectively, even) if the dimension of $H^0(C; L)$ is odd (respectively, even). The situation with moduli spaces of spin curves is similar as with moduli spaces of curves: they are natural objects that originate from classical constructions and whose study has also a string-theoretical motivation. Furthermore, by work of Cornalba [Co89], these moduli spaces admit a natural modular compactification called the moduli space of stable spin curves. Unfortunately, a complete description of the cohomology of these moduli spaces is available only for genus $g \leq 2$. In these cases, the descriptions rely heavily on the fact that the curves of genus ≤ 2 are either rational, or elliptic or hyperelliptic. This allows to get a combinatorical description of theta characteristics on smooth curves starting from configurations of points on $\mathbf{P}^1$.

Genus 3 is the lowest genus such that the general curve is no longer hyperelliptic. This makes the study of moduli of spin curves more interesting. In chapter II we compute the cohomology of the moduli space of smooth non-hyperelliptic odd spin curves of genus 3. This gives an independent and more explicit proof of a result of Looijenga [Lo93, Cor. 4.5], obtained by a completely different construction of this space in terms of the complement of an arrangement of divisors on an algebraic torus.

The starting point of our approach is the classical construction that identifies non-hyperelliptic odd spin curves of genus 3 with plane quartic curves with a marked bitangent line up to the action of PGL(3). The strategy of the proof is similar to that of the first chapter, but is more involved, partially due to the fact that one needs to distinguish between the cases in which the bitangent is a proper one (i.e. with two distinct contact points) or whether it is a flex bitangent (i.e. with one contact point with multiplicity 4).

Orbifold cohomology of moduli of curves

The subject of chapter III is the orbifold cohomology of moduli spaces of curves. Orbifold cohomology, also known as Chen–Ruan cohomology, has been introduced by Chen and Ruan in [CR04] as part of the project of extending Gromov–Witten invariants to orbifolds.

As a vector space, the orbifold cohomology with $\mathbf{Q}$-coefficients of an orbifold X coincides with the ordinary cohomology of the inertia stack of X, which is (loosely speaking) the stack parametrizing pairs (x, g) where x is an object of X and $g: x \to x$ is an automorphism of x. The grading of the orbifold cohomology is obtained by shifting the grading of the ordinary cohomology of each connected component of the inertia stack by a rational number, called *age* or *fermionic twist*.

In chapter III, which is joint work with Nicola Pagani (KTH, Stockholm), we study the inertia stack and the orbifold cohomology of the moduli space $\mathcal{M}_g$ of smooth genus g curves. In this case, the inertia stack $I(\mathcal{M}_g)$ can be viewed as the moduli space of pairs (C, α) where C is a smooth genus g and $\alpha: C \to C$ an automorphism. Following an idea of Fantechi, we describe all components of the inertia stack of $\mathcal{M}_g$ by associating to each of them certain numerical invariants of the cyclic cover $C \to C/\alpha$ associated. These invariants arise from the classical theory of abelian covers developed by Pardini in [Par91]. Furthermore, by a recent result of Catanese [Ca10] the invariants of the cover identify the connected components of $I(\mathcal{M}_g)$.

Subsequently, we apply this theory and work out the details in the case of curves of genus 3. In this special case we give a complete description of the additive structure of orbifold cohomology of $\mathcal{M}_3$ by identifying all components of $I(\mathcal{M}_3)$ and computing the rational cohomology of each of them by a variety of techniques. Furthermore, we consider the closure of the components $I(\mathcal{M}_3)$ in the inertia stack of the Deligne–Mumford compactification $\overline{\mathcal{M}}_3$. We calculate the cohomology of these components and the orbifold cohomology of the subring they generate inside the orbifold cohomology of $\overline{\mathcal{M}}_3$.

Moduli of abelian varieties

In chapter IV, which is based on joint work with Klaus Hulek, we deal with the cohomology of the moduli space of abelian varieties of dimension 4 and, more specifically, with the cohomology of two of its toroidal compactifications: the perfect cone compactification $\mathcal{A}_4^{\mathrm{perf}}$ and the second Voronoi compactification $\mathcal{A}_4^{\mathrm{Vor}}$. In the specific case of genus 4, these two compactifications are strictly related, because $\mathcal{A}_4^{\mathrm{Vor}}$, which is a smooth stack, can be obtained as the blow-up of $\mathcal{A}_4^{\mathrm{perf}}$ in one point.

The aim of our work is to gain cohomological information on the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ and $\mathcal{A}_4^{\mathrm{perf}}$ starting from known information coming either from moduli spaces of abelian varieties of smaller dimension or from moduli spaces of curves. To this end, we exploit the combinatorial structure of toroidal compactifications, which allows to stratify the boundary of them into locally closed subvarieties that are fibred over moduli spaces of abelian varieties of smaller dimension. A geometric analysis of these strata allows to compute their cohomology using certain Leray spectral sequences from the cohomology of $\mathcal{A}_k$ for $k \leq 3$ with values in certain symplectic local systems of small weight. In turn, the cohomology of these local systems can be computed using information of moduli spaces of pointed curves, such as Theorem I.1.1 in this thesis.

As for the interior $\mathcal{A}_4$, we do not deal directly with its cohomology, but only with the cohomology of the Zariski closure of the image of the Torelli map $\mathcal{M}_4 \to \mathcal{A}_4$, which is an irreducible hypersurface. The knowledge of the cohomology of this divisor combined with the information from the cohomology of the boundary suffices to compute the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ in all degrees different from the middle one and to compute the cohomology of $\mathcal{A}_4^{\mathrm{perf}}$ in degree ≤ 9.

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Chapter I

Rational cohomology of $\mathcal{M}_{3,2}$

I.1 Introduction

Let us denote by $\mathcal{M}_{3,2}$ the moduli space of non-singular complex projective curves with two marked points, and by $\mathcal{Q}_2$ the moduli space of plane quartic curves with two marked points. In this paper, we prove

Theorem I.1.1. *The rational cohomology groups of $\mathcal{Q}_2$ and $\mathcal{M}_{3,2}$, with their mixed Hodge structures and their structures as $\mathfrak{S}_2$-representations, are as follows.*

1. $$H^k(\mathcal{Q}_2;\mathbf{Q}) = \begin{cases} \mathbf{S}_2 \otimes \mathbf{Q} & k=0,\\ (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-1) & k=2,\\ \mathbf{S}_2\otimes\mathbf{Q}(-3) & k=5,\\ \mathbf{S}_2\otimes\mathbf{Q}(-6) & k=6,\\ (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-7)+\mathbf{S}_{1,1}\otimes\mathbf{Q}(-8) & k=8,\\ 0 & \text{otherwise.}\end{cases}$$

2. $$H^k(\mathcal{M}_{3,2};\mathbf{Q}) = \begin{cases} \mathbf{S}_2 \otimes \mathbf{Q} & k=0,\\ (\bigoplus^2\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-1) & k=2,\\ (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-2) & k=4,\\ \mathbf{S}_2\otimes\mathbf{Q}(-3) & k=5,\\ \mathbf{S}_2\otimes\mathbf{Q}(-6) & k=6,\\ (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-7) & k=8,\\ 0 & \text{otherwise.}\end{cases}$$

Note that the Euler characteristic of $\mathcal{M}_{3,2}$ in the Grothendieck group of mixed Hodge structures was computed in [BT07], exploiting Jonas Bergström's count of the number of points of $\mathcal{M}_{3,2}$ defined over finite fields ([Ber]). Another computation of this Euler characteristic can be found in [Tom05b, Chapter III]. All these results agree with the topological Euler characteristic of $\mathcal{M}_{3,2}$ as calculated in [BH].

O. Tommasi, *Rational cohomology of $\mathcal{M}_{3,2}$*. Compos. Math. **143** (2007), no. 4, 986–1002.

It is well known that the canonical model of a non-hyperelliptic curve of genus 3 is a smooth quartic in the projective plane. Hence $\mathcal{Q}_2$ is the complement of the hyperelliptic locus $\mathcal{H}_{3,2}$ inside $\mathcal{M}_{3,2}$. Let us start by considering the moduli space $\mathcal{Q}$ of smooth quartic curves in the projective plane. Quartic curves are defined by the vanishing of polynomials of degree four in three indeterminates, i.e., by elements of $S_4^2 := \mathbf{C}[x_0, x_1, x_2]_4$. Clearly, not every element of S_4^2 defines a non-singular curve, but we have to exclude the locus $\Sigma_4^2 \subset S_4^2$ of singular polynomials. The action of $G = \mathrm{GL}(3)$ on the coordinates x_0, x_1, x_2 induces an action on $S_4^2 \setminus \Sigma_4^2$, and $\mathcal{Q}$ is the the geometric quotient of $S_4^2 \setminus \Sigma_4^2$ by the action of G.

The rational cohomology of $S_4^2 \setminus \Sigma_4^2$ was computed by Vassiliev in [Vas99]. Comparing this result with the rational cohomology of the moduli space $\mathcal{Q}$, as computed by Looijenga in [Loo93], one observes that the cohomology of the space of non-singular polynomials in S_4^2 is isomorphic (as graded vector space) to the tensor product of the cohomology of the moduli space $\mathcal{Q}$ and that of $G = \mathrm{GL}(3)$. Indeed, Peters and Steenbrink [PS03] proved that this is always the case when comparing the rational cohomology of the space of non-singular homogeneous polynomials with the cohomology of the corresponding moduli space of smooth hypersurfaces.

As explained in [BT07, § 5], Peters–Steenbrink's result can be adapted to moduli spaces of smooth hypersurfaces with m marked points, when m is small enough. This requires to replace the space S_d^n of homogeneous polynomials of degree d in $x_0, \ldots, x_n$ with a certain incidence correspondence. In our case ($n = 2, d = 4, m = 2$) we set

$$\mathcal{I}_2 := \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) \times (S_4^2 \setminus \Sigma_4^2) : f(\alpha) = f(\beta) = 0\},$$

where $F(\mathbf{P}^2, 2)$ denotes the complement of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$. The action of $G = \mathrm{GL}(3)$ on $\mathbf{P}^2$ and S_4^2 can be extended to $\mathcal{I}_2$, and the geometric quotient $\mathcal{I}_2/G$ is isomorphic to $\mathcal{Q}_2$. Then the following isomorphism of graded vector spaces with mixed Hodge structures holds:

$$H^\bullet(\mathcal{I}_2; \mathbf{Q}) \cong H^\bullet(\mathcal{Q}_2; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3); \mathbf{Q}). \tag{I.1.1}$$

This follows from [PS03], in view of [BT07, Theorem 5.2]. As a consequence, we have that determining the rational cohomology of $\mathcal{I}_2$ immediately yields the rational cohomology of $\mathcal{Q}_2$. Note that the isomorphism (I.1.1) is compatible with the action of the symmetric group $\mathfrak{S}_2$ on the cohomology groups of $\mathcal{Q}_2$ and $\mathcal{I}_2$ induced by the involution interchanging the two marked points.

We compute $H^\bullet(\mathcal{I}_2; \mathbf{Q})$ by studying the natural projection $\pi_2\colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. The map π_2 is a locally trivial fibration, whose fibre is the complement of Σ_4^2 in a linear subspace of S_4^2. Therefore, we can compute the cohomology of this fibre with Vassiliev–Gorinov's method for the cohomology of complements of discriminants. The study of the Leray spectral sequence associated to the fibration π_2

allows to determine the cohomology of $\mathcal{I}_2$. In this last step, we will use very often the relation (I.1.1).

The plan of the paper is as follows. In §§ I.2 and I.3 we compute the rational cohomology of $\mathcal{Q}_2$ and prove Theorem I.1.1 by the methods explained above. We conclude the paper with a concise review of Vassiliev–Gorinov's method.

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Notation

$\mathfrak{S}_n$	the symmetric group in n letters.
S^n_d	vector space of homogeneous polynomials of degree d in $n+1$ indeterminates $x_0, \dots, x_n$.
Σ^n_d	locus of singular polynomials in S^n_d.
$K_0(\mathsf{HS}_{\mathbf{Q}})$	Grothendieck group of rational (mixed) Hodge structures over $\mathbf{Q}$.
$K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$	Grothendieck group of rational (mixed) Hodge structures endowed with an $\mathfrak{S}_n$-action.
$\mathbf{Q}(m)$	Tate Hodge structure of weight $-2m$.
$\mathbf{L}$	class of $\mathbf{Q}(-1)$ in $K_0(\mathsf{HS}_{\mathbf{Q}})$.
$\mathbf{S}_\lambda$	$\mathbf{Q}$-representation of $\mathfrak{S}_n$ indexed by the partition $\lambda \vdash n$.
s_λ	Schur polynomial indexed by the partition $\lambda \vdash n$.
Δ_j	j-dimensional closed simplex.
Δ_j	interior of the j-dimensional closed simplex.
$F(Z,k)$	space of ordered configurations of k distinct points on the variety Z (see Def. I.4.5).
$B(Z,k)$	space of unordered configurations of k distinct points on the variety Z (see Def. I.4.5).

Throughout this paper we will make an extensive use of Borel–Moore homology, i.e., homology with locally finite support. A reference for its definition and the properties we use is for instance [Ful84, Chapter 19].

To write the results on cohomology and Borel–Moore homology groups in a compact way, we will express them by means of polynomials, in the following way. Let $T_\bullet$ denote a graded $\mathbf{Q}$-vector space with mixed Hodge structures. For every $i \in \mathbf{Z}$, we can consider the class $[T_i]$ in the Grothendieck group of rational Hodge structures. We define the Hodge–Grothendieck polynomial (for short, HG polynomial) of $T_\bullet$ to be the polynomial

$$\wp(T_\bullet) := \sum_{i\in\mathbf{Z}} [T_i]t^i \in K_0(\mathsf{HS}_{\mathbf{Q}})[t].$$

If moreover a symmetric group $\mathfrak{S}_n$ acts on $T_\bullet$ respecting the grading and the mixed Hodge structures on $T_\bullet$, we define the $\mathfrak{S}_n$-*equivariant* Hodge Grothendieck polynomial (for short, $\mathfrak{S}_n$-HG polynomial) $\wp^{\mathfrak{S}_n}(T_\bullet)$ by replacing $K_0(\mathsf{HS}_{\mathbf{Q}})$ by $K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$ in the definition of the HG polynomial.

I.2 Rational cohomology of $\mathcal{Q}_2$

Consider the space S_4^2 of homogeneous quartic polynomials in x_0, x_1, x_2, and denote by $\Sigma := \Sigma_4^2$ the discriminant, i.e., the locus of singular quartic polynomials. For every $p \in \mathbf{P}^2$, denote by V_p the linear subspace of S_4^2 of polynomials vanishing at p. The aim of this section is to calculate the rational cohomology of the incidence correspondence $\mathcal{I}_2 = \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) \times (S_4^2 \setminus \Sigma) : f(\alpha) = f(\beta) = 0\}$. Note that knowing the cohomology of $\mathcal{I}_2$ is equivalent to knowing the cohomology of its projectivization

$$\mathcal{P}_2 = \{(\alpha, \beta, [f]) \in F(\mathbf{P}^2, 2) \times \mathbf{P}(S_4^2 \setminus \Sigma) : f(\alpha) = f(\beta) = 0\},$$

as the rational cohomology of $\mathcal{I}_2$ is isomorphic to the tensor product of the cohomology of $\mathcal{P}_2$ and $H^\bullet(\mathbf{C}^*; \mathbf{Q})$.

We will start by applying Vassiliev–Gorinov's method (see § I.4) to the calculation of the cohomology of $(V_p \cap V_q) \setminus \Sigma$, where p and q are two fixed distinct points in $\mathbf{P}^2$. Next, we will consider the Leray spectral sequence for the natural projection $\pi_2\colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. Note that the map π_2 is a locally trivial fibration with fibre isomorphic to $V_p \cap V_q \setminus \Sigma$.

By Alexander's duality between reduced cohomology and Borel–Moore homology, we have

$$\tilde{H}^\bullet((V_p \cap V_q) \setminus \Sigma; \mathbf{Q}) \cong \bar{H}_{25-\bullet}(V_p \cap V_q \cap \Sigma; \mathbf{Q})(-13). \tag{I.2.1}$$

To apply Vassiliev–Gorinov's method to $V_p \cap V_q \cap \Sigma$, we need an ordered list of all possible singular sets of the elements in $V_p \cap V_q \cap \Sigma$. We can easily obtain such a list by an adaptation of the list of possible singular configurations of quartic curves (like the one in [Vas99, Proposition 6]). For every configuration in the list, one has to distinguish further whether the singular points are or are not in general position with respect to p and q (for instance, if p or q are or are not contained in the singular configuration). This procedure yields a complete list of singular sets of elements of $V_p \cap V_q$; let us denote by R the number of types of configurations in the list. As recalled in § I.4, Vassiliev–Gorinov's method gives a recipe to construct spaces $|\mathcal{X}|$, $|\Lambda|$ and a map

$$|\epsilon|\colon\ |\mathcal{X}| \longrightarrow V_p \cap V_q \cap \Sigma$$

inducing an isomorphism on Borel–Moore homology. The Borel–Moore homology of $|\mathcal{X}|$ (respectively, $|\Lambda|$) can be computed by considering the stratification $\{F_j\}_{j=1,\dots,R}$ (resp., $\{\Phi_j\}_{j=1,\dots,R}$). The properties of F_j and Φ_j are explained in Proposition I.4.3. Recall in particular that F_j is the total space of a vector bundle over Φ_j, and that for finite configurations the Borel–Moore homology of Φ_j coincides (after a shift in the indices) with the Borel–Moore homology of the space of configurations of type j with coefficients in a rank 1 local system changing its orientation every time two points in a configuration are interchanged.

In our case, for most indices $j \in \{1,\dots,R\}$ the space of singular configurations of type j has trivial Borel–Moore homology in the appropriate system of coefficients. Hence, the strata F_j have trivial Borel–Moore homology. In view of Lemma I.4.6, this is the case for configurations with too many points lying on the same rational curve. Furthermore, the same occurs for configurations containing rational curves as components (see [Tom05a, Lemma 2.17] and following remarks).

In Table I.2 we list all remaining configurations, i.e., all singular configurations indexing strata that give a non-trivial contribution to the Borel–Moore homology of $\Sigma \cap V_p \cap V_q$. In the same table, we also give a description of the strata of $|\Lambda|$ and $|\mathcal{X}|$ corresponding to each configuration. From the descriptions, it is straightforward to compute the Borel–Moore homology of the strata Φ_j and F_j for $1 \le j \le 7$. The most difficult strata (corresponding to configurations of type 8, 9 and 10) are studied separately in § I.3. The results there, together with the description of the strata given in Table I.2, allow to compute the E^1 terms of the spectral sequences in Borel–Moore homology converging to $|\Lambda|$ and $|\mathcal{X}|$, induced by the filtrations associated, respectively, with $\{\Phi_j\}$ and $\{F_j\}$.

The columns of the spectral sequences converging to the Borel–Moore homology of $|\mathcal{X}|$ and $|\Lambda|$ can be divided into two blocks: one with the first seven columns, the other with columns 8, 9 and 10. Looking at Hodge weights, one can easily prove that all differentials in the spectral sequence between columns in the block 1–7 and in the block 8–10 are trivial. Furthermore, this behaviour carries on when one investigates the Leray spectral sequence associated to the fibration π_2. Therefore, we will consider the two blocks separately. The contribution of columns 1–7 is computed below. The contribution of columns 8–10 to the rational cohomology of $\mathcal{I}_2$ is computed in § I.3.

In the spectral sequence converging to the Borel–Moore homology of $|\Lambda|$ all terms in the first seven columns are killed by differentials, with the exception of an $\mathfrak{S}_2$-invariant 1-dimensional homology group in degree 0. This follows from dimensional reasons: If these classes were not killed, they would give rise to

Table I.2: Singular configurations and their contribution

1. The point p or the point q.
   Stratum: F_1 is a $\mathbf{C}^{11}$-bundle over $\Phi_1 = \{p, q\}$.
2. Any point different from p, q.
   Stratum: F_2 is a $\mathbf{C}^{10}$-bundle over $\Phi_2 \cong \mathbf{P}^2 \setminus \{p, q\}$.
3. The pair $\{p, q\}$.
   Stratum: F_3 is a $\mathbf{C}^9$-bundle over $\Phi_3 \cong \Delta_1$.

4a. Pairs of points on the line pq, different from $\{p, q\}$.
Stratum: F_{4a} is a $\mathbf{C}^8$-bundle over Φ_{4a}, which is a non-orientable Δ_1-bundle over a space which can be decomposed as the disjoint union of $\mathbf{C}^*$ and $B(\mathbf{C}, 2)$.

4b. Pairs of points $\{a, b\}$ with $a \in \{p, q\}$, $b \notin pq$.
Stratum: F_{4b} is a $\mathbf{C}^8$-bundle over Φ_{4b}, which is a non-orientable Δ_1-bundle over the disjoint union of two copies of $\mathbf{C}^2$.

5. Pairs of points $\{a, b\}$ with $a \in (pq \setminus \{p, q\})$, $b \notin pq$.
   Stratum: F_5 is a $\mathbf{C}^7$-bundle over Φ_5, which is a non-orientable Δ_1-bundle over $\mathbf{C}^* \times \mathbf{C}^2$.
6. Triplets consisting of p, q and another point outside pq.
   Stratum: F_6 is a $\mathbf{C}^6$-bundle over Φ_6, which is a Δ_2-bundle over $\mathbf{C}^2$.
7. Triplets with two points on pq (not both in $\{p, q\}$) and another point outside pq.
   Stratum: F_7 is a $\mathbf{C}^5$-bundle over Φ_7, which is a non-orientable Δ_2-bundle over a space that can be decomposed as the disjoint union of $\mathbf{C}^* \times \mathbf{C}^2$ and $B(\mathbf{C}, 2) \times \mathbf{C}^2$.
8. Five points $a, b, c, d, e \in \mathbf{P}^2$, such that a, b, d, e, p, q lie on a conic different from $ab \cup de$, $\{c\} = ab \cap de \not\subset \{p, q\}$ and $\{p, q\} \not\subset \{a, b, d, e\}$.
   Stratum: F_8 is a $\mathbf{C}$-bundle over Φ_8, which is a Δ_4-bundle over the configuration space X_8 of § I.3.
9. Six points that are the pairwise intersection of four lines ℓ_i $(1 \leq i \leq 4)$ in general position, such that $\{p, q\} \subset \bigcup_i \ell_i$.
   Stratum: F_9 is a $\mathbf{C}$-bundles over Φ_9, which is a Δ_5-bundle over the configuration spaces X_9 studied in § I.3. The simplices bundle does not change its orientation when two lines ℓ_i, ℓ_j are interchanged.
10. The entire $\mathbf{P}^2$.
    Stratum: F_{10} is an open cone over the space $|\Lambda|$, which is the union of all strata Φ_j with $j \leq 9$.

Table I.3: First seven columns of the spectral sequence converging to the Borel–Moore homology of $V_p \cap V_q \cap \Sigma$

22	0	$\mathbf{S}_2 \otimes \mathbf{Q}(12)$	0	0	0	0	0
21	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(11)$	0	0	0	0	0	0
20	0	$\mathbf{S}_2 \otimes \mathbf{Q}(11)$	0	0	0	0	0
19	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(10)$	0	0	0	0	0
18	0	0	0	0	0	0	0
17	0	0	0	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(10)$	0	0	0
16	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(9)$	0	$\mathbf{S}_2 \otimes \mathbf{Q}(10)$	0	0
15	0	0	0	$\mathbf{S}_2 \otimes \mathbf{Q}(9)$	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(9)$	0	0
14	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(8)$	0	0	0
13	0	0	0	0	0	0	0
12	0	0	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(8)$	0
11	0	0	0	0	0	0	$\mathbf{S}_2 \otimes \mathbf{Q}(8)$
10	0	0	0	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(7)$
	1	2	3	4	5	6	7

cohomology classes of degree ≥ 14 in the cohomology of $(V_p \cap V_q) \setminus \Sigma$, and this is impossible because the latter is affine of dimension 13. As a consequence, the strata 1–7 do not contribute to the Borel–Moore homology of the open cone F_{10}.

The first seven columns of the spectral sequence converging to the Borel–Moore homology of $\mathrm{Fil}_j \left|\mathcal{X}\right|$ are given in Table I.3. Note that the description of the strata of the domain $\left|\mathcal{X}\right|$ of the geometric realization given in Table I.2 allows us to study the behaviour of each Borel–Moore homology class with respect to the interchange of the points p, q. Table I.3 includes also the information on the $\mathfrak{S}_2$-action generated by this involution.

In the spectral sequence in Table I.3, the only possibly non-trivial differential is $d_2\colon E^2_{5,15} \to E^2_{3,16}$. This is certainly zero, because otherwise we would get a contradiction with the isomorphism

$$H^\bullet((V_p \cap V_q) \setminus \Sigma; \mathbf{Q}) \cong H^\bullet(\mathbf{C}^*; \mathbf{Q}) \otimes H^\bullet(\mathbf{P}((V_p \cap V_q) \setminus \Sigma); \mathbf{Q}).$$

In view of Alexander's duality (I.2.1), we have that the part of the rational cohomology of $V_p \cap V_q \setminus \Sigma$ that comes from the first seven columns of Vassiliev–Gorinov's spectral sequence has $\mathfrak{S}_2$-HG polynomial

$$(1 + \mathbf{L}t)\big(s_2 + \mathbf{L}^2t^3(2s_2 + s_{1,1}) + \mathbf{L}^3t^4 s_{1,1} + \mathbf{L}^4t^6(s_2 + s_{1,1}) + \mathbf{L}^5t^7 s_{1,1}\big), \quad \text{(I.2.2)}$$

where the $\mathfrak{S}_2$-action is generated by the involution interchanging p and q. Note that the second factor of (I.2.2) is the $\mathfrak{S}_2$-HG polynomial of the cohomology of the projectivization of $V_p \cap V_q \setminus \Sigma$.

Next, we study the contribution of this part of the cohomology of $V_p \cap V_q \setminus \Sigma$ to the Leray spectral sequence for the fibration $\pi_2\colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. It is simpler to consider the $\mathbf{C}^*$-quotient and study the fibration $\pi_2'\colon \mathcal{P}_2 \to F(\mathbf{P}^2, 2)$, which is a locally trivial fibration with fibre $\mathbf{P}(V_p \cap V_q \setminus \Sigma)$. The E_2 terms of the Leray spectral sequence are written in Table I.4. Note that the space $F(\mathbf{P}^2, 2)$ is

Table I.4: First block of the Leray spectral sequence in cohomology associated to π_2'

	0	1	2	3	4	5	6
7	$\mathbf{S}_{1,1}\otimes\mathbf{Q}(-5)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-6)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-7)$	0	$\mathbf{S}_2\otimes\mathbf{Q}(-8)$
6	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-4)$	0	$(\bigoplus^2\mathbf{S}_2+\bigoplus^2\mathbf{S}_{1,1})\otimes\mathbf{Q}(-5)$	0	$(\bigoplus^2\mathbf{S}_2+\bigoplus^2\mathbf{S}_{1,1})\otimes\mathbf{Q}(-6)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-7)$
5	0	0	0	0	0	0	0
4	$\mathbf{S}_{1,1}\otimes\mathbf{Q}(-3)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-4)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-5)$	0	$\mathbf{S}_2\otimes\mathbf{Q}(-6)$
3	$(\bigoplus^2\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-2)$	0	$(\bigoplus^3\mathbf{S}_2+\bigoplus^3\mathbf{S}_{1,1})\otimes\mathbf{Q}(-3)$	0	$(\bigoplus^3\mathbf{S}_2+\bigoplus^3\mathbf{S}_{1,1})\otimes\mathbf{Q}(-4)$	0	$(\mathbf{S}_2+\bigoplus^2\mathbf{S}_{1,1})\otimes\mathbf{Q}(-5)$
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	$\mathbf{S}_2$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-1)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(-2)$	0	$\mathbf{S}_{1,1}\otimes\mathbf{Q}(-3)$

simply connected and that its cohomology has $\mathfrak{S}_2$-HG polynomial $(s_2+\mathbf{L}ts_{1,1})(1+\mathbf{L}t^2+\mathbf{L}^2t^4)$ with respect to the natural action of $\mathfrak{S}_2$ generated by the involution $(\alpha,\beta)\leftrightarrow(\beta,\alpha)$.

The proof of the following lemma is based on a suggestion by Alexei Gorinov.

Lemma I.2.1. *In the spectral sequence associated to π_2', the differential $d_4\colon E_4^{0,3}\to E_4^{4,0}$ has rank two.*

Proof. Denote by $\mathcal{P}_1\subset\mathbf{P}^2\times\mathbf{P}(S_4^2\setminus\Sigma)$ the variety of pairs $(\xi,[f])$ such that $f(\xi)=0$. Consider the inclusion $i\colon\mathcal{P}_2\to\mathcal{P}_1\times\mathcal{P}_1$ defined by $i(\alpha,\beta,[f])=((\alpha,[f]),(\beta,[f]))$. There is a commutative diagram

$$\begin{array}{ccc}\mathcal{P}_2 & \xrightarrow{i} & \mathcal{P}_1\times\mathcal{P}_1\\ {\scriptstyle\pi_2'}\big\downarrow & & \big\downarrow{\scriptstyle\pi_1\times\pi_1}\\ F(\mathbf{P}^2,2) & \xrightarrow{\text{inclusion}} & \mathbf{P}^2\times\mathbf{P}^2,\end{array}\tag{I.2.3}$$

where $\pi_1\colon\mathcal{P}_1\to\mathbf{P}^2$ denotes the natural projection.

In particular, the differentials of the spectral sequences associated to π_2' and $\pi_1\times\pi_1$ commute with the maps induced by i on the E_4 terms of the spectral sequences.

Recall from [BT07, Proposition 1] that the differential

$$E_4^{0,3}(\pi_1) \xrightarrow{d_4} E_4^{4,0}(\pi_1)$$

has rank 1. This implies that the differential $E_4^{0,3}(\pi_1 \times \pi_1) \xrightarrow{d_4} E_4^{4,0}(\pi_1 \times \pi_1)$ has rank 2. Since

$$E_4^{4,0}(\pi_1 \times \pi_1) \cong E_2^{4,0}(\pi_1 \times \pi_1) \cong H^4(\mathbf{P}^2 \times \mathbf{P}^2; \mathbf{Q}) \otimes H^0(V_p \setminus \Sigma; \mathbf{Q})$$

and

$$E_4^{4,0}(\pi_2') \cong E_2^{4,0}(\pi_2') \cong H^4(F(\mathbf{P}^2, 2); \mathbf{Q}) \otimes H^0(V_p \cap V_q \setminus \Sigma; \mathbf{Q}),$$

one can verify directly that the composition of $d_4\colon\ E_4^{0,3}(\pi_1 \times \pi_1) \to E_4^{4,0}(\pi_1 \times \pi_1)$ and the map $E_4^{4,0}(\pi_1 \times \pi_1) \to E_4^{4,0}(\pi_2')$ is surjective. Then the claim follows from the commutativity of the diagram I.2.3. $\square$

Proof of Theorem I.1.1. Comparing the Leray–Hirsch isomorphism (I.1.1) with Table I.4 implies that the entire contribution of the first block to the cohomology of $\mathcal{I}_2$ is determined by the cohomology of $\mathcal{I}_2$ in degree ≤ 5. This follows from the fact that the cohomology of GL(3) is trivial in degree $k \geq 10$. Then Lemma I.2.1, together with the structure of $H^\bullet(\mathcal{I}_2; \mathbf{Q})$ as tensor product of GL(3), yields that the first block contributes

$$(1 + \mathbf{L}t)(1 + \mathbf{L}^2t^3)(1 + \mathbf{L}^3t^5)(s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3t^5s_2).$$

to the $\mathfrak{S}_2$-HG polynomial of $H^\bullet(\mathcal{I}_2; \mathbf{Q})$. This implies that the $\mathfrak{S}_2$-HG polynomial of the cohomology of the moduli space $\mathcal{Q}_2$ of smooth quartic curves with two marked points is

$$s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3t^5s_2, \tag{I.2.4}$$

plus the term coming from singular configurations of type 8–10. In the next section (see page 20), we will prove that this term equals

$$\mathbf{L}^6t^6s_2 + \mathbf{L}^7t^8(s_2 + s_{1,1}) + \mathbf{L}^8t^8s_{1,1}. \tag{I.3.9}$$

Summing the contributions of the two block of columns, we get that the $\mathfrak{S}_2$-HG polynomial of the cohomology of $\mathcal{Q}_2$ is

$$s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3t^5s_2 + \mathbf{L}^6t^6s_2 + \mathbf{L}^7t^8(s_2 + s_{1,1}) + \mathbf{L}^8t^8s_{1,1}.$$

This establishes the first part of Theorem I.1.1. To prove the second part of the theorem, recall from [Tom05b, Corollary III.2.2] that the $\mathfrak{S}_2$-HG polynomial of the cohomology of the hyperelliptic locus $\mathcal{H}_{3,2} \subset \mathcal{M}_{3,2}$ is $s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^7t^7s_{1,1}$, and consider the long exact sequence associated to the inclusion $\mathcal{H}_{3,2} \hookrightarrow \mathcal{M}_{3,2}$:

$$\to H^k(\mathcal{M}_{3,2}; \mathbf{Q}) \to H^k(\mathcal{Q}_2; \mathbf{Q}) \to H^{k-1}(\mathcal{H}_{3,2}; \mathbf{Q}) \otimes \mathbf{Q}(-1) \to H^{k+1}(\mathcal{M}_{3,2}; \mathbf{Q}) \to$$

which can be rephrased in Borel–Moore homology as

$$\cdots \to \bar{H}_k(\mathcal{M}_{3,2};\mathbf{Q}) \to \bar{H}_k(\mathcal{Q}_2;\mathbf{Q}) \xrightarrow{d_k} \bar{H}_{k-1}(\mathcal{H}_{3,2};\mathbf{Q}) \to \bar{H}_{k-1}(\mathcal{M}_{3,2};\mathbf{Q}) \to \cdots .$$

If $k \neq 8$, the differentials d_k are always zero for Hodge-theoretic reasons. If $k = 8$, both $\bar{H}_8(\mathcal{Q}_2;\mathbf{Q})$ and $\bar{H}_7(\mathcal{H}_{3,2};\mathbf{Q})$ have a one-dimensional summand of Hodge weight 0, on which $\mathfrak{S}_2$ acts as the sign representation. Hence, a priori d_8 can have either rank 0 or 1. To determine the rank of d_8, we observe that both the cohomology of $\mathcal{H}_{3,2}$ and $\mathcal{Q}_2$ were computed using Vassiliev–Gorinov's method. In particular, both $\bar{H}_8(\mathcal{Q}_2;\mathbf{Q})$ and $\bar{H}_7(\mathcal{H}_{3,2};\mathbf{Q})$ are related to configurations of at least 4 singular points. Moreover, there configurations correspond to strata of the geometric realizations that have Borel–Moore homology which is a tensor product of that of the group acting. This means that both Borel–Moore homology groups can be interpreted as Borel–Moore homology groups of certain moduli spaces.

Specifically, consider the moduli space $\mathcal{N}$ whose elements are isomorphism classes of triples (C,p,q), where C is the union of two smooth rational curves intersecting transversally at 4 distinct points and p,q are any distinct (but possibly singular) points on C. Note that the arithmetic genus of such a curve C is 3. Denote by S the rank 1 local system on $\mathcal{N}$ changing its orientation every time a pair of nodes on C is interchanged, and denote by $\mathcal{N}_h$ the closed subset of $\mathcal{N}$ such that the four nodes have the same moduli on both rational components.

Observe that the problem with the determination of d_8 only concerns the Hodge weight 0 summands of the Borel-Moore homology groups. For this reason, in the rest of the proof we will restrict to the Hodge weight 0 summands of each homology group we consider.

The space $\mathcal{N}$ can be written as the disjoint union of locally closed strata, each of them isomorphic to the quotient by the action of a finite group, of a product of moduli spaces $\mathcal{M}_{0,n}$ with $4 \leq n \leq 6$, whose cohomology groups are completely known (see e.g. [Get95]). Investigating this stratification, one gets that the only Borel–Moore homology group of Hodge weight 0 is $\bar{H}_4(\mathcal{N};S) = \mathbf{S}_{1,1}$. Analogous considerations also apply to $\mathcal{N}_h$. In that case, one has $\bar{H}_3(\mathcal{N}_h;S) \cong \mathbf{S}_{1,1}$ as only Borel–Moore homology group with Hodge weight 0. By the constructions in [Tom05b, III.2], there is a natural isomorphism $\bar{H}_3(\mathcal{N}_h;S) \cong \bar{H}_7(\mathcal{H}_{3,2};\mathbf{Q})$.

The weight 0 part of the Borel–Moore homology of $\mathcal{N} \setminus \mathcal{N}_h$ can also be computed directly with Vassiliev–Gorinov's method. This yields again that the only non-trivial Borel–More homology group with S-coefficient of $\mathcal{N} \setminus \mathcal{N}_h$ is $\bigoplus_2 \mathbf{S}_{1,1}$ in degree 4. Moreover, the direct computation shows that $\bar{H}_4(\mathcal{N} \setminus \mathcal{N}_h;S)$ is generated by two classes, both related to configurations of type (9) in Table I.2. This allows to define a surjective map $\bar{H}_4(\mathcal{N} \setminus \mathcal{N}_h;S) \to \bar{H}_8(\mathcal{Q}_2;\mathbf{Q})$ making the following diagram commute:

$$\begin{array}{ccccccccc}
 & & \mathbf{S}_{1,1} & & \bigoplus_2 \mathbf{S}_{1,1} & & \mathbf{S}_{1,1} & & \\
 & & \| \wr & & \| \wr & & \| \wr & & \\
0 & \longrightarrow & \bar{H}_4(\mathcal{N};S) & \longrightarrow & \bar{H}_4(\mathcal{N}\setminus\mathcal{N}_h;S) & \xrightarrow{\gamma} & \bar{H}_3(\mathcal{N}_h;S) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bar{H}_8(\mathcal{Q}_2;\mathbf{Q}) & \xrightarrow{d_8} & \bar{H}_7(\mathcal{H}_{3,2};\mathbf{Q}). & &
\end{array}$$

The commutativity of the diagram immediately yields $\operatorname{rank} d_8 = \operatorname{rank}\gamma = 1$. $\square$

I.3 Configurations of five and six points

The aim of this section is to compute the contribution of singular configurations of type 8, 9 and 10 (see Table I.2) to the rational cohomology of $\mathcal{I}_2$ and $\mathcal{Q}_2$. For these configuration, it seems more natural to work directly with the cohomology of $\mathcal{I}_2$, without having to pass through the study of the fibre of π_2. This is indeed possible. Namely, consider the space

$$\mathcal{D} := \{(\alpha,\beta,f)\in F(\mathbf{P}^2,2)\times\Sigma : f(\alpha)=f(\beta)=0\}.$$

Note that $\mathcal{D}$ is a closed subset of $\mathcal{V} := \{(\alpha,\beta,f)\in F(\mathbf{P}^2,2) : f(\alpha)=f(\beta)=0\}$. The space $\mathcal{V}$ is the total space of a vector bundle over $\mathbf{P}^2$, and $\mathcal{I}_2=\mathcal{V}\setminus\mathcal{D}$. Vassiliev–Gorinov's method can be exploited to compute the Borel–Moore homology of $\mathcal{D}$. This is done by defining the singular locus of an element (α,β,f) in $\mathcal{D}$ as the subset $\{(\alpha,\beta)\}\times K_f$ of $F(\mathbf{P}^2,2)\times\mathbf{P}^2$, where K_f denotes the singular locus of the polynomial f. In particular, the classification of singular sets of elements of $\mathcal{D}$ is obtained from the classification of singular sets of elements of $V_p\cap V_q\setminus\Sigma$ by allowing the pair (p,q) to move in $F(\mathbf{P}^2,2)$.

Even though this is no longer the original setting of Vassiliev–Gorinov's method, one can mimic the construction of the cubical spaces Λ and $\mathcal{X}$ (see § I.4), and obtain cubical spaces Λ' and $\mathcal{X}'$ that play an analogous role. In particular, the map $\left|\mathcal{X}'\right|\to\mathcal{D}$ induces an isomorphism on the Borel–Moore homology of these spaces, because it is a proper map with contractible fibres. Moreover, for the stratifications Φ' and F' obtained from the construction of Λ' and $\mathcal{X}'$, we have natural maps $\Phi'_k\to F(\mathbf{P}^2,2)$ and $F'_k\to F(\mathbf{P}^2,2)$ which are locally trivial fibrations with fibre isomorphic to Φ_k, respectively, F_k.

In view of the considerations above, computing the rational Borel–Moore homology of the spaces Φ'_j and F'_j for $j\in\{8,9,10\}$ is enough to get the contribution of configurations of type 8–10 to the cohomology of $\mathcal{I}_2$. We start by determining the twisted Borel–Moore homology of the underlying families of configurations X'_8 and X'_9.

Define $Y_8\subset F(\mathbf{P}^2,2)\times F(\mathbf{P}^2,5)$ to be the space of configurations (p,q,e_1,e_2,e_3,e_4,c) such that

- $\{c\} = e_1e_2 \cap e_3e_4 \not\subset \{p,q\}$;
- p,q,e_1,e_2,e_3,e_4 lie on a conic different from the reducible conic $e_1e_2 \cup e_3e_4$;
- $\{p,q\} \not\subset \{e_1,e_2,e_3,e_4\}$.

Then $X_8' \subset F(\mathbf{P}^2,2)\times B(\mathbf{P}^2,5)$ is isomorphic to the quotient of Y_8 by the action of the subgroup G of $\mathfrak{S}_4$ generated by the permutations $(1,2)$, $(3,4)$ and $(1,3)(2,4)$. The action of G on Y_8 is given by permuting the four points (e_1,e_2,e_3,e_4) in the configurations. Since G is a subgroup of $\mathfrak{S}_4$, it makes sense to restrict the sign representation to it.

Furthermore, note that the conic passing through the points p,q,e_1,e_2,e_3,e_4 is uniquely determined for every configuration in Y_8. Therefore, Y_8 can be embedded in the space $W \subset F(\mathbf{P}^2,2) \times F(\mathbf{P}^2,5) \times \mathbf{P}(S_2^2)$ of configurations $(p,q,e_1,e_2,e_3,e_4,c,C)$, such that

- $\{c\} = e_1e_2 \cap e_3e_4 \not\subset \{p,q\}$;
- the points p,q,e_1,e_2,e_3,e_4 lie on the conic C;
- the conic C is distinct from the reducible conic $e_1e_2 \cup e_3e_4$.

Hence, we have the chain of inclusions

$$Y_8 \hookrightarrow W \hookrightarrow F(\mathbf{P}^2,2) \times F(\mathbf{P}^2,5) \times \mathbf{P}(S_2^2).$$

Lemma I.3.1. *Denote by S the local system of coefficients induced on W/G and X_8' by the sign representation on G. Consider the $\mathfrak{S}_2$-action generated by the involution interchanging the points $(p,q) \in F(\mathbf{P}^2,2)$. Then one has*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X_8';S)) = (\mathbf{L}^{-1}t^3 + \mathbf{L}^{-2}t^4)s_2 \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})).$$

To prove Lemma I.3.1, we will consider the quotient of Y_8 and W by the action of PGL(3). Since every configuration in W contains points e_1,e_2,e_3,e_4 which are in general position, the group PGL(3) acts freely and transitively on W, hence W is isomorphic to the product of PGL(3) and the quotient $W/\mathrm{PGL}(3)$. Recall that PGL(3) is isomorphic to the configuration space of four ordered points in general position in $\mathbf{P}^2$. This yields a natural identification between the quotient $W/\mathrm{PGL}(3)$ and the space

$$W_E := W \cap (F(\mathbf{P}^2,2) \times \{(E_1,E_2,E_3,E_4,E_5)\}),$$

where

$$E_1 = [1,0,0], \quad E_2 = [0,1,0], \quad E_3 = [0,0,1], \quad E_4 = [1,1,1], \quad E_5 = [1,1,0].$$

In the following, we identify each element σ of G with the automorphism of $\mathbf{P}^2$ mapping E_i to $E_{\sigma(i)}$. This allows to consider G as a subgroup of $\mathrm{Aut}(\mathbf{P}^2)$, and induces an action of G on W_E that makes the isomorphism $W \cong W_E \times \mathrm{PGL}(3)$ is G-equivariant. Note that the action of G on $\mathrm{PGL}(3)$ is defined by restricting to G the natural action of the symmetric group $\mathfrak{S}_4$ on $\mathrm{PGL}(3) \hookrightarrow F(\mathbf{P}^2, 4)$ permuting the four points in the configuration. It is not difficult to prove that the rational Borel–Moore homology of $\mathrm{PGL}(3)$ is $\mathfrak{S}_4$-invariant and hence also G-invariant.

By applying the Künneth formula to the Borel–Moore homology of $W \cong W_E \times \mathrm{PGL}(3)$, and considering the part of the Borel–Moore homology which has the wished behaviour for the action of G, one gets

$$\bar{H}_\bullet(W/G; S) \cong \bar{H}_\bullet(W_E/G; S) \otimes \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}), \tag{I.3.1}$$

where S denotes the local system of rank 1 induced by the restriction of the sign representation to $G \subset \mathfrak{S}_4$.

The reasoning above applies to Y_8 as well as W. This yields the isomorphism

$$\bar{H}_\bullet(X_8'; S) \cong \bar{H}_\bullet((Y_8 \cap W_E)/G; S) \times \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}). \tag{I.3.2}$$

The space W_E can be described in the following way. Denote by $\mathcal{L}$ the space of conics passing through the E_i's and distinct from the reducible conic $(x_0-x_1)x_2 = 0$. Note that $\mathcal{L}$ is isomorphic to an affine line. Then we have

$$W_E = \{(p, q, C) \in F(\mathbf{P}^2, 2) \times \mathcal{L} : p, q \in C\}.$$

Lemma I.3.2. *In the notation of Lemma I.3.1, we have* $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(W_E/G; S)) = \mathbf{L}^{-2}t^4 s_2$.

Proof. The space $Q = W_E/G$ can be decomposed as the union of a closed locus K containing all equivalence classes of triples (p, q, C) such that C is a singular conic, and an open part U where the conic C is always non-singular.

We compute the Borel–Moore homology of K first. The locus K has two components, according to the position of the two points p, q. We denote by M the component of K such that p, q lie on the same irreducible component of C, and N the component in which p, q lie on two different components of C. The elements of the intersection $M \cap N$ are the configurations in which the singular point of C is either p or q.

Up to the G-action, the space M can be identified with the space of ordered configurations of two points on the projective line $x_1 = 0$, hence the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(M; S)$ is $(\mathbf{L}^{-1}t^2 + \mathbf{L}^{-2}t^4)s_2$.

Next, we identify the space $N \setminus M$ with a $\mathfrak{S}_2$-quotient of the space of pairs (p, q) where p lies on $x_1 = 0$, the point q lie on $x_0 - x_2 = 0$ and both points are distinct from the intersection point of these lines. The $\mathfrak{S}_2$-action interchanges E_1 and E_3, and E_2 and E_4, and we have to take invariant classes with respect to it. This implies that the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(N \setminus M; S)$ is $(\mathbf{L}^{-2}t^4)s_2$. Then, from the long exact sequence in Borel–Moore homology associated with the closed inclusion $M \hookrightarrow K$ we can conclude that the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(K;S)$ is $(\mathbf{L}^{-1}t^2 + 2\mathbf{L}^{-2}t^4)s_2$.

Subsequently, we compute the Borel–Moore homology of U by lifting U to a G-invariant subset $U' \subset W_E$, and looking for the part of the Borel–Moore homology of U' that has the wished behaviour with respect to the G-action. We have that U' projects to the locus of non-singular conics in $\mathcal{L}$, which is isomorphic to $\mathbf{C} \setminus \{\pm 1\}$. Note that the action of $(1,2) \in G$ on $\mathcal{L} \cong \mathbf{C}$ interchanges the two singular conics. The projection $U' \to \mathbf{C} \setminus \{\pm 1\}$ is a locally trivial fibration with fibre isomorphic to the space $F(R,2)$, where R is a chosen non-singular conic through the E_i's. In order to study the action of G on the Borel–Moore homology of $F(R,2)$, we assume that the conic R is fixed by all automorphisms in $G \subset \mathrm{Aut}(\mathbf{P}^2)$. If we fix an isomorphism $R \cong \mathbf{P}^1$, we have that taking the quotient by G gives finite maps $R \cong \mathbf{P}^1 \to \mathbf{P}^1$ and $F(R,2) \cong F(\mathbf{P}^1,2) \to F(\mathbf{P}^1,2)$. In particular, the Borel–Moore homology with standard coefficients of $F(R,2)$ is isomorphic to that of its quotient by G, hence all Borel–Moore homology classes of $F(R,2)$ are G-invariant. Hence, the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(U;S)$ is the product of $\wp(F(\mathbf{P}^1,2))s_2$ and the HG polynomial of the part of the Borel–Moore homology of $\mathbf{C} \setminus \{\pm 1\}$ which is anti-invariant for the involution $\xi \leftrightarrow -\xi$, which equals t.

To compute $\bar{H}_\bullet(Q;S)$, we can now use the long exact sequence in Borel–Moore homology associated to the closed inclusion $K \hookrightarrow Q$:

$$\cdots \to \bar{H}_k(K;S) \to \bar{H}_k(Q;S) \to \bar{H}_k(U;S) \to \bar{H}_{k-1}(K;S) \to \cdots$$

This yields immediately $\bar{H}_k(Q;S) = 0$ if $k > 5$ or $k < 2$. Moreover, we have

$$0 \to \bar{H}_5(Q;S) \to \mathbf{Q}(2) \xrightarrow{\delta_5} \mathbf{Q}(2)^2 \xrightarrow{\delta'_5} \bar{H}_4(Q;S) \to 0,$$

$$0 \to \bar{H}_3(Q;S) \to \mathbf{Q}(1) \xrightarrow{\delta_3} \mathbf{Q}(1) \xrightarrow{\delta'_3} \bar{H}_2(Q;S) \to 0.$$

Then the claim follows from the fact that both δ_5 and δ_3 are injections. As we will see, the subset $M \cup U$ has trivial Borel–Moore homology with S-coefficients, hence $\bar{H}_k(M;S)$ is contained in the kernel of δ'_k for every k.

To compute the Borel–Moore homology of $M \cup U$, consider the surjective map $\pi\colon M \cup U \to \mathcal{L}/G$ obtained by restricting the natural projection $W_E \to \mathcal{L}$. The map π is clearly locally trivial on U. We claim that π is also locally trivial in a neighborhood of the point w_0 in $\mathcal{L}/G$ parametrizing singular conics. Up to the G-action, and possibly the choice of a sufficiently small neighbourhood U_0, we can assume that this singular conic is $Y\colon x_1(x_0 - x_2) = 0$, and identify $\pi^{-1}(w)$ $(w \in U_0)$ with the locus of triples (Y, α, β) such that α and β lie on the line $x_1 = 0$. Then the fibre of π near w_0 can be identified with $F(\{x_1 = 0\}, 2)$ by considering the projection from the point E_4, which maps every non-singular conic in $\mathcal{L}$ onto the line $x_1 = 0$. This construction yields a map from the preimage in π of a neighbourhood of w_0 to $F(\mathbf{P}^1, 2) \cong F(\{x_1 = 0\}, 2)$, which admits a section. Hence, the map π is a locally trivial fibration over $\mathcal{L}/G$.

Note that this implies that the Borel–Moore homology of $M \cup U$ in the local system S is trivial. The Borel–Moore homology of $\mathcal{L}$ is clearly G-invariant, hence the elements of $H_\bullet(M \cup U; S)$ have to come from Borel–Moore homology classes of the fibre of π, in a local system different from the standard one. The fibre of π is isomorphic to a G-quotient of $F(\mathbf{P}^1, 2)$, and the whole Borel–Moore homology of $F(\mathbf{P}^1, 2)$ is G-invariant. For this reason, the fibre of π has trivial Borel–Moore homology in all local systems different from the standard one. $\square$

Proof of Lemma I.3.1. We start by investigating the space $W_E \setminus Y_8$ and its quotient Q' by the action of G. Recall that a configuration $(p, q, C) \in Q$ lies in Q' if and only if $\{p, q\} \subset \{E_1, E_2, E_3, E_4\}$. It is easy to see that Q' has two components, according to whether p and q lie both on the same component of the reducible conic $x_2(x_0 - x_1) = 0$, or not. Denote by Q_a the component corresponding to the first case and by Q_b the component corresponding to the second case. Up to the action of G, we may assume that for every configuration in Q_a we have $p = E_1$, $q = E_2$. Hence, the space Q_a is isomorphic to the quotient $\mathcal{L}/\iota$, where the involution ι is $(3, 4) \in G \subset \mathrm{Aut}(\mathbf{P}^2)$. Since $\mathcal{L}/\iota$ is isomorphic to $\mathbf{C}$, the Borel–Moore homology of Q_a with S-coefficients is isomorphic to the Borel–Moore homology of $\mathbf{C}$ induced by the sign representation on $\langle\iota\rangle = \mathfrak{S}_2$, which is trivial.

Analogously, up to the G-action one can assume that $p = E_1$, $q = E_3$ hold for every configuration in Q_b. In particular, Q_b is isomorphic to $\mathcal{L} \cong \mathbf{C}$ and is invariant for the involution interchanging p and q. This proves that $\bar{H}_\bullet(Q'; S)$ is isomorphic to $\bar{H}_\bullet(\mathbf{C})$ and is invariant for the involution $p \leftrightarrow q$. Then the claim follows from the long exact sequence in Borel–Moore homology associated to the closed inclusion $Q' \hookrightarrow Q$ and isomorphism (I.3.2). $\square$

Recall from Table I.2 that X'_9 is the locus

$$X'_9 := \left\{ \begin{array}{l} (p, q, S) \in F(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 6) : \\ \exists \{r_i\}_{1 \le i \le 4} \in B(\mathbf{P}^{2\vee}, 4)\big(S = Sing(\bigcup_i r_i), p, q \in \bigcup_i r_i\big) \end{array} \right\}.$$

Observe that giving six points that are the pairwise intersection of four lines in general position is equivalent to giving the configuration of four lines. Denote by $\tilde{F}(\mathbf{P}^{2\vee}, 4)$ the space of ordered configurations of lines in general position (i.e., such that no three of them pass through the same point), and by $\tilde{B}(\mathbf{P}^{2\vee}, 4)$ the analogous space of unordered configurations. Then we have

$$X'_9 \cong \left\{ (p, q, \{r_1, r_2, r_3, r_4\}) \in F(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^{2\vee}, 4) : p, q \in \bigcup_i r_i \right\}.$$

We start by investigating the closed subset X'_{9a} of configurations $(p, q, \{r_i\}_i) \in X'_9$ such that p and q lie on the same line r_j for some index j.

Lemma I.3.3.

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X'_{9a};\pm\mathbf{Q})) = \wp^{\mathfrak{S}_2}(\bar{H}_\bullet(F(\mathbf{P}^1,2);\mathbf{Q}))\cdot\wp(\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})).$$

Proof. Consider the variety

$$A := \{(p,q,r_1,r_2,r_3,r_4)\in F(\mathbf{P}^2,2)\times\tilde{F}(\mathbf{P}^{2\vee},4) : p,q\in r_4\}.$$

Note that X'_{9a} is the quotient of A by the action of $\mathfrak{S}_3$ interchanging r_1, r_2 and r_3. On the other hand, we have $A\cong F(\mathbf{P}^1,2)\times\mathrm{PGL}(3)$, where we used the fact that PGL(3) is isomorphic to the space of four lines in general position, and chosen an isomorphism $\mathbf{P}^1\cong r_4$ (for instance, the one mapping 0 to $r_1\cap r_4$, 1 to $r_2\cap r_4$ and ∞ to $r_3\cap r_4$). Hence, we can obtain the Borel–Moore homology of X'_{9a} by taking the $\mathfrak{S}_3$-invariant part of the Borel–Moore homology of A. This establishes the claim. ☐

Next, we consider $X'_{9b} := X'_9\setminus X'_{9a}$.

Lemma I.3.4.

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X'_{9b};\pm\mathbf{Q})) = (t^2s_{1,1}+\mathbf{L}^{-2}t^4s_2)\cdot\wp(\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})).$$

Proof. Consider the space

$$Y_9 := \{(p,q,r_1,r_2,r_3,r_4)\in F(\mathbf{P}^2,2)\times\tilde{F}(\mathbf{P}^{2\vee},4) : p,q\in\bigcup_i r_i\}.$$

Observe that $\tilde{F}(\mathbf{P}^{2\vee},4)$ is isomorphic to PGL(3), and that the group PGL(3) acts freely and transitively on Y_9. The quotient of this action is isomorphic to the fibre of the projection $Y_9\to\tilde{F}(\mathbf{P}^{2\vee},4)$ at the configuration (l_1,l_2,l_3,l_4), where

$$l_1\colon\ x_0=0,\quad l_2\colon\ x_1=0,\quad l_3\colon\ x_2=0,\quad l_4\colon\ x_0+x_1+x_2=0.$$

If we pose $L\colon\ x_0x_1x_2(x_0+x_1+x_2)=0$, this implies that $Y_9/\mathrm{PGL}(3)$ is isomorphic to $F(L,2)$, and we have an isomorphism

$$Y_9\cong F(L,2)\times\mathrm{PGL}(3). \tag{I.3.3}$$

Consider the action of $\mathfrak{S}_4$ on L and $F(L,2)$ defined by identifying every permutation $\sigma\in\mathfrak{S}_4$ with the automorphism of $\mathbf{P}^2$ sending the line l_i to l_{σ_i} for all i, $1\le i\le 4$. The natural action of $\mathfrak{S}_4$ on $\tilde{F}(\mathbf{P}^2,4)$ defines an action on PGL(3) via the isomorphism $\mathrm{PGL}(3)\cong\tilde{F}(\mathbf{P}^{2\vee},4)$, making isomorphism (I.3.3) $\mathfrak{S}_4$-equivariant. Applying Künneth formula and taking the $\mathfrak{S}_4$-invariant part of the Borel–Moore homology of Y_9 yields

$$\bar{H}_\bullet(X'_9;\mathbf{Q})\cong\bar{H}_\bullet(F(L,2);\mathbf{Q})^{\mathfrak{S}_4}\otimes\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q}),$$

where we used the fact that X'_9 is the quotient of Y_9 by the action of $\mathfrak{S}_4$, and that the whole Borel–Moore homology of PGL(3) is $\mathfrak{S}_4$-invariant.

Let us see what these considerations tell us about the Borel–Moore homology of X'_{9b}. The space X'_{9b} is isomorphic to the $\mathfrak{S}_4$-quotient of the product of PGL(3) and the locus of configurations of two points $(a,b) \in F(L,2)$ not lying on the same component of L. This locus can be decomposed according to whether a and b are or are not singular points of L into the loci

$$S_1 := \{(a,b) \in F(L,2) : a \text{ and } b \text{ are both singular points}\},$$

$$S_2 := \{(a,b) \in F(L,2) : \text{only one of the points } a \text{ and } b \text{ is a singular point of } L\},$$

$$S_3 := \{(a,b) \in F(L,2) : a \text{ and } b \text{ are non-singular points of } L\}.$$

The quotient $S_1/\mathfrak{S}_4$ consists of only one point, the class of the pair ($[1,0,0]$, $[0,1,-1]$). The quotient $S_2/\mathfrak{S}_4$ has two isomorphic components, according to which point (a or b) is a singular point of L. Consider the case in which a is singular. Up to the action of $\mathfrak{S}_4$, we can assume that a is the point $[1,0,0]$ and b lies on $x_0 = 0$. By the definition of S_2 we know that b is different from the points $[0,1,-1]$, $[0,0,1]$ and $[0,1,0]$. Note that, since we are working modulo $\mathfrak{S}_4$, the coordinates of b are defined up to the involution interchanging x_1 and x_2. This proves that both components of $S_2/\mathfrak{S}_4$ are isomorphic to $\mathbf{C}^*$.

Finally, we determine the Borel–Moore homology of the quotient of S_3 by the action of $\mathfrak{S}_4$. Up to the action of the group, we can assume that a lies on the line l_3 and b on l_4. The position of both points is determined up to the involution interchanging the lines l_1 and l_2. If we identify l_3 and l_4 with $\mathbf{P}^1$, and $l_3 \cap l_4$ with the point at infinity of the projective line, we have that $S_3/\mathfrak{S}_4$ can be embedded into the quotient of $(\mathbf{C} \setminus \{\pm 1\})^2$ by the relation $(t,s) \sim (-t,-s)$. The complement of $S_3/\mathfrak{S}_4$ in this quotient is the locus such that either t or s are equal to ± 1. We can study $(\mathbf{C} \setminus \{\pm 1\})^2/\sim$ as follows:

$$\begin{array}{ccccc}
\mathbf{C}^2 & \xrightarrow{\text{mod } \sim} & \{(x,y,z) \in \mathbf{C}^3 : y^2 = xz\} & \xrightarrow{\text{mod } \mathfrak{S}_2} & \mathbf{C}^2 \\
(t,s) & \longmapsto & (t^2, ts, s^2) & \longmapsto & (t^2+s^2, ts) \\
(1,s) & \longmapsto & (1, s, s^2) & \longmapsto & (s^2+1, s) \\
(t,1) & \longmapsto & (t^2, t, 1) & \longmapsto & (t^2+1, t),
\end{array}$$

where the second map denotes the quotient by the action of $\mathfrak{S}_2$ interchanging t and s. Concluding, the spectral sequence associated to this stratification has E^1 term as in Table I.5 (where we have taken into account the $\mathfrak{S}_2$-action interchanging a and b).

We can use the geometric description of S_1, S_2 and S_3 to determine all differentials of the spectral sequence above. In particular, the the 0-th row is exact, and both differentials in the row of index -1 have rank 1. Then the claim follows from the fact that $\bar{H}_\bullet(X'_{9b};\mathbf{Q})$ is isomorphic to $\bar{H}_\bullet(S_1 \cup S_2 \cup S_3;\mathbf{Q})^{\mathfrak{S}_4} \otimes \bar{H}_\bullet(\mathrm{PGL}(3))$. $\square$

Table I.5:

$$\begin{array}{r|lcc}
1 & 0 & 0 & \mathbf{S}_2\otimes\mathbf{Q}(2)\\
0 & 0 & (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(1) & (\mathbf{S}_2+\mathbf{S}_{1,1})\otimes\mathbf{Q}(1)\\
-1 & \mathbf{S}_2 & \mathbf{S}_2+\mathbf{S}_{1,1} & \bigoplus^2\mathbf{S}_{1,1}\\
\hline
 & 1 & 2 & 3
\end{array}$$

Proposition I.3.5. *The Borel–Moore homology groups of the unions of strata $\Phi'_8\cup\Phi'_9\subset\left|\Lambda'\right|$ and $F'_8\cup F'_9\cup F'_{10}\subset\left|\mathcal{X}'\right|$ are as follows:*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(\Phi'_8\cup\Phi'_9;\mathbf{Q}))=(\mathbf{L}^{-2}t^9s_2+(\mathbf{L}^{-1}s_2+\mathbf{L}^{-1}s_{1,1}+s_{1,1})t^7)\cdot\wp(\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})). \tag{I.3.4}$$

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(F'_8\cup F'_9\cup F'_{10};\mathbf{Q}))=(\mathbf{L}^{-2}t^9s_2+(\mathbf{L}^{-1}s_2+\mathbf{L}^{-1}s_{1,1}+s_{1,1})t^7)\cdot\wp(\bar{H}_\bullet(\mathrm{GL}(3);\mathbf{Q})). \tag{I.3.5}$$

Proof. Lemmas I.3.3 and I.3.4 imply that $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X'_9;\pm\mathbf{Q}))$ equals $(1+\mathbf{L}^2t^{-3})(1+\mathbf{L}^3t^{-5})\mathbf{L}^{-8}t^{16}\cdot(2\mathbf{L}^{-2}t^4s_2+(\mathbf{L}^{-1}+1)t^2s_{1,1})$. Recall from Table I.2 (page 6) that Φ'_9 is a simplices bundle with 5-dimensional fibre. Hence, the $\mathfrak{S}_2$-HG polynomial of the Borel–Moore homology of Φ'_9 equals that of X'_9 multiplied by t^5. Analogously, Lemma I.3.1 and Table I.2 yield that $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(\Phi'_8;\mathbf{Q}))$ is $(1+\mathbf{L}^2t^{-3})(1+\mathbf{L}^3t^{-5})\mathbf{L}^{-8}t^{16}\cdot(\mathbf{L}^{-2}t^8+\mathbf{L}^{-1}t^7)s_2$.

We compute the Borel–Moore homology of $\Psi:=\Phi'_8\cup\Phi'_9$ by exploiting the long exact sequence

$$\cdots\to\bar{H}_k(\Phi'_8;\mathbf{Q})\to\bar{H}_k(\Psi;\mathbf{Q})\to\bar{H}_k(\Phi'_9;\mathbf{Q})\xrightarrow{\delta_k}\bar{H}_{k-1}(\Phi'_8;\mathbf{Q})\to\cdots \tag{I.3.6}$$

Both the Borel–Moore homology of Φ'_8 and Φ'_9 are tensor products of the Borel–Moore homology of PGL(3). The question is whether their structure as tensor products of $\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$ is respected by the maps in (I.3.6) or not.

We computed in § I.2 that the strata 1–7 do not contribute to the Borel–Moore homology of F'_{10}. Recall that $F'_8\cup F'_9$ is a vector bundle of rank 1 over Ψ. By comparing the geometry of $\mathcal{D}$ and its projectivization, we can conclude that

$$\bar{H}_\bullet(\bigcup_{i=8}^{10}F'_i;\mathbf{Q})\cong\bar{H}_\bullet(\Psi;\mathbf{Q})\otimes\bar{H}_\bullet(\mathbf{C}^*;\mathbf{Q}). \tag{I.3.7}$$

Isomorphism (I.1.1), together with the computation of the Borel–Moore homology of $\bigcup_{i=1}^7F'_i$ in § I.2, yields that the Borel–Moore homology of $\bigcup_{i=8}^{10}F'_i$ is a tensor product of $\bar{H}_\bullet(\mathrm{GL}(3);\mathbf{Q})\cong\bar{H}_\bullet(\mathbf{C}^*;\mathbf{Q})\otimes\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$. In view of (I.3.7), this property implies that the Borel–Moore homology of Ψ is a tensor product of $\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$. The only possibility for this is that the maps of the exact sequence (I.3.6) respect the structure of Φ'_8 and Φ'_9 as tensor products of

$\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$. This is important, because it implies that all differentials δ_k in (I.3.6) are determined, once one knows the rank of

$$\delta_{25}\colon\ \bar{H}_{25}(\Phi'_9;\mathbf{Q})\cong\bigoplus^{2}\mathbf{S}_2\otimes\mathbf{Q}(2)\to\bar{H}_{24}(\Phi'_8;\mathbf{Q})\cong\mathbf{S}_2\otimes\mathbf{Q}(2).$$

We claim that δ_{25} has rank one. This would yield part (I.3.4) in the claim. Note that, in view of (I.3.7), equality (I.3.4) implies (I.3.5).

Define $B\subset X'_9$ as the locus of configurations $(p,q,\{r_i\})$ such that

$$p,q\notin Sing\left(\bigcup_i r_i\right),\quad pq\notin\{r_1,r_2,r_3,r_4\}.$$

Denote by $\mathcal{B}\to B$ the restriction of the bundle $\Phi'_9\to X'_9$ to B. Next, consider the locus $A\subset X'_8$ of configurations $(p,q,\{e_1,e_2,e_3,e_4\})$ such that $p\in e_1e_3$, $q\in e_2e_4$, $\{p,q\}\cap(\{e_1,e_2,e_3,e_4\}\cup(e_1e_2\cap e_3e_4)=\emptyset$. Denote by $\mathcal{A}\to A$ the restriction of the bundle $\Phi'_8\to X'_8$ to A. Note that for every element $a=(p,q,\{e_i\})$ of A, the configuration $C_a:=(p,q,\{e_1e_2,e_1e_3,e_2e_4,e_3e_4\})$ is an element of B. This means that the face of the 4-dimensional open simplex lying above a is identified (in $|\Lambda'|$) with one of the external faces of the 5-dimensional simplex contained in $\mathcal{B}$ which lies above $C_a\in B$. Moreover, this 4-dimensional open simplex is the only face of C_a which lies in $\mathcal{A}$. Recall that the Borel–Moore homology of the union of an open simplex and one of its open faces is trivial, because of the characterization of Borel–Moore homology as the relative homology of the one-point compactification of a space modulo the added point. This implies that the Borel–Moore homology of $\mathcal{A}\cup\mathcal{B}$ is trivial.

We have the following chains of inclusions:

$$\begin{array}{ccc}
\mathcal{A} & \underset{\text{open}}{\hookrightarrow} & \Phi'_8\\
\text{closed}\cap & & \cap\,\text{closed}\\
\mathcal{A}\cup\mathcal{B} & \underset{\text{open}}{\hookrightarrow} & \Psi\\
\text{open}\cup & & \cup\,\text{open}\\
\mathcal{B} & \underset{\text{open}}{\hookrightarrow} & \Phi'_9.
\end{array}$$

In particular, if we consider the long exact sequence in Borel–Moore homology associated to the closed inclusion $\mathcal{A}\hookrightarrow\mathcal{A}\cup\mathcal{B}$, we have that the map $\bar{H}_{25}(\mathcal{B};\mathbf{Q})\to\bar{H}_{24}(\mathcal{A};\mathbf{Q})$ is an isomorphism. By the computation of the Borel–Moore homology of X'_8 in Lemmas I.3.1 and I.3.2 we have that $\bar{H}_{24}(\mathcal{A};\mathbf{Q})\cong\bar{H}_{24}(\Phi'_8;\mathbf{Q})$ and $\bar{H}_{25}(\mathcal{B};\mathbf{Q})\subset\bar{H}_{25}(\Phi'_9;\mathbf{Q})$ are both one-dimensional. □

Now that the contribution of strata 8–10 to the Borel–Moore homology of $\mathcal{D}$ is known, we want to deduce their contribution to the rational cohomology of $\mathcal{I}_2$.

Then the closed inclusion $\mathcal{D} \to \mathcal{V}$ (for the definition, see the beginning of the present section) induces a long exact sequence

$$\cdots \to H^k(\mathcal{V};\mathbf{Q}) \to H^k(\mathcal{I}_2;\mathbf{Q}) \to \bar{H}_{33-k}(\mathcal{D};\mathbf{Q})(-k) \to H^{k+1}(\mathcal{V};\mathbf{Q}) \to \cdots,$$

which in the case of the part of Borel–Moore homology of $\mathcal{D}$ that comes from strata 8–10, gives the following contribution to the $\mathfrak{S}_2$-HG polynomial of the rational cohomology of $\mathcal{I}_2$:

$$(1+\mathbf{L}t)(1+\mathbf{L}^2t^3)(1+\mathbf{L}^3t^5)\big(\mathbf{L}^6t^6s_2+\mathbf{L}^7t^8(s_2+s_{1,1})+\mathbf{L}^8t^8s_{1,1}\big), \qquad \text{(I.3.8)}$$

hence the contribution of these strata to the HG polynomial of the cohomology of $\mathcal{Q}_2$ is

$$\mathbf{L}^6t^6s_2+\mathbf{L}^7t^8(s_2+s_{1,1})+\mathbf{L}^8t^8s_{1,1}. \qquad \text{(I.3.9)}$$

I.4 Vassiliev–Gorinov's method

In order to make the article as self-contained as possible, we include here an introduction to Vassiliev–Gorinov's method for computing the cohomology of complements of discriminants, following [Tom05a] and [Tom05b]. This review of the method is by no means complete, and we encourage the interested reader to consult [Vas99], [Gor05] and [Tom05a].

Let Z be a projective variety, $\mathcal{F}$ a vector bundle on Z and V the space of global sections of $\mathcal{F}$. Define the discriminant $\Sigma \subset V$ as the locus of sections with a vanishing locus which is either singular or not of the expected dimension. Assume that Σ is a subvariety of V of pure codimension 1. Our aim is to compute the rational cohomology of the complement of the discriminant, $X = V \setminus \Sigma$. This is equivalent to determining the Borel–Moore homology of the discriminant, because there is an isomorphism between the reduced cohomology of X and Borel–Moore homology of Σ. If we denote by M the dimension of V, this isomorphism can be formulated as

$$\tilde{H}^\bullet(X;\mathbf{Q}) \cong \bar{H}_{2M-\bullet-1}(\Sigma;\mathbf{Q})(-M).$$

Definition I.4.1. A subset $S \subset Z$ is called a *configuration* in Z if it is compact and non-empty. The space of all configurations in Z is denoted by $\mathrm{Conf}(Z)$.

Proposition I.4.2 ([Gor05])**.** *The Fubini–Study metric on Z induces in a natural way on* $\mathrm{Conf}(Z)$ *the structure of a compact complete metric space.*

To every element in $v \in V$, we can associate its singular locus $K_v \in \mathrm{Conf}(Z) \cup \{\emptyset\}$. We have that K_0 equals Z, and that $L(K) := \{v \in V : K \subset K_v\}$ is a linear space for all $K \in \mathrm{Conf}(Z)$.

Vassiliev–Gorinov's method is based on the choice of a collection of families of configurations $X_1, \ldots, X_R \subset \mathrm{Conf}(Z)$, satisfying some axioms ([Gor05, 3.2], [Tom05a, List 2.1]). Intuitively, we have to start by classifying all possible singular loci of elements of V. Note that singular loci of the same type have a space $L(K)$ of the same dimension. We can put all singular configurations of the same type in a family. Then we order all families we get according to the inclusion of configurations. In this way we obtain a collection of families of configurations which may already satisfy Gorinov's axioms. If this is not the case, the problem can be solved by adding new families to the collection. Typically, the elements of these new families will be degenerations of configurations already considered. For instance, configurations with three points on the same projective line and a point outside it can degenerate into configurations with four points on the same line, even if there is no $v \in V$ which is only singular at four collinear points.

Once the existence of a collection $X_1, \ldots, X_R$ satisfying Gorinov's axioms is established, Vassiliev–Gorinov's method gives a recipe for constructing a space $\left|\mathcal{X}\right|$ and a map

$$|\epsilon|\colon \left|\mathcal{X}\right| \longrightarrow \Sigma,$$

called *geometric realization*, which is a homotopy equivalence and induces an isomorphism on Borel–Moore homology. The original construction by Vassiliev and Gorinov uses topological joins to construct $\left|\mathcal{X}\right|$. This construction was reformulated in [Tom05a] by using the language of cubical spaces. This ensures in particular that the map induced by $|\epsilon|$ on Borel–Moore homology respects mixed Hodge structures.

Vassiliev–Gorinov's method provides also a stratification $\{F_j\}_{j=1,\ldots,N}$ on $\left|\mathcal{X}\right|$. Each F_j is locally closed in $\left|\mathcal{X}\right|$, hence one gets a spectral sequence converging to $\bar{H}_\bullet(\Sigma; \mathbf{Q}) \cong \bar{H}_\bullet(\left|\mathcal{X}\right|; \mathbf{Q})$, with $E^1_{p,q} \cong \bar{H}_{p+q}(F_p)$. To compute the Borel–Moore homology of F_j for all $j = 1, \ldots, R$, it is helpful to use an auxiliary space $\left|\Lambda\right|$, whose construction depends only on the geometry of the families $X_1 \ldots, X_R$, and which is covered by locally closed subsets $\{\Phi_j\}_{j=1,\ldots,N}$.

Proposition I.4.3 ([Gor05]). *1. For every $j = 1, \ldots, R$, the stratum F_j is a complex vector bundle over Φ_j. The space Φ_j is in turn a fiber bundle over the configuration space X_j.*

2. If X_j consists of configurations of m points, the fiber of Φ_j over any $x \in X_j$ is an $(m-1)$-dimensional open simplex, which changes its orientation under the homotopy class of a loop in X_j interchanging a pair of points in x_j.

3. If $X_R = \{Z\}$, F_R is the open cone with vertex a point (corresponding to the configuration Z), over $\left|\Lambda\right| \setminus \Phi_R$.

We recall here the topological definition of an open cone.

Definition I.4.4. Let B be a topological space. Then a space is said to be an *open cone* over B with vertex a point if it is homeomorphic to the space $B \times [0, 1)/R$, where the equivalence relation is $R = (B \times \{0\})^2$.

The fiber bundle $\Phi_j \to X_j$ of Proposition I.4.3 is in general non-orientable. As a consequence, we have to consider the homology of X_j with coefficients not in $\mathbf{Q}$, but in some local system of rank one. Therefore we recall some constructions concerning Borel–Moore homology of configuration spaces with twisted coefficients.

Definition I.4.5. Let Z be a topological space. Then for every $k \geq 1$ we have the space of ordered configurations of k points in Z,

$$F(Z,k) = Z^k \setminus \bigcup_{1 \leq i < j \leq k} \{(z_1, \dots, z_k) \in Z^k : z_i = z_j\}.$$

There is a natural action of the symmetric group $\mathfrak{S}_k$ on $F(k,Z)$. The quotient is called the space of unordered configurations of k points in Z,

$$B(Z,k) = F(Z,k)/\mathfrak{S}_k.$$

The *sign representation* $\pi_1(B(Z,k)) \to \mathrm{Aut}(\mathbf{Z})$ maps the paths in $B(Z,k)$ defining odd (respectively, even) permutations of k points to multiplication by -1 (respectively, 1). The local system $\pm\mathbf{Q}$ over $B(Z,k)$ is the one locally isomorphic to $\mathbf{Q}$, but with monodromy representation equal to the sign representation of $\pi_1(B(Z,k))$. We will often call $\bar{H}_\bullet(B(Z,k), \pm\mathbf{Q})$ the *Borel–Moore homology of* $B(Z,k)$ *with twisted coefficients*, or, simply, the *twisted Borel–Moore homology of* $B(Z,k)$.

The following is Lemma 2 in [Vas99].

Lemma I.4.6. *1. If $N \geq 1$, $k \geq 2$, the twisted Borel–Moore homology of $B(\mathbf{C}^N, k)$ is trivial.*

2. If $N \geq 1$, we have isomorphisms

$$\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) \cong \bar{H}_{\bullet - k(k-1)}(\mathbf{G}(k-1, \mathbf{P}^N); \mathbf{Q})$$

for every $k \geq 1$, where $\mathbf{G}(k-1, \mathbf{P}^N)$ denotes the Grassmann variety of $(k-1)$-dimensional linear subspaces in $\mathbf{P}^N$. In particular, $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) = 0$ if $k \geq N+2$.

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Chapter II

Cohomology of the moduli space of smooth plane quartic curves with an odd theta characteristic

II.1 Introduction

The subject of this chapter is the rational cohomology of the moduli space $\mathcal{Q}^-$ of smooth plane quartic curves with an odd theta characteristic. In other words, the elements of $\mathcal{Q}^-$ are isomorphism classes of pairs $(C, \mathcal{L})$ where C is a smooth plane quartic curve and $\mathcal{L}$ is a line bundle on C such that $\mathcal{L}^{\otimes 2} \cong \omega_C$ and the dimension of $H^0(C, L)$ is odd. It is a classical results that for a fixed quartic curve C such theta characteristics $\mathcal{L}$ correspond to the divisors on C cut by the 28 bitangents of C. Therefore, we can equivalently interpret $\mathcal{Q}^-$ as the moduli space of pairs (C, τ) where C is a smooth quartic curve and τ is a bitangent line to C. Note that there are two possibilities for line τ to be a bitangent line to a smooth plane quartic C: either τ intersects C in two distinct points with multiplicity two, in which case we will call τ a *proper bitangent*, of τ intersects C in one point with multiplicity 4, in which case we will call τ a *flex bitangent* of C.

Our interest in the cohomology of $\mathcal{Q}^-$ arises from a more general interest in the moduli space $\mathcal{S}_g$ of genus g curves with a theta characteristics (also known as smooth spin curves) and in its compactification, the moduli space $\overline{\mathcal{S}}_g$ of stable spin curves, constructed in [Co89]. This space has been extensively studied in the last years from several points of view, including its birational geometry [Lu10], combinatorial aspects [CC03] and intersection theory [FSZ10], mainly motivated by applications to mathematical physics. Nevertheless, the cohomology of $\mathcal{S}_g$ and $\overline{\mathcal{S}_g}$ is only known for genus $g \leq 2$ [BF07, Kr]. In particular, the computation in genus 2 heavily depends on the fact that all genus 2 curves are hyperelliptic, so that theta characteristics can be expressed as linear combinations of Weierstrass points. In this way, the coarse moduli space of spin curves can be interpreted as a moduli space of partially ordered configurations of points on rational curves. However, these results do not generalise to genus 3.

In the pursuit of cohomological information about $\mathcal{M}_3$, Looijenga studied the moduli space $\mathcal{Q}^-$ and identified it with a quotient of the complement of an arrangement of divisors on an algebraic torus. Using a relation between this arrangement and the root system E_6, he proved in [Lo93, Cor. 4.5] the following result:

Theorem II.1.1. *The cohomology with rational coefficients of $\mathcal{Q}^-$ is non-trivial only in degree $k \in \{0,5,6\}$. All cohomology groups carry pure Hodge structures. Specifically, one has $H^5(\mathcal{Q}^-;\mathbf{Q}) = \mathbf{Q}(-5)$ and $H^6(\mathcal{Q}^-;\mathbf{Q}) = \mathbf{Q}(-6)^{\oplus 2}$.*

Our aim is to give a completely independent proof of Theorem II.1.1 in which the cohomology classes are directly related to degenerations of the quartic curve considered. This kind of approach is interesting because it can be generalized to an open subset of component of the boundary $\overline{\mathcal{S}}_3^- \setminus \mathcal{S}_3^-$ corresponding to irreducible nodal curves. The cohomology of this boundary component is unknown at present and is the main obstruction to the computation of the cohomology of $\overline{\mathcal{S}}_3^-$. Furthermore, our approach could be more suitable for understanding the map in cohomology associated with the inclusion of $\mathcal{Q}^-$ into the moduli space $\mathcal{S}_3^-$ of smooth genus 3 curves with an odd theta characteristic.

To explain our approach, let us start by considering the moduli space $\mathcal{Q}$ of smooth quartic curves in the projective plane. Forgetting the chosen bitangent yields a map $o\colon \mathcal{Q}^- \to \mathcal{Q}$, which is finite of degree 28. Quartic curves are defined by the vanishing of polynomials of degree four in three indeterminates, i.e. by elements of the vector space $V := \mathbf{C}[x_0,x_1,x_2]_4$. Clearly, not every element of V defines a non-singular curve, but we have to exclude the locus $\Sigma \subset V$ of singular polynomials. The action of GL(3) on $\mathbf{P}^2$ and $\mathbf{C}[x_0,x_1,x_2]$ preserves Σ, thus inducing an action on $V \setminus \Sigma$. The moduli space $\mathcal{Q}$ is the geometric quotient of $V \setminus \Sigma$ by the action of GL(3).

The rational cohomology of $V\setminus\Sigma$ was computed by Vassiliev in [V99], using his method for the computation of the cohomology of complements of discriminants. Comparing this result with the rational cohomology of the moduli space $\mathcal{Q}$, as computed by Looijenga in [Lo93], one observes that the cohomology of the space of non-singular polynomials in V is isomorphic (as graded vector space) to the tensor product of the cohomology of the moduli space $\mathcal{Q}$ and that of GL(3). Indeed, Peters and Steenbrink [PS03] proved that this is always the case when comparing the rational cohomology of the space of non-singular homogeneous polynomials with the cohomology of the corresponding moduli space of smooth hypersurfaces.

In this chapter we use an analogous construction, in which we replace the vector space V of homogeneous polynomials of degree 4 with a certain incidence correspondence. This follows the approach of [T07], where we considered quartic curves with two marked points. A bitangent line τ to a fixed smooth plane quartic C is always uniquely determined by the scheme-theoretic intersection of C and τ, which is a subscheme $P \subset \mathbf{P}^2$ of length 2. Note that any $P \in \mathrm{Hilb}_2(\mathbf{P}^2)$ spans a uniquely defined line $\ell_P \subset \mathbf{P}^2$. If P is the intersection of C with a bitangent line, then this bitangent line is exactly ℓ_P.

Therefore, we consider the incidence correspondence

$$\mathcal{I}^- := \left\{(P,f) \in \mathrm{Hilb}_2(\mathbf{P}^2) \times (V \setminus \Sigma) | f|_{\ell_P} \in \mathbf{I}(P)^2\right\}.$$

The action of GL(3) on $\mathbf{P}^2$ and V extends to $\mathcal{I}^-$ and the geometric quotient $\mathcal{I}^-/\mathrm{GL}(3)$ is isomorphic to $\mathcal{Q}^-$. Then the following isomorphism of graded vector spaces with mixed Hodge structures holds:

$$H^\bullet(\mathcal{I}^-;\mathbf{Q}) \cong H^\bullet(\mathcal{Q}^-;\mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3);\mathbf{Q}).$$

This follows from [PS03], in view of [BT07, Theorem 5.2]. As a consequence, we have that determining the rational cohomology of $\mathcal{I}^-$ immediately yields the rational cohomology of $\mathcal{Q}^-$.

A natural way to investigate $\mathcal{I}^-$ and its cohomology is to use the natural projection $\pi^-\colon \mathcal{I}^- \to \mathrm{Hilb}_2(\mathbf{P}^2)$. First one observes that all fibres of π^- lying over reduced subschemes in $\mathrm{Hilb}_2(\mathbf{P}^2)$ are isomorphic. Analogously, all fibres of π^- lying over fat points $P \in \mathrm{Hilb}_2$ are isomorphic. Furthermore, in both cases the fibres are the complement of Σ in a linear subspace of V. This enable us to apply Vassiliev–Gorinov's method for the cohomology of complements of discriminants [V99, Go05, T05a]) to compute the cohomology of these fibres. The study of the Leray spectral sequence associated to the restriction of π^- to the stratum of $\mathcal{I}^-$ corresponding to proper bitangent, respectively, to the stratum of $\mathcal{I}^-$ corresponding to flex bitangent allows us to determine the cohomology of $\mathcal{I}^-$.

The structure of the chapter is as follows. In Section II.2 we set up our notation and we prove the relationship between the rational cohomology of the incidence correspondences we deal with and the cohomology of their GL(3)-quotients. In Section II.3 we compute the cohomology of the moduli space of plane smooth quartic curves with a proper bitangent. The proof of this result relies on the analysis of singular configurations performed in Sections II.5–II.10. Finally, in Section II.11 we prove that the moduli space of smooth plane quartic curves with a flex bitangent has the rational cohomology of a point. For Vassiliev–Gorinov's method, we refer to Section I.4 in the previous chapter.

Notation

V	vector space of homogeneous polynomials of degree 4 in x_0, x_1, x_2.
Σ	locus of singular polynomials in V.
$\mathfrak{S}_n$	the symmetric group in n letters.
$\mathbf{V}(f)$	vanishing locus of f.

$K_0(\mathsf{HS}_{\mathbf{Q}})$	Grothendieck group of rational (mixed) Hodge structures over $\mathbf{Q}$.
$K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$	Grothendieck group of rational (mixed) Hodge structures endowed with an $\mathfrak{S}_n$-action.
$\mathbf{Q}(m)$	Tate Hodge structure of weight $-2m$.
$\mathbf{L}$	class of $\mathbf{Q}(-1)$ in $K_0(\mathsf{HS}_{\mathbf{Q}})$.
$\mathbf{S}_\lambda$	$\mathbf{Q}$-representation of $\mathfrak{S}_n$ indexed by the partition $\lambda \vdash n$.
s_λ	Schur polynomial indexed by the partition $\lambda \vdash n$.
Δ_j	j-dimensional closed simplex.
$\mathring{\Delta}_j$	interior of the j-dimensional closed simplex.
$F(Z,k)$	space of ordered configurations of k distinct points on the variety Z (see Def. I.4.5).
$B(Z,k)$	space of unordered configurations of k distinct points on the variety Z (see Def. I.4.5).
$\pm\mathbf{Q}$	the twisted local system over $B(Z,k)$ induced by the sign representation on $\pi_1(B(Z,k))$. I.e. the local system $\pm\mathbf{Q}$ is the rank one local system that changes its orientation under paths inducing an odd permutation of the points in the configuration.
$\tilde{F}(\mathbf{P}^2,4)$	open subset of $F(\mathbf{P}^2,4)$ such that no three points in the configuration are collinear.
$\tilde{B}(\mathbf{P}^2,4)$	open subset of $B(\mathbf{P}^2,4)$ such that no three points in the configuration are collinear.
$\mathbf{P}^{2\vee}$	the dual projective plane, parametrizing all projective lines in $\mathbf{P}^2$.

Throughout this chapter we will make an extensive use of Borel–Moore homology, i.e. homology with locally finite support, which we will denote by the symbol $\bar{H}_\bullet$. A reference for its definition and the properties we use is for instance [F84, Chapter 19].

To write the results on cohomology and Borel–Moore homology groups in a compact way, we will express them by means of polynomials, in the following way. Let $T_\bullet$ denote a graded $\mathbf{Q}$–vector space with mixed Hodge structures. For every $i \in \mathbf{Z}$, we can consider the class $[T_i]$ in the Grothendieck group of rational Hodge structures. We define the Hodge–Grothendieck polynomial (for short, HG polynomial) of $T_\bullet$ to be the polynomial

$$\wp(T_\bullet) = \sum_{i\in\mathbf{Z}} [T_i]t^i \in K_0(\mathsf{HS}_{\mathbf{Q}})[t].$$

If moreover the symmetric group $\mathfrak{S}_n$ acts on $T_\bullet$ respecting the grading and the mixed Hodge structures on $T_\bullet$, we define the *$\mathfrak{S}_n$-equivariant HG polynomial* $\wp^{\mathfrak{S}_n}(T_\bullet)$ of $T_\bullet$ by replacing $K_0(\mathsf{HS}_{\mathbf{Q}})$ by $K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$ in the definition of the HG polynomial.

II.2 Setup

In this section, we establish the notation we will use in the next sections. The main ingredient of our construction is the incidence correspondence $\mathcal{I}^-$ parametrizing pairs (P, f) such that f is a polynomial defining a smooth plane quartic curve and P is the length two subscheme in $\mathbf{P}^2$ cut on the zero locus $\mathbf{V}(f)$ by a bitangent line. We consider two natural maps on $\mathcal{I}^-$, namely the projection $\pi^-\colon \mathcal{I}^- \to \mathrm{Hilb}_2(\mathbf{P}^2)$ and the quotient map $\mathcal{I}^- \to \mathcal{Q}^-$ by the action of $\mathrm{GL}(3)$.

As explained in the introduction, we stratify $\mathcal{I}^-$ into two strata, according to whether the bitangent line ℓ_P is a proper bitangent or a flex bitangent. For a pair (P, f) in $\mathcal{I}^-$, the bitangent line ℓ_P is a proper bitangent if and only if P is a reduced subscheme. The locus in $\mathrm{Hilb}_2(\mathbf{P}^2)$ parametrizing reduced subschemes can be identified with $B(\mathbf{P}^2, 2)$, the configuration space of unordered pairs of points in $\mathbf{P}^2$. The complement $\mathrm{Hilb}_2(\mathbf{P}^2) \setminus B(\mathbf{P}^2, 2)$ is the locus of fat points of multiplicity two, which is naturally isomorphic to the total space of the projectivized tangent bundle $\mathbf{P}(T_{\mathbf{P}^2})$.

Therefore, we define the open stratum $\mathcal{I}_0^- \subset \mathcal{I}^-$ to be the preimage of $B(\mathbf{P}^2, 2)$ under π^-, and the stratum $\mathcal{I}_\delta^-$ to be the preimage of $\mathbf{P}(T_{\mathbf{P}^2})$. When restricted to these two strata, the map π^- is a locally trivial fibration. We will denote the restriction of π^- to the preimages of these two strata of $\mathrm{Hilb}_2(\mathbf{P}^2)$ by

$$\pi_0^-\colon \mathcal{I}_0^- \to B(\mathbf{P}^2, 2), \qquad \pi_\delta^-\colon \mathcal{I}_\delta^- \to \mathbf{P}(T_{\mathbf{P}^2}).$$

Note that the quotient $\mathcal{Q}_0^- = \mathcal{I}_0^-/\mathrm{GL}(3)$ is a well-defined open subset of $\mathcal{Q}^-$, with complement the divisor $\mathcal{Q}_\delta^- = \mathcal{Q}^- \setminus \mathcal{Q}_0^- = \mathcal{I}_\delta^-/\mathrm{GL}(3)$. The quotient $\mathcal{Q}_0^-$ is the moduli space of smooth quartic curves with a marked proper bitangent, whereas $\mathcal{Q}_\delta^-$ is the moduli space of smooth quartic curves with a marked flex bitangent, i.e. a flex line with contact of order 4 with the curve.

In the next sections, we will compute the cohomology of $\mathcal{I}_0^-$ and $\mathcal{I}_\delta^-$ by using their structure as fibrations given by the maps π_0^-, respectively, π_δ^-. This will allow us to obtain the cohomology of the moduli spaces $\mathcal{Q}_0^-$ and $\mathcal{Q}_\delta^-$ by means of the following lemma.

Lemma II.2.1. *The following isomorphisms of graded vector spaces with mixed Hodge structures hold:*

$$H^\bullet(\mathcal{I}_0^-; \mathbf{Q}) \cong H^\bullet(\mathcal{Q}_0^-; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3); \mathbf{Q}). \tag{II.2.1}$$

$$H^\bullet(\mathcal{I}_\delta^-; \mathbf{Q}) \cong H^\bullet(\mathcal{Q}_\delta^-; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3); \mathbf{Q}). \tag{II.2.2}$$

We recall Peters-Steenbrink's generalization of the Leray-Hirsch theorem:

Theorem II.2.2 ([PS03])**.** *Let $\varphi : X \to Y$ be a geometric quotient for the action of a connected group G, such that for all $x \in X$ the connected component of the stabilizer of x is contractible. Consider the orbit inclusion*

$$\begin{array}{rccc} \rho_{x_0} : & G & \longrightarrow & X \\ & g & \longmapsto & gx_0, \end{array}$$

where $x_0 \in X$ is a fixed point. Suppose that for all $k > 0$ there exist classes $e_1^{(k)}, \dots, e_{n(k)}^{(k)} \in H^k(X;\mathbf{Q})$ that restrict to a basis for $H^k(G;\mathbf{Q})$ under the map induced by ρ_{x_0} on cohomology. Then the map

$$a \otimes \rho_{x_0}^*(e_i^{(k)}) \longmapsto \varphi^* a \cup e_i^{(k)}$$

extends linearly to an isomorphism of graded linear spaces

$$H^\bullet(Y;\mathbf{Q}) \otimes H^\bullet(G;\mathbf{Q}) \cong H^\bullet(X;\mathbf{Q})$$

that respects the rational mixed Hodge structures of the cohomology groups.

Proof of Lemma II.2.1. Recall from [PS03] that the assumption of Theorem II.2.2 are satisfied for the action of GL(3) on the space $X := V \setminus \Sigma$ of non-singular quartic polynomials. In particular, the map $\rho_f^*\colon H^\bullet(X;\mathbf{Q}) \to H^\bullet(\mathrm{GL}(3);\mathbf{Q})$ associated to the orbit inclusion $\rho_f\colon \mathrm{GL}(3) \to X$ is surjective for any choice of a base point $f \in X$.

Next, let (P, f) be a point of $\mathcal{I}_0^-$ and let us denote by $p_0\colon \mathcal{I}_0^- \to X$ the natural projection, which is clearly $GL(3)$-equivariant. Then the orbit inclusion ρ_f is the composition of p_0 and the orbit inclusion $\rho_{(P,f)}\colon \mathrm{GL}(3) \to \mathcal{I}_0^-$. Hence, also the induced map ρ_f^* in cohomology is the composition of $\rho_{(P,f)}^*$ and p_0^*. This implies that $\rho_{(P,f)}^*$ is surjective, so in particular it satisfies the assumptions of Theorem II.2.2. This establishes the isomorphism (II.2.1).

The proof of the isomorphism (II.2.2) is analogous, and requires to consider the orbit map $\rho_{(P,f)}\colon \mathrm{GL}(3) \to \mathcal{I}_\delta^-$ associated with a point $(P, f) \in \mathcal{I}_\delta^-$. □

II.3 Quartic curves with a proper bitangent

In this section, we compute the rational cohomology of the moduli space $\mathcal{Q}_0^-$ of pairs (C, τ) such that C is a smooth quartic curve and τ a proper bitangent. This is the main ingredient in the proof of Theorem II.1.1.

Theorem II.3.1. *The HG polynomial of the rational cohomology of $\mathcal{Q}^-$ is equal to $1 + t\mathbf{L} + t^5\mathbf{L}^5 + 2t^6\mathbf{L}^6$.*

We start by considering the fibre of the map $\pi_0^-\colon \mathcal{I}_0^- \to B(\mathbf{P}^2, 2)$ over a configuration $\{p, q\}$ of distinct points in $\mathbf{P}^2$. Denote by t the line pq and set $t^* = t \setminus \{p, q\}$. Consider the 11-dimensional complex vector space

$$V_{p,q} := \left\{ f \in V \;\middle|\; \begin{array}{c} \text{the line } pq \text{ is either contained in } \mathbf{V}(f) \text{ or it} \\ \text{is tangent to } \mathbf{V}(f) \text{ at the points } p \text{ and } q \end{array} \right\}.$$

Then the fibre $(\pi_0^-)^{-1}(\{p, q\})$ is equal to $V_{p,q} \setminus \Sigma$. Hence, the fibre of π_0^- can be viewed as the complement of the discriminant in the vector space $V_{p,q}$. In particular, its cohomology can be computed using Vassiliev–Gorinov's method.

To apply Vassiliev–Gorinov's method to $V_{p,q} \cap \Sigma$ we need an ordered list of all possible singular sets of the elements in $V_{p,q} \cap \Sigma$. We obtain such a list by refining the list of possible singular configurations of quartic curves in [V99, Prop. 6]. For the convenience of the reader, we copied this list in Table II.2. In the right-hand side column of that table one can read the dimension of the space of quartic polynomials which are singular at any fixed configuration of the corresponding type.

For every configuration in Table II.2, one has to distinguish further whether the singular points are or are not in general position with respect to p and q (for instance, if the singular configuration intersects or not the line $t := pq$). This procedure yields a complete list of singular sets of elements of $V_{p,q} \cap \Sigma$, which we will describe in Sections II.5 and II.6. For every type j of singular configurations, we will denote by X_j the space of all configurations of type j.

As recalled in Section I.4, Vassiliev–Gorinov's method gives a recipe to construct spaces $|\mathcal{X}|$ and $|\Lambda|$ and a map

$$|\epsilon|\colon\ |\mathcal{X}| \to V_{p,q} \cap \Sigma$$

inducing an isomorphism in rational Borel–Moore homology. In the version of the method we use, the spaces $|\mathcal{X}|$ and $|\Lambda|$ are constructed as the geometric realizations of certain cubical spaces associated to the ordered list of singular sets. The Borel–Moore homology of $|\mathcal{X}|$ (respectively, $|\Lambda|$) can be computed by considering the stratification $F_\bullet$ (resp. $\Phi_\bullet$), which is indexed by the types of configurations in the list. The properties of F_j and Φ_j associated to the configuration type j are explained in Proposition I.4.3. Recall in particular that F_j is the total space of a vector bundle over Φ_j, and that for finite configurations Φ_j is the total space of a (non-orientable) bundle in open simplices over the configuration space X_j. As a consequence, for finite configurations the Borel–Moore homology of Φ_j coincides (after a shift in the indices) with the Borel–Moore homology of X_j with coefficients in a rank 1 local system changing its orientation every time two points in a configuration are interchanged. We will call this local system the *twisted local system* $\pm\mathbf{Q}$.

It is also possible to compute directly the cohomology of $\mathcal{I}_0^-$, without having to pass through the study of the fibre of π_0^-. Namely, consider the space

$$\mathcal{D}_0^- := \left\{ \begin{array}{c} (\{\alpha,\beta\}, f) \in F(\mathbf{P}^2, 2) \times \Sigma:\ \text{the line } \alpha\beta \text{ is either contained} \\ \text{in } \mathbf{V}(f) \text{ or it is tangent to it at the points } \alpha \text{ and } \beta \end{array} \right\}.$$

Note that $\mathcal{D}_0^-$ is a closed subset of

$$\mathcal{V}_0^- := \left\{ \begin{array}{c} (\{\alpha,\beta\}, f) \in B(\mathbf{P}^2, 2):\ \text{the line } \alpha\beta \text{ is either contained} \\ \text{in } \mathbf{V}(f) \text{ or it is tangent to it at the points } \alpha \text{ and } \beta \end{array} \right\}.$$

The space $\mathcal{V}_0^-$ is the total space of a vector bundle over $\mathbf{P}^2$, and $\mathcal{I}_0^- = \mathcal{V}_0^- \setminus \mathcal{D}_0^-$. Vassiliev-Gorinov's method can be exploited to compute the Borel–Moore

Table II.2: Singular sets in $\mathbf{P}^2$ of quartic homogeneous polynomials according to [V99, Prop. 6].

1	Any point in $\mathbf{P}^2$	$\mathbf{C}^{12}$
2	Any pair of points in $\mathbf{P}^2$	$\mathbf{C}^9$
3	Any three points on the same line in $\mathbf{P}^2$	$\mathbf{C}^7$
4	Any triple of non-collinear points in $\mathbf{P}^2$	$\mathbf{C}^6$
5	Any line in $\mathbf{P}^2$	$\mathbf{C}^6$
6	Any three points on the same line ℓ plus a point outside ℓ	$\mathbf{C}^4$
7	Any quadruple of points, no three of them collinear	$\mathbf{C}^3$
8	The union of a line in $\mathbf{P}^2$ and a point outside it	$\mathbf{C}^3$
9	Five points $\{a, b, c, d, e\}$ such that $\{e\} = ab \cap cd$.	$\mathbf{C}^2$
10	Six points which are the pairwise intersection of four lines in general position	$\mathbf{C}$
11	Any non-singular quadric in $\mathbf{P}^2$	$\mathbf{C}$
12	The union of two lines in $\mathbf{P}^2$	$\mathbf{C}$
13	The entire projective plane	0

homology of $\mathcal{D}_0^-$. This is done by defining the singular locus of an element $(\{\alpha, \beta\}, f)$ in $\mathcal{D}_0^-$ as the subset $\{\{\alpha, \beta\}\} \times K_f$ of $B(\mathbf{P}^2, 2) \times \mathbf{P}^2$, where K_f denotes the singular locus of the polynomial f. In particular, the classification of singular sets of elements of $\mathcal{D}_0^-$ is obtained by the classification of singular sets of elements of $V_{p,q} \setminus \Sigma$ by allowing the pair $\{p, q\}$ to move in $B(\mathbf{P}^2, 2)$.

Even though this is no longer the original setting of Vassiliev-Gorinov's method, one can mimic the construction of the cubical spaces Λ and $\mathcal{X}$ (see Section I.4) and obtain cubical spaces Λ' and $\mathcal{X}'$ that play an analogous role. In particular, the map $|\mathcal{X}'| \to \mathcal{D}_0^-$ induces an isomorphism on the Borel–Moore homology of these spaces, because it is a proper map with contractible fibres. Moreover, for the stratifications $\Phi'_\bullet$ and $F'_\bullet$ obtained from the construction of Λ' and $\mathcal{X}'$ we have natural maps $\Phi'_k \to B(\mathbf{P}^2, 2)$ and $F'_k \to B(\mathbf{P}^2, 2)$ which are locally trivial fibrations with fibre isomorphic to Φ_k, respectively, F_k.

In the next sections, we proceed by giving the classification of the singular sets in $\mathbf{P}^2$ of quartic curves that are tangent to t at p and q. These are exactly the singular sets of the elements of $V_{p,q} \cap \Sigma$. In view of the discussion above, this classification also yields the classification of the singular sets in $B(\mathbf{P}^2, 2) \times \mathbf{P}^2$ of the elements of $\mathcal{D}_0^-$.

Before giving the refined list, we briefly comment about which types of singular configurations will arise. A first distinction is between configurations containing a finite number of points versus configurations containing curves. In the specific case of plane quartics, singular curves are always rational. In particular, one can apply Lemma [T05a, 2.17] (and the remarks following it) to conclude that all strata Φ_j and F_j have trivial Borel–Moore homology for j a type of configuration which contain rational curves. Hence, it is important to concentrate on finite configurations.

A further distinction is whether the stabilizer of a general configuration of type j in PGL(3) is finite or not. We will call configuration types with finite (resp., infinite) stabilizer *rigid configurations* (resp., *non-rigid* configurations). Typically, non-rigid finite configurations will contain few singular points which will be relatively free to move. Anyway, it is important to notice that non-rigid configurations will give a non-trivial contribution only if they contain very few points. This follows from Lemma I.4.6, which ensures that the twisted Borel–Moore homology of configurations of more than one point in affine space vanishes in all degrees, and that the same is true for $B(\mathbf{P}^1, k)$ for $k \geq 3$ and $B(\mathbf{P}^2, k)$ for $k \geq 4$. We will deal with non-rigid configurations in Section II.5.

The main result is the following:

Proposition II.3.2. *Let us denote by $F_{\mathrm{nrig}} \subset |\mathcal{X}|$ the union of the strata corresponding to non-rigid configurations (for the precise definition of these, see Sect. II.4). Then the $\mathfrak{S}_2$-equivariant HG polynomial of F_{nrig} with respect to the $\mathfrak{S}_2$-action generated by the interchange of the points p and q is given by*

$$(3s_2 + s_{1,1})t^{20}\mathbf{L}^{-10} + (3s_2 + 3s_{1,1})t^{19}\mathbf{L}^{-9} + (s_2 + 3s_{1,1})t^{18}\mathbf{L}^{-8} + s_{1,1}t^{20}\mathbf{L}^{-7}.$$

At the other end of the spectrum one finds rigid configurations. As we will see in Section II.6, if configurations of type j are rigid, then the Borel–Moore homology of the strata Φ'_j and F'_j is automatically a tensor product of the Borel–Moore homology of PGL(3). For this reason, for rigid configurations it is practical to work directly with the configuration space $X'_j \subset B(\mathbf{P}^2, 2) \times \mathbf{P}^2$ rather than with the configuration space $X_j \subset \mathbf{P}^2$. As we explained above, the relationship between the two is that X'_j is fibred over $B(\mathbf{P}^2, 2)$ with fibre isomorphic to X_j. We will investigate the contribution of rigid configurations in Section II.6, where we will prove the following

Proposition II.3.3. *The HG polynomial of $F'_{\mathrm{rig}} := |\mathcal{X}'| \setminus F'_{\mathrm{nrig}}$ equals*

$$t^5(1 + 2t\mathbf{L}) \cdot \wp(\bar{H}_\bullet(\mathrm{GL}(3); \mathbf{Q})).$$

Lemma II.3.4. *The Borel–Moore homology with constant (respectively, twisted) rational coefficients of the space $B(\mathbf{P}^2, 2)$ of unordered configurations of 2 distinct points on $\mathbf{P}^2$ is given by*

$$\wp(\bar{H}_\bullet(B(\mathbf{P}^2, 2); \mathbf{Q})) = (1 + t^2\mathbf{L}^{-1} + t^4\mathbf{L}^{-2})t^4\mathbf{L}^{-2},$$

respectively, by

$$\wp(\bar{H}_\bullet(B(\mathbf{P}^2, 2); \pm\mathbf{Q})) = (1 + t^2\mathbf{L}^{-1} + t^4\mathbf{L}^{-2})t^2\mathbf{L}^{-1}.$$

Proof. The space $B(\mathbf{P}^2, 2)$ is fibred over the space $\mathbf{P}^{2\vee}$ of lines in $\mathbf{P}^2$ by the map $\{p, q\} \to pq$, which is $\mathfrak{S}_2$-equivariant. The fibre is isomorphic to the configuration space $B(\mathbf{P}^1, 2)$. Then the claim follows from $\wp(\bar{H}_\bullet(B(\mathbf{P}^1, 2); \mathbf{Q})) = t^4\mathbf{L}^{-2}$ and $\wp(\bar{H}_\bullet(B(\mathbf{P}^1, 2); \pm\mathbf{Q})) = t^2\mathbf{L}^{-1}$ (see Lemma I.4.6). ☐

The proof of Theorem II.3.1 follows from the last two parts of the following lemma.

Lemma II.3.5. *1. The differentials δ_k in the long exact sequence in Borel–Moore homology*

$$\cdots \to \bar{H}_{k+1}(\mathcal{D}_0^-; \mathbf{Q}) \to \bar{H}_{k+1}(F'_{\mathrm{rig}}; \mathbf{Q}) \xrightarrow{\delta_k} \bar{H}_k(F'_{\mathrm{nrig}}; \mathbf{Q}) \to \bar{H}_k(\mathcal{D}_0^-; \mathbf{Q}) \to \cdots \tag{II.3.1}$$

associated with the inclusion $F'_{\mathrm{nrig}} \subset |\mathcal{X}'|$ and the augmentation $\epsilon' \colon |\mathcal{X}'| \to \mathcal{D}_0^-$ vanish for all indices k.

2. *The contribution of non-rigid configurations to the cohomology of $\mathcal{I}_0^-$ has HG polynomial $1 + t\mathbf{L}$.*

3. *The contribution of rigid configurations to the cohomology of $\mathcal{I}_0^-$ has HG polynomial $t^5\mathbf{L}^5 + 2t^6\mathbf{L}^6$.*

Proof. Recall from Lemma II.2.1 that the cohomology of $\mathcal{I}_0^-$ is a tensor product of the cohomology of GL(3). There are two equivalent ways to compute the cohomology of $\mathcal{I}_0^-$. One possibility is to compute the Borel–Moore homology of $\mathcal{D}_0^-$ by using the long exact sequence (II.3.1) and successively calculate the Borel–Moore homology of $\mathcal{I}_0^-$ by the long exact sequence

$$\cdots \to \bar{H}_k(\mathcal{D}_0^-; \mathbf{Q}) \to \bar{H}_k(\mathcal{V}_0^-; \mathbf{Q}) \to \bar{H}_k(\mathcal{I}_0^-; \mathbf{Q}) \to \bar{H}_{k-1}(\mathcal{D}_0^-; \mathbf{Q}) \to \cdots \tag{II.3.2}$$

Since $\mathcal{V}_0^-$ is a complex vector bundle of rank 11 over $B(\mathbf{P}^2, 2)$, its Borel–Moore homology is equal to $\bar{H}_{\bullet-22}(B(\mathbf{P}^2, 2); \mathbf{Q}) \otimes \mathbf{Q}(11)$. In particular, from Lemma II.3.4 it follows that $\bar{H}_k(\mathcal{I}_0^-; \mathbf{Q}) \to \bar{H}_{k-1}(\mathcal{D}_0^-; \mathbf{Q})$ is an isomorphism for $k \leq 25$. If we compare this with the Borel–Moore homology of F'_{rig} as given in Proposition II.3.2, we have that all Borel–Moore homology classes of $\mathcal{D}_0^-$ coming from $\bar{H}_\bullet(F'_{\mathrm{rig}}; \mathbf{Q})$ via the long exact sequence (II.3.1) are in this range. Finally, since $\mathcal{I}_0^-$ is smooth and 15-dimensional, its cohomology and Borel–Moore homology are related by

$$H^\bullet(\mathcal{I}_0^-; \mathbf{Q}) = \bar{H}_{30-\bullet}(\mathcal{I}_0^-; \mathbf{Q}) \otimes \mathbf{Q}(-15). \tag{II.3.3}$$

Applying this to $\bar{H}_\bullet(F'_{\mathrm{rig}}; \mathbf{Q})$, one gets that its contribution to the cohomology of $\mathcal{I}_0^-$ is as described in (3), provided δ_k is trivial for all $k \leq 25$.

Another way to compute the cohomology of $\mathcal{I}_0^-$ is to use Vassiliev–Gorinov's method to compute the Borel–Moore homology of the discriminant $V_{p,q} \cap \Sigma$, then Alexander's duality (I.2.1) to deduce from this the cohomology of its complement

Table II.3: E_2 and E_3 terms of the Leray spectral sequence of the fibration $\mathcal{I} \to B(\mathbf{P}^2, 2)$ contributed from non-rigid configurations

q							
4	0	0	$\mathbf{Q}(-5)$	0	$\mathbf{Q}(-6)$	0	$\mathbf{Q}(-7)$
3	$\mathbf{Q}(-3)$	0	$\mathbf{Q}(-4)^4$	0	$\mathbf{Q}(-5)^4$	0	$\mathbf{Q}(-6)^3$
2	$\mathbf{Q}(-2)^3$	0	$\mathbf{Q}(-3)^6$	0	$\mathbf{Q}(-4)^6$	0	$\mathbf{Q}(-5)^3$
1	$\mathbf{Q}(-1)^3$	0	$\mathbf{Q}(-2)^4$	0	$\mathbf{Q}(-3)^4$	0	$\mathbf{Q}(-4)$
0	$\mathbf{Q}$	0	$\mathbf{Q}(-1)$	0	$\mathbf{Q}(-2)$	0	0
	0	1	2	3	4	5	6 p

q							
4	0	0	0	0	0	0	$\mathbf{Q}(-7)$
3	0	0	$\mathbf{Q}(-4)$	0	$\mathbf{Q}(-5)$	0	$\mathbf{Q}(-6)^2$
2	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)^2$	0	$\mathbf{Q}(-4)^2$	0	$\mathbf{Q}(-5)$
1	$\mathbf{Q}(-1)^2$	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$	0	0
0	$\mathbf{Q}$	0	0	0	0	0	0
	0	1	2	3	4	5	6 p

$V_{p,q} \setminus \Delta$, and finally compute the cohomology of $\mathcal{I}_0^-$ using the Leray spectral sequence associated to $\pi_0^- \colon \mathcal{I}_0^- \to B(\mathbf{P}^2, 2)$. If we follow this program for the contribution of rigid configurations, the E_2 term of the Leray spectral sequence in cohomology associated to π_0^- is as given in the first part of Table II.3.

The information given so far determines the contribution of rigid and non-rigid configurations, up to the d_2 differentials of the Leray spectral sequence associated with π_0^- and the computation of the kernel of the maps δ_k of (II.3.1). At this point, it is important to keep in mind that the rational cohomology of $\mathcal{I}_0^-$ has to be a tensor product of the cohomology of GL(3), whose HG polynomial is $(1-t\mathbf{L})(1-t^3\mathbf{L}^2)(1-t^5\mathbf{L}^3)$. Then one discovers that the only possibility to obtain $H^\bullet(\mathcal{I}_0^-;\mathbf{Q})$ with a structure as tensor product of $H^\bullet(\mathrm{GL}(3);\mathbf{Q})$ is that all maps δ_k are 0 and that all d_2 differentials in the Leray spectral sequence in Table II.3 have the maximal possible rank. The triviality of the maps δ_k yields part (1) of the claim. The result on the rank of the differentials of the Leray spectral sequence associated to π_0^- implies that the contribution of non-rigid configurations to the E_3 term of this spectral sequence is as given in the second part of Table II.3. In particular, this yields that the contribution of non-rigid configurations to the cohomology of $\mathcal{I}_0^-$ is as described in part (2) of the claim. □

Proof of Theorem II.3.1. The previous Lemma implies that the cohomology of $\mathcal{I}_0^-$ is the direct sum of the contribution of non-rigid and of rigid configurations, and that its HG polynomials is $(1+t\mathbf{L}+t^5\mathbf{L}^5+2t^6\mathbf{L}^6)\cdot\wp(H^\bullet(\mathrm{GL}(3);\mathbf{Q}))$. Then the claim follows from the isomorphism (II.2.1) in Lemma II.2.1. □

II.4 Proof of the main theorem

Proof of Theorem II.1.1. In Theorem II.11.1 we will prove that $H^0(\mathcal{Q}^-_\delta;\mathbf{Q}) = \mathbf{Q}$ is the only non-trivial rational cohomology group of $\mathcal{Q}^-_\delta$. Therefore, the cohomology of $\mathcal{Q}^-$ and $\mathcal{Q}^-_0$ coincide in degree larger than 2 and the latter is known from Theorem II.3.1.

To complete the proof of Theorem II.1.1, we only need to show that the cohomology group $H^1(\mathcal{Q}^-_0;\mathbf{Q}) = \mathbf{Q}(-1)$ is killed by $H^0(\mathcal{Q}^-_\delta;\mathbf{Q}) = \mathbf{Q}$. This is equivalent to show that $H^1(\mathcal{Q}^-;\mathbf{Q})$ vanishes.

The most direct way to show this is to consider the projectivization $\mathbf{P}^(\mathcal{I}^-)$ of $\mathcal{I}^-$, i.e. the space

$$\mathbf{P}(\mathcal{I}^-) = \{(P,t,C) \in \operatorname{Sym}^2\mathbf{P}^2 \times \mathbf{P}^{2\vee} \times \mathbf{P}(V\setminus\Sigma) \times \mathbf{P}^{2\vee} :\ C\cap t = P\}.$$

Then $\mathcal{Q}^-$ is the quotient of $\mathbf{P}(\mathcal{I}^-)$ by the natural action of PGL(3) and the Leray spectral sequence associated with $\mathbf{P}(\mathcal{I}^-) \to \mathcal{Q}^-$ degenerates at E_2 by Lemma II.2.1. As the cohomology of PGL(3) vanishes in degree 1 and 2, this implies that the first cohomology groups of $\mathbf{P}(\mathcal{I}^-)$ and $\mathcal{Q}^-$ are isomorphic.

To prove the vanishing of the cohomology of $\mathbf{P}(\mathcal{I}^-)$ in degree 1, we proceed as follows. First, we compactify $\mathbf{P}(\mathcal{I}^-)$ by considering the space

$$\mathcal{W} = \{(P,t,C) \in \operatorname{Sym}^2\mathbf{P}^2 \times \mathbf{P}^{2\vee} \times \mathbf{P}(V) \times \mathbf{P}^{2\vee} :\ C\cap t \supset P\}.$$

Considering the projection $\mathcal{W} \to \mathbf{P}^{2\vee}$ yields that $\mathcal{W}$ is a smooth variety with cohomology isomorphic to that of $\mathbf{P}^{10}\times\mathbf{P}^2\times\mathbf{P}^{2\vee}$ as graded vector spaces with pure Hodge structures. In particular, one has $H^2(\mathcal{W};\mathbf{Q}) = \mathbf{Q}(-1)^{\oplus 3}$.

The complement $\mathcal{W}\setminus\mathbf{P}(\mathcal{I}^-)$ is the union of three divisors: the divisor Σ_{a} whose general element is a triple (p_1+p_2,t,C) with C singular in either p_1 or p_2, the divisor Σ_{b} corresponding to curves C containing t and the divisor Σ_{c} whose general element is a curve C singular outside t. Thus, what we need to prove is that the fundamental classes of $\Sigma_{\mathrm{a}}, \Sigma_{\mathrm{b}}$ and Σ_{c} generate $H^2(\mathcal{W};\mathbf{Q})$. As $\mathcal{W}$ is smooth, we prove this by exploiting Poincaré duality and intersecting the Σ_{k} with the classes of three curves contained in $\mathcal{W}$. Specifically, we consider the class C_1 of a general pencil of curves such that all elements are bitagent to a fixed line t_0 in two prescribed points p,q; the class C_2 of a rational family $(\{x_2=0\},[1,0,0]+[t_0,t_1,0],C_{[t_0,t_1]})$ where the equation of $C_{[t_0,t_1]}$ is $x_1^2(t_1x_0 - t_0x_1)^2 + t_0^2x_2g(x_0,x_1,x_2)$ with g a fixed cubic polynomial; the class C_3 of triples $(\{t_0x_2 - t_1x_1 = 0\},[1,0,0]+[0,t_0,t_1],D_{[t_0,t_1]})$ where the equation of $D_{[t_0,t_1]}$ is $t_0x_0^2x_1^2 + (t_0x_2 - t_1x_1)h(x_0,x_1,x_2)$ with h a fixed cubic polynomial. Then the claim follows from the fact that the intersection matrix $(\Sigma_{\mathrm{k}}\cdot C_i)$ has maximal rank. □

II.5 Non-rigid configurations

In this section we deals with the configuration types between 1 and 6 in Vassiliev's list (Table II.2). We need to refine these configuration types to get the classification of singular configurations of elements in $V_{p,q} \cap \Sigma$. For this first group of configuration types, one gets the cases which we list in Table II.4. In that list, we maintain the reference to the corresponding types in Vassiliev's list (Table II.2) by indicating the refined strata by roman letters. Furthermore, when it is convenient to group refined strata together, we will denote them collectively by the letter x.

In Table II.4 we also describe the configuration spaces $X_{j\mathrm{k}}$ corresponding to each refined configuration type jk and the associated strata $\Phi_{j\mathrm{k}} \subset \left|\Lambda\right|$ and $F_{j\mathrm{k}} \subset \left|\mathcal{X}\right|$. From the this description one finds that configurations of types from 1a to 6x either are non-rigid, or give strata $F_{j\mathrm{k}}$ and $\Phi_{j\mathrm{k}}$ which have trivial Borel–Moore homology.

Next, we compute the contribution of non-rigid configurations to the spectral sequence e^r converging to $\bar{H}_\bullet(\left|\Lambda\right| \setminus \Phi_{13}; \mathbf{Q})$ associated with the stratification $\Phi_\bullet$ indexed by the configuration types. Hence, the e^1 term of this spectral sequence is given by $\mathrm{e}^1_{u,v} = \bar{H}_{u+v}(\Phi_u; \mathbf{Q})$, where u refers to the uth configuration type in our list. Rigid configurations contribute the first nine non-trivial columns, and specifically, to the configuration types 1a, 1b, 1c, 2a, 2b, 2c, 2d, 4a and 4b. For the sake of simplicity, we will omit from the spectral sequence all configuration types jk such that the Borel–Moore homology of $\Phi_{j\mathrm{k}}$ is trivial.

Then one gets from the description of the strata given in Table II.4 that $\mathrm{e}^1_{u,v}$ for $1 \le u \le 9$ is as in the first part of Table II.5.

Lemma II.5.1. *The spectral sequence $\mathrm{e}^r_{u,v} \Rightarrow \bar{H}_{u+v}(\left|\Lambda\right| \setminus \Phi_{13}; \mathbf{Q})$ associated with the stratification $\Phi_\bullet$ and converging to the Borel–Moore homology of $\left|\Lambda\right| \setminus \Phi_{13}$ satisfies*

$$\mathrm{e}^\infty_{1,-1} = \mathbf{S}_2,\ \mathrm{e}^\infty_{u,v} = 0 \text{ for } (u,v) \neq (1,-1)$$

for $-3 \le v \le 1$, $1 \le u \le 9$ (i.e. for all terms coming from non-rigid configurations).

Proof. This is based on the fact that for every $j = 1, \dots, 4$, the union of the spaces $\Phi_{j\mathrm{k}}$ in $\left|\Lambda\right|$ coincides with the contribution of configurations of type 1–4 to the auxiliary Vassiliev spectral sequence in the case of unmarked quartic curves treated in [V99, Thm 3]. Then the claim follows from Vassiliev's proof that configurations of type 1–4 contribute only trivially to the Borel–Moore homology of the open stratum of the spectral sequence converging to the Borel–Moore homology of the discriminant of unmarked plane curves. □

Remark II.5.2. In view of Lemma I.4.3, the stratum F_{13} corresponding to the configuration $\mathbf{P}^2$ is an open cone over $\left|\Lambda\right| \setminus \Phi_{13}$. Then Lemma II.5.1 proves that the only contribution of configurations of type 1–4 to the Borel–Moore homology

Table II.4: Configurations of type 1–6 (non-rigid configurations) and the associated strata.

1a The point p or the point q.
Stratum: F_{1a} is a $\mathbf{C}^{10}$-bundle over $\Phi_{1a} = \{p, q\}$.

1b Any point in t^*.
Stratum: F_{1b} is a $\mathbf{C}^9$-bundle over $\Phi_{1b} \cong \mathbf{C}^*$.

1c Any point in $\mathbf{P}^2 \setminus t$.
Stratum: F_{1c} is a $\mathbf{C}^8$-bundle over $\Phi_{1c} \cong \mathbf{C}^2$.

2a The pair $\{p, q\}$.
Stratum: F_{2a} is a $\mathbf{C}^9$-bundle over $\Phi_{2a} \cong \mathring{\Delta}_1$.

2b Any other pair of points on t.
Stratum: F_{2b} is a $\mathbf{C}^8$-bundle over Φ_{2b}, which is a non-orientable $\mathring{\Delta}_1$–bundle over $B(t, 2) \setminus \{\{p, q\}\}$.

2c One point in $\{p, q\}$ and any point outside t.
Stratum: F_{2c} is a $\mathbf{C}^7$-bundle over Φ_{2c}, which is a non-orientable $\mathring{\Delta}_1$-bundle over the disjoint union of two copies of $\mathbf{C}^2$.

2d A point on t^* and any point outside t.
Stratum: F_{2d} is a $\mathbf{C}^6$-bundle over Φ_{2d}, which is a non-orientable $\mathring{\Delta}_1$-bundle over $\mathbf{C}^* \times \mathbf{C}^2$.

2e Any pair of points in $\mathbf{P}^2 \setminus t$.
Stratum: F_{2e} is a $\mathbf{C}^5$-bundle over Φ_{2e}, which is a non-orientable $\mathring{\Delta}_1$-bundle over $B(\mathbf{C}^2, 2)$. Therefore, the Borel–Moore homology of Φ_{2e} and F_{2e} is trivial.

3x Any three points on the same line ℓ in $\mathbf{P}^2$.
There are different strata to consider ($\ell = t$; $p \in \ell \neq t$ and p is one of the singular points; $q \in \ell \neq t$ and q is one of the singular points; the three points do not contain p or q but ℓ do; $\ell \neq \{p, q\} = \emptyset$). In view of Lemma I.4.6, the Borel–Moore homology of the configuration space $B(\ell, 3)$ is trivial. Hence, all these strata contribute trivially.

4a p, q and a further point outside t.
Stratum: F_{4a} is a $\mathbf{C}^6$-bundle over Φ_{4a}, which is a non-orientable $\mathring{\Delta}_2$-bundle over $\mathbf{C}^2$.

4b Any other pair of points on t and a point outside it.
Stratum: F_{4b} is a $\mathbf{C}^5$-bundle over Φ_{4b}, which is a non-orientable $\mathring{\Delta}_2$–bundle over $\mathbf{C}^2 \times (B(t, 2) \setminus \{\{p, q\}\})$.

4x Any other triple of non-collinear points in $\mathbf{P}^2$.
There are several cases to consider (p or q and two more points lying outside t; one point on t and two points outside; three points outside t). All of them contribute trivially in view of Lemma I.4.6.1.

5x Any line ℓ in $\mathbf{P}^2$.
There are several cases to be considered, according to whether the line ℓ is t, it passes through p of q, or it is in general position with respect to p, q. Observe that for every line ℓ in $\mathbf{P}^2$, all subsets of ℓ of cardinality at most 3 belong to configurations of type 1–4. This allows us to apply Lemma [T05a, 2.17] and conclude that the contribution of all strata of type 5x is trivial.

6x Any three points on the same line ℓ plus a point outside ℓ.
Several cases, all of them do not contribute. The proof is analogous to case 3x.

Table II.5: E^1 terms of the spectral sequences $\mathrm{e}^r_{u,v} \Rightarrow \bar{H}_{u+v}(\lvert\Lambda\rvert \setminus \Phi_{13}; \mathbf{Q})$ and $\mathrm{E}^r_{u,v} \Rightarrow \bar{H}_\bullet(\lvert\mathcal{X}\rvert; \mathbf{Q}) = \bar{H}_\bullet(V_{p,q} \cap \Sigma; \mathbf{Q})$.

$\mathrm{e}^1_{u,v}$ for $1 \leq u \leq 9$.

v									
1	0	0	$\mathbf{S}_2(2)$	0	0	0	0	0	0
0	0	$\mathbf{S}_2(1)$	0	0	0	0	$\mathbf{S}_2(3)$	0	0
-1	$\mathbf{S}_2 + \mathbf{S}_{1,1}$	$\mathbf{S}_{1,1}$	0	0	0	$(\mathbf{S}_2 + \mathbf{S}_{1,1})(2)$	$\mathbf{S}_{1,1}(2)$	0	$\mathbf{S}_2(3)$
-2	0	0	0	0	$\mathbf{S}_2(1)$	0	0	$\mathbf{S}_{1,1}(2)$	$\mathbf{S}_{1,1}(2)$
-3	0	0	0	$\mathbf{S}_{1,1}$	$\mathbf{S}_{1,1}$	0	0	0	0
u	1	2	3	4	5	6	7	8	9
type	(1a)	(1b)	(1c)	(2a)	(2b)	(2c)	(2d)	(4a)	(4b)

$\mathrm{E}^1_{u,v}$ for $1 \leq u \leq 9$.

v									
19	$(\mathbf{S}_2 + \mathbf{S}_{1,1})(10)$	0	0	0	0	0	0	0	0
18	0	$\mathbf{S}_2(10)$	0	0	0	0	0	0	0
17	0	$\mathbf{S}_{1,1}(9)$	$\mathbf{S}_2(10)$	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0
15	0	0	0	$\mathbf{S}_{1,1}(9)$	0	0	0	0	0
14	0	0	0	0	$\mathbf{S}_2(9)$	0	0	0	0
13	0	0	0	0	$\mathbf{S}_{1,1}(8)$	$(\mathbf{S}_2 + \mathbf{S}_{1,1})(9)$	0	0	0
12	0	0	0	0	0	0	$\mathbf{S}_2(9)$	0	0
11	0	0	0	0	0	0	$\mathbf{S}_{1,1}(8)$	0	0
10	0	0	0	0	0	0	0	$\mathbf{S}_{1,1}(8)$	0
9	0	0	0	0	0	0	0	0	$\mathbf{S}_2(8)$
8	0	0	0	0	0	0	0	0	$\mathbf{S}_{1,1}(7)$
u	1	2	3	4	5	6	7	8	9
type	(1a)	(1b)	(1c)	(2a)	(2b)	(2c)	(2d)	(4a)	(4b)

of $|\Lambda| \setminus \Phi_{13}$ is to $\bar{H}_0(|\Lambda| \setminus \Phi_{13}; \mathbf{Q}) = 0$. If we decompose the open cone F_{13} over $|\Lambda| \setminus \Phi_{13}$ as the union of its vertex and a $\mathring{\Delta}_1$-bundle over $|\Lambda| \setminus \Phi_{13}$, we see that the Borel–Moore homology group $\bar{H}_1(\mathring{\Delta}_1; \mathbf{Q}) \otimes \bar{H}_0(|\Lambda| \setminus \Phi_{13}; \mathbf{Q})$ is killed by the Borel–Moore homology of the vertex. In other words, this implies that configurations of type 1–4 contribute trivially to the Borel–Moore homology of the stratum $F_{13} \subset |\mathcal{X}|$.

We compute the contribution of non-rigid configurations to the spectral sequence $\mathrm{E}^r_{u,v} \Rightarrow \bar{H}_{u+v}(|\mathcal{X}|; \mathbf{Q}) \cong \bar{H}_{u+v}(V_{p,q} \cap \Sigma; \mathbf{Q})$ associated with the stratification $F_\bullet$. Again, this will give the first 9 columns of the spectral sequence. If we restrict our considerations to these first 9 columns, we obtain a spectral sequence converging to the Borel–Moore homology of the space $F_{\mathrm{nrig}} := \bigcup F_{j\mathrm{k}}$ where the union is over all configurations jk between 1a and 6x.

Lemma II.5.3.

The E^1 *terms of the spectral sequence* $\mathrm{E}^r_{u,v} \Rightarrow \bar{H}_{u+v}(V_{p,q} \cap \Sigma; \mathbf{Q})$ *associated with the stratification* $F_\bullet$ *for* $1 \leq u \leq 9$ *are as given in the second part of Table II.5.*

Proof. In this spectral sequence, the E^1 term is given by $\mathrm{E}^1_{u,v} = \bar{H}_{u+v}(F_u; \mathbf{Q})$, where u refers to the uth configuration type in our list. Since F_u is a vector bundle of a certain rank k_u over Φ_u, one has $\mathrm{E}^1_{u,v} = \mathrm{e}^1_{u,v-2k_u} \otimes \mathbf{Q}(k_u)$. □

Proof of Proposition II.3.2. Since F_{nrig} is the union of the strata $F_{j\mathrm{k}}$ with $j \leq 6$, its Borel–Moore homology can be computed by a spectral sequence whose E^1 term coincides with $\mathrm{E}^1_{u,v}$ if $u \leq 9$ and is 0 if $u \geq 10$. Hence, the E^1 term coincides with the $\mathrm{E}^1_{u,v}$ in the second part of Table II.5.

We observe that for $1 \leq u \leq 9$ the Hodge structure in $\mathrm{E}^1_{u,v}$ is pure of weight $-2(u+v-10)$. This implies that for every u, r such that $1 \leq u < u+r \leq 9$, the Hodge weight of $\mathrm{E}^r_{u,v}$ and $\mathrm{E}^r_{u+r,v-r+1}$ are different, hence all d_r differentials vanish in this range. From this the claim follows. □

II.6 Rigid configurations

In this section, we refine the second part of List II.2 (i.e. configuration types from 7 to 13) to complete the list of singular configurations we need to apply Vassiliev–Gorinov's method to $V_{p,q} \cap \Sigma$ and the incidence correspondence $\mathcal{D}_0^-$. As we have briefly explained in Section II.3, the configuration spaces associated with the refinements of configuration types 7–12 give a non-trivial contribution unless they correspond to rigid configurations, i.e. finite configurations $(\{p,q\}, \{s_1, \dots, s_r\}) \subset \mathbf{P}^2 \times \mathbf{P}^2$ with finite stabilizer in PGL(3). For such configurations, the computation of the Borel–Moore homology is easier for the "fibred" configuration space X'_j than for the configuration space X_j where we assume the bitangent to be fixed.

Table II.6: Rigid configurations of type 7

7a Any quadruple of points containing p and q. No three points in the configuration are allowed to be collinear.
Stratum: $F_{7\mathrm{a}}$ is a $\mathbf{C}^3$-bundle over $\Phi_{7\mathrm{a}}$, which in turn is a non-orientable $\mathring{\Delta}_3$-bundle over the configuration space $X_{7\mathrm{a}}$. The space $X_{7\mathrm{a}}$ is isomorphic to

$$\left\{\{a,b\} \in B(\mathbf{P}^2 \setminus t, 2) \mid ab \cap \{p,q\} = \emptyset\right\}.$$

The complement $\mathbf{P}^2 \setminus t$ is isomorphic to $\mathbf{C}^2$, hence the twisted Borel–Moore homology of $B(\mathbf{P}^2 \setminus t, 2)$ vanishes by Lemma I.4.6. Analogously, also the twisted Borel–Moore homology of $\left\{\{a,b\} \in B(\mathbf{P}^2 \setminus t, 2) \mid ab \cap \{p,q\} \neq \emptyset\right\}$ vanishes, since it is a $B(\mathbf{C},2)$-bundle over $\mathbf{C} \sqcup \mathbf{C}$. Hence the twisted Borel–Moore homology of $X_{7\mathrm{a}}$ is trivial.

7b Any quadruple of points of which exactly two lie on t. No three points are allowed to be collinear and $\{p,q\}$ cannot be contained in the configuration.
Stratum: $F_{7\mathrm{b}}$ is a $\mathbf{C}^2$-bundle over $\Phi_{7\mathrm{b}}$, which has trivial Borel–Moore homology (the proof is analogous to that for case 7a).

7c Any quadruple $\{a,b,c,d\}$ of points lying outside t, such that no three points in the configuration lie on the same line, and t is a common bitangent to two distinct quadrics in the pencil passing through $\{a,b,c,d\}$.
Stratum: $F'_{7\mathrm{c}}$ is a $\mathbf{C}$-bundle over the space $\Phi'_{7\mathrm{c}}$, which is a non-orientable $\mathring{\Delta}_3$-bundle over the configuration space $X'_{7\mathrm{c}} \subset B(\mathbf{P}^2,2) \times B(\mathbf{P}^2,4)$. In Lemma II.7.1 we will prove that the twisted Borel–Moore homology of $X'_{7\mathrm{c}}$ vanishes.

7d Quadruples $\{a,b,c,d\} \not\supset \{p,q\}$ of points in general linear position such that there is a conic $C \not\supset t$ passing through p,q,a,b,c,d. The conic C is allowed to be singular.
The stratum $F'_{7\mathrm{d}}$ is a $\mathbf{C}$-bundle over the space $\Phi'_{7\mathrm{d}}$, which is a non-orientable $\mathring{\Delta}_3$-bundle over the configuration space $X'_{7\mathrm{d}}$ studied in Section II.8. In Lemma II.8.1 we will prove that the twisted Borel–Moore homology of $X'_{7\mathrm{d}}$ vanishes.

The refinement of configuration type 7 (four points in general position) gives the four configuration types described in Table II.6. For all of these configurations 7k, $\mathrm{k} \in \{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\}$ we can prove that the twisted Borel–Moore homology of the associated configuration space $X'_{7\mathrm{k}}$ vanishes (see in particular Lemma II.7.1 and II.8.1). Here we abuse notation and we define the twisted local system $\pm\mathbf{Q}$ for a fibred configuration space $S \subset B(\mathbf{P}^2,2) \times B(\mathbf{P}^2,k)$ as the restriction to S of the pull-back of the twisted local system $\pm\mathbf{Q}$ under the projection $B(\mathbf{P}^2,2) \times B(\mathbf{P}^2,k) \to B(\mathbf{P}^2,k)$.

Configuration type 8 corresponds to the union of a line ℓ in $\mathbf{P}^2$ and a point s outside it. This type gives rise to several refined configuration types. For instance, one has to distinguish if ℓ coincides with the bitangent t, if it passes through one of the bitangency points p,q or through none of them. Also the point s may lie on t, coincide with either p or q or simply lie on t. For every refined substratum 8k one has that the Borel–Moore homology of the space $\Phi_{8\mathrm{k}}$ vanishes, and hence the same holds for $F_{8\mathrm{k}}$. This follows from Lemma [T05a, 2.17] and following remarks). To apply that lemma, we have to check that for every $\ell \cup \{s\} \in X_{8\mathrm{k}}$ the space $B(\ell, 4) \times \{s\}$ was contained in one of the preceding configurations 1a–7d. Moreover, one has to check that for a fixed $\ell \cup \{s\} \in X_{8\mathrm{k}}$ the vector subspaces $L(K) = \{f \in \Sigma \cap V_{p,q} | K_f \supset K\}$ for every $K \in B(\ell, 4) \times \{s\}$ defines a vector bundle. This ensures the vanishing of the Borel–Moore homology of $\Phi_{8\mathrm{k}}$ and $F_{8\mathrm{k}}$ for all k.

The refined singular configurations of type 9 and 10 are described in Table II.7. We will calculate the contribution of these configuration types in Sections II.9 and II.10. Configurations of type 11 (non-singular conics in $\mathbf{P}^2$ passing through p and q) and of type 12 (the union of two lines containing p and q) correspond to strata Φ_{11} and Φ_{12} with trivial Borel–Moore homology. In both cases the singular configurations are (possibly reducible) rational curves. The vanishing of Borel–Moore homology follows from Lemma [T05a, 2.17] in the case of Φ_{11} and from Lemma [T05a, 2.17] in the case of Φ_{12}.

The only remaining stratum is the stratum $F'_{13} \subset \left|\mathcal{X}'\right|$ corresponding to the configuration $\mathbf{P}^2$. As explained in Proposition I.4.3, the stratum F'_{13} is a topological open cone with vertex a point over the space $\left|\Lambda'\right| \setminus \Phi'_{13}$, which is the union of all $\Phi'_{j\mathrm{x}}$ with $j \leq 12$.

Lemma II.6.1. *The HG polynomial of the Borel–Moore homology of F'_{13} equals*

$$t^6 \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})).$$

Proof. Let us denote by $\mathrm{B} := \left|\Lambda'\right| \setminus \Phi'_{13}$ the base of the open cone F'_{13}. We intend to compute its Borel–Moore homology by using the spectral sequence associated to the stratification $\Phi_\bullet$. We have already proved that the Borel–Moore homology of $\Phi'_{j\mathrm{x}}$ with $5 \leq j \leq 8$ or $11 \leq j \leq 12$ is trivial, either in a straightforward way because these configurations contain too many points on the same rational curve, or in the Lemmas II.7.1 and II.8.1. The union $\bigcup_{\mathrm{k} \in \{\mathrm{a,b,c,d}\}} \Phi'_{9\mathrm{k}}$ has trivial Borel–Moore homology in view of Lemma II.9.5. Furthermore, as we explained in Remark II.5.2, configurations of type 1–4 contribute trivially to the Borel–Moore homology of F'_{13}. From this it follows that the only strata contributing to the Borel–Moore homology of the basis of the open cone F'_{13} are $\Phi'_{9\mathrm{e}}$ and Φ'_{10}. Therefore, there is a long exact sequence

$$\cdots \to \bar{H}_k(\Phi'_{9\mathrm{e}}; \mathbf{Q}) \to \bar{H}_k(\mathrm{B}; \mathbf{Q}) \to \bar{H}_k(\Phi'_{10}; \mathbf{Q}) \xrightarrow{\delta_k} \bar{H}_{k-1}(\Phi'_{9\mathrm{e}}; \mathbf{Q}) \to \cdots$$

in Borel–Moore homology. In Lemma II.9.4 and Lemma II.10.1 we prove that both the Borel–Moore homology of $\Phi'_{9\mathrm{e}}$ and of Φ'_{10} are tensor products of the Borel–Moore homology of PGL(3). This is a consequence of the fact that these configurations are rigid. Since PGL(3) acts equivariantly on the whole of $\mathcal{D}_0^-$ and $\left|\Lambda'\right|$, the differentials δ_k have to respect this structure as tensor products of

Table II.7: Rigid configurations of type 9 and 10.

9a Configurations of five points $\{a,b,c,d,e\}$ such that $a,b \in t$ and $t \cap cd = \{e\}$.
Stratum: $F_{9\mathrm{a}}$ is a $\mathbf{C}^2$-bundle over $\Phi_{9\mathrm{a}}$, which has trivial Borel–Moore homology. This follows from the fact that the map $X_{9\mathrm{a}} \to B(\mathbf{C}^2, 2)$ mapping $\{a,b,c,d,e\}$ as above to $\{c,d\}$ is a locally trivial fibration with fibre isomorphic to $B(\mathbf{C}, 2)$. Since $\bar{H}_\bullet(B(\mathbf{C},2); \pm\mathbf{Q})$ vanishes, the twisted Borel–Moore homology of $X_{9\mathrm{a}}$ must vanish as well.

9b Configurations of five points $\{p,q,a,b,c\}$ with $pa \cap qb = \{c\}$.
Stratum: $F_{9\mathrm{b}}$ is a $\mathbf{C}^2$-bundle over $\Phi_{9\mathrm{b}}$, which is a $\mathring{\Delta}_4$-bundle over the space $X_{9\mathrm{b}}$ studied in Section II.9.

9c Configurations of five points $\{a,b,c,d,e\}$ such that $ab \cap cd = \{e\}$, $p \in ab \setminus \{e\}$, $q \in cd \setminus \{e\}$ and $\{p,q\} \not\subset \{a,b,c,d\}$.
Stratum: $F'_{9\mathrm{c}}$ is a $\mathbf{C}$-bundle over $\Phi'_{9\mathrm{c}}$, which is a non-orientable $\mathring{\Delta}_4$-bundle over the space $X'_{9\mathrm{c}}$ studied in Section II.9. Note that for a configuration of type 9c the unique quartic curve with the prescribed singularities and with $t = pq$ as bitangent is the degenerate double conic $ab \cup cd$.

9d Configurations of five points $\{a,b,c,d,e\}$ with $e \in \{p,q\}$, $a,b,c,d \notin t$ such that $ab \cap cd = \{e\}$ and t is tangent to the conic passing through a,b,c,d,q for $e = p$, and to the conic through a,b,c,d,p for $e = q$.
Stratum: $F_{9\mathrm{d}}$ is a $\mathbf{C}$-bundle over $\Phi_{9\mathrm{d}}$, which is a non-orientable $\mathring{\Delta}_4$-bundle over the space $X_{9\mathrm{d}}$ studied in Section II.9. In Section II.9 we will prove that the fibred configuration space $X'_{9\mathrm{c}}$ has trivial twisted Borel–Moore homology.

9e Configurations of five points $\{a,b,c,d,e\}$ with $a,b \in t$, $\{a,b\} \neq \{p,q\}$ such that $ac \cap bd = \{e\}$.
Stratum: $F_{9\mathrm{e}}$ is a $\mathbf{C}$-bundle over $\Phi_{9\mathrm{e}}$, which is a non-orientable $\mathring{\Delta}_4$-bundle over the space $X_{9\mathrm{e}}$ studied in Section II.9. Note that for a configuration of type 9e, the unique quartic curve with the prescribed singularities with $t = pq$ as bitangent is the union of lines $ab \cup ac \cup bd \cup cd$.

10 Six points which are the pairwise intersection of four lines in general position.
Stratum: F'_{10} is a $\mathbf{C}$-bundle over Φ'_{10}, which is a non-orientable $\mathring{\Delta}_5$-bundle over the space X'_{10}. The Borel–Moore homology of the stratum F'_{10} will be computed in Section II.10.

$\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$. In particular, in our specific case this implies that all δ_k are induced from the differential δ_{25} between the non-trivial Borel–Moore homology classes of Φ_{10} and $\Phi_{9\mathrm{e}}$ in top degree. Furthermore, the claim is equivalent to showing that the differential δ_{25} is an isomorphism.

Assume by contradiction that δ_{25} were the 0 map. Then we would have $\bar{H}_{25}(\mathrm{B};\mathbf{Q}) = \mathbf{Q}(10)$ and thus $\bar{H}_{26}(F'_{13};\mathbf{Q}) = \mathbf{Q}(10)$ for the open cone over B. By briefly comparing this with the Borel–Moore homology of the strata $F'_{j\mathrm{x}}$ with $j \leq 12$, we find that the contribution of $\bar{H}_{26}(F'_{13};\mathbf{Q})$ to the spectral sequence $\mathbf{E}^r_{p,q} \Rightarrow \bar{H}_\bullet(\mathcal{D}_0^-;\mathbf{Q})$ cannot be killed by any differential of that spectral sequence. In particular, this means that $\bar{H}_{26}(\mathcal{D}_0^-;\mathbf{Q})$ is an extension of $\mathbf{Q}(10)$. By duality (see (II.3.2) and (II.3.3)), this would imply that $H^3(\mathcal{I}_0^-;\mathbf{Q})$ is an extension of $\mathbf{Q}(-5)$, which is clearly impossible since the Hodge weight of $\mathbf{Q}(-5)$ is $10 > 2\cdot 3$, whereas Hodge weights in cohomology can never be larger than twice the degree.

From this it follows that δ_{25} must have rank 1 and

$$\bar{H}_\bullet(\mathrm{B};\mathbf{Q}) = \bar{H}_{\bullet-5}(\mathrm{PGL}(3);\mathbf{Q}).$$

Then the claim follows from the structure of F'_{13} as an open cone over B. □

Remark II.6.2. One can also give a direct proof of the non-vanishing of δ_{25} based on geometric considerations on the configuration spaces involved.

Proof of Proposition II.3.3. We consider the spectral sequence

$$\mathbf{E}^r_{p,q} \Rightarrow \bar{H}_{p+q}(\mathcal{D}_0^-;\mathbf{Q}),\ \mathbf{E}^1_{p,q} = \bar{H}_{p+q}(F'_p;\mathbf{Q}).$$

We concentrate on the rigid configuration types, i.e. those of type jx with $7 \leq j \leq 13$. Their union is the space F'_{rig} of which we want to compute the Borel–Moore homology.

In view of the results in this section and in Sections II.8–II.10, the only strata $F'_{j\mathrm{x}}$ with non-trivial Borel–Moore homology are those of type 9x, 10 and 13, whose Borel–Moore homology is computed in Lemmas II.9.1–II.9.4, II.10.1 and II.6.1. Furthermore, the Borel–Moore homology of each of these strata is a tensor product of the Borel–Moore homology of PGL(3). Hence, the E^1 terms coming from such configurations are of the form $\mathbf{E}^1_{p,\bullet} = \hat{\mathbf{E}}^1_{p,\bullet} \otimes \bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q})$. We give the $\hat{\mathbf{E}}^1$ terms in Table II.8. Note that, by construction, differentials should respect the structure of the columns of the spectral sequence as tensor products of the Borel–Moore homology of PGL(3). This implies that the only differential that can possibly be non-trivial is $\hat{d}^1\colon \hat{\mathbf{E}}^1_{1,10} = \mathbf{Q}(3) \to \mathbf{E}^1_{1,9} = \mathbf{Q}(3)$. From the definition of $\mathbf{E}^1_{p,q}$ we have $\hat{\mathbf{E}}^1_{1,10} \otimes \bar{H}_{16}(\mathrm{PGL}(3);\mathbf{Q}) = \bar{H}_{27}(F'_{10};\mathbf{Q})$ and $\hat{\mathbf{E}}^1_{1,9} \otimes \bar{H}_{16}(\mathrm{PGL}(3);\mathbf{Q}) = \bar{H}_{26}(F'_{9\mathrm{e}};\mathbf{Q})$. This means that d^1 is induced by the differential $\bar{H}_{27}(F'_{10};\mathbf{Q}) \to \bar{H}_{26}(F'_{9\mathrm{e}};\mathbf{Q})$ of the long exact sequence associated to the inclusion of $F'_{9\mathrm{e}}$ as a closed subset of $F'_{9\mathrm{e}} \cup F'_{10}$.

We claim that d^1 is an isomorphism or, equivalently, that the Borel–Moore homology of $F'_{9\mathrm{e}} \cup F'_{10}$ vanishes in degree 27. Indeed, the union $F'_{9\mathrm{e}} \cup F'_{10}$ is a

Table II.8: $\hat{\mathbf{E}}^1$ terms associated with the spectral sequence converging to the Borel–Moore homology of $\mathcal{D}_0^-$ coming from configurations of type 5–13.

q				
9	$\mathbf{Q}(3)$	$\mathbf{Q}(3)$	0	
8	0	0	0	
7	$\mathbf{Q}(2)$	0	0	
6	$\mathbf{Q}(1)^{\oplus 2}$	0	0	
5	0	$\mathbf{Q}(1)$	0	
4	0	0	0	
3	0	0	$\mathbf{Q}^{\oplus 2}$	
	1	2	3	p
type	(9x)	(10)	(13)	

$\mathbf{C}$-bundle over $\Phi'_{9e} \cup \Phi'_{10}$, whose Borel–Moore homology in degree 25 vanishes by the proof of Lemma II.6.1. From this the claim follows. In particular, the spectral sequence $\mathbf{E}^r_{p,q}$ restricted to rigid configurations types degenerates at E^2. $\square$

II.7 Configuration type 7c — Pencils of conics

The aim of this section is to compute the contribution of singular configurations of type 7c (see page 41) to the Vassiliev spectral sequence converging to the Borel–Moore homology of the incidence correspondence $\mathcal{D}_0^-$. In other words, we will compute the rational Borel–Moore homology of the spaces Φ'_{7c} and F'_{7c}.

Lemma II.7.1. *The stratum $\Phi'_{7c} \subset \left|\Lambda'\right|$ and of the stratum F'_{7c} of $\left|\mathcal{X}'\right|$ have trivial Borel–Moore homology.*

Proof. We start by determining the twisted Borel–Moore homology of the underlying family of configurations X'_{7c}. Denote by $\tilde{B}(\mathbf{P}^2, 4)$ the space of quadruples of points in general position, i.e. such that no three of the points lie on the same line. For every element K of $\tilde{B}(\mathbf{P}^2, 4)$ there is exactly one pencil of conics $M_K \subset V$ with base locus K. For every point $p \in \mathbf{P}^2 \setminus K$ we denote by $Q_{K,p}$ the unique conic in K passing through p.

The family of configurations $X_{7c} \subset B(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^2, 4)$ is the locus of configurations (α, β, K) such that $K \cap \alpha\beta = \emptyset$ and the pencil M_K contains a conic tangent to $\tau = \alpha\beta$ in α and a conic tangent to τ in β. We can rephrase this by saying that the space $X'_{7c} \subset B(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^2, 4)$ is the space of configurations $(\{\alpha, \beta\}, K)$ such that the tangent line at α to the conic $Q_{K,\alpha}$ and the tangent line at β to the conic $Q_{K,\beta}$ are both equal to $\tau := \alpha\beta$. Note that two general conics admit exactly four common tangents of this type. If we fix the base locus K of the pencil of conics and a line τ disjoint from K, we have that exactly two conics in M_K are tangent to τ. These two conics coincide if and only if τ intersects the base locus of the pencil M_K.

From the discussion above, it follows that X'_{7c} is isomorphic to the locus $Y_{7c} \subset \mathbf{P}^{2\vee} \times \tilde{B}(\mathbf{P}^2, 4)$ of configurations (τ, K) where the line τ does not contain any of the points in K. The natural map $Y_{7c} \xrightarrow{\sim} X'_{7c} \to B(\mathbf{P}^2, 2)$ is defined by associating to (τ, K) the set of two points at which τ is tangent to a conic in M_K.

Let us start from the case in which the configuration $K = \{a_1, a_2, a_3, a_4\}$ is fixed. Then the space of lines τ such that $\tau \cap K = \emptyset$ is the complement in $\mathbf{P}^{2\vee}$ of the union of four lines $a_i{}^\vee$ in general position, given by that pencil of lines passing through each of the a_i. To proceed we need to know the Borel–Moore homology of $\mathcal{U} := \mathbf{P}^{2\vee} \setminus \bigcup a_i{}^\vee$ and its structure as representation of the symmetric group $\mathfrak{S}_4$ given by the natural action of $\mathfrak{S}_4$ permuting the points in K. This is computed in Lemma II.7.2 below. In particular, the Borel–Moore homology of $\mathcal{U}$ does not contain any alternating classes. Since $\mathcal{U}$ is isomorphic to the fibre of $X'_{7c} \to \tilde{B}(\mathbf{P}^2, 4)$ and the whole Borel–Moore homology of $\tilde{F}(\mathbf{P}^2, 4) \cong \mathrm{PGL}(3, \mathbf{C})$ is $\mathfrak{S}_4$-invariant, this implies that also the Borel–Moore homology of X'_{7c} does not contain any non-trivial $\mathfrak{S}_4$-alternating class.

In view of the structure of Φ'_{7c} as non-orientable simplicial bundle over X'_{7c}, and of F'_{7c} as vector bundle over Φ'_{7c}, the vanishing of the twisted Borel–Moore homology of X'_{7c} implies the vanishing of the Borel–Moore homology of Φ'_{7c} and F'_{7c} is trivial as well. $\square$

Lemma II.7.2. *The $\mathfrak{S}_4$-equivariant HG polynomial of $\bar{H}_\bullet(\mathbf{P}^{2\vee} \setminus \bigcup a_i{}^\vee; \mathbf{Q})$ is equal to $t^4\mathbf{L}^2 s_4 + t^3 s_{3,1}\mathbf{L} + t^2 s_{3,1}$.*

Proof. We start by computing the Borel–Moore homology of $\mathcal{C} := \bigcup a_i{}^\vee$. First we consider the singular locus of $\mathcal{C}$, which is the union of six points on which $\mathfrak{S}_4$ acts as the representation $\mathbf{S}_4 \oplus \mathbf{S}_{2,2} \oplus \mathbf{S}_{2,1,1}$. For each i, the locus $a_i{}^\vee \setminus \mathcal{C}_{\mathrm{sing}}$ is isomorphic to $\mathbf{P}^1$ minus three points. To determine the $\mathfrak{S}_4$-action on the Borel–Moore homology of $\mathcal{C} \setminus \mathcal{C}_{\mathrm{sing}}$, we start by observing that the group $\mathfrak{S}_3$ permuting the three singular points on $a_4{}^\vee$ acts as $\mathbf{S}_3$ on the Borel–Moore homology in degree 2 and as $\mathbf{S}_{2,1}$ in degree 1. By extending these representations to representations of $\mathfrak{S}_4$ we get that the Borel–Moore homology of $\mathcal{C} \setminus \mathcal{C}_{\mathrm{sing}}$ is $(\mathbf{S}_4 \oplus \mathbf{S}_{3,1})(1)$ in degree 2 and $\mathbf{S}_{3,1} \oplus \mathbf{S}_{2,2} \oplus \mathbf{S}_{2,1,1}$ in degree 1. In all other degrees the Borel–Moore homology is trivial.

The closed inclusion $\mathcal{C}_{\mathrm{sing}} \to \mathcal{C}$ induces a long exact sequence in Borel–Moore homology which yields that the $\mathfrak{S}_4$-equivariant HG-polynomial of $\mathcal{C}$ is $s_4 + s_{3,1}t + (s_4 + s_{3,1})t^2\mathbf{L}$. Here we used the fact that $\bar{H}_0(\mathcal{C}; \mathbf{Q})$ is 1-dimensional to compute the rank of the only non-trivial differential of the long exact sequence, i.e. $\bar{H}_1(\mathcal{C} \setminus \mathcal{C}_{\mathrm{sing}}; \mathbf{Q}) \to \bar{H}_0(\mathcal{C}_{\mathrm{sing}}; \mathbf{Q})$. Then the claim follows from the long exact sequence associated to the closed inclusion $\mathcal{C} \to \mathbf{P}^{2\vee}$, with complement $\mathcal{U}$. $\square$

II.8 Configuration type 7d — Conics through 6 points

In this section we deal with the configuration space $X'_{7\mathrm{d}}$ of semi-ordered configurations $(\{\alpha,\beta\}, P = \{p_1,p_2,p_3,p_4\})$ of points in $\mathbf{P}^2$, satisfying

- the points p_1,p_2,p_3,p_4 are in general position, i.e. $P \in \tilde{B}(\mathbf{P}^2,4)$;
- there is a conic $C \in \mathcal{K} := \mathbf{P}(\mathbf{C}[x_0,x_1,x_2]_2)$ containing $\{\alpha,\beta\} \cup P$;
- $\{\alpha,\beta\} \not\subset P$, $\alpha\beta \not\subset C$.

In the following, we will always use the notation $\mathcal{K}$ for the projective space of conic curves in $\mathbf{P}^2$.

We will prove the following result:

Lemma II.8.1. *Consider the rank 1 local system of coefficients induces by the sign representation of the symmetric group $\mathfrak{S}_4$ on the points in the configuration $P \in \tilde{B}(\mathbf{P}^2,4)$. Then the Borel–Moore homology of $X'_{7\mathrm{d}}$ with S-coefficients vanishes.*

We start with the observation that the conic C on which the points $\alpha,\beta,p_1,\dots,$ p_4 lie is unique for all $(\alpha,\beta,P) \in X_{7\mathrm{d}}$. Therefore, one can view $X_{7\mathrm{d}}$ as a subset of $B(\mathbf{P}^2,2) \times \tilde{B}(\mathbf{P}^2,4) \times \mathcal{K}$. A partial compactification is given by considering the space $Y'_{7\mathrm{d}}$ of configurations

$$(\{\alpha,\beta\}, P, C) \in B(\mathbf{P}^2,2) \times \tilde{B}(\mathbf{P}^2,4) \times \mathcal{K}$$

such that $\{\alpha,\beta\} \cup P$ lie on the conic C. Note that the local system S extends to $Y'_{7\mathrm{d}}$.

Then Lemma II.8.1 follows from the following lemma:

Lemma II.8.2. *The Borel–Moore homology with S-coefficients of both $Y'_{7\mathrm{d}}$ and $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$ vanishes.*

Proof. We can stratify $Y'_{7\mathrm{d}}$ as the disjoint union of the subset $Y'_{7\mathrm{d},1}$ where the conic C is singular and the locus $Y'_{7\mathrm{d},2}$ of $Y'_{7\mathrm{d}}$ such that the conic C has maximal rank. Then the proof of the first part of the claim consists of showing that the Borel–Moore homology of each of the strata $Y'_{7\mathrm{d},j}$ vanishes. For instance, the open stratum $Y'_{7\mathrm{d},2}$ is fibred over the space of all non-singular conics in $\mathbf{P}^2$ with two distinct marked points α,β, and the fibre is isomorphic to $B(C,4)$. Then the claim follows from the fact that $C \cong \mathbf{P}^1$ and S restricts to the local system $\pm\mathbf{Q}$ on the fibre. Recall from Lemma I.4.6 that the twisted Borel–Moore homology $\bar{H}_\bullet(B(\mathbf{P}^1,k);\pm\mathbf{Q})$ vanishes for $k \geq 3$. The proof of the vanishing for $Y_{7\mathrm{d},1}$ is similar, and is based on the fact that $Y_{7\mathrm{d},1}$ can be realised as a fibration with fibres isomorphic to $B(\mathbf{C},2)$. From this the vanishing of the twisted Borel–Moore homology of $Y'_{7\mathrm{d}}$ follows.

Next, let us consider the complement $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$. Since for these configurations $(\{\alpha,\beta\},P,C)$ one has $\alpha,\beta \in P$, one can view $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$ as the set of partially ordered configurations $(\{\alpha,\beta\},\{q_1,q_2\},C)$ in $(\tilde{F}(\mathbf{P}^2,4)/\sim) \times \mathcal{K}$ such that C contains $\{\alpha,\beta,q_1,q_2\}$. The relation $\sim$ on $\tilde{F}(\mathbf{P}^2,4)$ is generated by $(\alpha,\beta,q_1,q_2) \sim (\beta,\alpha,q_1,q_2)$ and $(\alpha,\beta,q_1,q_2) \sim (\alpha,\beta,q_2,q_1)$ and the local system S is the local system induced by the sign representation of the $\mathfrak{S}_2$-action interchanging q_1 and q_2.

Hence, $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$ is a finite quotient of the subset Z of $\tilde{F}(\mathbf{P}^2,4)\times\mathcal{K}$ of configurations (α,β,q_1,q_2,C) such that C contains the points α,β,q_1,q_2 but is different from the rank 1 conic $C_0 := \alpha\beta \cup q_1q_2$. This space Z is a rank 1 affine bundle over $\tilde{F}(\mathbf{P}^2,4)$, the fibre over a configuration of four points being the pencil of conics passing through the them, minus C_0. From the isomorphism $\tilde{F}(\mathbf{P}^2,4) \cong \mathrm{PGL}(3)$ one gets that the whole Borel–Moore homology of $\tilde{F}(\mathbf{P}^2,4)$ is invariant under the interchange of two points in the configuration. Moreover, also the whole Borel–Moore homology of the fibres of the $\mathbf{C}$-bundle is invariant under such an interchange. Hence the Borel–Moore homology of $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$ with constant coefficients, which equals the part of the Borel–Moore homology of Z which is invariant under the interchange of the third and four point in the configuration, is equal to the Borel–Moore homology of Z. From the construction of $Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}}$ as the quotient of Z under an involution, and from the definition of S, we get $\bar{H}_\bullet(Z;\mathbf{Q}) = \bar{H}_\bullet(Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}};\mathbf{Q}) \oplus \bar{H}_\bullet(Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}};S)$. From this the vanishing of $\bar{H}_\bullet(Y'_{7\mathrm{d}} \setminus X'_{7\mathrm{d}};S)$ follows. □

II.9 Type 9

In this section we compute the Borel–Moore homology of the strata $F'_{9\mathrm{k}} \subset |\mathcal{X}'|$ with $\mathrm{k} \in \{\mathrm{b},\mathrm{c},\mathrm{d},\mathrm{e}\}$ that correspond to singular configurations containing the union of four points $\{a,b,c,d\}$ in general position with the point e which is the intersection of the lines ab and cd. The configuration spaces $X_{9\mathrm{k}}$ were described in Table II.7 on page 43.

We will prove the following results:

Lemma II.9.1. *For configurations of type* 9b *one has:*

$$\begin{aligned}\bar{H}_\bullet(X'_{9\mathrm{b}};\pm\mathbf{Q}) &\cong \bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q}),\\ \bar{H}_\bullet(\Phi'_{9\mathrm{b}};\mathbf{Q}) &\cong \bar{H}_{\bullet-4}(\mathrm{PGL}(3);\mathbf{Q}),\\ \bar{H}_\bullet(F'_{9\mathrm{b}};\mathbf{Q}) &\cong \bar{H}_{\bullet-8}(\mathrm{PGL}(3);\mathbf{Q}) \otimes \mathbf{Q}(2).\end{aligned}$$

Lemma II.9.2. *For configurations of type* 9c *one has:*

$$\begin{aligned}\bar{H}_\bullet(X'_{9\mathrm{c}};\pm\mathbf{Q}) &\cong \bar{H}_{\bullet-1}(\mathrm{PGL}(3);\mathbf{Q}),\\ \bar{H}_\bullet(\Phi'_{9\mathrm{c}};\mathbf{Q}) &\cong \bar{H}_{\bullet-5}(\mathrm{PGL}(3);\mathbf{Q}),\\ \bar{H}_\bullet(F'_{9\mathrm{c}};\mathbf{Q}) &\cong \bar{H}_{\bullet-7}(\mathrm{PGL}(3);\mathbf{Q}) \otimes \mathbf{Q}(1).\end{aligned}$$

Lemma II.9.3. *For configurations of type* 9d *one has:*

$$\bar{H}_\bullet(X'_{9\mathrm{d}}; \pm\mathbf{Q}) = \bar{H}_\bullet(\Phi'_{9\mathrm{d}}; \mathbf{Q}) = \bar{H}_\bullet(F'_{9\mathrm{d}}; \mathbf{Q}) = 0.$$

Lemma II.9.4. *For configurations of type* 9e *one has:*

$$\begin{aligned}
\wp(\bar{H}_\bullet(X'_{9\mathrm{e}}; \pm\mathbf{Q})) &= (t^4\mathbf{L}^{-2} + t) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})) \\
\wp(\bar{H}_\bullet(\Phi'_{9\mathrm{e}}; \mathbf{Q})) &= (t^8\mathbf{L}^{-2} + t^5) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})) \\
\wp(\bar{H}_\bullet(F'_{9\mathrm{e}}; \mathbf{Q})) &= (t^{10}\mathbf{L}^{-3} + t^7\mathbf{L}^{-1}) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}))
\end{aligned}$$

Lemma II.9.5. *1. The Borel–Moore homology of the union U of the configuration spaces $X'_{9\mathrm{b}}$ and $X'_{9\mathrm{c}}$ inside $B(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 6)$ has trivial twisted Borel–Moore homology.*

2. The Borel–Moore homology of the union of the strata $\Phi'_{9\mathrm{b}}$ and $\Phi'_{9\mathrm{c}}$ inside $|\Lambda'|$ is trivial.

Proof of Lemma II.9.1. Recall that the configuration space $X'_{9\mathrm{b}}$ is a finite quotient of the space $\tilde{F}(\mathbf{P}^2, 4)$ of ordered configurations of points in general position, via the map

$$\begin{array}{ccc}
\tilde{F}(\mathbf{P}^2, 4) & \longrightarrow & X'_{9\mathrm{b}} \\
(a_1, a_2, a_3, a_4) & \longmapsto & (\{a_1, a_2\}, \{a_1, a_2, a_3, a_4, a_5\}),
\end{array}$$

where the point a_5 is the intersection point of the lines a_1a_3 and a_2a_4. This map can be identified with the quotient map of $\tilde{F}(\mathbf{P}^2, 4)$ by the involution $(1,2)(3,4)$. Note that the involution $(1,2)(3,4)$ has even sign, so that the restriction to $X'_{9\mathrm{b}}$ of the local system $\pm\mathbf{Q}$ (defined by the sign representation of the action of $\mathfrak{S}_5$ on the 5 singular points) equals the constant local system $\mathbf{Q}$. Then the claim follows from the fact that $\tilde{F}(\mathbf{P}^2, 4)$ is isomorphic to $\mathrm{PGL}(3)$ and that the whole of its Borel–Moore homology is invariant under permutation of the points.

The result over the Borel–Moore homology of $\Phi'_{9\mathrm{b}}$ follows from the fact that $\Phi'_{9\mathrm{b}}$ is a $\mathring{\Delta}_4$-bundle over $X'_{9\mathrm{b}}$. Note that the involution $(1,2)(3,4)$ does not change the orientation of the simplex with vertices a_1, a_2, a_3, a_4, a_5, so that the simplicial bundle $\Phi'_{9\mathrm{b}} \to X'_{9\mathrm{b}}$ is orientable in this case. The result over the Borel–Moore homology of $F'_{9\mathrm{b}}$ follows from the fact that $F'_{9\mathrm{b}} \to \Phi'_{9\mathrm{b}}$ is a complex vector bundle of rank 1. □

Proof of Lemma II.9.5. The union U of the configuration spaces $X'_{9\mathrm{b}}$ and $X'_{9\mathrm{c}}$ inside $B(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 5)$ is the locus of configurations $(\{\alpha, \beta\}, \{a_i\}_{1 \le i \le 5})$ such that a_1, a_2, a_3, a_4 are in general linear position, the point α belongs to a_1a_2, the point β belongs to a_3a_4 and furthermore $a_1a_2 \cap a_3a_4 = \{a_5\} \not\subset \alpha\beta$.

Consider the configuration space

$$Y := \left\{ (\alpha, \beta, a_1, a_2, a_3, a_4) \in F(\mathbf{P}^2, 2) \times \tilde{F}(\mathbf{P}^2, 4) \;\middle|\; \begin{array}{c} \alpha \in a_1a_2, \beta \in a_3a_4, \\ \alpha, \beta \notin (a_1a_2 \cap a_3a_4) \end{array} \right\}$$

of ordered configurations of six points, such that the last four points a_1, a_2, a_3, a_4 are in general position, the first point α lies on $a_1a_2 \setminus a_3a_4$ and the second point β lies on $a_3a_4 \setminus a_1a_2$. Notice that interchanging the configurations $(\alpha, \beta, a_1, a_2, a_3, a_4)$ and $(\alpha, \beta, a_2, a_1, a_3, a_4)$ gives a well defined involution on Y. It is easy to prove that the whole Borel–Moore homology of Y is invariant with respect to this involution. Namely, the space Y is fibred over $\tilde{F}(\mathbf{P}^2, 4) \cong \mathrm{PGL}(3)$, whose Borel–Moore homology is invariant under the involution interchanging a_1 and a_2. The involution interchanging a_1 and a_2 induces a trivial action also on the Borel–Moore homology of the fibre of $Y \to \tilde{F}(\mathbf{P}^2, 4)$, which is isomorphic to $\mathbf{C}^2 \cong (a_1a_2 \setminus \{\mathrm{pt}\}) \times (a_3a_4 \setminus \{\mathrm{pt}\})$.

The space U is the quotient of Y by the group generated by the involutions

$$\begin{aligned}(\alpha, \beta, a_1, a_2, a_3, a_4) &\leftrightarrow (\alpha, \beta, a_2, a_1, a_3, a_4),\\ (\alpha, \beta, a_1, a_2, a_3, a_4) &\leftrightarrow (\alpha, \beta, a_1, a_2, a_4, a_3),\\ (\alpha, \beta, a_1, a_2, a_3, a_4) &\leftrightarrow (\beta, \alpha, a_3, a_4, a_1, a_2).\end{aligned}$$

Thus, the twisted Borel–Moore homology of U is contained in the part of the Borel–Moore homology of Y which is alternating under the involution interchanging a_1 and a_2 in the configurations. This proves that the twisted Borel–Moore homology of U is trivial.

The second part of the claim follows from the fact that $\Phi'_{9b} \cup \Phi'_{9c}$ is a $\mathring{\Delta}_4$-bundle over U. $\square$

Proof of Lemma II.9.2. We start by observing that the part of the claim on the twisted Borel–Moore homology of X'_{9c} implies the claimsthe result for $\bar{H}_\bullet(\Phi'_{9c}; \mathbf{Q})$ and $\bar{H}_\bullet(F'_{9c}; \mathbf{Q})$. This follows from the structure of Φ'_{9c} as $\mathring{\Delta}_4$-bundle over X'_{9c} and from the structure of F'_{9c} as rank 1 complex vector bundle over Φ'_{9c}.

The configuration space X'_{9c} is an open subset of the space U of Lemma II.9.5, with complement the configuration space X'_{9b}. The twisted Borel–Moore homology of U is trivial, whereas the twisted Borel–Moore homology of X'_{9b} is isomorphic to the Borel–Moore homology of PGL(3) by Lemma II.9.1. Then the claim follows from the long exact sequence in Borel–Moore homology with $\pm\mathbf{Q}$-coefficients associated to the closed inclusion $X'_{9b} \hookrightarrow U$. $\square$

Proof of Lemma II.9.3. Consider the space

$$Y_{9d} := \{(a_1, a_2, a_3, a_4, \tau) \in \tilde{F}(\mathbf{P}^2, 4) \times \mathbf{P}^{2\vee} | \tau \cap a_1a_2 = \tau \cap a_3a_4 = a_1a_2 \cap a_3a_4\}$$

of ordered configurations of four points a_1, a_2, a_3, a_4 in general position, together with a line τ passing through the common point of the lines a_1a_2 and a_3a_4, and different from these two lines. The natural map $Y_{9d} \to \tilde{F}(\mathbf{P}^2, 4)$ gives Y_{9d} the structure of a $\mathbf{C}^*$-bundle over $\tilde{F}(\mathbf{P}^2, 4)$. Note that the whole Borel–Moore homology of Y_{9d} is invariant under the interchange of the points a_1, a_2. This involution fixes the fibres of the $\mathbf{C}^*$-bundle (considered as a subset of $\mathbf{P}^{2\vee}$), and it also acts trivially on the Borel–Moore homology of the basis $\tilde{F}(\mathbf{P}^2, 4) \cong \mathrm{PGL}(3)$.

Next, fix a configuration $\underline{y} = (a_1, a_2, a_3, a_4, \tau) \in Y_{9\mathrm{d}}$ and consider the pencil of quadrics through the points a_1, a_2, a_3, a_4. Every quadric in the pencil intersects the line τ in a subscheme of length 2, and exactly two quadrics in the pencil are tangent to τ, namely, the reduced conic $a_1a_2 \cup a_3a_4$ and a further conic, which we will denote by $Q_{\underline{y}}$. Note that the tangency points of the two conics are distinct points on τ, otherwise we would get a contradiction with the assumption that the a_i are in general position.

Consider next the map

$$\begin{array}{ccc} Y_{9\mathrm{d}} & \longrightarrow & X'_{9\mathrm{d}} \\ \underline{y} = (a_1, a_2, a_3, a_4, \tau) & \longmapsto & (\{\alpha, \beta\}, \{a_1, a_2, a_3, a_4, \alpha\}) \end{array}$$

where $\{\alpha\} = a_1a_2 \cap a_3a_4$ and β is the intersection point of $Q_{\underline{y}}$ and τ. This map is surjective with finite fibres, and allows to identify $X'_{9\mathrm{b}}$ with the quotient of $Y_{9\mathrm{d}}$ by the subgroup of $\mathfrak{S}_4$ generated by $(1,2)$, $(3,4)$ and $(1,3)(2,4)$. We are interested in the local system of coefficients of $X'_{9\mathrm{d}}$ induced by the sign representation on the 5 singular points. Since the involution $(1,2)$ interchanges exactly 2 singular points, the twisted Borel–Moore homology of $X'_{9\mathrm{d}}$ is contained in the part of the Borel–Moore homology of $Y_{9\mathrm{d}}$ which is alternating under $(1,2)$. Therefore, as the whole Borel–Moore homology of $Y_{9\mathrm{d}}$ is invariant, the twisted Borel–Moore homology of $X'_{9\mathrm{d}}$ must vanish. Furthermore, the structure of $\Phi'_{9\mathrm{d}}$ and $F'_{9\mathrm{d}}$ as fibrations over $X'_{9\mathrm{d}}$ also implies that the Borel–Moore homology of these spaces vanishes. □

Proof of Lemma II.9.4. Let us consider the configuration space

$$Y_{9\mathrm{e}} := \left\{ (\{\alpha, \beta\}, a_1, a_2, a_3, a_4) \in B(\mathbf{P}^2, 2) \times \tilde{F}(\mathbf{P}^2, 4) \;\middle|\; \begin{array}{c} \dim\langle \alpha, \beta, a_1, a_2 \rangle = 1, \\ \{\alpha, \beta\} \neq \{a_1, a_2\} \end{array} \right\}.$$

Let us choose a standard frame $(e_1, e_2, e_3, e_4) \in \tilde{F}(\mathbf{P}^2, 4)$ and identify $\mathbf{P}^1$ with the line e_1e_2. Then $Y_{9\mathrm{e}}$ is isomorphic to the product $(B(\mathbf{P}^1, 2) \setminus \{\mathrm{pt}\}) \times \mathrm{PGL}(3)$ by the map sending $(\{\alpha, \beta\}, a_1, a_2, a_3, a_4)$ to $(\{\varphi(\alpha), \varphi(\beta)\}, \varphi)$ where φ is the unique automorphism of $\mathbf{P}^2$ such that $\varphi(a_i) = e_i$ for all $i = 1, \ldots, 4$. In particular, this implies that the Borel–Moore homology of $Y_{9\mathrm{e}}$ with constant coefficients has HG polynomial $t^4\mathbf{L}^{-2} - t$.

The map $Y_{9\mathrm{e}} \to X'_{9\mathrm{e}}$ given by

$$(\{\alpha, \beta\}, a_1, a_2, a_3, a_4) \mapsto (\{\alpha, \beta\}, \{a_1, a_2, a_3, a_4, a_5\})$$

with a_5 the intersection point of the lines a_1a_3 and a_2a_4 allows to identify $X'_{9\mathrm{e}}$ with the quotient of $Y_{9\mathrm{e}}$ by the involution

$$i\colon\ (\{\alpha, \beta\}, a_1, a_2, a_3, a_4) \longmapsto (\{\alpha, \beta\}, a_2, a_1, a_4, a_3).$$

Since i interchanges two pairs of singular points, the twisted Borel–Moore homology of $X'_{9\mathrm{e}}$ coincides with the part of the Borel–Moore homology which is invariant under i. Then the claim follows from the fact that the Borel–Moore homology of both factors $\mathrm{PGL}(3)$ and $B(\mathbf{P}^1, 2) \setminus \{\mathrm{pt}\}$ of $Y_{9\mathrm{e}}$ is invariant under the $\mathfrak{S}_2$-action induced by i. □

II.10 Configurations of type 10

In this section, we deal with the singular sets of singular quartics with a marked bitangent that are the union of four lines in general position. Such a quartic has 6 distinct singular points, hence a singular set of the configuration space X'_{10} will be an element of $B(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 6)$. As always, we will denote by $\pm\mathbf{Q}$ the pull-back of the local system $\pm\mathbf{Q}$ under the forgetful map $X'_{10} \to B(\mathbf{P}^2, 6)$.

Lemma II.10.1.

$$\begin{aligned}
&\wp(\bar{H}_\bullet(X'_{10}; \pm\mathbf{Q})) = (\mathbf{L}^{-2}t^4 + 1) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})),\\
&\wp(\bar{H}_\bullet(\Phi'_{10}; \mathbf{Q})) = t^5(\mathbf{L}^{-2}t^4 + 1) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})),\\
&\wp(\bar{H}_\bullet(F'_{10}; \mathbf{Q})) = t^7\mathbf{L}^{-1}(\mathbf{L}^{-2}t^4 + 1) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})).
\end{aligned} \tag{II.10.1}$$

Proof. We start by observing that it suffices to prove the description of the Borel–Moore homology of the configuration space X'_{10}. The results on Φ'_{10} and F'_{10} will immediately follows from their structures as simplicial bundle and vector bundle, respectively.

Recall that the elements of X'_{10} are configurations $(\{\alpha, \beta\}, K)$ such that K is the singular set of $\mathcal{C} = \bigcup_i \ell_i \subset \mathbf{P}^2$ for a configuration of 4 lines $\ell_1, \ell_2, \ell_3, \ell_4$ in general position and the line $\tau = \alpha\beta$ is either tangent to $\mathcal{C}$ at the points α and β, or it is contained in $\mathcal{C}$. This implies that we may view X'_{10} as a subset of $B(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^{2\vee}, 4)$, and that X'_{10} has two connected components:

$$X'_{10\mathrm{a}} = \{(\{\alpha, \beta\}, \{\ell_1, \ldots, \ell_4\}) \in X'_{10} | \alpha\beta \subset \bigcup_i \ell_i\}$$

and

$$X'_{10\mathrm{b}} = \{(\{\alpha, \beta\}, \{\ell_1, \ldots, \ell_4\}) \in X'_{10} | \ell_1 \cap \ell_2 = \{\alpha\}, \ell_3 \cap \ell_4 = \{\beta\}\},$$

because the only bitangent lines to the singular quartic $\mathcal{C}$ are the components of $\mathcal{C}$ and the lines joining the intersection points of two disjoint pairs of components of $\mathcal{C}$.

We need to compute the Borel–Moore homology of X'_{10} in the twisted local system of coefficients $\pm\mathbf{Q}$. This local system of coefficients coincides with the restriction to X'_{10} of the trivial local system on $B(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^{2\vee}, 4)$ under the inclusion $X'_{10} \hookrightarrow B(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^{2\vee}, 4)$. This follows from the fact that interchanging two lines ℓ_i, ℓ_j interchanges two pairs of singular points of $\mathcal{C}$, thus inducing a permutation of even sign in the configuration of six singular points.

We proceed to consider configurations of type 10a. Notice that without loss of generality we can always assume that the marked points α, β lie on the line ℓ_4. In other words, we can obtain $X'_{10\mathrm{a}}$ as the quotient of the space

$$Y_{10\mathrm{a}} = \{(\{\alpha,\beta\},(\ell_1,\dots,\ell_4)) \in B(\mathbf{P}^2,2)\times \tilde{B}(\mathbf{P}^{2\vee},4) | \alpha,\beta\in\ell_4\}$$

by the action of $\mathfrak{S}_3$ permuting the lines ℓ_1,ℓ_2,ℓ_3.

The space $Y_{10\mathrm{a}}$ is fibred over $\tilde{F}(\mathbf{P}^2,4)\cong \mathrm{PGL}(3)$ with fibre isomorphic to $B(\ell_4,2)\cong B(\mathbf{P}^1,2)$. The space $B(\mathbf{P}^1,2)$ is isomorphic to $\mathrm{Sym}^2\mathbf{P}^1$ with the diagonal removed, i.e. to the complement of a smooth conic in $\mathbf{P}^2$. Hence we have

$$\wp(\bar{H}_\bullet(Y_{10\mathrm{a}};\mathbf{Q})) = t^4\mathbf{L}^{-2}\wp(\bar{H}_\bullet(\mathrm{PGL}(3);\mathbf{Q}),$$

and since the whole Borel–Moore homology of $Y_{10\mathrm{a}}$ is invariant under the $\mathfrak{S}_3$-action, this yields the Borel–Moore homology of $X'_{10\mathrm{a}}$ as well.

Analogously, we realize $X'_{10\mathrm{b}}$ as the quotient of $\tilde{B}(\mathbf{P}^{2\vee},4)$ the action of the group generated by the involution interchanging $\ell_1\leftrightarrow\ell_2$ and the involution $\ell_1\leftrightarrow\ell_3$, $\ell_2\leftrightarrow\ell_4$. Since the Borel–Moore homology of $\tilde{B}(\mathbf{P}^{2\vee},4)\cong\mathrm{PGL}(3)$ is invariant under any permutation of the points, the Borel–Moore homology of $X'_{10\mathrm{b}}$ coincides with that of PGL(3). Then the claim follows from the fact that the Borel–Moore homology of X'_{10} is the direct sum of the Borel–Moore homology of its two components $X'_{10\mathrm{a}}$ and $X'_{10\mathrm{b}}$. ☐

II.11 Quartic curves with a flex bitangent

In this section, we will compute the rational cohomology of the moduli space $\mathcal{Q}_\delta^-$ of pairs (C,τ) such that C is a smooth quartic curve and τ a flex bitangent.

Theorem II.11.1. *The rational cohomology of $\mathcal{Q}_\delta^-$ is one-dimensional and concentrated in degree* 0.

Recall that $\mathcal{I}_\delta^-$ is fibred over the space $\mathbf{P}(T_{\mathbf{P}^2})$, which can be viewed as the incidence correspondence

$$\{(\alpha,\tau)\in\mathbf{P}^2\times\mathbf{P}^{2\vee}|\alpha\in\tau\}.$$

We start by considering the fibre of the map $\pi_\delta^-\colon\mathcal{I}_\delta^-\to\mathbf{P}(T_{\mathbf{P}^2})$ over $(p,t)\in\mathbf{P}(T_{\mathbf{P}^2})$. Set $t'=t\setminus\{p\}$. Consider the 11-dimensional complex vector space

$$V_{p,t} := \left\{ f\in V \;\middle|\; \begin{array}{c}\text{the line t is either contained in } \mathbf{V}(f) \text{ or it}\\ \text{or is a flex bitangent to } t \text{ at the point } p\end{array}\right\}.$$

Then the fibre $(\pi_\delta^-)^{-1}(p,t)$ is equal to $V_{p,t}\setminus\Sigma$. Hence, the fibre of π_δ^- can be viewed as the complement of a discriminant in the vector space $V_{p,t}$. In particular, its cohomology can be computed using Vassiliev–Gorinov's method.

Table II.9: Singular configurations of type 1–3 and associated strata.

1a The point p.
Stratum: F_{1a} is isomorphic to $\mathbf{C}^{10}$.

1b One point on t'.
Stratum: F_{1b} is a $\mathbf{C}^9$-bundle over $t' \cong \mathbf{C}$.

1c A points outside t.
Stratum: F_{1c} is a $\mathbf{C}^8$-bundle over the affine space $\mathbf{C}^2$.

2a Two points on t.
Stratum: F_{2a} is a $\mathbf{C}^8$-bundle over Φ_{2a}, which is a non-orientable $\mathring{\Delta}_1$-bundle over $X_{2a} \cong B(\mathbf{P}^1, 2)$.

2b The point p and a point outside t.
Stratum: F_{2b} is a $\mathbf{C}^7$-bundle over Φ_{2b}, which is a non-orientable $\mathring{\Delta}_1$-bundle over $X_{2b} \cong \mathbf{C}^2$.

2c A point on t' and a point outside t.
Stratum: F_{2c} is a $\mathbf{C}^6$-bundle over Φ_{2c}, which is a non-orientable $\mathring{\Delta}_1$-bundle over $X_{2c} \cong \mathbf{C}^3$.

2d Two points outside t.
The Borel–Moore homology of F_{2d} and Φ_{2d} vanishes, because they are fibred over the configuration space $B(\mathbf{C}^2, 2)$, whose twisted Borel–Moore homology vanishes.

3x Configurations with three collinear points.
To get all strata, we have to distinguish whether the line is t, if one of the singular points coincide with p or lies on t or if the configuration is general. In each of these cases the space X_{3x} admits a locally fibration with fibre isomorphic to either $B(\mathbf{P}^1, 3)$ or $B(\mathbf{C}^2, 2)$. Since the twisted Borel–Moore homology of both these configuration spaces vanishes, configurations of type 3 do not contribute to the Borel–Moore homology of $\mathcal{D}_\delta^-$.

We proceed by giving the classification of the singular sets in $\mathbf{P}^2$ of quartic curves that pass through p and have the line $t \ni p$ as flex bitangent. These are exactly the singular sets of the elements of $V_{p,t} \cap \Sigma$. First, we classify in Tables II.9 and II.10 the singular sets that come from refining singular configurations of type 1–6 in Vassiliev's list (Table II.2).

For configurations K of type 7–13 in Table II.2 we can distinguish whether the general curve singular at K will not contain the line t (in which case we will call it a configuration of the first kind) or if every curve singular at K will contain t (configuration of the second kind).

It is easy to see that if the $\mathbf{V}(f)$ does not contain t, and its singular locus contains a configuration K of type 7–12 in Vassiliev's classification, then the singular locus $K_f \subset \mathbf{P}^2$ is either a line through p, or a conic (possibly singular) tangent to t at the point p. Hence, a configuration K of the first kind will either contain a rational curve, or a finite number of point lying on a rational curve. In this way we can prove that configurations of the first kind do not contribute to the spectral sequence converging to the Borel–Moore homology of $V_{p,t} \cap \Sigma$.

Table II.10: Singular configurations of type 4–6 and associated strata.

4a Two points on t and one point outside t.
The stratum $F_{4\mathrm{a}}$ is a $\mathbf{C}^5$-bundle over $\Phi_{4\mathrm{a}}$, which is a $\mathring{\Delta}_2$-bundle over the configuration space $X_{4\mathrm{a}} = B(\mathbf{P}^1, 2) \times \mathbf{C}^2$.

4x All other configurations of three points in linear general position.
We have to consider the following subcases: one of the points is p and the other two lie outside t ($\mathbf{C}^4$-bundle), a singular point lies on t' and the other two lie outside t ($\mathbf{C}^3$-bundle), all three points lie outside t (case 4^*, $\mathbf{C}^2$-bundle). All these subcases correspond to configuration spaces with trivial twisted Borel–Moore homology.

5x A line in $\mathbf{P}^2$.
We have to distinguish whether the line equals t, passes through p of not. In each case x the Borel–Moore homology of $\Phi_{5\mathrm{x}}$ is trivial, because the singular configuration contains a rational curve.

6x Three collinear points p_1, p_2, p_3 and a fourth point q in general linear position.
We have to distinguish between the following subcases: the three collinear points belong to t ($\mathbf{C}^4$-bundle), a point p_i and q both lies on t ($\mathbf{C}^3$-bundle), p_1 belongs to t but and p_2, p_3 and q do not ($\mathbf{C}^3$-bundle), none of the p_i lies on t but $q \in t$ ($\mathbf{C}^2$-bundle), $p_1 \in t$ but all other point lie outside t (case 6^*, $\mathbf{C}$-bundle). In each of these cases the configuration space is fibred over a base space with fibre isomorphic to either $B(\mathbf{C}^2, 2)$, $B(\mathbf{C}, 2)$ or $B(\mathbf{P}^1, 3)$. For this reason, all configurations of type 6 contribute trivially to the Borel–Moore homology of $\mathcal{D}_\delta^-$.

Therefore, it suffices for us to consider the strata associated to configurations of the second kind, which we list in Table II.11.

In view of the description of the strata given there, it suffices to deal with the configuration types 9b and $10'$. The Borel–Moore homology of $X_{9\mathrm{b}}$ and $X_{10'}$ is not difficult to compute. However, we will not need this result, because it is possible to prove that the contributions of these two strata kill each other in the spectral sequences associated to the stratifications $\Phi_\bullet$ and $F_\bullet$.

Lemma II.11.2. *The Borel–Moore homology of $\Phi_{9\mathrm{b}} \cup \Phi_{10'} \subset \left|\Lambda\right|$ is trivial, and the same holds for $F_{9\mathrm{b}} \cup F_{10'} \subset \left|\mathcal{X}\right|$.*

Proof. The stratum $\Phi_{10'}$ is a simplicial bundle over $X_{10'}$. Its fibre over a configuration $K \in X_{10'}$ is an open 5-dimensional simplex whose vertices are in canonical correspondence with the six points in K. We can partially compactify $\Phi'_{10'}$ by considering the simplicial bundle Ψ over $X_{10'}$ whose fibres are closed 5-dimensional simplices, in such a way that the fibres of $\Phi_{10'}$ coincide with the interiors of the fibres of $\Psi \to X_{10'}$. The simplicial bundle Ψ is contained $\left|\Lambda\right|$, where it can be realized as the union of the simplices corresponding to subsets of the configurations in $X_{10'}$.

Observe that every configuration $K \in X_{10'}$ contains exactly three points lying

Table II.11: Singular configurations of the second kind and associated strata.

$7'$	Two points on t and two other points (no three points in the configurations are allowed to be collinear). Stratum: The general curve singular in such a configuration K is the union of the line t, the line passing through the two points in $K \setminus t$ and a conic passing through the points of K. The configuration space $X_{7'}$ is contained in $B(t,2) \times B(\mathbf{C}^2,2)$ with complement a fibration over $F(\mathbf{P}^1,2)$ with fibre isomorphic to $B(\mathbf{C},2)$. Since the twisted Borel–Moore homology of both $B(\mathbf{C}^2,2)$ and $B(\mathbf{C},2)$ vanish by Lemma I.4.6, the twisted Borel–Moore homology of $X_{7'}$ and the Borel–Moore homology of $\Phi_{7'}$ and $F_{7'}$ are trivial as well.
8a	The union of the line t and a point outside t. The strata $F_{8\mathrm{a}}$ and $\Phi_{8\mathrm{a}}$ have trivial Borel–Moore homology, because the singular configurations of this type contain a rational curve.
8b	The union of a line different from t and a point on t. The Borel–Moore homology of the strata $\Phi_{8\mathrm{b}}$ and $F_{8\mathrm{b}}$ is trivial, because configurations of type 8b always contain a rational curve.
9a	Five points $\{a,b,c,d,e\}$ with $a,b \in t$, $c,d \in (\mathbf{P}^2 \setminus t)$ and $\{e\} = t \cap cd$. Stratum: The configuration space $X_{9\mathrm{a}}$ is isomorphic to $X_{7'}$, hence its twisted Borel–Moore homology vanishes. Therefore also the Borel–Moore homology of $\Phi_{9\mathrm{a}}$ and of $F_{9\mathrm{a}}$ is trivial.
9b	Five points $\{a,b,c,d,e\}$ with $a,b \in t$, $c,d \in (\mathbf{P}^2 \setminus t)$ and $\{e\} = t \cap cd$. Stratum: The stratum $F_{9\mathrm{b}}$ is a $\mathbf{C}$-bundle over $\Phi_{9\mathrm{b}}$, which in turn is a $\mathring{\Delta}_4$-bundle over the configuration space $X_{9\mathrm{b}}$, which is the quotient of the space $\{(a,b,c,d) \in \tilde{F}(\mathbf{P}^2,4) \vert a,b \in t\}$ by the equivalence relation generated by $(a,b,c,d) \sim (b,a,d,c)$.
$10'$	Six points which are the pairwise intersection of four lines in general position, one of which is t. Stratum: The stratum $F_{10'}$ is a $\mathbf{C}$-bundle over $\Phi_{10'}$, which in turn is a $\mathring{\Delta}_5$-bundle over the configuration space $X_{10'}$ which is isomorphic to the configuration space of three unordered lines in general position and such that their union intersects t in three distinct points.
$12'$	The union of t and another line. The strata $\Phi_{12'}$ and $F_{12'}$ have trivial Borel–Moore homology. This is a consequence of the fact that the singular locus contains a rational curve.
13	The whole projective plane.

outside t. They correspond to a 2-dimensional face D_K of the fibre Ψ_K of $\Psi \to X_{10'}$ lying over K. Let us define X_K to be the union of the interior of all the faces of the 5-dimensional simplex Ψ_K that contain the interior of D_K. The complement of X_K in Ψ_K is the union of all closed faces that do not contain the three vertices of D_K. The Borel–Moore homology of X_K coincides with the relative homology of the pair $(\Psi_K, \Psi_K \setminus \mathrm{X}_K)$, which is trivial because both spaces can be contracted to the same point.

Let us consider the subset $\mathrm{X} \subset \Psi$ given by the union of the X_K for all $K \in X_{10'}$. Then the Borel–Moore homology of X is trivial as well. On the other hand, we can view X as the disjoint union of open simplices of dimension varying from two to five. For $k = 2, \dots, 5$, denote by $\mathrm{X}^{(k)}$ the union of the interior of all k-dimensional faces of simplices contained in X.

The space $\mathrm{X}^{(2)}$ is fibred over $X_{10'}$ with fibre the interior of D_K. It coincides with the stratum Φ_{4^*} coming from configurations of type 4^*, containing three points in general linear position not lying on t (see Table II.10). In particular, the Borel–Moore homology of $\mathrm{X}^{(2)}$ is trivial. Analogously, the stratum $\mathrm{X}^{(3)}$ coincides with the stratum Φ_{6^*} corresponding to configurations of type 6^* (three collinear points of which exactly one lies on t and a fourth point not lying on t and not collinear with the others). As we explained in Table II.10, the Borel–Moore homology of $\mathrm{X}^{(3)}$ is trivial.

Hence, the Borel–Moore homology of X coincides with the Borel–Moore homology of the union of its strata $\mathrm{X}^{(4)} = \Phi_{9\mathrm{b}}$ and $\mathrm{X}^{(5)} = \Phi_{10'}$. This proves that the Borel–Moore homology of $\Phi_{9\mathrm{b}} \cup \Phi_{10'}$ is trivial. As to the second part of the claim, it suffices to observe that $F_{9\mathrm{b}} \cup F_{10'}$ is a complex line bundle over $\Phi_{9\mathrm{b}} \cup \Phi_{10'}$. $\square$

Furthermore, also the configuration space $X_{13} = \{\mathbf{P}^2\}$ contributes trivially to the Borel–Moore homology of $V_{p,t} \cap \Sigma$.

Lemma II.11.3. *1. The* e^1 *terms of the spectral sequence*

$$\mathrm{e}^r_{u,v} \Rightarrow \bar{H}_{u+v}(\left|\Lambda\right| \setminus \Phi_{13}; \mathbf{Q})$$

associated with the stratification $\Phi_\bullet$ *are as given in Table II.12.*

2. The Borel–Moore homology of F_{13} *is trivial.*

Proof. The terms of this spectral sequence are given by $\mathrm{e}^1_{u,v} = \bar{H}_v(\Phi_u; \mathbf{Q})$ for all configuration types u such that the Borel–Moore homology of Φ_u is non-trivial. Furthermore, we can omit all configurations with more than 4 singular points in view of Lemma II.11.2. Then the first part of the claim follows from the description of the strata $\Phi_{j\mathrm{x}}$ given in Table II.9 and II.10.

We can observe that only configurations of type 1–4 contribute non-trivial e^1 terms. Furthermore, the union of the configuration spaces $X_{j\mathrm{x}}$ with $j = 1, \dots, 4$ gives all configurations of ≤ 4 points in $\mathbf{P}^2$. Hence, the reasoning in the proof of Lemma II.5.1 applies also in this case, thus yielding

$$\mathrm{e}^\infty_{1,-1} = 0,\ \mathrm{e}^\infty_{u,v} = 0 \text{ for } (u,v) \neq (1,-1).$$

Table II.12: e^1 terms of the spectral sequence $\mathrm{e}^r_{u,v} \Rightarrow \bar{H}_{u+v}(|\Lambda| \setminus \Phi_{13}; \mathbf{Q})$.

v								
1	0	0	$\mathbf{Q}(2)$	0	0	$\mathbf{Q}(3)$	$\mathbf{Q}(3)$	
0	0	$\mathbf{Q}(1)$	0	0	$\mathbf{Q}(2)$	0	0	
-1	$\mathbf{Q}$	0	0	$\mathbf{Q}(1)$	0	0	0	
	1	2	3	4	5	6	7	u
type	(1a)	(1b)	(1c)	(2a)	(2b)	(2c)	(4a)	

Table II.13: E^1 terms of the spectral sequence $\mathrm{E}^r_{u,v} \Rightarrow \bar{H}_{u+v}(|\mathcal{X}|; \mathbf{Q}) = \bar{H}_{u+v}(V_{p,t} \cap \Sigma; \mathbf{Q})$.

v				
19	$\mathbf{Q}(10)^{\oplus 3}$	0	0	
18	0	0	0	
17	0	$\mathbf{Q}(9)^{\oplus 3}$	0	
16	0	0	0	
15	0	0	$\mathbf{Q}(8)$	
	1	2	3	u
type	(1x)	(2x)	(4x)	

As a consequence, the Borel–Moore homology of $|\Lambda| \setminus \Phi_{13}$ is 1-dimensional and concentrated in degree 0. Then the second part of the claim follows from the fact that the F_{13} is an open cone over $|\Lambda| \setminus \Phi_{13}$ in view of Proposition I.4.3. $\square$

We are ready to calculate the Borel–Moore homology of $V_{p,t} \cap \Sigma$.

Lemma II.11.4. *1. The* E^1 *terms of the spectral sequence*

$$\mathrm{E}^r_{u,v} \Rightarrow \bar{H}_{u+v}(V_{p,t} \cap \Sigma; \mathbf{Q})$$

are as given in Table II.13. This spectral sequence degenerates at E^1*.*

2. The rational cohomology of $V_{p,t} \setminus \Sigma$ *has HG polynomial* $(1 - t\mathbf{L})^3$*.*

Proof. We have $\mathrm{E}^1_{u,v} = \bar{H}_{u+v}(F_u; \mathbf{Q})$, where u refers to the uth configuration type in our list. Since F_u is a vector bundle of a certain rank k_u over Φ_u, one has $\mathrm{E}^1_{u,v} = \mathrm{e}^1_{u,v-2k_u} \otimes \mathbf{Q}(k_u)$. This allows to compute the $\mathrm{E}^1_{u,v}$ as in Table II.13. Degeneracy at E^1 follows immediately from the shape of the spectral sequence. The result on the cohomology of the complement of the discriminant $V_{p,t} \setminus \Sigma$ follows from Alexander's duality (I.2.1). $\square$

Table II.14: Leray spectral sequence of the fibration $\mathcal{I}_\delta^- \to \mathbf{P}(T_{\mathbf{P}^2})$

$E^2_{p,q}$

q							
3	$\mathbf{Q}(-3)$	0	$\mathbf{Q}(-4)^2$	0	$\mathbf{Q}(-5)^2$	0	$\mathbf{Q}(-6)$
2	$\mathbf{Q}(-2)^3$	0	$\mathbf{Q}(-3)^6$	0	$\mathbf{Q}(-4)^6$	0	$\mathbf{Q}(-5)^3$
1	$\mathbf{Q}(-1)^3$	0	$\mathbf{Q}(-2)^6$	0	$\mathbf{Q}(-3)^6$	0	$\mathbf{Q}(-4)^3$
0	$\mathbf{Q}$	0	$\mathbf{Q}(-1)^3$	0	$\mathbf{Q}(-2)^3$	0	$\mathbf{Q}(-3)$
	0	1	2	3	4	5	6 $\quad p$

$E^3_{p,q}$

q							
3	0	0	0	0	0	0	$\mathbf{Q}(-6)$
2	0	0	$\mathbf{Q}(-3)$	0	$\mathbf{Q}(-4)$	0	$\mathbf{Q}(-5)$
1	$\mathbf{Q}(-1)$	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$	0	0
0	$\mathbf{Q}$	0	0	0	0	0	0
	0	1	2	3	4	5	6 $\quad p$

Proof of Theorem II.11.1. We want to compute the cohomology of $\mathcal{I}_\delta^-$ by using the Leray spectral sequence associated to the fibration $\mathcal{I}_\delta^- \to \mathbf{P}(T_{\mathbf{P}^2})$ with fibre $V_{p,t}\setminus\Sigma$. From the fact that $\mathbf{P}(T_{\mathbf{P}^2})$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$ and from the computation of the cohomology of the fibre given in Lemma II.11.4 above we get that the E_2 term of the Leray spectral sequence is as given in the first part of Table II.14.

To compute the differentials of the spectral sequence, we keep in mind that we proved in Lemma II.2.1 that it has to be a tensor product of the cohomology of GL(3). There is only one possible behaviour of the differentials that would ensure such a divisibility: This is the case in which the E^3 term is as in the second part of Table II.14 and the spectral sequence degenerates at E^3. This yields

$$H^\bullet(\mathcal{I}_\delta^-;\mathbf{Q}) \cong H^\bullet(\mathrm{GL}(3);\mathbf{Q}).$$

This isomorphism implies the claim, since by Lemma II.2.1 we also have

$$H^\bullet(\mathcal{I}_\delta^-;\mathbf{Q}) \cong H^\bullet(\mathcal{Q}_\delta^-;\mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3);\mathbf{Q}).$$

□

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Chapter III

The orbifold cohomology of moduli of genus 3 curves

III.1 Introduction

It was a remarkable discovery of the beginning of this century, anticipated in physics in the nineties, that the degree zero small quantum cohomology of a smooth Deligne–Mumford stack is a (proper) ring extension of its ordinary cohomology ring: its definition was recognized and given in symplectic geometry by Chen and Ruan in [CR04]. The algebraic counterpart of this theory was developed by Abramovich–Graber–Vistoli in [AGV02], [AGV08].

The Chen–Ruan cohomology of a smooth Deligne–Mumford stack X is, by definition, the degree zero part of the small quantum cohomology ring of X, and the orbifold cohomology of X is the rationally graded vector space that underlies the Chen–Ruan cohomology algebra. The general idea, coming from stringy geometry, is that an important role in the study of X is played by the so-called *inertia stack* of X. When X is a moduli space for certain geometric objects, the inertia stack of X parametrizes the same geometric objects, together with the choice of an automorphism on them. The stack X itself appears as the connected component of its inertia stack associated with the trivial automorphism, but in general there are other connected components, usually called the *twisted sectors* of X, a terminology that originates from physics. Orbifold cohomology is simply the ordinary cohomology of the inertia stack, endowed with a different grading. Each twisted sector is assigned a rational number, called (depending on the author) *degree shifting number*, *age* or *fermionic shift*: this number depends on the action of the given automorphism on the normal bundle to the twisted sector in X. Then the degree of each cohomology class of the twisted sector is shifted by twice this rational number.

In this chapter, we study the inertia stack of moduli spaces $\mathcal{M}_g$ of smooth

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genus g curves. The starting point of our construction is that one can associate with each object (C, α) of the inertia stack the cover given by quotienting C by the cyclic group generated by α. Following an idea of Fantechi [Fan08], we exploit this correspondence to tackle the problem of the identification of the twisted sectors of $\mathcal{M}_g$ by using the classical theory of cyclic (possibly ramified) covers of algebraic varieties, as developed in [Par91]. We identify some discrete data in order to separate the inertia stack of $\mathcal{M}_g$ in its connected components. The first data are the genus of the quotient curve and the order N of the automorphism; the latter is a general invariant of twisted sectors as it appears already in the definition of the inertia stack. Finally, the branch locus of the covering can be split in $N-1$ parts according to the local monodromy around each of its points. The last invariants are simply the degrees of each of these $N-1$ divisors. It is a recent result of Catanese ([Ca10]) that these numerical data single out a connected component of the moduli space of connected cyclic covers.

Thus the topology of the moduli space of cyclic covers of curves, which was of classical interest, plays a central role in the study of the stringy geometry of the moduli spaces of curves, their Gromov–Witten theory and quantum cohomology. The construction outlined in the last part of the previous paragraph describes the connected components of this moduli space. For low values of g, it is possible to study explicitly the topology of the twisted sectors of $\mathcal{M}_{g,n}$. This has been done with elementary techniques in [Pag08], [S06] and [Pag10a] for genus 1 and 2.

In this chapter, we work out the details of the theory of twisted sectors of moduli of curves explained above in the case when g equals 3. We first solve the simple combinatorics of the numerical discrete data of genus 3, and then study the geometry and topology of the resulting twisted sectors of $\mathcal{M}_3$. In most cases, the cohomology of the twisted sector is computed in a rather straightforward way. The main exceptions are the twisted sectors corresponding to bielliptic and to quadrielliptic genus 3 curves, which require a more detailed analysis. In particular, our computation of the cohomology of the moduli space of bielliptic genus 3 curves is achieved by using a combination of Vassiliev–Gorinov's method for the computation of the cohomology of complements of discriminants with the study of certain Leray spectral sequences, following the approach of [T05], [T07]. We expect that these techniques could be applied also in other cases of moduli spaces of cyclic covers, at least for small values of g. Finally, we partially extend our investigation to the orbifold cohomology of the Deligne–Mumford compactification $\overline{\mathcal{M}}_3$ of $\mathcal{M}_3$. Specifically, we study the Zariski closure of $I(\mathcal{M}_3)$ inside the inertia stack $I(\overline{\mathcal{M}}_3)$. The connected components of this compactification are precisely the connected components of $I(\overline{\mathcal{M}}_3)$ whose general element is a smooth curve. We can think of this situation in analogy with what happens in the theory of moduli of stable maps: it is often the case that compactifying the space of maps one introduces "extraneous" components, and that the main interest is focused on the connected components of the moduli space whose general element is a map from a smooth curve. In the present work, the study of the compactification of the inertia stack is performed using moduli stacks of admissible covers (we refer for the general theory to [ACV03]). The cohomology of the twisted sectors contained in this compactification contributes what we call the compactified orbifold cohomology of $\mathcal{M}_3$.

Our main results are the description of the connected components of the twisted sectors of $\mathcal{M}_g$ (Sect. III.2.1), the age for each of them (Prop. III.5.6) and the explicit computation of the orbifold cohomology of $\mathcal{M}_3$, which we recollect in the following two theorems:

Theorem III.1.1 (Theorem III.5.7)**.** *The orbifold Poincaré polynomial of $\mathcal{M}_3$ is:*

$$\begin{aligned} &1 + t + 2t^2 + t^3 + t^{\frac{10}{3}} + t^{\frac{7}{2}} + 4t^4 + 2t^{\frac{9}{2}} + 2t^{\frac{14}{3}} + t^{\frac{33}{7}} + 5t^5 + t^{\frac{46}{9}} + t^{\frac{36}{7}} \\ &+ 3t^{\frac{16}{3}} + t^{\frac{38}{7}} + 4t^{\frac{11}{2}} + t^{\frac{50}{9}} + t^{\frac{39}{7}} + t^{\frac{17}{3}} + t^{\frac{40}{7}} + t^{\frac{52}{9}} + t^{\frac{41}{7}} + 10t^6 + t^{\frac{43}{7}} \\ &+ t^{\frac{56}{9}} + t^{\frac{44}{7}} + t^{\frac{19}{3}} + t^{\frac{45}{7}} + t^{\frac{58}{9}} + 3t^{\frac{13}{2}} + t^{\frac{46}{7}} + 2t^{\frac{20}{3}} + t^{\frac{48}{7}} + t^{\frac{62}{9}} + t^{\frac{51}{7}}. \end{aligned}$$

Theorem III.1.2 (Theorem III.5.9)**.** *The compactified orbifold Poincaré polynomial (see Definition III.5.5) of $\mathcal{M}_3$ is:*

$$\begin{aligned} &1 + t + 4t^2 + 4t^3 + t^{\frac{10}{3}} + t^{\frac{7}{2}} + 16t^4 + t^{\frac{9}{2}} + 2t^{\frac{14}{3}} + t^{\frac{33}{7}} + 12t^5 \\ &+ t^{\frac{46}{9}} + t^{\frac{36}{7}} + 5t^{\frac{16}{3}} + t^{\frac{38}{7}} + 5t^{\frac{11}{2}} + t^{\frac{50}{9}} + t^{\frac{39}{7}} + t^{\frac{40}{7}} + t^{\frac{52}{9}} + t^{\frac{41}{7}} \\ &+ 31t^6 + t^{\frac{43}{7}} + t^{\frac{56}{9}} + t^{\frac{44}{7}} + t^{\frac{45}{7}} + t^{\frac{58}{9}} + 5t^{\frac{13}{2}} + t^{\frac{46}{7}} + 5t^{\frac{20}{3}} + t^{\frac{48}{7}} + t^{\frac{62}{9}} \\ &+ 12t^7 + t^{\frac{51}{7}} + 2t^{\frac{22}{3}} + t^{\frac{15}{2}} + 16t^8 + t^{\frac{17}{2}} + t^{\frac{26}{3}} + 4t^9 + 4t^{10} + t^{11} + t^{12}. \end{aligned}$$

The ordinary additive cohomology of $\mathcal{M}_3$ and $\overline{\mathcal{M}}_3$ was studied by Looijenga and Getzler, respectively in [Lo91, (4.7)] and [Ge98, Prop. 16]: our results are an extension of theirs.

The chapter is developed on three levels. On the first level, we introduce the theory of inertia stack, age grading, orbifold cohomology, and orbifold Poincaré polynomial for a general smooth Deligne–Mumford stack X as developed by Chen–Ruan in [CR04] and Abramovich–Graber–Vistoli in [AGV02], [AGV08]. The compactified orbifold cohomology is introduced for any choice of a smooth compactification $X \subset \overline{X}$. On the second level we introduce the theory and prove some general results for the inertia stack of the moduli stack of curves $\mathcal{M}_g$ and its Deligne–Mumford compactification $\overline{\mathcal{M}}_g$. Finally the third level is devoted to work out all the details in the case $g = 3$.

We observe that we work with cohomology, as in the seminal paper of Chen–Ruan. Nevertheless, the techniques we use are algebraic, thus in this sense our work is closer to Abramovich–Graber–Vistoli's approach.

III.1.1 Acknowledgments

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III.1.2 Notation

We work over the field of complex numbers, and cohomology is always taken with rational coefficients. By a stack, we shall always mean a Deligne–Mumford stack, of finite type over $\mathbb{C}$. In this context, the canonical map from a stack to its coarse moduli space induces an isomorphism in cohomology: we will often identify the two cohomologies by means of this isomorphism. We adopt the convention that orbifold cohomology is the graded vector space underlying Chen–Ruan cohomology, where the latter carries the additional ring structure.

In our work, we shall consider the cohomology with its mixed Hodge structures. We shall denote by $\mathbf{Q}(-k)$ the Hodge structure of Tate of weight $2k$. The class of $\mathbf{Q}(-1)$ in the Grothendieck group $K_0(\mathsf{HS}_{\mathbf{Q}})$ of rational Hodge structures will be denoted by $L = [\mathbf{Q}(-1)]$.

Results on the cohomology with compact support of a quasi-projective variety (or stack with quasi-projective coarse moduli space) X shall often be expressed by means of its Euler characteristic in $K_0(\mathsf{HS}_{\mathbf{Q}})$. Following [PS08, § 5.5.2], we call this Euler characteristic the *Hodge–Grothendieck character for compact support* of X and denote it by

$$\chi^c_{\mathrm{Hdg}}(X) = \sum_{i \in \mathbb{N}} (-1)^i [H^i_c(X; \mathbf{Q})] \in K_0(\mathsf{HS}_{\mathbf{Q}}).$$

Hodge–Grothendieck characters for compact support are sometimes called Serre characteristics in the literature.

Similarly, to state results on cohomology with compact support in a concise way, we shall express them as polynomials with coefficients in the Grothendieck group of Hodge structures:

$$P_X(t) = \sum_{i \in \mathbb{N}} [H^i_c(X; \mathbf{Q})] t^i \in K_0(\mathsf{HS}_{\mathbf{Q}})[t].$$

We work with cyclic covers $f : C \to C'$, where C and C' are, respectively, the covering and the covered space. If the cyclic cover is not étale, it contains, respectively, ramification and branch points.

We shall denote the symmetric group in d letters by $\mathfrak{S}_d$ and the group of kth roots of unity by μ_k.

III.2 The inertia stacks

III.2.1 Definition of the inertia stack

In this section we recollect some basic notions concerning the inertia stack. For a more detailed study of this topic, we address the reader to [AGV08, Section 3].

We introduce the following natural stack associated to a stack X, which points to where X fails to be an algebraic space.

Definition III.2.1. ([AGV02, 4.4], [AGV08, Definition 3.1.1]) Let X be a stack. The *inertia stack* $I(X)$ of X is defined as:

$$I(X) := \coprod_{N \in \mathbb{N}_{>0}} I_N(X)$$

where $I_N(X)(S)$ is the following groupoid:

1. The objects are pairs (ξ, α), where ξ is an object of X over S, and $\alpha : \mu_N \to \mathrm{Aut}(\xi)$ is an injective homomorphism;

2. The morphisms are the morphisms $g : \xi \to \xi'$ of the groupoid $X(S)$, satisfying $g \cdot \alpha(1) = \alpha'(1) \cdot g$.

The inertia stack comes with a natural forgetful map $f : I(X) \to X$.

We also define $I_{TW}(X) := \coprod_{N>1} I_N(X)$. The connected components of $I_{TW}(X)$ are called *twisted sectors* of the inertia stack of X, or simply twisted sectors of X.

We remark that, by its very definition, $I_N(X)$ is an open and closed substack of $I(X)$, but it rarely happens that it is connected. One special case is when N equals 1: in this case the map f restricted to $I_1(X)$ induces an isomorphism of the latter with X. The connected component $I_1(X)$ will be referred to as the *untwisted sector*. We also observe that after the choice of a generator of μ_N, we obtain an isomorphism of $I(X)$ with $I'(X)$, where the latter is defined as the (2-)fiber product $X \times_{X \times X} X$ where both morphisms $X \to X \times X$ are the diagonals.

Remark III.2.2. There is an involution $\iota : I_N(X) \to I_N(X)$, which is induced by the map $\iota' : \mu_N \to \mu_N$ given by $\iota'(\zeta) := \zeta^{-1}$.

The inertia stack, which we have just defined, is the fundamental ingredient in the definition of orbifold cohomology (Chen–Ruan cohomology as a vector space). We observe that, at this level, we do not need X to be smooth nor proper.

Definition III.2.3. ([CR04]) Let X be a stack. The *orbifold cohomology* (with rational coefficients) of X is defined as a vector space as:

$$H^\bullet_{CR}(X) := H^\bullet(I(X); \mathbf{Q}).$$

Now if $X \hookrightarrow \overline{X}$ is an open dense embedding, we can define an intermediate space between $I(X)$ and $I(\overline{X})$, namely:

Definition III.2.4. Given a compactification of a stack $i: X \to \overline{X}$, we define the *compactified inertia stack* of X as the stack

$$\overline{I}(X) = \coprod_{N \in \mathbb{N}} \overline{I}_N(X)$$

where $\overline{I}_N(X)$ is the stack of all connected components Y of $I_N(\overline{X})$ such that $i^* Y \neq \emptyset$. We can thus define the *compactified orbifold cohomology* as the following vector space:

$$\overline{H}^{\bullet}(X) := H^{\bullet}(\overline{I}(X); \mathbf{Q}).$$

In the following sections, we will study the inertia stack for moduli of smooth genus g curves and its compactified inertia stack with respect to the Deligne–Mumford compactification. In the first case we will use the theory of cyclic covers of smooth curves (see [Par91]), in the second we will use the theory of admissible covers developed by various authors.

Remark III.2.5. The orbifold cohomology of X only depends upon the topological space (coarse moduli space) underlying $I(X)$. In [AGV08], the authors introduce two notions related to the inertia stack: the *stack of cyclotomic gerbes* ([AGV08, Definition 3.3.6]) and the *rigidified inertia stack* ([AGV08, 3.4]), showing in [AGV08, 3.4.1] that they are equivalent. It is relevant to observe that all these different notions of inertia stacks share the same coarse moduli space, and therefore they give rise to the same orbifold cohomology theory.

III.2.2 The inertia stack of moduli of genus g smooth curves

We want to study the twisted sectors of the inertia stack of moduli of smooth genus g curves. For this, we study the moduli stacks of cyclic ramified covers of curves of genus $g' < g$. This approach is due to Fantechi [Fan08], and builds on the theory of abelian covers of algebraic varieties (see Pardini [Par91]). A description of the theory of abelian covers in the case of curves and of cyclic groups that is closely related to the one we use can be found in [Ca10, Sections 1 and 2]. A similar construction for covers of prime order was studied in [Co87]. We start by summarizing informally the description of cyclic covers we will use in our constructions.

Fact III.2.6. *([Par91, Proposition 2.1]) Let C' be a smooth genus g' curve. Then the following data are equivalent:*

- *A cyclic (possibly ramified) μ_N-cover $\psi : C \to C'$, where C is a smooth curve, possibly disconnected;*

- *A sequence of $N-1$ smooth effective divisors $D_1, \dots, D_{N-1}$ (with pairwise disjoint support, possibly empty), a line bundle L on C' together with an isomorphism $\varphi : L^{\otimes N} \to \mathcal{O}_{C'}(\sum_i iD_i)$.*

With this result in mind, let us define:

Definition III.2.7. Let $g > 1$ be an integer. A g-admissible datum is an $(N+1)$-tuple of nonnegative integers $A = (g', N; d_1, \dots, d_{N-1})$ with $N \geq 2$ and $g' \leq g$, satisfying the following conditions:

- Riemann–Hurwitz formula

$$2g-2 = N(2g'-2) + \left(\sum d_i \ \gcd(i,N)\left(\frac{N}{\gcd(i,N)} - 1\right)\right); \quad \text{(III.2.1)}$$

- the structural equation of abelian covers

$$\sum i \ d_i = 0 \mod N. \quad \text{(III.2.2)}$$

The integers N and g' will be called respectively the *order* and the *base genus* of the g-admissible datum A.

Note that, for a fixed g, the set of all g-admissible data A is finite. With every admissible g-datum, we associate the integers $d = \sum d_i$, and a disjoint union decomposition $\{1, \dots, d\} = \coprod_{i=1}^{N-1} J_i$ by:

$$J_i := \left\{ j| \sum_{l<i} d_l < j < \sum_{l \leq i} d_l \right\}.$$

Moreover, we will denote by S_A the subgroup of $\mathfrak{S}_d$ (the symmetric group on d elements) defined by $S_A := \{\sigma| \ \sigma(J_i) = J_i\}$.

We shall now construct the twisted sectors of $\mathcal{M}_g$ as stacks of cyclic N-covers of curves of genus g', with branch locus of type $d_1, \dots, d_{N-1}$.

Definition III.2.8. Let A be a g-admissible datum. We define the stack $\mathcal{M}_A$ whose objects over a scheme S are $(N+2)$-tuples $(C, D_1, \dots, D_{N-1}, L, \varphi)$, where C is a smooth family of genus g' curves over S, the D_i are sections of $(\mathrm{Sym}^{d_i}_S C \setminus \Delta_{d_i}) \to S$ (where Δ_{d_i} denotes the big diagonal) defining disjoint divisors on C, L is a line bundle and $\varphi : L^{\otimes N} \to \mathcal{O}_C(\sum iD_i)$ is an isomorphism.

The stack $\mathcal{M}'_A$ is defined as the open and closed substack of $\mathcal{M}_A$ whose objects under the correspondence III.2.6 correspond to connected covers.

Remark III.2.9. Let us be more explicit about the morphisms of $\mathcal{M}_A(S)$. Let (C, D_i, L, φ) and (C', D'_i, L', φ') be two objects as in Definition III.2.8. Then a morphism between them is a couple of isomorphisms $(\sigma : \ C \to C', \tau : \sigma^* L' \to L)$ satisfying the following conditions: The map σ is an isomorphism of curves such that $\sigma^*(D'_i) = D_i$ and τ is an isomorphism of line bundles that makes the following diagram commute:

$$\begin{tikzcd} \sigma^*(L'^{\otimes N}) \arrow[r, "\sigma^*(\varphi')"] \arrow[d, "\tau^{\otimes N}"] & \sigma^*\left(\mathcal{O}_{C'}(\sum iD'_i)\right) \arrow[d, "\gamma"] \\ L^{\otimes N} \arrow[r, "\varphi"] & \mathcal{O}_C(\sum iD_i), \end{tikzcd} \tag{III.2.3}$$

where we denoted by γ the isomorphism induced by σ. A different definition for a morphism between the two families (C, D_i, L, φ) and (C', D'_i, L', φ') is the following: a single isomorphism $\sigma : C \to C'$ satisfying $\sigma^*(D'_i) = D_i$ and such that there exists an isomorphism $\tau : \ \sigma^* L' \to L$ making diagram (III.2.3) commute. These two different definitions of morphisms give rise to two different stacks, which share the same coarse moduli space.

Remark III.2.10. Let us denote by $\mathcal{M}_{g',d}(B\mu_N)$ the open substack of the moduli stack of stable maps $\mathcal{K}_{g',d}(B\mu_N)$ whose source curve is smooth. The moduli stack $\mathcal{K}_{g',d}(B\mu_N)$ is defined in [AV02]; see also [ACV03], where $\mathcal{K}_{g',d}(B\mu_N)$ is denoted $\mathcal{B}_{g',d}(\mu_N)$. We observe that the stack $\mathcal{M}_A$ we have just defined (with the first definition of morphisms in III.2.9) is an open and closed substack of the quotient stack $[\mathcal{M}_{g',d}(B\mu_N)/S_A]$ prescribed by the assignment of the ramifications $d_1, \dots, d_{N-1}$.

Remark III.2.11. If A is a g-admissible datum, then disconnected covers appear exactly when N is not prime and the greatest common divisor of N and all the i with $d_i \neq 0$ is strictly bigger than 1. If this condition is satisfied, and moreover $g' = 0$, then $\mathcal{M}'_A = \emptyset$, as $\mathcal{M}_A$ only parametrizes disconnected covers.

We can see that the moduli stacks $\mathcal{M}'_A$ we have just constructed constitute open and closed substacks of the inertia stack of $\mathcal{M}_g$:

Corollary III.2.12. *Let us fix $g, N > 1$. Then the stack $I_N(\mathcal{M}_g)$ of Definition III.2.1 is isomorphic to the disjoint union of all nonempty stacks $\mathcal{M}'_A$ for all g-admissible data $A = (g', m, d_1, \dots, d_{m-1})$ with order m equal to N.*

Proof. Follows from Definitions III.2.7, III.2.8 and by adapting the proof of [Par91, Theorem 2.1, Proposition 2.1] to this relative case (cf. Fact III.2.6). Indeed, there is a base-preserving equivalence of categories:

$$I_N(\mathcal{M}_g)(S) \to \coprod_A \mathcal{M}'_A(S), \tag{III.2.4}$$

where in the right hand side the disjoint union is taken over all g-admissible data with order N. We sketch the proof of this well–known fact, by explicitly defining the (functorial) correspondence (III.2.4). Let us assume that a μ_N-cover $\psi : X \to C$ is given (over a base S). Over each point $s \in S$, the branch divisor D_s of the cover $X_s \to C_s$ can be split according to local monodromy in smooth effective divisors $D_{1,s}, \dots, D_{N-1,s}$, having pairwise disjoint support. Identifying these divisors with sections of the appropriate symmetric product of $C \to S$ gives the D_i. Each D_i defines a codimension 1 subscheme of C which does not contain any fibre of $C \to S$, hence they give rise to effective Cartier divisors on C. At this point, we only need to construct the line bundle L together with the isomorphism $\varphi :\ L^{\otimes N} \to \mathcal{O}_C(\sum_i iD_i)$. Since the cover is nontrivial, the action of μ_N on the push-forward sheaf $\psi_*(\mathcal{O}_X)$ defines a splitting as a direct sum of line bundles:

$$\psi_*\mathcal{O}_X = L_0 \oplus \ldots \oplus L_{N-1}$$

where L_i is the subsheaf of $\psi_*(\mathcal{O}_X)$ of sections where μ_N acts with weight i. The line bundle L is then defined as $L_1^\vee$. By viewing the sections of $L = L_1^\vee$ as functions on the total space of the line bundle L_1, and hence also as functions on its trivial section C, we obtain an identification

$$\varphi :\ L_1^{\otimes N} \otimes \mathcal{O}_C(\sum_i iD_i) \to \mathcal{O}_C.$$

The correspondence just defined is essentially surjective. Indeed, if the line bundle L is given over C', we set $L_1 := L^\vee$. The line bundles L_a can then be defined as:

$$L_a := L_1^{\otimes a} \otimes \bigotimes_{i=1}^{N-1} \mathcal{O}_C(D_i)^{-[\frac{ai}{N}]} \quad a = 0, \ldots, N-1 \tag{III.2.5}$$

Now the number $[\frac{ai}{N}] + [\frac{bi}{N}] - [\frac{(a+b)i}{N}]$ can either be 0 or 1. In both cases, the canonical sections of the line bundles:

$$\mathcal{O}_C(D_i)^{[\frac{ai}{N}]+[\frac{bi}{N}]-[\frac{(a+b)i}{N}]}$$

permit the definition of a ring structure over $R := \bigoplus_{i=0}^{N-1} L_i$. Now the normalization of the spectrum of R reconstructs the smooth μ_N-cover of C' up to isomorphism of μ_N-cover. If one considers the definition of morphisms in the groupoid $\mathcal{M}'_A(S)$ given in Remark III.2.9, one can also check that the correspondence is fully faithful, hence an equivalence of categories. $\square$

Recently, Catanese proved that the moduli spaces $\mathcal{M}'_A$ are indeed connected:

Theorem III.2.13. *([Ca10, Theorem 2.4]) Let A be a g-admissible datum. Then the stack $\mathcal{M}'_A$ is connected (although possibly empty).*

In particular, as a consequence of Catanese's connectedness result (Theorem III.2.13), the nonempty moduli stacks $\mathcal{M}'_A$ give all the twisted sectors of the inertia stack of $\mathcal{M}_g$.

Remark III.2.14. (see Remark III.2.9) Working with the second definition of morphism in the definition of the stack $\mathcal{M}_A$ one obtains a decomposition of the rigidified inertia stack (see Remark III.2.5) of $\mathcal{M}_g$.

A	$\chi^c_{\mathrm{Hdg}}(\mathcal{M}_A)$	$\chi^c_{\mathrm{Hdg}}(\overline{\mathcal{M}}_A)$	$a(\mathcal{M}_A)$
$(2;8)$	L^5	$L^5+3L^4+6L^3+6L^2+3L+1$	$\frac{1}{2}$
$(3;4,1)$	L^2	L^2+2L+1	$\frac{5}{3}$
$(3;1,4)$	L^2	L^2+2L+1	$\frac{7}{3}$
$(4;4,0,0)$	L	$L+1$	2
$(4;0,0,4)$	L	$L+1$	3
$(4;2,3,0)$	L^2-L	L^2+2L+1	$\frac{7}{4}$
$(4;0,3,2)$	L^2-L	L^2+2L+1	$\frac{9}{4}$
$(4;2,0,2)$	$L-1$	$L+1$	$\frac{5}{2}$
$(6;1,0,2,0,1)$	$L-1$	$L+1$	$\frac{5}{2}$
$(6;1,0,1,2,0)$	$L-1$	$L+1$	$\frac{8}{3}$
$(6;0,2,1,0,1)$	$L-1$	$L+1$	$\frac{7}{3}$

Table III.1: Positive-dimensional twisted sectors. For the sake of brevity we omit $g'=0$ from the notation of the admissible datum.

III.3 The inertia stack of $\mathcal{M}_3$

In this section, we study the geometry of the twisted sectors of the inertia stack of the moduli space of smooth, genus 3 curves. We determine the cohomology of all these twisted sectors as a graded vector space with Hodge structures. We shall state these results in the form of polynomials with coefficients in $K_0(\mathsf{HS}_{\mathbf{Q}})$ (see Section III.1.2).

Our approach is based on the correspondence between twisted sectors and g-admissible data introduced in the previous section. Of course, in the case of genus 3 also a direct approach is possible by classifying all automorphisms of plane quartic curves (as in e.g. [Do10, Lemma 6.5.1]) and of all hyperelliptic genus 3 curves. However, our approach seems more suitable for cohomological computations and has the advantage that it generalizes to higher genus.

If X is a twisted sector of $I(\mathcal{M}_3)$, we have seen in the previous section that $X \cong \mathcal{M}_A$ for A a certain 3-admissible datum. We start by considering the admissible data with $g'=0$.

Proposition III.3.1. *There are 43 different* 3*-admissible data A with $g'=0$ that parametrize connected covers. The complete list of these admissible data and of the Hodge–Grothendieck characters for compact support of the associated twisted sectors $\mathcal{M}_A$ is given in Tables III.1 and III.2.*

Proof. If A is a g-admissible datum with $g'=0$, then it is easy to see that $\mathcal{M}_A \cong [\mathcal{M}_{0,d}/S_A]$. Therefore $H^\bullet_c(\mathcal{M}_A) \cong H^\bullet_c(\mathcal{M}_{0,d})^{S_A}$, the S_A-invariant part of

3-admissible with $g'=0$	Age	3-admissible with $g'=0$	Age
$(7;2,0,0,0,1,0)$	$\frac{20}{7}$	$(9;1,0,1,0,1,0,0,0)$	$\frac{26}{9}$
$(7;0,1,0,0,0,2)$	$\frac{22}{7}$	$(9;0,0,0,1,0,1,0,1)$	$\frac{28}{9}$
$(7;1,1,0,1,0,0)$	3	$(9;0,1,1,1,0,0,0,0)$	$\frac{29}{9}$
$(7;0,0,1,0,1,1)$	3	$(9;0,0,0,0,1,1,1,0)$	$\frac{25}{9}$
$(7;1,0,2,0,0,0)$	$\frac{23}{7}$	$(12;10100001000)$	$\frac{10}{3}$
$(7;0,0,0,2,0,1)$	$\frac{19}{7}$	$(12;00010000101)$	$\frac{8}{3}$
$(7;0,2,1,0,0,0)$	$\frac{24}{7}$	$(12;10001100000)$	$\frac{13}{4}$
$(7;0,0,0,1,2,0)$	$\frac{18}{7}$	$(12;00000110001)$	$\frac{11}{4}$
$(8;2,0,0,0,0,1,0)$	$\frac{13}{4}$	$(12;00111000000)$	$\frac{8}{3}$
$(8;0,1,0,0,0,0,2)$	$\frac{11}{4}$	$(12;00000011100)$	$\frac{10}{3}$
$(8;1,1,0,0,1,0,0)$	3	$(14;1000011000000)$	$\frac{51}{14}$
$(8;0,0,1,0,0,1,1)$	3	$(14;0000001100001)$	$\frac{33}{14}$
$(8;0,1,2,0,0,0,0)$	$\frac{11}{4}$	$(14;0100101000000)$	$\frac{41}{14}$
$(8;0,0,0,0,2,1,0)$	$\frac{13}{4}$	$(14;0000001010010)$	$\frac{43}{14}$
$(9;1,1,0,0,0,1,0,0)$	$\frac{31}{9}$	$(14;0011001000000)$	$\frac{45}{14}$
$(9;0,0,1,0,0,0,1,1)$	$\frac{23}{9}$	$(14;0000001001100)$	$\frac{39}{14}$

Table III.2: 0-dimensional twisted sectors. For the sake of brevity we omit $g'=0$ from the notation of the admissible datum.

the cohomology with compact support of $\mathcal{M}_{0,d}$. Hence, for every A, the cohomology with compact support of $\mathcal{M}_A$ can be computed from the description of the cohomology of $\mathcal{M}_{0,d}$ as a representation of the symmetric group $\mathfrak{S}_d$, which is known for every $d \geq 3$ by work of Getzler [Ge94, 5.6] (see also [KL02, Theorem 2.9]).

In our case, we need to work with connected covers, i.e. we restrict to 3-admissible data with $g' = 0$ that satisfy condition (III.2.11). We obtain their list (which we give in Table III.1 and III.2) by finding all solutions of equations (III.2.1), (III.2.2) and (III.2.11) for $g = 3$, $g' = 0$. Using Getzler's formulas we compute their Hodge–Grothendieck characters for compact support, i.e. the Euler characteristic of their cohomology with compact support in the Grothendieck group of Hodge structures. $\square$

Remark III.3.2. If A is an admissible datum with $g' = 0$, the Hodge–Grothendieck character for compact support of $\mathcal{M}_A$ determines uniquely the cohomology of $\mathcal{M}_A$, because its k-th compactly supported cohomology carries a pure Hodge structure of weight $2\dim(\mathcal{M}_A) - k$. This property holds for the space $\mathcal{M}_{0,d}$ and follows from the structure of $\mathcal{M}_{0,d}$ as a complement of hyperplanes in $\mathbf{C}^{d-3}$. Since $H^\bullet_c(\mathcal{M}_A;\mathbf{Q})$ is a subring of $H^\bullet_c(\mathcal{M}_{0,d};\mathbf{Q})$, it holds for $\mathcal{M}_A$ as well.

It is easy to see that there are exactly four 3-admissible data with $g' > 0$. Following Definition III.2.8, they correspond to the four moduli stacks of cyclic covers:

$$\mathcal{M}_{(1,2;4)},\quad \mathcal{M}_{(1,3;1,1)},\quad \mathcal{M}_{(1,4;0,2,0)}\ \text{ and } \mathcal{M}_{(2,2;0)}. \tag{III.3.1}$$

In view of Definition III.2.8, the moduli stacks $\mathcal{M}_{(g',N,d_1,\dots,d_{N-1})}$ parametrize objects of type $(C, L, D_1, \dots, D_{N-1}, \varphi)$, where C is a curve of genus g', the D_i are disjoint effective divisors of prescribed degrees d_i and $\varphi:\ L^{\otimes N} \to \mathcal{O}_C(\sum_i iD_i)$ is an isomorphism. Hence, it suffices to compute the cohomology of the following four stacks:

$$\begin{aligned}
\mathcal{A} &= \left\{(C, D_1, L)|\ g(C) = 1,\ \deg(D_1) = 4,\ L^{\otimes 2} \cong \mathcal{O}_C(D_1)\right\},\\
\mathcal{B} &= \left\{(C, D_1, D_2, L)\ \middle|\ \begin{array}{l} g(C) = 1, \deg(D_1) = \deg(D_2) = 1,\\ L^{\otimes 3} \cong \mathcal{O}_C(D_1 + 2D_2)\end{array}\right\},\\
\mathcal{C} &= \left\{(C, D_2, L)|\ g(C) = 1,\ \deg(D_2) = 2,\ L^{\otimes 4} \cong \mathcal{O}_C(2D_2)\right\},\\
\mathcal{D} &= \left\{(C, D_1, L)|\ g(C) = 2,\ L^{\otimes 2} \cong \mathcal{O}_C\right\}.
\end{aligned}$$

It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are connected, while $\mathcal{C}$ and $\mathcal{D}$ are not. The stack $\mathcal{C}$ has two open and closed substacks $\mathcal{C}'$ and $\mathcal{C}''$ that correspond, respectively, to the two conditions $L^{\otimes 2} \not\cong \mathcal{O}(D_2)$ and $L^{\otimes 2} \cong \mathcal{O}(D_2)$. The stack $\mathcal{D}$ has two open and closed substacks $\mathcal{D}'$ and $\mathcal{D}''$ that correspond, respectively, to the conditions $L \not\cong \mathcal{O}_C$ and $L \cong \mathcal{O}_C$. Observe that $\mathcal{C}''$ and $\mathcal{D}''$ parametrize *disconnected* covers.

In the remainder of this section, we study the cohomology of $\mathcal{B}$ and $\mathcal{D}'$, while we postpone the analogous computation for $\mathcal{A}$ and $\mathcal{C}'$ to the following section.

In the following lemma, we let $X_1(3)$ be the closed substack in $\mathcal{M}_{1,2}$ of curves (C, p_1, p_2) such that p_2 is a point of 3-torsion for the elliptic curve (C, p_1).

Lemma III.3.3. *The coarse moduli space of $\mathcal{B}$ is isomorphic to the coarse moduli space of $\mathcal{M}_{1,2} \setminus X_1(3)$.*

Proof. The moduli stack $\mathcal{B}$ parametrizes curves C of genus $g' = 1$, two distinct points $x, y \in C$ and a line bundle L of degree 1 on C. Let $\mathcal{C}_{1,2}$ be the universal curve over $\mathcal{M}_{1,2}$: it parametrizes genus 1 curves C with three points x_1, x_2, q such that $x_1 \neq x_2$. We define $\tilde{\mathcal{B}}$ to be the irreducible codimension 1 substack of $\mathcal{C}_{1,2}$ defined by the following constraint on the three points:

$$\tilde{\mathcal{B}} = \{(C, x_1, x_2, q)|\ 3q \equiv x_1 + 2x_2\}$$

Now it is clear that $\mathcal{B}$ and $\tilde{\mathcal{B}}$ share the same coarse moduli space. This can be checked by associating to a triple (C, x, y, L) the triple (C, x_1, x_2, q) where q is the point on the curve C determined by the isomorphism class of the line bundle L. The diagram:

$$\begin{tikzcd}
\tilde{\mathcal{B}} = \{3q = x_1 + 2x_2\} \arrow[r, hook] \arrow[d] & \mathcal{C}_{1,2} \arrow[d, "\pi_x"] & \ni (x, y, q) \arrow[d, mapsto] \\
\mathcal{M}_{1,2} \arrow[r, hook] & \overline{\mathcal{M}}_{1,2} & \ni (y, q)
\end{tikzcd}$$

is cartesian. Indeed, the restriction of π_x to the closed locus $\tilde{\mathcal{B}} \subset \mathcal{C}_{1,2}$ takes values in $\mathcal{M}_{1,2}$, since $q = y, 3q = x + 2y \implies x = y$. Furthermore, the restriction of π_x to $\tilde{\mathcal{B}}$ is an isomorphism onto the image locus, *i.e.* the points where $3q \neq 3y$. From this the claim follows. $\square$

From this description, we can deduce the cohomology with compact support of $\mathcal{B}$.

Corollary III.3.4. *The cohomology with compact support of $\mathcal{B}$ is given by*

$$P_{\mathcal{B}}(t) := \sum_{i \in \mathbb{N}} [H^i_c(\mathcal{B}; \mathbf{Q})] t^i = L^2 t^4 + L t^3 + t^2.$$

Proof. The cohomology with compact support of $\mathcal{M}_{1,2}$ is concentrated in degree 4, while the cohomology with compact support of $X_1(3)$ can be deduced from the fact that its coarse moduli space is a $\mathbb{P}^1$ minus two points. This classical result can be proved directly by considering the coarse moduli space of $X_1(3)$ as the quotient of the pointed rational curve $\mathbf{P}^1 \setminus \{[0,1], [1, -3\zeta_3^k]\}$ parametrizing the Hesse pencil $\lambda(x_0^3 + x_1^3 + x_3^3) + \mu x_0 x_1 x_2 = 0$ by the action of μ_3 generated by $[\lambda, \mu] \mapsto [\lambda, \zeta_3 \mu]$. Then the result follows from the long exact sequence of compactly supported cohomology, associated to the inclusion of $X_1(3)$ in $\mathcal{M}_{1,2}$ with complement isomorphic to $\mathcal{B}$. $\square$

Now we study the stack $\mathcal{D}'$. It turns out that we can describe it as a quotient of a moduli stack of genus 0 curves with marked points, by the action of a subgroup of the symmetric group that symmetrizes some of the points. This enables us to compute its cohomology from Getzler's formulas ([Ge94]).

Lemma III.3.5. *The coarse moduli space of $\mathcal{D}'$ is $\mathcal{M}_{0,6}/\mathfrak{S}_4 \times \mathfrak{S}_2$.*

Proof. The claim follows from the well known fact that every nontrivial square root of the structure sheaf on a smooth genus 2 curve C is of the form $\mathcal{O}(x_1 - x_2)$ where x_1 and x_2 are distinct Weierstrass points of C, and that this expression is unique up to changing the order of x_1 and x_2. □

Corollary III.3.6. *The cohomology with compact support of $\mathcal{D}'$ is given by*

$$P_{\mathcal{D}}(t) = L^3t^6 + L^2t^5.$$

III.3.1 The geometry of the twisted sectors $\mathcal{A}$ and $\mathcal{C}'$

The aim of this section is to prove the following two results:

Proposition III.3.7. *The cohomology with compact support of $\mathcal{A}$ is expressed by*

$$P_{\mathcal{A}}(t) = L^4t^8 + L^2t^5.$$

Proposition III.3.8. *The cohomology with compact support of $\mathcal{C}'$ is given by*

$$P_{\mathcal{C}'}(t) = L^2t^4 + Lt^3 + t^2.$$

This completes our cohomological analysis of the inertia stack of $\mathcal{M}_3$. In particular, we are now able to produce the dimension of the orbifold cohomology as a vector space $H^\bullet_{CR}(\mathcal{M}_3)$ from the Propositions III.3.1, III.3.7, III.3.8 and the Corollaries III.3.4 and III.3.6.

Corollary III.3.9. *The orbifold cohomology of $\mathcal{M}_3$ has dimension* 62*.*

In Section III.5.2 we shall describe the **Q**-graded structure of this vector space in Theorem III.5.7.

The cohomology of the moduli space $\mathcal{A}$

Recall that the moduli space $\mathcal{A}$ parametrizes bielliptic genus 3 curves. We described it as the moduli stack of genus 1 curves with a set $\{x_1, x_2, x_3, x_4\}$ of (unordered) marked points and a line bundle L such that $L^{\otimes 2} = \mathcal{O}(x_1+x_2+x_3+x_4)$.

The divisor $D_1 := x_1 + x_2 + x_3 + x_4$ defines an embedding of C in $\mathbf{P}^3$; the image is the complete intersection of two quadrics. If we consider C as a curve in $\mathbf{P}^3$, the square roots of $\mathcal{O}(D_1)$ correspond to divisors cut by planes in $\mathbf{P}^3$ that are tangent to C at two points (possibly coinciding) with multiplicity 2. For a fixed C this gives four distinct line bundles. They can be constructed explicitly as the g^1_2 cut by the ruling of each of the four singular quadrics in $\mathbf{P}^3$ lying in the ideal of C.

One can see this geometrically by considering that divisors linearly equivalent to D_1 are cut by 2-planes in $\mathbf{P}^3$. Hence, a square root of $\mathcal{O}(D_1)$ must correspond to a bitangent 2-plane and the only planes of this form are the tangent planes to the singular quadrics containing C.

From this it follows that $\mathcal{A}$ can be viewed as the moduli space of pairs $(C, \{x_1, \dots, x_4\})$ where the curve C is a smooth genus 1 curve lying on a fixed quadric cone $Q \subset \mathbf{P}^3$ and the x_i are 4 distinct unordered points on C lying on the same plane section $H \subset Q$. If the hyperplane section H is reducible, it is the union of two lines of the ruling of Q. In this case, the involution of C interchanging each pair of points lying on the same line of the ruling of Q lifts to an involution of the double cover $\tilde{C} \to C$; this involution gives C' a hyperelliptic structure. In particular, as shown in [Co87, Corollary 1], the composition of these two involutions on C gives a fixed-point-free involution on $\tilde{C}$.

Therefore, the coarse moduli space of the closed substack $\mathcal{A}_h$ of $\mathcal{A}$ corresponding to bielliptic structures on genus 3 hyperelliptic curves is isomorphic to the coarse moduli space of $\mathcal{D}'$.

At this point, it only remains to calculate the cohomology of the complement $\mathcal{A}_{nh} = \mathcal{A} \setminus \mathcal{A}_h$.

Lemma III.3.10. $P_{\mathcal{A}_{nh}}(t) = L^4t^8 + L^3t^7$.

Proof. First we note that there is a map $\mathcal{A}_{nh} \to \mathcal{M}_{0,4}/\mathfrak{S}_4$ associating to $(C, \{x_i\})$ the configuration $\{x_1, \dots, x_4\}$ of four points on the curve $H \cong \mathbf{P}^1$. To make explicit computations, we view Q as the weighted projective plane $\mathbf{P}(1, 1, 2)$ and choose coordinates u_0, u_1, w on Q such that H is defined by the equation $w = 0$.

Every element of $\mathcal{A}_{nh}$ has an equation of the form

$$\varphi_{\alpha,\epsilon,t}(u_0, u_1, w) := w^2 - \alpha(u_0, u_1)w + \epsilon u_0u_1(u_1 - u_0)(u_1 - tu_0) = 0$$

with $\alpha \in \mathbf{C}[u_0, u_1]_2 \cong \mathbf{C}^3$, $\epsilon \in \mathbf{C}$ and $[t] := \{0, 1, \infty, t\} \in \mathcal{M}_{0,4}/\mathfrak{S}_4$. This equation is uniquely defined up the action of $\mathbf{C}^*$ on (α, ϵ) by scaling:

$$s(\alpha, \epsilon) = (s^2\alpha, s^4\epsilon).$$

Hence, we study the incidence correspondence:

$$\mathcal{I} := \left\{ (\alpha, \epsilon, [t]) \in (\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4 \;\middle|\; \begin{array}{c} \varphi_{\alpha,\epsilon,t}(u_0, u_1, w) = 0 \text{ defines} \\ \text{a nonsingular curve} \end{array} \right\}.$$

To compute the cohomology of $\mathcal{I}$, we apply Vassiliev–Gorinov's method to the complement Σ of $\mathcal{I}$ inside $(\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$. Specifically, we use the version of the method developed for the case of curves with marked points given in [T07], to which we refer for technical details on the construction. Note that the original construction of Vassiliev–Gorinov's methods is based on the study of Borel–Moore homology (i.e. homology theory with compact support). For stylistic reasons, in this chapter we will use cohomology with compact support instead of Borel–Moore homology. All results can be easily adapted by duality.

The first step of Vassiliev–Gorinov's method consists in classifying all possible singular loci of elements of Σ. This classification is then used to define a cubical space $\mathcal{X}$ whose geometric realization $|\mathcal{X}|$ has the same cohomology with compact support as Σ. This geometric realization has a natural stratification $\{F_i\}$ by locally closed subsets that are indexed by the types of singular sets that arise in the classification. Each stratum F_i can be explicitly described as a bundle over the space

$$B_i := \left\{ (\varphi, [t], K) \,\middle|\, \begin{array}{c} (\varphi, [t]) \in \Sigma, K \subset Q \text{ is a singular configuration} \\ \text{of type } i \text{ containing the singular locus of } (\varphi, [t]) \end{array} \right\}.$$

If the configurations of type i are finite sets, then $F_i \to B_i$ is a nonorientable simplicial bundle; otherwise, the stratum F_i is a union of simplicial bundles.

The Gysin spectral sequence $E_r^{p,q} \Rightarrow H_c^{p+q}(\Sigma; \mathbf{Q})$ with $E_1^{p,q} = H_c^p(F_q; \mathbf{Q})$ associated with the stratification $\{F_i\}$ is called the *Vassiliev's spectral sequence.*

(1) *One singular point on H.* In this case the curve is of the form $w(w + au_0 + bu_1) = 0$ and both components pass through the singular point. The stratum F_1 is a $\mathbf{C}^2$-bundle over $H \times \mathcal{M}_{0,4}/\mathfrak{S}_4$.

(2) *Two singular points on H.* The curve is of the form $w(w + au_0 + bu_1) = 0$ and both components have to pass through the singular points. If we fix the two distinct singular points s_1, s_2 on H, then the $\varphi = (\alpha, \epsilon) \in \mathbf{C}^4$ giving a curve singular at s_1 and s_2 form a 1-dimensional subspace. This yields the following description for the stratum F_2: It is the quotient of a $\mathbf{C}$-bundle over $(0, 1) \times (F(H, 2) \times \mathcal{M}_{0,4})/\mathfrak{S}_4$ by the involution $(\tau, (s_1, s_2), [t], (\alpha, \epsilon)) \mapsto (1 - \tau, (s_2, s_1), [t], (\alpha, \epsilon))$. From this it follows that the cohomology with compact support of F_2 is concentrated in degree 7 and carries a Tate Hodge structure of weight 6.

(3) *The curve H.* The stratum F_3 has trivial cohomology with compact support because H is a smooth rational curve (see e.g. [T05, Lemma 2.19]).

(4) *One point P outside H.* Having a singular point outside H imposes three conditions on the equation, hence we get that F_4 is a $\mathbf{C}$-bundle over the space of configurations $(P, \{x_1, x_2, x_3, x_4\})$. Note that P is not allowed to lie on the same line of the ruling as any of the x_i. Hence the configuration space is a $\mathbf{C}^*$-bundle over $\mathcal{M}_{0,5}/\mathfrak{S}_4$, whose cohomology with compact support is concentrated in degree 4 by the results in [Ge94].

Table III.3: Spectral sequence converging to $H_c^\bullet((\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4 \setminus \mathcal{I}; \mathbf{Q})$.

q				
7	$\mathbf{Q}(-4)$	0	0	
6	0	0	0	
5	$\mathbf{Q}(-3)$	$\mathbf{Q}(-3)$	$\mathbf{Q}(-4)$	
4	0	0	$\mathbf{Q}(-3)$	
	1	2	3	p
type	(1)	(2)	(4)	

(5) *Two points outside H.* In this case the singular curve is the union of two irreducible plane sections H_1, H_2 of Q which are different from H and not tangent to each other. Each of the components passes through exactly two of the x_i. Up to reordering the points we may assume that H_1 passes through x_1 and x_2 and H_2 passes through x_3 and x_4.

It is important to observe that such a curve $H_1 \cup H_2$ is uniquely identified by the partially ordered configuration $(\{\{x_1, x_2\}, \{x_3, x_4\}\}, \{s_1, s_2\})$ of points on H, where s_1, s_2 denote the projections on H of the singular points of the curve. Furthermore, a configuration $(x_1, \ldots, x_4, s_1, s_2)$ comes from a singular curve $H_1 \cup H_2$ if and only if there is an automorphism of $H \cong \mathbf{P}^1$ that interchanges the following pairs of points: $x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_4$ and $s_1 \leftrightarrow s_2$. This condition defines a codimension 1 subset $N \subset \mathcal{M}_{0,6}$.

Thus, one can study the cohomology with compact support of the stratum F_5 by taking the part of the cohomology with compact support of N such that the symmetric group $\mathfrak{S}_4$ acts on the first two points as the representation $\mathbf{S}_4 \oplus \mathbf{S}_{2,2}$ and the symmetric group $\mathfrak{S}_2$ interchanging s_1 and s_2 acts as the alternating representation. Then a direct computation shows that the only cohomology class with this behaviour is the trivial class. From this it follows that this stratum contributes trivially to the Vassiliev's spectral sequence.

From this classification it follows that only configurations of type (1), (2) and (4) contribute to Vassiliev's spectral sequence. We give the Vassiliev's spectral sequence associated to this classification of the singularities in Table III.3. Our description of the singularities also shows that Σ has two irreducible components: namely, the divisor D_1 of curves with singularities of type (1) and the divisor D_2 of curves with singularities of type (3). By the long exact sequence associated to the inclusion $\Sigma \hookrightarrow (\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$, they give two classes δ_1, δ_2 in the first cohomology group of $\mathcal{I}$.

Consider the Leray spectral sequence associated to the quotient map $q : \mathcal{I} \to \mathcal{A}_{nh}$. The action of $\mathbf{C}^*$ on $\mathcal{I}$ can be extended to an action of $\mathbf{C}$ on $(\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$.

In this extended action, one can see that $0 \in \mathbf{C}$ maps surjectively to the locus $0 \times \mathcal{M}_{0,4}$, which is contained in the intersection of D_1 and D_2. If we denote by h a generator of $H^1(\mathbf{C}^*; \mathbf{Q})$, this shows that the image of h in the pull-back of the orbit map $\mathbf{C}^* \to (\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$ is a linear combination of the classes $\delta_1, \delta_2 \in H^1(\mathcal{I}; \mathbf{Q})$. This can be used to prove that the Leray spectral sequence in cohomology associated to q degenerates at E_2, i.e., by Poincaré duality, we have $H_c^\bullet(\mathcal{I}; \mathbf{Q}) \cong H_c^\bullet(\mathbf{C}^*; \mathbf{Q}) \otimes H_c^\bullet(\mathcal{A}_{nh}; \mathbf{Q})$ for cohomology with compact support.

Now let us go back to the spectral sequence in Table III.3. Since $(\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$ has cohomology with compact support concentrated in degree 10, the Gysin long exact sequence associated to the inclusion $\Sigma \hookrightarrow (\mathbf{C}^4 \times \mathcal{M}_{0,4})/\mathfrak{S}_4$ yields isomorphisms

$$H_c^{k-1}(\Sigma; \mathbf{Q}) \cong H_c^{k+1}(\mathcal{I}; \mathbf{Q})$$

for all $k \leq 9$. In view of the structure of $H_c^\bullet(\mathcal{I}; \mathbf{Q})$ as a tensor product, we have that the d^1-differential $E_{1,5}^1 \to E_{2,5}^1$ must be an isomorphism. All other differentials are necessarily 0 by Hodge-theoretic reasons, since they are maps between pure Hodge structures with different weights. This implies that the cohomology of $\mathcal{I}$ is isomorphic (as a graded vector space with mixed Hodge structures) to the cohomology of $\mathbf{C}^3 \times \mathbf{C}^* \times \mathbf{C}^*$ and that the cohomology of $\mathcal{A}_{nh}$ is isomorphic to that of $\mathbf{C}^3 \times \mathbf{C}^*$. $\square$

Proof of Proposition III.3.7. We want to compute the cohomology with compact support of $\mathcal{A}$ by using the Gysin long exact sequence

$$H_c^k(\mathcal{A}; \mathbf{Q}) \to H_c^k(\mathcal{A}_h; \mathbf{Q}) \xrightarrow{d_k} H_c^{k+1}(\mathcal{A}_{nh}; \mathbf{Q}) \to H_c^{k+1}(\mathcal{A}; \mathbf{Q})$$

associated to the inclusion $\mathcal{A}_h \hookrightarrow \mathcal{A}$. Since the cohomology with compact support of $\mathcal{A}_h$ (resp. $\mathcal{A}_{nh}$) is nontrivial only in degree 5 and 6 (resp. 7 and 8) the only differential which may be nontrivial is d_6. To prove the claim, we need to show that d_6 is an isomorphism or, equivalently, that the cohomology with compact support of $\mathcal{A}$ vanishes in degree 7. To prove this, we are allowed to discard in our configurations all subvarieties of $\mathcal{A}$ of codimension larger than 1, since they cannot possibly contribute to the cohomology with compact support in such a high degree.

Recall that $\mathcal{A}$ is the moduli space of pairs (C, H) where C is a smooth genus 1 curve lying on a fixed quadric cone Q and H is a reduced plane section of Q that intersects C in four distinct points. In particular, the fact that C lies on Q endows C with a natural structure as double cover of $\mathbf{P}^1$ ramified at 4 points, giving rise to a natural map $\mathcal{A} \to \mathcal{M}_{0,4}/\mathfrak{S}_4$. Note that, once a configuration $(0, \infty, 1, t) \in \mathcal{M}_{0,4}$ is chosen, there is a canonical form for the genus 1 curve in Q lying over $\{0, \infty, 1, t\}$, by taking the equation $w^2 - u_0u_1(u_0 - u_1)(tu_0 - u_1) = 0$. Therefore, it only remains to describe which reduced plane sections of Q are not tangent to a fixed smooth curve $C \subset Q$. Planes in projective three-space are parametrized by a $\mathbf{P}^3$; reduced plane sections H come from a rational curve in this $\mathbf{P}^3$, which we can discard since it has codimension > 1. The condition that H is not tangent to the fixed C defines an irreducible hypersurface in the $\mathbf{P}^3$ parametrizing plane sections. Here we only need to deal with irreducible plane sections because the locus of reducible plane sections has already codimension 1 in $\mathbf{P}^3$ and any special sublocus of it we could have to discard would not influence $H^7_c(\mathcal{A};\mathbf{Q})$.

From this description, it follows that the cohomology with compact support of $\mathcal{A}$ in degree ≥ 7 coincides with that of the $\mathfrak{S}_4$-quotient of a fibration over $\mathcal{M}_{0,4}$ having the complement of a $\mathfrak{S}_4$-invariant hypersurface in $\mathbf{P}^3$ as fibre. Then the claim follows from the fact that the cohomology with compact support of the complement of an irreducible hypersurfaces is 0 in degree 6. $\square$

The cohomology of the moduli space $\mathcal{C}'$

The space $\mathcal{C}'$ is the moduli space of genus 1 curves C with a set $\{x_1, x_2\}$ of (unordered) marked points and a line bundle L such that $L^{\otimes 4} = \mathcal{O}(x_1+x_2)^{\otimes 2}$ but $L^{\otimes 2} \neq \mathcal{O}(x_1 + x_2)$. In this section, we give a more explicit geometric description of $\mathcal{C}'$ that enables one to compute its cohomology.

First we observe that the line bundle $L^{\otimes 4}$ defines an embedding of C into $\mathbf{P}^3$; in the following, we shall identify C with its image in $\mathbf{P}^3$. Then all divisors linearly equivalent to $L^{\otimes 4}$ are cut by plane sections of C. In particular, the line bundle L itself has to be cut by a plane in $\mathbf{P}^3$ tangent to C with multiplicity 4 at one point q. This means that there is a quadric cone $Q \subset \mathbf{P}^3$ containing C, whose ruling cuts the divisor $L^{\otimes 2}$ on C. Then q is one of the ramification points of this g^1_2. Equivalently, the pair (Q, q) identifies the divisor L. The points x_1, x_2 are contact points of a bitangent plane to C. In other words, we can describe $\mathcal{C}'$ as the moduli space of triples (C, q, Π) where C is a smooth genus 1 curve lying on a fixed quadric cone Q, the point q is a ramification point of the g^1_2 cut by the ruling of Q and $\Pi \subset \mathbf{P}^3$ is a plane tangent to C at two distinct points.

Instead than working directly with $\mathcal{C}'$, we work with the moduli space $\tilde{\mathcal{C}}$ of sequences $(C, p_1, p_2, p_3, q, \Pi)$ with C and Π are as above and p_1, p_2, p_3, q are the ramification points of the g^1_2. The forgetful map $\tilde{\mathcal{C}} \to \mathcal{C}'$ can be interpreted as the quotient by the action of the symmetric group permuting the three points p_1, p_2, p_3. There is also a natural map $\varphi:\ \tilde{\mathcal{C}} \to \mathcal{M}_{0,4}$ that maps $(C, p_1, p_2, p_3, q, \Pi)$ to the moduli of the ordered branch locus of the g^1_2.

Proof of Proposition III.3.8. The structure of $\tilde{\mathcal{C}}$ as an $\mathfrak{S}_3$-cover of $\mathcal{C}'$ ensures that the rational cohomology of the latter space coincides with the $\mathfrak{S}_3$-invariant part of the cohomology of $\tilde{\mathcal{C}}$. We want to compute it by exploiting the Leray spectral sequence associated to the forgetful map $\varphi:\ \tilde{\mathcal{C}} \to \mathcal{M}_{0,4}$. To this end, we need to describe the fibers of φ and to calculate their cohomology.

The fibre of φ over a 4-tuple (p_1, p_2, p_3, q) is the space of all bitangents to the curve C obtained as a double cover of $\mathbf{P}^1$ ramified at p_1, p_2, p_3 and q. We need to parametrize all bitangent planes of the curve C that give rise to reduced plane sections. Since we are studying rational cohomology, which only depends on the coarse moduli space of the stack considered, it is enough to describe all reduced bitangents up to the action of the elliptic involution of (C, q).

Then an explicit computation yields that the space of all reduced bitangents to C has three distinct irreducible components, isomorphic to $\mathbf{P}^1$ and depending on the choice of one of the points p_i. In particular, the three components are permuted by the $\mathfrak{S}_3$-action. In the description of the components, we shall denote the line of the ruling of Q passing through q (respectively, through p_i for $1 \leq i \leq 3$) by ℓ, respectively, ℓ_i.

Then the point p_3 corresponds to the family of bitangents containing the reducible bitangent $\ell_1 \cup \ell_2$ and $\ell_3 \cup \ell$. This family contains exactly two flex bitangents, i.e. planes tangent to C at one point with multiplicity 4. The contact points on these flex bitangents are the points of C lying over the points of C lying in the fixed locus of the involution $\ell_1 \leftrightarrow \ell_2$, $\ell_3 \leftrightarrow \ell$ on the ruling of Q. In our description, we have to take only the bitangents with two distinct tangency points x_1, x_2, i.e. only proper bitangents, hence we need to discard these two points of the family. Hence each irreducible component of the fibre of φ over (p_1, p_2, p_3, q) is isomorphic to $\mathbf{C}^*$. If we consider the action of the involution $p_1 \leftrightarrow p_2$ on the cohomology of this component of the fibre, we get that it acts trivially in degree 0 and as the sign representation in degree 1.

We obtain the other two components by taking the action of the symmetric group $\mathfrak{S}_3$ into account. Then the cohomology of $\varphi^{-1}(p_1, p_2, p_3, q)$, with its structure as $\mathfrak{S}_3$-representation and its mixed Hodge structures, is given by $\mathbf{S}_3 + \mathbf{S}_{2,1}$ in degree 0 and $\mathbf{S}_{2,1} + \mathbf{S}_{1^3}$ in degree 1.

At this point, recall that $\mathcal{M}_{0,4}$ is isomorphic to $\mathbf{P}^1$ minus 3 points, and in particular, that its cohomology with the action of $\mathfrak{S}_3$ permuting the first three marked points is given by $\mathbf{S}_3$ in degree 0 and $\mathbf{S}_{2,1}$ in degree 1. The map φ is $\mathfrak{S}_3$-equivariant, hence the $\mathfrak{S}_3$-invariant part of the E_r terms of the Leray spectral sequence associated to φ converges to the cohomology of $\mathcal{C}'$. From the description the $\mathfrak{S}_3$-action on the basis and the fibre of φ, one gets that the only nontrivial E_2 terms of this Leray spectral sequence are $(E_2^{0,0})^{\mathfrak{S}_3} = \mathbf{Q}$, $(E_2^{1,0})^{\mathfrak{S}_3} = \mathbf{Q}(-1)$ and $(E_2^{1,1})^{\mathfrak{S}_3} = \mathbf{Q}(-2)$. Then the claim follows by Poincaré duality. □

III.4 The compactification of the inertia stack of $\mathcal{M}_g$

In Section III.2.2 we studied the twisted sectors of $\mathcal{M}_g$ as moduli stacks of cyclic covers. In the present section we consider the compactification of these twisted sectors inside the inertia stack of $\overline{\mathcal{M}}_g$. After developing the general theory, we study in detail the compactification of the moduli stacks of admissible covers that correspond to twisted sectors of $\mathcal{M}_3$.

Recall that in Section III.2.2 we defined the concept of g-admissible datum and saw that a g-admissible datum $(g', N, d_1, \ldots, d_{N_1})$ always singles out a connected component of the inertia stack, described as a moduli stack of μ_N-ramified covers of curves of genus g'. To compactify such moduli stacks of μ_N-covers, we rely on the general theory of *twisted stable map*, developed by Abramovich–Vistoli ([AV02]), which in our case specializes to the theory of *admissible covers*, as developed in [ACV03]. In the language of twisted stable maps, we are studying *balanced twisted stable maps* with value in the trivial gerbe $B\mu_N$. Equivalently, we are studying μ_N-admissible covers ([ACV03, Theorem 4.3.2]).

Definition III.4.1. Let A be a g-admissible datum and let us denote as usual the associated moduli stack by $\mathcal{M}_A$, the component consisting of connected covers by $\mathcal{M}'_A$ and the corresponding subgroup of the symmetric group $\mathfrak{S}_d$ by S_A. We define $\overline{\mathcal{M}}_A$ (respectively, $\overline{\mathcal{M}}'_A$) as the closure of $\mathcal{M}_A$ (respectively, $\mathcal{M}'_A$) inside $[\mathcal{K}_{g',d}(B\mu_N)/S_A]$, where $\mathcal{K}_{g',d}(B\mu_N)$ is the proper moduli stack of twisted stable d-pointed maps of genus g' to $B\mu_N$ defined in [AV02, Definition 4.3.1] (see also [ACV03, Section 2] for the specific case of $B\mu_N$).

The stacks $\overline{\mathcal{M}}_A$ give connected components of the inertia stack of $\overline{\mathcal{M}}_g$ by associating to each admissible cover in $\overline{\mathcal{M}}_A$ the pair $(C^{\mathrm{stab}}, \varphi)$ where C^{stab} is obtained by stabilizing the source curve C of the cover, and φ is the automorphism on C^{stab} induced by the action of μ_N on C. Conversely, since the moduli space of admissible covers is proper and contains the smooth ones, each smoothable cyclic cover $X \to C$ where X is a stable curve has an associated admissible cover by repeatingly blowing up the two curves X and C.

Proposition III.4.2. *Let us fix $g, N > 1$. Then the compactified inertia stack $I_N(\mathcal{M}_g)$ of Definition III.2.4 is isomorphic to the disjoint union of all nonempty stacks $\overline{\mathcal{M}}'_A$ for all g-admissible data $A = (g', m; d_1, \ldots, d_{m-1})$ with order m equal to N.*

Proof. The proof of the proposition follows by adapting the proof of Proposition III.2.12. In the case when the covering curve C' turns out to be unstable, one applies the usual stabilization procedure ([Kn83]). $\square$

In other words, in Definition III.4.1, we have described all the twisted sectors of $I(\overline{\mathcal{M}}_g)$ that do not come from the boundary.

Remark III.4.3. It is clear that $\overline{I}(\mathcal{M}_g) \neq I(\overline{\mathcal{M}}_g)$, i.e. that there are twisted sectors of $\mathcal{M}_g$ that do not contain any smooth curves. To see this, take two smooth, 1-pointed curves, one of genus $g' \geq 1$ and the other of genus $g - g'$, each admitting an automorphism of different order that fixes the marked point. Now the curve C obtained gluing the two curves at the marked points, with the automorphism induced by the two automorphisms, is a point in the inertia stack of $\overline{\mathcal{M}}_g$ that is not in $\overline{I}(\mathcal{M}_g)$.

Next, we turn our attention to the cohomology of the moduli stacks $\overline{\mathcal{M}}_A$ and, more specifically, to the compactified twisted sectors of $\mathcal{M}_g$ whose general object is a curve that is described as the cyclic cover of a genus 0 curve. For combinatorial reasons, the large majority of cases fall into this class. We can reduce the problem of computing the cohomology groups of the $\overline{\mathcal{M}}_A$ with $g' = 0$ to the problem of computing the part of the cohomology of $\overline{\mathcal{M}}_{0,d}$ under the action of the subgroup $S_A \subset \mathfrak{S}_n$, which is then known (see [Ge94, 5.8]). This relies on the construction of the $\overline{\mathcal{M}}_A$ as stack quotients of a connected substack of $\mathcal{K}_{g',d}(B\mu_N)$.

If X is a scheme, D is an effective Cartier divisor, and r is a natural number, then [Ca07] and [AGV08] introduced the stack $X_{D,r}$, called the *root of a line bundle with a section*. The following result is essential for our application:

Proposition III.4.4. *([Ca07, Corollary 2.3.7]) Let X be a scheme. If $X_{D,r}$ is obtained from X by applying the root construction, the canonical map $X_{D,r} \to X$ exhibits X as the coarse moduli space of $X_{D,r}$.*

Theorem III.4.5. *([BC07, p.2]) Let A be a g-admissible datum (see Definition III.2.7), with g' equal to 0. The space $\overline{\mathcal{M}}_A$ is then a μ_N-gerbe over the quotient stack $[X/S_A]$, where X is a stack constructed starting from $\overline{\mathcal{M}}_{0,\sum d_i}$ by successively applying the root construction (see [BC07, Section 2]).*

By combining these two results, we obtain a simple description of the cohomology of the twisted sectors of $\overline{I}(\mathcal{M}_g)$ whose general element is a cyclic cover of a genus 0 curve:

Corollary III.4.6. *If A is a g-admissible datum with base genus g' equal to 0, then the stack $\overline{\mathcal{M}}_A$ has the same rational Chow groups and rational cohomology groups as $\overline{\mathcal{M}}_{0,\sum d_i}/S_A$.*

Using the results of [Ge94, 5.8], we can now determine the rational cohomology of the positive-dimensional twisted sectors $\overline{\mathcal{M}}'_A$ whose general object covers a genus 0 curve. The Hodge–Grothendieck characters of these spaces is listed in the third column of tables III.1 and III.2. The twisted sectors are proper smooth stacks, hence their k-cohomology group carries a pure Hodge structure of weight k. For this reason, the Hodge–Grothendieck characters determine the rational cohomology as vector space with Hodge structures.

III.4.1 The compactification of the inertia stack of $\mathcal{M}_3$

There are only four twisted sectors in $\mathcal{M}_3$ whose general object covers curves of genus 1 or 2. They are the spaces that we called $\mathcal{A}, \mathcal{B}, \mathcal{C}'$ and $\mathcal{D}'$ in section III.3. The remainder of the present section is thus devoted to investigating the geometry of their compactifications, in order to compute their rational cohomology.

The general strategy here is the following. Since we deal with proper smooth stacks, their cohomology is determined uniquely by the Hodge–Grothendieck character. To compute this, we exploit the additivity of Hodge–Grothendieck characters for compact support.

As we already know the Hodge–Grothendieck characters for compact support of the open parts as a consequence of the Propositions III.3.1, III.3.7, III.3.8 and the Corollaries III.3.4 and III.3.6, we need to study the irreducible components of

$$\overline{\mathcal{A}} \setminus \mathcal{A},\ \overline{\mathcal{B}} \setminus \mathcal{B},\ \overline{\mathcal{C}'} \setminus \mathcal{C}',\ \overline{\mathcal{D}'} \setminus \mathcal{D}'.$$

Furthermore, by Poincaré duality, all we need to know are the coefficients of the Hodge–Grothendieck character with degree greater than or equal to half the complex dimension of the stack considered.

We describe in detail the case of $\mathcal{A}$ (the most complicated), and we sketch the proofs of the other cases.

Proposition III.4.7. *The Hodge–Grothendieck character of $\overline{\mathcal{A}}$ is*

$$\chi^c_{\mathrm{Hdg}}(\overline{\mathcal{A}}) = L^4 + 6L^3 + 9L^2 + 6L + 1.$$

Proof. We have seen in III.3.7 that the Hodge–Grothendieck character for compact support of $\mathcal{A}$ is $L^4 - L^2$. The moduli stack $\mathcal{A}$ admits a finite étale map onto $[\mathcal{M}_{1,4}/\mathfrak{S}_4]$. This map extends to a finite map $t : \overline{\mathcal{A}} \to [\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$ (see [ACV03, 3.0.5]) on the compactification $\overline{\mathcal{A}}$ by means of admissible covers.

The stratification of $\overline{\mathcal{M}}_{1,4}$ by topological type induces a stratification on $\overline{\mathcal{A}}$. We need to study its strata of codimension 1 and 2. The quotient stack $[\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$ has four boundary divisors, and their general element is as in Figure III.1. We denote by $D_1, \dots, D_4$ the associated locally closed codimension 1 strata, obtained by removing all curves with more than one node.

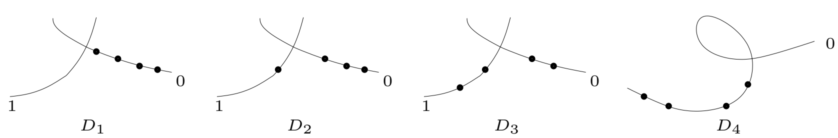

Figure III.1: The four boundary strata of codimension 1 in $[\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$

Now we describe one by one the irreducible components of $\overline{\mathcal{A}}$ that map onto the four codimension 1 boundary strata we have just pictured:

1. There are two irreducible components D_1', D_1'' lying over D_1. They parametrize admissible double covers $C \to C'$ such that the restriction to the preimage of the genus 1 component is, respectively, a trivial μ_2-cover in the case of D_1' and a nontrivial μ_2-cover in the case of D_1''. The coarse moduli space of D_1' is isomorphic to $\mathcal{M}_{1,1} \times \mathcal{M}_{0,5}/\mathfrak{S}_4$, and its Hodge–Grothendieck character for compact support is then L^3. The moduli space of D_1'' is isomorphic to $X_1(2) \times \mathcal{M}_{0,5}/\mathfrak{S}_4$ and $\chi^c_{\mathrm{Hdg}}(D_1'') = L^3 - L^2$.

2. Over D_2 there is one irreducible component D_2', whose moduli space is isomorphic to $II_1 \times \mathcal{M}_{0,4}/\mathfrak{S}_3$. Here II_1 is the moduli space of bielliptic curves with a choice of a distinguished bielliptic involution, and an ordering of the ramification points (see [Pag10a, Proposition 2.25]). As shown there, the stack II_1 has $\mathcal{M}_{0,5}/\mathfrak{S}_3$ as coarse moduli space. Hence, one has $\chi^c_{\mathrm{Hdg}}(D_2') = L^3 - L^2 + L$;

3. Over D_3 there is one component D_3'. Its moduli space is $II^1/\mathfrak{S}_2$, where II^1 is the moduli space of bielliptic curves C with a distinguished bielliptic involution α and a point p not fixed by α, and the involution on II^1 is defined by sending (C, p, α) to $(C, \alpha(p), \alpha)$. Using the construction of [Pag10a, Proposition 2.22], it is easy to show that the coarse moduli space of $II^1/\mathfrak{S}_2$ is a $\mathbf{C}^*$-bundle over $\mathcal{M}_{0,5}/\mathfrak{S}_3$. In particular, one has $\chi^c_{\mathrm{Hdg}}(D_3') = \chi^c_{\mathrm{Hdg}}(II^1/\mathfrak{S}_2) = L^3 - 2L^2 + 2L - 1$;

4. Finally, over D_4 there are two components, one whose general element is a cover unramified over the node, and the other one whose general element is totally ramified over the node. Both these moduli spaces are isomorphic to $\mathcal{M}_{0,6}/\mathfrak{S}_4 \times \mathfrak{S}_2$, and their Hodge–Grothendieck character for compact support is then $L^3 - L^2$.

The second step is to study the number of irreducible components of the preimages in $\overline{\mathcal{M}}_A$ of each of the codimension 2 strata of $[\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$. There are exactly nine codimension 2 strata $F_1, \dots, F_9$ in $[\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$. We describe them in Figure III.2 by drawing their general element.

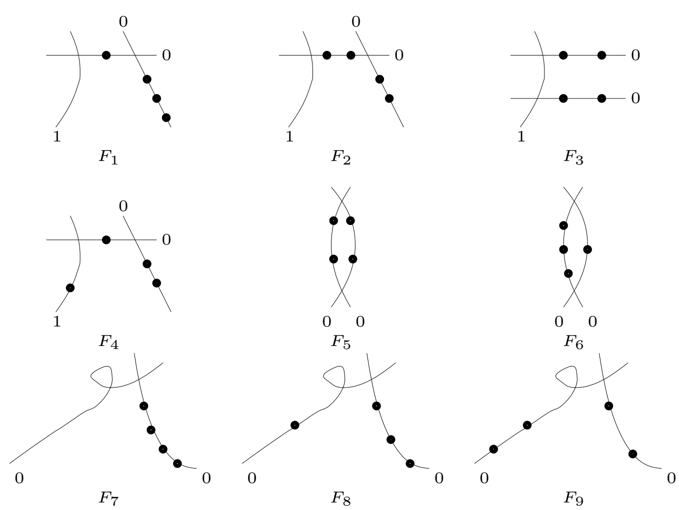

Figure III.2: The nine strata of codimension 2 in $[\overline{\mathcal{M}}_{1,4}/\mathfrak{S}_4]$

- The strata F_1, F_2 and F_3 lie in the closure of the codimension 1 stratum D_1. Again, the preimage of each of them under t has two components, corresponding respectively to trivial and nontrivial μ_2-covers of the irreducible component of genus 1;

- The stratum F_4 is contained in the closure of D_2 and its preimage under t is irreducible;
- The preimage $t^{-1}(F_5)$ has two components, corresponding to whether the two nodes are contained or not in the branch locus;
- The preimage $t^{-1}(F_6)$ is irreducible;
- The strata F_7, F_8 and F_9 lie in the closure of D_4 and the preimage of each of them has 2 irreducible components. Indeed, while the fact that the separating node is or not part of the branch locus depends on the disposition of the other branch points, the irreducible node can be either a branch point or a regular point, which gives rise to two different components, exactly as it was the case for D_4.

So, putting everything together, we have:

$$\begin{aligned}\chi^c_{\mathrm{Hdg}}(\overline{\mathcal{A}}) &= \chi^c_{\mathrm{Hdg}}(\mathcal{A}) + \chi^c_{\mathrm{Hdg}}(\text{codim } 1) + \chi^c_{\mathrm{Hdg}}(\text{codim } 2) + \dots \\ &= L^4 - L^2 + (6L^3 - 6L^2) + (16L^2) + \dots \end{aligned} \tag{III.4.1}$$

and now, using the fact that Poincaré duality holds, we get to the conclusion:

$$\chi^c_{\mathrm{Hdg}}(\overline{\mathcal{A}}) = L^4 + 6L^3 + 9L^2 + 6L + 1. \tag{III.4.2}$$

□

Remark III.4.8. In the proof of the above result we have used Poincaré duality. It is possible to reach the same conclusion by means of a more refined analysis of the boundary strata. This gives a non-trivial check on the whole result.

More precisely, the quantity $L^4 + 6L^3 + 9L^2 + 6L + 1$ is obtained as the sum of the following contributions:

$$(L^4 - L^2) + (6L^3 - 6L^2 + 3L - 1) + (16L^2 - 15L + 5) + (18L - 13) + 10,$$

where we have collected the terms according to the codimension of the corresponding stratum (from codimension 0 to codimension 4).

Proposition III.4.9. *The Hodge–Grothendieck character of $\overline{\mathcal{B}}$ is $L^2 + 3L + 1$.*

Sketch of proof. We have seen in Corollary III.3.4 that $\chi^c_{\mathrm{Hdg}}(\mathcal{B})$ equals $L^2 - L + 1$. The moduli stack $\mathcal{B}$ admits a finite étale map onto $\mathcal{M}_{1,2}$. On its compactification $\overline{\mathcal{B}}$ by means of admissible covers this map extends to a finite map $t : \overline{\mathcal{B}} \to \overline{\mathcal{M}}_{1,2}$. The boundary $\partial\mathcal{M}_{1,2}$ has two irreducible components: namely, a component

δ_1 whose general element is of compact type, and another component δ_0 whose general element is an irreducible curve of geometric genus 0 with one node. It is easy to see that the preimage of both δ_0 and δ_1 under t has two irreducible components. By the additivity of Hodge–Grothendieck characters for compact support this ensures that the coefficients of L and L^2 in the Hodge–Grothendieck character of $\overline{\mathcal{B}}$ are as in the claim. The constant coefficient is equal to the degree 2 coefficient by Poincaré duality. ☐

Proposition III.4.10. *The Hodge–Grothendieck character of $\overline{\mathcal{C}}'$ is $L^2 + 3L + 1$.*

Sketch of proof. We have seen in III.3.8 that $\chi^c_{\mathrm{Hdg}}(\mathcal{C}')$ equals $L^2 - L + 1$. The moduli stack $\mathcal{C}'$ admits a finite étale map onto $[\mathcal{M}_{1,2}/\mathfrak{S}_2]$. This map extends to a finite map $t : \overline{\mathcal{C}'} \to [\overline{\mathcal{M}}_{1,2}/\mathfrak{S}_2]$ on the compactification $\overline{\mathcal{C}'}$ by means of admissible covers. There are two irreducible components of codimension 1 in $[\partial\mathcal{M}_{1,2}/\mathfrak{S}_2]$, one whose general element is of compact type, and the other one whose general element is an irreducible curve of geometric genus 0 with one node. We call them respectively δ_1 and δ_0. Now the fiber of t over δ_1 is made of one irreducible component, while the fiber of t over δ_0 is made of three irreducible components, one for each possible way of prescribing a balanced ramification over the node. ☐

Proposition III.4.11. *The stack $\overline{\mathcal{D}'}$ has Hodge–Grothendieck character for compact support $L^3 + 4L^2 + 4L + 1$.*

One can compute the Hodge–Grothendieck character of $\overline{\mathcal{D}'}$ with the same strategy used for the Propositions III.4.7–III.4.10, but also a more geometric proof is possible:

Remark III.4.12*.* As shown by Bernstein in [Be99, Sections 2.5–2.6], the coarse moduli space of admissible étale double covers in genus g coincides with the coarse *moduli space of Prym curves*, i.e. of quasistable curves together with a line bundle which is isomorphic to $\mathcal{O}(1)$ on the exceptional components and to a square root of the trivial bundle on the nonexceptional ones. Indeed the latter coarse moduli space, usually denoted by $\overline{R}_g$, has been extensively studied. The results of Bernstein imply that $\overline{\mathcal{D}'}$ has coarse moduli space $\overline{R}_2$, and by [Kr10, Lemma 20] the latter coincides with $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_4 \times \mathfrak{S}_2$. The combination of these results implies Proposition III.4.11.

III.5 The Age Grading

III.5.1 Definition of Chen–Ruan degree

To complete our computation of the orbifold cohomology of $\mathcal{M}_3$ we still need to consider its structure as a $\mathbf{Q}$-graded vector space. To compute the new grading, the degree of each cohomology class of the inertia stack of $\mathcal{M}_3$ has to be shifted by the *age* of the twisted sector. We recall from the introduction that the motivation for the new grading is that orbifold cohomology can be endowed with a natural product, the Chen–Ruan product, which is not compatible with the "naive" grading of the cohomology of the inertia stack, but turns it into a $\mathbf{Q}$-graded algebra, once it is endowed with the new grading.

In the following, we denote by $R\mu_N$ the representation ring of μ_N, and by ζ_N a chosen generator for the group μ_N of N-roots of 1.

Definition III.5.1. [AGV08, Section 7.1] Let $\rho : \mu_N \to \mathbb{C}^*$ be a group homomorphism. It is determined by an integer $0 \leq k \leq N-1$ as $\rho(\zeta_N) = \zeta_N^k$. We define the *age* of ρ by:

$$\operatorname{age}(\rho) = k/N.$$

The age extends to a unique additive homomorphism $\operatorname{age} : R\mu_N \to \mathbb{Q}$.

Next, we define the age of a twisted sector Y. In the following definition, we let f be the restriction to the twisted sector Y of the natural map $I(X) \to X$.

Definition III.5.2. ([CR04, Section 3.2], [AGV08, Definition 7.1.1]) Let Y be a twisted sector and $g : \operatorname{Spec}\mathbb{C} \to Y$ a point. Then the pull-back via $f \circ g$ of the tangent sheaf, $(f \circ g)^*(T_X)$, is a representation of μ_N on a finite-dimensional vector space. We define:

$$a(Y) := \operatorname{age}((f \circ g)^*(T_X))$$

We are ready to define the orbifold, or Chen–Ruan, degree.

Definition III.5.3. ([CR04, Definition 3.2.2]) We define the dth degree orbifold cohomology group of X as follows:

$$H^d_{CR}(X) := \bigoplus_Y H^{d-2a(Y)}(Y; \mathbf{Q})$$

where the sum runs over all components sectors Y of $I(X)$.

Definition III.5.4. We define the *orbifold Poincaré polynomial* of X as:

$$P^{CR}(X) := \sum_{i \in [0, \dim_{\mathbb{C}}(X)] \cap \mathbb{Q}} \dim H^i_{CR}(X) t^i$$

Note that the degree of $H^\bullet_{CR}$ is given by the unconventional grading defined in Definition III.5.3.

In Definition III.2.4 we have introduced a compactification of the inertia stack. The connected components of $\overline{I}(X)$ can be assigned the age grading as in this section, simply using the fact that every connected component of $\overline{I}(X)$ contains, by its very definition, a connected component of $I(X)$. Taking cohomology, one obtains a linear subspace of the orbifold cohomology $H^*_{CR}(\overline{\mathcal{M}}_g)$. Although we shall not deal with this in the present work, it is actually possible to prove that the orbifold cohomology classes coming from $\overline{I}(X)$ form a subalgebra of the Chen–Ruan cohomology ring.

Definition III.5.5. We define the compactified orbifold cohomology $\overline{H}^{\bullet}_{CR}(X)$ as $H^{\bullet}(\overline{I}(X))$ with the grading induced from $H^{\bullet}_{CR}(X)$. The additive structure of this vector space can be recollected in the polynomial:

$$\overline{P}^{CR}(X) := \sum_{i \in [0, \dim_{\mathbb{C}}(X)] \cap \mathbb{Q}} \dim \overline{H}^{i}_{CR}(X) t^{i}$$

We call this polynomial the *compactified orbifold Poincaré polynomial.*

We observe that Poincaré duality holds for the orbifold cohomology of X if X is a proper smooth stack, and for the compactified orbifold cohomology of X with respect to the compactification $X \subset \overline{X}$, if $\overline{X}$ is smooth.

III.5.2 The orbifold Poincaré polynomials of $\mathcal{M}_3$

To compute the age of the twisted sectors, we will use the following Proposition, suggested to us by Fantechi [Fan08], which builds on [Par91, Proposition 4.1].

Proposition III.5.6. *([Fan08]) Let $g > 1$ and let Y be the twisted sector of $\mathcal{M}_g$ corresponding to the discrete datum $(g', N, d_1, \ldots, d_{N-1})$ (Definition III.2.7, Proposition III.2.12). Then its age is equal to:*

$$a(Y) = \frac{(3g'-3)(N-1)}{2} + \frac{1}{N} \sum_{i=1}^{N-1} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k,i) \right) \qquad \text{(III.5.1)}$$

where $\sigma(k,i) = 0$ if $ki + \gcd(i, N) \equiv 0 \pmod{N}$ and 1 otherwise.

Proof. A point of $Y = \mathcal{M}'_{(N, g', d_1, \ldots, d_{N-1})}$ is a μ_N-cover $C \to C'$. By standard infinitesimal deformation theory, the fiber of the tangent sheaf $T\mathcal{M}_g$ at C is $H^1(C, T_C)$. Thus the cyclic group μ_N acts on the vector space $H^1(C, T_C)$, and the latter splits in a direct sum of eigenspaces as

$$H^1(C, T_C) = \bigoplus H^1(C, T_C)^{\chi_k},$$

where $H^1(C, T_C)^{\chi_k}$ denotes the subspace where μ_N acts with weight k. According to Definition III.5.2, we have:

$$a(Y) = \sum_{k=1}^{N-1} \frac{k}{N} h^1(C, T_C)^{\chi_k}. \qquad \text{(III.5.2)}$$

Now by stability we can substitute $h^1(C, T_C)$ with $-\chi(C, T_C)$. Moreover, since π is finite and T_C is coherent, we have that all higher invariant direct images $R^i \pi_*^{\mu_N}(T_C)$ vanish (see [Gr57, Chapter V, Corollary p. 202]), and thus:

$$H^i(C, T_C)^{\chi_k} = H^i(C', (\pi_* T_C))^{\chi_k} = H^i(C', (\pi_* T_C)^{\chi_k}), \quad i = 0, 1.$$

So (III.5.2) becomes

$$a(Y) = \sum_{k=1}^{N-1} -\frac{k}{N}\chi((\pi_* T_C)^{\chi_k}). \tag{III.5.3}$$

The sheaf $\pi_*(T_C)$ is studied in [Par91, Proposition 4.1,(a)]. In particular, one has

$$\deg\left(\pi_*(T_C)^{\chi_k}\right) = \deg T_{C'} - \sum_{\sigma(k,i)=1} d_i - \deg L_k \tag{III.5.4}$$

where the degree of the line bundles L_k can be computed via [Par91, Proposition 2.1] (see also (III.2.5)):

$$\deg L_k = \sum_i \left\{\frac{ki}{N}\right\} d_i.$$

Now, if we combine (III.5.4) with Riemann-Roch to compute the Euler characteristic in (III.5.3), we obtain

$$a(Y) = \frac{1}{N}\sum_{k=1}^{N-1} k\left(3g' - 3 + \sum_{\sigma(k,i)=1} d_i + \sum_{i=1}^{N-1}\left\{\frac{ki}{N}\right\} d_i\right) \tag{III.5.5}$$

and from this last equation the statement follows. □

The results we have seen so far enable us to compute the orbifold Poincaré polynomial of $\mathcal{M}_3$ and its compactified orbifold Poincaré polynomial.

Theorem III.5.7. *The orbifold Poincaré polynomial (III.5.4) of $\mathcal{M}_3$ is:*

$$\begin{aligned} 1 &+ t + 2t^2 + t^3 + t^{\frac{10}{3}} + t^{\frac{7}{2}} + 4t^4 + 2t^{\frac{9}{2}} + 2t^{\frac{14}{3}} + t^{\frac{33}{7}} + 5t^5 + t^{\frac{46}{9}} + t^{\frac{36}{7}} \\ &+3t^{\frac{16}{3}} + t^{\frac{38}{7}} + 4t^{\frac{11}{2}} + t^{\frac{50}{9}} + t^{\frac{39}{7}} + t^{\frac{17}{3}} + t^{\frac{40}{7}} + t^{\frac{52}{9}} + t^{\frac{41}{7}} + 10t^6 + t^{\frac{43}{7}} \\ &+t^{\frac{56}{9}} + t^{\frac{44}{7}} + t^{\frac{19}{3}} + t^{\frac{45}{7}} + t^{\frac{58}{9}} + 3t^{\frac{13}{2}} + t^{\frac{46}{7}} + 2t^{\frac{20}{3}} + t^{\frac{48}{7}} + t^{\frac{62}{9}} + t^{\frac{51}{7}}. \end{aligned}$$

Proof. The ordinary Poincaré polynomial of $\mathcal{M}_3$ is calculated in [Lo91, (4.7)]. Then the result follows from the determination of the Poincaré polynomials of the moduli stacks $\mathcal{M}'_A$ carried out in Section III.3 for any 3-admissible datum A. See in particular Table III.1 and III.2, Proposition III.3.7 and III.3.8 and Corollary III.3.4 and III.3.6.

Finally, the age of the twisted sectors is computed using Proposition III.5.1. □

Remark III.5.8. In our work we have computed the Hodge structures on the twisted sectors of $\mathcal{M}_3$, which turned out to be always pure and of Tate type. Therefore, if following [CR04, Def. 3.2.4] one defines

$$H^{p,q}(H^d_{CR}(\mathcal{M}_3)) = \bigoplus_Y H^{p-a(Y),q-a(Y)}(H^{d-2a(Y)}(Y;\mathbf{Q})),$$

where the sum runs over all sectors Y of $\mathcal{M}_3$, one can endow $H^d_{CR}(\mathcal{M}_3)$ with a canonical structure of direct sum of pure Hodge structures of Tate type whose Hodge filtrations have been shifted by a rational number. Under this convention, what we actually prove is the following result:

$$\begin{aligned}
&\sum_{p,d\in\mathbf{Q}} \dim(H^{p,p}(H^d_{CR}(\mathcal{M}_3)))L^{2p}t^d = \\
&1 + L^{1/2}t + 2Lt^2 + L^{3/2}t^3 + (3L^2 + L^{5/2})t^4 \\
&\quad + (2L^{5/2} + 3L^3)t^5 + (5L^3 + 2L^{7/2} + 2L^4 + L^6)t^6 \\
&\quad + L^{7/4}t^{7/2}(1+Lt) + L^{9/4}t^{9/2}(1+Lt) + 3L^{11/4}t^{11/2} + 3L^{13/4}t^{13/2} \\
&\quad + L^{5/3}t^{10/3} + L^{7/3}t^{14/3}(2+Lt) + L^{8/3}t^{16/3}(3+Lt) + 2L^{10/3}t^{20/3} \\
&\quad + L^{33/14}t^{33/7} + L^{18/7}t^{36/7} + L^{19/7}t^{38/7} + L^{39/14}t^{39/7} + L^{20/7}t^{40/7} \\
&\quad + L^{41/14}t^{41/7} + L^{43/14}t^{43/7} + L^{22/7}t^{44/7} + L^{45/14}t^{45/7} + L^{23/7}t^{46/7} \\
&\quad + L^{24/7}t^{48/7} + L^{51/14}t^{51/7} + L^{23/9}t^{46/9} + L^{25/9}t^{50/9} \\
&\quad + L^{26/9}t^{52/9} + L^{28/9}t^{56/9} + L^{29/9}t^{58/9} + L^{31/9}t^{62/9}.
\end{aligned}$$

Theorem III.5.9. *The compactified orbifold Poincaré polynomial (see Definition III.5.5) of $\mathcal{M}_3$ is:*

$$\begin{aligned}
&1 + t + 4t^2 + 4t^3 + t^{\frac{10}{3}} + t^{\frac{7}{2}} + 16t^4 + t^{\frac{9}{2}} + 2t^{\frac{14}{3}} + t^{\frac{33}{7}} + 12t^5 \\
&\quad + t^{\frac{46}{9}} + t^{\frac{36}{7}} + 5t^{\frac{16}{3}} + t^{\frac{38}{7}} + 5t^{\frac{11}{2}} + t^{\frac{50}{9}} + t^{\frac{39}{7}} + t^{\frac{40}{7}} + t^{\frac{52}{9}} + t^{\frac{41}{7}} \\
&\quad + 31t^6 + t^{\frac{43}{7}} + t^{\frac{56}{9}} + t^{\frac{44}{7}} + t^{\frac{45}{7}} + t^{\frac{58}{9}} + 5t^{\frac{13}{2}} + t^{\frac{46}{7}} + 5t^{\frac{20}{3}} + t^{\frac{48}{7}} + t^{\frac{62}{9}} \\
&\quad + 12t^7 + t^{\frac{51}{7}} + 2t^{\frac{22}{3}} + t^{\frac{15}{2}} + 16t^8 + t^{\frac{17}{2}} + t^{\frac{26}{3}} + 4t^9 + 4t^{10} + t^{11} + t^{12}.
\end{aligned}$$

Remark III.5.10. From the description of the cohomology of the compactified twisted sectors it also follows that the whole compactified orbifold cohomology of $\mathcal{M}_3$ is additively generated by algebraic classes.

Proof. The ordinary Poincaré polynomial of $\overline{\mathcal{M}}_3$ was first computed in [Ge98, Prop. 16]. The proof of this theorem follows again as a recollection of the results obtained in Section III.4, in particular the Propositions III.4.7, III.4.9, III.4.10, III.4.11 for the Poincaré polynomials of $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$, $\overline{\mathcal{C}}'$, $\overline{\mathcal{D}}'$ respectively. For the remaining twisted sectors, the Poincaré polynomial is computed applying Corollary III.4.6 and the results are summarized in Table III.1 and III.2.

Finally, for the degree shifting numbers, one can compute directly the age of the twisted sectors using Proposition III.5.1. $\square$

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Chapter IV

Cohomology of the second Voronoi compactification of $\mathcal{A}_4$

IV.1 Introduction and plan

The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of genus g are much studied objects in algebraic geometry. Although much progress has been made in understanding the geometry of these spaces, we still know relatively little about the cohomology or the Chow groups of $\mathcal{A}_g$ and its compactifications. These are difficult questions even for low genus. Mumford in his seminal paper [Mu2] computed the Chow ring of $\overline{\mathcal{M}}_2$, or what is the same, of the second Voronoi compactification $\mathcal{A}_2^{\mathrm{Vor}}$. It was also in this paper that he laid the foundations for the study of the Chow ring of $\overline{\mathcal{M}}_g$ in general. Lee and Weintraub [LW] have investigated the cohomology of certain level covers of $\mathcal{A}_2^{\mathrm{Vor}}$. The cohomology of $\mathcal{A}_3$ and the Satake compactification $\mathcal{A}_3^{\mathrm{Sat}}$ was determined by Hain [Ha], while the Chow group of the second Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}$ had earlier been computed by van der Geer [vdG]. The authors of this paper proved in [HT] that the Chow ring and the cohomology ring of $\mathcal{A}_g^{\mathrm{Vor}}$ are isomorphic for $g = 2, 3$.

Very little is known about the topology of $\mathcal{A}_g$ and its compactifications in general. A positive exception is given by stable cohomology, which is defined in terms of the natural maps $\mathcal{A}_{g'} \hookrightarrow \mathcal{A}_g$ for $g' < g$ given by multiplication with a fixed abelian variety of dimension $g - g'$. The stable cohomology of $\mathcal{A}_g$ is known: it coincides with the stable group cohomology of $\mathrm{Sp}(2g, \mathbf{Z})$ and is generated by the odd Hodge classes λ_{2i+1} by a classical result by Borel [Bo]. Also the stable cohomology of the Satake compactification is known ([CL]), whereas the corresponding result for toroidal compactifications of $\mathcal{A}_g$ is posed as an open problem in [Gr].

In this paper we investigate the case of genus 4 of whose cohomology very little is known. More precisely, we investigate the cohomology of toroidal com-

Joint with Klaus Hulek (Leibniz Universität Hannover) pactifications of $\mathcal{A}_4$. In general there are several meaningful compactifications of $\mathcal{A}_g$. Besides the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ there is the perfect compactification $\mathcal{A}_g^{\mathrm{perf}}$, given by the second Voronoi decomposition and the perfect cone (or first Voronoi) decomposition respectively, as well as the Igusa compactification $\mathcal{A}_g^{\mathrm{Igu}}$. It was shown by Alexeev [Al] and Olsson [Ol] that (at least up to normalization) $\mathcal{A}_g^{\mathrm{Vor}}$ represents a geometric functor given by stable semi-abelic varieties. On the other hand $\mathcal{A}_g^{\mathrm{perf}}$ is, as was proved by Shepherd-Barron [S-B], a canonical model in the sense of Mori theory, i.e. its canonical bundle is ample, if $g \geq 12$. Finally, Igusa constructed $\mathcal{A}_g^{\mathrm{Igu}}$ as a partial blow-up of $\mathcal{A}_g^{\mathrm{Sat}}$ and it was shown by Namikawa [Nam] that Igusa's model is the toroidal compactification defined by the central cone decomposition. In genus $g \leq 3$ all of the above toroidal compactifications coincide. In genus 4 the Igusa and the perfect cone decomposition coincide and the second Voronoi compactification $\mathcal{A}_4^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_4^{\mathrm{perf}}$. However, for general g all three compactifications are different.

The main result of our paper is the determination of the Betti numbers of $\mathcal{A}_4^{\mathrm{perf}}$ of degree less than or equal to 9 and of all Betti numbers of $\mathcal{A}_4^{\mathrm{Vor}}$ with the exception of the middle Betti number b_{10}.

The starting point of our investigations is the fact that every toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ admits a map $\varphi_g\colon \mathcal{A}_g^{\mathrm{tor}} \longrightarrow \mathcal{A}_g^{\mathrm{Sat}}$. We recall that

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0,$$

which allows us to construct a stratification of $\mathcal{A}_g^{\mathrm{tor}}$ by considering the closed loci $\beta_i^{(g)} = \beta_i = \varphi_g^{-1}(\mathcal{A}_{g-i}^{\mathrm{Sat}})$ and their open parts $\beta_i^0 = \beta_i \setminus \beta_{i+1} = \varphi_g^{-1}(\mathcal{A}_{g-i})$. Each stratum β_i^0 is itself the disjoint union of locally closed substrata that are quotients of torus bundles over the product of a certain number of copies of the universal family $\mathcal{X}_{g-i}$ over $\mathcal{A}_{g-i}$ by finite groups. The strategy is then to compute the cohomology with compact support of each of these substrata using Leray spectral sequences and then to glue these strata by Gysin spectral sequences to compute the cohomology with compact support of β_i^0.

The use of Leray spectral sequences requires to know the cohomology with compact support of $\mathcal{A}_{g-i}$ not only with constant coefficients, but also with coefficients in certain symplectic local systems of low weight. In the case of $i = 1, 2$ we deduce this information from results on the cohomology of moduli spaces of pointed curves. Passing from the moduli space of curves to the moduli space of abelian varieties produces a small ambiguity, which does not influence our final result, mainly because it disappears at the level of Euler characteristics. Up to this ambiguity, we are able to obtain complete results for the cohomology with compact support of all strata contained in the boundary as well as of the closure $\overline{\mathcal{J}}_4$ of the Jacobian locus in $\mathcal{A}_4$ and we believe that this is of some independent interest.

Unfortunately, we do not know the cohomology of $\mathcal{A}_4$ itself. However, there are two facts which help us. The first is that the complement in $\mathcal{A}_4$ of the closure of the locus of jacobians has a smooth affine variety as coarse moduli space. This implies that its cohomology with compact support is trivial if the degree is smaller than 9 and thus that the cohomology with compact support of $\mathcal{A}_4$ agrees with that of $\overline{\mathcal{J}}_4$ in degree ≤ 9. The second is that $\mathcal{A}_4^{\mathrm{Vor}}$ is (globally) the quotient of a smooth projective scheme by a finite group. This implies that its cohomology satisfies Poincaré duality, and, more specifically, that its cohomology in degree k carries a pure Hodge structure of weight k.

For $\mathcal{A}_4^{\mathrm{Vor}}$, putting the cohomological information from all strata β_i^0 together yields Table IV.1, from which we can deduce Theorem IV.2.1 by using the Gysin spectral sequence associated to the stratification given by the β_i. We finally obtain the Betti numbers for $\mathcal{A}_4^{\mathrm{perf}}$ in Theorem IV.2.2 by using the fact that $\mathcal{A}_4^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_4^{\mathrm{perf}}$ in one point.

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Notation

$\mathcal{A}_g$	moduli stack of principally polarized abelian varieties of genus g
$\mathcal{X}_g$	universal family over $\mathcal{A}_g$
$\mathbf{V}_{\lambda_1,\dots,\lambda_g}$	rational local system on $\mathcal{A}_g$ induced by the $\mathrm{Sp}(2g,\mathbf{Q})$-representation indexed by the partition $(\lambda_1,\dots,\lambda_g)$
$\mathcal{A}_g^{\mathrm{Sat}}$	Satake compactification of $\mathcal{A}_g$
$\mathcal{A}_g^{\mathrm{Vor}}$	Voronoi compactification of $\mathcal{A}_g$
$\mathcal{X}_g^{\mathrm{Vor}}$	universal family over $\mathcal{A}_g^{\mathrm{Vor}}$
$\mathcal{A}_g^{\mathrm{perf}}$	perfect cone compactification of $\mathcal{A}_g$
$\mathcal{A}_g^{\mathrm{Igu}}$	Igusa compactification of $\mathcal{A}_g$
$\mathcal{M}_{g,n}$	moduli stack of non-singular curves of genus g with n marked points
$\mathcal{M}_g := \mathcal{M}_{g,0}$	
$\mathfrak{S}_d$	symmetric group in d letters
$\mathrm{Sym}^2_{\geq 0}(\mathbf{R}^g)$	space of real positive semidefinite quadratic forms in $\mathbf{R}^g$
$\langle\varphi_1,\dots,\varphi_r\rangle$	convex cone generated by the half rays $\mathbf{R}_{\geq 0}\varphi_1$, …, $\mathbf{R}_{\geq 0}\varphi_r$

For every g, we denote by $\varphi_g\colon \mathcal{A}_g^{\mathrm{Vor}} \to \mathcal{A}_g^{\mathrm{Sat}}$ (respectively, $\psi_g\colon \mathcal{A}_g^{\mathrm{perf}} \to \mathcal{A}_g^{\mathrm{Sat}}$) the natural map from the Voronoi (respectively, perfect cone) to the Satake compactification. Let $\pi_g\colon \mathcal{X}_g^{\mathrm{Vor}} \to \mathcal{A}_g^{\mathrm{Vor}}$ be the universal family, $q_g\colon \mathcal{X}_g^{\mathrm{Vor}} \to \mathcal{X}_g^{\mathrm{Vor}}/\pm 1$ the quotient map from the universal family to the universal Kummer family and $k_g\colon \mathcal{X}_g^{\mathrm{Vor}}/\pm 1 \to \mathcal{A}_g^{\mathrm{Vor}}$ the universal Kummer morphism.

For $0 \leq i \leq g$, we set $\beta_i^0 = \varphi_g^{-1}(\mathcal{A}_{g-i}) \subset \mathcal{A}_g^{\mathrm{Vor}}$, $\beta_i = \varphi_g^{-1}(\mathcal{A}_{g-i}^{\mathrm{Sat}}) \subset \mathcal{A}_g^{\mathrm{Vor}}$ and $\beta_i^{\mathrm{perf}} = \psi_g^{-1}(\mathcal{A}_{g-i}^{\mathrm{Sat}}) \subset \mathcal{A}_g^{\mathrm{perf}}$.

We denote the Torelli map in genus g by $\tau_g\colon \mathcal{M}_g \to \mathcal{A}_g$ its image, the Jacobian locus, by $\mathcal{J}_g = \tau_g(\mathcal{M}_g)$ and closure of the image in $\mathcal{A}_g$ by $\overline{\mathcal{J}_g}$.

Throughout the paper, we work over the field $\mathbf{C}$ of complex numbers. All cohomology groups we consider will have rational coefficients. Since the rational cohomology of a Deligne–Mumford stack coincides with the rational cohomology of its coarse moduli space, we will sometimes abuse notation and denote stack and coarse moduli space with the same symbol.

In this paper, we make extensive use of mixed Hodge structures, focussing mainly on their weight filtration. We will denote by $\mathbf{Q}(-k)$ the Hodge structure of Tate of weight $2k$. For two mixed Hodge structures A, B we will denote by $A \oplus B$ their direct sum and by $A + B$ any extension

$$0 \to B \to E \to A \to 0.$$

Furthermore, we will denote Tate twists of mixed Hodge structures by $A(-k) = A \otimes \mathbf{Q}(-k)$.

IV.2 Main theorem

Theorem IV.2.1. *The cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ vanishes in odd degree and is algebraic in all even degrees, with the only possible exception of degree* 10. *The Betti numbers are given by*

i	0	2	4	6	8	10	12	14	16	18	20
b_i	1	3	5	11	17	*(not known)*	17	11	5	3	1

Theorem IV.2.2. *The Betti numbers of $\mathcal{A}_4^{\mathrm{perf}}$ in degree ≤ 9 are given by*

i	0	1	2	3	4	5	6	7	8	9
b_i	1	0	2	0	3	0	8	0	13	0

Moreover, all cohomology classes of degree ≤ 9 are algebraic.

The only missing information needed to compute all Betti numbers of $\mathcal{A}_4^{\mathrm{Vor}}$ is the Euler number. As we shall see, we are able to compute the Euler numbers of all strata β_i^0 for $i \geq 1$, and thus it would suffice to compute the Euler number of the space $\mathcal{A}_4$ itself. Indeed, one can compute the Euler number of level covers

$\mathcal{A}_4(n)$ for $n \geq 3$ by Hirzebruch–Mumford proportionality. From this one could compute $e(\mathcal{A}_4)$ if one had a complete classification of torsion elements in the group $\mathrm{Sp}(8,\mathbf{Z})$. Although this is not known, it does not seem an impossible task to obtain such a classification. We will, however, not approach this problem in this paper.

Proof of Theorem IV.2.1. To compute the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$, we study the Gysin spectral sequence $E_r^{p,q} \Rightarrow H^{p+q}(\mathcal{A}_4^{\mathrm{Vor}};\mathbf{Q})$ associated with the filtration $\{T_i\}_{i=1,\dots,6}$ such that

- $T_i = \beta_{5-i}$, $i = 1,\dots,4$;
- $T_5 = \overline{\mathcal{J}_4} \cup T_4$;
- $T_6 = \mathcal{A}_4^{\mathrm{Vor}}$.

The E_1 term of this spectral sequence has the form $E_1^{p,q} = H_c^{p+q}(T_p \setminus T_{p-1};\mathbf{Q})$. For $p = 1,\dots,4$ the strata $T_p \setminus T_{p-1}$ coincide with the strata of $\mathcal{A}_4^{\mathrm{Vor}}$ of semi-abelic varieties of torus rank $5-p$; their cohomology with compact support is computed in the next sections by combining combinatorial information on the toroidal compactification with the geometry of fibrations on moduli spaces of abelian varieties (see Propositions IV.4.1, IV.5.1, IV.6.11 and Theorem IV.7.2). The stratum $T_5 \setminus T_4$ is the closure inside $\mathcal{A}_4$ of the locus of jacobians. Its cohomology with compact support is computed in Lemma IV.3.1.

The only remaining stratum is the open stratum $T_6 \setminus T_5$. Let $\mathcal{J}_4^{\mathrm{Sat}}$ be the closure of $\mathcal{J}_4$ in $\mathcal{A}_4^{\mathrm{Sat}}$. Since this contains the entire boundary of $\mathcal{A}_4^{\mathrm{Sat}}$ it follows that

$$T_6 \setminus T_5 = \mathcal{A}_4 \setminus \overline{\mathcal{J}}_4 = \mathcal{A}_4^{\mathrm{Sat}} \setminus \mathcal{J}_4^{\mathrm{Sat}}.$$

The latter set is affine since it is the complement of an ample hypersurface on $\mathcal{A}_4^{\mathrm{Sat}}$ (see [HaHu]). In particular, its cohomology with compact support can be non-trivial only if the degree lies between 10 and 20.

From this it follows that the E_1 term of the Gysin spectral sequence associated with the filtration $\{T_i\}$ is as given in Table IV.1. For the sake of simplicity, in that table we have denoted $H_c^3(\mathcal{A}_2;\mathbf{V}_{2,2})$ and $H_c^4(\mathcal{A}_2;\mathbf{V}_{2,2})$ with the same symbol H, even though a priori they are only isomorphic after passing to the associated graded piece with respect to the weight filtration.

Since the terms in the sixth column are only known for $q \leq 3$, in the following we will only deal with the terms of the spectral sequence that are independent of them, that is, the $E_r^{p,q}$ terms with $p+q \leq 8$.

Let us recall that $\mathcal{A}_4^{\mathrm{Vor}}$ is a smooth Deligne–Mumford stack which is globally the quotient of a smooth proper variety by a finite group. From this it follows that the cohomology groups of $\mathcal{A}_4^{\mathrm{Vor}}$ carry pure Hodge structures of weight equal to the degree. Therefore, the Hodge structures on $E_\infty^{p,q}$ have to be pure of weight $p+q$. This means that for all p,q, the graded pieces of $E_1^{p,q}$ of weight different

Table IV.1: E_1 term of the Gysin spectral sequence associated with the filtration T_i.

q						
17	$\mathbf{Q}(-9)$	0	0	0	0	0
16	0	0	0	0	0	0
15	$\mathbf{Q}(-8)$	0	0	0	0	0
14	0	0	0	$\mathbf{Q}(-9)$	0	$\mathbf{Q}(-10)$
13	$\mathbf{Q}(-7)^{\oplus 2}$	0	$\mathbf{Q}(-8)$	0	$\mathbf{Q}(-9)$	?
12	0	$\mathbf{Q}(-7)$	0	$\mathbf{Q}(-8)^{\oplus 2}$	0	?
11	$\mathbf{Q}(-6)^{\oplus 4}$	0	$\mathbf{Q}(-7)^{\oplus 3}$	0	$\mathbf{Q}(-8)$	?
10	0	$\mathbf{Q}(-6)^{\oplus 3}$	0	$\mathbf{Q}(-7)^{\oplus 3}$	0	?
9	$\mathbf{Q}(-5)^{\oplus 6}$	0	$\mathbf{Q}(-6)^{\oplus 5}$	$\mathbf{Q}(-6)^{\oplus \epsilon}$	$\mathbf{Q}(-7)^{\oplus 2}$	?
8	0	$\mathbf{Q}(-5)^{\oplus 4}$	$\mathbf{Q}(-3)$	$\mathbf{Q}(-6)^{\oplus(4+\epsilon)} + \mathbf{Q}(-3)$	0	?
7	$\mathbf{Q}(-4)^{\oplus 7}$	0	$\mathbf{Q}(-5)^{\oplus 5} + H(-1)$	$\mathbf{Q}(-5)^{\oplus \epsilon}$	$\mathbf{Q}(-6)$	?
6	0	$\mathbf{Q}(-4)^{\oplus 4}$	$\mathbf{Q}(-2) + H(-1)$	$\mathbf{Q}(-5)^{\oplus(3+\epsilon)} + \mathbf{Q}(-2)$	0	?
5	$\mathbf{Q}(-3)^{\oplus 6} + \mathbf{Q}(-1)$	$\mathbf{Q}(-1)$	$\mathbf{Q}(-4)^{\oplus 3} + \mathbf{Q}(-2) + H$	$\mathbf{Q}(-2)$	$\mathbf{Q}(-5)$	?
4	0	$\mathbf{Q}(-3)^{\oplus 2}$	$\mathbf{Q}(-1) + H$	$\mathbf{Q}(-4)^{\oplus 2} + \mathbf{Q}(-1)$	0	?
3	$\mathbf{Q}(-2)^{\oplus 3}$	$\mathbf{Q}$	$\mathbf{Q}(-3)^{\oplus 2}$	$\mathbf{Q}(-1)$	$\mathbf{Q}(-4) + \mathbf{Q}(-1)$	0
2	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3) + \mathbf{Q}$	0	0
1	$\mathbf{Q}(-1)^{\oplus 2}$	0	$\mathbf{Q}(-2)$	0	0	0
0	0	$\mathbf{Q}(-1)$	0	0	0	0
-1	$\mathbf{Q}$	0	0	0	0	0
	1	2	3	4	5	6 p

$H = H^3_c(\mathcal{A}_2; \mathbf{V}_{2,2}) \cong H^4_c(\mathcal{A}_2; \mathbf{V}_{2,2})$ (up to grading), $\epsilon := \operatorname{rank} H^9_c(\mathcal{A}_3; \mathbf{V}_{1,1,0})$.

Table IV.2: $E_\infty^{p,q}$ in the range $p+q \leq 9$.

q						
8	0					
7	$\mathbf{Q}(-4)^{\oplus 7}$	0				
6	0	$\mathbf{Q}(-4)^{\oplus 4}$	0			
5	$\mathbf{Q}(-3)^{\oplus 6}$	0	$\mathbf{Q}(-4)^{\oplus 3}$	0		
4	0	$\mathbf{Q}(-3)^{\oplus 2}$	0	$\mathbf{Q}(-4)^{\oplus 2}$	0	
3	$\mathbf{Q}(-2)^{\oplus 3}$	0	$\mathbf{Q}(-3)^{\oplus 2}$	0	$\mathbf{Q}(-4)$	0
2	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$	0	0
1	$\mathbf{Q}(-1)^{\oplus 2}$	0	$\mathbf{Q}(-2)$	0	0	0
0	0	$\mathbf{Q}(-1)$	0	0	0	0
-1	$\mathbf{Q}$	0	0	0	0	0
	1	2	3	4	5	6 p

from $p+q$ are killed by differentials. In particular, if we restrict to the range $p+q \leq 9$, this gives that the E_∞ terms are as given in Table IV.2. Of course, this does not describe precisely at which E_r the spectral sequence degenerates, or what exactly is the rank of the differentials. For instance, if one assumes $H=0$, a natural thing to expect is that the d_1-differentials $E_1^{1,5} \to E_1^{2,5}$, $E_1^{3,5} \to E_1^{4,5}$, $E_1^{3,4} \to E_1^{4,4}$ and $E_1^{4,3} \to E_1^{5,3}$, as well as the d_2-differential $E_2^{2,3} \to E_2^{4,2}$ have rank 1, but this is not the only possibility. The claim on the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ in degree ≤ 9 follows from the E_∞ term in Table IV.2. The claim on the cohomology in degree ≥ 11 follows by Poincaré duality.

□

Proof of Theorem IV.2.2. The proof is analogous to that of Theorem IV.2.1. Rather than working with the filtration $\{T_i\}$, we will consider the stratification $\{T_i^{\mathrm{perf}}\}$ defined analogously by $T_i^{\mathrm{perf}} = \beta_{5-i}^{\mathrm{perf}}$ for $1 \leq i \leq 4$ and $T_5^{\mathrm{perf}} = \overline{\mathcal{J}}_4 \cup T_4^{\mathrm{perf}}$, $T_6^{\mathrm{perf}} = \mathcal{A}_4^{\mathrm{perf}}$. The closed stratum T_1 is the the locus β_4^{perf} of torus rank 4 inside $\mathcal{A}_4^{\mathrm{perf}}$. Hence $E_1^{1,q} = H^{q+1}(\beta_4^{\mathrm{perf}}; \mathbf{Q})$ can be obtained from Theorem IV.7.1.

Since the exceptional divisor of the blow-up map $q\colon \mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{perf}}$ is contained in T_1, we have $(\mathcal{A}_4^{\mathrm{perf}} \setminus q(T_1)) \cong (\mathcal{A}_4^{\mathrm{Vor}} \setminus T_1)$. In particular, the Gysin spectral sequence associated with the stratification of $\mathcal{A}_4^{\mathrm{perf}}$ has $E_1^{p,q}$ terms that coincide with those of Table IV.1 for $p \geq 2$. Moreover, also the rank of all differentials $E_r^{p,q} \to E_r^{p+r,q-r+1}$ coincide with those for the filtration $\{T_i\}$ as long as no $E_r^{1,q}$-terms are involved. This already implies the claim for all degrees different from 6. In degree 6, it is necessary to decide whether the class of Hodge weight 2 in $E_1^{5,1}$ is killed by differentials of the spectral sequence or not. If we consider the map $\mathcal{A}_4^{\mathrm{Vor}} \supset \beta_4 \to \beta_4^{\mathrm{perf}} \subset \mathcal{A}_4^{\mathrm{perf}}$, we have that the weight 2 class on β_4^{perf} lies in the image of the weight 2 class in the cohomology of β_4, which was killed by differentials for purity reasons on $\mathcal{A}_4^{\mathrm{Vor}}$. This implies that this must be the case also on $\mathcal{A}_4^{\mathrm{perf}}$. From this the claim follows. □

Remark IV.2.3. Comparing Table IV.1 with the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ of degree ≥ 11 suggests that the cohomology of the open stratum $\mathcal{A}_4 \setminus \overline{\mathcal{J}}_4$ could vanish in all positive degrees, with the exception of degree 10 on which Poincaré duality yields no information. Note that the vanishing of the cohomology of $\mathcal{A}_4 \setminus \overline{\mathcal{J}}_4$ would imply that the cohomology of $\mathcal{A}_4$ coincides with the stable cohomology (i.e. the subring generated by the chern classes λ_1, λ_3 of the Euler bundle) in degree ≤ 10, while a priori this is known only in degree $\leq g-2=2$.

To use Table IV.1 to prove the vanishing of $H^k(\mathcal{A}_4\setminus\overline{\mathcal{J}}_4;\mathbf{Q})$, one needs to prove that all (algebraic) classes of weight $10-k$ that occur in $E_1^{p,q}$ with $p+q=10-k$ give rise to cohomology classes in $H_c^{10-k}(\mathcal{A}_4^{\mathrm{Vor}};\mathbf{Q})$ that are linearly independent. This is known for divisors (ensuring the vanishing of $H^k(\mathcal{A}_4\setminus\overline{\mathcal{J}}_4;\mathbf{Q})$ for $k=1,2$). It would be interesting to investigate it for classes of higher codimension.

Note that, if one knew that $H^k(\mathcal{A}_4 \setminus \overline{\mathcal{J}}_4;\mathbf{Q})$ vanishes for all $1\leq k\leq 9$, then this would yields the following result for the Betti numbers of $\mathcal{A}_4^{\mathrm{perf}}$ in degree ≥ 11:

i	12	14	16	18	20
b_i	14	9	4	2	1

as well as the vanishing of all odd Betti numbers of $\mathcal{A}_4^{\mathrm{perf}}$.

IV.3 Torus rank 0

We start by considering $T_5 \setminus T_4$, which is the Zariski closure $\overline{\mathcal{J}}_4$ of the locus of jacobians $\mathcal{J}_4=\tau_4(\mathcal{M}_4)$ inside $\mathcal{A}_4$.

Lemma IV.3.1. *The only non-zero Betti numbers with compact support of $\overline{\mathcal{J}}_4$ are as follows:*

i	18	16	14	12	10	8
b_i	1	1	2	1	1	2

In particular, all odd Betti numbers vanish.

Furthermore, all cohomology groups with compact support are generated by algebraic classes, with the only exception of $H_c^8(\overline{\mathcal{J}}_4;\mathbf{Q})$, which is an extension of $\mathbf{Q}(-4)$ by $\mathbf{Q}(-1)$.

Proof. We compute the cohomology with compact support of $\overline{\mathcal{J}}_4$ by recalling that the Zariski closure of the locus of jacobians in $\mathcal{A}_4$ is the union of the image of the Torelli map and the locus of abelian fourfolds that are products of abelian varieties of dimension ≤ 3. This allows to cover $\overline{\mathcal{J}_4}$ by the following locally closed disjoint strata:

$$S_1 = \mathrm{Sym}^4\mathcal{A}_1,\ S_2=\tau_2(\mathcal{M}_2)\times\mathrm{Sym}^2\mathcal{A}_1,\ S_3=\mathrm{Sym}^2\tau(\mathcal{M}_2),$$
$$S_4=\tau_3(\mathcal{M}_3)\times\mathcal{A}_1,\ S_5=\tau_4(\mathcal{M}_4).$$

Table IV.3: E_1 term of the Gysin spectral sequence converging to the cohomology with compact support of the closure of the locus of jacobians in $\mathcal{A}_4$

q					
13	0	0	0	0	$\mathbf{Q}(-9)$
12	0	0	0	0	0
11	0	0	0	0	$\mathbf{Q}(-8)$
10	0	0	0	$\mathbf{Q}(-7)$	0
9	0	0	$\mathbf{Q}(-6)$	0	$\mathbf{Q}(-7)$
8	0	$\mathbf{Q}(-5)$	0	$\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$
7	$\mathbf{Q}(-4)$	0	0	0	0
6	0	0	0	0	0
5	0	0	0	0	0
4	0	0	0	$\mathbf{Q}(-1)$	0
	1	2	3	4	5 p

Furthermore, the Torelli map in all genera induces an isomorphism in cohomology with rational coefficients between $\mathcal{M}_g$ and its image $\tau_g(\mathcal{M}_g)$. This allows to compute the cohomology with compact support of all strata from previously known results on the cohomology of $\mathcal{M}_g$ with $g \leq 4$ ([Mu2],[Lo],[T1]). These yield that the E_1 term $E_1^{p,q} = H_c^{p+q}(S_p;\mathbf{Q})$ of the Gysin exact sequence of the filtration associated with the stratification S_j is as in Table IV.3.

In view of Table IV.3, to calculate the cohomology with compact support of $\overline{\mathcal{J}}_4$ it is sufficient to know the rank of the differential

$$d:\ H_c^{12}(\overline{\mathcal{J}}_4^{\mathrm{red}};\mathbf{Q}) \cong \mathbf{Q}(-6)^{\oplus 2} \longrightarrow H_c^{13}(\mathcal{J}_4;\mathbf{Q}) \cong \mathbf{Q}(-6)$$

in the Gysin long exact sequence associated with the closed inclusion of the locus $\overline{\mathcal{J}}_4^{\mathrm{red}} = \overline{\mathcal{J}}_4 \setminus \mathcal{J}_4 = \overline{S}_3 \cup \overline{S}_4 \subset \mathcal{A}_4$ of reducible abelian fourfolds in the Zariski closure in $\mathcal{A}_4$ of the locus of jacobians $\mathcal{J}_4 = \tau_4(\mathcal{M}_4)$.

We observe that $H_c^{12}(\overline{\mathcal{J}}_4^{\mathrm{red}};\mathbf{Q})$ is generated by two 6-dimensional algebraic cycles C_1 and C_2, where C_1 is the fundamental class of $\overline{S_3}$ and C_2 the fundamental class of $\overline{\tau(\mathcal{H}_3)\times\mathcal{A}_1}$, where $\mathcal{H}_3$ is the hyperelliptic locus. Therefore, the surjectivity of d is equivalent to the existence of a relation between C_1 and C_2 viewed as elements of the Chow group of $\overline{\mathcal{J}}_4$.

Let us denote by $\mathcal{M}_4^{\mathrm{ct}}$ the moduli space of stable genus 4 curves of compact type, i.e. such that that their generalized Jacobian is compact. Then the Torelli map extends to a proper morphism

$$\tau^{\mathrm{ct}}\colon\ \mathcal{M}_4^{\mathrm{ct}} \longrightarrow \overline{\mathcal{J}}_4.$$

From the geometric description of the map τ^{ct} it follows that the image under τ^{ct} of the Chow group of dimension 6 cycles supported on the boundary $\mathcal{M}_4^{\mathrm{ct}} \setminus \mathcal{M}_4$

coincides with $\langle C_1, C_2\rangle$. Indeed, let D_1 be the closure of the locus of stable curves consisting of two genus 2 curves intersecting in a Weierstrass point and let D_2 be the closure of the locus of stable curves consisting of elliptic curves intersecting a hyperelliptic genus 3 curve in a Weierstrass point. Then D_1 and D_2 map to C_1 and C_2 respectively. It is known that the dimension 6 classes in $\mathcal{M}_4^{\mathrm{ct}}$ fulfill a relation, given by the restriction of the relation on $\overline{\mathcal{M}}_4$ of [Y, Prop. 2]. When pushed forward via τ^{ct}, this relation gives a non-trivial relation between C_1 and C_2. Thus the differential d has to be surjective and the claim follows. □

IV.4 Torus rank 1

Next, we deal with the locus β_0^1 of semi-abelic varieties of torus rank 1.

Proposition IV.4.1. *The rational cohomology with compact support of β_1^0 is as follows: the non-zero Betti numbers are*

i	6	7	8	9	10	11	12	13	14	16	18
b_i	2	1	3	1	$4+\epsilon$	ϵ	$5+\epsilon$	ϵ	3	2	1

where $\epsilon = \operatorname{rank} H_c^9(\mathcal{A}_3; \mathbf{V}_{1,1,0})$. The cohomology groups of even degree $2k$ are algebraic for $k \geq 7$; for $k \leq 6$ they are extensions of pure Hodge structures of the form $H_c^{2k}(\beta_2^0; \mathbf{Q}) = \mathbf{Q}(-k)^{\oplus(b_{2k}-1)} + \mathbf{Q}(k-3)$. The Hodge structures in odd degree are given by $H_c^{2k+1}(\beta_2^0; \mathbf{Q}) = \mathbf{Q}(2-k)$ for $k = 7, 9$ and $H_c^{2k+1}(\beta_2^0; \mathbf{Q}) = \mathbf{Q}(-k)^{\oplus\epsilon}$ for $k = 11, 13$.

Proof. To compute the cohomology with compact support of β_1^0 we will use the map $k_3\colon \beta_1^0 \to \mathcal{A}_3$ realizing β_1^0 as the universal Kummer variety over $\mathcal{A}_3$. The fibre of β_1^0 over a point parametrizing an abelian surface S is $K := S/\pm 1$.

Note that the cohomology of K vanishes in odd degree because of the Kummer involution. The cohomology of K is one-dimensional in degree 0 and 6 and induces trivial local systems on $\mathcal{A}_3$. The cohomology group $H^2(K; \mathbf{Q}) \cong \bigwedge^2 H^1(S; \mathbf{Q})$ is 15-dimensional and induces the local system $\mathbf{V}_{1,1,0} \oplus \mathbf{Q}(-1)$ on $\mathcal{A}_3$. By Poincaré duality we have $H^4(K; \mathbf{Q}) \cong H^2(K; \mathbf{Q}) \otimes \mathbf{Q}(-1)$, inducing the local system $\mathbf{V}_{1,1,0}(-1) \oplus \mathbf{Q}(-2)$ on $\mathcal{A}_3$.

The cohomology with compact support of $\mathcal{A}_3$ in the local system $\mathbf{V}_{1,1,0}$ is calculated in Lemma IV.8.6. We refer to Theorem IV.8.2 for the cohomology with compact support of $\mathcal{A}_3$ with constant coefficients, which was calculated by Hain in [Ha]. These results allow to compute $E_2^{p,q} = H_c^p(\mathcal{A}_3; R_!^q k_{3*}(\mathbf{Q}))$ for the Leray spectral sequence $E_\bullet^{p,q} \Rightarrow H_c^{p+q}(\beta_1^0; \mathbf{Q})$ associated with $k_3\colon \beta_1^0 \to \mathcal{A}_3$. This E_2 term is given in Table IV.4. Note that all differentials of this Leray spectral sequence vanish for Hodge-theoretic reasons, so that $E_2 = E_\infty$. Specifically, all differentials must involve one $E_2^{p,q}$ term with $p+q$ odd, but there are only two such terms, namely $E_2^{5,2}$ and $E_2^{5,4}$. It follows from Table IV.4 that for both these terms, any differential d_k with $k \geq 2$ which involves one of them will map either

Table IV.4: E_2 term of the Leray spectral sequence converging to $H^\bullet_c(\beta_1^0;\mathbf{Q})$.

q								
6	0	$\mathbf{Q}(-6)+\mathbf{Q}(-3)$	0	$\mathbf{Q}(-7)$	0	$\mathbf{Q}(-8)$	0	$\mathbf{Q}(-9)$
5	0	0	0	0	0	0	0	0
4	$\mathbf{Q}(-2)$	$\mathbf{Q}(-5)+\mathbf{Q}(-2)$	0	$\mathbf{Q}(-6)^{\oplus(1+\epsilon)}$	$\mathbf{Q}(-6)^{\oplus\epsilon}$	$\mathbf{Q}(-7)$	0	$\mathbf{Q}(-8)$
3	0	0	0	0	0	0	0	0
2	$\mathbf{Q}(-1)$	$\mathbf{Q}(-4)+\mathbf{Q}(-1)$	0	$\mathbf{Q}(-5)^{\oplus(1+\epsilon)}$	$\mathbf{Q}(-5)^{\oplus\epsilon}$	$\mathbf{Q}(-6)$	0	$\mathbf{Q}(-7)$
1	0	0	0	0	0	0	0	0
0	0	$\mathbf{Q}(-3)+\mathbf{Q}$	0	$\mathbf{Q}(-4)$	0	$\mathbf{Q}(-5)$	0	$\mathbf{Q}(-6)$
	5	6	7	8	9	10	11	12 p

$\epsilon = \operatorname{rank} H^9_c(\mathcal{A}_3;\mathbf{V}_{1,1,0}) \in \{0,1\}$.

to 0 or to a E_2 term that carries a pure Hodge structure of different weight. In both cases the differential has to be 0. $\square$

IV.5 Torus rank 2

In this section we compute the cohomology with compact support of the stratum β_2^0 of $\mathcal{A}_4^{\mathrm{Vor}}$ of rank 2 degenerations of abelian fourfolds. For this purpose we recall first the known global construction of β_0^2 as the quotient of a $\mathbf{P}^1$-bundle of a fibre product of the universal family over $\mathcal{A}_2$.

Furthermore, let us recall that the restriction of the Voronoi fan in genus g to $\mathrm{Sym}^2_{\geq 0}(\mathbf{R}^{g'})$ for genus $g \geq g'$ coincides with the Voronoi fan in genus g'. This implies that the geometric constructions of the fibrations $\beta_2^0 \to \mathcal{A}_2$ and $\beta_3^0 \to \mathcal{A}_1$ we give in this section and in the following one, respectively, are actually independent of the choice of $g=4$ but extend to analogous descriptions of the fibres of fibrations $\beta_2^0 \to \mathcal{A}_{g-2}$ and $\beta_3^0 \to \mathcal{A}_{g-3}$ that exist for $\beta_2^0, \beta_3^0 \subset \mathcal{A}_g^{\mathrm{Vor}}$ independently of g. In particular, the geometric construction of β_2^0 explained in this section coincides with the construction used in [HT, §4] to compute the cohomology with compact support of the corresponding locus in $\mathcal{A}_3^{\mathrm{Vor}}$.

Proposition IV.5.1. *The rational cohomology with compact support of β_2^0 is as follows: the non-zero Betti numbers are*

i	4	6	7	8	9	10	11	12	14	16
b_i	1	2	$1+r$	$4+r$	$1+r$	$5+r$	1	5	3	1

where $r = \operatorname{rank} H^3_c(\mathcal{A}_2;\mathbf{V}_{2,2})$. If we assume $r=0$, then all cohomology groups of even degree are algebraic, with the exception of $H^8_c(\beta_2^0;\mathbf{Q}) = \mathbf{Q}(-4)^{\oplus 3} + \mathbf{Q}(-2)$ which is an extension of Hodge structures of Tate type. The Hodge structure in odd degree $2k+1$ with $k=3,4,5$ is pure of Tate type of weight $2k-4$.

In the previous section, we calculated the cohomology with compact support of β_1^0 using the map $k_3\colon\ \beta_1^0 \to \mathcal{A}_3$ given by the universal Kummer variety. This map extends to the stratum β_2^0 of degenerations of abelian fourfolds of torus rank 2, giving a map $k_3\colon\ (\beta_1 \setminus \beta_3) \to \mathcal{A}_3^{\mathrm{Vor}}$. Under this map, the elements of $\mathcal{A}_4^{\mathrm{Vor}}$ with torus rank 2 are mapped to elements of $\mathcal{A}_3^{\mathrm{Vor}}$ of torus rank 1. If we denote by ${\beta'_t}^0$ the stratum of $\mathcal{A}_3^{\mathrm{Vor}}$ of semi-abelian varieties of torus rank exactly t, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_4^{\mathrm{Vor}} & \mathcal{A}_3^{\mathrm{Vor}} & \mathcal{A}_2^{\mathrm{Vor}} \\
\beta_2^0 \arrow[u, hook] \arrow[r, "k_3"] & {\beta'_1}^0 \arrow[u, hook] \arrow[r, "k_2"] & \mathcal{A}_2 \arrow[u, hook] \\
\pi_3^{-1}({\beta'_1}^0) \arrow[u, "q_3"] \arrow[ur, "\pi_3"'] & \mathcal{X}_2 \arrow[u, "q_2"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

The map π_3 is the restriction of the universal family over $\mathcal{A}_3^{\mathrm{Vor}}$. In particular, the fibres of π_3 over points of ${\beta'_1}^0$ are rank 1 degenerations of abelian threefolds, i.e. compactified $\mathbf{C}^*$-bundles over abelian surfaces. A geometric description of these compactified $\mathbf{C}^*$-bundles is given in [Mu1]. They are obtained by taking the $\mathbf{P}^1$-bundle associated to the $\mathbf{C}^*$-bundle and then gluing the 0- and the ∞-section with a shift, defined by a point of the underlying abelian surface that is uniquely determined by the line bundle associated to the $\mathbf{C}^*$-bundle. In particular, this shift is 0 for the fibres of the π_3 over the 0-section of the Kummer fibration ${\beta'_1}^0 \cong (\mathcal{X}_2/\pm 1) \xrightarrow{k_2} \mathcal{A}_2$, which are thus products of a nodal curve and an abelian surface.

We want to describe the situation in more detail. For this, consider the universal Poincaré bundle $\mathcal{P} \to \mathcal{X}_2 \times_{\mathcal{A}_2} \hat{\mathcal{X}}_2$, normalized so that the restriction to the zero section $\hat{\mathcal{X}}_2 \to \mathcal{X}_2 \times_{\mathcal{A}_2} \hat{\mathcal{X}}_2$ is trivial. Let $\overline{U} = \mathbf{P}(\mathcal{P} \oplus \mathcal{O}_{\mathcal{X}_2 \times_{\mathcal{A}_2} \hat{\mathcal{X}}_2})$ be the associated $\mathbf{P}^1$-bundle. Using the principal polarization we can naturally identify $\hat{\mathcal{X}}_2$ and $\mathcal{X}_2$, which we will do from now on. We denote by Δ the union of the 0-section and the ∞-section of this bundle. Set $U = \overline{U} \setminus \Delta$, which is simply the $\mathbf{C}^*$-bundle given by the universal Poincaré bundle $\mathcal{P}$ with the 0-section removed and denote the bundle map by $f\colon\ U \to \mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$. Then there is a map $\rho\colon\ \overline{U} \to \beta_2^0$ with finite fibres. Note that the two components of Δ are identified under the map ρ. The restriction of ρ to both U and to Δ is given by a finite group action, although the group is not the same in the two cases (see the discussion below).

IV.5.1 Geometry of the $\mathbf{C}^*$-bundle

We now consider the situation over a fixed point $[S] \in \mathcal{A}_2$. For a fixed degree 0 line bundle $\mathcal{L}_0$ on S the preimage $f^{-1}(S \times \{\mathcal{L}_0\})$ is a semi-abelian threefold, namely the $\mathbf{C}^*$-bundle given by the extension corresponding to $\mathcal{L}_0 \in \hat{S}$. This semi-abelian threefold admits a Kummer involution ι which acts as $x \mapsto -x$ on the base S and by $t \mapsto 1/t$ on the fibre over the zero section. The Kummer involution ι is defined universally on U.

Consider the two involutions i_1, i_2 on $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$ defined by

$$i_1(S,p,q) = (S,-p,-q) \text{ and } i_2(S,p,q) = (S,q,p)$$

for every abelian surface S and every $p,q \in S$. These two involutions lift to involutions j_1 and j_2 on U that act trivially on the fibre of $f\colon U \to \mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$ over the zero section.

The following lemma can also be proved directly from the toroidal construction of $\mathcal{A}_4^{\mathrm{Vor}}$ using the approach of [S-B, Lemma 2.4].

Lemma IV.5.2. *The diagram*

$$\begin{tikzcd} U \arrow[r, "f"'] \arrow[d, "\rho|_U"] \arrow[rr, bend left, "g"] & \mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2 \arrow[r] \arrow[d, "\rho'"] & \mathcal{A}_2 \\ \beta_2^0 \setminus \rho(\Delta) \arrow[r] & \mathrm{Sym}^2_{\mathcal{A}_2}(\mathcal{X}_2/\pm 1) \arrow[ur] & \end{tikzcd} \qquad \text{(IV.5.1)}$$

where $\rho'\colon \mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2 \to \mathrm{Sym}^2_{\mathcal{A}_2}(\mathcal{X}_2/\pm 1)$ is the natural map, is commutative. Moreover $\rho|_U\colon U \to \rho(U) = \beta_2^0 \setminus \rho(\Delta)$ is the quotient of U by the subgroup of the automorphism group of U generated by ι, j_1 and j_2.

Proof. Since the map ρ' in the diagram (IV.5.1) has degree 8 and ι, j_1, j_2 generate a subgroup of order 8 of the automorphism group of U, it suffices to show that the map $\rho|_U$ factors through each of the involutions ι and j_1, j_2.

Recall that the elements of β_2^0 correspond to rank 2 degenerations of abelian fourfolds. More precisely, every point of $\rho(U)$ corresponds to a degenerate abelian fourfold X whose normalization is a $\mathbf{P}^1 \times \mathbf{P}^1$-bundle, namely the compactification of a product of two $\mathbf{C}^*$-bundles on the abelian surface S given by $k_1 \circ k_2([X])$. The degenerate abelian threefold itself is given by identifying the 0-sections and the ∞-sections of the $\mathbf{P}^1 \times \mathbf{P}^1$-bundle. This identification is determined by a complex parameter, namely the point on a fibre of $f\colon U \to \mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$.

Since a degree 0 line bundle $\mathcal{L}_0$ and its inverse define isomorphic semi-abelian threefolds and since the role of the two line bundles is symmetric, the map $\rho|_U$ factors through ι and j_2. Since j_1 is the commutator of ι and j_2 the map $\rho|_U$ also factors through j_1. $\square$

We will compute the cohomology with compact support of β_2^0 by considering the Leray spectral sequence associated with the fibration $k_2 \circ k_3\colon \beta_2^0 \to \mathcal{A}_2$. This requires to compute the cohomology with compact support of the fibre $(k_2 \circ k_3)^{-1}([S])$ over a point $[S] \in \mathcal{A}_2$. To this end, we decompose $(k_2 \circ k_3)^{-1}([S])$ into an open part given by its intersection with $\rho(U)$ and a closed part given by its complement.

k	$H^k(S\times S;\mathbf{Q})^{(i_1,i_2)}$	κ-invariant	κ-alternating
8	$\mathbf{Q}(-4)$	$\mathbf{Q}(-4)$	0
6	$(\bigwedge^2\Lambda)(-2)^{\oplus 2}$	$(\bigwedge^2\Lambda)(-2)$	$(\bigwedge^2\Lambda)(-2)$
4	$\mathbf{Q}(-2)\oplus\Lambda^{\otimes 2}(-1)\oplus\mathrm{Sym}^2(\bigwedge^2\Lambda)$	$\mathbf{Q}(-2)\oplus\mathrm{Sym}^2(\bigwedge^2\Lambda)$	$\Lambda^{\otimes 2}(-1)$
2	$\bigwedge^2\Lambda^{\oplus 2}$	$\bigwedge^2\Lambda$	$\bigwedge^2\Lambda$
0	$\mathbf{Q}$	$\mathbf{Q}$	0

Table IV.5: Cohomology of $S\times S/(i_1,i_2)$

IV.5.2 Cohomology of the open part of the fibre

The fibration $g\colon U\to\mathcal{A}_2$ obtained by composing the $\mathbf{C}^*$-bundle $f\colon U\to\mathcal{X}_2\times_{\mathcal{A}_2}\mathcal{X}_2$ with the natural map $\mathcal{X}_2\times_{\mathcal{A}_2}\mathcal{X}_2\to\mathcal{A}_2$ plays an important role in the study of the restriction of $k_2\circ k_3$ to $\rho(U)$. Namely, the fibre of $(k_2\circ k_3)|_{\rho(U)}$ over $[S]\in\mathcal{A}_2$ coincides with the quotient of the fibre of g under the automorphism group generated by j_1, j_2 and ι. Therefore, the cohomology of the fibre of $k_2\circ k_3$ restricted to $\rho(U)$ is the part of the cohomology of $g^{-1}([S])$ that is invariant under j_1, j_2 and ι.

We start by computing the actions of i_1, i_2 and of the involution $\kappa:\ (p,q)\mapsto(-p,q)$ induced by the Kummer involution of semi-abelian threefolds of torus rank 1 on the cohomology of $S\times S$. Recall that the cohomology of S is isomorphic to the exterior algebra generated by the 6-dimensional space $\Lambda := H^1(S;\mathbf{Q})$ and that $H^\bullet(S\times S;\mathbf{Q})\cong H^\bullet(S;\mathbf{Q})^{\otimes 2}$ by the Künneth formula. Using this description, one can calculate the part of the cohomology of $S\times S$ which is invariant under i_1 and i_2. In particular, since all cohomology in odd degree is alternating under the involution i_1, the only non-trivial invariant cohomology groups are in even degree. We give the description of the invariant cohomology groups in the second column of Table IV.5. One then proceeds to investigate their structure with respect to κ. For instance one can use the isomorphism $H^k(S\times S;\mathbf{Q})^{(i_1,i_2,\kappa)}\cong H^k(S\times S/(i_1,i_2,\kappa);\mathbf{Q})$, together with the fact that the quotient of $S\times S$ by the subgroup generated by i_1, i_2 and κ is the second symmetric product of $S/\pm 1$. In this way one proves that the behaviour of the cohomology with respect to κ is as given in the last two columns of Table IV.5.

Lemma IV.5.3. *The (i_1,i_2,ι)-invariant part of the Leray spectral sequence associated with the $\mathbf{C}^*$-bundle $g^{-1}([S])\to S\times S$ gives rise to a spectral sequence $E_\bullet^{p,q}\Rightarrow H^{p+q}((k_2\circ k_3|_{\rho(U)})^{-1}([S]);\mathbf{Q})$ which behaves as follows:*

- *$E_2^{p,q}$ vanishes for $q\neq 0,1$;*

- *$E_2^{p,0}$ is the part of $H^k(S\times S;\mathbf{Q})$ which is invariant under i_1, i_2 and κ;*

q										
1	0	0	$\bigwedge^2 \Lambda(-1)$	0	$\Lambda^{\otimes 2}(-2)$	0	$(\bigwedge^2 \Lambda)(-3)$	0	0	
0	$\mathbf{Q}$	0	$(\bigwedge^2 \Lambda)$	0	$\mathbf{Q}(-2) \oplus \mathrm{Sym}^2(\bigwedge^2 \Lambda)$	0	$(\bigwedge^2 \Lambda)(-2)$	0	$\mathbf{Q}(-4)$	
	0	1	2	3	4	5	6	7	8	p

Table IV.6: E_2 term of the spectral sequence converging to $H^k(g^{-1}([S]);\mathbf{Q})^{(i_1,i_2,\iota)}$

q										
1	0	0	0	0	$\mathbf{V}_{2,0}(-2)$	0	$\mathbf{V}_{1,1}(-3)$	0	0	
0	$\mathbf{Q}$	0	$\mathbf{Q}(-1) \oplus \mathbf{V}_{1,1}$	0	$\mathbf{Q}(-2)^{\oplus 2} \oplus \mathbf{V}_{2,2}$	0	0	0	0	
	0	1	2	3	4	5	6	7	8	p

Table IV.7: E_∞ term of the spectral sequence converging to $H^k(g^{-1}([S]);\mathbf{Q})^{(i_1,i_2,\iota)}$

- $E_2^{p,1}$ *is the part of* $H^k(S \times S;\mathbf{Q})$ *which is invariant under* i_1, i_2 *and alternating under* κ, *tensored with the Tate Hodge structure* $\mathbf{Q}(-1)$.

Furthermore, the E_∞ *term of this spectral sequence, together with its structure as* $\mathrm{Sp}(4,\mathbf{Q})$*-representation, is as given in Table IV.7.*

Proof. Let us consider the $\mathbf{C}^*$-bundle $f_S := f|_{g^{-1}([S])}\colon\ g^{-1}([S]) \to S \times S$. The Leray spectral sequence in cohomology associated with f_S converges to the cohomology of $g^{-1}([S])$ and has E_2 term $E_2^{p,q} \cong H^p(S \times S;\mathbf{Q}) \otimes H^q(\mathbf{C}^*;\mathbf{Q})$. However, we are only interested in the part of the cohomology of $g^{-1}([S])$ which is invariant under j_1, j_2 and ι. Since the actions of j_1, j_2 and ι respect the map $g^{-1}([S]) \to S \times S$, they act also on the terms of the Leray spectral sequence associated with f_S. In particular, the spectral sequence whose E_r terms are the (j_1, j_2, ι)-invariant part of the terms of the Leray spectral sequence associated with f_S converges to $H^k(g^{-1}([S]);\mathbf{Q})^{j_1,j_2,\iota}$.

In particular, the E_2 term of this spectral sequence is given by the (j_1, j_2, ι)-invariant part of $H^p(S \times S;\mathbf{Q}) \otimes H^q(\mathbf{C}^*;\mathbf{Q})$. We have already determined the behaviour of the projection of these involutions to $S \times S$ in Table IV.5, so it remains only to determine their action on the fibre $\mathbf{C}^*$. Since j_1 and j_2 both fix the fibre of f over the origin, they act trivially on the cohomology of $\mathbf{C}^*$. Instead, the Kummer involution ι acts as the identity on $H^0(\mathbf{C}^*;\mathbf{Q})$ and as the alternating representation on $H^1(\mathbf{C}^*;\mathbf{Q})$. From this the first part of the claim follows. For the convenience of the reader, we have written the E_2 term of the spectral sequence in Table IV.6. Notice that this spectral sequence has only two non-trivial rows. Therefore, it could be written equivalently as a long exact sequence. In particular, the only differentials one needs to study are the d_2-differentials.

These differentials are given by restriction of the differentials of the Leray spectral sequence associated with the $\mathbf{C}^*$-bundle f_S. Recall that f_S is the $\mathbf{C}^*$-bundle obtaining by subtracting the 0-section from the Poincaré bundle over $S \times S$. Therefore (see e.g. [Hu, XVI.7.5]) the d_2-differentials are given by taking the intersection product with the first Chern class of the Poincaré bundle, which is known to be equal to $[\mathrm{diag}(S)] - [S \times \{0\}] - [\{0\} \times S]$, where $[\cdot]$ denotes the fundamental class and $\mathrm{diag}\colon S \to S \times S$ is the diagonal map. An explicit computation of the intersections of this class with the κ-alternating classes in $H^k(S \times S; \mathbf{Q})^{i_1,i_2}$ yields the description of $E_3 = E_\infty$ given in Table IV.7. Here we have used the fact that $\mathrm{Sym}^2 \Lambda$ is the irreducible $\mathrm{Sp}(4, \mathbf{Q})$ -representation $\mathbf{V}_{2,0}$, whereas $\bigwedge^2 \Lambda$ decomposes into irreducible $\mathrm{Sp}(4, \mathbf{Q})$-representations as $\mathbf{Q}(-1) \oplus \mathbf{V}_{1,1}$ and $\mathrm{Sym}^2(\bigwedge^2 \Lambda)$ decomposes as $\mathbf{Q}(-2)^{\oplus 2} \oplus \mathbf{V}_{1,1}(-1) \oplus \mathbf{V}_{2,2}$. In the notation, Tate twists are only relevant for the Hodge structure.

□

IV.5.3 Geometry of $\rho(\Delta)$

The map ρ identifies the two components of Δ, each of which is isomorphic to $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$. In particular, the space $\rho(\Delta)$ can be realized as a finite quotient of $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$. This can be read off from the construction of the toroidal compactification, as in [S-B, Lemma 2.4]. See also [HS1, Section I] for an outline of this construction. Also note that the stratum Δ corresponds to the stratum in the partial compactification in the direction of the 2-dimensional cusp associated with a maximal-dimensional cone in the second Voronoi decomposition for $g = 4$. A detailed description can be found in [HKW, Part I, Chapter 3].

Specifically, the stratum $\rho(\Delta)$ corresponds to the $\mathrm{GL}(2, \mathbf{Z})$-orbit of the cone $\langle x_1^2, x_2^2, (x_1 - x_2)^2 \rangle$. Hence, the map $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2 \to \rho(\Delta)$ is the quotient map with respect to the stabilizer G of the cone $\langle x_1^2, x_2^2, (x_1 - x_2)^2 \rangle$ in $\mathrm{Sym}^2(\mathbf{Z}^2)$. This is generated by three involutions: the multiplication map by -1, the involution interchanging x_1 and x_2 and the involution $x_1 \mapsto x_1,\ x_2 \mapsto x_1 - x_2$.

These generators of G act on $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$ by the following three involutions: the involution i_1 which acts by $(x, y) \mapsto (-x, -y)$ on each fibre $S \times S$, the involution i_2 which interchanges the two factors of $\mathcal{X}_2 \times_{\mathcal{A}_2} \mathcal{X}_2$ and finally the involution i_3 which acts by $(x, y) \mapsto (x + y, -y)$.

From this description, it follows that there is a fibration $g'\colon\ \rho(\Delta) \to \mathcal{A}_2$ whose fibre over $[S] \in \mathcal{A}_2$ is isomorphic to the quotient of $S \times S$ by the subgroup of $\mathrm{Aut}(S \times S)$ generated by the three involutions i_1, i_2 and i_3 introduced above.

If we write $\Lambda := H^1(S \times \{0\}; \mathbf{Q})$ and $\Lambda' := H^1(\{0\} \times S; \mathbf{Q})$, then the cohomology of $S \times S$ is the exterior algebra of $H^1(S \times S; \mathbf{Q}) = \Lambda \oplus \Lambda'$. If we denote by $f_1, \ldots, f_4$, resp. $f_5, \ldots, f_8$ the generators of Λ, resp. Λ', the three involutions act of $H^1(S \times S; \mathbf{Q})$ as follows:

$$f_i \mapsto -f_i,\ i = 1, \ldots, 8, \tag{IV.5.2}$$

$$f_i \leftrightarrow f_{i+4},\ i = 1, \ldots, 4, \tag{IV.5.3}$$

$$f_i \mapsto f_i,\ f_{i+4} \mapsto f_i - f_{i+4},\ i = 1, \ldots, 4. \tag{IV.5.4}$$

Then one proceeds to determine the invariant part of the exterior algebra of $\Lambda \oplus \Lambda'$ under these involution. Moreover, to determine the local systems $R^q_! g'_*(\mathbf{Q})$ that appear in the Leray spectral sequence associated with $g'\colon\ \rho(\Delta) \to \mathcal{A}_2$, one needs to investigate the structure of the invariant subspaces as representations of $\mathrm{Sp}(4, \mathbf{Q})$. An explicit calculation of the invariant classes yields the results which we summarize in the following lemma.

Lemma IV.5.4. *The rational cohomology groups the fibre of $g'\colon\ \rho(\Delta) \to \mathcal{A}_2$ over a point $[S] \in \mathcal{A}_2$, with their mixed Hodge structures and structure as* $\mathrm{Sp}(4, \mathbf{Q})$*-representations, are given by*

$$H^k(g'^{-1}([S]); \mathbf{Q}) = \begin{cases} \mathbf{Q} & k = 0, \\ \bigwedge^2 \mathbf{V}_{1,0} = \mathbf{Q}(-1) \oplus \mathbf{V}_{1,1} & k = 2, \\ \mathbf{Q}(-2)^{\oplus 2} \oplus \mathbf{V}_{1,1}(-1) \oplus \mathbf{V}_{2,2} & k = 4, \\ (\bigwedge^2 \mathbf{V}_{1,0})(-2) = \mathbf{Q}(-3) \oplus \mathbf{V}_{1,1}(-2) & k = 6, \\ \mathbf{Q}(-4) & k = 8, \\ 0 & \textit{otherwise.} \end{cases}$$

IV.5.4 Proof of Proposition IV.5.1

We will prove Proposition IV.5.1 by investigating the Leray spectral sequence associated with the fibration $k_2 \circ k_3\colon\ \beta_2^0 \to \mathcal{A}_2$. As explained at the beginning of this section, the fibre of $k_2 \circ k_3$ over a point $[S] \in \mathcal{A}_2$ is the disjoint union of an open part, which is $(k_2 \circ k_3|_{\rho(U)})^{-1}([S])$, and a closed part, which is the fibre of $g'\colon\ \rho(\Delta) \to \mathcal{A}_2$. The cohomology of the fibre of $k_2 \circ k_3|_{\rho(U)}$ was determined in Lemma IV.5.3, whereas the cohomology of the fibre of g' was computed in Lemma IV.5.4. Notice that $(k_2 \circ k_3|_{\rho(U)})^{-1}([S]) = g^{-1}([S])/(j_1, j_2, \iota)$ is the finite quotient of a smooth quasi-projective variety, so that we can use Poincaré duality to obtain its cohomology with compact support from its cohomology. Furthermore, since $g'^{-1}([S]) = S^2/(i_1, i_2, i_3)$ is compact, its cohomology with compact support coincides with its cohomology.

To compute the cohomology with compact support of the fibre of $k_2 \circ k_3$ one can use the Gysin long exact sequence associated with the inclusion $g'^{-1}([S]) \hookrightarrow (k_2 \circ k_3)^{-1}([S])$:

$$\cdots \to H^k_c(g^{-1}([S]); \mathbf{Q})^{(j_1,j_2,\iota)} \to H^k_c((k_2 \circ k_3)^{-1}([S]); \mathbf{Q}) \to \\ H^k_c(S \times S; \mathbf{Q})^{(i_1,i_2,i_3)} \xrightarrow{\delta_k} H^{k+1}_c(g^{-1}([S]); \mathbf{Q})^{(j_1,j_2,\iota)} \to \cdots \tag{IV.5.5}$$

Notice that all differentials δ_k in (IV.5.5) have to respect the structure of the cohomology groups as representations of $\mathrm{Sp}(4, \mathbf{Q})$. In this specific case, this

$R^q_!(k_2 \circ k_3)_*(\mathbf{Q})$	q					
$\mathbf{Q}(-5)$	10	0	$\mathbf{Q}(-7)$	0	$\mathbf{Q}(-8)$	
0	9	0	0	0	0	
$\mathbf{Q}(-4)^{\oplus 2} \oplus \mathbf{V}_{1,1}(-3)$	8	$\mathbf{Q}(-3)$	$\mathbf{Q}(-6)^{\oplus 2}$	0	$\mathbf{Q}(-7)^{\oplus 2}$	
0	7	0	0	0	0	
$\mathbf{Q}(-3)^{\oplus 3} \oplus \mathbf{V}_{1,1}(-2) \oplus \mathbf{V}_{2,2}(-1)$	6	$\mathbf{Q}(-2) \oplus H(-1)$	$\mathbf{Q}(-5)^{\oplus 3} \oplus H(-1)$	0	$\mathbf{Q}(-6)^{\oplus 3}$	
$\mathbf{V}_{2,0}(-1)$	5	$\mathbf{Q}(-2)$	0	0	0	
$\mathbf{Q}(-2)^{\oplus 2} \oplus \mathbf{V}_{1,1}(-1) \oplus \mathbf{V}_{2,2}$	4	$\mathbf{Q}(-1) \oplus H$	$\mathbf{Q}(-4)^{\oplus 2} \oplus H$	0	$\mathbf{Q}(-5)^{\oplus 2}$	
0	3	0	0	0	0	
$\mathbf{Q}(-1)$	2	0	$\mathbf{Q}(-3)$	0	$\mathbf{Q}(-4)$	
0	1	0	0	0	0	
$\mathbf{Q}$	0	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$	
		3	4	5	6	p

Table IV.8: E_2 term of the Leray spectral sequence converging to the cohomology with compact support of β_2^0. We denote $H = H^3_c(\mathcal{A}_2; \mathbf{V}_{2,2}) \cong H^4_c(\mathcal{A}_2; \mathbf{V}_{2,2})$ (up to grading).

implies that all δ_k with $k \neq 2$ vanish, whereas

$$\delta_2 \colon \mathbf{Q}(-1) \oplus \mathbf{V}_{1,1} \longrightarrow \mathbf{V}_{1,1}$$

is surjective by Lemma IV.5.5 below.

The above determines the cohomology with compact support of the fibre of $k_2 \circ k_3$. In particular, it also determines the local systems $R^q_!(k_2 \circ k_3)_*(\mathbf{Q})$ occurring in the Leray spectral sequence in cohomology with compact support associated with the fibration $k_2 \circ k_3$. These local systems are given in the first column of Table IV.8.

Recall that the E_2 term of the Leray spectral sequence $E_r^{p,q} \Rightarrow H_c^{p+q}(\beta_2^0; \mathbf{Q})$ associated with $k_2 \circ k_3$ are of the form $E_2^{p,q} = H^p_c(\mathcal{A}_2; R^q_!(k_2 \circ k_3)_*(\mathbf{Q}))$. From the decomposition into symplectic local systems of the $R^q_!(k_2 \circ k_3)_*(\mathbf{Q})$, one gets the E_2 term of the Leray spectral sequence as in Table IV.8. Here we used the description of the cohomology with compact support of $\mathcal{A}_2$ with coefficients in the local systems $\mathbf{V}_{1,1}$, $\mathbf{V}_{2,0}$ and $\mathbf{V}_{2,2}$ from Lemma IV.8.5 and IV.8.7.

To prove the claim, it remains to show that the Leray spectral sequence degenerates at E_2. From the shape of the spectral sequence, it follows that all d_2 differentials, and all differentials d_r with $r \geq 4$ are necessarily trivial. The only differentials one needs to investigate are the d_3-differentials $E^3_{3,q} \to E^3_{6,q-2}$. These are necessarily 0 by Hodge-theoretic reasons, because morphisms of Hodge structures between pure Hodge structures of different weights are necessarily trivial. From this the claim follows. □

Lemma IV.5.5. *The differential* $\delta_2: \ H^2_c(S\times S;\mathbf{Q})^{(i_1,i_2,i_3)} \to H^3_c(g^{-1}([S]);\mathbf{Q})^{(j_1,j_2,\iota)}$ *is surjective.*

Proof. We shall prove the claim by an explicit computation on the generators of the groups involved. Since in the proofs of Lemma IV.5.3 and Lemma IV.5.4 we described the cohomology of the fibres of E rather than those of the cohomology with compact support, to compute the rank of δ_2 we shall compute the rank of the map induced by δ_2 on cohomology by Poincaré duality

$$\delta_2^*: \ H^6(S\times S;\mathbf{Q})^{(i_1,i_2,i_3)}\otimes\mathbf{Q}(-1) \longrightarrow H^7(g^{-1}([S]);\mathbf{Q})^{(j_1,j_2,\iota)},$$

which can be described explicitly as the composition of the map

$$\begin{array}{ccc} H^6(S\times S;\mathbf{Q})^{(i_1,i_2,i_3)} & \longrightarrow & H^7(g^{-1}([S]);\mathbf{Q}) \\ \alpha & \longmapsto & Q\otimes\alpha, \end{array}$$

where Q denotes the image of the generator of $H^1(\mathbf{C}^*;\mathbf{Q})$ inside the cohomology of $g^{-1}([S])$, and the symmetrization with respect to the group G generated by j_1, j_2 and ι.

A direct computation yields that the classes

$$v_{i,j,k,l} = f_i\wedge f_j\wedge f_{i+4}\wedge f_{j+4}\wedge(2f_k\wedge f_l + 2f_{k+4}\wedge f_{l+4} + f_k\wedge f_{l+4} + f_{k+4}\wedge f_l)$$

with $\{i,j,k,l\}=\{1,2,3,4\}$ form a basis of $H^6(S\times S;\mathbf{Q})^{(i_1,i_2,i_3)}$. Here $f_1,\ldots,f_8$ denote the basis of $f_1,\ldots,f_8$ described in Section IV.5.3. Then we have

$$\delta_2^*(v_{i,j,k,l}) = Q\otimes f_i\wedge f_j\wedge f_{i+4}\wedge f_{j+4}\wedge(f_k\wedge f_{l+4} + f_{k+4}\wedge f_l)$$

and these classes generate $H^7(g^{-1}([S]);\mathbf{Q})$. From this the claim follows. □

IV.6 Torus rank 3

In this section we compute the cohomology with compact support of the stratum with torus rank 3. As in the previous section, our strategy is based on a detailed geometric analysis of the fibration $\beta_3^0\to\mathcal{A}_1$ whose toric part is actually independent of the choice of $g=4$.

IV.6.1 Description of the geometry

We first note that the spaces $\mathcal{A}_4^{\mathrm{Igu}}$ and $\mathcal{A}_4^{\mathrm{Vor}}$ only differ over $\mathcal{A}_0$ and hence $\beta_3^{\mathrm{perf}}\setminus\beta_4^{\mathrm{perf}} = \beta_3^{\mathrm{Vor}}\setminus\beta_4^{\mathrm{Vor}} =: \beta_3^0$. In this section we want to compute $H^\bullet_c(\beta^3_0;\mathbf{Q})$. For this we first give a geometric description.

In order to compactify $\mathcal{A}_4$ we start with the lattice $\mathbf{Z}^4$. The choice of a toroidal compactification corresponds to the choice of an admissible fan Σ_4 in the cone of semi-positive forms in $\mathrm{Sym}^2(\mathbf{Z}^4)$. One possible choice for such a fan is given by the perfect cone decomposition Σ_4^{perf}. A cusp of $\mathcal{A}_4$ corresponds to the choice of an isotropic subspace $U \subset \mathbf{Q}^4$. In our case, for the stratum over $\mathcal{A}_1$ we take $U = \langle e_1, e_2, e_3 \rangle$ where the e_i $(1 \leq i \leq 4)$ are the standard basis of $\mathbf{Z}^4$. This defines an embedding $\mathrm{Sym}^2(\mathbf{Z}^3) \subset \mathrm{Sym}^2(\mathbf{Z}^4)$ and, by restriction of Σ_4^{perf}, also a fan in $\mathrm{Sym}^2(\mathbf{Z}^3)$ which is nothing but Σ_3^{perf}. The stratum β_3^0 itself consists of different strata which are in $1:1$-correspondence with the $\mathrm{GL}(3,\mathbf{Z})$-orbits of the cones σ in Σ_3^{perf} whose interior contains rank 3 matrices. Up to the action of $\mathrm{GL}(3,\mathbf{Z})$ there is a unique minimal cone with this property, namely the cone $\sigma^{(3)} = \langle x_1^2, x_2^2, x_3^2 \rangle$. Beyond that there are (again up to group action) 4 further cones. In dimension 4 there are two cones, namely $\sigma_I^{(4)} = \langle x_1^2, x_2^2, x_3^2, (x_2 - x_3)^2 \rangle$ and $\sigma_{II}^{(4)} = \langle x_1^2, x_2^2, (x_2 - x_3)^2, (x_1 - x_3)^2 \rangle$. In dimensions 5 and 6 there are one cone each, namely $\sigma^{(5)} = \langle x_1^2, x_2^2, x_3^2, (x_2 - x_3)^2, (x_1 - x_3)^2 \rangle$ and $\sigma^{(6)} = \langle x_1^2, x_2^2, x_3^2, (x_2 - x_3)^2, (x_1 - x_3)^2, (x_1 - x_2)^2 \rangle$. Note that all cones are contained in $\sigma^{(6)}$. In fact the perfect cone decomposition in genus 3 (where it coincides with the second Voronoi decomposition) is obtained by taking the $\mathrm{GL}(3,\mathbf{Z})$-orbit of $\sigma^{(6)}$ and all its faces.

To describe the various strata let $\mathcal{X}_1 \to \mathcal{A}_1$ be the universal elliptic curve and let $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$ be the triple product with itself over $\mathcal{A}_1$. Let $T = \mathrm{Sym}^2(\mathbf{Z}^3) \otimes \mathbf{C}^*$ be the 6-dimensional torus associated with $\mathrm{Sym}^2(\mathbf{Z}^3)$. Every cone σ in Σ_3^{perf} is basic (i.e. the generators of the rays are part of a $\mathbf{Z}$-basis of $\mathrm{Sym}^2(\mathbf{Z}^3)$) and defines a subtorus $T^\sigma \subset T$ of rank $\dim(\sigma)$. We can now give a description of β_3^0.

Proposition IV.6.1. *The variety β_3^0 admits a stratification into strata as follows:*

(i) there are 6 *strata of β_3^0, corresponding to the cones $\sigma^{(3)}$, $\sigma_I^{(4)}$, $\sigma_{II}^{(4)}$, $\sigma^{(5)}$ and $\sigma^{(6)}$.*

(ii) Each stratum is the finite quotient of a torus bundle over $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$ with fibre T/T^σ.

Proof. See [S-B, Lemma 2.4]. □

We shall now compute the cohomology with compact support for each of these strata and then use a spectral sequence argument to compute the cohomology with compact support of β_3^0. We denote the substratum of β_3^0 associated with a cone σ by $\beta_3^0(\sigma)$ and the total space of the torus bundle by $\mathcal{T}(\sigma)$.

Before we state the results we have to give a brief outline of the construction of the stratum $\beta_3^0(\sigma)$ with a view towards describing suitable coordinates in which our calculations can be done. Consider a point in Siegel space of genus 4:

$$\tau = \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\ \tau_{1,2} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\ \tau_{1,3} & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4} \end{pmatrix} \in \mathbf{H}_4.$$

Going to the cusp over $\mathcal{A}_1$ means sending the top left hand 3×3 block of this matrix to $i\infty$. We shall make this more precise. We consider the basis of $\mathrm{Sym}^2(\mathbf{Z}^3)$ given by $U_{i,j}^* = (2 - \delta_{i,j})x_i x_j$. Let $t_{i,j}$ $(1 \le i, j \le 3)$ be the dual basis. Setting

$$t_{i,j} = e^{2\pi\sqrt{-1}\tau_{i,j}} \ (1 \le i, j \le 3)$$

defines a map

$$\mathbf{H}_4 \to T \times \mathbf{C}^3 \times \mathbf{H}_1 \tag{IV.6.1}$$

$$\tau \mapsto ((t_{i,j}), \tau_{1,4}, \tau_{2,4}, \tau_{3,4}, \tau_{4,4}).$$

This corresponds to taking the partial quotient $X(U) = P'(U) \backslash \mathbf{H}_4$ with respect to the center $P'(U)$ of the unipotent radical of the parabolic subgroup $P(U)$ associated with the cusp U. We denote $P''(U) = P(U)/P'(U)$. The partial quotient $X(U)$ can be considered as an open set of the trivial torus bundle $\mathcal{X}(U)$ (with fibre T) over $\mathbf{C}^3 \times \mathbf{H}_1$. Using the fan Σ_3^{perf} one constructs $\mathcal{X}_{\Sigma_3^{\mathrm{perf}}}(U)$ by taking a fibrewise toric embedding. Let $X_{\Sigma_3^{\mathrm{perf}}}(U)$ be the interior of the closure of $X(U)$ in $\mathcal{X}_{\Sigma_3^{\mathrm{perf}}}(U)$. The action of the group $P''(U)$ on $X(U)$ extends to an action on $X_{\Sigma_3^{\mathrm{perf}}}(U)$ and one obtains the partial compactification in the direction of the cusp U by $Y_{\Sigma_3^{\mathrm{perf}}}(U) = P''(U) \backslash X_{\Sigma_3^{\mathrm{perf}}}(U)$.

Every cone $\sigma \in \Sigma_3^{\mathrm{perf}}$ defines an affine toric variety X_σ. Since all cones σ are basic one has $X_\sigma = \mathbf{C}^k \times (\mathbf{C}^*)^{6-k}$ where k is the number of generators of σ. Every inclusion $\sigma \subset \sigma'$ induces an inclusion $X_\sigma \subset X_{\sigma'}$. Note that $X_{(0)} = T$ and, in particular we obtain an inclusion $X_{(0)} = T \subset X_{\sigma^{(6)}} \cong \mathbf{C}^6$. Let $T_1, \ldots, T_6$ be the coordinates on $X_{\sigma^{(6)}} \cong \mathbf{C}^6$ corresponding to the generators of $\sigma^{(6)}$ which form a basis of $\mathrm{Sym}^2(\mathbf{Z}^3)$. Computing the dual basis of this basis one finds that this inclusion is given by

$$\begin{array}{lll} T_1 = t_{1,1}t_{1,3}t_{1,2}, & T_2 = t_{2,2}t_{2,3}t_{1,2}, & T_3 = t_{3,3}t_{1,3}t_{2,3}, \\ T_4 = t_{2,3}^{-1}, & T_5 = t_{1,3}^{-1}, & T_6 = t_{1,2}^{-1}. \end{array} \tag{IV.6.2}$$

The relation to the strata $\beta_3^0(\sigma)$ is then the following. The coordinate $\tau_{4,4}$ defines a point in $\mathcal{A}_1$ and the coordinates $\tau_{1,4}, \tau_{2,4}, \tau_{3,4}$ define a point in the fibre of $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$ over $[\tau_{4,4}] \in \mathcal{A}_1$ which is $E_{\tau_{4,4}} \times E_{\tau_{4,4}} \times E_{\tau_{4,4}}$, where $E_{\tau_{4,4}} = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_{4,4})$ is the elliptic curve defined by $\tau_{4,4}$. The fibres of $\beta_3^0(\sigma) \to \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$ are isomorphic to the torus T/T^σ.

Finally, we have to make some comments on the structure of the parabolic subgroup $P(U)$. This group is generated by four types of matrices. The first type are block matrices of the form

$$g_1 = \begin{pmatrix} \mathbf{1} & 0 & S & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } S = {}^tS \in \mathrm{Sym}^2(\mathbf{Z}^3).$$

These matrices generate the center $P'(U)$ of the unipotent radical and act by

$$\begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\ \tau_{1,2} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\ \tau_{1,3} & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4} \end{pmatrix} \to \begin{pmatrix} \tau_{1,1}+s_{1,1} & \tau_{1,2}+s_{1,2} & \tau_{1,3}+s_{1,3} & \tau_{1,4} \\ \tau_{1,2}+s_{1,2} & \tau_{2,2}+s_{2,2} & \tau_{2,3}+s_{2,3} & \tau_{2,4} \\ \tau_{1,3}+s_{1,3} & \tau_{2,3}+s_{2,3} & \tau_{3,3}+s_{3,3} & \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4} \end{pmatrix}$$

giving rise to the partial quotient $\mathbf{H}_4 \to T \times \mathbf{C}^3 \times \mathbf{H}_1$ described above.

The second set of generators is of the form

$$g_2 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & c & 0 & d \end{pmatrix}, \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}),$$

resp.

$$g_3 = \begin{pmatrix} \mathbf{1} & M & 0 & N \\ 0 & 1 & {}^tN & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & -{}^tM & 1 \end{pmatrix}, \text{ where } M, N \in \mathbf{Z}^3.$$

Note that the elements of type g_2, g_3 generate a Jacobi group, which, in particular, acts on the base $\mathbf{C}^3 \times \mathbf{H}_1$ of the partial quotient by $P'(U)$ given by the map $\mathbf{H}_4 \to \mathbf{C}^3 \times \mathbf{H}_1$ giving rise to the triple product $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$.

Finally we have matrices of the form

$$g_4 = \begin{pmatrix} {}^tQ^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } Q \in \mathrm{GL}(3, \mathbf{Z}).$$

These matrices are of particular importance to us as they operate on the space $\mathrm{Sym}^2(\mathbf{Z}^3)$ by

$$\mathrm{GL}(3, \mathbf{Z}) \ni g: \; M \mapsto {}^tQ^{-1} M Q^{-1}.$$

IV.6.2 The cohomology of $\beta_3^0(\sigma^{(3)})$

In this section we will prove

Lemma IV.6.2. *The rational cohomology groups with compact support of $\beta_3^0(\sigma^{(3)})$ are given by*

$$H_c^k(\beta_3^0(\sigma^{(3)});\mathbf{Q}) = \begin{cases} \mathbf{Q}(-k/2) & k = 12, 14 \\ \mathbf{Q}((5-k)/2) & k = 9, 7 \\ 0 & \textit{otherwise.} \end{cases}$$

We start by giving an explicit description of the torus bundle $\mathcal{T}(\sigma^{(3)})$ defined by the cone $\sigma^{(3)}$.

Lemma IV.6.3. *Let $q_{\sigma^{(3)}}\colon \mathcal{T}(\sigma^{(3)}) \to \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$ be the rank 3 torus bundle associated with $\sigma^{(3)}$. Then over each fibre $E\times E\times E$ of $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$ we have*

$$\mathcal{T}(\sigma^{(3)})|_{E\times E\times E} \cong p_{2,3}^*(\mathcal{P}^0) \oplus p_{1,3}^*(\mathcal{P}^0) \oplus p_{1,2}^*(\mathcal{P}^0)$$

where $\mathcal{P}^0$ is the Poincaré bundle over the product $E\times E$ with the 0-section removed, and $p_{i,j}\colon E\times E\times E \to E\times E$ is the projection to the ith and jth factor.

Proof. We first recall the following description of the Poincaré bundle over $E\times E$ where $E = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$. Consider the action of the group $\mathbf{Z}^4$ on the trivial rank-1 bundle on $\mathbf{C}\times\mathbf{C}$ given by

$$(n_1,n_2,m_1,m_2)\colon (z_1,z_2,w) \mapsto \tag{IV.6.3}$$

$$(z_1+n_1+m_1\tau, z_2+n_2+m_2\tau, we^{-2\pi i(m_1z_2+m_2z_1+m_1m_2\tau)})$$

(where the z_i are the coordinates on the base and w is the fibre coordinate). We claim that the quotient line bundle on $E\times E$ is the Poincaré bundle. For this it is enough to see that this line bundle is trivial on $E\times\{0\}$ and $\{0\}\times E$ (which is obvious) and that it is isomorphic to $\mathcal{O}_E(O-P)$ on $E\times\{P\}$. The latter can be checked by comparing the transformation behaviour of (IV.6.3) to the transformation behaviour of the theta function $\vartheta(z,\tau)$ in one variable (see e.g. [La, 15.1.3.]).

We have to compare this to our situation. In this case we have an action of the group generated by the matrices g_3 with $M,N\in\mathbf{Z}^3$. For $N=(n_1,n_2,n_3)$ we have $\tau_{i,4}\mapsto\tau_{i,4}+n_i$ and for $M=(m_1,m_2,m_3)$ we have $\tau_{i,j}\mapsto\tau_{i,j}+m_j\tau_{i,4}+m_i\tau_{j,4}+m_im_j\tau_{4,4}$ for $1\le i,j\le 3$ and $\tau_{i,4}\mapsto\tau_{i,4}+m_i\tau_{4,4}$. Recall that the entries $\tau_{i,4}$ for $i=1,2,3$ are coordinates on the factors of $E\times E\times E$ and that it follows from (IV.6.2) that we can choose $t_{i,j}^{-1}$ with $t_{i,j}=e^{2\pi i\tau_{i,j}}$ for $(i,j)=(1,2),(1,3),(2,3)$ as coordinates on the torus $\mathcal{T}(\sigma^{(3)})$. Comparing this to the transformation (IV.6.3) gives the claim. ∎

Proof of Lemma IV.6.2. Recall that the stratum $\beta_3^0(\sigma^{(3)})$ is a finite quotient of the rank 3 torus bundle $q_{\sigma^{(3)}}\colon \mathcal{T}(\sigma^{(3)}) \to \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$. This enables us to calculate its rational cohomology by exploiting Leray spectral sequences.

Notice that the base of $q_{\sigma^{(3)}}$ is the total space of the fibration $p\colon \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$. Over a point $[E] \in \mathcal{A}_1$, the fibre of p is $p^{-1}([E]) \cong E \times E \times E$ and the fibre of $p \circ q_{\sigma^{(3)}}$ over $[E]$ is the total space of the rank 3 torus bundle $q_{\sigma^{(3)}}|_{E\times E\times E} : \mathcal{T}(\sigma^{(3)})|_{E\times E\times E} \to E \times E \times E$ described in Lemma IV.6.3. The cohomology of $(p \circ q_{\sigma^{(3)}})^{-1}([E])$ can be computed by the Leray spectral sequence associated with this rank 3 torus bundle:

$$E_2^{p,q}(q_{\sigma^{(3)}}) = H^q(T/T^{(\sigma^{(3)})}; \mathbf{Q}) \otimes H^p(E \times E \times E; \mathbf{Q}) \Longrightarrow H^{p+q}((p \circ q_{\sigma^{(3)}})^{-1}([E]); \mathbf{Q}). \quad \text{(IV.6.4)}$$

Note that the cohomology of $E \times E \times E$ (respectively, the torus $T/T^{(\sigma^{(3)})}$) is an exterior algebra generated by $H^1(E \times E \times E; \mathbf{Q})$ (resp. $H^1(T/T^{(\sigma^{(3)})}; \mathbf{Q})$). We denote by Q_1, Q_2 and Q_3, respectively, the generators of $H^1(T/T^{\sigma^{(3)}}; \mathbf{Q}) \cong H^1((\mathbf{C}^*)^3; \mathbf{Q}) \cong \mathbf{Q}^3$ defined by integrating along the loop around 0 defined, respectively, by $|t_{2,3}^{-1}| = 1$, $|t_{1,3}^{-1}| = 1$ or $|t_{1,2}^{-1}| = 1$.

We can write each copy of E as a quotient $E = \mathbf{C}/(\mathbf{Z}e_{2i-1} + \mathbf{Z}e_{2i})$; $i = 1, 2, 3$. Then $e_1, \ldots, e_6$ give rise to a basis of the first homology group of $E \times E \times E$. We will denote by $f_1, \ldots, f_6$ the elements of the basis of $H^1(E \times E \times E; \mathbf{Q})$ dual to $e_1, \ldots, e_6$. Notice that the transformation behaviour of the f_{2i-1} and of the f_{2i} for $1 \le i \le 3$ agrees with the transformation behaviour of the coordinates $\{\tau_{i,4} | \ 1 \le i \le 3\}$ of $\mathbf{C}^3 \cong (\mathbf{Z}e_1 + \mathbf{Z}e_2 + \cdots + \mathbf{Z}e_6) \otimes_{\mathbf{Z}} \mathbf{C}$ (and that of the differentials $d\tau_{i,4}$ which give rise to classes in cohomology).

As we are interested in the quotient of $\mathcal{T}(\sigma^{(3)})$ by the finite group $G(\sigma^{(3)})$, we shall compute the invariant cohomology with respect to this group. This is done in Lemma IV.6.4 for the invariant cohomology of the fibre $\mathcal{T}(\sigma^{(3)})|_{E\times E\times E} = (p \circ q_{\sigma^{(3)}})^{-1}([E])$ using a Leray spectral sequence argument. It remains to determine the local systems $R_!^i(p \circ q_{\sigma^{(3)}})_*(\mathbf{Q})$ over $\mathcal{A}_1$ defined by the fibration $p \circ q_{\sigma^{(3)}} \colon \beta_3^0(\sigma^{(3)}) = \mathcal{T}(\sigma^{(3)}) \to \mathcal{A}_1$. This is quite straightforward, since the cohomology with compact support of the fibre is constant in degrees 12 and 10, and since $\operatorname{Sym}^2 H^1(E; \mathbf{Q})$ induces the symplectic local system $\mathbf{V}_2$ on $\mathcal{A}_1$.

Recall that the cohomology with compact support of $\mathcal{A}_1$ with constant coefficients is concentrated in degree 2, and that the only non-trivial cohomology group of $\mathcal{A}_1$ with coefficients in $\mathbf{V}_2$ is $H_c^1(\mathcal{A}_1; \mathbf{V}_2) = \mathbf{Q}$ (see e.g. [G1, Thm. 5.3]). In particular, it then follows from Lemma IV.6.4 that the Leray spectral sequence associated with $p \circ q_{\sigma^{(3)}}$ has only two columns containing non-trivial E_2 terms, so it has to degenerate at E_2. From this the claim follows. $\square$

Lemma IV.6.4. *For every $[E] \in \mathcal{A}_1$, the rational cohomology with compact support of the fibre of $\beta_3^0(\sigma^{(3)}) \to \mathcal{A}_1$, with its Hodge structures, coincides with the $G(\sigma^{(3)})$-invariant part of the cohomology with compact support of the rank* 3 *torus bundle $\mathcal{T}(\sigma^{(3)})|_{E\times E\times E}$ and is given by*

$$\left(H_c^k(\mathcal{T}(\sigma^{(3)})|_{E\times E\times E};\mathbf{Q})\right)^{G(\sigma^{(3)})} = \begin{cases} \mathbf{Q}(-6) & k=12,\\ \mathbf{Q}(-5) & k=10,\\ \operatorname{Sym}^2(H^1(E;\mathbf{Q}))\otimes\mathbf{Q}(-2) & k=8,\\ \operatorname{Sym}^2(H^1(E;\mathbf{Q}))\otimes\mathbf{Q}(-1) & k=6,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. The stabilizer $G(\sigma^{(3)})$ of $\sigma^{(3)}$ in $\mathrm{GL}(3,\mathbf{Z})$ is an extension of the symmetric group $\mathfrak{S}_3$ (permuting the coordinates x_1,x_2,x_3) by $(\mathbf{Z}/2\mathbf{Z})^3$ (acting by involutions $(x_1,x_2,x_3,x_4)\mapsto(\pm x_1,\pm x_2,\pm x_3,x_4)$).

The interchange of two coordinates (say, x_i and x_j) acts on $H^1(E\times E\times E;\mathbf{Q})$ by interchanging f_{2i-1} with f_{2j-1}, f_{2i} with f_{2j} and leaving all other generators invariant. The action on $H^1(T/T^{(\sigma^{(3)})};\mathbf{Q})$ interchanges Q_i and Q_j and leaves the third generator invariant.

The automorphism mapping x_i to $-x_i$ acts on $H^1(E\times E\times E;\mathbf{Q})$ as multiplication by -1 on the generators f_{2i-1},f_{2i} and on $H^1(T/T^{(\sigma^{(3)})};\mathbf{Q})$ as multiplication by -1 on Q_k with $k\neq i$. All other generators are invariant.

We can compute the $G(\sigma^{(3)})$-invariant part of the rational cohomology of the rank 3 torus bundle $\mathcal{T}(\sigma^{(3)})|_{E\times E\times E}$ by restricting to the $G(\sigma^{(3)})$-invariant part of the Leray spectral sequence (IV.6.4) associated with $q_{\sigma^{(3)}}$. This yields a spectral sequence $E_2^{p,q}$ converging to the $G(\sigma^{(3)})$-invariant part of $H^{p+q}(\mathcal{T}(\sigma^{(3)})|_{E\times E\times E};\mathbf{Q})$.

A computation of the part of the tensor product $\bigwedge^\bullet H^1(E\times E\times E;\mathbf{Q})\otimes\bigwedge^\bullet H^1(T/T^{(\sigma^{(3)})};\mathbf{Q})$ which is invariant under $G(\sigma^{(3)})$ yields that $E_2^{p,q}$ is non-zero only for $(p,q)\in\{(2,0),(2,1),(4,0),((2,2),(4,1),(6,0)\}$. A precise description of the generators of the non-trivial E_2 terms is given in Table IV.9. Note that the spaces for $p=q=2$ and $p=4,q=2$ are both isomorphic to $\operatorname{Sym}^2 H^1(E;\mathbf{Q})$ as $\mathrm{Sp}(2,\mathbf{Q})$-representations.

Next, one investigates the differentials of the spectral sequence. As differentials have to occur between $E_2^{p,q}$ terms such that the two $p+q$ have different parity, an inspection of the spectral sequence quickly reveals that all differentials have to be trivial, with the possible exception of

$$d_2^{2,1}\colon\ E_2^{2,1}\to E_2^{4,0} \tag{IV.6.5}$$

and

$$d_2^{4,1}\colon\ E_2^{4,1}\to E_2^{6,0}. \tag{IV.6.6}$$

We can determine their rank by exploiting the description of the restriction to $E\times E\times E$ of the torus bundle $\mathcal{T}(\sigma^{(3)})$ given in Lemma IV.6.3 as a direct sum of pull-backs of the Poincaré bundle with the 0-section removed. This description implies that one can employ the usual description of d_2 differentials of $\mathbf{C}^*$-bundles to investigate $d_2^{2,1}$ and $d_2^{4,1}$. In particular, each of these differentials is given by formally replacing each generator Q_k of $H^1(T/T^{\sigma^{(3)}};\mathbf{Q})$ by the first Chern class of

p	q	dim.	generators
0	0	1	1
2	0	1	$\sum_i f_{2i-1} \wedge f_{2i}$
2	1	1	$\sum_{i<j,k\neq i,j} Q_k \otimes (f_{2i-1} \wedge f_{2j} - f_{2i} \wedge f_{2j-1})$
2	2	3	$\sum_{i<j,k\neq i,j} Q_i \wedge Q_j \otimes W_k^{(m)},\ m = 1,2,3$
4	0	1	$\sum_{i<j} f_{2i-1} \wedge f_{2i} \wedge f_{2j-1} \wedge f_{2j}$
4	1	1	$\sum_{i<j,k\neq i,j} Q_k \otimes (f_{2i-1} \wedge f_{2j} - f_{2i} \wedge f_{2j-1}) \wedge f_{2k-1} \wedge f_{2k}$
4	2	3	$\sum_{i<j,k\neq i,j} Q_i \wedge Q_j \otimes W_k^{(m)} \wedge f_{2k-1} \wedge f_{2k},\ m = 1,2,3$
6	0	1	$f_1 \wedge f_2 \wedge f_3 \wedge f_4 \wedge f_5 \wedge f_6$.

All indices i, j, k are between 1 and 3. For indices $i < j$ we set $W_k^{(1)} = f_{2i-1} \wedge f_{2j} + f_{2i} \wedge f_{2j-1}$, $W_k^{(2)} = f_{2i-1} \wedge f_{2j-1}$ and $W_k^{(3)} = f_{2i} \wedge f_{2j}$ for $k \neq i, j$.

Table IV.9: Description of the generators of the E_2 terms of the $G(\sigma^{(3)})$-invariant part of the spectral sequence associated with $q_{\sigma^{(3)}}$.

the bundle $p_{i,j}^*(\mathcal{P})$, where $1 \leq i < j \leq 3$ are chosen such that $\{i,j,k\} = \{1,2,3\}$. Recall that on the product $E \times E$ the Poincaré bundle $\mathcal{P} \cong \mathcal{O}_{E\times E}(E \times \{0\} + \{0\} \times E - \Delta)$, where Δ is the diagonal. From this one concludes that $c_1(\mathcal{P}) = f_1 \wedge f_2 + f_3 \wedge f_4 - (f_1 + f_3) \wedge (f_2 + f_4) = f_2 \wedge f_3 - f_1 \wedge f_4$. It is then a straightforward calculation to prove that both differentials are isomorphisms.

It remains to pass from cohomology to cohomology with compact support, which we can do by Poincaré duality, using the fact that $\mathcal{T}(\sigma^{(3)})|_{E\times E\times E}$ is smooth of complex dimension 6. Finally, we can identify the $G(\sigma^{(3)})$-invariant part of the cohomology with compact support of $\mathcal{T}(\sigma^{(3)})|_{E\times E\times E}$ with the cohomology with compact support of its finite quotient $\left(\mathcal{T}(\sigma^{(3)})|_{E\times E\times E}\right)/G(\sigma^{(3)})$, which coincides with the fibre of $\beta_3^0(\sigma^{(3)}) \to \mathcal{A}_1$ over $[E]$. $\square$

IV.6.3 The cohomology of $\beta_3^0(\sigma_I^{(4)})$

In this section we will prove

Lemma IV.6.5. *The rational cohomology groups with compact support of* $\beta_3^0(\sigma_I^{(4)})$

are given by

$$H_c^k(\beta_3^0(\sigma_I^{(4)});\mathbf{Q}) = \begin{cases} \mathbf{Q}(-6) & k=12 \\ \mathbf{Q}(-5)^{\oplus 2} & k=10 \\ \mathbf{Q}(-4)+\mathbf{Q}(-2) & k=8 \\ \mathbf{Q} & k=5 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We shall make again use of the twofold fibre structure of this stratum. The stratum $\beta_3^0(\sigma_I^{(4)})$ is a finite quotient of a rank 2 torus bundle $q_{\sigma_I^{(4)}}\colon \mathcal{T}(\sigma_I^{(4)}) \to \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1$ with fibres isomorphic to $T/T^{(\sigma_I^{(4)})}$. Note that the generators of $\sigma_I^{(4)}$ correspond to the first four generators of the cone $\sigma^{(6)}$. Comparing this to the embedding described in (IV.6.2) we find that we can choose $t_{1,3}^{-1}, t_{1,2}^{-1}$ as coordinates on $T/T^{(\sigma_I^{(4)})}$. As before we denote $p\colon \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$.

As we are interested in the quotient of $\mathcal{T}(\sigma_I^{(4)})$ by the finite group $G(\sigma_I^{(4)})$, we shall compute the invariant cohomology with respect to this group. Thus we first have to describe the automorphism group $G(\sigma_I^{(4)})$ of the cone $\sigma_I^{(4)}$, i.e. all elements of the form $g_3 \in \mathrm{GL}(3,\mathbf{Z})$ which fix this cone. We have already discussed this in [HT, Section 3]. The result is that the automorphism group is generated by the following four transformations:

$$x_1 \mapsto x_1, \quad x_2 \mapsto x_2 - x_3, \quad x_3 \mapsto -x_3 \tag{IV.6.7}$$

$$x_1 \mapsto -x_1, \quad x_2, x_3 \mapsto x_2, x_3 \tag{IV.6.8}$$

$$x_1 \mapsto x_1, \quad x_2 \leftrightarrow x_3. \tag{IV.6.9}$$

$$x_i \mapsto -x_i; \quad i = 1,2,3. \tag{IV.6.10}$$

Note that these automorphisms act trivially on the base of the fibration $\mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \times_{\mathcal{A}_1} \mathcal{X}_1 \to \mathcal{A}_1$.

Again we shall determine the invariant cohomology of the fibre $(q_{\sigma_I^{(4)}} \circ p)^{-1}([E])$ using the Leray spectral sequence with terms $E_2^{p,q} = H^q(T/T^{(\sigma_I^{(4)})},\mathbf{Q}) \otimes H^p(E \times E \times E, \mathbf{Q})$. The result is given by:

Lemma IV.6.6. *For every $[E] \in \mathcal{A}_1$, the rational cohomology with compact support of the fibre of $\beta_3^0(\sigma_I^{(4)}) \to \mathcal{A}_1$, with its Hodge structures, is given by*

$$\left(H_c^k(\mathcal{T}(\sigma_I^{(4)})|_{E\times E\times E};\mathbf{Q})\right)^{G(\sigma_I^{(4)})} = \begin{cases} \mathbf{Q}(-5) & k=10, \\ \mathbf{Q}(-4)^{\oplus 2} & k=8 \\ \mathrm{Sym}^2(H^1(E;\mathbf{Q}))\otimes\mathbf{Q}(-2) & k=7, \\ \mathbf{Q}(-3) & k=6 \\ \mathrm{Sym}^2(H^1(E;\mathbf{Q})) & k=4, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We denote the generators of $H^1(T/T^{(\sigma_I^{(4)})};\mathbf{Q}) \cong H^1((\mathbf{C}^*)^2;\mathbf{Q}) \cong \mathbf{Q}^2$ corresponding to $t_{1,3}^{-1}, t_{1,2}^{-1}$ by Q_2, Q_3. The $f_i, i = 1, \dots, 6$ are, as before, a basis of the cohomology of the triple product $E \times E \times E$.

We must now compute the action on (co)homology of the automorphisms of $\sigma_I^{(4)}$. As a non-trivial example we shall do this in detail in the case of the transformation given in (IV.6.7), the computations in the other cases are analogous.

The action of this transformation on Siegel space is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\ \tau_{1,2} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\ \tau_{1,3} & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & -\tau_{1,2} - \tau_{1,3} & \tau_{1,4} \\ \tau_{1,2} & \tau_{2,2} & -\tau_{2,2} - \tau_{2,3} & \tau_{2,4} \\ -\tau_{1,2} - \tau_{1,3} & -\tau_{2,2} - \tau_{2,3} & \tau_{2,2} + 2\tau_{2,3} + \tau_{3,3} & -\tau_{2,4} - \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & -\tau_{2,4} - \tau_{3,4} & \tau_{4,4} \end{pmatrix}.$$

From this we conclude that under this transformation:

$$Q_2 \mapsto -Q_2 - Q_3; \quad Q_3 \mapsto Q_3; \tag{IV.6.11}$$

$$f_i \mapsto f_i, \quad i = 1, \dots, 4 \quad f_i \mapsto -f_{i-2} - f_i; \quad i = 5, 6,$$

Note that the latter coincides with the transformation behaviour of the differentials $d\tau_{i,4}, i = 1, 2, 3$, and the former with the transformation behaviour of $-\tau_{1,3}, -\tau_{1,2}$.

An analogous computation for the other automorphisms gives the following results:

$$Q_2, Q_3 \mapsto -Q_2, -Q_3; \tag{IV.6.12}$$

$$f_1, f_2 \mapsto -f_1, -f_2, \quad f_i \mapsto f_i, \quad i = 3, \dots, 6.$$

$$Q_2, Q_3 \mapsto Q_3, Q_2; \tag{IV.6.13}$$

$$f_1, f_2 \mapsto f_1, f_2, \quad f_3 \leftrightarrow f_5, \quad f_4 \leftrightarrow f_6,$$

$$Q_2, Q_3 \mapsto Q_2, Q_3; \tag{IV.6.14}$$

$$f_i \mapsto -f_i, \quad i = 1, \dots, 6.$$

Now we must compute the invariant cohomology with respect to $G(\sigma_I^{(4)})$. This can either be done by a (lengthy) computation by hand or a standard computer algebra system.

The $E_2^{p,0}$ terms of the spectral sequence can be computed as follows. The invariant part $E_2^{2,0}$ of the cohomology group $H^0(T/T^{\sigma_I^{(4)}};\mathbf{Q}) \otimes H^2(E \times E \times E;\mathbf{Q})$ is two-dimensional and generated by the tensors

$$I_1 = f_1 \wedge f_2, \ I_2 = 2(f_3 \wedge f_4 + f_5 \wedge f_6) + (f_3 \wedge f_6 + f_5 \wedge f_4).$$

Table IV.10: E_2 term of the spectral sequence converging to the cohomology of $\beta_3^0(\sigma_I^{(4)})$

q							
2	0	0	$\mathbf{V}_2(-2)$	0	$\mathbf{V}_2(-3)$	0	0
1	0	0	$\mathbf{V}_2(-1) \oplus \mathbf{Q}(-2)$	0	$\mathbf{V}_2(-2) \oplus \mathbf{Q}(-3)$	0	0
0	$\mathbf{Q}$	0	$\mathbf{Q}(-1)^{\oplus 2}$	0	$\mathbf{Q}(-2)^{\oplus 2}$	0	$\mathbf{Q}(-3)$
	0	1	2	3	4	5	6 p

The term $E_2^{4,0}$ is also two-dimensional, with generators $I_1 \wedge I_2$ and $I_2 \wedge I_2$. The terms $E_2^{0,0} \cong H^0(T/T^{\sigma_I^{(4)}};\mathbf{Q}) \otimes H^0(E \times E \times E;\mathbf{Q})$ and $E_2^{6,0} \cong H^0(T/T^{\sigma_I^{(4)}};\mathbf{Q}) \otimes H^6(E \times E \times E;\mathbf{Q})$ are one-dimensional and generated by fundamental classes.

The term $E_2^{2,1}$, which is the invariant part of $H^1(T/T^{\sigma_I^{(4)}};\mathbf{Q}) \otimes H^2(E \times E \times E;\mathbf{Q})$ is 4-dimensional, with generators

$$g_{i,j} = ((Q_2 + 2Q_3) \otimes f_{j+2} + (2Q_2 + Q_3) \otimes f_{j+4}) \wedge f_i,\ i,j = 1,2.$$

In particular, it is isomorphic to $H^1(E;\mathbf{Q}) \otimes H^1(E;\mathbf{Q}) = \mathrm{Sym}^2(H^1(E;\mathbf{Q})) \oplus \bigwedge^2 H^1(E;\mathbf{Q})$. The term $E_2^{4,1}$ is also four-dimensional and generated by $(g_{i,j} \wedge I_2)$. All other $E_2^{p,1}$ vanish.

Finally, the only non-trivial terms of the form $E_2^{p,2}$ are those with $p = 2$ and $p = 4$. The subspace $E_2^{2,2} \subset H^2(T/T^{\sigma_I^{(4)}};\mathbf{Q}) \otimes H^2(E \times E \times E;\mathbf{Q})$ is isomorphic to $\mathrm{Sym}^2(H^1(E;\mathbf{Q}))$ and is generated by the invariant tensors

$$Q_2 \wedge Q_3 \otimes f_3 \wedge f_5,\ Q_2 \wedge Q_3 \otimes (f_3 \wedge f_6 + f_4 \wedge f_5),\ Q_2 \wedge Q_3 \otimes f_4 \wedge f_6.$$

Finally, the subspace $E_2^{4,2}$ is 4-dimensional and equal to $E_2^{2,2} \wedge I_1$.

In terms of local systems this gives rise to the Table IV.10. We claim that that the differentials $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ for $(p,q) = (2,1),(2,2)$ and $(4,1)$ are of maximal rank. Indeed, by Schur's lemma it is enough to prove that they are non-zero. To check this it is enough to recall that the torus bundle is isomorphic to $p_{1,3}^*(\mathcal{P}^0) \oplus p_{1,2}^*(\mathcal{P}^0)$. In particular, for every class $\alpha \in E_2^{p,1}$ we obtain $d_2^{p,1}(\alpha)$ by replacing Q_2 with $c_1(p_{1,3}^*(\mathcal{P})) = -(f_1 \wedge f_6 + f_5 \wedge f_2)$ and Q_3 with $c_1(p_{1,3}^*(\mathcal{P})) = -(f_1 \wedge f_4 + f_3 \wedge f_2)$ in the expression of α. Analogously, for every class $\beta \in E_2^{2,2}$ we get $d_2^{2,2}(\beta)$ by replacing $Q_2 \wedge Q_3$ with $Q_2 \otimes c_1(p_{1,2}^*(\mathcal{P})) - Q_3 \otimes c_1(p_{1,3}^*(\mathcal{P}))$. Then the claim follows from a straightforward calculation. $\square$

To complete the proof of Lemma IV.6.5 is now an easy consequence of the Leray spectral sequence of the fibration $p \circ q_{\sigma_I^{(4)}} : \mathcal{T}(\sigma_I^{(4)}) \to \mathcal{A}_1$. Looking at the weights of the Hodge structures, we see immediately that all differentials must vanish and thus the result follows. $\square$

IV.6.4 The cohomology of $\beta_3^0(\sigma_{II}^{(4)})$

Before we can describe the cohomology of this stratum we must identify the toric bundle $\mathcal{T}(\sigma_{II}^{(4)})$.

Lemma IV.6.7. *Let $p_{1,2}: E\times E\times E \to E\times E$ be the projection onto the first two factors and let $q: E\times E\times E \to E\times E$ be the map given by $q(x,y,z) = (x+y+z, z)$. Then*

$$\mathcal{T}(\sigma_{II}^{(4)})|_{E\times E\times E} \cong p_{1,2}^*(\mathcal{P}^0) \oplus q^*((\mathcal{P}^{-1})^0).$$

where $\mathcal{P}^0$ is the Poincaré bundle over the product $E\times E$ with the 0-section removed.

Proof. Since the generators of the cone $\sigma_{II}^{(4)}$ correspond to T_1, T_2, T_4, T_5 we can take $T_6 = t_{1,2}^{-1}$ and $T_3 = t_{3,3}t_{1,3}t_{2,3}$ as coordinates on the torus $T/T^{(\sigma_{II}^{(4)})}$. In Lemma IV.6.4 we had seen that the action of the group generated by the matrices g_3 with $M, N \in \mathbf{Z}^3$ is as follows. For $N = (n_1, n_2, n_3)$ we have $\tau_{i,4} \mapsto \tau_{i,4} + n_i$ and for $M = (m_1, m_2, m_3)$ we have $\tau_{i,j} \mapsto \tau_{i,j} + m_j\tau_{i,4} + m_i\tau_{j,4} + m_im_j\tau_{4,4}$ for $1 \le i, j \le 3$ and $\tau_{i,4} \mapsto \tau_{i,4} + m_i\tau_{4,4}$. In particular

$$\tau_{1,2} \mapsto \tau_{1,2} + m_2\tau_{1,4} + m_1\tau_{2,4} + m_1m_2\tau_{4,4}$$

whereas

$$(\tau_{1,3} + \tau_{2,3} + \tau_{3,3}) \mapsto (\tau_{1,3} + \tau_{2,3} + \tau_{3,3}) +$$
$$m_3(\tau_{1,4} + \tau_{2,4} + \tau_{3,4}) + (m_1 + m_2 + m_3)\tau_{3,4} + m_3(m_1 + m_2 + m_3)\tau_{4,4}.$$

A comparison with the transformation behaviour for the Poincaré bundle described in Lemma IV.6.4 gives the claim. ☐

Lemma IV.6.8. *The rational cohomology groups with compact support of $\beta_3^0(\sigma_{II}^{(4)})$ are given by*

$$H_c^k(\beta_3^0(\sigma_{II}^{(4)}); \mathbf{Q}) = \mathbf{Q}(-k/2), \quad k = 10, 12.$$

Proof. As in the previous case we first have to describe the automorphism $G(\sigma_{II}^{(4)})$ of the cone $\sigma_{II}^{(4)}$. This group is the symmetric group S_4 permuting the generators of the cone together with the map $x_i \mapsto -x_i$. Hence we can work with the following generators:

$$x_i \mapsto -x_i, \quad i = 1, \dots, 6 \tag{IV.6.15}$$

$$x_1 \leftrightarrow x_2, \quad x_3 \mapsto x_1 + x_2 - x_3 \tag{IV.6.16}$$

$$x_1 \mapsto x_1 - x_3, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3 \tag{IV.6.17}$$

$$x_1 \mapsto x_3 - x_2, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto x_1 - x_2. \tag{IV.6.18}$$

We now have to compute the induced action of these automorphisms on the cohomology groups $H^\bullet(T/T^{(\sigma_{II}^{(4)})}; \mathbf{Q}) \otimes H^\bullet(E\times E\times E; \mathbf{Q})$. To this end, we denote by Q_3 (respectively, R) the generator of $H^1(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q})$ corresponding to the parameter $T_6 = t_{1,2}^{-1}$ (respectively, to $T_3 = t_{1,3}t_{2,3}t_{3,3}$).

It is immediately clear that in the case of (IV.6.15) the action is given by

$$Q_3, R \mapsto Q_3, R; \tag{IV.6.19}$$

$$f_i \mapsto -f_i, \quad i = 1, \dots, 6.$$

We note that this implies that there can be no non-trivial invariant cohomology classes involving terms of odd degree in $H^\bullet(E \times E \times E)$. Next we claim that the action on cohomology of (IV.6.16) is given by

$$Q_3 \mapsto Q_3 - R, \ R \mapsto -R \tag{IV.6.20}$$

$$f_i \mapsto f_{i+2} + f_{i+4}, \quad i = 1, 2; \quad f_i \mapsto f_{i-2} + f_{i+2}, \quad i = 3, 4; \quad f_i \mapsto -f_i, \quad i = 5, 6.$$

To see this we compute

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\ \tau_{1,2} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\ \tau_{1,3} & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\ \tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} * & \tau_{1,2} + \tau_{2,3} + \tau_{1,3} + \tau_{3,3} & -\tau_{2,3} - \tau_{3,3} & \tau_{2,4} + \tau_{3,4} \\ * & * & -\tau_{1,3} - \tau_{3,3} & \tau_{1,4} + \tau_{3,4} \\ * & * & \tau_{3,3} & -\tau_{3,4} \\ * & * & * & \tau_{4,4} \end{pmatrix}.$$

This immediately gives the claim for the f_i. For Q_3, R we observe that the action induced on the homology is dual to the action on the subspace $\langle -\tau_{1,2}, \tau_{1,3} + \tau_{2,3} + \tau_{3,3}$. Since cohomology is dual to homology, the action on Q_3, R agrees with that on $-\tau_{1,2}, \tau_{1,3} + \tau_{2,3} + \tau_{3,3}$.

A similar calculation gives the following results in the remaining cases:

$$Q_3 \mapsto -Q_3, \ R \mapsto -Q_3 + R; \tag{IV.6.21}$$

$$f_i \mapsto f_i, \quad i = 1, 2; \quad f_i \mapsto -f_i, \quad i = 3, 4; \quad f_i \mapsto -f_{i-4} - f_i, \quad i = 5, 6.$$

$$Q_3 \leftrightarrow R; \tag{IV.6.22}$$

$$f_i \mapsto f_{i+4}, \quad i = 1, 2; \quad f_i \mapsto -f_{i-2} - f_i - f_{i+2}, \quad i = 3, 4; \quad f_i \mapsto f_{i-4}, \quad i = 5, 6.$$

It is now straightforward to compute the invariants under $G(\sigma_{II}^{(4)})$. In the cohomology group $H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^2(E \times E \times E;\mathbf{Q})$ we find one invariant tensor, namely

$$I_1 = 3(f_1 \wedge f_2 + f_3 \wedge f_4) + 2\varphi + \psi,$$

Table IV.11: E_2 term of the spectral sequence converging to the cohomology of $\beta_3^0(\sigma_{II}^{(4)})$

q							
2	0	0	0	0	0	0	0
1	0	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$	0	0
0	$\mathbf{Q}$	0	$\mathbf{Q}(-1)$	0	$\mathbf{Q}(-2)$	0	$\mathbf{Q}(-3)$
	0	1	2	3	4	5	6 p

where we denoted $\varphi = (f_1+f_3+f_5)\wedge f_6+f_5\wedge(f_2+f_4+f_6)$ and $\psi = f_1\wedge f_4+f_3\wedge f_2$. In $H^1(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^2(E\times E\times E;\mathbf{Q})$ we also obtain one invariant tensor, namely

$$I_2 = -R\otimes(2\varphi+\psi)+Q\otimes(\varphi+2\psi).$$

The invariant class in $H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^4(E\times E\times E;\mathbf{Q})$ is $I_1\wedge I_1$ and in $H^1(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^4(E\times E\times E;\mathbf{Q})$ it is $I_2\wedge I_1$. This together with the fundamental classes in $H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^0(E\times E\times E;\mathbf{Q})$ and $H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q}) \otimes H^6(E\times E\times E;\mathbf{Q})$ are the only invariants.

As before we now look at the Leray spectral sequence in cohomology associated with $p\circ q_{\sigma_{II}^{(4)}}\colon \mathcal{T}(\sigma_{II}^{(4)}) \to \mathcal{A}_1$. Since all representations are trivial we thus obtain Table IV.11. Hence, we have two differentials which could be non-zero, namely

$$d_2^{2,1}\colon\ H^1(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q})\otimes H^2(E\times E\times E;\mathbf{Q}) \to H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q})\otimes H^4(E\times E\times E;\mathbf{Q}),$$

resp.

$$d_2^{4,1}\colon\ H^1(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q})\otimes H^4(E\times E\times E;\mathbf{Q}) \to H^0(T/T^{(\sigma_{II}^{(4)})};\mathbf{Q})\otimes H^6(E\times E\times E;\mathbf{Q}).$$

Indeed we claim that they do not vanish. For this we use the description of $T(\sigma_{II}^{(4)})|_{E\times E\times E}$ given in Lemma IV.6.7. It follows from this description that this bundle splits into the product of two factors with Euler classes $-\varphi$ and ψ. The claim that the first differential is non-zero is now equivalent to

$$\varphi\wedge(2\varphi+\psi)+\psi\wedge(\varphi+2\psi)\neq 0.$$

For the second differential we must check that

$$(\varphi\wedge(2\varphi+\psi)+\psi\wedge(\varphi+2\psi))\wedge I_1\neq 0.$$

This can be checked by direct calculation. At the same time this proves that the first differential does not vanish. The claim of the lemma now follows immediately after converting to cohomology with compact support. $\square$

IV.6.5 The cohomology of $\beta_3^0(\sigma^{(5)})$

Lemma IV.6.9. *The rational cohomology groups with compact support of $\beta_3^0(\sigma^{(5)})$ are given by*

$$H_c^k(\beta_3^0(\sigma^{(5)}); \mathbf{Q}) = \begin{cases} \mathbf{Q}(-k/2) & k = 6, 10 \\ \mathbf{Q}(-k/2)^{\oplus 2} & k = 8 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We first have to compute the automorphism group $G(\sigma^{(5)})$. It is not hard to see that this group is generated by the transformations

$$x_i \mapsto -x_i, \quad i = 1, 2, 3 \tag{IV.6.23}$$

$$x_1 \leftrightarrow x_2, \quad x_3 \mapsto x_3 \tag{IV.6.24}$$

$$x_1 \mapsto x_1 - x_3, \quad x_2 \mapsto x_2 - x_3, \quad x_3 \mapsto -x_3 \tag{IV.6.25}$$

$$x_1 \mapsto x_1 - x_3, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3. \tag{IV.6.26}$$

A computation analogous to that in Lemma IV.6.5 shows that this results in the following action on cohomology, where again we denoted by f_i the generators of the cohomology of $E \times E \times E$ and by Q_3 the generator of the cohomology of the fibre of the torus bundle:

$$f_i \mapsto -f_i, i = 1, \ldots, 6; \quad Q_3 \mapsto Q_3 \tag{IV.6.27}$$

$$f_i \leftrightarrow f_{i+2}, i = 1, 2; \quad f_j \mapsto f_j, j = 5, 6; \quad Q_3 \mapsto Q_3 \tag{IV.6.28}$$

$$f_i \mapsto f_i, i = 1, \ldots, 4; \quad f_k \mapsto -f_{k-4} - f_{k-2} - f_k, k = 5, 6; \quad Q_3 \mapsto Q_3 \tag{IV.6.29}$$

$$f_i \mapsto f_i, i = 1, 2; \quad f_j \mapsto -f_j, j = 3, 4; \quad f_k \mapsto -f_{k-4} - f_k, k = 5, 6; \quad Q_3 \mapsto -Q_3. \tag{IV.6.30}$$

Next, we compute the invariant cohomology in $H^0(\mathbf{C}^*; \mathbf{Q}) \otimes H^{2k}(E \times E \times E; \mathbf{Q})$. Clearly this is 1-dimensional for $k = 0, 6$. By duality it is enough to do the computation for $k = 2$. Here we find a 2-dimensional invariant subspace generated by $i_1 := f_1 \wedge f_2 + f_3 \wedge f_4$ and $i_2 := f_1 \wedge f_4 + f_3 \wedge f_2 + 2(f_1 + f_3 + f_5) \wedge f_6 + 2f_5 \wedge (f_2 + f_4 + f_6)$.

In this situation we also have invariant cohomology in $H^1(\mathbf{C}^*; \mathbf{Q}) \otimes H^2(E \times E \times E; \mathbf{Q})$. This is 1-dimensional and generated by $Q_3 \otimes (f_1 \wedge f_4 + f_3 \wedge f_2)$. By duality we also have a 1-dimensional invariant subspace in $H^1(\mathbf{C}^*) \otimes H^4(E \times E \times E)$. A standard calculation shows that this is generated by $Q_3 \wedge (f_1 \wedge f_4 + f_3 \wedge f_2) \wedge i_2$.

In this case the differentials in the Leray spectral sequence are not automatically 0. The situation is described in Table IV.12. here are two differentials which we have to consider. These are:

$$d_2^{2,1} : H^1(\mathbf{C}^*; \mathbf{Q}) \otimes H^2(E \times E \times E; \mathbf{Q}) \to H^0(\mathbf{C}^*; \mathbf{Q}) \otimes H^4(E \times E \times E; \mathbf{Q}),$$

Table IV.12: E_2 term of the spectral sequence converging to the cohomology of $\beta_3^0(\sigma^{(5)})$

$$
\begin{array}{c|cccccccc}
q & & & & & & & & \\
1 & 0 & 0 & \mathbf{Q}(-2) & 0 & \mathbf{Q}(-3) & 0 & 0 & \\
0 & \mathbf{Q} & 0 & \mathbf{Q}(-1)^{\oplus 2} & 0 & \mathbf{Q}(-2)^{\oplus 2} & 0 & \mathbf{Q}(-3) & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & p
\end{array}
$$

resp.

$$d_2^{4,1}: H^1(\mathbf{C}^*;\mathbf{Q}) \otimes H^4(E\times E\times E;\mathbf{Q}) \to H^0(\mathbf{C}^*;\mathbf{Q}) \otimes H^6(E\times E\times E;\mathbf{Q}).$$

We claim that both differentials are non-zero, i.e. they have rank 1. We first treat $d_2^{1,2}$. The differential is given by taking the cup-product with the first Chern class of the vector bundle spanned by the torus bundle $\mathcal{T}(\sigma^{(5)})|_{E\times E\times E}$. As in previous cases on can see that $\mathcal{T}(\sigma^{(5)})|_{E\times E\times E} \cong p_{12}^*(\mathcal{P}^0)$. This shows that

$$d_2^{1,2}\colon\ Q_3 \otimes (f_1\wedge f_4 + f_3\wedge f_2) \mapsto$$

$$(f_1\wedge f_4 + f_3\wedge f_2)\wedge(f_1\wedge f_4 + f_3\wedge f_2) = 2f_1\wedge f_2\wedge f_3\wedge f_4 \neq 0.$$

The argument for $d_2^{1,4}$ is analogous. Finally we use the duality $H_c^k(\beta_3^0(\sigma^{(5)});\mathbf{Q}) = H^{10-k}(\beta_3^0(\sigma^{(5)});\mathbf{Q})^* \otimes \mathbf{Q}(-5)$ (which holds on finite smooth covers) to obtain the claim.

□

IV.6.6 The cohomology of $\beta_3^0(\sigma^{(6)})$

Lemma IV.6.10. *The rational cohomology groups with compact support of $\beta_3^0(\sigma^{(6)})$ are given by*

$$H_c^k(\beta_3^0(\sigma^{(6)});\mathbf{Q}) = \begin{cases} \mathbf{Q}(-k/2) & k = 2,4,6,8 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof of this lemma is analogous to the other cases. We first note that the automorphism group of $G(\sigma^{(6)})$ is generated by the symmetric group in three variables permuting the coordinates x_i $(i = 1,2,3)$ and the transformations (IV.6.23) and (IV.6.25) already considered in the previous section. In this case the torus rank is 0 and hence it suffices to compute the action on the cohomology of the triple product $E\times E\times E$. In view of the transformation (IV.6.23) there is no invariant in odd degree. By duality it is enough to compute the invariant cohomology in $H^2(E\times E\times E;\mathbf{Q})$. A straightforward calculation shows that this is 1-dimensional with generator $2v_1+v_2+v_3$, with $v_1 = f_1\wedge f_2 + f_3\wedge f_4 + f_5\wedge f_6$, $v_2 = f_1\wedge f_4 + f_3\wedge f_6 + f_5\wedge f_2$ and $v_3 = f_1\wedge f_6 + f_3\wedge f_2 + f_5\wedge f_4$. □

Table IV.13: Gysin spectral sequence converging to the cohomology with compact support of β_3^0

q				
10	0	0	0	$\mathbf{Q}(-7)$
9	0	0	$\mathbf{Q}(-6)^{\oplus 2}$	0
8	0	$\mathbf{Q}(-5)$	0	$\mathbf{Q}(-6)$
7	$\mathbf{Q}(-4)$	0	$\mathbf{Q}(-5)^{\oplus 3}$	0
6	0	$\mathbf{Q}(-4)^{\oplus 2}$	0	0
5	$\mathbf{Q}(-3)$	0	$\mathbf{Q}(-4)+\mathbf{Q}(-2)$	$\mathbf{Q}(-2)$
4	0	$\mathbf{Q}(-3)$	0	0
3	$\mathbf{Q}(-2)$	0	0	$\mathbf{Q}(-1)$
2	0	0	$\mathbf{Q}$	0
1	$\mathbf{Q}(-1)$	0	0	0
	1	2	3	4 p

IV.6.7 The cohomology of β_3^0

In this section, we will use the computations on the strata of β_3^0 to prove the following result.

Proposition IV.6.11. *The rational cohomology with compact support of β_3^0 is as follows: the non-zero Betti numbers are*

i	2	4	5	6	7	8	10	12	14
b_i	1	1	1	2	1	4	4	3	1

One has $H^7_c(\beta_3^0;\mathbf{Q})=\mathbf{Q}(-1)$ and $H^5_c(\beta_3^0;\mathbf{Q})=\mathbf{Q}$. Furthermore all cohomology groups of even degree are algebraic.

Proof. We consider the Gysin spectral sequence associated with the stratification of β_3^0 given by the locally closed strata $W_1=\beta_3^0(\sigma^{(6)})$, $W_2=\beta_3^0(\sigma^{(5)})$, $W_3=\beta_3^0(\sigma_I^{(4)})\cup\beta_3^0(\sigma_{II}^{(4)})$ and $W_4=\beta_3^0(\sigma^{(3)})$. We set $Y_p=\overline{W}_p$. This is the spectral sequence $E_\bullet^{p,q}\Rightarrow H_c^{p+q}(\beta_3^{(0)};\mathbf{Q})$ with $E_1^{p,q}=H_c^{p+q}(Y_p\setminus Y_{p-1};\mathbf{Q})=H_c^{p+q}(W_p;\mathbf{Q})$. The cohomology with compact support of the strata W_i was computed in the Lemmas IV.6.2, IV.6.5, IV.6.8, IV.6.9 and IV.6.10. In view of these results, the E_1 term of the Gysin spectral sequence is as given in Table IV.13. We consider the differentials $d_r^{p,q}\colon E_r^{p,q}\to E_r^{p+r,q-r+1}$. Inspection of Table IV.13 shows that the only possible non-zero differential is $d_1^{3,5}\colon E_1^{3,5}=H_c^8(\beta_3^0(\sigma_I^{(4)});\mathbf{Q})\to E_1^{4,5}=H_c^9(\beta_3^0(\sigma^{(3)});\mathbf{Q})$. We can interpret this differential as arising from the Gysin long exact sequence associated with the inclusion of $\beta_3^0(\sigma_I^{(4)}$ in the partial compactification $\beta_3^0(\sigma^{(3)})\cup\beta_3^0(\sigma_I^{(4)})$ of $\beta_3^0(\sigma^{(3)})$. Let us denote by Ψ_E the fibre of the fibration $\beta_3^0(\sigma^{(3)}) \cup \beta_3^0(\sigma_I^{(4)}) \to \mathcal{A}_1$ over a point $[E] \in \mathcal{A}_1$. Thanks to the Leray spectral sequence associated with that fibration, all we need to know is that the cohomology with compact support of Ψ_E vanishes in degree 7. This requires to prove that the differential

$$d_7: \ H_c^7((p \circ q_{\sigma_I^{(4)}})^{-1}([E]); \mathbf{Q}) \to H_c^8((p \circ q_{\sigma^{(3)}})^{-1}([E]); \mathbf{Q})$$

in the Gysin long exact sequence associated with $(p \circ q_{\sigma_I^{(4)}})^{-1}([E]) \subset \Psi_E$ is an isomorphism.

Since in the proofs of Lemma IV.6.4 and Lemma IV.6.6 we described the generators of the cohomology of the fibres of E rather than those of the cohomology with compact support, we shall analyze the map induced by d_7 on cohomology by Poincaré duality, whose rank coincides with that of d_7. Let us recall that the inclusion of $\{0\} \times (\mathbf{C}^*)^2$ in $\mathbf{C} \times (\mathbf{C}^*)^2$ induces a Gysin long exact sequence whose differentials define the maps

$$\begin{array}{ccc} H^k((\mathbf{C}^*)^2; \mathbf{Q}) \otimes \mathbf{Q}(-1) & \longrightarrow & H^{k+1}((\mathbf{C}^*)^3; \mathbf{Q}) \\ T_i & \longmapsto & T_i \wedge T_1 \qquad\qquad i = 2, 3. \end{array}$$

As a consequence, the differential $H^3((p \circ q_{\sigma_I^{(4)}})^{-1}([E]); \mathbf{Q}) \otimes \mathbf{Q}(-1) \to H^4((p \circ q_{\sigma^{(3)}})^{-1}([E]); \mathbf{Q})$ maps each of the generators $g_{i,j}$ described in the proof of Lemma IV.6.6 to the class obtained by replacing T_2 by $T_2 \wedge T_1$ in the expression, and then symmetrizing for the action of the group $G(\sigma^{(3)})$. This yields:

$$g_{i,j} \longmapsto \frac{2}{3} \sum_{0 \le k, l \le 2} Q_{k+1} \wedge Q_{l+1} \otimes f_{2k+i} \wedge f_{2l+j},$$

hence in particular the differential is surjective. From this the claim follows. □

IV.7 Torus rank 4

In this section we compute the cohomology of the closed strata $\beta_4 \subset \mathcal{A}_4^{\mathrm{Vor}}$ and $\beta_4^{\mathrm{perf}} \subset \mathcal{A}_4^{\mathrm{perf}}$ of torus rank 4 in the second Voronoi and the perfect cone compactification, respectively.

We shall first state the main results:

Theorem IV.7.1. *The cohomology groups with rational coefficients of the closed stratum $\beta_4^{\mathrm{perf}} \subset \mathcal{A}_4^{\mathrm{perf}}$ of the perfect cone compactification of the moduli space of abelian varieties of dimension 4 are non-zero only in even degree. The only non-zero Betti numbers are $b_0 = b_2 = b_4 = 1$, $b_6 = b_8 = 4$, $b_{10} = 3$ and $b_{12} = 1$.*

The cohomology is algebraic in all degrees different from 6, whereas $H^6(\beta_4^{\mathrm{perf}}; \mathbf{Q})$ is an extension of $\mathbf{Q}(-3)^{\oplus 3}$ by $\mathbf{Q}(-1)$.

The closed stratum $\beta_4 \subset \mathcal{A}_4^{\mathrm{Vor}}$ has two irreducible components: a nine-dimensional component E, which is the exceptional divisor of the blow-up $q\colon \mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{perf}}$, and a six-dimensional component, which is the proper transform of β_4^{perf} under q.

Theorem IV.7.2. *1. The rational cohomology of E is all algebraic. The only non-zero Betti numbers are $b_0 = b_2 = b_{16} = b_{18} = 1$, $b_4 = b_{14} = 2$ and $b_6 = b_8 = b_{10} = b_{12} = 3$.*

2. The rational cohomology of β_4 is non-trivial only in even degree. The non-zero Betti numbers are

i	0	2	4	6	8	10	12	14	16	18
b_i	1	2	3	7	7	6	4	2	1	1

All cohomology groups are algebraic, with the exception of $H^6(\beta_4;\mathbf{Q})$, which is an extension of $\mathbf{Q}(-3)^{\oplus 6}$ by $\mathbf{Q}(-1)$.

IV.7.1 Cone decompositions

It is in this section that we require full information about the perfect cone or first Voronoi and the second Voronoi decomposition in $\mathrm{Sym}^2_{\geq 0}(\mathbf{R}^4)$. Details concerning these decompositions can be found in [ER1], [ER2], [Val] and [Vor]. We start by recalling the perfect cone decomposition. The starting point is two 10-dimensional cones, namely the principal cone $\Pi_1(4)$ and the second perfect cone $\Pi_2(4)$. These cones are given by

$$\begin{aligned}\Pi_1(4) = \langle & x_1^2, x_2^2, x_3^2, x_4^2, (x_1-x_2)^2, (x_1-x_3)^2, (x_1-x_4)^2, (x_2-x_3)^2,\\ & (x_2-x_4)^2, (x_3-x_4)^2\rangle\end{aligned}$$

and

$$\begin{aligned}\Pi_2(4) = \langle & x_1^2, x_2^2, x_3^2, x_4^2, (x_1-x_3)^2, (x_1-x_4)^2, (x_2-x_3)^2, (x_2-x_4)^2,\\ & (x_3-x_4)^2, (x_1+x_2-x_3)^2, (x_1+x_2-x_4)^2, (x_1+x_2-x_3-x_4)^2\rangle\end{aligned}$$

respectively. The perfect cone decomposition consists of all $\mathrm{GL}(4,\mathbf{Z})$-translates of these cones and their faces. While the cone $\Pi_1(4)$ is basic, the cone $\Pi_2(4)$ is not, hence it defines a singular point $P_{\mathrm{sing}} \in \mathcal{A}_4^{\mathrm{perf}}$. Nevertheless, all 9-dimensional faces of $\Pi_2(4)$ are basic. Modulo the action of $\mathrm{GL}(4,\mathbf{Z})$ these 9-dimensional faces define two orbits. Traditionally these are called RT (red triangle) and BF (black face) respectively (see [ER2]).

In genus 4 and 5 (but not in general) the second Voronoi decomposition is a subdivision of the perfect cone decomposition. In our case it is the refinement of the perfect cone decomposition obtained by adding all cones that arise as spans of the 9-dimensional faces of $\Pi_2(4)$ with the central ray generated by

$$e = \frac{1}{3}\left[x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 \right.$$
$$\left. +(x_3 - x_4)^2 + (x_1 + x_2 - x_3)^2 + (x_1 + x_2 - x_4)^2 + (x_1 + x_2 - x_3 - x_4)^2\right]. \tag{IV.7.1}$$

In particular, all perfect cones, with the exception of $\Pi_2(4)$, belong to the second Voronoi decomposition. Geometrically this means that $\mathcal{A}_4^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_4^{\mathrm{perf}} = \mathcal{A}_4^{\mathrm{Igu}}$ in the singular point P_{sing}. Since all cones on the second Voronoi decomposition are basic $\mathcal{A}_4^{\mathrm{Vor}}$ is smooth (as a stack). Moreover, the exceptional divisor E is irreducible and smooth (again as a stack). (For a discussion of this see also [HS2].

A description of representatives of all $\mathrm{GL}(4, \mathbf{Z})$-orbits of cones in the second Voronoi, and hence also the perfect cone decomposition, can be found in [Val, Chapter 4]. For cones with extremal rays spanned by quadratic forms of rank 1 the list is given in [Val, S.4.4.4]. Note that in this list K_5 denotes the cone $\Pi_2(4)$, and the 9-dimensional cones $K_5 - 1$ and $K_{3,3}$ correspond to the equivalence classes BF, respectively, RT of [ER2]. The remaining cones are listed in [Val, S.4.4.5]. The following list gives the number of $\mathrm{GL}(4, \mathbf{Z})$-orbits of cones in each dimension for the two decompositions.

dimension	1	2	3	4	5	6	7	8	9	10
# perfect cones	1	1	2	3	4	5	4	2	2	2
# second Voronoi cones	2	2	4	7	9	11	11	7	4	3

From this we see that the perfect cone decomposition has 26 different cones, whereas the second Voronoi decomposition has 60 different cones. The lists in [Val] also allow us to write down generators for the extremal rays of representatives in all cases.

IV.7.2 Plan for computation

We briefly recall the structure of β_4 and β_4^{perf} which comes from the toroidal construction. More generally, let β_4^{Σ} be the stratum of any admissible fan Σ (in our case either the perfect cone or the second Voronoi fan), then each cone $\sigma \in \Sigma$ defines a torus orbit T_σ of dimension $10 - k$ where k is the dimension of σ. Let $G_\sigma \subset \mathrm{GL}(4, \mathbf{Z})$ be the stabilizer of σ with respect to the natural action of $\mathrm{GL}(4, \mathbf{Z})$ on $\mathrm{Sym}^2_{\geq 0}(\mathbf{R}^4)$. Then G_σ acts on T_σ and β_4^{Σ} is the disjoint union of the quotients $Z_\sigma = T_\sigma / G_\sigma$ where σ runs through a set of representatives of all cones in Σ which contain a form of rank 4 in their interior. We then define a stratification by defining S_p as the union of all Z_σ where $\dim \sigma \geq 10 - p$. In particular, $S_p \setminus S_{p-1}$ is the union of all Z_σ with $\dim \sigma = 10 - p$.

The Gysin spectral sequence $E_{\bullet}^{p,q} \Rightarrow H_c^{p+q}(\beta_4^{\mathrm{perf}}; \mathbf{Q}) = H^{p+q}(\beta_4^{\mathrm{perf}}; \mathbf{Q})$ associated with the filtration S_p has E_1 term given by

$$E_1^{p,q} = H_c^{p+q}(S_p \setminus S_{p-1}; \mathbf{Q}).$$

Since $S_p \setminus S_{p-1}$ is the disjoint union of the Z_σ with $\dim \sigma = 10 - p$ it follows that

$$H_c^{\bullet}(S_p \setminus S_{p-1}; \mathbf{Q}) = \bigoplus_{\dim \sigma} H_c^{\bullet}(Z_\sigma; \mathbf{Q}).$$

In our situation we have considerably more information. In particular we know that, with the exception of $\Pi_2(4)$, all cones in both the perfect cone and the second Voronoi decomposition, are basic. In particular all strata S_p with $p \leq 9$ are locally quotients of a smooth variety by a finite group. Moreover $T_\sigma = (\mathbf{C}^*)^{10-\dim \sigma}$ and $Z_\sigma = (\mathbf{C}^*)^{10-\dim\sigma}/G_\sigma$. The torus orbit of $\Pi_2(4)$ is a point. Thus we have to compute for each cone σ the cohomology of the torus T_σ with respect to G_σ. Recall that $H^{\bullet}((\mathbf{C}^*)^k; \mathbf{Q})$ is the exterior algebra generated by the k-dimensional vector space $H^1((\mathbf{C}^*)^k; \mathbf{Q})$. Moreover, a basis of the vector space $H^1((\mathbf{C}^*)^k; \mathbf{Q})$ can be obtained by taking the Alexander dual classes of the fundamental classes of the components

$$\{(y_1, \dots, y_k) | y_i = 0\}, \ i = 1, \dots, k$$

of the complement of $(\mathbf{C}^*)^k$ in $\mathbf{C}^k$. This means that, once the generators of the cone σ and of the group G_σ are known, the computation of the cohomology of Z_σ reduces to a linear algebra problem, which can be solved using computational tools. In our case, the generators of the stabilizers G_σ were calculated with Magma ([BCP]) and the invariant part of the algebra $\bigwedge^{\bullet} H^1((\mathbf{C}^*)^k; \mathbf{Q})$ with Singular ([GPS]).

IV.7.3 Perfect cones

We shall now perform the programme outlined above for the perfect cone compactification, which coincides with the Igusa compactification in genus 4. We have already mentioned that a list of representatives of all cones in the perfect cone decomposition, together with their generators, can be found in [Val, Ch. 4]. This enables us to compute the stabilizer groups G_σ as well as the invariant cohomology of the torus orbits $T_\sigma = (\mathbf{C}^*)^k$ where $k = 10 - \dim \sigma$. The results so obtained are listed in Table IV.14, where the notation for the cones is the one of [Val, §4]. The information on the cohomology of the strata is given in the form of Hodge Euler characteristics, i.e. what is given is the Euler characteristic of $H_c^{\bullet}(Z_\sigma; \mathbf{Q})$ in the Grothendieck group of Hodge structures. The symbol $\mathbf{L}$ denotes the class of the weight 2 Tate Hodge structure $\mathbf{Q}(-1)$ in the Grothendieck group.

Table IV.14: $\mathrm{GL}(4,\mathbf{Z})$-orbits of perfect cones

Σ	$\dim\Sigma$	$\mathbf{e}_{\mathrm{Hdg}}(Z_\Sigma)$	Σ	$\dim\Sigma$	$\mathbf{e}_{\mathrm{Hdg}}(Z_\Sigma)$
$K_5=\Pi_1(4)$	10	1	K_4+1	7	$\mathbf{L}^3$
$\Pi_2(4)$	10	1	C_{222}	6	$\mathbf{L}^4-\mathbf{L}^3$
K_5-1	9	$\mathbf{L}$	C_{321}	6	$\mathbf{L}^4+1$
$K_{3,3}$	9	$\mathbf{L}-1$	$C_{221}+1$	6	$\mathbf{L}^4$
K_5-2	8	$\mathbf{L}^2$	C_3+C_3	6	$\mathbf{L}^4$
K_5-1-1	8	$\mathbf{L}^2-\mathbf{L}$	C_5	5	$\mathbf{L}^5-1$
K_5-2-1	7	$\mathbf{L}^3-\mathbf{L}^2$	C_4+1	5	$\mathbf{L}^5$
C_{2221}	7	$\mathbf{L}^3$	C_3+1+1	5	$\mathbf{L}^5+\mathbf{L}$
K_5-3	7	$\mathbf{L}^3$	$1+1+1+1$	4	$\mathbf{L}^6$

Table IV.15: E_1 term of the spectral sequence converging to $H^\bullet(\beta_4^{\mathrm{perf}};\mathbf{Q})$.

q							
6	0	0	0	0	0	0	$\mathbf{Q}(-6)$
5	0	0	0	0	0	$\mathbf{Q}(-5)^3$	0
4	0	0	0	0	$\mathbf{Q}(-4)^4$	0	0
3	0	0	0	$\mathbf{Q}(-3)^4$	$\mathbf{Q}(-3)$	0	0
2	0	0	$\mathbf{Q}(-2)^2$	$\mathbf{Q}(-2)$	0	0	0
1	0	$\mathbf{Q}(-1)^2$	$\mathbf{Q}(-1)$	0	0	$\mathbf{Q}(-1)$	0
0	$\mathbf{Q}^2$	$\mathbf{Q}$	0	0	$\mathbf{Q}$	$\mathbf{Q}$	0
	0	1	2	3	4	5	6 p

The relationship between cohomology and cohomology with compact support is given by Poincaré duality:

$$H^l_c(Z_\sigma;\mathbf{Q})=\mathrm{Hom}(H^{2k-l}(Z_\sigma;\mathbf{Q}),\mathbf{Q}(-k)),$$

which holds since the Z_σ are finite quotients of the smooth varieties T_σ.

In view of the information on the cohomology of the Z_σ given in Table IV.14, this yields that the E_1 terms of the spectral sequence $E_\bullet^{p,q}\Rightarrow H^{p+q}(\beta_4^{\mathrm{perf}};\mathbf{Q})$ are as shown in Table IV.15.

To establish Theorem IV.7.1, we need to determine the rank of all differentials in the spectral sequence. As morphisms between pure Hodge structures of different weights are necessarily trivial, one remains with five differentials to investigate, all of the form $d_1^{p,q}\colon E_1^{p,q}\to E_1^{p+1,q}$. We will denote them by

$$\delta_0\colon E_1^{0,0}\longrightarrow E_1^{1,0},\quad \delta_0'\colon E_1^{4,0}\longrightarrow E_1^{5,0},$$
$$\delta_1\colon E_1^{1,1}\longrightarrow E_1^{2,1},\quad \delta_2\colon E_1^{2,2}\longrightarrow E_1^{3,2},\quad \delta_3\colon E_1^{3,3}\longrightarrow E_1^{4,3}.$$

Lemma IV.7.3. *All the differentials $\delta_0,\delta_0',\delta_1,\delta_2$ and δ_3 have rank* 1.

Proof. Since β_4^{perf} is connected, one has $H^0(\beta_4^{\mathrm{perf}};\mathbf{Q})=\mathbf{Q}$. This implies that δ_0 has rank 1.

Next, we consider the differential $\delta_0'\colon E_1^{4,0}\cong\mathbf{Q}\longrightarrow E_1^{5,0}\cong\mathbf{Q}$. From the description of the strata given in Table IV.14, we have $E_1^{4,0}=H_c^4(Z_{C_{321}};\mathbf{Q})$ and $E_1^{5,0}=H_c^5(Z_{C_5};\mathbf{Q})$ for the cones

$$C_5=\langle x_1^2,x_2^2.(x_1-x_4)^2,(x_2-x_3)^2,(x_3-x_4)^2\rangle,$$
$$C_{321}=\langle x_1^2,x_2^2,x_4^2,(x_1-x_4)^2,(x_2-x_3)^2,(x_3-x_4)^2\rangle.$$

The cone C_5 is contained in C_{321}, hence $Z_{C_{321}}$ is contained in the closure of Z_{C_5}. Furthermore, the rank of δ_0' must coincide with the rank of the differential $\eta_0\colon H_c^4(Z_{C_{321}};\mathbf{Q})\to H_c^5(Z_{C_5};\mathbf{Q})$ of the Gysin long exact sequence associated with the inclusion of $Z_{C_{321}}$ in the partial compactification $Z_{C_5}\cup Z_{C_{321}}$ of Z_{C_5}.

If one considers the stabilizers, one observes that $G_{C_{321}}$ is a subgroup of G_{C_5}. Therefore, one can view η_0 as a map from the cohomology of $(\mathbf{C}^*)^4$ to the cohomology of $(\mathbf{C}^*)^5$ in the following way:

$$\begin{tikzcd}
H_c^4(Z_{C_{321}};\mathbf{Q}) \arrow[d, "\eta_0"] \arrow[r, equal] & H_c^4((\mathbf{C}^*)^4;\mathbf{Q})^{G_{C_{321}}} \arrow[dr, "\eta_0"] & \\
H_c^5(Z_{C_5};\mathbf{Q}) \arrow[r, equal] & H_c^5((\mathbf{C}^*)^5;\mathbf{Q})^{G_{C_5}} \arrow[r, equal] & H_c^5((\mathbf{C}^*)^5;\mathbf{Q})^{G_{C_{321}}},
\end{tikzcd}$$

where we used the fact that the $G_{C_{321}}$-invariant part of $H_c^5((\mathbf{C}^*)^5;\mathbf{Q})$ coincides with the G_{C_5}-invariant part. This new interpretation relates the map η_0 to the differential

$$H_c^4((\mathbf{C}^*)^4;\mathbf{Q})\cong\mathbf{Q}\longrightarrow H_c^5((\mathbf{C}^*)^5;\mathbf{Q})\cong\mathbf{Q}\tag{IV.7.2}$$

of the Gysin exact sequence of an inclusion $(\mathbf{C}^*)^4\hookrightarrow\mathbf{C}\times(\mathbf{C}^*)^4$, with complement isomorphic to $(\mathbf{C}^*)^5$. In particular, since $H_c^k(\mathbf{C}\times(\mathbf{C}^*)^4;\mathbf{Q})$ vanishes for $k\le 5$, the differential (IV.7.2) is an isomorphism, and the same holds for η_0.

Let us consider the differential

$$\delta_1\colon\ E_1^{1,1}\cong H_c^2(Z_{K_5-1};\mathbf{Q})\oplus H_c^2(Z_{K_{3,3}};\mathbf{Q})\to E_1^{2,1}\cong H_c^3(Z_{K_5-1-1};\mathbf{Q}).$$

Note that both Z_{K_5-1} and $Z_{K_{3,3}}$ are contained in the closure of $Z_{K_5-1-1}\subset\beta_4^{\mathrm{perf}}$. We choose to investigate the inclusion $i_{3,3}$ of $Z_{K_{3,3}}$ in the partial compactification $Z_{K_{3,3}}\cup Z_{K_5-1-1}$ of Z_{K_5-1-1}. Then the rank of δ_1 cannot be smaller than the rank of the differential

$$\eta_1\colon\ H_c^2(Z_{K_{3,3}};\mathbf{Q})\longrightarrow H_c^3(Z_{K_5-1-1};\mathbf{Q})$$

in the Gysin long exact sequence associated with $i_{3,3}$, even though there is no canonical isomorphism between the kernel of η_1 and that of δ_1.

In Vallentin's notation, the cone $K_{3,3}$ is given by

$$\begin{aligned}K_{3,3}=\ &\langle x_1^2,x_2^2,x_3^2,x_4^3,(x_1-x_3)^2,(x_1-x_4)^2,(x_2-x_3)^2,(x_2-x_4)^2,\\ &(x_1+x_2-x_3-x_4)^2\rangle.\end{aligned}$$

In particular, its subcone

$$K_5 - 1 - 1b = \langle x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2 \rangle$$

belongs to the same $\mathrm{GL}(4, \mathbf{Z})$-orbit as $K_5 - 1 - 1$, so that $Z_{K_5-1-1b} \subset \beta_4^{\mathrm{perf}}$ coincides with Z_{K_5-1-1}. The stabilizer G_{K_5-1-1b} of the cone $K_5 - 1 - 1b$ is generated by $-\mathrm{Id}_{\mathbf{Z}^4}$ and by the two automorphisms

$$\begin{cases} x_1 \mapsto x_3 \\ x_2 \mapsto x_4 \\ x_3 \mapsto x_2 \\ x_4 \mapsto x_1 \end{cases} \text{and} \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_4 \\ x_4 \mapsto x_3 \end{cases}.$$

In particular, one can check that the group G_{K_5-1-1b} is contained in the stabilizer $G_{K_{3,3}}$ of the cone $K_{3,3}$.

Analogously to the case of η_0, we can reduce the study of η_1 to the study of the long exact sequence of an inclusion $\mathbf{C}^* \hookrightarrow \mathbf{C} \times \mathbf{C}^*$ with complement isomorphic to $(\mathbf{C}^*)^2$, by exploiting the diagram

$$\begin{tikzcd} H_c^2(Z_{K_{3,3}}; \mathbf{Q}) \arrow[r, equal] \arrow[d, "\eta_1"] & H_c^2(\mathbf{C}^*; \mathbf{Q})^{G_{K_{3,3}}} \arrow[r, equal] & H_c^2(\mathbf{C}^*; \mathbf{Q})^{G_{K_5-1-1b}} \arrow[d, "\eta_1"] \\ H_c^3(Z_{K_5-1-1b}; \mathbf{Q}) \arrow[rr, equal] & & H_c^3((\mathbf{C}^*)^2; \mathbf{Q})^{G_{K_5-1-1b}}. \end{tikzcd}$$

Then the claim follows from the fact that the differential $H_c^2(\mathbf{C}^*; \mathbf{Q}) \cong \mathbf{Q}(-1) \to H_c^3((\mathbf{C}^*)^2; \mathbf{Q})$ in the Gysin long exact sequence associated with the inclusion $\mathbf{C}^* \hookrightarrow \mathbf{C} \times \mathbf{C}^*$ has rank 1.

The proof for δ_2 and δ_3 is completely analogous to that for δ_1. In the case of δ_2 one considers the inclusion of the 2-dimensional stratum Z_{K_5-2} in the 3-dimensional stratum Z_{K_5-2-1}, given by the inclusion of the cone

$$K_5 - 2 - 1b = \langle x_1^2, x_2^2, x_3^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2 \rangle,$$

which lies in the same $\mathrm{GL}(4, \mathbf{Z})$-orbit as $K_5 - 2 - 1$, in

$$K_5 - 2 = \langle x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2 \rangle.$$

In this case, the stabilizers of $K_5 - 2$ and of $K_5 - 2 - 1b$ coincide as subgroups of $\mathrm{GL}(4, \mathbf{Z})$.

In the case of δ_3, one considers the inclusion of the 3-dimensional stratum $Z_{C_{2221}}$ in the 4-dimensional stratum $Z_{C_{222}}$, given by the inclusion of the cone

$$C_{222} = \langle x_1^2, x_2^2, x_3^2, (x_1 - x_4)^2, (x_2 - x_4)^2, (x_3 - x_4)^2 \rangle$$

in

$$C_{2221} = \langle x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_4)^2, (x_2 - x_4)^2, (x_3 - x_4)^2 \rangle.$$

Again, the stabilizers of C_{222} and C_{2221} coincide. □

IV.7.4 Cones containing e

We shall now prove Theorem IV.7.2.

Proof of (1)⇒(2) *in Theorem IV.7.2.* Assume that the cohomology with compact support of the exceptional divisor E of the blow-up $\mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{perf}}$ is as stated in (1). The Gysin long exact sequence associated with the closed inclusion $E \subset \beta_4$ is as follows:

$$\cdots \to H_c^{k-1}(E;\mathbf{Q}) \xrightarrow{d_k} H_c^k(\beta_4 \setminus E;\mathbf{Q}) \to H_c^k(\beta_4;\mathbf{Q}) \to H_c^k(E;\mathbf{Q}) \to \cdots \quad \text{(IV.7.3)}$$

Similarly the Gysin sequence of the pair $\{P_{\mathrm{sing}}\} \subset \beta_4^{\mathrm{perf}}$ reads

$$\cdots \to H_c^{k-1}(P_{\mathrm{sing}};\mathbf{Q}) \to H_c^k(\beta_4^{\mathrm{perf}} \setminus \{P_{\mathrm{sing}}\};\mathbf{Q}) \to$$

$$\to H_c^k({\beta_4}^{\mathrm{perf}};\mathbf{Q}) \to H_c^k(P_{\mathrm{sing}};\mathbf{Q}) \to \cdots$$

Since $\mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{perf}}$ is an isomorphism outside E, the complement $\beta_4 \setminus E$ is isomorphic to $\beta_4^{\mathrm{perf}} \setminus \{P_{\mathrm{sing}}\}$. By Theorem IV.7.1 the odd cohomology with compact support of β_4^{perf} vanishes and hence $H_c^k(\beta_4^{\mathrm{perf}} \setminus \{P_{\mathrm{sing}}\};\mathbf{Q}) = H_c^k(\beta_4 \setminus E;\mathbf{Q}) = 0$ for odd $k \geq 3$. Moreover, $H_c^1(\beta_4^{\mathrm{perf}} \setminus \{P_{\mathrm{sing}}\};\mathbf{Q}) = H_c^1(\beta_4 \setminus E;\mathbf{Q}) = 0$ since P_{sing} is a point and β_4^{perf} is compact (which implies that cohomology with compact support and ordinary cohomology coincide).

Furthermore by the description of $H_c^\bullet(E;\mathbf{Q})$ from (1) we know that all odd cohomology of E vanishes. This ensures that all differentials $d_k, k \geq 1$ are zero. This implies that the Betti numbers b_k of β_4 with $k \geq 1$ are as stated in Theorem IV.7.2. Also the description of the Hodge structures follows from Theorem IV.7.1 and from part (1) in view of the long exact sequence (IV.7.3). Finally, the fact that $H_c^0(\beta_4;\mathbf{Q})$ is one-dimensional follow from the connectedness of β_4. To complete the proof, recall that β_4 is compact, so that cohomology and cohomology with compact support agree. □

Lemma IV.7.4. *For every k, the cohomology group $H^k(E;\mathbf{Q})$ carries a pure Hodge structure of weight k.*

Proof. To prove the claim, we consider the second Voronoi compactification $\mathcal{A}_4^{\mathrm{Vor}}(n)$ of the moduli space of principally polarized abelian fourfolds with a level-n structure ($n \geq 3$). Recall that $\mathcal{A}_4^{\mathrm{Vor}}(n)$ is a smooth projective scheme and that the map $\pi(n)\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \to \mathcal{A}_4^{\mathrm{Vor}}$ is a finite group quotient. The preimage $\pi(n)^{-1}(E)$ of E is the union of finitely many irreducible components, all of which are smooth and pairwise disjoint. This follows from the toric description, since these components are themselves toric varieties given by the star $\mathrm{Star}(\langle e\rangle)$ in the lattice $\mathrm{Sym}^2(\mathbf{Z}^4)/\mathbf{Z}e$.

Table IV.16: $\mathrm{GL}(4,\mathbf{Z})$-orbits of cones of dimension ≥ 6 containing e

σ	$\dim\sigma$	$\mathbf{e}_{\mathrm{Hdg}}(Z_\sigma)$	σ	$\dim\sigma$	$\mathbf{e}_{\mathrm{Hdg}}(Z_\sigma)$
$111+$	10	1	$321-$	7	$\mathbf{L}^3-2\mathbf{L}^2+\mathbf{L}$
$111-$	10	1	$222'$	7	$\mathbf{L}^3-\mathbf{L}^2$
$211+$	9	$\mathbf{L}-1$	$22'2''$	7	$\mathbf{L}^3$
$211-$	9	$\mathbf{L}$	$222+$	7	$\mathbf{L}^3$
$311+$	8	$\mathbf{L}^2-\mathbf{L}$	$222-$	7	$\mathbf{L}^3$
$311-$	8	$\mathbf{L}^2$	421	6	$\mathbf{L}^4-\mathbf{L}^3+\mathbf{L}^2-\mathbf{L}$
$22'1$	8	$\mathbf{L}^2-\mathbf{L}$	$331+$	6	$\mathbf{L}^4+1$
$221+$	8	$\mathbf{L}^2$	$331-$	6	$\mathbf{L}^4-\mathbf{L}^3-\mathbf{L}+1$
$221-$	8	$\mathbf{L}^2+\mathbf{L}$	$322+$	6	$\mathbf{L}^4-\mathbf{L}^3$
411	7	$\mathbf{L}^3-\mathbf{L}^2$	$322-$	6	$\mathbf{L}^4-\mathbf{L}^3$
$321+$	7	$\mathbf{L}^3-\mathbf{L}^2+\mathbf{L}-1$	$322'$	6	$\mathbf{L}^4-2\mathbf{L}^3+2\mathbf{L}^2-2\mathbf{L}+1$

In particular, this implies that the Hodge structures on the cohomology groups of $\pi(n)^{-1}(E)$ are pure of weight equal to the degree. As $\pi(n)$ is finite, the pullback map

$$\pi(n)|^*_{\pi(n)^{-1}(E)}\colon\ H^k(E;\mathbf{Q})\to H^k(\pi(n)^{-1}(E);\mathbf{Q})$$

is injective. This implies that each cohomology group $H^k(E;\mathbf{Q})$ is a Hodge substructure of $H^k(\pi(n)^{-1}(E);\mathbf{Q})$, thus yielding the claim. $\square$

Proof of (1) *in Theorem IV.7.2.* In view of Lemma IV.7.4, determining the cohomology of E is equivalent to computing its Hodge Euler characteristics, i.e.

$$\mathbf{e}_{\mathrm{Hdg}}(E)=\sum_{k\in\mathbf{Z}}(-1)^k[H^k_c(E;\mathbf{Q})],$$

where $[\cdot]$ denotes the class in the Grothendieck group $K_0(\mathsf{HS}_{\mathbf{Q}})$ of Hodge structures. Hodge Euler characteristics are additive, so we are going to work with a locally closed stratification of E and add up the Hodge Euler characteristics to get the result.

The toroidal construction of $\mathcal{A}_4^{\mathrm{Vor}}$ yields that E is the union of toric strata Z_σ for all cones σ belonging to the second Voronoi decomposition but not to the perfect cone decomposition. Note that for such σ the variety Z_σ automatically maps to $\mathcal{A}_0$ under the map $\mathcal{A}_4^{\mathrm{Vor}}\to\mathcal{A}_4^{\mathrm{Sat}}$. Furthermore, up to the action of $\mathrm{GL}(4,\mathbf{Z})$, one can assume that these cones contain the extremal ray $\langle e\rangle$ defined in (IV.7.1) as extremal ray.

Since cones that lie in the same $\mathrm{GL}(4,\mathbf{Z})$-orbit give the same variety Z_σ, we have to work with a list of representatives of all $\mathrm{GL}(4,\mathbf{Z})$-orbits of cones fulfilling our conditions. Such a list is given in [Val, §4.4.5]. As in the proof of Theorem IV.14, we compute for each cone σ in Vallentin's list the generators of its

Table IV.17: $\mathrm{GL}(4,\mathbf{Z})$-orbits of cones of dimension ≤ 5 containing e

Σ	$\dim\Sigma$	$\mathbf{e}_{\mathrm{Hdg}}(Z_\sigma)$
$422'$	5	$\mathbf{L}^5+\mathbf{L}^3-\mathbf{L}^2+\mathbf{L}$
$332-$	5	$\mathbf{L}^5-\mathbf{L}^4+\mathbf{L}^3-3\mathbf{L}^2+2\mathbf{L}$
431	5	$\mathbf{L}^5-\mathbf{L}^4+\mathbf{L}^3-\mathbf{L}^2+\mathbf{L}-1$
422	5	$\mathbf{L}^5-\mathbf{L}^4$
$332+$	5	$\mathbf{L}^5-2\mathbf{L}^4+\mathbf{L}^3-\mathbf{L}^2+2\mathbf{L}-1$
432	4	$\mathbf{L}^6-2\mathbf{L}^5+2\mathbf{L}^4-4\mathbf{L}^3+5\mathbf{L}^2-2\mathbf{L}$
$333-$	4	$\mathbf{L}^6+2\mathbf{L}^2$
441	4	$\mathbf{L}^6+\mathbf{L}^2$
$333+$	4	$\mathbf{L}^6-\mathbf{L}^5-\mathbf{L}^3+2\mathbf{L}^2-\mathbf{L}$
433	3	$\mathbf{L}^7-\mathbf{L}^6+\mathbf{L}^5-\mathbf{L}^4+4\mathbf{L}^3-4\mathbf{L}^2$
442	3	$\mathbf{L}^7+2\mathbf{L}^3-\mathbf{L}^2$
443	2	$\mathbf{L}^8+2\mathbf{L}^4-3\mathbf{L}^3$
444	1	$\mathbf{L}^9-\mathbf{L}^4$
TOT.		$\mathbf{L}^9+\mathbf{L}^8+2\mathbf{L}^7+3\mathbf{L}^6+3\mathbf{L}^5+3\mathbf{L}^4+3\mathbf{L}^3+2\mathbf{L}^2+\mathbf{L}+1$

stabilizer G_σ in $\mathrm{GL}(4,\mathbf{Z})$, as well as their action on $H^1((\mathbf{C}^*)^{10-\dim\sigma};\mathbf{Q})$. Then we use the computer algebra program Singular [GPS] to calculate all positive Betti numbers of the quotient $Z_\sigma=(\mathbf{C}^*)^{10-\dim\sigma}/G_\sigma$. The results are given in Tables IV.16 and IV.17, where we list all cones and the Hodge Euler characteristics of the corresponding strata of E.

As already explained, the Hodge Euler characteristic of E is the sum for the Euler characteristics of all strata Z_σ and is computed at the bottom of Table IV.17. In view of Lemma IV.7.4, and recalling that $\mathbf{L}$ is the notation of the weight 2 Tate Hodge structure $\mathbf{Q}(-1)$ in the Grothendieck group of rational Hodge structures, we can conclude that the Betti numbers of E agree with those given in the statement of Theorem IV.7.2. □

Remark IV.7.5. Note that the Betti numbers of E satisfy Poincaré duality. Indeed, this must be the case as E is smooth up to finite group action.

IV.8 Cohomology of $\mathcal{A}_3$ with coefficients in symplectic local systems

In this section, we recollect the information on the cohomology of local systems on $\mathcal{A}_2$ and $\mathcal{A}_3$ that we used in the course of the paper. Let us recall that the cohomology of local systems of odd weight on $\mathcal{A}_g$ vanishes because it is killed by the abelian involution. Therefore, we only need to deal with local systems of even rank.

The cohomology of $\mathcal{A}_2$ and $\mathcal{A}_3$ with constant coefficients is known. The moduli space $\mathcal{A}_2$ is the disjoint union of the moduli space $\mathcal{M}_2$ of genus 2 curves and the locus $\mathrm{Sym}^2 \mathcal{A}_1$ of products. Since it is known that the rational cohomology of both these spaces vanishes in positive degree, we have

Lemma IV.8.1. *The only non-trivial rational cohomology groups with compact support of $\mathcal{A}_2$ are $H^4_c(\mathcal{A}_2;\mathbf{Q}) = \mathbf{Q}(-2)$ and $H^2_c(\mathcal{A}_2;\mathbf{Q}) = \mathbf{Q}(-1)$.*

The rational cohomology of $\mathcal{A}_3$ was computed by Hain ([Ha]). We state below his result in terms of cohomology with compact support.

Theorem IV.8.2 (Hain)**.** *The non-trivial Betti numbers with compact support of $\mathcal{A}_3$ are*

i	12	10	8	6
b_i	1	1	1	2

Furthermore, all cohomology groups are algebraic with the exception of $H^6_c(\mathcal{A}_3;\mathbf{Q})$, which is an extension of $\mathbf{Q}(-3)$ by $\mathbf{Q}$.

We deduce the results we need on non-trivial symplectic local systems with weight ≤ 2 from results on moduli spaces of curves ([BT],[T3]). Note that the result for $\mathbf{V}_{1,1}$ was already proven in [HT, Lemma 3.1].

Lemma IV.8.3. *The cohomology groups with compact support of the weight 2 symplectic local systems on $\mathcal{M}_2$ are as follows: the cohomology of $\mathbf{V}_{2,0}$ vanishes in all degrees, whereas the only non-zero cohomology group with compact support of $\mathbf{V}_{1,1}$ is $H^3_c(\mathcal{M}_2;\mathbf{V}_{1,1}) = \mathbf{Q}$.*

Lemma IV.8.4. *The rational cohomology of $\mathcal{M}_3$ with coefficients in $\mathbf{V}_{1,0,0}$ and $\mathbf{V}_{2,0,0}$ is 0 in all degrees. The only non-trivial cohomology group with compact support of $\mathcal{M}_3$ with coefficients in $\mathbf{V}_{1,1,0}$ is $H^9_c(\mathcal{M}_3;\mathbf{V}_{1,1,0}) = \mathbf{Q}(-5)$.*

Proof of Lemma IV.8.4. Following the approach of [G2], we use the forgetful maps $p_1\colon\ \mathcal{M}_{3,1} \to \mathcal{M}_3$ and $p_2\colon\ \mathcal{M}_{3,2} \to \mathcal{M}_3$ to obtain information. Note that p_1 is the universal curve over $\mathcal{M}_3$ and that the fibre of p_2 is the configuration space of 2 distinct points on a genus 3 curve.

According to [BT, Cor. 1], there is an isomorphism $H^\bullet(\mathcal{M}_{3,1};\mathbf{Q}) \cong H^\bullet(\mathcal{M}_3;\mathbf{Q}) \otimes H^\bullet(\mathbf{P}^1;\mathbf{Q})$ as vector spaces with mixed Hodge structures. If we compare this with the Leray spectral sequence in cohomology associated with p_1, we get that the cohomology of $\mathcal{M}_3$ with coefficients in $\mathbf{V}_{1,0,0}$ must vanish.

Next, we analyze the Leray spectral sequence in cohomology associated with p_2. Taking the $\mathfrak{S}_2$-action into account, the cohomology of the fibre of p_2 induces the following local systems on $\mathcal{M}_3$:

deg.	local system: invariant part	local system: alternating part
0	$\mathbf{Q}$	0
1	$\mathbf{V}_{1,0,0}$	$\mathbf{V}_{1,0,0}$
2	$\mathbf{Q}(-1) \oplus \mathbf{V}_{1,1,0}$	$\mathbf{Q}(-1) \oplus \mathbf{V}_{2,0,0}$
3	0	$\mathbf{V}_{1,0,0}(-1)$

This implies that the cohomology of $\mathcal{M}_3$ with coefficients in $\mathbf{V}_{2,0,0}$ (respectively, in $\mathbf{V}_{1,1,0}$) is strictly related to the $\mathfrak{S}_2$-alternating (resp. $\mathfrak{S}_2$-invariant) part of the cohomology of $\mathcal{M}_{3,2}$. The rational cohomology of $\mathcal{M}_{3,2}$ is described with its mixed Hodge structures and the action of the symmetric group in [T3, Thm 1.1]. By comparing this with the E_2-term of the Leray spectral sequence associated with p_2, one obtains that the cohomology of $\mathbf{V}_{2,0,0}$ vanishes and that the only non-trivial cohomology group of $\mathbf{V}_{1,1,0}$ is $H^3(\mathcal{M}_3; \mathbf{V}_{1,1,0}) = \mathbf{Q}(-3)$. Then the claim follows from Poincaré duality. □

Proof of Lemma IV.8.3. The proof is analogous to that of Lemma IV.8.4. In this case, one needs to compare the Leray spectral sequence associated with $p_2 : \mathcal{M}_{2,2} \to \mathcal{M}_2$ with the cohomology of $\mathcal{M}_{2,2}$ computed in [T2, II,2.2]. Note that in this case the cohomology of $\mathbf{V}_{1,0}$ vanishes because it is killed by the hyperelliptic involution on the universal curve over $\mathcal{M}_2$. □

Next, we compute the cohomology of the weight 2 local systems on $\mathcal{A}_2$ and $\mathcal{A}_3$ we are interested in, by using Gysin long exact sequences in cohomology with compact support and the stratification $\mathcal{A}_2 = \tau_2(\mathcal{M}_2) \sqcup \operatorname{Sym}^2 \mathcal{A}_1$ of $\mathcal{A}_2$, respectively, the stratification $\mathcal{A}_3 = \tau_3(\mathcal{M}_3) \sqcup \tau_2(\mathcal{M}_2) \times \mathcal{A}_1 \sqcup \operatorname{Sym}^3 \mathcal{A}_1$ of $\mathcal{A}_3$. The result on the cohomology with compact support of $\mathbf{V}_{1,1}$ was already proved in [HT, Lemma 3.1].

Lemma IV.8.5. *The only non-trivial cohomology groups of $\mathcal{A}_2$ with coefficients in a local system of weight 2 are $H^3_c(\mathcal{A}_2; \mathbf{V}_{1,1}) = \mathbf{Q}$ and $H^3_c(\mathcal{A}_2; \mathbf{V}_{2,0}) = \mathbf{Q}(-1)$.*

Proof. Using branching formulae as in [BvdG, §§7–8], one proves that the restriction of $\mathbf{V}_{2,0}$ to $\operatorname{Sym}^2 \mathcal{A}_1 \subset \mathcal{A}_2$ coincides with the symmetrization of $\mathbf{V}_2 \times \mathbf{V}_0$ on $\mathcal{A}_1 \times \mathcal{A}_1$. Its cohomology with compact support is then $\mathbf{Q}(-1)$ in degree 3 and trivial in all other degrees by e.g. [G1, Thm. 5.3]. Analogously, one shows that the cohomology of $\operatorname{Sym}^2 \mathcal{A}_1$ with coefficients in the restriction of the local system $\mathbf{V}_{1,1}$ is trivial. Then the claim follows from the Gysin long exact sequence associated with the inclusion $\operatorname{Sym}^2 \mathcal{A}_1 \subset \mathcal{A}_2$. □

Lemma IV.8.6. *The cohomology with compact support of $\mathcal{A}_3$ in the local system $\mathbf{V}_{1,1,0}$ is non-trivial only in degree 5 and possibly in degrees 8 and 9 and is given in these degrees by $H^5_c(\mathcal{A}_3; \mathbf{V}_{1,1,0}) = \mathbf{Q}(-1)$ and $H^8_c(\mathcal{A}_3; \mathbf{V}_{1,1,0}) \cong H^9_c(\mathcal{A}_3; \mathbf{V}_{1,1,0}) = \mathbf{Q}(-4)^{\oplus \epsilon}$ with $\epsilon \in \{0, 1\}$.*

Proof. Branching formulae yield that the cohomology with compact support of the restriction of $\mathbf{V}_{1,1,0}$ to $\tau_2(\mathcal{M}_2)\times\mathcal{A}_1$ is equal to $\mathbf{Q}(-5)$ (coming from the local system $\mathbf{V}_{1,1}\otimes\mathbf{V}_0$ on $\mathcal{M}_2\times\mathcal{A}_1$) in degree 8, to $\mathbf{Q}(-1)$ in degree 5 (coming from the local system $\mathbf{V}_0\otimes\mathbf{V}_0(-1)$) and is trivial in all other degrees. Moreover, the restriction of $\mathbf{V}_{1,1,0}$ to $\operatorname{Sym}^3\mathcal{A}_1$ is trivial, as is easy to prove if one looks at the cohomology of the restriction of the universal abelian variety over $\mathcal{A}_3$ to $\operatorname{Sym}^3\mathcal{A}_1$.

It remains to consider the Gysin long exact sequence associated with the closed inclusion $\mathcal{A}_3^{\mathrm{red}}\subset\mathcal{A}_3$. The only differential which can possibly be non-trivial is

$$\mathbf{Q}(-5)=H^8_c(\mathcal{A}_3^{\mathrm{red}};\mathbf{V}_{1,1,0})\longrightarrow H^9_c(\mathcal{M}_3;\mathbf{V}_{1,1,0})=\mathbf{Q}(-5).$$

From this the claim follows. ☐

In the investigation of the cohomology with compact support of the locus β_2^0 of semi-abelic varieties of torus rank 2 we also need to consider the cohomology with compact support of the weight 4 local system $\mathbf{V}_{2,2}$ on $\mathcal{A}_2$. For our application, we do not need a complete result in this case. The following lemma suffices:

Lemma IV.8.7. *The cohomology with compact support of $\mathcal{A}_2$ with coefficients in the local system $\mathbf{V}_{2,2}$ is 0 in all degrees different from $3,4$. Furthermore, for every weight k there is an isomorphism*

$$\operatorname{Gr}^W_k(H^3_c(\mathcal{A}_2;\mathbf{V}_{2,2}))\cong\operatorname{Gr}^W_k(H^4_c(\mathcal{A}_2;\mathbf{V}_{2,2}))$$

between the graded pieces of the weight filtration.

Proof. First, we prove that the result holds in the Grothendieck group of rational Hodge structures. This requires to prove that the Euler characteristic of $H^\bullet_c(\mathcal{A}_2;\mathbf{V}_{2,2})$ in the Grothendieck group of rational Hodge structures vanishes. By branching formulae, the cohomology with compact support of the restriction of $\mathbf{V}_{2,2}$ to $\operatorname{Sym}^2\mathcal{A}_1$ is equal to the cohomology of the local system $\mathbf{V}_0\otimes\mathbf{V}_0(-2)$, which is equal to $\mathbf{Q}(-4)$ in degree 4 and trivial otherwise. On the other hand, the Euler characteristic of $H^\bullet_c(\mathcal{A}_2;\mathbf{V}_{2,2})$ was proved in [Ber, Theorem 11.6] to be equal to $-[\mathbf{Q}(-4)]$. Then the additivity of Euler characteristics ensures that the Euler characteristic of $\mathbf{V}_{2,2}$ vanishes. This means that the Euler characteristic of each graded piece of the weight filtration on $H^\bullet_c(\mathcal{A}_2;\mathbf{V}_{2,2})$ is 0.

More generally, the fact that $\mathcal{M}_2$ and $\operatorname{Sym}^2\mathcal{A}_1$ are affine of dimension 3 and 2 respectively, combined with the Gysin long exact sequence associated to $\operatorname{Sym}^2\mathcal{A}_1\hookrightarrow\mathcal{A}_2$ implies that the cohomology of $\mathcal{A}_2$ with values in any local system is trivial in degree greater than 3. Thus, by Poincaré duality, the cohomology with compact support of $\mathcal{A}_2$ can be non-trivial only in degree larger than or equal to 3. Furthermore, for non-trivial irreducible local systems H^0 (and hence H^6_c) vanishes, whereas H^1 (and hence H^5_c) is always zero by the Raghunathan rigidity theorem [R]. This means that the cohomology with compact support of $\mathbf{V}_{2,2}$ on $\mathcal{A}_2$ can be non-zero only in degrees 3 and 4. The cohomology groups in these degrees are then isomorphic when passing to the associated graded pieces of the weight filtration as a consequence of the vanishing of the Euler characteristic in the Grothendieck group of Hodge structures. ☐

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