Induced representations of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and the transfer operator for its subgroups of finite index

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Abstract

In the transfer operator approach to Selberg's zeta function for a subgroup Γ of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and character χ the space $F(s;\Gamma;\chi)$ of eigenfunctions of the transfer operator $\mathcal{L}_s^{\Gamma,\chi}$ for the eigenvalue 1 with certain asymptotics at zero is in bijection with the space $\mathcal{S}(s;\Gamma,\chi)$ of Maass cusp forms of Γ and χ respectively the space $\mathfrak{S}(s;\Gamma;\chi)$ of period functions of Lewis and Zagier. Selberg's zeta function $Z(s;\Gamma;\chi)$ gets thereby expressed in terms of the Fredholm determinant of $\mathcal{L}_s^{\Gamma,\chi}$. We extend this approach to the case χ an arbitrary unitary representation of Γ. We show that also the symmetries of the transfer operator with unitary representation can be related to the automorphisms of the corresponding Maass cusp forms.

According to the Jacquet-Langlands correspondence the space $\mathcal{S}(s;\mathcal{O}_n)$ of Maass cusp forms for $\mathcal{O}_n$, the unit group of an indefinite quaternion devision algebra of discriminant n, is in bijection with the space $\mathcal{S}^{new}(s;\Gamma_0(n))$ of new forms for the Hecke congruence group $\Gamma_0(n)$, when n is square free and a product of an even number of primes. The expectation, that the restriction of the transfer operator to the corresponding subspace $F^{new}(s;\Gamma_0(n))$ of the eigenfunctions, coincides with this operator corresponding to one of the irreducible components of $U_{\Gamma_0(n)}$ is not true as we show in the case $\Gamma_0(6)$.

By a result of Millington, the representation $U_{\Gamma_0(n)}$ of $\mathrm{PSL}(2,\mathbb{Z})$ appearing in the transfer operator $\mathcal{L}_s^{\Gamma_0(n)}$ and induced from the trivial representation of $\Gamma_0(n)$, can be identified with a permutation representation of the finite group $Q(n) = \mathrm{PSL}(2,\mathbb{Z})/H(n)$ with $H(n) = \ker U_{\Gamma_0(n)}$. By applying results of the theory of finite groups it follows that for coprime n and m, $U_{\Gamma_0(nm)} = U_{\Gamma_0(n)} \otimes U_{\Gamma_0(m)}$ and for p prime $U_{\Gamma_0(p)} \cong U_t \oplus U_p$ where U_t is the trivial 1-dim. representation and U_p is a certain p-dimensional irreducible representation of $\mathrm{PSL}(2,\mathbb{Z})$. This leads to a decomposition of the transfer operator and the space of its eigenfunctions. In the special case $n = 6$ the representation $U_{\Gamma_0(6)}$ contains the irreducible representation $U_2 \otimes U_3$, which characterizes a subspace of $F(s;\Gamma_0(6))$ describing a mixture of old and new eigenfunctions including the space $F^{new}(s;\Gamma_0(6))$. In the case of the principal congruence subgroup $\Gamma(2)$ the induced representation $U_{\Gamma(2)}$ on the other hand contains a 1-dim. subrepresentation U_{sgn} characterizing exactly the space $F^{new}(s;\Gamma(2))$.

Finally it is proved, that a character χ on Γ is congruence iff the representation ρ_χ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from χ is congruence. Applying this to the representation ρ_{χ_α} induced from Selberg's family of characters χ_α, for $\Gamma_0(4)$, yields besides the α-values for which χ_α is congruence, also certain non-congruence subgroups to which Zograf's criterion can not be applied.

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CHAPTER 1

Introduction

In the theory of quantum chaos the main problems are to quantize chaotic classical Hamiltonian systems in analogy to the Bohr, Sommerfeld, Einstein, Keller, Maslov (BSEKM) semiclassical quantization of completely integrable systems and to find some connections between the classical and quantum descriptions of such systems (see [**2**] and the references there).

The motion of a free particle on a surface of constant negative curvature of the form $\Gamma\backslash\mathbb{H}$, $\mathbb{H}$ the Poincaré upper half-plane and Γ a Fuchsian group of the first kind, is a highly chaotic classical dynamical system [**21**] which is described by the geodesic flow on the unit tangent bundle of the surface, that is,

$$\phi_t : S(\Gamma\backslash\mathbb{H}) \to S(\Gamma\backslash\mathbb{H}). \tag{1.1}$$

Selberg's zeta function and Selberg's trace formula, which are well known tools in the "spectral theory of automorphic functions", provide partial answers to these questions of quantum chaos for the aforementioned system [**30**]. Later, Mayer developed another approach to the theory of quantum chaos on the modular surface via his transfer operator which is assigned to the geodesic flow (1.1) for Γ a finite index subgroup of the projective modular group [**32, 12**].

The quantum description of a free particle on the quotient surface $\Gamma\backslash\mathbb{H}$ with a unitary representation χ of Γ is mathematically closely related to the "spectral theory of automorphic functions" which originated in the work of Selberg (see [**57**] and the references there). In this theory, the discontinuous action of Γ on $\mathbb{H}$ leads to automorphic functions and operators on $\mathbb{H}$ defined through their transformation properties under the action of Γ.

The automorphic Laplacian $\Delta(\Gamma;\chi)$, acting on the Hilbert space $\mathcal{H}(\Gamma;\chi)$ of automorphic functions with respect to Γ and χ and square integrable on $\mathbb{H}$, has in the regular case (see Definition 2.4) a continuous and maybe a discrete spectrum [**57, 16**]. The continuous part of the spectrum is described by the Eisenstein series, automorphic eigenfunctions of the Laplacian on $\Gamma\backslash\mathbb{H}$ not belonging to the Hilbert space $\mathcal{H}(\Gamma;\chi)$, which are the scattering states of the system [**57, 55**]. Part of the discrete spectrum is described by the Maass cusp forms, which are the solutions in the Hilbert space $\mathcal{H}(\Gamma;\chi)$ of the Schrödinger equation on $\Gamma\backslash\mathbb{H}$

$$\Delta(\Gamma;\chi)\mathfrak{f} = s(1-s)\mathfrak{f}, \tag{1.2}$$

and vanish exponentially fast at each open cusp (see Definition 2.2) of the surface $\Gamma\backslash\mathbb{H}$. For such solutions the spectral parameters s define a discrete set of points lying on the line $\Re(s) = \frac{1}{2}$ and may be on the segment $[\frac{1}{2}, 1)$ [**57, 28, 23**]. In addition to this, the discrete spectrum of the automorphic Laplacian can contain finitely many other points $\lambda = s(1-s)$ for which the spectral parameter s is a pole of an Eisenstein series in the interval $(\frac{1}{2}, 1]$. The residue of the Eisenstein series at this pole s is an eigenfunction of $\Delta(\Gamma;\chi)$ in the Hilbert space with eigenvalue $s(1-s)$. The eigenfunctions of $\Delta(\Gamma;\chi)$ in the Hilbert space are often called Maass wave forms [**57, 23**], which describe the bound states of the system.

On the other hand, Selberg's trace formula relates the spectrum of the automorphic Laplacian $\Delta(\Gamma;\chi)$ to the geometry of the surface $\Gamma\backslash\mathbb{H}$. More precisely, Selberg's trace formula is an identity, which for certain test functions h (see [**57**] and the references there) has the form

$$\sum_{k=0}^{\infty} h(\lambda_k) + C = I + H + E + P \tag{1.3}$$

where the λ_k's are the eigenvalues of $\Delta(\Gamma;\chi)$ and C denotes the explicitly known contribution of the continuous spectrum. The terms I, H, E, and P denote the explicitly known contributions of the identity element, the hyperbolic conjugacy classes, the elliptic conjugacy classes, and the parabolic conjugacy classes, respectively (see theorem 2.3).

Selberg's zeta function for Γ and χ is defined in the domain $\Re(s) > 1$ by an absolutely convergent infinite product given by ([**49**], see also [**57**])

$$Z(s;\Gamma;\chi) = \prod_{k=0}^{\infty} \prod_{\{\gamma\}_\Gamma} \det(1_V - \chi(\gamma)\mathcal{N}(\gamma)^{-k-s}) \tag{1.4}$$

where $\{\gamma\}$ runs over all primitive hyperbolic conjugacy classes of Γ, $\mathcal{N}(\gamma) > 1$ denotes the norm of γ and 1_V denotes the identity operator on the representation space of χ. A part of the nontrivial zeros of Selberg's zeta function coincides with the spectral parameters of the Maass wave forms and hence yields the quantized energy levels $\lambda = s(1-s)$ of a free particle on $\Gamma\backslash\mathbb{H}$.

Coming back to the chaotic classical Hamiltonian system in (1.1), it is known that there is a one to one correspondence between the primitive hyperbolic conjugacy classes γ of Γ of norm $\mathcal{N}(\gamma)$ and the prime periodic orbits γ_ϕ of ϕ_t of period $l(\gamma_\phi)$ where $\mathcal{N}(\gamma) = \exp(l(\gamma_\phi))$ (see [**6**], Chapter 6 and references there). Therefore, Selberg's zeta function can be written in terms of the length spectrum of the closed orbits of the geodesic flow, that is,

$$Z(s;\Gamma;\chi) = \prod_{\gamma_\phi} \prod_{k=0}^{\infty} \det\left[1_V - \chi(g_{\gamma_\phi}) \exp(-(s+k)l(\gamma_\phi))\right], \tag{1.5}$$

where γ_ϕ is a periodic orbit of the geodesic flow ϕ_t of prime period $l(\gamma_\phi)$ and $g_{\gamma_\phi} \in \Gamma$ is an hyperbolic element with $g_{\gamma_\phi}\gamma_\phi = \gamma_\phi$ [**14**]. Hence, in this classical context Selberg's zeta function can be interpreted as a dynamical zeta function of the classical dynamical system described by the geodesic flow in (1.1). Hence, Selberg's zeta function connects the quantum spectrum of the system (nontrivial zeros) to the length spectrum of ϕ_t which is a classical object and therefore this function can be considered as some kind of quantization procedure for this chaotic system analogous to the BSEKM (Bohr, Sommerfeld, Einstein, Keller, Maslov) quantization of integrable Hamiltonian systems [**30**].

The contribution of the hyperbolic classes in the right hand side of Selberg's trace formula can be written for a certain test function and some regularization as the logarithmic derivative of Selberg's zeta function (see [**57**], Chapter 7). Selberg's trace formula hence connects also the eigenvalues of the Laplacian and the length spectrum of ϕ_t [**30**].

In [**32**], D. Mayer introduced a new dynamical approach to Selberg's zeta function for the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ by applying the transfer operator method of the thermodynamic formalism in the theory of dynamical system, which has its roots in classical statistical mechanics (see [**6**], Chapter 7 and [**47**]). Thereby one tries to express the Ruelle dynamical zeta function

$$\zeta(z) = \exp(\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in Fix\tau^m} \prod_{k=0}^{m-1} g(\tau^k x)) \tag{1.6}$$

for a weighted dynamical system $\tau : M \to M$ with weight function $g : M \to \mathbb{R}$, where $Fix\,\tau^m$ is the set of fixed points of τ^m, in terms of the Fredholm determinant of the Ruelle transfer operator $\mathcal{L}$ defined by

$$(\mathcal{L}f)(x) = \sum_{y\in\tau^{-1}\{x\}} g(y)f(y) \tag{1.7}$$

and acting on functions on M. This way the spectral properties of the transfer operator could yield, among other things, results on the analytic properties of the dynamical zeta function and also some information about ergodic properties of the dynamical system $\tau : M \to M$.

It turns out that this idea works very well for the geodesic flow on the surfaces of constant negative curvature. In the case of the geodesic flow on the modular surface

$$\phi_t : S(\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}) \to S(\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}), \tag{1.8}$$

D. Mayer considers Ruelle's transfer operator for the Poincaré map on a certain Poincaré section of the unit tangent bundle $S(\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H})$ and the weight function $g(x) = x^{2s}$, which for $\Re(s) > \frac{1}{2}$ is explicitly given by [**32**]

$$\widetilde{\mathcal{L}}_s := \begin{pmatrix} 0 & \mathcal{L}_s \\ \mathcal{L}_s & 0 \end{pmatrix} : B(D) \oplus B(D) \to B(D) \oplus B(D) \tag{1.9}$$

with

$$\mathcal{L}_s f(\zeta) = \sum_{n=1}^{\infty} \left(\frac{1}{\zeta+n}\right)^{2s} f\left(\frac{1}{\zeta+n}\right), \tag{1.10}$$

where $B(D)$ denotes the Banach space of holomorphic functions on the disc

$$D = \left\{ z \in \mathbb{C} \mid |z-1| < \frac{3}{2} \right\} \tag{1.11}$$

and continuous on the closure of D. This operator turns out to be trace class and it extends to a meromorphic family of operators on the entire complex s-plane [**32**]. Selberg's zeta function $Z(s)$ for the modular group is expressed in terms of the Fredholm determinant of the transfer operator $\widetilde{\mathcal{L}}_s$ [**32**] as

$$Z(s) = \det(1 - \widetilde{\mathcal{L}}_s) = \det(1 - \mathcal{L}_s)\det(1 + \mathcal{L}_s), \tag{1.12}$$

where the determinant is defined in the sense of Grothendieck (see [**6**], page 218). Therefore, the s-values for which $\mathcal{L}_s$ has an eigenfunction $f \in B(D)$ with eigenvalues $\lambda = \pm 1$,

$$\mathcal{L}_s f(z) = \pm f(z), \quad f \in B(D), \tag{1.13}$$

determine the zeros of $Z(s)$. Hence condition (1.13), which is related to the classical system of the geodesic flow, determines the spectral parameters for the Maass forms, that is, the quantization of the energy levels of the system.

In addition to this, the eigenfunctions $f \in B(D)$ in (1.13) for the spectral parameters s are directly related to the corresponding Maass forms. This is a consequence of the Lewis-Zagier theory of period functions [**29**]. By definition, a period function $\psi : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ is a solution of the functional equation

$$\psi(\zeta) - \psi(\zeta+1) \mp (\zeta+1)^{-2s}\psi\left(\frac{\zeta}{\zeta+1}\right) = 0 \tag{1.14}$$

which in the case of Maass cusp forms fulfils the asymptotics $\psi(\zeta) = o(\zeta^{-\min\{1,2\Re(s)\}})$ as $\zeta \downarrow 0$ and $\psi(\zeta) = o(\zeta^{-\min\{0,2\Re(s)-1\}})$ as $\zeta \to \infty$, where the limits are taken along the real axis. It is known, that there is a one to one correspondence between Maass cusp forms with spectral parameter s and the period functions ψ_s in (1.14).

The Lewis-Zagier theory of period functions can be extended to the space of all automorphic eigenfunctions of the Laplacian with polynomial growth at infinity ([**29**], Chapter III, page 226). This is the space spanned by Maass wave forms and Eisenstein series. The solutions of the functional equation (1.14) corresponding to automorphic eigenfunctions of the Laplacian not belonging to the Hilbert space are called periodlike functions. Moreover, the period(like) function ψ_s is given in terms of an integral transform of the corresponding Maass cusp form u,

$$(\mathcal{I}u)(\zeta) := \int_{L_{0,\infty}} \eta(u, R_\zeta^s)(z), \tag{1.15}$$

where $R_\zeta(z) = \dfrac{y}{(\zeta - x)^2 + y^2}$ with $z = x + iy$ is the hyperbolic Poisson kernel, η is an explicitly known closed 1-form (see (3.47)) and $L_{0,\infty}$ denotes a path homotopic to the path from zero to infinity along the imaginary axis.

For the group $\mathrm{PSL}(2, \mathbb{Z})$ there is one Eisenstein series $E(s, z)$ which has a pole at $s = 1$ with the constant function as its residue. In [**12**], by using this integral transformation it is shown that the constant function corresponds to the period function $\psi(\zeta) = \frac{1}{\zeta}$ and also the periodlike function corresponding to the Eisenstein series $E(s, z)$ is given explicitly.

On the other hand, for the spectral parameters s of the Maass cusp forms the eigenfunctions $f \in B(D)$ in (1.13) are in bijection with the period functions in (1.14). Indeed, for a period function $\psi(\zeta)$, the function $f(\zeta) = \psi(\zeta + 1)$ restricted to D is an eigenfunction of Mayer's transfer operator $\mathcal{L}_s$ in $B(D)$ with eigenvalues ± 1 and vice versa.

This way the transfer operator describes a classical correspondence for the eigenvalues and eigenfunctions of the Laplacian and hence yields in principle a full description of the quantum behaviour of this classical system of a free particle. In the more general case of subgroups of finite index in $\mathrm{PSL}(2, \mathbb{Z})$ with some representation, the transfer operator approach has lead to other usefull results in the spectral theory of automorphic functions. For example for the Hecke congruence subgroups $\Gamma_0(n)$ with the trivial representation it lead to a new proof of the Venkov-Zograf factorization of Selberg's zeta function [**14**], or allowed to calculate Selberg's zeta function numerically [**18**], which has recently also been applied to the Phillips-Sarnak conjecture on the existence of Maass cusp forms for arbitrary cofinite groups [**41**].

To formulate the problems we consider in this thesis, we need some further notations. For a subgroup Γ of finite index μ in the projective modular group with a unitary representation χ of Γ, Mayer's transfer operator is defined by [**14**]

$$\mathcal{L}_s^{\Gamma,\chi} = \begin{pmatrix} 0 & \mathcal{L}_s^{\Gamma,\chi,+} \\ \mathcal{L}_s^{\Gamma,\chi,-} & 0 \end{pmatrix} : \bigoplus_{i=1}^{2\mu \dim \chi} B(D) \to \bigoplus_{i=1}^{2\mu \dim \chi} B(D), \tag{1.16}$$

with $\mathcal{L}_s^{\Gamma,\chi,\pm}$ given in the domain $\Re(s) > 1/2$ by

$$\begin{aligned} &\mathcal{L}_s^{\Gamma,\chi,\pm} : \bigoplus_{i=1}^{\mu \dim \chi} B(D) \to \bigoplus_{i=1}^{\mu \dim \chi} B(D) \\ &\mathcal{L}_s^{\Gamma,\chi,\pm} f(\zeta) = \sum_{n=1}^{\infty} (\frac{1}{\zeta+n})^{2s} \rho_\chi(ST^{\pm n}) f(\frac{1}{\zeta+n}) \end{aligned} \tag{1.17}$$

where ρ_χ is the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from χ. Selberg's zeta function for Γ and representation χ has a determinant expression in terms of this transfer operator [**14**] as

$$Z(s;\Gamma;\chi) = \det(1_{V\oplus V} - \mathcal{L}_s^{\Gamma,\chi}), \tag{1.18}$$

where the determinant is defined in the sense of Grothendieck (see [**6**], page 218) and $1_{V\oplus V}$ is the identity operator on $V \oplus V$ with V the representation space of χ.

For a symmetry operator $P : V \to V$ with $P\mathcal{L}_s^{\Gamma,\chi,+} = \mathcal{L}_s^{\Gamma,\chi,-}P$ and $P^2 = id_V$, Selberg's zeta function $Z(s;\Gamma;\chi)$ in (1.18) can be written as

$$Z(s;\Gamma;\chi) = \det(1_V - P\mathcal{L}_s^{\Gamma,\chi,+}) \det(1_V + P\mathcal{L}_s^{\Gamma,\chi,+}). \tag{1.19}$$

We extend the notion of the Lewis-Zagier period functions to a subgroup Γ of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and a unitary representation χ of Γ from [**15**]. The period functions $\psi : \mathbb{C}\backslash(-\infty, 0] \to V$ for Γ and χ are the holomorphic solutions of the functional equation

$$\psi(\zeta) - \rho_\chi(T^{-1})\psi(\zeta+1) - (\zeta+1)^{-2s}\rho_\chi(STS)\psi(\frac{\zeta}{\zeta+1}) = 0 \tag{1.20}$$

whose components ψ_i fulfil the asymptotics $\psi_i(\zeta) = o(\zeta^{-\min\{1,2\Re(s)\}})$ as $\zeta \downarrow 0$ and $\psi_i(\zeta) = o(\zeta^{-\min\{0,2\Re(s)-1\}})$ as $\zeta \to \infty$, where the limits are taken along the real axis (see Definition 3.6). We denote the space of these period functions by $\mathfrak{S}(s;\Gamma;\chi)$. Then we show that this space is in bijection with the space $F(s;\Gamma;\chi)$ of eigenfunctions $f \in \oplus_{i=1}^{\mu \dim \chi} B(D)$ of the transfer operator $P\mathcal{L}_s^{\Gamma,\chi,+}$ with eigenvalue $\lambda = \pm 1$. This last bijection is described by the map (see theorem 3.10)

$$\mathcal{P} : \mathfrak{S}(s;\Gamma;\chi) \to F(s;\Gamma;\chi) \tag{1.21}$$

given explicitly by

$$f(\zeta) = \mathcal{P}\psi(\zeta) = \rho_\chi(ST^{-1})\psi(\zeta+1)|_D. \tag{1.22}$$

Next, we show that the space $\mathcal{S}(s;\Gamma;\chi)$ of Maass cusp forms for Γ and χ is isomorphic to the space $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ of V-valued Maass cusp forms for $\mathrm{PSL}(2,\mathbb{Z})$ and the representation ρ_χ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from χ. This isomorphism is described by the map

$$B : \mathcal{S}(s;\Gamma;\chi) \to \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi), \tag{1.23}$$

given by

$$B\mathfrak{f}(z) = (\mathfrak{f}_1(z), \mathfrak{f}_2(z), \dots, \mathfrak{f}_\mu(z))^t, \quad \mathfrak{f}_i(z) := \mathfrak{f}(r_i z) \tag{1.24}$$

with ^{t} denoting the vector transpose and $\{r_i \mid i = 1, \dots, \mu\}$ is a set of representatives of the right cosets of Γ in $\mathrm{PSL}(2,\mathbb{Z})$. By extending the integral transform $\mathcal{I}$ in (1.15) to act on vectors componentwise, it follows that the map

$$\mathcal{I} \circ B : \mathcal{S}(s;\Gamma;\chi) \to \mathfrak{S}(s;\Gamma;\chi) \tag{1.25}$$

or equivalently the map

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi) \to \mathfrak{S}(s;\Gamma;\chi) \tag{1.26}$$

is well-defined (see lemma 3.5).

Let us recall now the generalized version of Lewis-Zagier correspondence for period functions. We extend a result of Hilgert and Deitmar (see theorem 3.6) on the Lewis-Zagier correspondence to the case of a representation χ of Γ, for which the kernel of the induced representation ρ_χ is of finite index: there is a bijection between the space of Maass cusp forms $\mathcal{S}(s;\Gamma;\chi)$ and the space of period functions $\mathfrak{S}(s;\Gamma;\chi)$, that is, the maps (1.25) and (1.26) are bijections (corollary 3.2). To explain this in more detail we need some notations. Let Λ be a normal subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and ρ_Λ be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial representation of Λ. This yields the regular representation ρ'_Λ of the finite group $\mathrm{PSL}(2,\mathbb{Z})/\Lambda$ by

$$\rho_\Lambda(g) = \rho'_\Lambda(g\Lambda), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{1.27}$$

Moreover, let η' be an irreducible subrepresentation of ρ'_Λ which gives an irreducible subrepresentation η of ρ_Λ by

$$\eta(g) = \eta'(g\Lambda), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{1.28}$$

Then according to Hilgert and Deitmar [**15**], the map

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta) \to \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta) \tag{1.29}$$

is a bijection. For an arbitrary unitary induced representation ρ_χ of $\mathrm{PSL}(2,\mathbb{Z})$, according to the “fundamental theorem on homomorphisms” ([**1**], page 10) the epimorphism

$$\rho_\chi : \mathrm{PSL}(2,\mathbb{Z}) \to \rho_\chi(\mathrm{PSL}(2,\mathbb{Z})) \tag{1.30}$$

yields the group isomorphism

$$\rho_\chi(\mathrm{PSL}(2,\mathbb{Z})) \cong \mathrm{PSL}(2,\mathbb{Z})/\Lambda, \quad \Lambda = \ker \rho_\chi. \tag{1.31}$$

We remark that this isomorphism has been already derived by Millington in [**34**] for ρ_χ the special case of the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial representation of any subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of finite index. Then the representation ρ'_χ of the finite group $\mathrm{PSL}(2,\mathbb{Z})/\Lambda$ is given by

$$\rho'_\chi(g\Lambda) = \rho_\chi(g), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{1.32}$$

It is decomposed into its irreducible subrepresentations as follows

$$\rho'_\chi = \oplus_{i=1}^N m_i \eta'_i \tag{1.33}$$

where N denotes the number of non-isomorphic irreducible subrepresentations and m_i denotes the multiplicity of the irreducible subrepresentation η'_i. This yields the decomposition of the induced representation ρ_χ into its irreducible subrepresentations,

$$\rho_\chi = \oplus_{i=1}^N m_i \eta_i \tag{1.34}$$

where N denotes the number of non-isomorphic irreducible subrepresentations and m_i denotes the multiplicity of the irreducible subrepresentation η_i and where

$$\eta_i(g) = \eta'_i(g\Lambda), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{1.35}$$

Then we have for the space of vector valued Maass cusp forms

$$\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi) \cong \oplus_{i=1}^N m_i \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i). \tag{1.36}$$

By applying the aforementioned result of Hilgert and Deitmar to each subspace in this decomposition and using the linearity of $\mathcal{I}$ and the bijection

$$\mathfrak{S}(s;\Gamma;\chi) \cong \oplus_{i=1}^N m_i \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i), \tag{1.37}$$

it follows that the map $\mathcal{I}$ in (1.26) and hence the map $\mathcal{I} \circ B$ in (1.25) is a bijection (see corollary 3.2). From this it follows that the map

$$\mathcal{P} \circ \mathcal{I} \circ B : S(s;\Gamma;\chi) \to F(s;\Gamma;\chi). \tag{1.38}$$

is also a bijection (see theorem 3.11).

In [**19**] it has been shown for χ a unitary character, that the integral transform $\mathcal{I} \circ B$ induces an automorphism of the space of period functions from the one of the space of Maass cusp forms and that this automorphism is related to a symmetry of Mayer's transfer operator. We extend this result to the case where χ is an arbitrary unitary representation. If J is an automorphism of the space $\mathcal{S}(s;\Gamma;\chi)$ of Maass cusp forms, given by

$$J\mathfrak{u}(z) = \mathfrak{u}(jz), \tag{1.39}$$

then the integral transformation $\mathcal{I} \circ B$ induces an automorphism $\mathcal{J}$ of the space $\mathfrak{S}(s;\Gamma;\chi)$ of period functions given by

$$\mathcal{J}\psi(\zeta) = \rho_\chi(S)\pi(J)\zeta^{-2s}\psi(\frac{1}{\zeta}) \tag{1.40}$$

where $\pi(J)$ is a certain matrix which in particular for a one dimensional representation χ is a monomial matrix. Moreover if $j^2 \in \ker\chi$ it turns out that $\pi(J)$ is indeed a symmetry of Mayer's transfer operator $\mathcal{L}_s^{\Gamma,\chi}$, that is, it fulfills the following conditions:

1) $\pi(J)^2 = id_{\mu_\Gamma \dim \chi}$, where id_n denotes the $n \times n$ identity matrix.
2) $\pi(J)\rho_\chi(S) = \rho_\chi(S)\pi(J)$.
3) $\pi(J)\rho_\chi(T) = \rho_\chi(T^{-1})\pi(J)$.

From these equalities one can show that $\pi(J)\mathcal{L}_s^{\Gamma,\chi,+} = \mathcal{L}_s^{\Gamma,\chi,-}\pi(J)$ (see Lemma 3.13).

The decomposition of the induced representation in the transfer operator $P\mathcal{L}_s^{\Gamma,\chi,+}$, with $\mathcal{L}_s^{\Gamma,\chi,+}$ given in (1.17) and P a symmetry, into its subrepresentations yields a decomposition of this operator into the transfer operators for $\mathrm{PSL}(2,\mathbb{Z})$ and the aforementioned subrepresentations (for an example see (5.18)). This leads for example to a factorization of Selberg's zeta function for Γ a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and χ a unitary representation of Γ (see (5.20)). In particular, we study the representation $U_{\Gamma_0(n)}$ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial representation of the Hecke congruence group $\Gamma_0(n)$ and the representation $U_{\overline{\Gamma}_0(n)}$ of $\mathrm{PGL}(2,\mathbb{Z})$ induced from the trivial representation of the extension $\overline{\Gamma}_0(n)$ of the Hecke congruence subgroups to $\mathrm{PGL}(2,\mathbb{Z})$. The representation $U_{\Gamma_0(n)}$ appears in Mayer's transfer operator $\mathcal{L}_s^{\Gamma_0(n)}$ for $\Gamma_0(n)$ and the trivial one dimensional representation, whereas $U_{\overline{\Gamma}_0(n)}$ appears in Mayer's transfer operator $P\mathcal{L}_s^{\Gamma_0(n),+}$ with the special symmetry operator P given by

$$P = U_{\overline{\Gamma}_0(n)}(M), \quad M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.41}$$

According to Millington [**34**], the group generated by the elements $U_{\Gamma_0(n)}(S)$ and $U_{\Gamma_0(n)}(T)$ with S and T the generators of $\mathrm{PSL}(2,\mathbb{Z})$ is isomorphic to the factor group $\mathrm{PSL}(2,\mathbb{Z})$ modulo the maximal normal subgroup $H(n)$ of $\mathrm{PSL}(2,\mathbb{Z})$ contained in $\Gamma_0(n)$, that is,

$$\mathrm{PSL}(2,\mathbb{Z})/H(n) \cong U_{\Gamma_0(n)}(\mathrm{PSL}(2,\mathbb{Z})). \tag{1.42}$$

It turns out that $H(n)$ is indeed the kernel of the representation $U_{\Gamma_0(n)}$.

Based on this we show that for $n \mid m$ there is a permutation representation $\pi_{Q(m),X(n)}$ of the finite group $Q(m) = \mathrm{PSL}(2,\mathbb{Z})/H(m)$ associated with its action on the finite set $X(n) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(n)$ such that for any g in $\mathrm{PSL}(2,\mathbb{Z})$ (see Lemma 4.5),

$$U_{\Gamma_0(n)}(g) = \pi_{Q(m),X(n)}(gH(m)), \quad n \mid m, \tag{1.43}$$

where in particular n can be equal to m. This way the decomposition of the representation $U_{\Gamma_0(n)}$ into its irreducible subrepresentations is reduced to that of $\pi_{Q(m),X(n)}$. Using the properties of $X(n)$ as a $Q(m)$-set we show that for $n \mid m$ $\pi_{Q(m),X(n)}$ is a subrepresentation of $\pi_{Q(m),X(m)}$ and therefore we conclude that $U_{\Gamma_0(n)}$ is a subrepresentation of $U_{\Gamma_0(m)}$. Moreover, for coprime m and n we show that $\pi_{Q(nm),X(nm)} \cong \pi_{Q(nm),X(n)} \otimes \pi_{Q(nm),X(m)}$ and hence $U_{\Gamma_0(nm)} = U_{\Gamma_0(n)} \otimes U_{\Gamma_0(m)}$. Thus the problem of decomposing $U_{\Gamma_0(n)}$ reduces to that of $U_{\Gamma_0(q)}$ with $q = p^e$ a prime power. With the same arguments we get similar results for $U_{\overline{\Gamma}_0(n)}$ (see Lemmas 4.6, 4.8, and 4.7).

In the particular case p a prime, we show that the group $K(p) = \Gamma_0(p)/H(p)$ acts doubly transitively on $X(p)$ (see Lemma 4.9). From this we conclude that the permutation representation $\pi_{Q(p),X(p)}$ associated with $X(p)$ contains two non-isomorphic irreducible subrepresentations, namely the trivial one dimensional representation and a certain p-dimensional representation (see Lemmas 4.9 and 4.11). This yields also the decomposition of $U_{\Gamma_0(p)}$ into its irreducible subrepresentations,

$$U_{\Gamma_0(p)} = U_t \oplus U_p \tag{1.44}$$

where U_t is the trivial one dimension representation of $\mathrm{PSL}(2,\mathbb{Z})$ and U_p is a certain p-dimensional representation of $\mathrm{PSL}(2,\mathbb{Z})$ (see Lemma 4.12). We obtain a similar decomposition for $U_{\overline{\Gamma}_0(q)}$ (see Lemma 4.13).

By applying the Gelfand pair method we prove that for a prime power $q = p^e$ with $e \geq 2$ the representation $\pi_{Q(q),X(q)}$ and hence $U_{\overline{\Gamma}_0(q)}$ is multiplicity-free (see lemma 4.27 and corollary 4.1). From this and by using Wielandt's theorem (see lemma 4.14) it follows that $U_{\Gamma_0(q)}$ and $\pi_{Q(q),X(q)}$ are also multiplicity-free (see lemma 4.29).

Since $U_{\Gamma_0(q)}$ and $\pi_{Q(q),X(q)}$ are multiplicity-free representations, according to Wielandt's theorem the number of non-isomorphic irreducible subrepresentations of $\pi_{Q(q),X(q)}$ and hence that of $U_{\Gamma_0(q)}$ is equal to the number of orbits of the group $K(q) = \Gamma_0(q)/H(q)$ in $X(q)$. Similarly, the number of non-isomorphic irreducible subrepresentations of $\overline{U}_{\Gamma_0(q)}$ is equal to the number of orbits of the group $\overline{K}(q) = \overline{\Gamma}_0(q)/M(q)$ in $\overline{X}(q) = \mathrm{PGL}(2,\mathbb{Z})/\overline{\Gamma}_0(q)$ where $M(q) = \ker U_{\overline{\Gamma}_0(q)}$ is a known group. We determine the number of these orbits by calculating the order of the stabilizer of each point of the sets $X(q)$ and $\overline{X}(q)$ respectively, from which we derive the number of non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(q)}$ respectively $U_{\overline{\Gamma}_0(q)}$ (see lemmas 4.23 and 4.26 and remark 4.2).

According to the Jacquet-Langlands correspondence the space of Maass cusp forms $\mathcal{S}(s;\mathcal{O}_n)$ for $\mathcal{O}_n$, the unit group of an indefinite quaternion devision algebra of discriminant n, is in bijection with the space $\mathcal{S}^{new}(s;\Gamma_0(n))$ of new forms of Atkin-Lehner for the Hecke congruence group $\Gamma_0(n)$ [**54, 7**] where the discriminant n is a product of an even number of different primes [**54**]. We introduce the subspace $F^{new}(s;\Gamma_0(n)) \subset F(s;\Gamma_0(n))$ of new eigenfunctions of the transfer operator $P\mathcal{L}_s^{\Gamma_0(n)}$ as $F^{new}(s;\Gamma_0(n)) = \mathcal{P} \circ \mathcal{I} \circ B\mathcal{S}^{new}(s;\Gamma_0(n))$. The expectation, that the restriction of the transfer operator to the corresponding subspace $F^{new}(s;\Gamma_0(n))$ of eigenfunctions coincides with this operator corresponding to one of the irreducible components of $U_{\Gamma_0(n)}$ with a singular representation (see Definition 2.3) is not true as we show in the case $\Gamma_0(6)$. Indeed, the representation $U_{\Gamma_0(6)}$ has a decomposition into irreducibles given by

$$M_{\Gamma_0(6)}U_{\Gamma_0(6)}M_{\Gamma_0(6)}^{-1} = U_t \oplus U_2 \oplus U_3 \oplus U_2 \otimes U_3 \tag{1.45}$$

where $M_{\Gamma_0(6)}$ is some known matrix and where the irreducible subrepresentations in the right hand side are all regular (see Definition 2.4). This yields the following decomposition of the transfer operator for $\Gamma_0(6)$

$$\begin{aligned} & M_{\Gamma_0(6)} P \mathcal{L}_s^{\Gamma_0(6),+} M_{\Gamma_0(6)}^{-1} = \\ & P_1 \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),+} \oplus P_2 \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2,+} \oplus P_3 \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_3,+} \oplus \\ & \oplus P_{2\times 3} \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2\otimes U_3,+}, \end{aligned} \tag{1.46}$$

where the symmetry P of the transfer operator $\mathcal{L}_s^{\Gamma_0(6)}$ corresponding to the automorphism $J\mathfrak{u}(z) = \mathfrak{u}(-\overline{z})$ of the Maass cusp forms of $\Gamma_0(6)$ is decomposed into the operators P_i as

$$M_{\Gamma_0(6)} P M_{\Gamma_0(6)}^{-1} = P_1 \oplus P_2 \oplus P_3 \oplus P_{2\times 3}, \tag{1.47}$$

which themselves are symmetry operators of the transfer operators $\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z})}$, $\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2}$, $\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_3}$, and $\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2\otimes U_3}$ respectively. The space of eigenfunctions $F(s;\Gamma_0(6))$ of $P\mathcal{L}_s^{\Gamma_0(6),+}$ is then decomposed as

$$\begin{aligned} & F(s;\Gamma_0(6)) \cong M_{\Gamma_0(6)} F(s;\Gamma_0(6)) = F(s;\mathrm{PSL}(2,\mathbb{Z})) \oplus \\ & F(s;\mathrm{PSL}(2,\mathbb{Z});U_2) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_3) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3). \end{aligned} \tag{1.48}$$

It turns out that the space $F(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$ is in bijection with a subspace of $F(s;\Gamma_0(6))$ which besides $F^{new}(s;\Gamma_0(6))$ contains part of the old subspace $F^{old}(s;\Gamma_0(6))$ in $F(s;\Gamma_0(6))$. Contrary to this, for the transfer operator $\mathcal{L}_s^{\Gamma(2)}$ for the principal congruence subgroup $\Gamma(2)$ the representation $U_{\Gamma(2)}$ contains a certain singular subrepresentation U_{sgn} such that the corresponding transfer operator $\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_{sgn}}$ describes exactly the space of new eigenfunctions, that is, $F(s;\mathrm{PSL}(2,\mathbb{Z});U_{sgn}) \cong F^{new}(s;\Gamma_0(6))$.

We introduce finally a method to determine the congruence property of a representation χ of a subgroup Γ of $\mathrm{PSL}(2,\mathbb{Z})$ via its induction to $\mathrm{PSL}(2,\mathbb{Z})$, which seems to be more suitable for the transfer operator approach to the spectral theory of automorphic functions. The representation χ is known to be congruence if its kernel is a congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, that is, its kernel contains a principal congruence subgroup $\Gamma(n)$ of $\mathrm{PSL}(2,\mathbb{Z})$ for some n. The spectrum of the automorphic Laplacian $\Delta(\Gamma;\chi)$ is known to depend strongly on such a congruence property of χ [**5, 40**]. On the other hand, this spectrum is connected to the spectrum of the transfer operator $\mathcal{L}_s^{\Gamma;\chi}$, and hence depends on the congruence property of the representation ρ_χ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from χ. Indeed, we prove that χ is congruence if and only if ρ_χ is congruence. By applying this to Selberg's family of characters χ_α

of $\Gamma_0(4)$ we rederive the result (see [**8**] and the references there), that in the interval $[0, \frac{1}{2}]$ χ_α is congruence only for $\alpha = 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$.

Moreover, for non-congruence representations ρ_{χ_α} we show that the Fuchsian group $\ker \rho_{\chi_\alpha}$ for infinitely many values of α does not satisfy Zograf's hypothesis, namely that the hyperbolic area of their fundamental domains is smaller than $32\pi(g_\alpha + 1)$, where g_α denotes the genus of the group.

In detail this thesis is organized as follows:

In chapter 2 we recall for a Fuchsian group Γ of the first kind with an arbitrary unitary representation χ the basics of the spectral theory of automorphic functions, namely

- The spectral decomposition of the automorphic Laplacian,
- the Eisenstein series,
- Selberg's trace formula,
- Selberg's zeta function.

In chapter 3 we recall a generalized version of Mayer's transfer operator for a subgroup Γ of finite index μ in $\mathrm{PSL}(2, \mathbb{Z})$ and an arbitrary finite dimensional unitary representation and introduce its symmetry operators. We recall a generalized version of the Lewis and Zagier theory of period functions for finite index subgroups Γ of the modular group and a unitary representation χ. We show that the period functions for Γ and χ can be expressed as an integral transform of the corresponding Maass cusp forms. Using a result by Hilgert and Deitmar we also show that the space of eigenfunctions for eigenvalue ± 1 of the generalized transfer operator of Mayer with a symmetry operator P, satisfying certain asymptotics at zero, is in bijection with the corresponding space of period functions and hence with the corresponding space of Maass cusp forms.

We show that the symmetry operators P of the space of period functions are induced from the automorphisms of the space of Maass cusp forms via the aforementioned integral transform.

In chapter 4, we study the representation $U_{\Gamma_0(n)}$ of $\mathrm{PSL}(2, \mathbb{Z})$ induced from the trivial representation of the Hecke congruence subgroup $\Gamma_0(n)$ and the representation $U_{\overline{\Gamma}_0(n)}$ of $\mathrm{PGL}(2, \mathbb{Z})$ induced from the trivial representation of the extension $\overline{\Gamma}_0(n)$ of the Hecke congruence subgroups to $\mathrm{PGL}(2, \mathbb{Z})$. In particular, we show that

- For coprime n and m,

$$U_{\Gamma_0(nm)} \cong U_{\Gamma_0(n)} \otimes U_{\Gamma_0(m)} \tag{1.49}$$

  respectively

$$U_{\overline{\Gamma}_0(nm)} \cong U_{\overline{\Gamma}_0(n)} \otimes U_{\overline{\Gamma}_0(m)} \tag{1.50}$$

  where $\otimes$ denotes the tensor product of representations.
- For $n \mid m$ the representation $U_{\Gamma_0(n)}$ and $U_{\overline{\Gamma}_0(n)}$ are subrepresentations of $U_{\Gamma_0(m)}$ and $U_{\overline{\Gamma}_0(m)}$, respectively.

- The representation $U_{\Gamma_0(n)}$ and $U_{\overline{\Gamma}_0(n)}$ are multiplicity-free, that is, each irreducible subrepresentation appears once in the representation. We determine also the number of non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(n)}$ respectively of $U_{\overline{\Gamma}_0(n)}$.

Concerning the Jacquet-Langlands correspondence, we decompose in chapter 5 the representations $U_{\Gamma_0(6)}$ and $U_{\overline{\Gamma}_0(6)}$ into their irreducible subrepresentations, which leads to a decomposition of Mayer's transfer operator for $\Gamma_0(6)$ respectively the space of its eigenfunctions with eigenvalues ± 1. In this decomposition each subspace is characterized by an irreducible subrepresentation of $U_{\Gamma_0(6)}$. We study the correspondence between these subspaces and Atkin-Lehner's theory of old and new Maass cusp forms. In particular we show with this example, that one cannot characterize the space of new Maass cusp forms respectively the corresponding space of new eigenfunctions of the transfer operator by an irreducible singular subrepresentation of $U_{\Gamma_0(6)}$. Contrary to this we recall in the case of the transfer operator for the principal congruence subgroup $\Gamma(2)$, that the space of new eigenfunctions of Mayer's transfer operator corresponding to the space of new Maass cusp forms is characterized by a singular subrepresentation of $U_{\Gamma(2)}$ which especially leads to a transfer operator holomorphic in the entire s-plane [**13**].

In chapter 6 we recall the notion of a congruence representation and study the congruence properties of a family of characters χ_α for $\Gamma_0(4)$, introduced by Selberg, via its induction to $\mathrm{PSL}(2,\mathbb{Z})$. We find, that $\ker \rho_{\chi_\alpha}$ for a non-congruence representation ρ_{χ_α} is a group which does not satisfy the hypothesis of Zograf's criterion for the existence of small eigenvalues of the automorphic Laplacian (see theorem 6.9), which together with Selberg's lower bound for the eigenvalues of the automorphic Laplacian for congruence subgroups showed immediately, that χ_α can be congruence only for a finite number of α-values.

CHAPTER 2

Spectral theory of automorphic functions

In this chapter we fix some notations and recall briefly some definitions and results from the spectral theory of the automorphic Laplacian which we need in the sequel.

2.1. The hyperbolic plane and its isometries

We denote the hyperbolic plane by $\mathbb{H}$ and realize it as the upper half plane,

$$\{x+iy \in \mathbb{C} \mid y>0\} \tag{2.1}$$

equipped with the Poincaré metric,

$$ds^2 = \frac{dx^2+dy^2}{y^2} \tag{2.2}$$

and corresponding Poincaré measure,

$$d\mu(z) = \frac{dxdy}{y^2}. \tag{2.3}$$

The Laplace operator $-\Delta$ on $\mathbb{H}$ associated to the Poincaré metric has in Cartesian coordinates the following explicit form,

$$-\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{2.4}$$

The operator Δ is called the hyperbolic Laplacian. The group of all orientation preserving isometries of $\mathbb{H}$ is identified with the group

$$\mathrm{PSL}(2,\mathbb{R}) := \left\{ g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det g = 1, \quad a,b,c,d \in \mathbb{R} \right\}, \tag{2.5}$$

acting on $\mathbb{H}$ by linear fractional transformations,

$$\begin{aligned} \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (g,z) &\mapsto gz = \frac{az+b}{cz+d}. \end{aligned} \tag{2.6}$$

The elements of $\mathrm{PSL}(2,\mathbb{R})$ different from the identity are classified into three disjoint classes according to their traces as matrices. An element $g \in \mathrm{PSL}(2,\mathbb{R})$ is called elliptic, parabolic or hyperbolic if $|\,tr(g)\,| < 2$, $|\,tr(g)\,| = 2$ or $|\,tr(g)\,| > 2$, respectively. Since the action of $\mathrm{PSL}(2,\mathbb{R})$ can be extended by continuity to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, this classification can be reformulated as follows: elliptic elements have only one fixed point in $\mathbb{H}$, the parabolic ones have one unique fixed point on $\mathbb{R} \cup \{\infty\}$, and the hyperbolic elements have two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

2.2. Fuchsian groups

A discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ is called a Fuchsian group. A cusp of a Fuchsian group Γ is defined to be the fixed point of a parabolic element of Γ. A fundamental domain F_Γ for a Fuchsian group Γ is defined to be the closure of a domain $U \subset \mathbb{H}$ containing all Γ-inequivalent points of $\mathbb{H}$, such that

$$\mathbb{H} = \underset{\gamma \in \Gamma}{\cup} \gamma F_\Gamma.$$

The volume of the quotient space $\Gamma \setminus \mathbb{H}$ or what is the same, the fundamental domain F_Γ is given by

$$vol(\Gamma \setminus \mathbb{H}) := |F_\Gamma| := \int_{F_\Gamma} d\mu(z). \tag{2.7}$$

A Fuchsian group of the first kind is a Fuchsian group Γ for which the volume of $\Gamma \setminus \mathbb{H}$ is finite. If the surface $\Gamma \setminus \mathbb{H}$ is compact, the group Γ is called cocompact. We refer to [**35**] for more details about the definitions above. A Fuchsian group Γ of the first kind is determined by [**55**]

1. a finite number $2g$ of hyperbolic generators, $A_1, B_1, \ldots, A_g, B_g$
2. a finite number l of elliptic generators, $R_1, \ldots, R_l$
3. a finite number h of parabolic generators, $S_1, \ldots, S_h$

such that the following relations hold

$$\begin{gathered}[A_1, B_1] \ldots [A_g, B_g] S_1 \ldots S_h R_1 \ldots R_l = id, \\ R_1^{m_1} = id, \ldots, R_l^{m_l} = id.\end{gathered} \tag{2.8}$$

Here id is the identity element of the group, $m_j \in \mathbb{N}$ with $m_j \geq 2$ is the order of the elliptic element R_j, and $[,]$ denotes the commutator which for two elements A and B of the group is defined by $[A, B] = ABA^{-1}B^{-1}$. The signature of a group Γ, determined by the set of generators, is defined to be the set of numbers

$$(g; m_1, \ldots, m_l; h), \tag{2.9}$$

which is a topological invariant of the group like the fundamental group of the corresponding surface. We note that the genus g of the surface $\Gamma \setminus \mathbb{H}$ is half the number of hyperbolic generators of Γ and h is the number of inequivalent cusps of the group Γ. Moreover, the group Γ is cocompact if and only if $h = 0$. For a Fuchsian group of the first kind with signature as in (2.9), the volume of $\Gamma \setminus \mathbb{H}$ is given by the Gauss-Bonnet formula [**57**],

$$|F_\Gamma| = 2\pi \left(2g - 2 + \sum_{j=1}^{l} (1 - \frac{1}{m_j}) + h \right). \tag{2.10}$$

The projective modular group is an example of a Fuchsian group of the first kind. It is defined by

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ a,b,c,d \in \mathbb{Z} \right\}. \tag{2.11}$$

This group has signature $(0;3,2;1)$ and it is generated by the parabolic element

$$T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

and the elliptic element

$$S = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.13}$$

with the relations $S^2 = (ST)^3 = \pm id$.

2.3. Spectral decomposition of the automorphic Laplacian

The notion of automorphic functions and operators on $\mathbb{H}$ is defined by their transformation property under the action of Fuchsian groups. Let V denote a finite dimensional Hermitian space and χ be a unitary representation of Γ on V. Then a vector valued function $\mathfrak{f} : \mathbb{H} \to V$ such that

$$\mathfrak{f}(\gamma z) = \chi(\gamma)\mathfrak{f}(z), \quad \forall \gamma \in \Gamma, \tag{2.14}$$

is called an automorphic function with respect to Γ and χ. We denote by $\mathcal{H}(\Gamma;\chi)$ the Hilbert space of square integrable automorphic functions on F_Γ with the scalar product given by

$$(\mathfrak{f},\mathfrak{h}) = \int_{F_\Gamma} \langle \mathfrak{f}(z), \mathfrak{h}(z)\rangle_V \, d\mu(z), \quad \mathfrak{f},\mathfrak{h} \in \mathcal{H}(\Gamma;\chi), \tag{2.15}$$

where F_Γ is the fundamental domain of Γ and $\langle , \rangle_V$ denotes the Hermitian form on V.

The action of the Laplace operator on a n-dimensional vector valued function $\mathfrak{f} \in C^\infty(\mathbb{H})$ is defined to be componentwise, namely $\Delta\mathfrak{f}$ is a n-dimensional vector valued function with components given by $(\Delta\mathfrak{f})_i = \Delta f_i$, $1 \le i \le n$. Let $\mathfrak{D}$ be the dense domain in $\mathcal{H}(\Gamma;\chi)$ consisting of the functions $\mathfrak{g}$ which are restrictions of functions $\mathfrak{f}$ on $\mathbb{H}$ to F_Γ, such that

- $\mathfrak{f} \in C^\infty(\mathbb{H})$
- $\mathfrak{f}(\gamma z) = \chi(\gamma)\mathfrak{f}(z),\ \gamma \in \Gamma$
- $\mathfrak{f}$ and $\Delta\mathfrak{f}$ both are uniformly bounded for $z \in \mathbb{H}$

Then an operator $\widetilde{\Delta}$ is defined by

$$\widetilde{\Delta}\mathfrak{g} = \Delta\mathfrak{f}, \quad \mathfrak{g} \in \mathfrak{D}, \quad \mathfrak{g} = \mathfrak{f}|_{F_\Gamma} \tag{2.16}$$

which is symmetric and non-negative definite. It turns out that this operator is essentially self adjoint [**55, 16**]. The automorphic Laplacian

$\Delta(\Gamma;\chi)$ is defined as the unique self-adjoint extension (Friedrichs extension) of the operator $\widetilde{\Delta}$ on $\mathcal{H}(\Gamma;\chi)$. Thus the automorphic Laplacian $\Delta(\Gamma;\chi)$ is a self-adjoint, non-negative definite, unbounded operator. For more details and proofs of these assertions see [**55, 16**].

Let Γ be a Fuchsian group of the first kind with inequivalent cusps x_α, $1 \leq \alpha \leq h$. An element $\gamma \in \Gamma$ is primitive if it can not be written as a power of another element of the group. Let S_α be a primitive parabolic element leaving the cusp x_α invariant, namely $S_\alpha x_\alpha = x_\alpha$. Then S_α is the generator of the stabilizer group of the cusp x_α, denoted by Γ_α.

Definition 2.1. *A finite dimensional representation χ of Γ is called singular in the cusp x_α if*

$$\dim \ker(\chi(S_\alpha) - 1_V) = 0, \tag{2.17}$$

where 1_V is the identity operator in V. Otherwise we say that the representation χ is regular in the cusp x_α.

Definition 2.2. *If χ is regular at a cusp x_α, we say that the cusp is open, otherwise the cusp is closed.*

Definition 2.3. *A representation χ is called singular if it is singular in all cusps.*

Definition 2.4. *The representation χ is called regular if it is regular at least in one cusp, that is, if at least one cusp is open.*

Remark 2.1. *The definition of singularity of a representation is opposite to Selberg's definition (see* [**49**], [**57**]*). It is more reasonable from the point of view of mathematical physics as argued by E. Balslev* [1]*: if the spectrum of the hyperbolic Laplacian on a non-compact surface is purely discrete, this situation is rather singular. In the regular situation, we have in general a continuous spectrum and maybe a discrete one.*

For cocompact groups and also for cofinite groups with singular representation the automorphic Laplacian $\Delta(\Gamma;\chi)$ has only a purely discrete spectrum in $\mathcal{H}(\Gamma;\chi)$ (see [**57**] pages 17 and 18).

For cofinite groups with a regular representation χ the automorphic Laplacian $\Delta(\Gamma;\chi)$ in $\mathcal{H}(\Gamma;\chi)$ has a continuous spectrum and maybe a discrete one. The continuous spectrum is described by the Eisenstein series [**57, 55**]. To define them, we need some notations. The subspace $V_\alpha \subset V$ for x_α a cusp with $S_\alpha x_\alpha = x_\alpha$ is defined by

$$V_\alpha := \{v \in V \mid \chi(S_\alpha)v = v\}\,, \tag{2.18}$$

its dimension is denoted by $k_\alpha := \dim V_\alpha$. We denote furthermore an orthonormal basis of V_α by $\{e_l(\alpha)\}_{l=1}^{k_\alpha}$.

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[1] Private communication between Alexei Venkov and Erik Balslev

DEFINITION 2.5. *The degree of regularity of the representation* χ *is defined as*

$$k := k(\Gamma;\chi) := \sum_{\alpha=1}^{h} k_\alpha. \tag{2.19}$$

Let P_α be the orthogonal projection of V onto V_α and χ^* the adjoint representation of χ in the Hermitian space V. For an open cusp x_α, take $\sigma_\alpha \in \mathrm{PSL}(2,\mathbb{R})$ such that $\sigma_\alpha\infty = x_\alpha$. The Eisenstein series $E_\alpha(.,s) : \mathbb{H} \to V_\alpha$ for $\Re(s) > 1$ is defined to be a k_α-dimensional vector whose components $E_{\alpha,l}$ are given by

$$E_{\alpha,l}(z,s) = \sum_{\sigma\in\Gamma_\alpha\backslash\Gamma} (\mathrm{Im}(\sigma_\alpha^{-1}\sigma z))^s \chi^*(\sigma) e_l(\alpha), \quad 1 \le l \le k_\alpha. \tag{2.20}$$

This is an absolutely convergent series in the domain $\Re(s) > 1$ and it is uniformly convergent in z on any compact subsets of $\mathbb{H}$. In the half-plane $\Re(s) > 1$, the Eisenstein series has the following properties [**57, 55**]:

1. $E_\alpha(z,s)$ is holomorphic in s.
2. For fixed s, $\Delta E_\alpha(z,s) = s(1-s)E_\alpha(z,s)$.
3. For fixed s, $E_\alpha(z,s)$ is automorphic relative to Γ and χ.
4. The zero-th order term of the Fourier expansion of $E_\alpha(z,s)$ at a cusp x_β, that is, the constant term of the Fourier expansion of $P_\beta E_\alpha(\sigma_\beta z, s)$ with $\sigma_\beta\infty = x_\beta$ has components given by

$$\delta_{\alpha,\beta} y^s e_l(\alpha) + \phi_{\alpha l,\beta}(s) y^{1-s}. \tag{2.21}$$

   For the complete description of the Fourier expansion of an Eisenstein series see ([**55**], page 45, Theorem 3.1.2 and page 47, Theorem 3.1.3).

The entries of the automorphic scattering matrix

$$\Phi(s) = \Phi(s;\Gamma;\chi) := \{\Phi_{bd}(s)\}_{b,d=1}^{k(\Gamma;\chi)} \tag{2.22}$$

are given by

$$\Phi_{bd}(s) = \Phi_{\alpha l,\beta k}(s) = \langle e_k(\beta), \phi_{\alpha l,\beta}(s)\rangle_V. \tag{2.23}$$

The indices b and d are defined by

$$b = k_1 + k_2 + \ldots + k_{\alpha-1} + l, \quad d = k_1 + k_2 + \ldots + k_{\beta-1} + k \tag{2.24}$$

such that

$$1 \le \alpha,\beta \le h, \quad 1 \le l \le k_\alpha, \quad 1 \le k \le k_\beta. \tag{2.25}$$

Then the following theorem holds [**57, 55**]

THEOREM 2.1. **a)** *The scattering matrix* $\Phi(s)$ *and the Eisenstein series* $E_\alpha(z,s)$ *admit meromorphic continuations to the entire* s*-plane.*

b) *In the half plane* $\Re(s) \geq \frac{1}{2}$, $\Phi(s)$ *and* $E_\alpha(z,s)$ *have only a finite number of common simple poles* $s_j \in (\frac{1}{2}, 1]$ *such that* $\lambda_j := s_j(1-s_j)$ *is a real eigenvalue of* $\Delta(\Gamma;\chi)$.

c) *The scattering matrix fulfills the functional equation*

$$\Phi(s)\Phi(1-s) = Id \tag{2.26}$$

it is unitary on $\Re(s) = \frac{1}{2}$ *and hermitian for real* s *which follows from* $\Phi(s) = \overline{\Phi(\overline{s})}^T$ *for arbitrary* s.

d) *The Eisenstein series satisfies the following functional equation,*

$$E(z,s) = \Phi(s)E(z,1-s) \tag{2.27}$$

where

$$E(z,s) = (E_1(z,s), \ldots, E_h(z,s))^t \tag{2.28}$$

and ^{t} *denotes the vector transpose.*

Definition 2.6. *An element* $\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$ *is called a cuspidal automorphic function with respect to* Γ *and* χ *iff for each open cusp* x_α

$$\int_0^1 \langle \mathfrak{f}(\sigma_\alpha z), v\rangle_V \, dx = 0 \tag{2.29}$$

for any $y > 0$ *and* $v \in V_\alpha$. *We denote by* $\mathcal{S}(\Gamma;\chi)$ *the space of all cusp forms of* Γ *and* χ.

Definition 2.7. *An element* $\mathfrak{u} \in \mathcal{S}(\Gamma;\chi)$ *is called a Maass cusp form with spectral parameter* s *of* Γ *and* χ *if it is an eigenfunction of the automorphic Laplacian with eigenvalue* $s(1-s)$, *that is,*

$$\Delta(\Gamma;\chi)\mathfrak{u} = s(1-s)\mathfrak{u}, \quad \mathfrak{u} \in \mathcal{S}(\Gamma;\chi). \tag{2.30}$$

We denote by $\mathcal{S}(s;\Gamma;\chi)$ *the space of Maass cusp forms with spectral parameter* s *of* Γ *and* χ.

The spectral parameters of the Maass cusp forms define a discrete set of points s, lying on the line $\Re(s) = \frac{1}{2}$ and in the interval $\left(\frac{1}{2}, 1\right]$ [**57, 55, 28**].

For the stabilizer group Γ_α of the cusp x_α, and $\sigma_\alpha \in \mathrm{PSL}(2,\mathbb{R})$ an element such that $\sigma_\alpha\infty = x_\alpha$, one has

$$\sigma_\alpha^{-1}\Gamma_\alpha\sigma_\alpha = \{\pm T^n \mid n \in \mathbb{Z}\}. \tag{2.31}$$

For simplicity we assume that (otherwise we should consider a matrix conjugation of χ for each cusp)

$$\chi(\sigma_\alpha T \sigma_\alpha^{-1}) = \mathrm{diag}(\mathrm{e}^{2\pi \mathrm{i}\xi_1(\alpha)}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\xi_{\dim\chi}(\alpha)}) \tag{2.32}$$

with $0 \leq \xi_j(\alpha) < 1$, $1 \leq j \leq \dim\chi$, and where $\mathrm{diag}(\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n})$ denotes the diagonal matrix with entries $a_1, \ldots, a_n$. Then for each component $\mathfrak{u}_j$, $1 \leq j \leq \dim\chi$, of $\mathfrak{u} \in \mathcal{S}(s;\Gamma;\chi)$ we have

$$\mathfrak{u}_j(\sigma_\alpha(z+1)) = \exp(2\pi i\xi_j(\alpha))\mathfrak{u}_j(\sigma_\alpha z). \tag{2.33}$$

Since $\mathfrak{u}$ is in the Hilbert space and the components $\mathfrak{u}_j$ are real analytic, $\mathfrak{u}_j$ has at the cusp x_α a Fourier type expansion of the form (see [**57**], Chapter 5)

$$\mathfrak{u}_j(\sigma_\alpha z) = \sum_{m-\xi_j(\alpha)\in\mathbb{Z}} \rho_j(m,\alpha)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|m|y)\exp(2\pi imx) \tag{2.34}$$

with $z = x+iy$ and where the $\rho_j(m,\alpha)$ are complex numbers and $K_\nu(r)$ is the K-Bessel function.

Since $\mathfrak{u} \in \mathcal{S}(s;\Gamma;\chi)$ is real analytic, it is bounded in any compact domain in the fundamental domain of Γ. But the exponential decay of the K-Bessel functions in the Fourier type expansion (2.34) as $y \to \infty$ shows that the domain on which $\mathfrak{u}$ is bounded can be extended to infinity and hence $\mathfrak{u}$ is bounded on the whole fundamental domain. Then from Theorem 5.5 in ([**57**], page 33) it follows that

$$\rho_j(m,\alpha) = O(|m|^{\frac{1}{2}}), \quad m \neq 0, \quad 1 \leq j \leq \dim\chi. \tag{2.35}$$

This estimate and the fact that the K-Bessel functions in the Fourier type expansion of $\mathfrak{u}$ decay exponentially fast at infinity yields the fact that the Maass cusp form $\mathfrak{u} \in \mathcal{S}(s;\Gamma,\chi)$ decays exponentially fast at each cusp of the group Γ.

For $\psi \in C_0^\infty([0,\infty))$ a smooth function with compact support on the positive real line the so called incomplete theta series assigned to an open cusp x_α is defined by

$$\theta_{\psi,\alpha l}(z) = \sum_{\sigma\in\Gamma_\alpha\backslash\Gamma} \psi(\Im(\sigma_\alpha^{-1}\sigma z))\chi^*(\sigma)e_l(\alpha) \tag{2.36}$$

where $z = x+iy$ and Γ_α is the stabilizer group of x_α. We denote by $\Theta_1(\Gamma;\chi)$ the closure of the subspace of $\mathcal{H}(\Gamma;\chi)$ spanned by all incomplete theta series. Moreover, we denote by $\Theta_0(\Gamma;\chi)$ the finite dimensional space spanned by the residues of Eisenstein series at their finitely many poles in $(\frac{1}{2},1]$. The following theorem describes then the spectral decomposition of the automorphic Laplacian [**57**]:

THEOREM 2.2. *The Hilbert space $\mathcal{H}(\Gamma;\chi)$ can be decomposed into three invariant subspaces defined above, namely*

$$\mathcal{H}(\Gamma;\chi) = \mathcal{S}(\Gamma;\chi) \oplus \Theta_0(\Gamma;\chi) \oplus \Theta_1(\Gamma;\chi). \tag{2.37}$$

The automorphic Laplacian $\Delta(\Gamma;\chi)$ has a purely discrete spectrum on the space $\mathcal{S}(\Gamma;\chi) \oplus \Theta_0(\Gamma;\chi) \subset \mathcal{H}(\Gamma;\chi)$. The spectrum of $\Delta(\Gamma;\chi)$ on $\Theta_1(\Gamma;\chi)$ is absolutely continuous and given by the semi-axis $[\frac{1}{4},\infty)$ with finite multiplicity. It is determined by the Eisenstein series $E_\alpha(z, s = \frac{1}{2}+it)$ with $t \in \mathbb{R}$ which are obviously not in $\Theta_1(\Gamma;\chi)$.

2.4. Selberg's trace formula

We recall Selberg's trace formula from [**57**]. First we need some notations. For a fixed α let $\{e_l(\alpha)\}_{l=1}^{\dim V}$ be a basis of V in which $\chi(S_\alpha)(1_V - P_\alpha)$ is diagonal,

$$\chi(S_\alpha)(1_V - P_\alpha)e_l(\alpha) = \nu_{\alpha l} e_l(\alpha). \tag{2.38}$$

Then the following holds

$$\nu_{\alpha l} = \begin{cases} 0 & e_l(\alpha) \in V_\alpha, \\ \exp(2\pi i \theta_{\alpha l}) & e_l(\alpha) \in V \ominus V_\alpha. \end{cases} \tag{2.39}$$

where $0 < \theta_{\alpha l} < 1$.

THEOREM 2.3. *Let $\widetilde{h}(r) := h(r^2 + \frac{1}{4})$ be a function of the complex variable r which satisfies the following conditions:*

- *as a function of r, $\widetilde{h}(r)$ is holomorphic in the strip $\left\{r \in \mathbb{C} : |Im(r)| < \frac{1}{2} + \varepsilon\right\}$ for some $\varepsilon > 0$.*
- *in that strip $\widetilde{h}(r) = O((1 + |r^2|)^{-1-\varepsilon})$, such that all series and integrals appearing below converge absolutely.*

Moreover, let Γ be a Fuchsian group of the first kind with a unitary representation χ. Then for cofinite respectively cocompact Γ the following identities hold,

$$\sum_{k=0}^{\infty} h(\lambda_k) + C = I + H + E + P \tag{2.40}$$

respectively

$$\sum_{k=0}^{\infty} h(\lambda_k) = I + H + E \tag{2.41}$$

where $\{\lambda_n \mid 0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$ is the ordered set of the discrete eigenvalues of $\Delta(\Gamma;\chi)$. Here C describes the contribution of the continuous part of the spectrum and is given by

$$C = C(\widetilde{h}(r);\Gamma;\chi) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\varphi'}{\varphi}(\frac{1}{2}+ir;\Gamma;\chi)\widetilde{h}(r)dr + \frac{K_0}{4}\widetilde{h}(0), \tag{2.42}$$

where φ denotes the determinant of the scattering matrix $\Phi(s;\Gamma;\chi)$ and $K_0 = tr(\Phi(\frac{1}{2};\Gamma;\chi))$.

On the right hand side of (2.40) I describes the contribution of the identity element of the group, which is given by

$$I = I(\widetilde{h}(r);\Gamma;\chi) = \frac{\dim\chi |F_\Gamma|}{4\pi} \int_{-\infty}^{\infty} r \tanh(\pi r)\, \widetilde{h}(r) dr. \tag{2.43}$$

The term H denotes the contribution of the hyperbolic conjugacy classes and is given by

(2.44)

$$H = H(\widetilde{h}(r); \Gamma; \chi) = \sum_{\{\gamma\}_\Gamma} \sum_{m=1}^{\infty} \frac{tr_V \chi^m(\gamma) log \mathcal{N}(\gamma)}{\mathcal{N}(\gamma)^{\frac{m}{2}} - \mathcal{N}(\gamma)^{-\frac{m}{2}}} g(m \log \mathcal{N}(\gamma))$$

where $\mathcal{N}(\gamma)$ denotes the norm of the element $\gamma \in \{\gamma\}_\Gamma$ (for the definition of $\mathcal{N}(\gamma)$ see formula (2.48) in the next section) and the function g the Selberg transform of $\widetilde{h}(r)$:

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} \widetilde{h}(r) dr. \tag{2.45}$$

The term E refers to the contribution of the elliptic elements, and is given by the sum over the primitive elliptic conjugacy classes $\{R\}_\Gamma$ of order ν_R

$$\begin{aligned} &E = E(\widetilde{h}(r); \Gamma; \chi) = \\ &\frac{1}{2} \sum_{\{R\}_\Gamma} \sum_{m=1}^{\nu_R - 1} \frac{tr_V \chi^m(R)}{\nu_R \sin \pi m / \nu_R} \int_{-\infty}^{\infty} \frac{exp(-2\pi r m / \nu_R)}{1 + exp(-2\pi r)} \widetilde{h}(r) dr. \end{aligned} \tag{2.46}$$

Finally, the last term P refers to the parabolic conjugacy classes and is given by

$$\begin{aligned} &P = P(\widetilde{h}(r); \Gamma; \chi) = \\ &-\left(k(\Gamma; \chi) \ln 2 + \sum_{\alpha=1}^{h} \sum_{l=k_\alpha+1}^{\dim \chi} \ln |1 - \exp(2\pi i \theta_{\alpha l})| \right) g(0) \\ &- \frac{k(\Gamma; \chi)}{2\pi} \int_{-\infty}^{\infty} \psi(1 + ir) \widetilde{h}(r) dr + \frac{k(\Gamma; \chi)}{4} \widetilde{h}(0) \end{aligned} \tag{2.47}$$

where h is the number of inequivalent cusps and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ with $\Gamma'(z) = \frac{d}{dz}\Gamma(z)$ the di-gamma function.

2.5. Selberg's zeta function

Let Γ be a Fuchsian group of the first kind with a unitary representation χ. The Selberg zeta function for Γ and χ is defined in the domain $\Re(s) > 1$ by an absolutely convergent infinite product given by (see [**49**])

$$Z(s; \Gamma; \chi) = \prod_{k=0}^{\infty} \prod_{\{\gamma\}_\Gamma} \det(1_V - \chi(\gamma) \mathcal{N}(\gamma)^{-k-s}) \tag{2.48}$$

where $\{\gamma\}$ runs over all primitive hyperbolic conjugacy classes of Γ and $\mathcal{N}(\gamma) > 1$ denotes the norm of γ. By definition every hyperbolic element γ of the group Γ can be conjugated by an element from $\mathrm{PSL}(2, \mathbb{R})$

to a matrix of the form

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \tag{2.49}$$

with $\rho > 1$. The norm of γ is then defined by $\mathcal{N}(\gamma) = \rho^2$.

The Selberg trace formula provides a huge amount of information about Selberg's zeta function. Let us choose the test function $h(r)$ in the left hand side of the trace formula in (2.40) such that $\widetilde{h}(r) = h(\frac{1}{4} + r^2)$ has the form

$$\widetilde{h}(r) = \frac{1}{r^2 + \frac{1}{4} + s(s-1)} - \frac{1}{r^2 + \beta^2}, \quad (\beta > \frac{1}{2}, \quad \Re(s) > 1). \tag{2.50}$$

Then according to (2.45) the function g is given by

$$g(u) = \frac{1}{2s-1} e^{(s-\frac{1}{2})|u|} - \frac{1}{2\beta} e^{-\beta|u|}, \tag{2.51}$$

leading to the following identity [**57**]

$$\frac{d}{ds} H(s;\Gamma;\chi) = \frac{d}{ds} \frac{1}{2s-1} \frac{d}{ds} \log Z(s;\Gamma;\chi), \tag{2.52}$$

where $H(s;\Gamma;\chi)$ denotes the contribution of the hyperbolic elements in the trace formula and the s-dependence comes from the s-dependence of the test function $\widetilde{h}(r)$ in (2.50).

Based on (2.52) one proves that $Z(s;\Gamma;\chi)$ has an analytic (meromorphic) continuation to the entire complex s-plane and satisfies the functional equation

$$Z(1-s;\Gamma;\chi) = \Psi(s;\Gamma;\chi)\varphi(s;\Gamma;\chi)Z(s;\Gamma;\chi), \tag{2.53}$$

where $\Psi(s;\Gamma;\chi)$ is an explicitly known function and $\varphi(s;\Gamma;\chi)$ is the determinant of the scattering matrix [**57**]. We note, that for cocompact groups there is no scattering matrix and $\varphi(s;\Gamma;\chi)$ in (2.53) is not present.

The nontrivial zeros of $Z(s;\Gamma;\chi)$ are related to the eigenvalues of the automorphic Laplacian $\Delta(\Gamma;\chi)$ and its resonances, that is, the poles of the determinant of the scattering matrix [**57**]. Thereby for a cofinite group Γ one distinguishes the following sets of zeros of Selberg's zeta function:

1) The discrete set S_1 of s-values which are the spectral parameters of cusp forms. These are located symmetrically relative to the real axis on the line $\Re(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$ or in the interval $(\frac{1}{2}, 1]$ such that $\lambda = s(1-s)$ is an eigenvalue of the automorphic Laplacian $\Delta(\Gamma;\chi)$. The order of the zero s is equal to the multiplicity of the corresponding eigenvalue $\lambda = s(1-s)$ [**28, 57**].

2) The discrete set S_2 containing the poles of the determinant $\varphi(s;\Gamma;\chi)$ of the scattering matrix in the half plane $\Re(s) < \frac{1}{2}$. The order of the zero at the point $s \in S_2$ is equal to the order of the pole of $\varphi(s;\Gamma;\chi)$ at this point [**57**].

3) The set S_3 consisting of the finitely many poles of the scattering matrix in the interval $(\frac{1}{2}, 1]$. For $s \in S_3$, $\lambda = s(1-s)$ is an eigenvalue of the automorphic Laplacian. The order of the zero $s \in S_3$ is identical to the multiplicity of the eigenvalue $\lambda = s(1-s)$ [**57**].

4) The set S_4^+ of zeros consisting of the integers $-j$, $j \in \mathbb{N} \cup 0$ such that n_j given by [**57**]

$$n_j := \frac{\dim V|F|}{\pi}(j + \frac{1}{2}) - \sum_{\{R\}_\Gamma} \sum_{k=1}^{\nu-1} \frac{tr_V \chi(R^k)}{\nu \sin \frac{k\pi}{\nu_R}} \sin(\frac{k\pi(2j+1)}{\nu_R}), \tag{2.54}$$

fulfills $n_j > 0$, which is the order of the zero of Selberg's zeta function at $s = -j$.

The poles of Selberg's zeta function on the other hand are given by [**57**]:

1 the point $s = \frac{1}{2}$ of order $\frac{1}{2}(k(\Gamma;\chi) - \mathrm{tr}\Phi(\frac{1}{2};\Gamma;\chi))$,

2 the set S_5 of so called trivial poles at the points $(-j + \frac{1}{2})$, $j \in \mathbb{N}$ of order $k(\Gamma;\chi)$,

3 the finite set S_4^- of poles at the non-positive integers $-j$, $j \in \mathbb{N} \cup 0$ of order $-n_j > 0$ where n_j is given in (2.54).

For a cocompact group Γ, Selberg's zeta function $Z(s;\Gamma;\chi)$ is holomorphic in the entire complex s-plane with the zeros described by the aforementioned sets of zeros in items 1) and 4).

CHAPTER 3

Mayer's Transfer operator approach to the spectral theory of automorphic functions

This chapter recalls briefly Mayer's transfer operator and its aspects concerning the spectral theory of automorphic functions. In particular we present a short review of the Lewis-Zagier correspondence connecting the eigenfunctions of Mayer's transfer operator to automorphic functions.

3.1. Mayer's transfer operator

To begin with, we recall the general notion of the Ruelle transfer operator from [**45**] and [**46**]. Consider a weighted dynamical system described by a map $\tau : M \to M$ with a weight function $g : M \to \mathbb{R}$ such that τ is not invertible with at most a countable set of inverse branches. The Ruelle dynamical zeta function assigned to this dynamical system is defined by

$$\zeta(z) = \exp\Big(\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in Fix\tau^m} \prod_{k=0}^{m-1} g(\tau^k x)\Big) \tag{3.1}$$

where $Fix\tau^m$ denotes the set of fixed points of the map τ^m. We assume that this sum is convergent in some domain of $z \in \mathbb{C}$. The Ruelle transfer operator for the aforementioned dynamical system is defined by

$$(\mathcal{L}f)(x) = \sum_{y \in \tau^{-1}\{x\}} g(y)f(y). \tag{3.2}$$

The analytic properties of the dynamical zeta function gives some information about the underlying dynamical system. In the transfer operator approach one tries to express dynamical zeta functions in terms of Fredholm determinants of some operator, which provides a framework to study their analytic properties via this operator. In particular, the spectrum of the transfer operator yields among other properties the position of the zeros of the dynamical zeta function. We note that there is not such an expression of the zeta function in terms of a Fredholm determinant for a general dynamical system and one has to treat each case separately.

One of the most important realizations of this program is Mayer's transfer operator approach to Selberg's zeta function. We recall it briefly from [**14**] and refer for more details to the references there. The motion of a free particle on the surface $\Gamma\backslash\mathbb{H}$, where Γ is for instance a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$, is described by the geodesic flow on the unit tangent bundle of the surface, that is,

$$\phi_t : S(\Gamma\backslash\mathbb{H}) \to S(\Gamma\backslash\mathbb{H}). \tag{3.3}$$

Mayer constructs a Poincaré section X_Γ in the unit tangent bundle $S(\Gamma\backslash\mathbb{H})$, which, in a suitable coordinate system, is given by $[0,1]\times[0,1]\times\mathbb{Z}_2\times\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z})$. With respect to this section, the Poincaré return map $P_\Gamma : X_\Gamma \to X_\Gamma$ for the geodesic flow is then given by

$$\begin{aligned} &P_\Gamma : [0,1]\times[0,1]\times\mathbb{Z}_2\times\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}) \\ &\ \to [0,1]\times[0,1]\times\mathbb{Z}_2\times\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}) \\ &(x,y,\epsilon,[r]_\Gamma) \mapsto (T_G x, \frac{1}{y+n}, -\epsilon, [rT^{n\epsilon}S]_\Gamma) \end{aligned} \tag{3.4}$$

with $n = n(x) = [\frac{1}{x}]$, where $[x]$ denotes the integer part of $x \in [0,1]$ and $T_G : [0,1] \to [0,1]$ is the Gauss map given by

$$T_G(x) = \begin{cases} \dfrac{1}{x} - [\dfrac{1}{x}], & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3.5}$$

Its action in the expanding direction is then

$$\begin{aligned} &P_{\Gamma,ex} : [0,1]\times\mathbb{Z}_2\times\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}) \to [0,1]\times\mathbb{Z}_2\times\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}) \\ &(x,\epsilon,[r]_\Gamma) \mapsto (T_G x, -\epsilon, [rT^{n\epsilon}S]_\Gamma). \end{aligned} \tag{3.6}$$

Mayer's transfer operator for $\mathrm{PSL}(2,\mathbb{Z})$ is defined as the Ruelle transfer operator given in (3.2) for the dynamical system described by this Poincaré map and the weight function $\exp(-sA(y))$ where $A(y) = \log|T'_G(y)|$, $s \in \mathbb{C}$ and $y = \frac{1}{x+n}$, $n \in \mathbb{N}$.

We recall briefly the explicit form of this operator as well as its domain of definition in the case of a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$ and a representation χ of Γ. For this we need to recall the notion of induced representation of a group. Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of finite index $\mu = [\mathrm{PSL}(2,\mathbb{Z}) : \Gamma]$ and $\chi : \Gamma \to \mathrm{Aut}\, W$ be a unitary representation of Γ in the Hermitian vector space W. Also, let V be the direct sum of μ copies of W, that is,

$$V = \oplus_{j=1}^{\mu} W. \tag{3.7}$$

Definition 3.1. *The representation $\rho_\chi : \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{Aut}V$ induced from the representation χ of Γ is defined by*

$$(\rho_\chi(g)v)_i = \sum_{j=1}^{\mu} \delta_{\Gamma,\chi}(r_i g r_j^{-1}) w_j, \quad v = \oplus_{j=1}^{\mu} w_j \in V, \quad w_j \in W, \tag{3.8}$$

where $\{r_1, r_2, \ldots, r_\mu\}$ *is a set of representatives of the right cosets of* Γ *in* $\mathrm{PSL}(2,\mathbb{Z})$*,*

$$\delta_{\Gamma,\chi}(\gamma) = \begin{cases} \chi(\gamma), & \gamma \in \Gamma, \\ 0_W, & \gamma \notin \Gamma, \end{cases} \tag{3.9}$$

and 0_W *is the* 0*-map in* W*.*

Let $D \subset \mathbb{C}$ be the following open disc

$$D = \left\{ z \in \mathbb{C} \mid |z-1| < \frac{3}{2} \right\}. \tag{3.10}$$

Then the Banach space $B(D)$ is defined by

$$B(D) = \left\{ \vartheta : D \to \mathbb{C} \mid \vartheta \text{ holomorphic on } D \text{ and continuous on } \overline{D} \right\} \tag{3.11}$$

with the sup norm $\|\vartheta\| = \sup_{z \in \overline{D}} |\vartheta|$.

Definition 3.2. *Mayer's transfer operator* $\mathcal{L}_s^{\Gamma,\chi}$ *for the group* Γ *and representation* χ *is defined to be the operator*

$$\mathcal{L}_s^{\Gamma,\chi} = \begin{pmatrix} 0 & \mathcal{L}_s^{\Gamma,\chi,+} \\ \mathcal{L}_s^{\Gamma,\chi,-} & 0 \end{pmatrix} : \bigoplus_{i=1}^{2\mu \dim \chi} B(D) \to \bigoplus_{i=1}^{2\mu \dim \chi} B(D), \tag{3.12}$$

with $\mathcal{L}_s^{\Gamma,\chi,\pm}$ *given in the domain* $\Re(s) > 1/2$ *by*

$$\begin{aligned} &\mathcal{L}_s^{\Gamma,\chi,\pm} : \bigoplus_{i=1}^{\mu \dim \chi} B(D) \to \bigoplus_{i=1}^{\mu \dim \chi} B(D) \\ &\mathcal{L}_s^{\Gamma,\chi,\pm} f(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2s} \rho_\chi(ST^{\pm n}) f(\frac{1}{z+n}). \end{aligned} \tag{3.13}$$

The family of operators $\mathcal{L}_s^{\Gamma,\chi}$ and $\mathcal{L}_s^{\Gamma,\chi,\pm}$ can be extended to a meromorphic family of operators in the entire complex s plane with possible poles at $s_k = \frac{1-k}{2}$, $k = 0, 1, 2, \ldots$ [**14**]. Moreover, Mayer's transfer operator is a nuclear operator of order zero and its Fredholm determinant is defined in the sense of Grothendieck by [**14**]

$$\det(1 - \mathcal{L}_s^{\Gamma,\chi}) = \exp\left(- \sum_{n=1}^{\infty} \frac{\mathrm{tr}\left[\mathcal{L}_s^{\Gamma,\chi}\right]^n}{n} \right). \tag{3.14}$$

On the other hand, it is known that Selberg's zeta function can be interpreted as a dynamical zeta function. In terms of the closed orbits of the geodesic flow in (3.3), it can be written as

$$Z(s;\Gamma;\chi) = \prod_{\gamma} \prod_{k=0}^{\infty} \det\left[1_W - \chi(g_\gamma) \exp(-(s+k) l(\gamma))\right] \tag{3.15}$$

where γ is a primitive periodic orbit of the geodesic flow $\phi_t : S(\Gamma\backslash\mathbb{H}) \to S(\Gamma\backslash\mathbb{H})$ and $g_\gamma \in \Gamma$ is an hyperbolic element with $g_\gamma \gamma = \gamma$. It turns out that this function coincides with Selberg's zeta function $Z(s;\mathrm{PSL}(2,\mathbb{Z}),\rho_\chi)$ for the modular group and induced representation ρ_χ:

(3.16)
$$Z(s;\mathrm{PSL}(2,\mathbb{Z}),\rho_\chi)=\prod_\gamma\prod_{k=0}^{\infty}\det\left[1_V-\rho_\chi(\sigma_\gamma)\exp(-(s+k)l(\gamma))\right]$$

where the product runs over all primitive periodic orbits γ of the geodesic flow (3.3) of period $l(\gamma)$ on the modular surface and $\sigma_\gamma\in\mathrm{PSL}(2,\mathbb{Z})$ is hyperbolic with $\sigma_\gamma\gamma=\gamma$ ([**57**], page 49, Theorem 7.2). Based on this form of Selberg's zeta function, Mayer proved

Theorem 3.1. *The Selberg zeta function $Z(s;\Gamma;\chi)$ can be expressed in terms of the Fredholm determinant of the transfer operator* (3.12)-(3.13) *as*

$$Z(s;\Gamma;\chi)=\det(1-\mathcal{L}_s^{\Gamma,\chi})\tag{3.17}$$

or

$$Z(s;\Gamma;\chi)=\det(1-\mathcal{L}_s^{\Gamma,\chi,+}\mathcal{L}_s^{\Gamma,\chi,-})=\det(1-\mathcal{L}_s^{\Gamma,\chi,-}\mathcal{L}_s^{\Gamma,\chi,+}).\tag{3.18}$$

Thereby the determinant is defined in the sense of Grothendieck.

Definition 3.3. *An operator $P:V\to V$ is called a symmetry operator of the transfer operator $\mathcal{L}_s^{\Gamma,\chi}$ if*

$$P^2=id_V,\quad \begin{pmatrix}0&P\\P&0\end{pmatrix}\mathcal{L}_s^{\Gamma,\chi}=\mathcal{L}_s^{\Gamma,\chi}\begin{pmatrix}0&P\\P&0\end{pmatrix}\tag{3.19}$$

respectively, if

$$P\rho_\chi(S)=\rho_\chi(S)P,\quad P\rho_\chi(T)=\rho_\chi(T^{-1})P.\tag{3.20}$$

From this definition it follows that for a symmetry P of $\mathcal{L}_s^{\Gamma,\chi}$ we have

$$P\mathcal{L}_s^{\Gamma,\chi,+}=\mathcal{L}_s^{\Gamma,\chi,-}P.\tag{3.21}$$

Consequently, Selberg's zeta function in (3.18) can in this case be written as

$$Z(s;\Gamma;\chi)=\det(1-(P\mathcal{L}_s^{\Gamma,\chi,+})^2)\tag{3.22}$$

and hence as

$$Z(s;\Gamma;\chi)=\det(1-P\mathcal{L}_s^{\Gamma,\chi,+})\det(1+P\mathcal{L}_s^{\Gamma,\chi,+}).\tag{3.23}$$

The equations (3.18) respectively (3.23) provide an alternative approach to the spectral theory of automorphic functions via Mayer's transfer operator. Indeed, according to these equations, the s-values for which the transfer operator $\mathcal{L}_s^{\Gamma,\chi}$ has an eigenvalue $\lambda=1$ respectively $P\mathcal{L}_s^{\Gamma,\chi,+}$ has an eigenvalue $\lambda=\pm1$, are zeros of Selberg's zeta function $Z(s;\Gamma;\chi)$. But the latter are related to the eigenvalues and resonances of the automorphic Laplacian $\Delta(\Gamma;\chi)$. Moreover, these eigenfunctions of the transfer operator with eigenvalues $\lambda = \pm 1$ are directly related to the automorphic functions with respect to Γ and χ. We explain this relation in more detail in the following sections.

3.2. The Lewis-Zagier correspondence

In [**29**], Lewis and Zagier introduced a 1-1 correspondence between the space of automorphic functions with respect to the modular group with trivial character and the space of the so called period(like) functions for this group. The period(like) functions are solutions of a three term functional equation (Lewis functional equation), holomorphic in some domain with certain asymptotics at boundaries. Moreover, they showed that the period(like) functions are obtained by an integral transform of automorphic functions. In [**15**], Deitmar and Hilgert extended this correspondence to the space of cusp forms for submodular groups of finite index and trivial character which however can be generalized to the case of an arbitrary unitary character [**19**]. The Lewis-Zagier theory is an intermediate step to connect the eigenfunctions of Mayer's transfer operator to automorphic functions. We explain this connection later in more detail. First we describe the passage from the Maass cusp forms to period functions via an integral transform for the case of a submodular group of finite index with a unitary representation and then we recall a version of the Lewis-Zagier correspondence from [**15**] and [**19**].

Let Γ be a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and χ be a unitary representation of Γ on the Hermitian vector space W. Also let ρ_χ denote the representation of $\mathrm{PSL}(2,\mathbb{Z})$ on the space V induced from the representation χ of Γ as defined in Definition 3.1.

Definition 3.4. *The Lewis three term functional equation for Γ and χ is defined to be*

$$\psi(\zeta) - \rho_\chi(T^{-1})\psi(\zeta+1) - (\zeta+1)^{-2s}\rho_\chi(STS)\psi(\frac{\zeta}{\zeta+1}) = 0 \tag{3.24}$$

where ψ is a function $\psi : \mathbb{C}\backslash(-\infty,0] \to V$.

Definition 3.5. *A periodlike function with respect to Γ and χ is a holomorphic function $\psi : \mathbb{C}\backslash(-\infty,0] \to V$ satisfying the Lewis functional equation for Γ and χ. We denote the space of such periodlike functions by $\mathcal{F}(s;\Gamma;\chi)$.*

Definition 3.6. *A period function with respect to Γ and χ is a periodlike function ψ with respect to Γ and χ which fulfils the asymptotics $\psi_i(\zeta) = o(\zeta^{-\min\{1,2\Re(s)\}})$ as $\zeta \downarrow 0$ and $\psi_i(\zeta) = o(\zeta^{-\min\{0,2\Re(s)-1\}})$ as $\zeta \to \infty$ where the limits are taken along the real axis. We denote the space of such period functions by $\mathfrak{S}(s;\Gamma;\chi)$.*

REMARK 3.1. *In* [**15**]*, for* $\Re s > 0$ *a period function is defined as a periodlike function* ψ *such that for* $z \in \mathbb{C}\backslash(-\infty, 0]$

$$\psi_i(z) = O(\min\{1, |z|^{-C}\}), \tag{3.25}$$

where $0 < C < \Re s$*. But as discussed in* [**15**]*, by the so called "Bootstrapping" argument of Lewis and Zagier* [**29**]*, this definition of period functions is equivalent to Definition 3.6.*

Next we are going to describe the passage from the space $\mathcal{S}(s;\Gamma;\chi)$ of Maass cusp forms to the space $\mathfrak{S}(s;\Gamma;\chi)$ of period functions. We follow the steps in [**36**] where Muehlenbruch described the passage from the Maass cusp forms for the Hecke congruence groups and trivial character to the period functions via an integral transform. First, we need some auxiliary results (see [**56**] and [**57**]).

LEMMA 3.2. *For* $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ *with* $\mu := [\mathrm{PSL}(2,\mathbb{Z}) : \Gamma] < \infty$ *let* $\{r_1, r_2, \ldots, r_\mu\}$ *be a set of representatives of* $\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z})$*. Then the map* $B : \mathcal{H}(\Gamma;\chi) \to \mathcal{H}(\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ *given by*

$$B\mathfrak{f}(z) = (\mathfrak{f}_1(z), \mathfrak{f}_2(z), \ldots, \mathfrak{f}_\mu(z))^t, \quad \mathfrak{f}_i(z) := \mathfrak{f}(r_i z) \tag{3.26}$$

with ^{t} *denoting the vector transpose, is an isomorphism. Hence, the spaces* $\mathcal{H}(\Gamma;\chi)$ *and* $\mathcal{H}(\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ *are isomorphic.*

PROOF. Let W and V be the representation spaces of the representations χ and ρ_χ respectively. Also let F and F_Γ be the fundamental domains of $\mathrm{PSL}(2,\mathbb{Z})$ respectively Γ. Then for an element $\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$ we have,

$$\begin{aligned}&\int_F \langle B\mathfrak{f}, B\mathfrak{f}\rangle_V \, d\mu(z) = \sum_{i=1}^{\mu} \int_F \langle \mathfrak{f}(r_i z), \mathfrak{f}(r_i z)\rangle_W \, d\mu(z) = \\ &\sum_{i=1}^{\mu} \int_{r_i F} \langle \mathfrak{f}(z), \mathfrak{f}(z)\rangle_W \, d\mu(z) = \int_{F_\Gamma} \langle \mathfrak{f}, \mathfrak{f}\rangle_W \, d\mu(z) < \infty\end{aligned} \tag{3.27}$$

where we used the invariance of $d\mu$ under the $\mathrm{PSL}(2,\mathbb{Z})$ action and $F_\Gamma = \cup_{i=1}^{\mu} r_i F$.

Now let

$$\delta_{\Gamma,\chi}(\gamma) = \begin{cases} \chi(\gamma) & \gamma \in \Gamma \\ 0_W & \gamma \notin \Gamma \end{cases} \tag{3.28}$$

where 0_W is the 0-map in W. Then for $g \in \mathrm{PSL}(2,\mathbb{Z})$ we have

(3.29)

$$\begin{aligned}&B\mathfrak{f}(gz) = (\mathfrak{f}(r_1 g z), \mathfrak{f}(r_2 g z), \ldots, \mathfrak{f}(r_\mu g z))^t = \\ &\sum_{j=1}^{\mu} (\delta_{\Gamma,\chi}(r_1 g r_j^{-1})\mathfrak{f}(r_j z), \delta_{\Gamma,\chi}(r_2 g r_j^{-1})\mathfrak{f}(r_j z), \ldots, \delta_{\Gamma,\chi}(r_\mu g r_j^{-1})\mathfrak{f}(r_j z))^t.\end{aligned}$$

By definition of the induced representation the right hand side of this equation is $\rho_\chi(g)B\mathfrak{f}(z)$ and hence we have

$$B\mathfrak{f}(gz) = \rho_\chi(g)B\mathfrak{f}(z). \tag{3.30}$$

This together with (3.27) shows that $B : \mathcal{H}(\Gamma;\chi) \to \mathcal{H}(\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ is well defined. Now we show that B is 1-1. Let $B\mathfrak{f} = B\mathfrak{f}'$ and assume that $r_1 = id$. Then it follows that $(B\mathfrak{f})_1 = (B\mathfrak{f}')_1$ or $\mathfrak{f} = \mathfrak{f}'$ and hence B is injective. Remains to prove that B is also onto. For this consider $\mathfrak{v} = (\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\mu) \in \mathcal{H}(\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ where $\mathfrak{f}_i$ with $1 \le i \le \mu$ is a W-valued function on $\mathbb{H}$. Since for any $g \in \mathrm{PSL}(2,\mathbb{Z})$ we have $\mathfrak{v}(gz) = \rho_\chi(g)\mathfrak{v}(z)$, assuming that $r_1 = id$, it follows from the definition of ρ_χ that

$$\mathfrak{f}_1(\gamma z) = \chi(\gamma)\mathfrak{f}_1(z), \quad \gamma \in \Gamma. \tag{3.31}$$

Since ρ_χ is unitary we have

$$\langle \mathfrak{v}, \mathfrak{v}\rangle_V = \langle \rho_\chi(r_i)\mathfrak{v}, \rho_\chi(r_i)\mathfrak{v}\rangle_V = \langle \mathfrak{v}(r_i z), \mathfrak{v}(r_i z)\rangle_V \tag{3.32}$$

and hence

$$\int_F \langle \mathfrak{v}, \mathfrak{v}\rangle_V \, d\mu = \frac{1}{\mu}\sum_{i=1}^{\mu} \int_F \langle \mathfrak{v}(r_i z), \mathfrak{v}(r_i z)\rangle_V \, d\mu. \tag{3.33}$$

Raplacing $r_i z$ with z in the right hand side of the above equation and using the invariance of the measure under the $\mathrm{PSL}(2,\mathbb{Z})$ action we get

$$\begin{aligned} &\int_F \langle \mathfrak{v}, \mathfrak{v}\rangle_V \, d\mu(z) = \frac{1}{\mu}\int_{\cup_{i=1}^{\mu} r_i F} \langle \mathfrak{v}, \mathfrak{v}\rangle_V \, d\mu(z) = \\ &\frac{1}{\mu}\int_{F_\Gamma} \langle \mathfrak{v}, \mathfrak{v}\rangle_V \, d\mu(z) = \frac{1}{\mu}\sum_{i=1}^{\mu}\int_{F_\Gamma} \langle \mathfrak{f}_i, \mathfrak{f}_i\rangle_W \, d\mu(z) \end{aligned} \tag{3.34}$$

where in the second equality we used the identity $F_\Gamma = \cup_{i=1}^{\mu} r_i F$. Hence

$$\sum_{i=1}^{\mu}\int_{F_\Gamma} \langle \mathfrak{f}_i, \mathfrak{f}_i\rangle_W \, d\mu(z) = \mu\int_F \langle \mathfrak{v}, \mathfrak{v}\rangle_V \, d\mu(z) < \infty \tag{3.35}$$

since $\mathfrak{v} \in \mathcal{H}(\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$. Therefore

$$\int_{F_\Gamma} \langle \mathfrak{f}_1, \mathfrak{f}_1\rangle_W \, d\mu(z) < \infty. \tag{3.36}$$

From this and (3.31) we conclude that $\mathfrak{f}_1 \in \mathcal{H}(\Gamma;\chi)$. Since we assumed that $r_1 = id$, we have $\mathfrak{v}_1 = (B\mathfrak{f})_1 = \mathfrak{f}_1$ and hence $(\mathfrak{v} - B\mathfrak{f}_1)_1 = 0$. Now we are going to show that $\mathfrak{v} = B\mathfrak{f}_1$. Let

$$\mathfrak{w} := \mathfrak{v} - B\mathfrak{f}_1. \tag{3.37}$$

Then obviously we have

$$\mathfrak{w}(gz) = \rho_\chi(g)\mathfrak{w}(z), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{3.38}$$

From this and the assumption $r_1 = id$, we get for $g = r_i$, $1 \le i \le \mu$,

$$0 = \mathfrak{w}_1(r_i z) = \sum_{j=1}^{\mu} \delta_{\Gamma,\chi}(r_1 r_i r_j^{-1})\mathfrak{w}_j(z) = \sum_{j=1}^{\mu} \delta_{ij}\mathfrak{w}_j(z) = \mathfrak{w}_i(z) \tag{3.39}$$

where in the third equality we used the fact that r_i and r_j are distinct representatives iff $r_i r_j^{-1} \notin \Gamma$. Thus $\mathfrak{w} = 0$ and hence $\mathfrak{v} = B\mathfrak{f}_1$. That means B is onto and this completes the proof. □

It is known that $\Delta(\Gamma; \chi)$ and $\Delta(\mathrm{PSL}(2,\mathbb{Z}), \rho_\chi)$ are unitary equivalent ([**57**], page 51). This is described by the operator $B : \mathcal{H}(\Gamma; \chi) \to \mathcal{H}(\mathrm{PSL}(2,\mathbb{Z}), \rho_\chi)$, indeed

$$B\Delta(\Gamma; \chi) = \Delta(\mathrm{PSL}(2,\mathbb{Z}), \rho_\chi)B. \tag{3.40}$$

From this and Theorem 2.2 we get the following corollary of the previous lemma:

Corollary 3.1. *The maps*

$$B : \mathcal{S}(\Gamma; \chi) \to \mathcal{S}(\mathrm{PSL}(2,\mathbb{Z}); \rho_\chi) \tag{3.41}$$

$$B : \Theta_0(\Gamma; \chi) \to \Theta_0(\mathrm{PSL}(2,\mathbb{Z}); \rho_\chi) \tag{3.42}$$

$$B : \Theta_1(\Gamma; \chi) \to \Theta_1(\mathrm{PSL}(2,\mathbb{Z}); \rho_\chi) \tag{3.43}$$

and also the maps

$$B : \mathcal{S}(s; \Gamma; \chi) \to \mathcal{S}(s; \mathrm{PSL}(2,\mathbb{Z}); \rho_\chi) \tag{3.44}$$

$$B : \Theta_0(s; \Gamma; \chi) \to \Theta_0(s; \mathrm{PSL}(2,\mathbb{Z}); \rho_\chi) \tag{3.45}$$

$$B : \Theta_1(s; \Gamma; \chi) \to \Theta_1(s; \mathrm{PSL}(2,\mathbb{Z}); \rho_\chi). \tag{3.46}$$

all are isomorphisms.

Before proceeding further, we need to recall some definitions from [**29**]. The 1-form $\eta(u, v)$ is defined for smooth functions $u, v : \mathbb{H} \to \mathbb{C}$ by

$$\begin{aligned}&\eta(u,v)(z) := \\ &[v(z)\partial_y u(z) - u(z)\partial_y v(z)]\,dx + [u(z)\partial_x v(z) - v(z)\partial_x u(z)]\,dy.\end{aligned} \tag{3.47}$$

It is known, that for u and v eigenfunctions of the hyperbolic Laplace operator with the same eigenvalue the 1-form $\eta(u, v)$ is closed. The hyperbolic Poisson kernel $R_\zeta(z)$, on the upper half plane given by

$$R_\zeta(z) = \frac{y}{(\zeta - x)^2 + y^2} = \frac{i}{2}\left(\frac{1}{z - \zeta} - \frac{1}{\overline{z} - \zeta}\right), \quad \zeta \in \mathbb{C}, \quad z = x + iy \in \mathbb{H}, \tag{3.48}$$

defines an eigenfunction of the hyperbolic Laplace operator with

$$\Delta R_\zeta^s(z) = s(1 - s)R_\zeta^s(z). \tag{3.49}$$

Moreover, under $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$ it transforms as

$$R_{g\zeta}(gz) = (c\zeta + d)^2 R_\zeta(z). \tag{3.50}$$

For $\vec{u} = (u_1, \ldots, u_n)^t$ with $\Delta u_i = \lambda u_i$ for $1 \leq i \leq n$ and v with $\Delta v = \lambda v$ a vector valued closed 1-form $\eta(\vec{u}, v)$ is defined by

$$\eta(\vec{u}, v) := (\eta(u_1, v), \eta(u_2, v), \ldots, \eta(u_n, v))^t \tag{3.51}$$

where ^{t} denotes again the vector transpose.

Lemma 3.3. *For an element $\mathfrak{u} \in \mathcal{S}(s; \Gamma; \chi)$ define*

$$(\mathcal{I} \circ B\mathfrak{u})(\zeta) := \int_{L_{0,\infty}} \eta(B\mathfrak{u}, R_\zeta^s)(z), \tag{3.52}$$

where $L_{0,\infty}$ denotes a path homotopic to the path from zero to infinity along the imaginary axis. Then $(\mathcal{I} \circ B\mathfrak{u})(\zeta)$ can be extended to a vector valued holomorphic function on $\mathbb{C}\backslash(-\infty, 0]$.

Proof. For an arbitrary component $\mathfrak{v}_i$ of $\mathfrak{v} = B\mathfrak{u}$, we have according to ([**29**], page 211),

$$\begin{aligned} &\eta(\mathfrak{v}_i, R_\zeta^s)(z) = \\ &\left(\frac{s y^{s-1}(y^2 - (x-\zeta)^2)}{((x-\zeta)^2 + y^2)^{s+1}} \mathfrak{v}_i(z) + \frac{y^s}{((x-\zeta)^2 + y^2)^s} \partial_y \mathfrak{v}_i(z) \right) dx + \\ &\left(\frac{-2s(x-\zeta) y^s}{((x-\zeta)^2 + y^2)^{s+1}} \mathfrak{v}_i(z) - \frac{y^s}{((x-\zeta)^2 + y^2)^s} \partial_x \mathfrak{v}_i(z) \right) dy. \end{aligned} \tag{3.53}$$

Let then $L_{0,\infty}$ be the path from 0 to ∞ along the imaginary axis. The ith component of (3.52) can be written as

$$(\mathcal{I} \circ \mathfrak{v}_i)(\zeta) = \int_0^{i\infty} \left(\frac{2s\zeta y^s}{(\zeta^2 + y^2)^{s+1}} \mathfrak{v}_i(z) - \frac{y^s}{(\zeta^2 + y^2)^s} \partial_x \mathfrak{v}_i(z) \right) dy. \tag{3.54}$$

Since $\mathfrak{v}$ belongs to $\mathcal{S}(s; \mathrm{PSL}(2,\mathbb{Z}); \rho_\chi)$, each component of $\mathfrak{v}$ decays exponentially fast as $y \to \infty$. Then $\mathfrak{v}(Sz) = \rho_\chi(S)\mathfrak{v}(z)$ yields the exponential decay of each component of $\mathfrak{v}$ as $y \to 0$. It follows from the Fourier type expansion of $\mathfrak{v}$ and estimates (2.35) that the same asymptotics holds for $\partial_x \mathfrak{v}_i(z)$. Hence the integral in (3.54) in the half plane $\Re\zeta > 0$ is convergent and it defines evidently a holomorphic function of ζ in this domain. Since $\Delta \mathfrak{v}_i = s(1-s)\mathfrak{v}_i$, the 1-form η is closed and an argument similar to the one in ([**29**], page 212) yields the analytic continuation of $(\mathcal{I} \circ \mathfrak{v}_i)(\zeta)$ to $\mathbb{C}\backslash(-\infty, 0]$. $\square$

Next we prove that $\mathcal{I} \circ B\mathfrak{u}(\zeta)$ is a periodlike function.

Lemma 3.4. *For $\zeta \in \mathbb{C}\backslash(-\infty, 0]$ the function $\mathcal{I} \circ B\mathfrak{u}(\zeta)$ with $\mathfrak{u} \in \mathcal{S}(s; \Gamma; \chi)$ satisfies the Lewis three term functional equation in Definition 3.4.*

PROOF. Let $\Re\zeta > 0$ and consider the integral transform

$$\mathcal{I} \circ B\mathfrak{u}(\zeta) = \int_{L_{0,\infty}} \eta(B\mathfrak{u}, R^s_\zeta) \tag{3.55}$$

where $L_{0,\infty}$ denotes the path from zero to ∞ along the imaginary axis. Moreover let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \tag{3.56}$$

be one of the following matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = ST^{-1}S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.57}$$

Then following the steps as in ([**36**], page 224) one gets

$$(c\zeta + d)^{-2s} \rho_\chi(\gamma^{-1}) \mathcal{I} \circ B\mathfrak{u}(\gamma\zeta) = \int_{L_{\gamma^{-1}0,\gamma^{-1}\infty}} \eta(B\mathfrak{u}, R^s_\zeta), \tag{3.58}$$

where $L_{\gamma^{-1}0,\gamma^{-1}\infty} = \gamma^{-1} L_{0,\infty}$. We note that for $\gamma = T$ and $\gamma = T'$ the integration path $L_{\gamma^{-1}0,\gamma^{-1}\infty}$ does not pass through any singularity of the integrand which are $z = \zeta$ and $z = \overline{\zeta}$. In fact, we have $\Re z \leq 0$ on the path $L_{\gamma^{-1}0,\gamma^{-1}\infty}$ and hence for $\Re\zeta > 0$ the integrals are well defined. For $\gamma = T$ we have

$$\rho_\chi(T^{-1}) \mathcal{I} \circ B\mathfrak{u}(\zeta + 1) = \int_{L_{T^{-1}0,T^{-1}\infty}} \eta(B\mathfrak{u}, R^s_\zeta) \tag{3.59}$$

and for $\gamma = T'$ we have

$$(\zeta + 1)^{-2s} \rho_\chi(STS) \mathcal{I} \circ B\mathfrak{u}(\frac{\zeta}{\zeta + 1}) = \int_{L_{T'^{-1}0,T'^{-1}\infty}} \eta(B\mathfrak{u}, R^s_\zeta). \tag{3.60}$$

From these identities and the fact that $L_{0,\infty} = L_{T^{-1}0,T^{-1}\infty} \cup L_{T'^{-1}0,T'^{-1}\infty}$ we get

$$\begin{aligned} &\rho_\chi(T^{-1}) \mathcal{I} \circ B\mathfrak{u}(\zeta + 1) + (\zeta + 1)^{-2s} \rho_\chi(STS) \mathcal{I} \circ B\mathfrak{u}(\frac{\zeta}{\zeta + 1}) \\ &= \int_{L_{0,\infty}} \eta(B\mathfrak{u}, R^s_\zeta) = \mathcal{I} \circ B\mathfrak{u}(\zeta). \end{aligned} \tag{3.61}$$

Thus for $\Re\zeta > 0$ we proved that

$$\mathcal{I} \circ B\mathfrak{u}(\zeta) - \rho_\chi(T^{-1}) \mathcal{I} \circ B\mathfrak{u}(\zeta + 1) - (\zeta + 1)^{-2s} \rho_\chi(STS) \mathcal{I} \circ B\mathfrak{u}(\frac{\zeta}{\zeta + 1}) = 0. \tag{3.62}$$

But $\mathcal{I} \circ B\mathfrak{u}(\zeta)$ can be extended to a holomorphic function on the cut complex plane $\mathbb{C}\backslash[0, -\infty)$ as we mentioned in the proof of Lemma 3.3. Hence we get equation (3.62) on the entire domain $\mathbb{C}\backslash[0, -\infty)$ and this completes the proof. □

We determine next the asymptotics of $\mathcal{I} \circ B\mathfrak{u}(\zeta)$ for $\mathfrak{u} \in \mathcal{S}(s;\Gamma;\chi)$ as $\zeta \downarrow 0$ and $\zeta \to \infty$ along the real axis. From (3.54) we easily get

$$\mathcal{I} \circ B\mathfrak{u}(\zeta) = \int_0^\infty (\frac{y}{\zeta^2+y^2})^s(\frac{2s\zeta B\mathfrak{u}}{\zeta^2+y^2} - \partial_x B\mathfrak{u})dy. \tag{3.63}$$

Then for $\zeta \in (0,+\infty)$ we have

$$|\mathcal{I} \circ B\mathfrak{u}(\zeta)| \leq \int_0^\infty |\frac{y}{\zeta^2+y^2}|^{\Re s}|\frac{2s\zeta B\mathfrak{u}}{\zeta^2+y^2} - \partial_x B\mathfrak{u}|dy. \tag{3.64}$$

From the Fourier type expansion of $B\mathfrak{u} \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi)$ and the estimates (2.35), it follows that both $B\mathfrak{u}(iy)$ and $\partial_x B\mathfrak{u}(iy)$ are bounded on $[0,\infty)$ and, as mentioned already, they decay exponentially fast at zero and infinity. Hence by using the estimates $|\frac{y}{\zeta^2+y^2}| \leq \zeta^{-2}y$ respectively $|\frac{y}{\zeta^2+y^2}| \leq y^{-1}$ for $y > 0$ and any $\zeta \in (0,+\infty)$ we get $|\mathcal{I} \circ B\mathfrak{u}| = O(\zeta^{-2\Re(s)})$ and $|\mathcal{I} \circ B\mathfrak{u}| = O(1)$ respectively. By choosing the stronger estimate for each value of $\Re(s)$ we get

$$|\mathcal{I} \circ B\mathfrak{u}(\zeta)| = \begin{cases} O(\zeta^{\max(0,-2\Re(s))}) & \zeta \downarrow 0, \\ O(\zeta^{\min(0,-2\Re(s))}) & \zeta \to \infty. \end{cases} \tag{3.65}$$

Now one can easily check that a function satisfying these estimates, satisfies also the estimates given in Definition 3.6. Thus we get

LEMMA 3.5. *Let* Γ *be a subgroup of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$ *and* χ *be a unitary representation of* Γ. *Then the integral transformation* $\mathcal{I} \circ B$ *defines a map from* $\mathcal{S}(s;\Gamma;\chi)$ *to* $\mathfrak{S}(s;\Gamma;\chi)$.

In [**15**], A. Deitmar and J. Hilgert generalized the Lewis-Zagier correspondence to the case of Maass cusp forms of any submodular group Σ of finite index with the one dimensional trivial representation. For this they considered first the isomorphism

$$B : \mathcal{S}(s;\Sigma) \to \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\Sigma) \tag{3.66}$$

where ρ_Σ denotes the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial representation of Σ. On the other hand, according to the "fundamental theorem on homomorphisms" the epimorphism

$$\rho_\Sigma : \mathrm{PSL}(2,\mathbb{Z}) \to \rho_\Sigma(\mathrm{PSL}(2,\mathbb{Z})) \tag{3.67}$$

yields the isomorphism of groups

$$\rho_\Sigma(\mathrm{PSL}(2,\mathbb{Z})) \cong \mathrm{PSL}(2,\mathbb{Z})/\ker\rho_\Sigma. \tag{3.68}$$

This isomorphism also follows from the work of Millington [**34**], which we explain in section 4.5 for a special induced representation. Assuming that $\ker\rho_\Sigma$ is of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ the representation ρ_Σ of $\mathrm{PSL}(2,\mathbb{Z})$ gives a representation ρ'_Σ of the finite group $\mathrm{PSL}(2,\mathbb{Z})/\ker\rho_\Sigma$ in the following way:

$$\rho'_\Sigma(g\ker\rho_\Sigma) = \rho_\Sigma(g), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{3.69}$$

Let

$$\rho'_\Sigma = \oplus_{i=1}^N m_i \eta'_i \tag{3.70}$$

be the decomposition of ρ'_Σ into its irreducible subrepresentations, where η'_i is an irreducible representation of the group $\mathrm{PSL}(2,\mathbb{Z})/\ker\rho_\Sigma$ with multiplicity m_i and N is the number of non-isomorphic irreducible subrepresentations. Then the decomposition of ρ_Σ into its irreducible subrepresentations is given by

$$\rho_\Sigma = \oplus_{i=1}^N m_i \eta_i \tag{3.71}$$

where η_i is an irreducible representation of $\mathrm{PSL}(2,\mathbb{Z})$ with multiplicity m_i and N is the number of non-isomorphic irreducible subrepresentations and we have

$$\eta_i(g) = \eta'_i(g\ker\rho_\Sigma), \quad g \in \mathrm{PSL}(2,\mathbb{Z}). \tag{3.72}$$

This decomposition of the representation ρ_Σ yields the decomposition

$$\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\Sigma) \cong \oplus_{i=1}^N m_i \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i). \tag{3.73}$$

Deitmar and Hilgert then proved ([**15**], page 1091, Theorem 3.3):

Theorem 3.6. *Let Λ be a normal subgroup of finite index in the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ and ρ_Λ be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the one dimensional trivial representation of Λ which yields the regular representation of the finite group $\mathrm{PSL}(2,\mathbb{Z})/\Lambda$. Moreover let η be an irreducible subrepresentation of ρ_Λ corresponding to an irreducible subrepresentation of the regular representation of $\mathrm{PSL}(2,\mathbb{Z})/\Lambda$. Then the space of Maass cusp forms $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta)$ is in one to one correspondence with the space of period functions $\mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta)$. Indeed, the map*

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta) \to \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta) \tag{3.74}$$

is a bijection. We note also that η has obviously a finite image as in the assumptions of Deitmar and Hilgert.

According to this theorem for each η_i in (3.71) with $\Lambda = \ker\rho_\Sigma$ the map

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i) \to \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i) \tag{3.75}$$

From this and (3.73) and by using the linearity of $\mathcal{I}$ it follows that

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\Sigma) \to \oplus_{i=1}^N m_i \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i) \tag{3.76}$$

is bijection. This and the bijection

$$\mathfrak{S}(s;\Sigma) \cong \oplus_{i=1}^N m_i \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i). \tag{3.77}$$

yield the bijection

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\Sigma) \to \mathfrak{S}(s;\Sigma) \tag{3.78}$$

or equivalently the bijection

$$\mathcal{I} \circ B : \mathcal{S}(s; \Sigma) \to \mathfrak{S}(s; \Sigma). \tag{3.79}$$

This is indeed the generalized version of Lewis-Zagier correspondence derived by Deitmar and Hilgert. But from theorem 3.6 one can get a more general result. As a corollary of this theorem we have

Corollary 3.2. *Let Γ be a subgroup of finite index in* $\mathrm{PSL}(2, \mathbb{Z})$ *and χ be a unitary representation of Γ such that the kernel of the representation ρ_χ of* $\mathrm{PSL}(2, \mathbb{Z})$ *has finite index in* $\mathrm{PSL}(2, \mathbb{Z})$. *Then for s a spectral parameter the space of Maass cusp forms $\mathcal{S}(s; \Gamma; \chi)$ is in one to one correspondence with the space of period functions $\mathfrak{S}(s; \Gamma; \chi)$. Indeed, the map*

$$\mathcal{I} \circ B : \mathcal{S}(s; \Gamma; \chi) \to \mathfrak{S}(s; \Gamma; \chi) \tag{3.80}$$

is a bijection.

Proof. We note that $\mathcal{S}(s; \Gamma; \chi)$ is isomorphic to $\mathcal{S}(s; \mathrm{PSL}(2, \mathbb{Z}); \rho_\chi)$ (see Corollary 3.1). The epimorphism

$$\rho_\chi : \mathrm{PSL}(2, \mathbb{Z}) \to \rho_\chi(\mathrm{PSL}(2, \mathbb{Z})) \tag{3.81}$$

yields the group isomorphism

$$\rho_\chi(\mathrm{PSL}(2, \mathbb{Z})) \cong \mathrm{PSL}(2, \mathbb{Z})/\Lambda, \tag{3.82}$$

where $\Lambda := \ker \rho_\chi$ is a normal subgroup of finite index in $\mathrm{PSL}(2, \mathbb{Z})$ and hence $\mathrm{PSL}(2, \mathbb{Z})/\Lambda$ is a finite group. Then the representation ρ_χ relates to the representation ρ'_χ of the finite group $\mathrm{PSL}(2, \mathbb{Z})/\Lambda$ by

$$\rho_\chi(g) = \rho'_\chi(g\Lambda), \quad g \in \mathrm{PSL}(2, \mathbb{Z}). \tag{3.83}$$

Consider the decomposition of the induced representation ρ'_χ into its irreducible subrepresentations,

$$\rho'_\chi = \oplus_{i=1}^{N} m_i \eta'_i \tag{3.84}$$

where N denotes the number of non-isomorphic irreducible subrepresentations and m_i denotes the multiplicity of the irreducible subrepresentation η'_i. This yields the decomposition of the induced representation ρ_χ into its irreducible subrepresentations,

$$\rho_\chi = \oplus_{i=1}^{N} m_i \eta_i \tag{3.85}$$

where N denotes the number of non-isomorphic irreducible subrepresentations and m_i denotes the multiplicity of the irreducible subrepresentation η_i and where

$$\eta_i(g) = \eta'_i(g\Lambda), \quad g \in \mathrm{PSL}(2, \mathbb{Z}). \tag{3.86}$$

Then we have for the space of vector valued Maass cusp forms

$$\mathcal{S}(s; \mathrm{PSL}(2, \mathbb{Z}); \rho_\chi) \cong \oplus_{i=1}^{N} m_i \mathcal{S}(s; \mathrm{PSL}(2, \mathbb{Z}); \eta_i). \tag{3.87}$$

Applying Theorem 3.6 to each of the representations η_i and s a spectral parameter and using the linearity of $\mathcal{I}$ and the fact that

$$\mathfrak{S}(s;\Gamma;\chi) \cong \oplus_{i=1}^{N} m_i \mathfrak{S}(s;\mathrm{PSL}(2,\mathbb{Z});\eta_i). \tag{3.88}$$

we get the bijection

$$\mathcal{I} : \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi) \to \mathfrak{S}(s;\Gamma;\chi). \tag{3.89}$$

From this and the isomorphism

$$B : \mathcal{S}(s;\Gamma;\chi) \to \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\rho_\chi) \tag{3.90}$$

it follows that

$$\mathcal{I} \circ B : \mathcal{S}(s;\Gamma;\chi) \to \mathfrak{S}(s;\Gamma;\chi) \tag{3.91}$$

is a bijection which completes the proof. □

3.3. Eigenfunctions of Mayer's transfer operator

We consider a subgroup Γ of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and a unitary representation χ of Γ which allows for a symmetry P. For simplicity reasons, we write $\mathcal{L}_s$ instead of $\mathcal{L}_s^{\Gamma,\chi,+}$. According to (3.22), the eigenfunctions of the transfer operator $P\mathcal{L}_s$ with eigenvalues $\lambda = \pm 1$ determine the s-values for which Selberg's zeta function vanishes. In this section we show that these eigenfunctions are directly related to the period(like) functions. This establishes a connection between the eigenfunctions of Mayer's transfer operator and the automorphic functions. If χ is a unitary character, this relation is known to be a 1-1 correspondence, which can be extended indeed to an arbitrary unitary representation [**19**].

We are going to show that every period function determines a certain eigenfunction of the transfer operator $P\mathcal{L}_s$ and vice versa. First, we show that every eigenfunction of the transfer operator $P\mathcal{L}_s$ with eigenvalue $\lambda = \pm 1$ determines a periodlike function.

Lemma 3.7. *For χ a finite dimensional unitary representation of the subgroup Γ of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$ *and P a symmetry let $P\mathcal{L}_s f = \lambda f$ with $f \in \bigoplus_{i=1}^{\mu \dim \chi} B(D)$ and $\lambda = \pm 1$. Define $\psi(\zeta) := \rho_\chi(TS) f(\zeta - 1)$. Then $\psi(\zeta)$ fulfils the functional equations*

$$\psi(\zeta) = \lambda \zeta^{-2s} P \rho_\chi(S) \psi(\frac{1}{\zeta}) \tag{3.92}$$

and

$$\psi(\zeta) - \rho_\chi(T^{-1})\psi(\zeta+1) - (\zeta+1)^{-2s}\rho_\chi(STS)\psi(\frac{\zeta}{\zeta+1}) = 0 \tag{3.93}$$

and hence is a periodlike function.

PROOF. For the proof we follow the arguments in [**19**] for unitary characters. Let $\Re(s) > \frac{1}{2}$. According to (3.13) we have

$$\mathcal{L}_s f(\zeta) = \sum_{n=1}^{\infty} (\frac{1}{\zeta+n})^{2s} \rho_\chi(ST^n) f(\frac{1}{\zeta+n}). \tag{3.94}$$

By a simple calculation we get

$$P\mathcal{L}_s f(\zeta) - \rho_\chi(ST^{-1}S) P\mathcal{L}_s f(\zeta+1) = (\frac{1}{\zeta+1})^{2s} P\rho_\chi(ST) f(\frac{1}{\zeta+1}), \tag{3.95}$$

or, since by assumption $P\mathcal{L}_s f = \lambda f$,

$$f(\zeta) - \rho_\chi(ST^{-1}S) f(\zeta+1) = \lambda(\frac{1}{\zeta+1})^{2s} P\rho_\chi(ST) f(\frac{1}{\zeta+1}). \tag{3.96}$$

Changing the variable $\zeta \to \zeta - 1$ and inserting the identity $f(\zeta) = \rho_\chi(ST^{-1})\psi(\zeta+1)$, one gets

$$\psi(\zeta) - \rho_\chi(T^{-1})\psi(\zeta+1) = \lambda\zeta^{-2s} P\rho_\chi(ST^{-1})\psi(\frac{\zeta+1}{\zeta}). \tag{3.97}$$

Next we replace the variable ζ by $1/\zeta$ and multiply both sides of the equation by $\lambda\zeta^{-2s}\rho_\chi(S)P$. This leads to

$$\lambda\zeta^{-2s}\rho_\chi(S)P\psi(\frac{1}{\zeta}) - \lambda\zeta^{-2s}P\rho_\chi(ST^{-1})\psi(\frac{\zeta+1}{\zeta}) = \rho_\chi(T^{-1})\psi(\zeta+1). \tag{3.98}$$

Subtracting this from (3.97) yields formula (3.92). Using this formula, for the right hand side of (3.97) we have

$$\lambda\zeta^{-2s}P\rho_\chi(ST^{-1})\psi(\frac{\zeta+1}{\zeta}) = (\zeta+1)^{-2s}\rho_\chi(STS)\psi(\frac{\zeta}{\zeta+1}). \tag{3.99}$$

Inserting this identity into the equation (3.97) yields equation (3.93). The equations (3.92) and (3.93) have been derived in the half-plane $\Re(s) > \frac{1}{2}$. Analytic continuation shows that they hold for arbitrary $s \in \mathbb{C}$. This completes the proof. □

For the following we need some auxiliary results on the asymptotic expansion of periodlike functions. We follow the same steps as in [**15**] and [**19**] where the asymptotics were obtained in the case of a unitary character. For $\Re(\zeta) > 0$ and $\Re(s) > \frac{1}{2}$ with $s \notin \mathbb{Z}$ define

$$Q_0(\zeta) := \zeta^{-2s}\psi(\frac{1}{\zeta}) - \sum_{n=0}^{\infty}(n+\zeta)^{-2s}\rho_\chi(TT'^n)^{-1}\psi(1+\frac{1}{n+\zeta}) \tag{3.100}$$

where $T' = ST^{-1}S$. Then we have ([**15**], page 1095)

$$Q_0(\zeta+1) = \rho_\chi(T')Q_0(\zeta). \tag{3.101}$$

Moreover $Q_0(\zeta)$ has an meromorphic continuation to the right half-plane $2\Re(s) > -M$ for any $M \in \mathbb{N}$. To obtain this analytic continuations we subtract and add for $\Re(s) > \frac{1}{2}$ the M-th Taylor polynomial of ψ at the point 1. This leads to the following identity for any $M > 0$
(3.102)

$$Q_0(\zeta) = \zeta^{-2s}\psi(\frac{1}{\zeta}) - \sum_{m=0}^{M} \zeta_{\rho_\chi}(m+2s,\zeta)C_m + \\ - \sum_{n=0}^{\infty}(n+\zeta)^{-2s}\rho_\chi(TT'^n)^{-1}\left(\psi(1+\frac{1}{n+\zeta}) - \sum_{m=0}^{M}\frac{C_m}{(n+\zeta)^m}\right)$$

where

$$\zeta_{\rho_\chi}(s,a) = \sum_{k=0}^{\infty}\frac{\rho_\chi(TT'^k)^{-1}}{(k+a)^s} \tag{3.103}$$

and

$$C_m = \frac{1}{m!}\frac{\partial^m}{\partial\zeta^m}\psi|_{\zeta=1} \tag{3.104}$$

is the m-th coefficient of the Taylor expansion of ψ at $\zeta = 1$. We note that (3.102) is well defined in the domain $2\Re(s) > -M \in \mathbb{N}$ with possible poles in the second term. The analytic continuation of the second term is obtained through its relation to Lerch's transcendent ([**39**], page 612)

$$\Phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(a+k)^s}, \quad |z| = 1, \tag{3.105}$$

which is absolutely convergent for $\Re(s) > 1$ and $\Re(a) > 0$. By a similarity transformation we have

$$\begin{aligned}&\zeta_{\rho_\chi}(s,a) = \\ &\mathcal{M}^{-1}\mathrm{diag}(\Phi(e^{2i\pi\lambda_1},s,a),\ldots,\Phi(e^{2i\pi\lambda_{\dim\rho_\chi}},s,a))\mathcal{M}\rho_\chi(T^{-1})\end{aligned} \tag{3.106}$$

where $\mathcal{M}$ is a matrix diagonalizing $\rho_\chi(T'^{-1})$:

$$\mathcal{M}\rho_\chi(T'^{-1})\mathcal{M}^{-1} = \mathrm{diag}(e^{2i\pi\lambda_1},\ldots,e^{2i\pi\lambda_{\dim\rho_\chi}}). \tag{3.107}$$

Then the analytic continuation of $\zeta_{\rho_\chi}(s,a)$ is obtained from the well known analytic continuation of Lerch's transcendent (see for example [**17**]).

We can now determine the asymptotic behaviour of the period-like functions at zero on the positive real axis as in ([**15**], page 1096). First we need to recall the asymptotic behaviour of Lerch's transcendent. For $\Re(a) > 0$ and $z = \exp(2i\pi\lambda_i)$ we have [**17, 15**]

$$\Phi(z,s,a) = \sum_{k=-1}^{N-1} b_k(z,s)a^{-s-k} + R_N(z,s,a) \tag{3.108}$$

where $R_N(z,s,a) = O(a^{-N-s})$ as $a \to \infty$ and $b_k(z,s)$ are explicitly known constants. For $z \neq 1$ these constants are given in ([**17**], page 214, Theorem 1) with $b_{-1}(z,s) \equiv 0$ and for $z = 1$ they are given for example in ([**15**], page 1093). We remind that for $z = 1$ Lerch's transcendent coincides with the Hurwitz zeta function.

Let $\psi : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}^{\mu \dim \rho_\chi}$ be a holomorphic solution of (3.93) with $s \notin \mathbb{Z}$. According to (3.102), we have

(3.109)

$$\psi(\zeta) = \zeta^{-2s} Q_0(\zeta^{-1}) + \sum_{m=0}^{M} \zeta^{-2s} \zeta_{\rho_\chi}(m+2s, \zeta^{-1}) C_m + \zeta^{-2s} \times$$
$$\sum_{n=0}^{\infty} (n+\zeta^{-1})^{-2s} \rho_\chi(TT'^n)^{-1} \left(\psi(1 + \frac{1}{n+\zeta^{-1}}) - \sum_{m=0}^{M} \frac{C_m}{(n+\zeta^{-1})^m} \right).$$

Each term in the last sum is of order $O(\zeta^M)$ ([**15**], page 1096) and hence for $1 \leq i \leq \dim \rho_\chi$ we have

(3.110)

$$\psi_i(\zeta) \underset{\zeta \to 0}{\sim} \zeta^{-2s} Q_{0i}(\zeta^{-1}) + \sum_{m=0}^{M} \zeta^{-2s} \left[\zeta_{\rho_\chi}(m+2s, \zeta^{-1}) C_m \right]_i + O(\zeta^M).$$

Inserting the estimate (3.108) into (3.106) and this into relation (3.110) leads to

$$\psi(\zeta) \underset{\zeta \to 0}{\sim} \zeta^{-2s} Q_0(\frac{1}{\zeta}) + \sum_{l=-1}^{\infty} C_l^* \zeta^l, \tag{3.111}$$

where the constants C_l^* can be calculated explicitly as in ([**15**], page 1094, formula 39), in particular C_{-1}^* is proportional to $\psi(1)$. Now we can prove the following Lemma.

LEMMA 3.8. *Let* Γ *be a subgroup of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$*,* χ *be a finite dimensional unitary representation of* Γ*,* P *be a symmetry of the transfer operator for* Γ *and* χ*, and let* $\psi : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}^{\mu \dim \rho_\chi}$ *be a solution of equations* (3.92) *and* (3.93) *with* $\psi_i(\zeta) = o(\zeta^{-\min\{1, 2\Re(s)\}})$ *as* $\zeta \downarrow 0$ *and* $\psi_i(\zeta) = o(\zeta^{-\min\{0, 2\Re(s)-1\}})$ *as* $\zeta \to \infty$*. Then* ψ *determines an eigenfunction of the transfer operator* $P\mathcal{L}_s$ *with eigenvalue* $\lambda = \pm 1$*.*

PROOF. Since $\psi_i(\zeta) = o(\zeta^{-\min\{1, 2\Re(s)\}})$ as $\zeta \downarrow 0$, it follows from the asymptotic expansion (3.111) that $\lim_{\zeta \to \infty} Q_0(\zeta) = 0$. Because of the periodicity of Q_0 given in (3.101), Q_0 must then be identically zero and $C_{-1}^* = 0$. Hence (3.100) reduces to

$$\zeta^{-2s} \psi(\frac{1}{\zeta}) = \sum_{n=0}^{\infty} (n+\zeta)^{-2s} \rho_\chi(TT'^n)^{-1} \psi(1 + \frac{1}{n+\zeta}) \tag{3.112}$$

or, by changing ζ to ζ^{-1}

$$\psi(\zeta) = \zeta^{-2s} \sum_{n=0}^{\infty} (n+\zeta^{-1})^{-2s} \rho_\chi(TT'^n)^{-1} \psi(1+\frac{1}{n+\zeta^{-1}}). \tag{3.113}$$

Since ψ fulfils equation (3.92), this equation can be rewritten as

$$\lambda \zeta^{-2s} P \rho_\chi(S) \psi(\frac{1}{\zeta}) = \zeta^{-2s} \sum_{n=0}^{\infty} (n+\zeta^{-1})^{-2s} \rho_\chi(TT'^n)^{-1} \psi(1+\frac{1}{n+\zeta^{-1}}). \tag{3.114}$$

Changing once more ζ to ζ^{-1} we get

$$\lambda P \rho_\chi(S) \psi(\zeta) = \sum_{n=0}^{\infty} (n+\zeta)^{-2s} \rho_\chi(TT'^n)^{-1} \psi(1+\frac{1}{n+\zeta}). \tag{3.115}$$

Then replace ζ by $\zeta+1$. This leads to

$$\lambda P \rho_\chi(S) \psi(\zeta+1) = \sum_{n=1}^{\infty} (n+\zeta)^{-2s} \rho_\chi(T(T')^{n-1})^{-1} \psi(1+\frac{1}{n+\zeta}). \tag{3.116}$$

Now we insert the identity $\psi(\zeta+1) = \rho_\chi(TS) f(\zeta)$ which leads to

$$\lambda P \rho_\chi(STS) f(\zeta) = \sum_{n=1}^{\infty} (n+\zeta)^{-2s} \rho_\chi(T')^{-n+1} \rho_\chi(T^{-1}) \rho_\chi(TS) f(\frac{1}{n+\zeta}). \tag{3.117}$$

Since $T' = ST^{-1}S$, a simple calculation shows

$$\lambda f(\zeta) = \sum_{n=1}^{\infty} (n+\zeta)^{-2s} P \rho_\chi(ST^n) f(\frac{1}{n+\zeta}). \tag{3.118}$$

which completes the proof. □

REMARK 3.2. *We note that in the proof of Lemma 3.8 we did not use the asymptotics of ψ at infinity. Indeed, this follows from the asymptotics of ψ at zero by using* (3.92).

In the following Lemma we give a sufficient condition for the eigenfunctions of the transfer operator $P\mathcal{L}_s$ to correspond to a period function.

LEMMA 3.9. *For χ a finite dimensional unitary representation of the subgroup Γ of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$ *and P a symmetry let $P\mathcal{L}_s f = \lambda f$ with $f \in \bigoplus_{i=1}^{\mu \dim \chi} B(D)$ and $\lambda = \pm 1$. If $f(0) = 0$ then the periodlike function $\psi(\zeta) = \rho_\chi(TS) f(\zeta - 1)$ is indeed a period function.*

PROOF. By inverting the steps in the proof of Lemma 3.8 from (3.118) to (3.112), it follows that for the periodlike function ψ the function $Q_0(\zeta)$ vanishes. On the other hand, if $f(0) = 0$ the corresponding periodlike function $\psi(\zeta)$ fulfills $\psi(1) = 0$. Hence, recalling that the coefficient C^*_{-1} in the asymptotics (3.111) is proportional to $\psi(1)$, it follows that $C^*_{-1} = 0$. Then the asymptotics (3.111) with $C^*_{-1} = 0$ and vanishing Q_0 yields the desired asymptotics for ψ to be a period function. This completes the proof. □

We summarize the above results in the following theorem:

THEOREM 3.10. *For χ a finite dimensional unitary representation of a subgroup Γ of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and P a symmetry, let $F(s;\Gamma;\chi)$ denote the space of eigenfunctions $f \in \oplus_{i=1}^{\mu \dim \chi} B(D)$ of the transfer operator $P\mathcal{L}_s^{\Gamma,\chi,+}$ with eigenvalue $\lambda = \pm 1$ and $f(0) = 0$. Let $\mathfrak{S}(s;\Gamma;\chi)$ be the space of period functions as defined in Definition 3.6. Then the map $\mathcal{P} : \mathfrak{S}(s;\Gamma;\chi) \to F(s;\Gamma;\chi)$ given by*

$$f(\zeta) = \mathcal{P}\psi(\zeta) = \rho_\chi(ST^{-1})\psi(\zeta+1)|_D \tag{3.119}$$

is a bijection.

From this and Corollary 3.2 follows

THEOREM 3.11. *Let Γ be a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and let χ be a finite dimensional unitary representation of Γ such that the kernel of the induced representation ρ_χ of $\mathrm{PSL}(2,\mathbb{Z})$ is of finite index in $\mathrm{PSL}(2,\mathbb{Z})$. Moreover, let χ allow for a symmetry P for Mayer's transfer operator $\mathcal{L}_s^{\Gamma,\chi}$ in (3.21). Then the map*

$$\mathcal{P} \circ \mathcal{I} \circ B : \mathcal{S}(s;\Gamma;\chi) \to F(s;\Gamma;\chi) \tag{3.120}$$

is a bijection.

PROOF. According to corollary 3.2

$$\mathcal{I} \circ B : \mathcal{S}(s;\Gamma;\chi) \to \mathfrak{S}(s;\Gamma;\chi) \tag{3.121}$$

is a bijection. On the other hand according to theorem 3.10

$$\mathcal{P} : \mathfrak{S}(s;\Gamma;\chi) \to F(s;\Gamma;\chi) \tag{3.122}$$

is a bijection. Composition of these two maps leads to the desired result. □

3.4. Automorphisms of periodlike functions

In this section we consider subgroups $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ with $\mu = [\mathrm{PSL}(2,\mathbb{Z}) : \Gamma] < \infty$ such that

$$M\Gamma M = \Gamma, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.123}$$

and χ a unitary representation of Γ in some μ-dimensional Hermitian space. By definition we have

$$\mathrm{PGL}(2,\mathbb{Z}) := \left\{ g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, \quad \det g = \pm 1 \right\}. \tag{3.124}$$

The action

$$\mathrm{PGL}(2,\mathbb{Z}) \times \mathbb{H} \to \mathbb{H} \tag{3.125}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{Z})$ is defined by

$$(g,z) = gz = \begin{cases} \dfrac{az+b}{cz+d}, & \det g = 1, \\ \dfrac{a\overline{z}+b}{c\overline{z}+d}, & \det g = -1. \end{cases} \tag{3.126}$$

DEFINITION 3.7. *Let $j \in \mathrm{PGL}(2,\mathbb{Z})$ and J be the transformation defined by*

$$J\mathfrak{f}(z) = \mathfrak{f}(jz), \quad \mathfrak{f} \in \mathcal{H}(\Gamma;\chi). \tag{3.127}$$

Then J is called an automorphism of $\mathcal{H}(\Gamma;\chi)$ if $J\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$ for all $\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$. The automorphism of any subspace of $\mathcal{H}(\Gamma;\chi)$ is defined in the same way.

For example the element $j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ defines an automorphism of Maass cusp forms of $\mathrm{PSL}(2,\mathbb{Z})$ with trivial representation whose eigenspaces for eigenvalues 1 or -1 correspond to even or odd Maass cusp forms.

LEMMA 3.12. *The transformation $J\mathfrak{f}(z) = \mathfrak{f}(jz)$ is an automorphism of $\mathcal{H}(\Gamma;\chi)$ iff j belongs to $N_{\mathrm{PGL}(2,\mathbb{Z})}(\Gamma)$, the normalizer of Γ in $\mathrm{PGL}(2,\mathbb{Z})$ and $\chi(j\gamma j^{-1}) = \chi(\gamma)$ for all $\gamma \in \Gamma$. In particular, an automorphism J of $\mathcal{H}(\Gamma;\chi)$ defines an automorphism of the space $\mathcal{S}(s;\Gamma;\chi)$.*

PROOF. First, we show that for F a fundamental domain of Γ, $F' := jF$ is also a fundamental domain of Γ. Let $F_{\mathrm{PSL}(2,\mathbb{Z})}$ denote the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$. Then we have

$$F = \bigcup_{r \in R(\Gamma\backslash \mathrm{PSL}(2,\mathbb{Z}))} rF_{\mathrm{PSL}(2,\mathbb{Z}))} \tag{3.128}$$

where $R(\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}))$ denotes a set of representatives of the right cosets of Γ in $\mathrm{PSL}(2,\mathbb{Z})$. Multiplying both side of this identity by j we get

$$F' = jF = \bigcup_{r \in R(\Gamma\backslash \mathrm{PSL}(2,\mathbb{Z}))} jrF_{\mathrm{PSL}(2,\mathbb{Z})} \tag{3.129}$$

or,

$$F' = \bigcup_{r \in R(\Gamma\backslash \mathrm{PSL}(2,\mathbb{Z}))} jrj^{-1}jF_{\mathrm{PSL}(2,\mathbb{Z})}. \tag{3.130}$$

But for a set of representatives r of the right cosets of Γ in $\mathrm{PSL}(2,\mathbb{Z})$, the set of elements jrj^{-1} is another set of representatives of the right cosets of Γ in $\mathrm{PSL}(2,\mathbb{Z})$ and hence we get

$$F' = \bigcup_{r' \in R(\Gamma\backslash \mathrm{PSL}(2,\mathbb{Z}))} r'jF_{\mathrm{PSL}(2,\mathbb{Z})}. \tag{3.131}$$

Let $F_{\mathrm{PSL}(2,\mathbb{Z})}$ be the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ which is symmetric relative to the y-axis, that is, $MF_{\mathrm{PSL}(2,\mathbb{Z})} = F_{\mathrm{PSL}(2,\mathbb{Z})}$ (see for example [**23**], page 44, Figure 9). Since $j = \gamma M$ where $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$, we get $jF_{\mathrm{PSL}(2,\mathbb{Z})} = \gamma F_{\mathrm{PSL}(2,\mathbb{Z})}$. Thus, obviously, $F_0 := jF_{\mathrm{PSL}(2,\mathbb{Z})}$ is a fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ and hence

$$F' = \bigcup_{r' \in R(\Gamma\backslash \mathrm{PSL}(2,\mathbb{Z}))} r' F_0. \tag{3.132}$$

Therefore $F' = jF$ is a fundamental domain of Γ. Now, from the facts that $j\Gamma j^{-1} = \Gamma$ and $\chi(j\gamma j^{-1}) = \chi(\gamma)$ for all $\gamma \in \Gamma$ it follows that if $\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$ then $J\mathfrak{f}$ has the same transformation property as an element of $\mathcal{H}(\Gamma;\chi)$. Moreover, by the variable substitution $z \to jz$ we get

$$\int_{jF} \langle \mathfrak{f}(jz), \mathfrak{f}(jz)\rangle \, d\mu(z) = \int_F \langle \mathfrak{f}(z), \mathfrak{f}(z)\rangle \, d\mu(z) < \infty. \tag{3.133}$$

That means $J\mathfrak{f}$ is square integrable iff $\mathfrak{f}$ is so and hence $J\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$. Conversely if both $\mathfrak{f}$ and $J\mathfrak{f}$ are in $\mathcal{H}(\Gamma;\chi)$ then for $\gamma \in \Gamma$ we have $(J\mathfrak{f})(\gamma z) = \chi(\gamma) J\mathfrak{f}(z)$ and hence

$$\mathfrak{f}(j\gamma z) = \chi(\gamma)\mathfrak{f}(jz). \tag{3.134}$$

But since $\mathfrak{f} \in \mathcal{H}(\Gamma;\chi)$ we have $\mathfrak{f}(\gamma z) = \chi(\gamma)\mathfrak{f}(z)$ and hence

$$\mathfrak{f}(\gamma jz) = \chi(\gamma)\mathfrak{f}(jz). \tag{3.135}$$

Therefore we must have $\mathfrak{f}(j\gamma z) = \mathfrak{f}(\gamma jz)$ which leads to

$$j\gamma j^{-1} = \theta\gamma, \quad \theta \in \ker \chi. \tag{3.136}$$

That is j belongs to normalizer of Γ and $\chi(j\gamma j^{-1}) = \chi(\gamma)$. Thus we proved the first assertion.

To prove the second part we follow the same lines as in [**19**]. Recall that a cusp $x_\alpha \in \mathbb{R} \cup \infty$ of Γ is defined by $S_\alpha x_\alpha = x_\alpha$ where S_α is a primitive parabolic element of Γ. Let x_α be a cusp of Γ, then $jS_\alpha j^{-1} jx_\alpha = jx_\alpha$. Evidently, $jS_\alpha j^{-1}$ is a primitive parabolic element of Γ and hence jx_α is a cusp of Γ.

For each cusp x_α of Γ let σ_α be an element in $\mathrm{PSL}(2,\mathbb{Z})$ such that $\sigma_\alpha\infty = x_\alpha$. Then for $\mathfrak{u} \in \mathcal{S}(s;\Gamma;\chi)$ we have

$$(J\mathfrak{u})(\sigma_\alpha z) = \mathfrak{u}(j\sigma_\alpha z). \tag{3.137}$$

Since $j\sigma_\alpha\infty$ is a cusp of Γ, $\mathfrak{u}(j\sigma_\alpha z)$ vanishes exponentially fast at infinity and hence $(J\mathfrak{u})(z)$ vanishes exponentially fast at the cusp x_α. Therefore $J\mathfrak{u}$ belongs to $\mathcal{S}(s;\Gamma;\chi)$. Thus, the proof of the second assertion is complete. $\square$

Next we are going to determine the automorphism of periodlike functions which corresponds to the automorphism J of the automorphic functions.

THEOREM 3.13. *Let* $\mathfrak{u} \in \mathcal{S}(s;\Gamma;\chi)$ *and* j *be an element of the normalizer* $N_{\mathrm{PGL}(2,\mathbb{Z})}(\Gamma)$ *of* Γ *in* $\mathrm{PGL}(2,\mathbb{Z})$ *with* $\det j = -1$ *such that* $\chi(j\gamma j^{-1}) = \chi(\gamma)$ *for all* $\gamma \in \Gamma$ *and the kernel of the induced representation* ρ_χ *has finite index in* $\mathrm{PSL}(2,\mathbb{Z})$. *Then the automorphism* $J : \mathcal{S}(s;\Gamma;\chi) \to \mathcal{S}(s;\Gamma;\chi)$, *given by*

$$J\mathfrak{u}(z) = \mathfrak{u}(jz), \tag{3.138}$$

induces via the integral transformation $\mathcal{I} \circ B$ *an automorphism*

$$\mathcal{J} : \mathfrak{S}(s;\Gamma;\chi) \to \mathfrak{S}(s;\Gamma;\chi) \tag{3.139}$$

given by

$$\mathcal{J}\psi(\zeta) = \rho_\chi(S)\pi(J)\zeta^{-2s}\psi(\frac{1}{\zeta}). \tag{3.140}$$

The matrix $\pi(J)$ *is defined by*

$$\pi(J)_{k,\alpha;l,\beta} = [\delta_\Gamma(jR_kMR_l^{-1})]_{\alpha,\beta}, \quad 1 \le k,l \le \mu_\Gamma, \quad 1 \le \alpha,\beta \le \dim\chi \tag{3.141}$$

where μ_Γ *is the index of* Γ *in* $\mathrm{PSL}(2,\mathbb{Z})$, $R_i \in R(\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z}))$, *and*

$$\delta_\Gamma(\gamma) := \begin{cases} \chi(\gamma), & \gamma \in \Gamma, \\ 0, & \gamma \notin \Gamma. \end{cases} \tag{3.142}$$

Moreover, the matrix $\pi(J)$ *is a symmetry operator of Mayer's transfer operator* $\mathcal{L}_s^{\Gamma,\chi}$. *(see Definition 3.3).*

PROOF. For $\psi(\zeta) = (\mathcal{I} \circ B\mathfrak{u})(\zeta)$ let

$$\mathcal{J}\psi(\zeta) := (\mathcal{I} \circ B \circ J\mathfrak{u})(\zeta). \tag{3.143}$$

Then we have

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = \int_0^{i\infty} \eta(B \circ J\mathfrak{u}, R_\zeta^s)(z) = \int_{S0}^{Si\infty} \eta(B \circ J\mathfrak{u}, R_\zeta^s)(Sz). \tag{3.144}$$

If $z \mapsto g(z)$ is a holomorphic change of the variable, then for two eigenfunctions of the hyperbolic Laplacian with the same eigenvalue the 1-form η transforms as ([**29**], page 210)

$$\eta(u \circ g, v \circ g) = \eta(u,v) \circ g. \tag{3.145}$$

Hence, observing that each component of $\mathfrak{u}$ and R_ζ^s are eigenfunctions of the hyperbolic Laplacian with eigenvalue $s(1-s)$ (see section 3.2), we get

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = -\int_0^{i\infty} \eta(B \circ J\mathfrak{u}(Sz), R_\zeta^s(Sz)). \tag{3.146}$$

where we used the fact that $S0 = i\infty$ and $Si\infty = 0$. Inserting the identity $B \circ J\mathfrak{u}(Sz) = \rho_\chi(S)B \circ J\mathfrak{u}(z)$ into (3.146) and using the linearity of η we get

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = -\rho_\chi(S) \int_0^{i\infty} \eta(B \circ J\mathfrak{u}(z), R_\zeta^s(Sz)). \tag{3.147}$$

But on $\Re z = 0$ the 1-form η fulfills

$$\begin{aligned} &\eta(J_M u(z), J_M v(z))|_{\Re z=0} = \\ &[u(Mz)\partial_x v(Mz) - v(Mz)\partial_x u(Mz)]_{\Re z=0}\, dy = \\ &-[u(z)\partial_x v(z) - v(z)\partial_x u(z)]_{\Re z=0}\, dy \end{aligned} \tag{3.148}$$

where $J_M \mathfrak{u}(z) := \mathfrak{u}(Mz)$. Consequently, (3.147) can be written as

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = \rho_\chi(S) \int_0^{i\infty} \eta(J_M \circ B \circ J\mathfrak{u}(z), R_\zeta^s(SMz)). \tag{3.149}$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$ the following identity holds [**29**]

$$R_{g\zeta}(gz) = (c\zeta + d)^2 R_\zeta(z). \tag{3.150}$$

Thus we have

$$R_\zeta(Sz) = \zeta^{-2} R_{S\zeta}(z). \tag{3.151}$$

Moreover, one can easily check that $R_{-\zeta}(Mz) = R_\zeta(z)$ and hence

$$R_\zeta(SMz) = \zeta^{-2} R_{S\zeta}(Mz) = \zeta^{-2} R_{-S\zeta}(z). \tag{3.152}$$

Inserting this into (3.149) and by using the linearity of η, we get

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = \rho_\chi(S)\zeta^{-2s} \int_0^{i\infty} \eta(J_M \circ B \circ J\mathfrak{u}(z), R_{-S\zeta}^s(z)). \tag{3.153}$$

Let R_i, $1 \le i \le \mu$, be a set of representatives of the right cosets of Γ in $\mathrm{PSL}(2,\mathbb{Z})$. Then we have

$$(J_M \circ B \circ J\mathfrak{u})_k(z) = \mathfrak{u}(jR_k Mz). \tag{3.154}$$

Evidently, the operator $J_M \circ B \circ J$ defines the matrix $\pi(J)$ which acts on the vector $B\mathfrak{u}$ and hence we get

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = \rho_\chi(S)\pi(J)\zeta^{-2s} \int_0^{i\infty} \eta(B\mathfrak{u}(z), R_{-S\zeta}^s(z)). \tag{3.155}$$

Since $-S\zeta = \frac{1}{\zeta}$ we get

$$\mathcal{I} \circ B \circ J\mathfrak{u}(\zeta) = \rho_\chi(S)\pi(J)\zeta^{-2s}\psi(\frac{1}{\zeta}) \tag{3.156}$$

which is the desired result.

To prove the last assertion we must show according to the definition 3.3 of a symmetry operator of $\mathcal{L}_s^{\Gamma,\chi}$ that $\pi(J)$ fulfils

1) $\pi(J)^2 = id_{\mu_\Gamma \dim\chi}$, where id_d denotes the $d \times d$ identity matrix.
2) $\pi(J)\rho_\chi(S) = \rho_\chi(S)\pi(J)$.
3) $\pi(J)\rho_\chi(T) = \rho_\chi(T^{-1})\pi(J)$.

The matrix $\pi(J)$ can be considered a monomial matrix whose entries are block matrices of dimension $n = \dim\chi$. That is, for $1 \leq k,l \leq \mu_\Gamma$ the component $[\pi(J)]_{k,l}$ of $\pi(J)$ is a matrix whose (α,β) components are defined by

$$[[\pi(J)]_{k,l}]_{\alpha,\beta} := \pi(J)_{k,\alpha;l,\beta}, \quad 1 \leq \alpha,\beta \leq \dim\chi. \tag{3.157}$$

With this interpretation of $\pi(J)$ as a monomial matrix it is obvious that for a nonzero component (k,l) of its square, namely

$$[\pi(J)^2]_{k,l} = \sum_{m=0}^{\mu_\Gamma} [\pi(J)]_{k,m}[\pi(J)]_{m,l}, \quad 1 \leq k,l \leq \mu_\Gamma \tag{3.158}$$

all terms except one of them vanish. That is, there is a unique $1 \leq m \leq \mu_\Gamma$ such that

$$[\pi(J)^2]_{k,l} = [\pi(J)]_{k,m}[\pi(J)]_{m,l}. \tag{3.159}$$

Thus according to the definition of $\pi(J)$ we have

$$[\pi(J)^2]_{k,l} = \chi(jR_kMR_m^{-1})\chi(jR_mMR_l^{-1}). \tag{3.160}$$

On the other hand, according to our assumption for any $\gamma \in \Gamma$ we have $\chi(j\gamma j^{-1}) = \chi(\gamma)$. Therefore, we get

$$[\pi(J)^2]_{k,l} = \chi(j^2R_kMR_m^{-1}j^{-1})\chi(jR_mMR_l^{-1}) = \chi(j^2R_kR_l^{-1}). \tag{3.161}$$

But $j^2 = id_2$ and hence we have

$$[\pi(J)^2]_{k,l} = \chi(j^2R_kR_l^{-1}) = \chi(R_kR_l^{-1}). \tag{3.162}$$

Since $R_kR_l^{-1} \in \Gamma$ iff $k = l$ it follows that $\pi(J)$ is a diagonal matrix whose diagonal elements are the identity matrix $\chi(id_2)$. This yields the first item, namely $\pi(J)^2 = id_{\mu_\Gamma \dim\chi}$. To prove the second item we consider the matrix $\rho_\chi(S)$ as a monomial matrix with block matrices of dimension $\dim\chi$ as its entries. Then similar to the previous case, there is an unique $1 \leq m \leq \mu_\Gamma$ such that a nonzero component (k,l) of $\pi(J)\rho_\chi(S)$ can be written as

$$[\pi(J)\rho_\chi(S)]_{k,l} = [\pi(J)]_{k,m}[\rho_\chi(S)]_{m,l}, \quad 1 \leq k,l \leq \mu_\Gamma. \tag{3.163}$$

Then according to the definitions of $\pi(J)$ and ρ_χ we get

$$[\pi(J)\rho_\chi(S)]_{k,l} = \chi(jR_kMR_m^{-1})\chi(R_mSR_l^{-1}) \tag{3.164}$$

or,

$$[\pi(J)\rho_\chi(S)]_{k,l} = \chi(jR_kMSR_l^{-1}), \quad 1 \leq k,l \leq \mu_\Gamma. \tag{3.165}$$

In (3.164) evidently we have

$$jR_kMR_m^{-1} \in \Gamma, \quad R_mSR_l^{-1} \in \Gamma. \tag{3.166}$$

From this and $j\Gamma j^{-1} = \Gamma$ respectively $MS = SM$ by a simple calculation it follows that

$$R_kSR_{m'}^{-1} \in \Gamma, \quad jR_{m'}MR_l^{-1} \in \Gamma \tag{3.167}$$

where $j^{-1}R_mMS$ determines uniquely a representative $R_{m'}$. Hence, the (k,l) component of $\rho_\chi(S)\pi(J)$ is nonzero and given by

$$[\rho_\chi(S)\pi(J)]_{k,l} = [\rho_\chi(S)]_{k,m'}[\pi(J)]_{m',l}, \quad 1 \le k, l \le \mu_\Gamma. \tag{3.168}$$

Then according to the definitions of $\pi(J)$ and ρ_χ we get

$$[\rho_\chi(S)\pi(J)]_{k,l} = \chi(R_kSR_{m'}^{-1})\chi(jR_{m'}MR_l^{-1}). \tag{3.169}$$

By using the fact that for each $\gamma \in \Gamma$, $\chi(j\gamma j^{-1}) = \chi(\gamma)$ we get

$$[\rho_\chi(S)\pi(J)]_{k,l} = \chi(jR_kSR_{m'}^{-1}j^{-1})\chi(jR_{m'}MR_l^{-1}) = \chi(jR_kSMR_l^{-1}). \tag{3.170}$$

Since $MS = SM$, we get

$$[\rho_\chi(S)\pi(J)]_{k,l} = \chi(jR_kMSR_l^{-1}), \quad 1 \le k, l \le \mu_\Gamma. \tag{3.171}$$

This and (3.165) yields the second item.

To prove the third item we consider the matrix $\rho_\chi(T)$ as a monomial matrix with block matrices of dimension $\dim\chi$ as its entries. Then similar to the previous case, there is an unique $1 \le m \le \mu_\Gamma$ such that a nonzero component (k,l) of $\pi(J)\rho_\chi(T)$ can be written as

$$[\pi(J)\rho_\chi(T)]_{k,l} = [\pi(J)]_{k,m}[\rho_\chi(T)]_{m,l}, \quad 1 \le k, l \le \mu_\Gamma. \tag{3.172}$$

Then according to the definitions of $\pi(J)$ and ρ_χ we get

$$[\pi(J)\rho_\chi(T)]_{k,l} = \chi(jR_kMR_m^{-1})\chi(R_mTR_l^{-1}) \tag{3.173}$$

or,

$$[\pi(J)\rho_\chi(T)]_{k,l} = \chi(jR_kMTR_l^{-1}), \quad 1 \le k, l \le \mu_\Gamma. \tag{3.174}$$

In (3.173) evidently we have

$$jR_kMR_m^{-1} \in \Gamma, \quad R_mTR_l^{-1} \in \Gamma. \tag{3.175}$$

From this and $j\Gamma j^{-1} = \Gamma$ respectively $MT = T^{-1}M$ it follows by a simple calculation that

$$R_kT^{-1}R_{m'}^{-1} \in \Gamma, \quad jR_{m'}MR_l^{-1} \in \Gamma \tag{3.176}$$

where $j^{-1}R_mMT^{-1}$ determines uniquely a representative $R_{m'}$. Hence, the (k,l) component of $\rho_\chi(T^{-1})\pi(J)$ is nonzero and given by

$$[\rho_\chi(T^{-1})\pi(J)]_{k,l} = [\rho_\chi(T^{-1})]_{k,m'}[\pi(J)]_{m',l}, \quad 1 \le k, l \le \mu_\Gamma. \tag{3.177}$$

Then according to the definitions of $\pi(J)$ and ρ_χ we get

$$[\rho_\chi(T^{-1})\pi(J)]_{k,l} = \chi(R_kT^{-1}R_{m'}^{-1})\chi(jR_{m'}MR_l^{-1}). \tag{3.178}$$

By using the fact that for each $\gamma \in \Gamma$, $\chi(j\gamma j^{-1}) = \chi(\gamma)$ we get

$$[\rho_\chi(T^{-1})\pi(J)]_{k,l} = \chi(jR_kT^{-1}R_{m'}^{-1}j^{-1})\chi(jR_{m'}MR_l^{-1}) = \chi(jR_kT^{-1}MR_l^{-1}). \tag{3.179}$$

Since $MT = T^{-1}M$, we get

$$[\rho_\chi(T^{-1})\pi(J)]_{k,l} = \chi(jR_kMTR_l^{-1}), \quad 1 \le k, l \le \mu_\Gamma. \tag{3.180}$$

This and (3.174) yields the third item. Thus the proof of the theorem is complete. □

CHAPTER 4

Induced representations of the projective modular group

In this chapter, we present some properties of the representation of the projective modular group, induced from the trivial character of the Hecke congruence subgroups. In particular, we discuss its decomposition into subrepresentations.

4.1. Preliminaries

Recall that the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ and its extension $\mathrm{PGL}(2,\mathbb{Z})$ are defined respectively by

$$\begin{aligned} \mathrm{PSL}(2,\mathbb{Z}) &= \mathrm{SL}(2,\mathbb{Z})/\{\pm 1\} = \\ &= \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, \quad ad-bc=1 \right\}. \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathrm{PGL}(2,\mathbb{Z}) &= \mathrm{GL}(2,\mathbb{Z})/\{\pm 1\} = \\ &= \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, \quad ad-bc=\pm 1 \right\}. \end{aligned} \tag{4.2}$$

As in [**25**], we consider the subgroups

(4.3)

$$\begin{aligned} &\Pi(n) = \{ g \in \mathrm{PGL}(2,\mathbb{Z}) \mid g \equiv \pm id \mod n \}, \\ &M(n) = \left\{ g \in \mathrm{PGL}(2,\mathbb{Z}) \mid g \equiv \alpha\, id \mod n, \quad \alpha^2 \equiv \pm 1 \mod n \right\}, \\ &\Gamma(n) = \{ g \in \mathrm{PSL}(2,\mathbb{Z}) \mid g \equiv \pm id \mod n \}, \\ &H(n) = \left\{ g \in \mathrm{PSL}(2,\mathbb{Z}) \mid g \equiv \alpha\, id \mod n, \quad \alpha^2 \equiv 1 \mod n \right\}. \end{aligned}$$

We recall some facts about these subgroups (see [**25**], page 29)

$$\Pi(n) \lhd \mathrm{PGL}(2,\mathbb{Z}), \quad M(n) \lhd \mathrm{PGL}(2,\mathbb{Z}) \tag{4.4}$$

and

$$\Gamma(n) \lhd \mathrm{PSL}(2,\mathbb{Z}), \quad H(n) \lhd \mathrm{PSL}(2,\mathbb{Z}), \tag{4.5}$$

where $K \lhd G$ means that K is a normal subgroup of G. For $n > 2$ one has, $\Pi(n) = \Gamma(n)$. The index of $H(n)$ in $M(n)$ is given by

$$[M(n) : H(n)] = \begin{cases} 2 & \text{if } -1 \text{ is a square} \mod n, \\ 1 & \text{otherwise.} \end{cases} \tag{4.6}$$

In particular, $M(n) = H(n)$ if n is divisible by 4 or by a prime p with $p \equiv 3 \mod 4$. Moreover, the following isomorphism holds,

$$H(n)/\Gamma(n) \cong \left\{a \in \mathbb{Z}_n | a^2 = 1\right\} / \{\pm 1\} . \tag{4.7}$$

In particular, for the cases $n = 1, 2, 4, 6$ or n an odd prime power, the group $H(n)$ coincides with $\Gamma(n)$.

The group $\Gamma(n)$ is called the principal congruence subgroup. The index of $\Gamma(n)$ in $\mathrm{PSL}(2, \mathbb{Z})$, is given by (see for example [**43**])

$$\mu_{\Gamma(n)} := [\mathrm{PSL}(2, \mathbb{Z}) : \Gamma(n)] = \begin{cases} \frac{1}{2} n^3 \prod_{p|n} (1 - \frac{1}{p^2}), & n > 2, \\ 6, & n = 2 \end{cases} \tag{4.8}$$

where the p run over all primes dividing n. The Hecke congruence group $\Gamma_0(n)$ of level n is a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$, defined by

$$\Gamma_0(n) = \{g \in \mathrm{PSL}(2, \mathbb{Z}) \,|\, c = 0 \mod n\} . \tag{4.9}$$

The index of $\Gamma_0(n)$ in $\mathrm{PSL}(2, \mathbb{Z})$ is given by [**43**]

$$\mu_{\Gamma_0(n)} = [\mathrm{PSL}(2, \mathbb{Z}) : \Gamma_0(n)] = n \prod_{p|n} (1 + \frac{1}{p}) \tag{4.10}$$

where the p run over all primes dividing n.

We denote by $R(\Gamma_0(n) \backslash \mathrm{PSL}(2, \mathbb{Z}))$ and $R(\mathrm{PSL}(2, \mathbb{Z}) / \Gamma_0(n))$ a set of representatives of the right cosets and the left cosets of $\Gamma_0(n)$ in $\mathrm{PSL}(2, \mathbb{Z})$, respectively.

Now we note that

$$\mathrm{PGL}(2, \mathbb{Z}) = \mathrm{PSL}(2, \mathbb{Z}) \rtimes C_2 \tag{4.11}$$

where C_2 is the cyclic group of order 2 generated by the element

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.12}$$

and $\rtimes$ denotes the semidirect product of groups.

For any subgroup $\Gamma \leqslant \mathrm{PSL}(2, \mathbb{Z})$ we define the extended subgroup $\overline{\Gamma} \leqslant \mathrm{PGL}(2, \mathbb{Z})$ by

$$\overline{\Gamma} = \Gamma \rtimes C_2. \tag{4.13}$$

Then the extended Hecke congruence subgroup of $\mathrm{PGL}(2, \mathbb{Z})$ is given by

$$\overline{\Gamma}_0(n) = \Gamma_0(n) \rtimes C_2 = \{g \in \mathrm{PGL}(2, \mathbb{Z}) \,|\, c = 0 \mod n\} . \tag{4.14}$$

The index $\mu_{\overline{\Gamma}_0(n)}$ of $\overline{\Gamma}_0(n)$ in $\mathrm{PGL}(2, \mathbb{Z})$ coincides with the index of $\Gamma_0(n)$ in $\mathrm{PSL}(2, \mathbb{Z})$.

We denote by $R(\overline{\Gamma}_0(n) \backslash \mathrm{PGL}(2, \mathbb{Z}))$ and $R(\mathrm{PGL}(2, \mathbb{Z}) / \overline{\Gamma}_0(n))$ a set of representatives of the right cosets and the left cosets of $\overline{\Gamma}_0(n)$ in $\mathrm{PGL}(2, \mathbb{Z})$, respectively.

Next, we define the factor groups

$$
\begin{aligned}
Q(n) &:= \mathrm{PSL}(2,\mathbb{Z})/H(n)\\
K(n) &:= \Gamma_0(n)/H(n)\\
G(n) &:= \mathrm{PGL}(2,\mathbb{Z})/M(n)\\
\overline{K}(n) &:= \overline{\Gamma}_0(n)/M(n).
\end{aligned}
\tag{4.15}
$$

In the following lemma we determine the order of $K(p^e)$.

Lemma 4.1. *For a prime power $q = p^e$, $e \geq 1$, the order of the group $K(q) = \Gamma_0(q)/H(q)$ is given by*

$$
|K(q)| = \begin{cases} \frac{1}{2}p^{2e-1}(p-1), & q = p^e, p \neq 2\\ 2, & q = 2,\\ 4, & q = 4,\\ 2^{2e-3}, & q = 2^e, e \geq 3. \end{cases}
\tag{4.16}
$$

Proof. First, we note that for $q = p^e$ we have

$$
[\Gamma_0(q) : \Gamma(q)] = \frac{[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(q)]}{[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_0(q)]}.
\tag{4.17}
$$

Then according to (4.8) and (4.10) we get

$$
[\Gamma_0(q) : \Gamma(q)] = \begin{cases} \frac{1}{2}p^{2e-1}(p-1), & q \neq 2,\\ 2, & q = 2. \end{cases}
\tag{4.18}
$$

On the other hand, we have

$$
|K(q)| = [\Gamma_0(q) : H(q)] = \frac{[\Gamma_0(q) : \Gamma(q)]}{[H(q) : \Gamma(q)]}.
\tag{4.19}
$$

For $q = 2$, $q = 4$, and q an odd prime power p^e with $e \geq 1$, as mentioned we have $H(q) = \Gamma(q)$ and hence the order of $K(p^e)$ is equal to the index of $\Gamma(q)$ in $\Gamma_0(q)$ as given in (4.18).

For $q = 2^e$ and $e \geq 3$, to calculate the index of $\Gamma(2^e)$ in $H(2^e)$ we note that

(4.20)

$$
H(2^e)/\Gamma(2^e) = \left\{ \delta\Gamma(2^e) \mid \delta \in H(2^e), \delta = \begin{pmatrix} a & 0\\ 0 & a \end{pmatrix} \mod 2^e, \quad a^2 \equiv 1 \mod 2^e \right\}.
$$

Let $(\mathbb{Z}/2^e\mathbb{Z})^\times$ be the multiplicative group of the invertible elements of $\mathbb{Z}/2^e\mathbb{Z}$ with the identity element $1 + 2^e\mathbb{Z}$ denoted by 1_{2^e} respectively the element $-1 + 2^e\mathbb{Z}$ denoted by -1_{2^e}. Then we have

$$
H(2^e)/\Gamma(2^e) \cong \left\{ a \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid a^2 \equiv 1 \mod 2^e \right\} / \left\{ \pm 1_{2^e} \right\}.
\tag{4.21}
$$

Observing that the elements of $(\mathbb{Z}/2^e\mathbb{Z})^\times$ modulo 2^e are odd integers, for $a \in (\mathbb{Z}/2^e\mathbb{Z})^\times$ one can easily show that if $a^2 \equiv 1 \mod 2^e$ then

$a \equiv \pm 1 \mod 2^e$ or $a \equiv \pm 1 + 2^{e-1} \mod 2^e$. Hence we have

(4.22)
$$\left\{a \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid a^2 \equiv 1 \mod 2^e\right\} = \left\{1 \mod 2^e, -1 \mod 2^e, 1 + 2^{e-1} \mod 2^e, -1 + 2^{e-1} \mod 2^e\right\}.$$

Therefore, we get

$$H(2^e)/\Gamma(2^e) \cong \left\{1 \mod 2^e, 1 + 2^{e-1} \mod 2^e\right\} \tag{4.23}$$

and hence for $e \geq 3$

$$[H(2^e) : \Gamma(2^e)] = 2. \tag{4.24}$$

This, (4.18), and (4.19) complete the proof. □

For the proof of the next lemma we need to introduce the notion of the product of subsets of a group. Let X and Y be subsets of a group G. Then the product of X and Y is defined to be

$$XY = \{xy \mid x \in X, \quad y \in Y\} \subseteq G \tag{4.25}$$

where xy is the product of x and y as elements of G.

Lemma 4.2. *The following isomorphism of groups holds:*

$$\begin{cases} G(n) \cong Q(n), & \textit{if } -1 \textit{ is a square} \mod n, \\ G(n) = Q(n) \rtimes \mathcal{C}_2, & \textit{otherwise}, \end{cases} \tag{4.26}$$

where $\mathcal{C}_2 < G(n)$ is the cyclic group of order two generated by $MH(n)$ with M as in (4.12). *In the case where -1 is a square modulo n the isomorphism $\imath : Q(n) \to G(n)$ is given explicitly by*

$$\imath(gH(n)) = gM(n). \tag{4.27}$$

Proof. For -1 a square modulo n, $H(n)$ is a subgroup of index 2 in $M(n)$ and we have

(4.28)
$$M(n) = H(n) \cup \gamma H(n), \quad \gamma \in M(n), \quad \gamma \equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mod n$$

where α is an integer such that $\alpha^2 \equiv -1 \mod n$. According to the " first isomorphism theorem" (see [**1**], page 11) we have

$$\mathrm{PSL}(2,\mathbb{Z})M(n)/M(n) \cong \mathrm{PSL}(2,\mathbb{Z})/(\mathrm{PSL}(2,\mathbb{Z}) \cap M(n)). \tag{4.29}$$

Since $\mathrm{PGL}(2,\mathbb{Z}) = \mathrm{PSL}(2,\mathbb{Z})M(n)$ and $\mathrm{PSL}(2,\mathbb{Z}) \cap M(n) = H(n)$, (4.29) reduces to

$$\mathrm{PGL}(2,\mathbb{Z})/M(n) \cong \mathrm{PSL}(2,\mathbb{Z})/H(n) \tag{4.30}$$

which is the desired result.

Next, we prove that this isomorphism is given by the map $\imath : Q(n) \to G(n)$. For two distinct elements $g_1H(n)$ and $g_2H(n)$ of $Q(n)$ we have $g_1^{-1}g_2 \notin H(n)$. From this and observing that $\det(g_1^{-1}g_2) = 1$ whereas γ in (4.28) has negative determinant, it follows that $g_1^{-1}g_2 \notin$

$M(n)$, that is, $g_1M(n)$ and $g_2M(n)$ are distinct elements of $G(n)$. Thus $\imath$ is injective. On the other hand, the cardinality of the image of $\imath$ coincides with that of $G(n)$ and hence $\imath$ is onto. Finally one can easily check that $\imath$ preserves the group structure. Thus the isomorphism $\imath : Q(n) \to G(n)$ is given by (4.27). For the case where $M(n) = H(n)$ we note that $Q(n) = \mathrm{PSL}(2,\mathbb{Z})/H(n)$ is a normal subgroup of index two in $G(n) = \mathrm{PGL}(2,\mathbb{Z})/H(n)$. Moreover, the cyclic group of order two, generated by $MH(n)$ is obviously a subgroup of $G(n)$ such that $\mathcal{C}_2 \cap Q(n) = \{id\}$ and $G(n) = Q(n)\mathcal{C}_2$. Hence, we get the desired result, that is, $G(n) = Q(n) \rtimes \mathcal{C}_2$. $\square$

It is straightforward to verify, that if A, B, and C are all subgroups of a group G which satisfy furthermore $A \leqslant C$, then

$$A(B \cap C) = AB \cap C. \tag{4.31}$$

We need this for the proof of the following Lemma.

Lemma 4.3. *Let $n, m \in \mathbb{N}$ be coprimes. Then*

$$Q(nm) \cong Q(n) \times Q(m), \tag{4.32}$$

$$K(nm) \cong K(n) \times K(m) \tag{4.33}$$

where $\times$ denotes the direct product of groups and $\cong$ denotes a group isomorphism.

Proof. For the first assertion we refer to ([**25**], page 30). To prove the second assertion, first we prove that

$$\begin{aligned}&\Gamma_0(nm)/H(nm) = \\ &\left[\left(\Gamma_0(nm) \cap H(m)\right)/H(nm)\right] \times \left[\left(\Gamma_0(nm) \cap H(n)\right)/H(nm)\right].\end{aligned} \tag{4.34}$$

Since

$$\Gamma_0(nm)/H(nm) \leq \mathrm{PSL}(2,\mathbb{Z})/H(nm) \tag{4.35}$$

and

$$H(n)/H(nm) \lhd \mathrm{PSL}(2,\mathbb{Z})/H(nm) \tag{4.36}$$

respectively

$$H(m)/H(nm) \lhd \mathrm{PSL}(2,\mathbb{Z})/H(nm), \tag{4.37}$$

we have (see for example [**1**], page 6, Proposition 7)

$$\begin{aligned}&[\Gamma_0(nm)/H(nm)] \cap [H(n)/H(nm)] = \\ &\left(\Gamma_0(nm) \cap H(n)\right)/H(nm) \lhd \Gamma_0(nm)/H(nm)\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}&[\Gamma_0(nm)/H(nm)] \cap [H(m)/H(nm)] = \\ &\left(\Gamma_0(nm) \cap H(m)\right)/H(nm) \lhd \Gamma_0(nm)/H(nm).\end{aligned} \tag{4.39}$$

Since $(m,n)=1$ obviously we have,

$$[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\cap[(\Gamma_0(nm)\cap H(n))\,/H(nm)]=\{Id\} \tag{4.40}$$

where Id denotes the identity element of $\mathrm{PSL}(2,\mathbb{Z})/H(nm)$, namely $H(nm)$. Moreover, by applying (4.31) with $A=(\Gamma_0(nm)\cap H(m))\,/H(nm)$, $B=H(n)/H(nm)$, and $C=\Gamma_0(nm)/H(nm)$ in the middle term below it follows that

$$\begin{aligned}&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[(\Gamma_0(nm)\cap H(n))\,/H(nm)]=\\&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,\{[H(n)/H(nm)]\cap[\Gamma_0(nm)/H(nm)]\}=\\&\{[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[H(n)/H(nm)]\}\cap[\Gamma_0(nm)/H(nm)]\,.\end{aligned} \tag{4.41}$$

Since $H(n)/H(nm)\lhd Q(nm)$ and $(\Gamma_0(nm)\cap H(m))\,/H(nm)\leqslant Q(nm)$ we have

$$\begin{aligned}&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[H(n)/H(nm)]=\\&[H(n)/H(nm)]\,[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,.\end{aligned} \tag{4.42}$$

But we have also $H(m)\subset\Gamma_0(m)$ and hence

$$\Gamma_0(nm)\cap H(m)=\Gamma_0(n)\cap\Gamma_0(m)\cap H(m)=\Gamma_0(n)\cap H(m). \tag{4.43}$$

Inserting this into the right hand side of (4.42) we get

$$\begin{aligned}&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[H(n)/H(nm)]=\\&[H(n)/H(nm)]\,[(\Gamma_0(n)\cap H(m))\,/H(nm)]=\\&[H(n)/H(nm)]\,\{[H(m)/H(nm)]\cap[\Gamma_0(n)/H(nm)]\}\end{aligned} \tag{4.44}$$

Then applying (4.31) to the last term in the above equality with $A=H(n)/H(nm)$, $B=H(m)/H(nm)$, and $C=\Gamma_0(n)/H(nm)$, we get

$$\begin{aligned}&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[H(n)/H(nm)]=\\&\{[H(n)/H(nm)]\,[H(m)/H(nm)]\}\cap[\Gamma_0(n)/H(nm)]\,.\end{aligned} \tag{4.45}$$

Recall that $H(n)$ and $H(m)$ are normal in $\mathrm{PSL}(2,\mathbb{Z})$ and for coprime m and n we have (an obvious consequence of Theorem 1.4.2 in [**43**], page 23)

$$\mathrm{PSL}(2,\mathbb{Z})=H(n)H(m),\quad H(n)\cap H(m)=H(nm). \tag{4.46}$$

Hence, according to (Lemma 7. in [**1**], page 19) it follows that

$$Q(nm)=H(m)/H(nm)\times H(n)/H(nm) \tag{4.47}$$

where $\times$ denotes the direct product of groups. By inserting this into (4.45), we get

$$\begin{aligned}&[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,[H(n)/H(nm)]=\\&[\mathrm{PSL}(2,\mathbb{Z})/H(nm)]\cap[\Gamma_0(n)/H(nm)]=\\&\Gamma_0(n)/H(nm).\end{aligned} \tag{4.48}$$

Now inserting this into the last term in equality (4.41) we get

$$\begin{aligned}[(\Gamma_0(nm)\cap H(m))\,/H(nm)]\,.\,[(\Gamma_0(nm)\cap H(n))\,/H(nm)] =\\ \Gamma_0(nm)/H(nm).\end{aligned} \tag{4.49}$$

This proves (4.34). Now consider the group homomorphism

$$\phi : Q(nm) \to Q(n) \tag{4.50}$$

given by

$$\phi(gH(nm)) = gH(n). \tag{4.51}$$

Evidently, we have $\ker\phi = H(n)/H(nm)$. By restricting the domain of ϕ to $\Gamma_0(mn)/H(mn)$, we get a homomorphism

$$\phi_1 : \Gamma_0(mn)/H(mn) \to \phi(\Gamma_0(mn)/H(mn)) \tag{4.52}$$

given by

$$\phi_1(\gamma H(nm)) = \gamma H(n), \quad \gamma \in \Gamma_0(mn). \tag{4.53}$$

Evidently, the kernel of ϕ_1 is given by

$$\ker\phi_1 = \ker\phi\cap[\Gamma_0(mn)/H(mn)] = (\Gamma_0(mn)\cap H(n))\,/H(mn). \tag{4.54}$$

According to the "Fundamental Theorem on Homomorphism" ([**1**], page 10) for ϕ_1 we have

$$\begin{aligned}\phi_1(\Gamma_0(mn)/H(mn)) = \phi(\Gamma_0(mn)/H(mn)) \cong\\ \{\Gamma_0(mn)/H(mn)\}\,/\,\{(\Gamma_0(mn)\cap H(n))\,/H(mn)\}\,.\end{aligned} \tag{4.55}$$

Then from the "Second Isomorphism Theorem" ([**1**], page 12) it follows that

$$\phi(\Gamma_0(mn)/H(mn)) \cong \Gamma_0(mn)/[\Gamma_0(mn)\cap H(n)]. \tag{4.56}$$

Now applying the "First Isomorphism Theorem" ([**1**], page 11) leads to

$$\phi(\Gamma_0(mn)/H(mn)) \cong [\Gamma_0(mn)H(n)]/H(n). \tag{4.57}$$

Since $\Gamma_0(nm)\Gamma(n) = \Gamma_0(n)$ ([**43**], page 26) and since $\Gamma(n) \leqslant H(n) < \Gamma_0(n)$, obviously we get also

$$\Gamma_0(nm)H(n) = \Gamma_0(n). \tag{4.58}$$

Inserting this into (4.57), leads to the following isomorphism

$$\phi(\Gamma_0(mn)/H(mn)) \cong \Gamma_0(n)/H(n). \tag{4.59}$$

On the other hand from (4.34) and (4.54) we have

$$\phi(\Gamma_0(mn)/H(mn)) = (\Gamma_0(nm)\cap H(m))\,/H(nm). \tag{4.60}$$

Therefore we get

$$(\Gamma_0(nm)\cap H(m))\,/H(nm) \cong \Gamma_0(n)/H(n). \tag{4.61}$$

In the same way we can show that

$$(\Gamma_0(nm)\cap H(n))\,/H(nm) \cong \Gamma_0(m)/H(m). \tag{4.62}$$

The equations (4.61), (4.62), and (4.34) give the desired result. □

Now we recall the notion of G-sets ([**1**], page 27)

DEFINITION 4.1. *The (left) action of a group G on a set X is a map*

$$G \times X \to X \qquad (4.63)$$
$$(g, x) \mapsto gx$$

such that

- $Id\, x = x$ *for all* $x \in X$ *where* Id *is the identity element of* G.
- $(g_1 g_2)x = g_1(g_2 x)$ *for all* $g_1, g_2 \in G$.

If there is an action of G on X we say that G acts on X or X is a G-set.

DEFINITION 4.2. *For two G-sets X and Y a map $\phi : X \to Y$ is said to be a G-homomorphism if it commutes with the action of G, namely if*

$$\phi(gx) = g\phi(x), \quad \forall g \in G, \quad \forall x \in X. \qquad (4.64)$$

If ϕ is also bijective, then ϕ is called a G-set isomorphism. In this case we say that X and Y are isomorphic G-sets and we write $X \cong_G Y$.

For $n \mid m$ an action of $Q(m)$ on $\mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(n)$ is defined by

$$\begin{aligned} &Q(m) \times \mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(n) \to \mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(n) \\ &(gH(m), x\Gamma_0(n)) \mapsto gH(m)(x\Gamma_0(n)) := gx\Gamma_0(n). \end{aligned} \qquad (4.65)$$

This is well defined since $H(m)$ is normal in $\mathrm{PSL}(2, \mathbb{Z})$ and $H(m) \subset \Gamma_0(n)$. For this action of $Q(m)$ we denote by $X(n)$ the set of left cosets $\mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(n)$ as a $Q(m)$-set. Furtheremore, this action is obviously transitive. In the same way, for $n \mid m$ an action of $G(m)$ on $\mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(n)$ is defined by

$$\begin{aligned} &G(m) \times \mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(n) \to \mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(n) \\ &(gM(m), x\overline{\Gamma}_0(n)) \mapsto gM(m)(x\overline{\Gamma}_0(n)) := gx\overline{\Gamma}_0(n). \end{aligned} \qquad (4.66)$$

This action is again well defined since $M(m)$ is normal in $\mathrm{PGL}(2, \mathbb{Z})$ and $M(m) \subset \overline{\Gamma}_0(n)$. Similarly it is also transitive. For this action of $G(m)$ we denote by $\overline{X}(n)$ the set of left cosets $\mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(n)$ as a $G(m)$-set.

4.2. The representations of the projective modular group induced from the Hecke congruence subgroups with trivial character

The representation $U_{\Gamma_0(n)}$ of $\mathrm{PSL}(2, \mathbb{Z})$ induced from the one dimensional trivial representation of $\Gamma_0(n)$, is defined by $\mu_{\Gamma_0(n)}$-dimensional permutation matrices whose entries are given by [**12**]

$$\left[U_{\Gamma_0(n)}(g)\right]_{ij} = \delta_{\Gamma_0(n)}(r_i g r_j^{-1}), \quad r_i \in R(\Gamma_0(n)\backslash \mathrm{PSL}(2, \mathbb{Z})) \qquad (4.67)$$

where

$$\delta_{\Gamma_0(n)}(\gamma) = \begin{cases} 1, & \gamma \in \Gamma_0(n), \\ 0, & \gamma \notin \Gamma_0(n). \end{cases} \tag{4.68}$$

In the same way, the representation $U_{\overline{\Gamma}_0(n)}$ of $\mathrm{PGL}(2,\mathbb{Z})$ induced from the one dimensional trivial representation of $\overline{\Gamma}_0(n)$ is defined by $\mu_{\overline{\Gamma}_0(n)}$-dimensional permutation matrices whose entries are given by

$$\left[U_{\overline{\Gamma}_0(n)}(g)\right]_{ij} = \delta_{\overline{\Gamma}_0(n)}(r_i g r_j^{-1}), \quad r_i \in R(\overline{\Gamma}_0(n)\backslash \mathrm{PGL}(2,\mathbb{Z})) \tag{4.69}$$

where

$$\delta_{\overline{\Gamma}_0(n)}(\gamma) = \begin{cases} 1, & \gamma \in \overline{\Gamma}_0(n), \\ 0, & \gamma \notin \overline{\Gamma}_0(n). \end{cases} \tag{4.70}$$

LEMMA 4.4. *The kernels of the induced representations* $U_{\Gamma_0(n)}$ *and* $U_{\overline{\Gamma}_0(n)}$ *coincide with the groups* $H(n)$ *and* $M(n)$*, respectively.*

PROOF. First we prove that $\ker U_{\overline{\Gamma}_0(n)} = M(n)$. Since $M(n) \lhd \mathrm{PGL}(2,\mathbb{Z})$ and $M(n) \leq \overline{\Gamma}_0(n)$, from the definition of $U_{\overline{\Gamma}_0(n)}$ evidently we get $M(n) \leq \ker U_{\overline{\Gamma}_0(n)}$. Now we prove the converse. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker U_{\overline{\Gamma}_0(n)}$. Then for all $r \in R(\overline{\Gamma}_0(n)\backslash \mathrm{PGL}(2,\mathbb{Z}))$ we have $\delta_{\overline{\Gamma}_0(n)}(rgr^{-1}) = 1$ or equivalently $rgr^{-1} \in \overline{\Gamma}_0(n)$. For $r = id$ it follows that $g \in \overline{\Gamma}_0(n)$. Then for $r = S$ we have

$$SgS^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \in \overline{\Gamma}_0(n) \tag{4.71}$$

and hence $g \in \overline{\Gamma}_0^0(n)$. For $r = ST$ we have

$$rgr^{-1} = \begin{pmatrix} d-c & -c \\ a+c-b-d & a+c \end{pmatrix} \in \overline{\Gamma}_0(n) \tag{4.72}$$

and hence $a \equiv d \mod n$. This together with the fact that $ad \equiv \pm 1 \mod n$ yields $g \in M(n)$. That means $\ker U_{\overline{\Gamma}_0(n)} \leq M(n)$ and hence $\ker U_{\overline{\Gamma}_0(n)} = M(n)$. To prove the other assertion we note that $\ker U_{\Gamma_0(n)} = \ker U_{\overline{\Gamma}_0(n)} \cap \mathrm{PSL}(2,\mathbb{Z})$ and hence $\ker U_{\Gamma_0(n)} = H(n)$. $\square$

Let $V_{\mu_{\Gamma_0(n)}}$ be a $\mu_{\Gamma_0(n)}$-dimensional Hermitian vector space and let $\left\{e_{x\Gamma_0(n)} \mid x\Gamma_0(n) \in X(n)\right\}$ be an orthonormal basis of $V_{\mu_{\Gamma_0(n)}}$ indexed by the elements of $X(n)$. Then for $n|m$ the map $\pi_{Q(m),X(n)} : Q(m) \to \mathrm{Aut}V_{\mu_{\Gamma_0(n)}}$ defined by

$$\pi_{Q(m),X(n)}(gH(m))e_{x\Gamma_0(n)} = e_{gx\Gamma_0(n)} \tag{4.73}$$

defines a permutation representation of $Q(m)$ associated with $X(n)$ (see for example [**52**], page 5). For $m = n$ we put for simplicity $\pi_{Q(n)} := \pi_{Q(n),X(n)}$.

LEMMA 4.5. *Let $U_{\Gamma_0(n)}$ be the induced representation of* $\mathrm{PSL}(2,\mathbb{Z})$ *defined in* (4.67) *and let $\pi_{Q(m),X(n)}$ be the permutation representation of the group $Q(m)$ as defined in* (4.73). *Then there are sets of representatives of the left cosets respectively right cosets* $R(\mathrm{PSL}(2,\mathbb{Z}))/\Gamma_0(n)$ *respectively* $R(\Gamma_0(n)\backslash\mathrm{PSL}(2,\mathbb{Z}))$ *such that with respect to these representatives for each* $g\in\mathrm{PSL}(2,\mathbb{Z})$ *we have*

$$U_{\Gamma_0(n)}(g)=\pi_{Q(m),X(n)}(gH(m)). \tag{4.74}$$

PROOF. The entries of the permutation matrix $\pi_{Q(m),X(n)}(gH(m))$ are given by

(4.75)
$$\left[\pi_{Q(m),X(n)}(gH(m))\right]_{xx'}=\left\langle e_{x\Gamma_0(n)},\pi_{Q(m),X(n)}(gH(m))e_{x'\Gamma_0(n)}\right\rangle= \left\langle e_{x\Gamma_0(n)},e_{gx'\Gamma_0(n)}\right\rangle$$

where $\langle,\rangle$ denotes the inner product in the Hermitian vector space $V_{\mu_{\Gamma_0(n)}}$. But we have

$$\left\langle e_{x\Gamma_0(n)},e_{gx'\Gamma_0(n)}\right\rangle=\begin{cases}1, & x\Gamma_0(n)=gx'\Gamma_0(n),\\ 0, & x\Gamma_0(n)\neq gx'\Gamma_0(n)\end{cases} \tag{4.76}$$

or

$$\left\langle e_{x\Gamma_0(n)},e_{gx'\Gamma_0(n)}\right\rangle=\begin{cases}1, & x^{-1}gx'\in\Gamma_0(n),\\ 0, & x^{-1}gx'\notin\Gamma_0(n)\end{cases}=\delta_{\Gamma_0(n)}(x^{-1}gx') \tag{4.77}$$

where $\delta_{\Gamma_0(n)}$ is given in (4.68). Thus we have

$$\left[\pi_{Q(m),X(n)}(gH(m))\right]_{xx'}=\delta_{\Gamma_0(n)}(x^{-1}gx'). \tag{4.78}$$

For the set of representatives of the left cosets $\mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(n)$, given by

$$R(\mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(n))=\{x_1,x_2,\ldots,x_\mu\}\,, \tag{4.79}$$

as a set of representatives of the right cosets $\Gamma_0(n)\backslash\mathrm{PSL}(2,\mathbb{Z})$ can be chosen

$$R(\Gamma_0(n)\backslash\mathrm{PSL}(2,\mathbb{Z}))=\left\{r_1=x_1^{-1},r_2=x_2^{-1},\ldots,r_\mu=x_\mu^{-1}\right\} \tag{4.80}$$

With this choice of representatives, we get

$$\begin{aligned}&\left[\pi_{Q(m),X(n)}(gH(m))\right]_{x_ix_j}=\\ &\delta_{\Gamma_0(n)}(x_i^{-1}gx_j)=\delta_{\Gamma_0(n)}(r_igr_j^{-1})=\left[U_{\Gamma_0(n)}(g)\right]_{r_ir_j}\end{aligned} \tag{4.81}$$

or

$$\left[\pi_{Q(m),X(n)}(gH(m))\right]_{ij}=\left[U_{\Gamma_0(n)}(g)\right]_{ij},\quad 1\leq i,j\leq\mu. \tag{4.82}$$

Thus the proof is complete. □

Let $V_{\mu_{\overline{\Gamma}_0(n)}}$ be a $\mu_{\overline{\Gamma}_0(n)}$-dimensional Hermitian vector space and let $\left\{e_x | x \in \overline{X}(n)\right\}$ be an orthonormal basis of $V_{\mu_{\overline{\Gamma}_0(n)}}$ indexed by the elements of $\overline{X}(n)$. Then for $n|m$ the map $\pi_{G(m),X(n)} : G(m) \to \mathrm{Aut} V_{\mu_{\overline{\Gamma}_0(n)}}$ defined by

$$\pi_{G(m),\overline{X}(n)}(gM(m))e_{x\overline{\Gamma}_0(n)} = e_{gx\overline{\Gamma}_0(n)} \tag{4.83}$$

is a permutation representation of $G(m)$ associated with $\overline{X}(n)$. For simplicity we put $\pi_{G(n)} := \pi_{G(n),\overline{X}(n)}$. Similar to the previous lemma one can prove:

LEMMA 4.6. *Let $U_{\overline{\Gamma}_0(n)}$ be the induced representation of* $\mathrm{PGL}(2,\mathbb{Z})$ *defined in* (4.69) *and let $\pi_{G(m),\overline{X}(n)}$ be the permutation representation of the group $G(m)$ as defined in* (4.83)*. Then there is a set of representatives of the right cosets $R(\overline{\Gamma}_0(n)\backslash\mathrm{PGL}(2,\mathbb{Z}))$ such that for each $g \in \mathrm{PGL}(2,\mathbb{Z})$ we have*

$$U_{\overline{\Gamma}_0(n)}(g) = \pi_{G(m),\overline{X}(n)}(gM(m)). \tag{4.84}$$

REMARK 4.1. *Since $Q(n)$ acts transitively on $X(n)$ and for this action $K(n)$ is the stabilizer of the point $\Gamma_0(n) \in X(n)$, $Q(n)/K(n)$ and $X(n)$ are isomorphic $Q(n)$-sets (*[**11**]*, page 80, Lemma 3.1.6). Hence, $\pi_{Q(n)}$ is isomorphic to the permutation representation of $Q(n)$ associated to $Q(n)/K(n)$. But the latter can be evidently considered as the representation of $Q(n)$, induced from the one dimensional trivial representation of $K(n) < Q(n)$, that is,*

$$\pi_{Q(n)}(gH(n)) = ind_{K(n)}^{Q(n)}(1)(gH(n)), \quad gH(n) \in Q(n). \tag{4.85}$$

Similarly, $G(n)/\overline{K}(n)$ is isomorphic to $\overline{X}(n)$ as $G(n)$-sets and hence $\pi_{G(n)}$ can be considered as the representation of $G(n)$, induced from the one dimensional trivial representation of $\overline{K}(n) < G(n)$, that is,

$$\pi_{G(n)}(gM(n)) = ind_{\overline{K}(n)}^{G(n)}(1)(gM(n)), \quad gM(n) \in G(n). \tag{4.86}$$

4.3. Tensor products of induced representations

In this section we prove the following lemma:

LEMMA 4.7. *Let m and n be coprime. Then up to conjugation we have*

$$U_{\Gamma_0(nm)} = U_{\Gamma_0(n)} \otimes U_{\Gamma_0(m)}. \tag{4.87}$$

and

$$U_{\overline{\Gamma}_0(nm)} = U_{\overline{\Gamma}_0(n)} \otimes U_{\overline{\Gamma}_0(m)}. \tag{4.88}$$

PROOF. Since the proofs of both assertions are similar, we only give the proof of the first one. Consider $X(n) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(n)$ as a $Q(nm)$-set. Let $V_{\mu_{\Gamma_0(n)}}$ be a $\mu_{\Gamma_0(n)}$-dimensional Hermitian vector space with an orthonormal basis given by $\{e_{x\Gamma_0(n)} \mid x\Gamma_0(n) \in X(n)\}$. Then a permutation representation

$$\pi_{Q(nm),X(n)} : Q(nm) \to AutV_{\mu_{\Gamma_0(n)}} \tag{4.89}$$

is defined by

$$\pi_{Q(nm),X(n)}(gH(nm))e_{x\Gamma_0(n)} = e_{gx\Gamma_0(n)}. \tag{4.90}$$

According to Lemma 4.5, up to a conjugation, we have

$$U_{\Gamma_0(n)}(g) = \pi_{Q(nm),X(n)}(gH(nm)). \tag{4.91}$$

In the same way, let $X(m) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(m)$ be a $Q(nm)$-set and let $V_{\mu_{\Gamma_0(m)}}$ be a $\mu_{\Gamma_0(m)}$-dimensional Hermitian vector space with an orthonormal basis given by $\{e_{y\Gamma_0(m)} \mid y\Gamma_0(m) \in X(m)\}$. Then a permutation representation

$$\pi_{Q(nm),X(m)} : Q(nm) \to AutV_{\mu_{\Gamma_0(m)}} \tag{4.92}$$

is defined by

$$\pi_{Q(nm),X(m)}(gH(nm))e_{y\Gamma_0(m)} = e_{gy\Gamma_0(m)} \tag{4.93}$$

and up to a conjugation we have

$$U_{\Gamma_0(m)}(g) = \pi_{Q(nm),X(m)}(gH(nm)). \tag{4.94}$$

We consider $X(n) \times X(m)$, with $\times$ denoting the Cartesian product, as a $Q(nm)$-set via the action

$$Q(nm) \times (X(n) \times X(m)) \to X(n) \times X(m) \tag{4.95}$$

given by

$$gH(nm)(x\Gamma_0(n), y\Gamma_0(m)) \mapsto (gx\Gamma_0(n), gy\Gamma_0(m)). \tag{4.96}$$

Now we show that this action is transitive. To this end, recall that according to (4.47) each element $gH(nm) \in Q(nm)$ can be written in terms of unique elements $g_mH(nm) \in H(m)/H(nm)$ and $g_nH(nm) \in H(n)/H(nm)$ as

$$gH(nm) = g_mH(nm)g_nH(nm) = g_nH(nm)g_mH(nm). \tag{4.97}$$

Then (4.96) can be written as

$$gH(nm)(x\Gamma_0(n), y\Gamma_0(m)) \mapsto (g_mg_nx\Gamma_0(n), g_ng_my\Gamma_0(m)). \tag{4.98}$$

Since $H(n)$ and $H(m)$ are normal subgroups of $\mathrm{PSL}(2,\mathbb{Z})$, there are $g'_n \in H(n)$ and $g'_m \in H(m)$ such that $g_nx = xg'_n$ and $g_my = yg'_m$. Hence, observing that $H(n)$ and $H(m)$ are subgroups of $\Gamma_0(n)$ and $\Gamma_0(m)$, respectively, (4.98) reduced to

$$gH(nm)(x\Gamma_0(n), y\Gamma_0(m)) \mapsto (g_mx\Gamma_0(n), g_ny\Gamma_0(m)). \tag{4.99}$$

On the other hand from the "First Isomorphism Theorem" ([**1**], page 11) and by using the identities (4.46) it follows that

$$Q(n) \cong H(m)/H(nm), \quad Q(m) \cong H(n)/H(nm). \tag{4.100}$$

Hence, observing that $Q(n)$ and $Q(m)$ act transitively on $X(n)$ and $X(m)$, respectively, the action of $Q(nm)$ on $X(n) \times X(m)$ given in (4.99) and consequentely the original action given in (4.96) are transitive.

Now let $V_{\mu_{\Gamma_0(n)} \cdot \mu_{\Gamma_0(m)}}$ be a $\mu_{\Gamma_0(n)} \cdot \mu_{\Gamma_0(m)}$-dimensional Hermitian vector space with an orthonormal basis given by

$$\left\{ e_{x\Gamma_0(n), y\Gamma_0(m)} \mid (x\Gamma_0(n), y\Gamma_0(m)) \in X(n) \times X(m) \right\}.$$

Then a permutation representation

$$\pi_{Q(nm), X(n)\times X(m)} : Q(nm) \to AutV_{\mu_{\Gamma_0(n)} \cdot \mu_{\Gamma_0(m)}} \tag{4.101}$$

is defined by

$$\pi_{Q(nm), X(n)\times X(m)}(gH(nm)) e_{x\Gamma_0(n), y\Gamma_0(m)} = e_{gx\Gamma_0(n), gy\Gamma_0(m)}. \tag{4.102}$$

It is known that, up to a conjugation,([**4**], page 26, Proposition 2.22)

$$\pi_{Q(nm), X(n)\times X(m)} = \pi_{Q(nm), X(n)} \otimes \pi_{Q(nm), X(m)}. \tag{4.103}$$

This formula also holds for m and n which are not necessarily coprime.

On the other hand let $X(nm) := \mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(nm)$ be a $Q(nm)$-set and let $V_{\mu_{\Gamma_0(nm)}}$ be a Hermitian vector space with an orthonormal basis $\left\{ e_{x\Gamma_0(nm)} \mid x\Gamma_0(nm) \in X(nm) \right\}$. Then a permutation representation

$$\pi_{Q(nm), X(nm)} : Q(nm) \to AutV_{\mu_{\Gamma_0(nm)}} \tag{4.104}$$

is defined by

$$\pi_{Q(nm), X(nm)}(gH(nm)) e_{x\Gamma_0(nm)} = e_{gx\Gamma_0(nm)}. \tag{4.105}$$

Since the actions of $Q(nm)$ on $X(nm)$ and $X(n) \times X(m)$ are transitive and $K(nm)$ is the stabilizer in $Q(nm)$ of both $(\Gamma_0(n), \Gamma_0(m)) \in X(n) \times X(m)$ and $\Gamma_0(nm) \in X(nm)$, we have ([**11**], page 80, Lemma 3.1.6)

$$X(nm) \cong Q(nm)/K(nm) \cong X(n) \times X(m) \tag{4.106}$$

where $\cong$ denotes isomorphism as $Q(nm)$-set. Hence, $\pi_{Q(nm), X(nm)}$ is isomorphic to $\pi_{Q(nm), X(n)\times X(m)}$. This together with (4.103) leads to the following isomorphism of representations

$$\pi_{Q(nm), X(nm)} \cong \pi_{Q(nm), X(n)} \otimes \pi_{Q(nm), X(m)}. \tag{4.107}$$

But according to Lemma 4.5 $\pi_{Q(nm), X(nm)}(gH(nm)) = U_{\Gamma_0(nm)}(g)$ (up to a conjugation), which together with (4.91) and (4.94) yields the desired result. $\square$

4.4. Subrepresentations of $U_{\Gamma_0(n)}$ and $U_{\overline{\Gamma}_0(n)}$

Let $V_{\mu_{\Gamma_0(m)}}$ be a $\mu_{\Gamma_0(m)}$-dimensional Hermitian vector space with an orthonormal basis $\left\{ e_{x\Gamma_0(m)} \mid x\Gamma_0(m) \in X(m) = \mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(m) \right\}$. As before, a permutation representation

$$\pi_{Q(m)} : Q(m) \to AutV_{\mu_{\Gamma_0(m)}} \tag{4.108}$$

is given by

$$\pi_{Q(m)}(gH(m))e_{x\Gamma_0(m)} = e_{gx\Gamma_0(m)}. \tag{4.109}$$

For $n|m$ let α_1 be the surjective map

$$\alpha_1 : X(m) \to X(n), \tag{4.110}$$

defined by

$$\alpha_1(x\Gamma_0(m)) = x\Gamma_0(n). \tag{4.111}$$

For each $y\Gamma_0(n) \in X(n)$ define

$$\widehat{e}_{y\Gamma_0(n)} := \sum_{x\Gamma_0(m):\alpha_1(x\Gamma_0(m))=y\Gamma_0(n)} \frac{e_{x\Gamma_0(m)}}{\| \sum_{\alpha_1(x\Gamma_0(m))=y\Gamma_0(n)} e_{x\Gamma_0(m)}\|} \tag{4.112}$$

where for an element v in the Hermitian vector space $V_{\mu_{\Gamma_0(m)}}$ with Hermitian form $\langle , \rangle$, $\|v\|^2 := \langle v, v\rangle$ is the norm of v. We denote by $V_{\mu_{\Gamma_0(n)}}$ the Hermitian vector subspace of $V_{\mu_{\Gamma_0(m)}}$ spanned by the orthonormal basis $\left\{\widehat{e}_{y\Gamma_0(n)} \mid y\Gamma_0(n) \in X(n)\right\}$. Now consider $X(n)$ as a $Q(m)$-set with respect to the action

$$\begin{aligned} Q(m) \times X(n) &\to X(n) \\ (gH(m), y\Gamma_0(n)) &\mapsto gy\Gamma_0(n). \end{aligned} \tag{4.113}$$

Then obviously $V_{\mu_{\Gamma_0(n)}}$ is an invariant subspace of $V_{\mu_{\Gamma_0(m)}}$ under the action of $\pi_{Q(m)}$ on $V_{\mu_{\Gamma_0(m)}}$ induced from the action of $Q(m)$ on $X(m)$ since $\widehat{e}_{gy\Gamma_0(n)}$ is again of the form (4.112). This way we obtain a subrepresentation of $\pi_{Q(m)}$ given by

$$\begin{aligned} &\pi_{Q(m),X(n)} : Q(m) \to AutV_{\mu_{\Gamma_0(n)}} \\ &\pi_{Q(m),X(n)}(gH(m))\widehat{e}_{y\Gamma_0(n)} = \widehat{e}_{gy\Gamma_0(n)}. \end{aligned} \tag{4.114}$$

Hence there exists a representation θ of $Q(m)$ such that ([**52**], page 6, Theorem 1)

$$\pi_{Q(m)} \cong \pi_{Q(m),X(n)} \oplus \theta \tag{4.115}$$

where $\cong$ denotes an isomorphism of representations. According to Lemma 4.5, for $g \in \mathrm{PSL}(2,\mathbb{Z})$ we have

$$U_{\Gamma_0(m)}(g) = \pi_{Q(m)}(gH(m)) \tag{4.116}$$

and

$$U_{\Gamma_0(n)}(g) = \pi_{Q(m),X(n)}(gH(m)). \tag{4.117}$$

From these identities and (4.115) it follows that, up to isomorphism, the representation $U_{\Gamma_0(n)}$ occurs in the representation $U_{\Gamma_0(m)}$, namely for some representation θ' of $\mathrm{PSL}(2,\mathbb{Z})$ we have

$$U_{\Gamma_0(m)} \cong U_{\Gamma_0(n)} \oplus \theta'. \tag{4.118}$$

Similarly, one can show that the representation $\pi_{G(m),\overline{X}(n)}$ is a subrepresentation of $\pi_{G(m)}$ and hence $U_{\overline{\Gamma}_0(n)}$ occurs in the representation $U_{\overline{\Gamma}_0(m)}$. We summarize these results in the following lemma:

LEMMA 4.8. *If $n|m$ the representations $U_{\Gamma_0(n)}$ and $U_{\overline{\Gamma}_0(n)}$ are subrepresentations of $U_{\Gamma_0(m)}$ and $U_{\overline{\Gamma}_0(m)}$, respectively.*

4.5. Irreducible decomposition of $U_{\Gamma_0(p)}$ and $U_{\overline{\Gamma}_0(p)}$

In this section we discuss the decomposition of the representations $U_{\Gamma_0(p)}$ and $U_{\overline{\Gamma}_0(p)}$ for p a prime into their irreducible subrepresentations. We consider first $U_{\Gamma_0(p)}$. Let $V_{\mu_{\Gamma_0(p)}}$ be the Hermitian vector space with basis $\left\{e_{x\Gamma_0(p)} \mid x\Gamma_0(p) \in X(p)\right\}$ and let $\pi_{Q(p)}$ denote the permutation representation defined on the space $V_{\mu_{\Gamma_0(p)}}$. Then according to Lemma 4.5, the decomposition into irreducible subrepresentations of $\pi_{Q(p)}$ yields that of $U_{\Gamma_0(p)}$. Hence we study the representation $\pi_{Q(p)}$.

LEMMA 4.9. *The action of $Q(p)$ on $X(p)$ is doubly transitive.*

PROOF. It is enough to show that the stabilizer $Stab_{Q(p)}(x\Gamma_0(p))$ of any element $x\Gamma_0(p) \in X(p)$ acts transitively on $X(p) - \{x\Gamma_0(p)\}$ (see [**1**], page 35, Exercise 2). A set of left cosets of $\Gamma_0(p)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by (see [**27**], page 107)

$$\mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(p) = \left\{\Gamma_0(p), S\Gamma_0(p), T^{-1}S\Gamma_0(p), \dots, T^{-(p-1)}S\Gamma_0(p)\right\}. \tag{4.119}$$

Let $x = id_{2\times 2}$ and

$$H := \left\{\Gamma(p), T^{-1}\Gamma(p), \dots, T^{-(p-1)}\Gamma(p)\right\}. \tag{4.120}$$

Since $T^{-p} \in \Gamma(p)$, H is a subgroup of $Stab_{Q(p)}(\Gamma_0(p))$. Hence, the action of $Stab_{Q(p)}(\Gamma_0(p))$ on the element $S\Gamma_0(p)$ yields all elements of $X(p) - \{\Gamma_0(p)\}$. Recall that for a G-set X if G_x is the stabilizer of $x \in X$ then the stabilizer of $gx \in X$ is given by gG_xg^{-1} (see for example [**1**], page 29, Lemma 2). Hence, for $T^{-k}S\Gamma_0(p) \in X(p)$ with $0 \le k \le p-1$, $H_k := T^{-k}SH(T^{-k}S)^{-1}$ is a subgroup of $Stab_{Q(p)}(T^{-k}S\Gamma_0(p))$. Then evidently, for a fixed k, the action of H_k on $\Gamma_0(p) \in X(p)$ yields

$$H_k\Gamma_0(p) := \left\{T^{-k}ST^{-i}S\Gamma_0(p) \mid i = 0,1,\dots,p-1\right\} \tag{4.121}$$

On the other hand, obviously we have

$$(T^{-k}ST^{-i'}S)^{-1}T^{-k}ST^{-i}S = ST^{i'-i}S \in \Gamma_0(p) \tag{4.122}$$

iff $i = i'$ and hence elements of $H_k\Gamma_0(p)$ are $p-1$ distinct left cosets of $\Gamma_0(p)$ in $\mathrm{PSL}(2,\mathbb{Z})$. Moreover, for each $0 \le k \le p-1$, one can easily check that $T^{-k}S\Gamma_0(p) \neq T^{-k}ST^{-i}S\Gamma_0(p)$, that is the orbit of $\Gamma_0(p)$ under H_k in $X(p)$ dose not pass through $T^{-k}S\Gamma_0(p)$. Thus $H_k\Gamma_0(p)$ coincides with $X(p) - \left\{T^{-k}S\Gamma_0(p)\right\}$. Therefore, for each $0 \le k \le p-1$, $Stab_{Q(p)}(T^{-k}S\Gamma_0(p))$ acts transitively on $X(p) - \left\{T^{-k}S\Gamma_0(p)\right\}$. Thus we proved that the stabilizer $Stab_{Q(p)}(x\Gamma_0(p))$ of any element $x\Gamma_0(p) \in X(p)$ acts transitively on $X(p) - \{x\Gamma_0(p)\}$ and hence $Q(p)$ acts doubly transitively on $X(p)$ which is the desired result. $\square$

We recall now a result on decomposition of permutation representations into irreducible subrepresentations (see [**52**], page 17, Exercise 2.6)

LEMMA 4.10. *Let X be a G-set such that the action G on X is doubly transitive and let π be a permutation representation of G associated with X. Then π is decomposed into two irreducible subrepresentations, that is,*

$$\pi = \pi_t \oplus \theta \tag{4.123}$$

where π_t denotes the trivial representation of G and θ is an irreducible representation of G.

Thus as a consequence of Lemma 4.9 we have

LEMMA 4.11. *The representation $\pi_{Q(p)}$ is decomposed into two irreducible representations, that is, there is a $(p+1)\times(p+1)$ matrix $M_{\Gamma_0(p)}$ such that*

$$M_{\Gamma_0(p)}\pi_{Q(p)}M^{-1}_{\Gamma_0(p)} = \rho_t \oplus \rho_p \tag{4.124}$$

where ρ_t is the one dimensional trivial representation and ρ_p is a p-dimensional irreducible representation of $Q(p)$.

A bijective map of a set X is called a permutation of X. The set of all permutations of X with the composition of maps as the binary operation is a group. For $X = \{1, 2, 3, \ldots, n\}$ with $n \in \mathbb{N}$, this group is called the symmetric group of degree n, denoted by S_n ([**1**], page 7). Let V be a Hermitian vector space with an orthonormal basis $\langle e_i \mid i = 1, 2, \ldots, n\}$. A representation π of S_n on the representation space V is defined by

$$\pi(\sigma)e_i = e_{\sigma(i)}, \quad \sigma \in S_n. \tag{4.125}$$

The one dimensional subspace V_1 of V spanned by the vector $e_1 + \ldots + e_n$ is invariant under the action of S_n. This is the representation space of the trivial representation $\pi_{trivial}$ of S_n which is then obviously a subrepresentation of π. It is known that the representation π is decomposed into two irreducible representations (see for example [**53**], page 64), that is,

$$\pi = \pi_{standard} \oplus \pi_{trivial}. \tag{4.126}$$

This decomposition defines the "standard representation" $\pi_{standard}$ of S_n which is an irreducible representation of S_n of degree $n-1$ with $V_{n-1} := V \ominus V_1$ as its representation space. According to a result due to Millington [**34**] the group $\langle U_{\Gamma_0(p)}(S), U_{\Gamma_0(p)}(T)\rangle$, generated by the permutation matrices $U_{\Gamma_0(p)}(S)$ and $U_{\Gamma_0(p)}(T)$, is isomorphic to the factor group of $\mathrm{PSL}(2,\mathbb{Z})$ modulo the maximal subgroup of $\Gamma_0(p)$ normal in $\mathrm{PSL}(2,\mathbb{Z})$. For p prime the principal congruence subgroup $\Gamma(p)$ is the maximal subgroup of $\Gamma_0(p)$ which is normal in $\mathrm{PSL}(2,\mathbb{Z})$. Hence, it follows that $\langle U_{\Gamma_0(p)}(S), U_{\Gamma_0(p)}(T)\rangle \cong \mathrm{PSL}(2,\mathbb{Z})/\Gamma(p) = Q(p)$. Thus the group $Q(p)$ is isomorphic to the subgroup $\langle U_{\Gamma_0(p)}(S), U_{\Gamma_0(p)}(T)\rangle$ of the group of all $(p+1)$ dimensional permutation matrices which is isomorphic to S_{p+1}. Hence, regarding the previous lemma, we can consider the representation ρ_p as the standard representation of S_{p+1} restricted to $\langle U_{\Gamma_0(p)}(S), U_{\Gamma_0(p)}(T)\rangle$. We also note that this subgroup obviously coincides with $\langle \pi_{Q(p)}(S\Gamma(p)), \pi_{Q(p)}(T\Gamma(p))\rangle$, the group generated by the elements $\pi_{Q(p)}(S\Gamma(p))$ and $\pi_{Q(p)}(T\Gamma(p))$. From Lemmas 4.5 and 4.11 follows

LEMMA 4.12. *The $p+1$ dimensional induced representation $U_{\Gamma_0(p)}$ of $\mathrm{PSL}(2,\mathbb{Z})$ can be decomposed into a trivial one dimensional representation U_t and a p-dimensional representation U_p, that is,*

$$M_{\Gamma_0(p)} U_{\Gamma_0(p)} M_{\Gamma_0(p)}^{-1} = U_t \oplus U_p \tag{4.127}$$

where for $g \in \mathrm{PSL}(2,\mathbb{Z})$, up to an isomorphism, $U_p(g) = \rho_p(gH(p))$.

By the same arguments we can obtain similar results for $\pi_{G(p)}$ and $U_{\overline{\Gamma}_0(p)}$.

LEMMA 4.13. *The representation $\pi_{G(p)}$ is decomposed into two irreducible representations, that is, there is a $(p+1)\times(p+1)$ matrix $M_{\overline{\Gamma}_0(p)}$ such that*

$$M_{\overline{\Gamma}_0(p)} \pi_{G(p)} M_{\overline{\Gamma}_0(p)}^{-1} = \overline{\rho}_t \oplus \overline{\rho}_p \tag{4.128}$$

where $\overline{\rho_t}$ is the one dimensional trivial representation and $\overline{\rho}_p$ is a p-dimensional irreducible representation of $G(p)$. Moreover, the $p+1$ dimensional induced representation $U_{\overline{\Gamma}_0(p)}$ of $\mathrm{PGL}(2,\mathbb{Z})$ is decomposed into a trivial one dimensional representation $\overline{U}_t$ and a p-dimensional representation $\overline{U}_p$, that is, there is a $(p+1)\times(p+1)$ matrix $M_{\overline{\Gamma}_0(p)}$ such that

$$M_{\overline{\Gamma}_0(p)} U_{\overline{\Gamma}_0(p)} M_{\overline{\Gamma}_0(p)} = \overline{U}_t \oplus \overline{U}_p. \tag{4.129}$$

Furtheremore, up to an isomorphism we have

$$\overline{U}_p(g) = \overline{\rho}_p(gM(p)), \quad \forall g \in \mathrm{PGL}(2,\mathbb{Z}) \tag{4.130}$$

*where $\overline{\rho}_p$ is the restriction of the standard representation of S_{p+1} to its subgroup $\langle \overline{U}_{\Gamma_0(p)}(S), \overline{U}_{\Gamma_0(p)}(T), \overline{U}_{\Gamma_0(p)}(M)\rangle$ which is isomorphic to $G(p)$ [**34**].*

Next we give some examples where all elements introduced above are given explicitly.

EXAMPLE 4.1. *Let $R(\overline{\Gamma}_0(2)\backslash\mathrm{PGL}(2,\mathbb{Z})) = R(\Gamma_0(2)\backslash\mathrm{PSL}(2,\mathbb{Z})) = \{id, S, ST\}$. Then the induced representation $U_{\Gamma_0(2)}$ of the generators*

S and T for $\mathrm{PSL}(2,\mathbb{Z})$ *is given by*

$$U_{\Gamma_0(2)}(T)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad U_{\Gamma_0(2)}(S)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}. \tag{4.131}$$

The matrix $M_{\Gamma_0(2)}$ can be chosen as

$$M_{\Gamma_0(2)}=\begin{pmatrix}1&1&1\\\omega^2&1&\omega\\\omega^2&\omega&1\end{pmatrix},\quad \omega=e^{2\pi i/3}. \tag{4.132}$$

We have then

$$M_{\Gamma_0(2)}U_{\Gamma_0(2)}M^{-1}_{\Gamma_0(2)}=U_t\oplus U_2 \tag{4.133}$$

where U_t is the trivial representation of $\mathrm{PSL}(2,\mathbb{Z})$ *and*

$$U_2(T)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad U_2(S)=\begin{pmatrix}0&\omega\\\omega^2&0\end{pmatrix} \tag{4.134}$$

For $U_{\overline{\Gamma}_0(2)}$ we have

$$U_{\overline{\Gamma}_0(2)}(T)=U_{\Gamma_0(2)}(T),\quad U_{\overline{\Gamma}_0(2)}=U_{\Gamma_0(2)}(S),\quad U_{\overline{\Gamma}_0(2)}(M)=Id_{3\times 3}. \tag{4.135}$$

Moreover, $M_{\overline{\Gamma}_0(2)}=M_{\Gamma_0(2)}$ and we have

$$M_{\overline{\Gamma}_0(2)}U_{\overline{\Gamma}_0(2)}M^{-1}_{\overline{\Gamma}_0(2)}=\overline{U}_t\oplus\overline{U}_2 \tag{4.136}$$

where $\overline{U}_t$ is the trivial representation of $\mathrm{PGL}(2,\mathbb{Z})$ *and*

$$\overline{U}_2(T)=U_2(T),\quad \overline{U}_2(S)=U_2(S),\quad \overline{U}_2(M)=Id_{2\times 2}. \tag{4.137}$$

EXAMPLE 4.2. *Let $R(\overline{\Gamma}_0(3)\backslash\mathrm{PGL}(2,\mathbb{Z}))=R(\Gamma_0(3)\backslash\mathrm{PSL}(2,\mathbb{Z}))=\{id,S,ST,ST^2\}$. Then the induced representation $U_{\Gamma_0(3)}$ for the generators S and T of* $\mathrm{PSL}(2,\mathbb{Z})$ *is given by*

$$U_{\Gamma_0(3)}(T)=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\quad U_{\Gamma_0(3)}(S)=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}. \tag{4.138}$$

The matrix $M_{\Gamma_0(3)}$ can be chosen as

$$M_{\Gamma_0(3)}=\begin{pmatrix}1&1&1&1\\1&1&1&-3\\1&1&-3&1\\1&-3&1&1\end{pmatrix}. \tag{4.139}$$

Then we have

$$M_{\Gamma_0(3)}U_{\Gamma_0(3)}M^{-1}_{\Gamma_0(3)}=U_t\oplus U_3 \tag{4.140}$$

where U_t is the trivial representation of $\mathrm{PSL}(2,\mathbb{Z})$ *and*

$$U_3(T) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_3(S) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}. \tag{4.141}$$

For $U_{\overline{\Gamma}_0(3)}$ we have

$$U_{\overline{\Gamma}_0(3)}(T) = U_{\Gamma_0(3)}(T), \quad U_{\overline{\Gamma}_0(3)}(S) = U_{\Gamma_0(3)}(S) \tag{4.142}$$

and

$$U_{\overline{\Gamma}_0(3)}(M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.143}$$

Then $M_{\overline{\Gamma}_0(3)} = M_{\Gamma_0(3)}$ and we have

$$M_{\Gamma_0(3)} U_{\Gamma_0(3)} M^{-1}_{\Gamma_0(3)} = \overline{U}_t \oplus \overline{U}_3 \tag{4.144}$$

where $\overline{U}_3(T) = U_3(T)$, $\overline{U}_3(S) = U_3(S)$, and

$$\overline{U}_3(M) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.145}$$

4.6. Number of irreducibles in the induced representation

In this section, for a prime power q, we determine the number of non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(q)}$ and $U_{\overline{\Gamma}_0(q)}$. According to Lemmas 4.5 and 4.6 the number of non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(q)}$ and $U_{\overline{\Gamma}_0(q)}$ is equal to that of $\pi_{Q(q)}$ and $\pi_{G(q)}$, respectively. For technical reasons, we consider the representations $\pi_{Q(q)}$ and $\pi_{G(q)}$ instead of $U_{\Gamma_0(q)}$ and $U_{\overline{\Gamma}_0(q)}$ themselves. To count the number of non-isomorphic irreducible subrepresentations of $\pi_{Q(q)}$ and $\pi_{G(q)}$ one needs a result due to Wielandt. We recall it from [**10**]:

LEMMA 4.14. *Let G be a finite group, $K \leq G$ a subgroup, $X = G/K$ a G-set, and $V = \oplus_{i=1}^{N} m_i V_i$ be the decomposition into irreducibles of the permutation representation of G on X, where m_i denotes the multiplicity of the irreducible representation V_i. Then*

$$\sum_{i=0}^{N} m_i^2 = |X/K| \tag{4.146}$$

where $|X/K|$ denotes the number of orbits of K in X.

In the next section, we show that the representations $\pi_{Q(q)}$ and $\pi_{G(q)}$ are multiplicity-free. Hence, the left hand side of (4.146) for $\pi_{Q(q)}$ and $\pi_{G(q)}$ gives the number of their non-isomorphic irreducible subrepresentations. Thus according to Wielandt's result, $|X(q)/K(q)|$ and

$|\overline{X}(q)/\overline{K}(q)|$ give the number of non-isomorphic irreducible subrepresentations of $\pi_{Q(q)}$ and $\pi_{G(q)}$, respectively. In the rest of this section we count the number of orbits of $K(q)$ and $\overline{K}(q)$ in $X(q)$ and $\overline{X}(q)$, respectively.

To begin with, for a prime power $q = p^e$ we fix the representatives of $X(q)$ as the following ([**27**], page 107)

$$id; \quad T^{-j}S, \quad j = 0, \ldots, p^e - 1; \quad ST^{-jp}S, \quad j = 1, 2, \ldots, p^{e-1} - 1. \tag{4.147}$$

We denote by $[x\Gamma_0(q)]$ the orbit of $K(q)$ in $X(q)$ passing through the point $x\Gamma_0(q) \in X(q)$. Evidently, $\Gamma_0(q) \in X(q)$ is fixed under the action of the whole group $K(q) = \Gamma_0(q)/H(q)$. Hence, $[\Gamma_0(q)]$ is an orbit consisting of only one element $\Gamma_0(q)$. Next we determine the orbit $[S\Gamma_0(q)]$.

Lemma 4.15. *The stabilizer group $Stab_{K(q)}(S\Gamma_0(q))$ in $K(q)$ of the point $S\Gamma_0(q) \in X(q)$ is given by*

$$Stab_{K(q)}(S\Gamma_0(q)) = K_0^0(q) := \Gamma_0^0(q)/H(q). \tag{4.148}$$

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} H(q) \in Stab_{K(q)}(S\Gamma_0(q))$, that is,

$$gS\Gamma_0(q) = S\Gamma_0(q). \tag{4.149}$$

Then a simple calculation shows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^0(q)$ and hence $g \in K_0^0(q)$. Conversely, one can easily check that for any $g \in K_0^0(q)$, $gS\Gamma_0(q) = S\Gamma_0(q)$. This completes the proof. □

According to the so called “orbit stabilizer theorem” ([**1**], page 30, Corollary 5), the length of the orbit $[S\Gamma_0(q)]$ is equal to the index of $Stab_{K(q)}(S\Gamma_0(q))$ in $K(q)$, that is,

$$|\,[S\Gamma_0(q)]\,| = [K(q) : Stab_{K(q)}(S\Gamma_0(q))]. \tag{4.150}$$

Thus, according to Lemma 4.15 we get

$$|\,[S\Gamma_0(q)]\,| = [K(q) : K_0^0(q)] = \frac{|K(q)|}{|K_0^0(q)|}. \tag{4.151}$$

Then by a simple index calculation we get

$$\begin{aligned} |\,[S\Gamma_0(q)]\,| &= \frac{[\Gamma_0(q) : H(q)]}{[\Gamma_0^0(q) : H(q)]} = \\ &= \frac{[\Gamma_0(q) : H(q)] \times [H(q) : \Gamma(q)]}{[\Gamma_0^0(q) : H(q)] \times [H(q) : \Gamma(q)]} = \frac{[\Gamma_0(q) : \Gamma(q)]}{[\Gamma_0^0(q) : \Gamma(q)]}. \end{aligned} \tag{4.152}$$

According to Rankin ([**43**], page 27) and taking into account that we consider the subgroups of the projective modular group, we have for

$q = p^e$

$$[\Gamma_0^0(q) : \Gamma(q)] = \begin{cases} \frac{1}{2}p^{e-1}(p-1), & q \neq 2, \\ 1, & q = 2. \end{cases} \tag{4.153}$$

Moreover, for $q = p^e$ we have

$$[\Gamma_0(q) : \Gamma(q)] = \begin{cases} \frac{1}{2}p^{2e-1}(p-1), & q \neq 2, \\ 2, & q = 2. \end{cases} \tag{4.154}$$

From this, (4.153), and (4.152) we get

$$|\,[S\Gamma_0(q)]\,| = q = p^e \tag{4.155}$$

Obviously, the action of the q elements $T^{-j}H(q) \in K(q)$, $0 \leq j < q-1$ on $S\Gamma_0(q)$ gives elements of the orbit $[S\Gamma_0(q)]$. Since $T^{-j}H(q) \neq T^{-j'}H(q)$ for $j \neq j'$ the number of these elements is equal to the length of the orbit $[S\Gamma_0(q)]$ and hence,

$$[S\Gamma_0(q)] = \left\{T^{-j}S\Gamma_0(q) \mid j = 0, \ldots, q-1\right\}. \tag{4.156}$$

Thus, for $q = p$, that is,

$$X(p) = \left\{\Gamma_0(p), S\Gamma_0(p), TS\Gamma_0(p), \ldots, T^{p-1}S\Gamma_0(p)\right\} \tag{4.157}$$

obviously we have

$$X(p) = [\Gamma_0(p)] \sqcup [S\Gamma_0(p)]. \tag{4.158}$$

That is, $|X(p)/K(p)| = 2$ as we expected from Lemma 4.12.

From now on we assume that $q = p^e$ with $e \geq 2$. We determine the length of the orbits of $K(q)$ when acting on the other elements of $X(q)$, namely on

$$Y(q) := \left\{y_k := ST^{-kp}S\Gamma_0(q), \quad 1 \leq k \leq p^{e-1} - 1\right\}. \tag{4.159}$$

For this we consider the following partition of $Y(q)$ into disjoint sets:

$$Y(q) := Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_{e-2} \tag{4.160}$$

where

$$Y_0 := \{y_k \in Y(q) \mid p \nmid k\}, \quad Y_i := \left\{y_k \in Y(q) \mid p^i \| k\right\}, \quad 1 \leq i \leq e-2. \tag{4.161}$$

Here, $p^i \| k$ means that $p^i | k$ but $p^{i+1} \nmid k$. In the following lemma we summarize properties of these subsets:

Lemma 4.16. *Each orbit of $K(q)$ in $Y(q)$ belongs to one of the subsets Y_i, $0 \leq i \leq e-2$. Their cardinality is given by*

$$|Y_i| = p^{e-i-2}(p-1), \quad 0 \leq i \leq e-2. \tag{4.162}$$

PROOF. Two elements $y_k, y_{k'} \in Y(q)$ belong to the same orbit iff there is an element $g = \gamma H(q) \in K(q)$ such that $gy_k = y_{k'}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ a simple calculation shows that y_k and $y_{k'}$ are in the same orbit iff

$$-ak' + dk - kk'bp \equiv 0 \mod p^{e-1}. \tag{4.163}$$

Assume that $y_k \in Y_i$ and $y_{k'} \in Y_{i'}$ with $i \neq i'$ belong to the same orbit, then it follows from (4.163) that $p \mid d$ or $p \mid a$ which is obviously a contradiction. Hence, each orbit must belong to one of the sets Y_i.

To prove the second part, we note that the set Y_0 contains the elements y_k with $1 \leq k \leq p^{e-1} - 1$ such that $(p, k) = 1$. The number of such elements is given in terms of Euler's function, that is, $|Y_0| = \phi(p^{e-1}) = p^{e-2}(p-1)$ (see for example [**44**], page 38, Definition 2). Hence, there are $|Y(q)| - |Y_0| = p^{e-2} - 1$ elements y_k in $Y(q)$ such that $p \mid k$. Let $Y_0' := Y(q) - Y_0$ and $\mathcal{Y}_1 = \left\{k'' = \frac{k'}{p} \mid y_{k'} \in Y_0'\right\} = \{1, 2, 3, \dots, p^{e-2} - 1\}$. Then $|Y_1|$ is equal to the number of elements $k'' \in \mathcal{Y}_1$ such that $(p, k'') = 1$, which by definition is again given in terms of Euler's function as $|Y_1| = \phi(p^{e-2}) = p^{e-3}(p-1)$. Repeating the same argument one obtains the cardinalities of the other Y_i's, namely $|Y_i| = \phi(p^{e-1-i}) = p^{e-i-2}(p-1)$. □

Next we show, that the stabilizer of any point in $Y(q)$ is a p-group, that is, its order is a power of the prime p.

LEMMA 4.17. *The stabilizer $Stab_{K(q)}(y_k)$ of each element $y_k \in Y(q)$, $q = p^e$, is a p-group.*

PROOF. Let $Stab_{K(q)}(y_k)$ be the stabilizer of an element $y_k \in Y_i$ where $0 \leq i \leq e-2$. Then for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ one has $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma(p^e) \in Stab_{K(q)}(y_k)$ iff $gy_k = y_k$ iff

$$-a + d - bkp \equiv 0 \mod p^{e-1-i}. \tag{4.164}$$

From this together with $ad \equiv 1 \mod p^e$ it follows that $a \equiv d \equiv \pm 1 \mod p$ and $ad \equiv 1 \mod p$. On the other hand $c \equiv 0 \mod p^e$ and hence $c \equiv 0 \mod p$. Therefore, observing that $\Gamma_0(p)$ is a subgroup of the projective modular group, we get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$ where

$$\Gamma_1(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n) \mid a \equiv d \equiv \pm 1 \mod n \right\}. \tag{4.165}$$

Consequently, it follows that $g \in \Gamma_1(p)/\Gamma(p^e)$. Indeed, we showed that if $g \in Stab_{K(q)}(y_k)$ then $g \in \Gamma_1(p)/\Gamma(p^e)$, that is, $Stab_{K(q)}(y_k) \leq$

$\Gamma_1(p)/\Gamma(p^e)$. But for the order of $\Gamma_1(p)/\Gamma(p^e)$ as a simple calculation shows we have

$$|\Gamma_1(p)/\Gamma(p^e)| = \frac{[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(p^e)]}{[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_1(p)]}. \tag{4.166}$$

On the other hand, Theorem 4.2.5. in Miyake ([**35**], page 106) yields

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_1(p)] = \begin{cases} \frac{1}{2}(p^2-1), & p > 2, \\ 3, & p = 2, \end{cases} \tag{4.167}$$

and

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(p^e)] = \begin{cases} \frac{1}{2}p^{3e-2}(p^2-1), & p^e > 2, \\ 6, & p = 2, e = 1. \end{cases} \tag{4.168}$$

Thus obviously we get

$$|\Gamma_1(p)/\Gamma(p^e)| = \begin{cases} p^{3e-2} & p > 2, e \geq 1 \\ 2 & p = 2, e = 1 \\ 2^{3e-3} & p = 2, e \geq 2 \end{cases} \tag{4.169}$$

and hence $\Gamma_1(p)/\Gamma(p^e)$ is a p-group. Therefore, $Stab_{K(q)}(y_k)$ also must be a p-group, since the order of a subgroup of a finite group divides the order of the group (see for example [**1**], page 3). □

For a prime p let p^n divide the order of the finite group G whereas p^{n+1} does not, that is $p^n \| |G|$. A subgroup $H \leq G$ is called a Sylow p-subgroup of G if it is a maximal p-subgroup of G, that is, if its order is $|G|_p := p^n$. We denote this group by $P_{Sylow}(G)$.

Lemma 4.18. *The Sylow p-subgroup $P_{Sylow}(K(p^e))$ of the group $K(p^e)$ is unique and hence normal in $K(p^e)$.*

Proof. Recall that according to Lemma 4.1 the order of $K(p^e)$ is given by

$$|K(q)| = \begin{cases} \frac{1}{2}p^{2e-1}(p-1), & q = p^e, p \neq 2, e \geq 1 \\ 2, & q = 2, \\ 4, & q = 4, \\ 2^{2e-3}, & q = 2^e, e \geq 3. \end{cases} \tag{4.170}$$

Thus for $q = 2^e$ and $e \geq 1$ the group $K(q)$ is itself a 2-group and hence $P_{Sylow}(K(2^e)) = K(2^e)$. Evidently, in this case the assertion is trivial. Next we assume $p > 2$. Let G be a finite group, P_{Sylow} a Sylow p-subgroup of G of order $|G|_p$, and n_P be the number of Sylow p-subgroups of G. Then $n_P \equiv 1 \mod p$ and n_P divides $(|G|/|G|_p)$ (see [**1**], page 64, Sylow's theorem and page 66, corollary 1). In the case of a Sylow p-subgroup P_{Sylow} of $K(p^e)$, we have $n_P \mid \frac{1}{2}(p-1)$ (see (4.170)) and $n_P \equiv 1 \mod p$. Hence, $n_P = 1$ that means the Sylow p-subgroup

$P_{Sylow}(K(p^e))$ is unique. On the other hand, all Sylow p-subgroups of a group are conjugate by elements of the group (see [**1**], page 64, Sylow's theorem). This together with the uniqueness of $P_{Sylow}(K(p^e))$ and the fact that any conjugate of a p-subgroup by elements of the group is again a p-subgroup of the group leads to normality of $P_{Sylow}(K(p^e))$ in $K(p^e)$. $\square$

LEMMA 4.19. *Let $q = p^e$ be a prime power with $p > 2$, $e \geq 2$, and $Stab_{K(q)}(y_{k_i})$ be the stabilizer of a point $y_{k_i} \in Y_i$. Then the order of this stabilizer is given by*

$$|Stab_{K(q)}(y_{k_i})| = p^{e+i+1}. \tag{4.171}$$

PROOF. First, we determine a lower bound for the order of the stabilizer groups. According to the "orbit stabilizer theorem", Lemma 4.16, and Lemma 4.1 the following holds

$$\left[K(p^e) : Stab_{K(p^e)}(y_{k_i})\right] = \frac{\frac{1}{2}(p-1)p^{2e-1}}{|Stab_{K(p^e)}(y_{k_i})|} \leq |Y_i| = (p-1)p^{e-i-2}. \tag{4.172}$$

By a simple calculation, this leads to

$$|Stab_{K(p^e)}(y_{k_i})| \geq \frac{1}{2}p^{e+i+1}. \tag{4.173}$$

Since according to Lemma 4.17 $|Stab_{K(p^e)}(y_{k_i})|$ must be a power of p, the latter yields

$$|Stab_{K(p^e)}(y_{k_i})| \geq p^{e+i+1}. \tag{4.174}$$

Consider then the following range of i values

$$\begin{cases} \left[\frac{e-2}{2}\right] < i \leq e-2, & \text{for odd } e, \\ \frac{e-2}{2} \leq i \leq e-2, & \text{for even } e, \end{cases} \tag{4.175}$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$. In these intervals, for any $y_{k_i} \in Y_i$ one can easily check that $pk_i \equiv 0 \mod p^{e-i-1}$ and hence according to (4.164) we get

$$\begin{aligned} &Stab_{K(p^e)}(y_{k_i}) = \\ &\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma(p^e) \in K(p^e) \mid -a + d \equiv 0 \mod p^{e-i-1} \right\}. \end{aligned} \tag{4.176}$$

Since $\Gamma(p^e)$ is a normal subgroup of $\Gamma_1(p^e)$ and $\Gamma_1(p^e) < \Gamma_0(p^e)$,

$$K_1(p^e) := \Gamma_1(p^e)/\Gamma(p^e) \tag{4.177}$$

is obviously a subgroup of $K(p^e)$. The index of $\Gamma(p^e)$ in $\Gamma_1(p^e)$ is given by (see for example [**27**], page 107, Problem 7 and page 231)

$$[\Gamma_1(p^e) : \Gamma(p^e)] = p^e. \tag{4.178}$$

From this and observing that for $1 \leq j, j' \leq p^e$, $T^j\Gamma(p^e) = T^{j'}\Gamma(p^e)$ if and only if $j = j'$ we get (see also [**35**], page 107)

$$K_1(p^e) = \left\{T^k\Gamma(p^e)|1 \leq k \leq p^e\right\}. \tag{4.179}$$

Evidently, $K_1(p^e)$ is a subgroup of $Stab_{K(p^e)}(y_{k_i})$ and hence we have

$$|Stab_{K(p^e)}(y_{k_i})| = |K_1(p^e)||Stab_{K(p^e)}(y_{k_i})/K_1(p^e)| \tag{4.180}$$

or,

$$|Stab_{K(p^e)}(y_{k_i})| = p^e|Stab_{K(p^e)}(y_{k_i})/K_1(p^e)|. \tag{4.181}$$

Furtheremore, we have

$$\begin{gathered} Stab_{K(p^e)}(y_{k_i})/K_1(p^e) = \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} K_1(p^e) \in K(p^e)/K_1(p^e) \mid -a + d \equiv 0 \mod p^{e-i-1}\right\}. \end{gathered} \tag{4.182}$$

On the other hand, $\Gamma_1(p^e)$ is a normal subgroup of $\Gamma_0(p^e)$ ([**27**], page 107, Problem 1) and hence $K_1(p^e)$ is a normal subgroup of $K(p^e)$. Indeed, according to the “Second Isomorphism Theorem” ([**1**], page 12) we have

$$K(p^e)/K_1(p^e) \cong \Gamma_0(p^e)/\Gamma_1(p^e). \tag{4.183}$$

Let $(\mathbb{Z}/p^e\mathbb{Z})^\times$ be the multiplicative group of the invertible elements of $\mathbb{Z}/p^e\mathbb{Z}$ with the identity element $1+p^e\mathbb{Z}$ denoted by 1_{p^e} respectively the element $-1+p^e\mathbb{Z}$ denoted by -1_{p^e}. Then according to Miyake ([**35**], page 105) and the fact that $\Gamma_0(p^e)$ is a subgroup of the projective modular group we get

$$K(p^e)/K_1(p^e) \cong \Gamma_0(p^e)/\Gamma_1(p^e) \cong (\mathbb{Z}/p^e\mathbb{Z})^\times / \left\{\pm 1_{p^e}\right\}. \tag{4.184}$$

From this and $ad \equiv 1 \mod p^e$ it follows from (4.182) that

$$\begin{gathered} Stab_{K(p^e)}(y_{k_i})/K_1(p^e) \cong \\ \left\{d \in (\mathbb{Z}/p^e\mathbb{Z})^\times \mid -d + d^{-1} \equiv 0 \mod p^{e-i-1}\right\} / \left\{\pm 1_{p^e}\right\}. \end{gathered} \tag{4.185}$$

or equivalently,

$$\begin{gathered} Stab_{K(p^e)}(y_{k_i})/K_1(p^e) \cong \\ \left\{d \in (\mathbb{Z}/p^e\mathbb{Z})^\times \mid d \equiv 1 \mod p^{e-i-1}\right\}. \end{gathered} \tag{4.186}$$

Let $h : (\mathbb{Z}/p^e\mathbb{Z})^\times \to (\mathbb{Z}/p^{e-i-1}\mathbb{Z})^\times$ be the homomorphism given by $h(a) = a \mod p^{e-i-1}$ which is evidently onto. Then according to (4.186) we have

$$|Stab_{K(p^e)}(y_{k_i})/K_1(p^e)| = |\ker h| = \frac{\phi(p^e)}{\phi(p^{e-i-1})} = p^{i+1}. \tag{4.187}$$

This and (4.181) yields

$$|Stab_{K(p^e)}(y_{k_i})| = p^{e+i+1}. \tag{4.188}$$

Finally, for $e \geq 3$ consider $i' = e - 2 - i$ with i in the range given in (4.175) and hence

$$\begin{cases} 0 \leq i' \leq \left[\frac{e-2}{2}\right], & \text{for odd } e, \\ 0 \leq i' < \frac{e-2}{2}, & \text{for even } e. \end{cases} \tag{4.189}$$

According to Lemma 4.17 the stabilizer group $Stab_{K(p^e)}(y_{k_{i'}})$ is a p-group and hence it must be a subgroup of the unique Sylow p-subgroup of $K(p^e)$ ([**1**], page 64). Then with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^e)$ the stabilizer group is given by

(4.190)

$$Stab_{K(p^e)}(y_{k_{i'}}) = \\ \left\{ g\Gamma(p^e) \in P_{Sylow}(K(p^e)) \mid -a + d - bpk_{i'} \equiv 0 \mod p^{e-i'-1} \right\}.$$

One can easily check that in the given range (4.175) of i for $i' = e-2-i$ one has

$$Stab_{K(p^e)}(y_{k_{i'}}) \leq Stab_{K(p^e)}(y_{k_i}). \tag{4.191}$$

Moreover, for the elements $T^j\Gamma(p^e) \in Stab_{K(p^e)}(y_{k_i})$ we have $T^j\Gamma(p^e) \notin Stab_{K(p^e)}(y_{k_{i'}})$ for $1 \leq j < p^{e-2(i'+1)} - 1$ and $T^j\Gamma(p^e) \in Stab_{K(p^e)}(y_{k_{i'}})$ for $j = p^{e-2(i'+1)}$ and hence

$$\left[Stab_{K(p^e)}(y_{k_i}) : Stab_{K(p^e)}(y_{k_{i'}})\right] \geq p^{e-2(i'+1)}. \tag{4.192}$$

On the other hand, evidently we have

(4.193)

$$\left[P_{Sylow}(K(p^e)) : Stab_{K(p^e)}(y_{k_{i'}})\right] = \\ \left[P_{Sylow}(K(p^e)) : Stab_{K(p^e)}(y_{k_i})\right] \left[Stab_{K(p^e)}(y_{k_i}) : Stab_{K(p^e)}(y_{k_{i'}})\right].$$

But according to (4.188) and (4.154) we have

$$\left[P_{Sylow}(K(p^e)) : Stab_{K(p^e)}(y_{k_i})\right] = p^{e-i-2} \tag{4.194}$$

and therefore we get

$$\left[P_{Sylow}(K(p^e)) : Stab_{K(p^e)}(y_{k_{i'}})\right] \geq p^{e-i-2}p^{e-2(i'+1)} = p^{e-i'-2} \tag{4.195}$$

where we used $i = e - 2 - i'$. Consequently, by a simple calculation we get

$$|Stab_{K(p^e)}(y_{k_{i'}})| \leq p^{e+i'+1} \tag{4.196}$$

which together with (4.174) yields

$$|Stab_{K(p^e)}(y_{k_{i'}})| = p^{e+i'+1}. \tag{4.197}$$

Thus the proof is complete. □

To treat the case $q = 2^e$ we need the following auxiliary result:

LEMMA 4.20. *For $e \geq 2$ let*

$$\begin{cases} \left[\frac{e-2}{2}\right] + 1 \leq i \leq e-2, & \textit{for odd } e, \\ \frac{e-2}{2} \leq i \leq e-2, & \textit{for even } e, \end{cases} \tag{4.198}$$

and

$$\begin{cases} 0 \leq i' \leq \left[\frac{e-2}{2}\right], & \textit{for odd } e, \\ 0 \leq i' \leq \frac{e-2}{2} - 1, & \textit{for even } e > 3, \\ 0 = i', & \textit{for } e = 2. \end{cases} \tag{4.199}$$

If $i + i' = e - 2$, then $Stab_{K(2^e)}(y_{k_{i'}})$ is a subgroup of $Stab_{K(2^e)}(y_{k_i})$ of index given by

$$\left[Stab_{K(2^e)}(y_{k_i}) : Stab_{K(2^e)}(y_{k_{i'}})\right] = 2^{e-2(i'+1)}. \tag{4.200}$$

PROOF. Similar to the case $p > 2$ where the stabilizers are given in (4.176) and (4.190) we have

$$\begin{gathered} Stab_{K(2^e)}(y_{k_i}) = \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) \in K(2^e) \mid -a + d \equiv 0 \mod 2^{e-i-1} \right\} \end{gathered} \tag{4.201}$$

and

$$\begin{gathered} Stab_{K(2^e)}(y_{k_{i'}}) = \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) \in K(2^e) \mid -a + d - 2bk_{i'} \equiv 0 \mod 2^{e-i'-1} \right\}. \end{gathered} \tag{4.202}$$

In the range of i and i' given in (4.198) respectively (4.199), obviously, we have

$$2^{e-i'-1} \geq 2^{e-i-1} \tag{4.203}$$

and

$$2k_{i'} \equiv 0 \mod 2^{e-i-1} = 0 \mod 2^{i'+1}. \tag{4.204}$$

Thus, if

$$-a + d - 2bk_{i'} \equiv 0 \mod 2^{e-i'-1} \tag{4.205}$$

then

$$-a + d \equiv 0 \mod 2^{e-i-1}. \tag{4.206}$$

Hence, according to (4.201) and (4.202) we get

$$Stab_{K(2^e)}(y_{k_{i'}}) \leq Stab_{K(2^e)}(y_{k_i}). \tag{4.207}$$

Next, we determine the index of $Stab_{K(2^e)}(y_{k_{i'}})$ in $Stab_{K(2^e)}(y_{k_i})$. Since $T^j H(2^e) \in Stab_{K(2^e)}(y_{k_i})$ whereas $T^j H(2^e) \notin Stab_{K(2^e)}(y_{k_{i'}})$ for $j = 1, 2, \ldots, 2^{e-2(i'+1)} - 1$ and $T^j H(2^e) \in Stab_{K(2^e)}(y_{k_{i'}})$ for $j = 2^{e-2(i'+1)}$, all these elements belong to the set of representatives of the cosets of $Stab_{K(2^e)}(y_{k_{i'}})$ in $Stab_{K(2^e)}(y_{k_i})$. To show that these are all the representatives, take an arbitrary element,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) \in Stab_{K(2^e)}(y_{k_i}), \tag{4.208}$$

Then it is enough to show that there is $t \in \mathbb{Z}$, $1 \leq t \leq 2^{e-2(i'+1)}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) Stab_{K(2^e)}(y_{k_{i'}}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} H(2^e) Stab_{K(2^e)}(y_{k_{i'}}). \tag{4.209}$$

This is equivalent to show that

$$\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) \in Stab_{K(2^e)}(y_{k_{i'}}). \tag{4.210}$$

Or,

$$\begin{pmatrix} a - tc & b - td \\ c & d \end{pmatrix} H(2^e) \in Stab_{K(2^e)}(y_{k_{i'}}). \tag{4.211}$$

This holds if and only if

$$-(a - tc) + d - 2(b - td)k_{i'} \equiv 0 \mod 2^{e-i'-1}. \tag{4.212}$$

Then by a simple calculation we get

$$-a + d - (b - td)\alpha' 2^{i'+1} \equiv 0 \mod 2^{e-i'-1} \tag{4.213}$$

where $\alpha' = \frac{k_{i'}}{2^{i'}}$. According to (4.208) we have

$$-a + d \equiv 0 \mod 2^{e-i-1} \tag{4.214}$$

or, since $i + i' = e - 2$,

$$-a + d = \alpha 2^{e-i-1} = \alpha 2^{i'+1} \tag{4.215}$$

for some $\alpha \in \mathbb{Z}$. Inserting this into (4.213) we get

$$\alpha - (b - td)\alpha' \equiv 0 \mod 2^{e-2(i'+1)}. \tag{4.216}$$

We note that α' and d are coprime to 2 and hence they can be considered as invertible elements of $\mathbb{Z}/2^{e-2(i'+1)}\mathbb{Z}$. Hence, as an equation in $\mathbb{Z}/2^{e-2(i'+1)}\mathbb{Z}$ we find the following solution for t,

$$t \equiv (b - \alpha\alpha'^{-1})d^{-1} \mod 2^{e-2(i'+1)}. \tag{4.217}$$

Hence, (4.209) holds for some $t \in \mathbb{Z}$. Therefore, a set of representatives of $Stab_{K(2^e)}(y_{k_{i'}})$ in $Stab_{K(2^e)}(y_{k_i})$ is given by

$$T^j H(2^e), \quad j = 1, 2, \ldots 2^{e-2(i'+1)}, \tag{4.218}$$

and

$$\left[Stab_{K(2^e)}(y_{k_i}) : Stab_{K(2^e)}(y_{k_{i'}}) \right] = 2^{e-2(i'+1)}. \tag{4.219}$$

$\square$

LEMMA 4.21. *For $e \geq 2$ and $0 \leq i \leq e-2$ let $Stab_{K(2^e)}(y_{k_i})$ denote the stabilizer group of a point $y_{k_i} \in Y_i \subset Y(2^e)$. Then for $e = 2$ we have*

$$|Stab_{K(2^e)}(y_{k_0})| = |K(2^e)| = 4. \tag{4.220}$$

For $e = 3$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^2 & i = 0 \\ 2^3 & i = 1. \end{cases} \tag{4.221}$$

For $e = 4$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^3 & i = 0 \\ 2^5 & 1 \leq i \leq 2. \end{cases} \tag{4.222}$$

For $e = 5$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^{e-1} & i = 0 \\ 2^{2e-3} & e-4 \leq i \leq e-2. \end{cases} \tag{4.223}$$

For $e = 6$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^{e-1} & i = 0 \\ 2^{e+1} & i = 1 \\ 2^{2e-3} & e-4 \leq i \leq e-2. \end{cases} \tag{4.224}$$

For $e = 7$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^{e-1} & i = 0 \\ 2^{e+1} & i = 1 \\ 2^{e+3} & i = 2 \\ 2^{2e-3} & e-4 \leq i \leq e-2. \end{cases} \tag{4.225}$$

Finally, for $e \geq 8$ we have

$$|Stab_{K(2^e)}(y_{k_i})| = \begin{cases} 2^{e-1} & i = 0 \\ 2^{e+1} & i = 1 \\ 2^{e+3} & i = 2 \\ 2^{e+i+1} & 3 \leq i \leq e-5 \\ 2^{2e-3} & e-4 \leq i \leq e-2. \end{cases} \tag{4.226}$$

PROOF. For $e = 2$, $Y(2^2) = Y_0$ contains only one element and hence $K(2^2)$ is its stabilizer which according to Lemma 4.1 yields the desired result (4.220). Next, for $e \geq 3$ we consider i in the range given by

$$\begin{cases} \left[\frac{e-2}{2}\right] + 1 \leq i \leq e-2, & \text{for odd } e, \\ \frac{e-2}{2} \leq i \leq e-2, & \text{for even } e. \end{cases} \tag{4.227}$$

Recall that

$$Stab_{K(2^e)}(y_{k_i}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) \in K(2^e) \mid -a + d \equiv 0 \mod 2^{e-i-1} \right\}. \tag{4.228}$$

Let us define

$$H_1(2^e) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2^e) \mid a \equiv d \mod 2^e \right\}. \tag{4.229}$$

By an easy calculation one can show that for an arbitrary element $g \in \Gamma_0(2^e)$ and $\gamma \in H_1(2^e)$, $g\gamma g^{-1} \in H_1(2^e)$ and hence $H_1(2^e)$ is normal in $\Gamma_0(2^e)$. Then, obviously,

$$K_1(2^e) := H_1(2^e)/H(2^e) \tag{4.230}$$

is a normal subgroup of $K(2^e)$. Next we determine the elements of $K_1(2^e)$, namely the left cosets of $H(2^e)$ in $H_1(2^e)$. For this we note that for $1 \le j, j' \le 2^e$, $T^j H(2^e) = T^{j'} H(2^e)$ if and only if $j = j'$ and hence

$$\left\{ T^j H(2^e) \mid 1 \le j \le 2^e \right\} \le K_1(2^e). \tag{4.231}$$

To prove equality we must show that for any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_1(2^e)$ there is $t \in \mathbb{Z}$, $1 \le t \le 2^e$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} H(2^e) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} H(2^e). \tag{4.232}$$

This is equivalent to show that

$$\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(2^e) \tag{4.233}$$

or,

$$\begin{pmatrix} a - tc & b - td \\ c & d \end{pmatrix} \in H(2^e). \tag{4.234}$$

Since $c \equiv 0 \mod 2^e$, (4.234) holds if and only if

$$b - td \equiv 0 \mod 2^e. \tag{4.235}$$

Since d is obviously an invertible element of $\mathbb{Z}/2^e\mathbb{Z}$, as an equation in $\mathbb{Z}/2^e\mathbb{Z}$ we get for $t \in \mathbb{Z}/2^e\mathbb{Z}$

$$t \equiv bd^{-1} \mod 2^e. \tag{4.236}$$

Thus we have

$$K_1(2^e) = \left\{ T^j H(2^e) \mid 1 \le j \le 2^e \right\}. \tag{4.237}$$

This is evidently a subgroup of $Stab_{K(2^e)}(y_{k_i})$ and since $K_1(2^e)$ is normal in $K(2^e)$, it is also normal in $Stab_{K(2^e)}(y_{k_i})$. Obviously

$$|Stab_{K(2^e)}(y_{k_i})| = |Stab_{K(2^e)}(y_{k_i})/K_1(2^e)||K_1(2^e)|. \tag{4.238}$$

But $|K_1(2^e)| = 2^e$ and hence

$$|Stab_{K(2^e)}(y_{k_i})| = 2^e |Stab_{K(2^e)}(y_{k_i})/K_1(2^e)|. \tag{4.239}$$

To calculate $|Stab_{K(2^e)}(y_{k_i})/K_1(2^e)|$ we note that

$$Stab_{K(2^e)}(y_{k_i})/K_1(2^e) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} K_1(2^e) \in K(2^e)/K_1(2^e) \mid -a+d \equiv 0 \mod 2^{e-i-1} \right\}. \tag{4.240}$$

For $e \geq 3$ we have

$$K(2^e)/K_1(2^e) \cong \Gamma_0(2^e)/H_1(2^e) \cong (\Gamma_0(2^e)/\Gamma_1(2^e))/(H_1(2^e)/\Gamma_1(2^e)). \tag{4.241}$$

Let us define

$$\Delta(2^e) := H_1(2^e)/\Gamma_1(2^e) \tag{4.242}$$

where the definition of $\Gamma_1(n)$ is given in (4.165). As in Corollary (4.2.2) in [**35**], the group $\Gamma_1(2^e)$ can namely be considered as the kernel of a homomorphism of $\Gamma_0(2^e)$ to a finite group and hence it is another normal subgroup of $\Gamma_0(2^e)$. Therefore $H(2^e)$ and $\Gamma_1(2^e)$ are normal in the subgroup $H_1(2^e)$ of $\Gamma_0(2^e)$. Then the following group isomorphism holds for $e \geq 3$:

$$Stab_{K(2^e)}(y_{k_i})/K_1(2^e) \cong \left\{ g\Gamma_1(2^e) \in \Gamma_0(2^e)/\Gamma_1(2^e) \mid -a+d \equiv 0 \mod 2^{e-i-1} \right\} / \Delta(2^e). \tag{4.243}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2^e)$. Obviously $H(2^e)\Gamma_1(2^e)$ is a subgroup of $H_1(2^e)$. Conversely, by a simple calculation one can show, that each element of $H_1(2^e)$ can be written as a product of an element of $H(2^e)$ and an element of $\Gamma_1(2^e)$, that means, $H_1(2^e)$ is a subgroup of $H(2^e)\Gamma_1(2^e)$. Therefore we get

$$H_1(2^e) = H(2^e)\Gamma_1(2^e). \tag{4.244}$$

On the other hand obviously we have

$$H(2^e) \cap \Gamma_1(2^e) = \Gamma(2^e). \tag{4.245}$$

Thus from the "First Isomorphism Theorem" ([**1**], page 11) it follows that

$$\Delta(2^e) = H_1(2^e)/\Gamma_1(2^e) \cong H(2^e)/\Gamma(2^e). \tag{4.246}$$

Consequently, according to (4.23) $\Delta(2^e)$ is a cyclic group of order two and we have

$$\Delta(2^e) \cong \left\{ 1 \mod 2^e, 1 + 2^{e-1} \mod 2^e \right\}. \tag{4.247}$$

Since we consider subgroups of the projective modular group not the modular group itself, it follows from Miyake ([**35**], page 105),

$$\Gamma_0(2^e)/\Gamma_1(2^e) \cong (\mathbb{Z}/2^e\mathbb{Z})^\times / \{\pm 1_{2^e}\} \tag{4.248}$$

with $\pm 1_{2^e}$ as in (4.21). If

$$\mathcal{G}_i(2^e) := \left\{ g\Gamma_1(2^e) \in \Gamma_0(2^e)/\Gamma_1(2^e) \mid -a+d \equiv 0 \mod 2^{e-i-1} \right\} \tag{4.249}$$

with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2^e)$, we can rewrite $\mathcal{G}_i(2^e)$ in terms of $(\mathbb{Z}/2^e\mathbb{Z})^\times$, as

$$\mathcal{G}_i(2^e) \cong \left\{ d \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid -d+d^{-1} \equiv 0 \mod 2^{e-i-1} \right\} / \{\pm 1_{2^e}\}. \tag{4.250}$$

Or,

$$\mathcal{G}_i(2^e) \cong \left\{ d \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid d^2 \equiv 1 \mod 2^{e-i-1} \right\} / \{\pm 1_{2^e}\}. \tag{4.251}$$

Let us define

$$\mathcal{A}_i(2^e) = \left\{ d \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid d^2 \equiv 1 \mod 2^{e-i-1} \right\}. \tag{4.252}$$

Thus for i in the range given in (4.227), according to (4.239) and (4.243) and recalling $\Delta(2^e)$ is of order 2, the cardinality of $Stab_{K(2^e)}(y_{k_i})$ in terms of the cardinality of $\mathcal{A}_i(2^e)$ is given obviously by

$$|Stab_{K(2^e)}(y_{k_i})| = 2^{e-2}|\mathcal{A}_i(2^e)|. \tag{4.253}$$

Since all elements of $(\mathbb{Z}/8\mathbb{Z})^\times$ are of second order, we get for $e=3$ and $i=1$

$$\mathcal{A}_1(2^3) = (\mathbb{Z}/8\mathbb{Z})^\times. \tag{4.254}$$

But $(\mathbb{Z}/8\mathbb{Z})^\times$ has four elements and hence we get $|Stab_{K(2^3)}(y_{k_1})| = 2^3$. Then from Lemma 4.20 we obtain $|Stab_{K(2^3)}(y_{k_0})| = 2^2$. Thus (4.221) has been proved.

For $e=4$ and $1 \le i \le 2$, by an easy calculation one can check that

$$\mathcal{A}_1(2^4) = \mathcal{A}_2(2^4) = (\mathbb{Z}/16\mathbb{Z})^\times \cong \{1,3,5,7,9,11,13,15\} \tag{4.255}$$

and hence $|Stab_{K(2^4)}(y_{k_1})| = |Stab_{K(2^4)}(y_{k_2})| = 2^5$. Then according to Lemma 4.20 we get $|Stab_{K(2^4)}(y_{k_0})| = 2^3$ which proves (4.222).

For $e \ge 5$ consider the homomorphisms

$$\begin{aligned} &h_k : (\mathbb{Z}/2^e\mathbb{Z})^\times \to (\mathbb{Z}/2^k\mathbb{Z})^\times \\ &h_k(\alpha) = \alpha \mod 2^k \end{aligned} \tag{4.256}$$

with $k = 1,2,3$. Since, as can be easily checked, each element of $(\mathbb{Z}/2^k\mathbb{Z})^\times$ for $k=1,2,3$ is of second order, we have

$$h_k(\alpha^2) = \alpha^2 \mod 2^k = h_k(\alpha)^2 \equiv 1 \mod 2^k, \quad k=1,2,3. \tag{4.257}$$

Thus for any $\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times$ we have

$$\alpha^2 \equiv 1 \mod 2^k, \quad k=1,2,3. \tag{4.258}$$

Hence, for $e \geq 5$ we get

$$\mathcal{A}_i(2^e) = (\mathbb{Z}/2^e\mathbb{Z})^\times, \quad e-4 \leq i \leq e-2. \tag{4.259}$$

Therefore, according to (4.253) and the fact that $|(\mathbb{Z}/2^e\mathbb{Z})^\times| = 2^{e-1}$ (see for example [**44**]) we get

$$|Stab_{K(2^e)}(y_{k_i})| = 2^{2e-3}, \quad e-4 \leq i \leq e-2. \tag{4.260}$$

This and Lemma 4.20 yield all assertions for $e \geq 5$ except the case $3 \leq i \leq e-5$ for $e \geq 8$ in (4.226).

As mentioned in (4.22), for $k \geq 3$ one has $\alpha^2 \equiv 1 \mod 2^k$ iff $\alpha \equiv \pm 1 \mod 2^k$ or $\alpha \equiv 2^{k-1} \pm 1 \mod 2^k$. Thus, for $e \geq 8$ and i values in the intersection of the range in (4.227) and the range $3 \leq i \leq e-5$, namely

$$\begin{cases} \left[\frac{e-2}{2}\right] + 1 \leq i \leq e-5, & \text{for odd } e, \\ \frac{e-2}{2} \leq i \leq e-5, & \text{for even } e \end{cases} \tag{4.261}$$

we get

$$\begin{aligned} &\mathcal{A}_i(2^e) = \\ &\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 1 \mod 2^{e-i-1}\right\} \sqcup \\ &\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv -1 \mod 2^{e-i-1}\right\} \sqcup \\ &\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 2^{e-i-2}+1 \mod 2^{e-i-1}\right\} \sqcup \\ &\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 2^{e-i-2}-1 \mod 2^{e-i-1}\right\}. \end{aligned} \tag{4.262}$$

Let $h : (\mathbb{Z}/2^e\mathbb{Z})^\times \to (\mathbb{Z}/2^{e-i-1}\mathbb{Z})^\times$ be the homomorphism given by

$$h(\alpha) = \alpha \mod 2^{e-i-1}. \tag{4.263}$$

Since h is onto we get

$$|\ker h| = \frac{|(\mathbb{Z}/2^e\mathbb{Z})^\times|}{|(\mathbb{Z}/2^{e-i-1}\mathbb{Z})^\times|} = \frac{2^{e-1}}{2^{e-i-2}} = 2^{i+1} \tag{4.264}$$

Then, evidently, we have

$$\begin{aligned} &\sharp\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 1 \mod 2^{e-i-1}\right\} = \\ &\sharp\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv -1 \mod 2^{e-i-1}\right\} = \\ &\sharp\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 2^{e-i-2}+1 \mod 2^{e-i-1}\right\} = \\ &\sharp\left\{\alpha \in (\mathbb{Z}/2^e\mathbb{Z})^\times \mid \alpha \equiv 2^{e-i-2}-1 \mod 2^{e-i-1}\right\} = |\ker h| = 2^{i+1} \end{aligned} \tag{4.265}$$

From this, (4.262) and (4.253) we get for the values of i in (4.261)

$$|Stab_{K(2^e)}(y_{k_i})| = 2^{e+i+1}. \tag{4.266}$$

To complete the proof we must still show the assertion for $i' = e-2-i$, that is, for i' in the range

$$\begin{cases} 3 \le i' \le \left[\frac{e-2}{2}\right], & \text{for odd } e, \\ 3 \le i' \le \frac{e-2}{2} - 1, & \text{for even } e. \end{cases} \tag{4.267}$$

For this, first we note that

$$\begin{aligned} &\left[K(2^e) : Stab_{K(2^e)}(y_{k_{i'}})\right] = \\ &\left[K(2^e) : Stab_{K(2^e)}(y_{k_i})\right] \times \left[Stab_{K(2^e)}(y_{k_i}) : Stab_{K(2^e)}(y_{k_{i'}})\right]. \end{aligned} \tag{4.268}$$

Then, from Lemma 4.20 it follows that

$$\left[K(2^e) : Stab_{K(2^e)}(y_{k_{i'}})\right] = \left[K(2^e) : Stab_{K(2^e)}(y_{k_i})\right] 2^{e-2(i'+1)}. \tag{4.269}$$

Hence we get

$$|Stab_{K(2^e)}(y_{k_{i'}})| = \frac{|Stab_{K(2^e)}(y_{k_i})|}{2^{e-2(i'+1)}}. \tag{4.270}$$

This together with the known values $|Stab_{K(2^e)}(y_{k_i})| = 2^{e+i+1} = 2^{2e-i'-1}$ for i in the range (4.261) completes the proof. □

LEMMA 4.22. *For $p > 2$ and $e \ge 1$ the number of orbits in $X(p^e)$ under the action of $K(p^e)$, is given by*

$$|X(p^e)/K(p^e)| = 2e. \tag{4.271}$$

Furtheremore, the number of orbits in $X(2^e)$ under the action of $K(2^e)$, is given by

$$|X(2^e)/K(2^e)| = \begin{cases} 2 & e = 1 \\ e+1 & 2 \le e \le 3 \\ 6 & e = 4 \\ 10 & e = 5 \\ 4(e-3) & e \ge 6. \end{cases} \tag{4.272}$$

PROOF. The assertion for $e = 1$ is already obtained in (4.158). In the rest of the proof we assume $e \ge 2$. For $p > 2$ consider an arbitrary point y_{k_i} in Y_i. As we mentioned already, the orbit $[y_{k_i}]$ of $y_{k_i} \in Y_i$ belongs to Y_i. According to the "orbit stabilizer theorem", the length of this orbit is given by

$$|[y_{k_i}]| = \frac{|K(p^e)|}{|Stab_{K(p^e)}(y_{k_i})|} = \frac{1}{2} p^{e-i-2}(p-1) \tag{4.273}$$

where we used Lemmas 4.1 and 4.19. Since, according to Lemma 4.19 the stabilizer of each point of Y_i has the same order, it follows that there must be another orbit in Y_i with the same length. These are the only orbits in Y_i because the sum of their length is equal to the cardinality of Y_i. Recall that besides the orbits in $Y(p^e)$ there are the two other orbits $[\Gamma_0(p^e)]$ and $[S\Gamma_0(p^e)]$ of lengths 1 and p^e, respectively in $X(p^e)$. Therefore if n_i denotes the number of orbits in Y_i then we have

$$|X(p^e)/K(p^e)| = 2 + \sum_{i=0}^{e-2} n_i. \tag{4.274}$$

But we already proved that for $0 \le i \le e-2$ we have $n_i = 2$ and hence

$$|X(p^e)/K(p^e)| = 2e. \tag{4.275}$$

Next, we consider the case of powers of $p = 2$. According to Lemma 4.1 we have

$$|K(2^e)| = \begin{cases} 2, & e = 1, \\ 4, & e = 2, \\ 2^{2e-3}, & e \ge 3. \end{cases} \tag{4.276}$$

To determine the number of orbits of $K(2^e)$ in $X(2^e)$ recall that for $e \ge 2$ besides the orbits $[\Gamma_0(2^e)]$ and $[S\Gamma_0(2^e)]$ there are the other orbits in the set

$$Y(2^e) = \sqcup_{i=0}^{e-2} Y_i \tag{4.277}$$

where each orbit belongs to a set Y_i. Hence, if n_i denotes the number of orbits in Y_i then the number of orbits of $K(2^e)$ in $X(2^e)$ is given by

$$|X(2^e)/K(2^e)| = 2 + \sum_{i=0}^{e-2} n_i. \tag{4.278}$$

We recall also that (Lemma 4.16)

$$|Y_i| = 2^{e-i-2}, \quad 0 \le i \le e-2. \tag{4.279}$$

Since Y_0 for $e = 2$ contains only one element, Y_0 itself is an orbit and hence $n_0 = 1$. Thus according to (4.278) for $e = 2$ we have $|X(2^2)/K(2^2)| = 3$.

For $e \ge 3$ according to Lemma 4.21 the stabilizer of each point in Y_0 is of order 2^{e-1}. Hence, from the orbit stabilizer theorem it follows that the length of an orbit in Y_0 coincides with the cardinality of Y_0 and consequentely for any $e \ge 3$ we have $n_0 = 1$. For $e \ge 3$, according to (4.279) Y_{e-2} contains only one element and therefore Y_{e-2} itself is an orbit. That is, for any $e \ge 3$ we have $n_{e-2} = |Y_{e-2}| = 1$. Hence for $e = 3$ according to (4.278) we get $|X(2^3)/K(2^3)| = 4$.

For $e \ge 4$ according to Lemma 4.21 $K(2^e)$ is the stabilizer of each point in Y_{e-3} and hence according to the orbit stabilizer theorem each point in Y_{e-3} is an orbit. Thus the number of orbits in Y_{e-3} coincides with its cardinality. Therefore we get $n_{e-3} = 2$. Hence, according to (4.278) we get $|X(2^4)/K(2^4)| = 6$.

With the same argument, for $e \ge 5$ we get $n_{e-4} = 4$. Hence, according to (4.278) we get $|X(2^5)/K(2^5)| = 10$.

Next, let $e \geq 6$. According to Lemma 4.21 the stabilizer of each point in Y_1 is of the order 2^{e+1} and hence according to the orbit stabilizer theorem each orbit in Y_1 is of the same length 2^{e-4}. Since the sum of the lengths of these orbits must coincide with the cardinality of Y_1, namely 2^{e-3}, it follows that Y_1 contains two orbits. That is, for $e \geq 6$ we have $n_1 = 2$. Hence, according to (4.278) we get $|X(2^6)/K(2^6)| = 12$. Similarly, for $e \geq 7$ we get $n_2 = 4$ and hence $|X(2^7)/K(2^7)| = 16$.

Finally, let $e \geq 8$ and $3 \leq i \leq e-5$. Then according to Lemma 4.21 the stabilizer of each point in Y_i is of the same order 2^{e+i+1} and hence according to orbit stabilizer theorem the orbits in Y_i are of the same length 2^{e-i-4}. Since the sum of the lengths of these orbits must be equal to the cardinality of Y_i, namely 2^{e-i-2}, it follows that Y_i contains four orbits. Thus for $e \geq 8$ and $3 \leq i \leq e-5$ we get $n_i = 4$ and hence according to (4.278) we get $|X(2^e)/K(2^e)| = 4(e-3)$. Thus the proof is complete. ☐

As a consequence of this lemma, according to Lemma 4.5, for the representation $U_{\Gamma_0(q)}$ we get

LEMMA 4.23. *Let $q = p^e$ be a prime power with $e \in \mathbb{N}$ and*

$$U_{\Gamma_0(q)} = \bigoplus_{i=1}^{N} m_i V_i \tag{4.280}$$

be the decomposition of the representation $U_{\Gamma_0(q)}$ into irreducible representations where m_i is the multiplicity of V_i in $U_{\Gamma_0(q)}$ and N is the number of non-isomorphic irreducible representations appearing in $U_{\Gamma_0(p^e)}$. Then we have

$$\sum_{i=1}^{N} m_i^2 = \begin{cases} 2e, & p > 2,\ e \geq 1, \\ 2 & p = 2,\ e = 1 \\ e+1 & p = 2,\ 2 \leq e \leq 3 \\ 6 & p = 2,\ e = 4 \\ 10 & p = 2,\ e = 5 \\ 4(e-3) & p = 2,\ e \geq 6. \end{cases} \tag{4.281}$$

We close this section by counting the number of irreducibles in $U_{\overline{\Gamma}_0(q)}$ for q a prime power. Recall that

$$X(q) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma_0(q), \quad \overline{X}(q) = \mathrm{PGL}(2,\mathbb{Z})/\overline{\Gamma}_0(q) \tag{4.282}$$

and that for $X(q)$ and $\overline{X}(q)$ we can chose the same set of representatives as given in (4.147). Indeed, there is a bijection

$$b : X(q) \to \overline{X}(q) \tag{4.283}$$

given by

$$b(x\Gamma_0(q)) = x\overline{\Gamma}_0(q). \tag{4.284}$$

Moreover, recall that

$$K(q) = \Gamma_0(q)/H(q), \quad \overline{K}(q) = \overline{\Gamma}_0(q)/M(q) \tag{4.285}$$

are subgroups of $Q(q) = \mathrm{PSL}(2,\mathbb{Z})/H(q)$ and $G(q) = \mathrm{PGL}(2,\mathbb{Z})/M(q)$, respectively (see (4.15)), where $H(q)$ and $M(q)$ are defined in (4.3). The action of $K(q)$ on $X(q)$ is given by

$$\begin{aligned} &K(q) \times X(q) \to X(q) \\ &(gH(q), x\Gamma_0(q)) \mapsto gH(q)(x\Gamma_0(q)) = gx\Gamma_0(q) \end{aligned} \tag{4.286}$$

and the action of $\overline{K}(q)$ on $\overline{X}(q)$ is given by

$$\begin{aligned} &\overline{K}(q) \times \overline{X}(q) \to \overline{X}(q) \\ &(gM(q), x\overline{\Gamma}_0(q)) \mapsto gM(q)(x\overline{\Gamma}_0(q)) = gx\overline{\Gamma}_0(q). \end{aligned} \tag{4.287}$$

Now we determine the cardinality of $\overline{X}(q)/\overline{K}(q)$, namely the number of orbits of $\overline{K}(q)$ in $\overline{X}(q)$. Evidently, $\overline{\Gamma}_0(q) \in \overline{X}(q)$ is invariant under $\overline{K}(q)$ and hence $\left[\overline{\Gamma}_0(q)\right]$ is an orbit consisting of one element, namely $\overline{\Gamma}_0(q)$. Using the same arguments as in the case of the orbit $[S\Gamma_0(q)] \in X(q)/K(q)$, we get for the orbit $\left[S\overline{\Gamma}_0(q)\right] \in \overline{X}(q)/\overline{K}(q)$

$$\left[S\overline{\Gamma}_0(q)\right] = \left\{T^{-j}S\overline{\Gamma}_0(q) \mid 0 \le j \le q-1\right\}. \tag{4.288}$$

Therefore, we have for $q = p$

$$\overline{X}(q) = [\overline{\Gamma}_0(q)] \sqcup [S\overline{\Gamma}_0(q)] \tag{4.289}$$

and hence

$$|\overline{X}(q)/\overline{K}(q)| = 2. \tag{4.290}$$

It remains to determine the orbits of $\overline{K}(q)$ in

$$\overline{Y}(q) := \{\overline{y} := b(y) \mid y \in Y(q)\}. \tag{4.291}$$

Similar to the action of $K(q)$ on $Y(q)$, the action of $\overline{K}(q)$ on $\overline{Y}(q)$ partitions this set into the subsets

$$\overline{Y}_i := \{\overline{y} := b(y) \mid y \in Y_i\}, \quad 0 \le i \le e-2. \tag{4.292}$$

If $\overline{n}_i$ denotes the number of orbits in the set $\overline{Y}_i \subset \overline{Y}(q)$, then the number of orbits of $\overline{K}(q)$ in $\overline{X}(q)$ is given by

$$|\overline{X}(q)/\overline{K}(q)| = 2 + \sum_{i=0}^{e-2} \overline{n}_i. \tag{4.293}$$

The number $\overline{n}_i$ can be determined from the number n_i of orbits in Y_i. For this we need the following lemma:

LEMMA 4.24. *For $e \ge 2$ and $0 \le i \le e-2$ let $[y_{k_i}]$ be the orbit of $K(p^e)$ in $Y_i \subset Y(p^e)$ through the point $y_{k_i} \in Y_i$. Similarly, let $\left[\overline{y}_{k_i}\right]$ be* *the orbit of* $\overline{K}(p^e)$ *in* $\overline{Y}_i \subset \overline{Y}(p^e)$ *through the point* $\overline{y}_{k_i} = b(y_{k_i}) \in \overline{Y}_i$. *Then we have for* $p \geq 2$, $1 \leq i \leq e-3$ *and* $e \geq 4$,

$$|\left[\overline{y}_{k_i}\right]| = \begin{cases} |\left[y_{k_i}\right]| & -1 \text{ is a square modulo } p^e \\ 2|\left[y_{k_i}\right]| & \text{otherwise.} \end{cases} \tag{4.294}$$

Moreover we have for $p \geq 2$, $e \geq 2$, *and* $i = 0$ *respectively* $i = e-2$

$$|\left[\overline{y}_{k_i}\right]| = |\left[y_{k_i}\right]|. \tag{4.295}$$

Proof. Let -1 be a square modulo $q = p^e$. Then according to (4.6) we have $[M(q) : H(q)] = 2$ where $H(q)$ and $M(q)$ are defined in (4.3). Since $M(q)$ is normal in $\overline{\Gamma}_0(q)$ and $\Gamma_0(q)$ is a subgroup of $\overline{\Gamma}_0(q)$, we have $\Gamma_0(q)M(q) \leq \overline{\Gamma}_0(q)$ (see for example [**1**], page 6, Proposition 7). Thus according to the "First Isomorphism Theorem" the following group isomorphism hold

$$(\Gamma_0(q)M(q))/M(q) \cong \Gamma_0(q)/(\Gamma_0(q) \cap M(q)). \tag{4.296}$$

On the other hand, for an arbitrary element $\overline{\gamma} \in \overline{\Gamma}_0(q)$ obviously there is an element $\delta \in M(q)$ such that $\overline{\gamma}\delta \in \Gamma_0(q)$. Hence, each element $\overline{\gamma} \in \overline{\Gamma}_0(q)$ can be written as $\gamma\delta'$ where $\gamma = \overline{\gamma}\delta \in \Gamma_0(q)$, $\delta \in M(q)$, and $\delta' = \delta^{-1} \in M(q)$. That means $\overline{\Gamma}_0(q) \leq \Gamma_0(q)M(q)$. From this and $\Gamma_0(q)M(q) \leq \overline{\Gamma}_0(q)$ we get

$$\overline{\Gamma}_0(q) = \Gamma_0(q)M(q). \tag{4.297}$$

On the other hand, according to definitions of $M(q)$ and $H(q)$ in (4.3) and since elements of $\Gamma_0(q)$ have positive determinant, obviously we get

$$H(q) = \Gamma_0(q) \cap M(q). \tag{4.298}$$

Inserting this and (4.297) into (4.296) yields the following group isomorphism,

$$\overline{\Gamma}_0(q)/M(q) \cong \Gamma_0(q)/H(q) \tag{4.299}$$

or

$$\overline{K}(q) \cong K(q). \tag{4.300}$$

Since $K(q)$ and $\overline{K}(q)$ are subgroups of $Q(q)$ and $G(q)$, respectively, it follows from Lemma 4.2 that this isomorphism is given explicitly by

$$\begin{aligned} &\imath : K(q) \to \overline{K}(q) \\ &\imath(gH(q)) = gM(q), \quad g \in \Gamma_0(q). \end{aligned} \tag{4.301}$$

Recall that the action of $K(q)$ on $X(q)$ is given by

$$\begin{aligned} &K(q) \times X(q) \to X(q) \\ &(gH(q), x\Gamma_0(q)) \mapsto gH(q)(x\Gamma_0(q)) = gx\Gamma_0(q) \end{aligned} \tag{4.302}$$

and the action of $\overline{K}(q)$ on $\overline{X}(q)$ is given by

$$\begin{aligned} &\overline{K}(q) \times \overline{X}(q) \to \overline{X}(q) \\ &(gM(q), x\overline{\Gamma}_0(q)) \mapsto gM(q)(x\overline{\Gamma}_0(q)) = gx\overline{\Gamma}_0(q). \end{aligned} \tag{4.303}$$

Let $b : X(q) \to \overline{X}(q)$ be the bijection given in (4.284). Then for $gH(q) \in K(q)$ and $x\Gamma_0(q) \in X(q)$ evidently we have

$$\begin{aligned} &b(gH(q)(x\Gamma_0(q))) = b(gx\Gamma_0(q)) = gx\overline{\Gamma}_0(q) = gM(q)x\overline{\Gamma}_0(q) = \\ &= \imath(gH(q))(x\overline{\Gamma}_0(q)) = \imath(gH(q))b(x\Gamma_0(q)) \in \overline{X}(q). \end{aligned} \tag{4.304}$$

That means the bijection b intertwines the action of $K(q)$ and the one of $\overline{K}(q)$ on $X(q)$ respectively $\overline{X}(q)$.

Hence, if $[y_{k_i}]$ is an orbit in $Y_i \subset Y(q)$ through the point $y_{k_i} \in Y_i$ then the orbit $[\overline{y}_{k_i}]$ in $\overline{Y}_i \subset \overline{Y}(q)$ through the point $\overline{y}_{k_i}$ is given by

$$[\overline{y}_{k_i}] = \{b(y) \mid y \in [y_{k_i}]\}. \tag{4.305}$$

Therefore, obviously we get the first part of (4.294).

Next, assume that -1 is not a square modulo q. Then according to (4.6) we have $M(q) = H(q)$. We note that obviously $K(q) = \Gamma_0(q)/H(q)$ is a normal subgroup of $\overline{K}(q) = \overline{\Gamma}_0(q)/H(q)$. Moreover, the cyclic group $\mathcal{C}_2$ of order two, generated by the element $MH(q)$, is a subgroup of $\overline{K}(q)$ such that $K(q) \cap \mathcal{C}_2 = \{H(q)\}$. On the other hand, from (4.14) it follows that $[\overline{K}(q) : K(q)] = 2$ and hence $\overline{K}(q) = K(q)\mathcal{C}_2$. Therefore, we have

$$\overline{K}(q) = K(q) \rtimes \mathcal{C}_2. \tag{4.306}$$

From this and the bijection (4.284) it follows that $Stab_{K(p^e)}(y_{k_i}) \leq Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ and that $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i}) \cap K(q) = Stab_{K(p^e)}(y_{k_i})$. Then from the first isomorphism theorem it follows that

$$Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})K/K \cong Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})/(Stab_{\overline{K}(p^e)}(\overline{y}_{k_i}) \cap K)$$

or

$$Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})K/K \cong Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})/Stab_{K(p^e)}(y_{k_i}). \tag{4.307}$$

If $MH(q) \notin Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ then $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})K = K$ and if $MH(q) \in Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ then $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})K = \overline{K}$.Therefore according to (4.307), either $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ coincides with $Stab_{K(p^e)}(y_{k_i})$ or $[Stab_{\overline{K}(p^e)}(\overline{y}_{k_i}) : Stab_{K(p^e)}(y_{k_i})] = 2$. Hence, the stabilizer $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ of a point $\overline{y}_{k_i} \in \overline{Y}_i$ coincides with either $Stab_{K(q)}(y_{k_i})$ or $Stab_{K(q)}(y_{k_i}) \rtimes \mathcal{C}_2$. For $p > 2$ and $0 \leq i \leq e-2$ respectively for $p = 2$ and $1 \leq i \leq e-3$ one can easily check that $MH(p^e)$ does not belong to $Stab_{\overline{K}(p^e)}(\overline{y}_{k_i})$ and hence

$$Stab_{\overline{K}(p^e)}(\overline{y}_{k_i}) = Stab_{K(p^e)}(y_{k_i}), \quad \overline{y}_{k_i} = b(y_{k_i}). \tag{4.308}$$

This and the orbit stabilizer theorem together with (4.306) yields the second part of (4.294). Recall that for $p = 2$, Y_0 and Y_{e-2} contain only one orbit under the action of $K(p^e)$ (see proof of Lemma 4.22). Hence, evidently, $\overline{Y}_0$ and $\overline{Y}_{e-2}$ also contain only one orbit under the action of $\overline{K}(p^e)$ which yields the second assertion. $\square$

Thus we get

LEMMA 4.25. *For $p > 2$ and $e \geq 1$ the number of orbits in $\overline{X}(p^e)$ under the action of $\overline{K}(p^e)$, is given by*

$$|\overline{X}(p^e)/\overline{K}(p^e)| = \begin{cases} 2e & -1 \text{ is a square modulo } p^e \\ e+1 & \text{otherwise.} \end{cases} \tag{4.309}$$

Furthermore, the number of orbits in $\overline{X}(2^e)$ under the action of $\overline{K}(2^e)$, is given by

$$|\overline{X}(2^e)/\overline{K}(2^e)| = \begin{cases} 2 & p=2,\ e=1 \\ e+1 & p=2,\ 2 \leq e \leq 4 \\ 7 & p=2,\ e=5 \\ 2(e-2) & p=2,\ e \geq 6. \end{cases} \tag{4.310}$$

PROOF. For $e = 1$ the assertion has been already proven in (4.290). Let $p > 2$ respectively $e \geq 2$ and -1 be a square modulo p^e. Then according to Lemma 4.24 obviously we have $\overline{n}_i = n_i$ and hence according to (4.293) and (4.274) we get

$$|\overline{X}(p^e)/\overline{K}(p^e)| = |X(p^e)/K(p^e)|. \tag{4.311}$$

Hence, Lemma 4.22 yields the first part of (4.309). Next, for an odd prime let -1 be not a square modulo p^e. Moreover let n_i and $\overline{n}_i$ be the number of orbits in Y_i respectively $\overline{Y}_i$. Then according to Lemma 4.24 for $1 \leq i \leq e-3$, $e \geq 4$ we have $\overline{n}_i = n_i/2 = 1$ and for $e \geq 2$ we have $\overline{n}_0 = n_0 = 1$, $\overline{n}_{e-2} = n_{e-2} = 1$ and hence

$$|\overline{X}(p^e)/\overline{K}(p^e)| = 2 + \sum_{i=0}^{e-2} \overline{n}_i = 2 + e - 1 = e + 1 \tag{4.312}$$

which yields the second part of (4.309). Finally, we consider powers of 2. We note that -1 is not a square modulo 2^e for $e \geq 2$ [**25**]. From Lemma 4.24 and the known values of n_i as given in the proof of Lemma 4.22 we have

$$\begin{cases} \overline{n}_0 = n_0 = 1 & e \geq 2 \\ \overline{n}_{e-2} = n_{e-2} = 1 & e \geq 3 \\ \overline{n}_{e-3} = \frac{n_{e-3}}{2} = 1 & e \geq 4 \\ \overline{n}_{e-4} = \frac{n_{e-4}}{2} = 2 & e \geq 5 \\ \overline{n}_1 = \frac{n_1}{2} = 1 & e \geq 6 \\ \overline{n}_2 = \frac{n_2}{2} = 2 & e \geq 7 \\ \overline{n}_i = \frac{n_i}{2} = 2 & e \geq 8, 3 \leq i \leq e-5 \end{cases} \tag{4.313}$$

This and formula (4.293) complete the proof. □

Since the number of non-isomorphic irreducible subrepresentations of $\pi_{G(p^e)}$ and their multiplicities coincide with that of $U_{\overline{\Gamma}_0(p^e)}$, we get as a consequence of this lemma and Lemma 4.14

LEMMA 4.26. *Let $q = p^e$ be a prime power with $e \in \mathbb{N}$ and moreover, let*

$$U_{\overline{\Gamma}_0(q)} = \bigoplus_{i=1}^{\overline{N}} m_i \overline{V}_i \tag{4.314}$$

be the decomposition of the representation $U_{\overline{\Gamma}_0(q)}$ into irreducible representations, where m_i is the multiplicity of $\overline{V}_i$ in $U_{\overline{\Gamma}_0(q)}$ and $\overline{N}$ the number of non-isomorphic irreducible representations appearing in $U_{\overline{\Gamma}_0(p^e)}$. Then we have for $p > 2$

$$\sum_{i=1}^{\overline{N}} m_i^2 = \begin{cases} 2e & -1 \text{ is a square modulo } p^e \\ e+1 & \text{otherwise.} \end{cases} \tag{4.315}$$

Moreover, we have for $p = 2$

$$\sum_{i=1}^{\overline{N}} m_i^2 = \begin{cases} 2 & p=2,\ e=1 \\ e+1 & p=2,\ 2 \le e \le 4 \\ 7 & p=2,\ e=5 \\ 2(e-2) & p=2,\ e \ge 6. \end{cases} \tag{4.316}$$

REMARK 4.2. *In the next section we show that the representations $U_{\Gamma_0(p^e)}$ and $U_{\overline{\Gamma}_0(p^e)}$ are multiplicity-free. Hence, $N = \sum_{i=1}^{N} m_i^2$ and $\overline{N} = \sum_{i=1}^{\overline{N}} m_i^2$ yield the number of the non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(p^e)}$ and $U_{\overline{\Gamma}_0(p^e)}$, respectively.*

4.7. Multiplicity-free property of induced representations

In this section we prove that for $n \in \mathbb{N}$, $n > 1$, each irreducible component in the decomposition of $U_{\Gamma_0(n)}$ occurs with multiplicity one. First we recall some definitions and notations.

DEFINITION 4.3 (multiplicity-free representation). *A representation of a finite group is called multiplicity-free if in its decomposition into irreducibles, each irreducible representation occurs only once (see for example* [**9**]*, page 375).*

DEFINITION 4.4. *Let G be a finite group and K be a subgroup of G. Then (G, K) is called a Gelfand pair if the representation $ind_K^G(1)$ of G induced from the one dimensional trivial representation of K is multiplicity-free (see* [**9**]*, page 376).*

We recall a criterion for being a Gelfand pair [**10, 9**]: let G be a finite group and $K \leq G$. If $g^{-1} \in KgK$ for all $g \in G$ that is, if $KgK = Kg^{-1}K$ for all $g \in G$, then (G, K) is a Gelfand pair.

LEMMA 4.27. *For a prime power $q = p^e$ with $e \in \mathbb{N}$, $(G(q), \overline{K}(q))$ is a Gelfand pair.*

PROOF. According to Lemma 4.13 the permutation representation $\pi_{G(p)}$ and hence the induced representation $ind_{\overline{K}(p)}^{G(p)}(1)$ is multiplicity-free which yields the desired assertion for $e = 1$. Now assume, that $e \geq 2$. We have to show that for every $gM(q) \in G(q)$,

$$g^{-1}M(q) \in \overline{K}(q)gM(q)\overline{K}(q). \tag{4.317}$$

That is, for all $gM(q) \in G(q)$, there exists $kM(q)$, $k \in \overline{\Gamma}_0(q)$ and $k'M(q)$, $k' \in \overline{\Gamma}_0(q)$ in $\overline{K}(q)$ such that

$$g^{-1}M(q) = kM(q)gM(q)k'M(q) \tag{4.318}$$

or

$$gkg \in k'^{-1}M(q) \subset \overline{\Gamma}_0(q). \tag{4.319}$$

Thus we need to prove the following statement:

$$\forall g \in \mathrm{PGL}(2, \mathbb{Z}), \quad \exists k \in \overline{\Gamma}_0(q) \colon gkg \in \overline{\Gamma}_0(q). \tag{4.320}$$

On the other hand, each $g \in \mathrm{PGL}(2, \mathbb{Z})$ can be written as $g = r\gamma$ with $r \in R(\mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(q))$ and $\gamma \in \overline{\Gamma}_0(q)$ and hence $gkg \in \overline{\Gamma}_0(q)$ iff $r\gamma k r\gamma \in \overline{\Gamma}_0(q)$ iff $rk''r \in \overline{\Gamma}_0(q)$ with $k'' = \gamma k \in \overline{\Gamma}_0(q)$. Therefore, condition (4.320) is equivalent to

$$\forall r \in R(\mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(q)), \quad \exists k'' \in \overline{\Gamma}_0(q) : rk''r \in \overline{\Gamma}_0(q). \tag{4.321}$$

We note that $R(\mathrm{PGL}(2, \mathbb{Z})/\overline{\Gamma}_0(q))$ can be chosen to be the same as the set of representatives $R(\mathrm{PSL}(2, \mathbb{Z})/\Gamma_0(q))$ whose elements are given by ([**27**], page 107)

$$id; \quad T^{-j}S, \quad j = 0, \ldots, p^e - 1; \quad ST^{-jp}S, \quad j = 1, 2, \ldots, p^{e-1} - 1. \tag{4.322}$$

For $r = id$ the requirement is obviously fulfilled with $k'' = id$. For $r = T^{-j}S$, the condition (4.321) is fulfilled with $k'' = T^j$. For $r = ST^{-jp}S$ this condition is fulfilled with

$$k'' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.323}$$

Thus the proof is complete. □

As a direct consequence of this lemma we have:

COROLLARY 4.1. *The permutation representation $\pi_{G(q)}$ and hence the induced representation $U_{\overline{\Gamma}_0(q)}$ are multiplicity-free.*

To prove the multiplicity-free property of $U_{\Gamma_0(q)}$ we need a result on the restriction of representations to subgroups ([**24**], pages 217 and 219).

LEMMA 4.28. *Let H be a normal subgroup of index 2 in a finite group G, and χ be an irreducible representation of G. Then either*

- *The restriction $\chi \downarrow H$ of χ to H is an irreducible representation of H or*
- *The restriction $\chi \downarrow H$ of χ to H is the direct sum of two non-isomorphic representations of H.*

Moreover, if $\chi \downarrow H = \chi_1 \oplus \chi_2$, where χ_1 and χ_2 are non-isomorphic representations of H, and if ϕ is an irreducible representation of G whose restriction to H has χ_1 or χ_2 as a subrepresentation, then χ is isomorphic to ϕ.

LEMMA 4.29. *For a prime power $q = p^e$ the representation $U_{\Gamma_0(q)}$ is multiplicity-free.*

PROOF. As shown in section 4.5, $U_{\Gamma_0(p)}$ is multiplicity-free. Let q be a prime power p^e with $p > 2$ and $e \geq 2$. Assume that $U_{\Gamma_0(q)}$ is multiplicity-free and has a decomposition into irreducible subrepresentations given by

$$U_{\Gamma_0(q)} = \bigoplus_{i=1}^{N} m_i V_i, \tag{4.324}$$

where $m_i = 1$ and, according to Lemma 4.23, the number N of non-isomorphic irreducible subrepresentations is given by $N = 2e$. We are going to prove, that $U_{\Gamma_0(qp)}$ is then also multiplicity-free. To this end, note that $U_{\Gamma_0(q)}$ occurs at least once in the representation $U_{\Gamma_0(qp)}$ as we proved in section 4.4. Thus the decomposition of $U_{\Gamma_0(qp)}$ can be written as

$$U_{\Gamma_0(qp)} = \bigoplus_{i=1}^{2e} m_i V_i \oplus \bigoplus_{i=2e+1}^{N'} m_i V_i \tag{4.325}$$

where N' denotes the number of non-isomorphic irreducible subrepresentations in $U_{\Gamma_0(qp)}$. According to Lemma 4.23 we have

$$\sum_{i=1}^{2e} m_i^2 + \sum_{i=2e+1}^{N'} m_i^2 = 2e + 2. \tag{4.326}$$

But $\sum_{i=1}^{2e} m_i^2 = 2e$ and hence

$$\sum_{i=2e+1}^{N'} m_i^2 = 2. \tag{4.327}$$

Obviously, this condition fixes $N' = 2e+2$ and $m_{2e+1} = m_{2e+2} = 1$ that means $U_{\Gamma_0(qp)}$ is multiplicity-free. Thus we proved that for an odd prime power q, $U_{\Gamma_0(q)}$ is multiplicity-free.

Next, we consider the case where q is a power of 2. Recall that for the decomposition

$$U_{\Gamma_0(q)} = \bigoplus_{i=1}^{N} m_i V_i \tag{4.328}$$

we have

$$|X(q)/K(q)| = \sum_{i=1}^{N} m_i^2. \tag{4.329}$$

Hence, observing that according to Lemma 4.22

$$|X(2^{e+1})/K(2^{e+1})| - |X(2^e)/K(2e)| \le 2, \quad 1 \le e \le 3, \tag{4.330}$$

with the same argument as in the case $p > 2$ it follows, that $U_{\Gamma_0(q)}$ for $q = 2^e$ and $1 \le e \le 4$ is multiplicity-free.

Finally, for $e \ge 5$ we show that $\pi_{Q(2^e)}$ and consequentely $U_{\Gamma_0(2^e)}$ is multiplicity-free. We conclude this from the multiplicity-free property of $\pi_{G(2^e)}$ which was proved in Corollary 4.1. To begin with, recall that the restriction of $\pi_{G(q)}$ to $Q(q)$ coincides evidently with $\pi_{Q(q)}$ and that according to Lemma 4.2 $Q(q)$ is a normal subgroup of $G(q)$ of index two for $q = 2^e$ and $e \ge 2$. Hence, according to Lemma 4.28 the restriction of each irreducible subrepresentation of $\pi_{G(q)}$ to $Q(q)$ is an irreducible subrepresentation of $\pi_{Q(q)}$ or it decomposes into two non-isomorphic irreducible subrepresentations of $\pi_{Q(q)}$.

As mentioned, $\pi_{G(2^3)}$ and $\pi_{Q(2^3)}$ are multiplicity-free and hence according to Lemmas 4.23 and 4.26 each of them contains four irreducible subrepresentations. Since $\pi_{G(2^3)}$ and $\pi_{Q(2^3)}$ have the same number of irreducible subrepresentations, the first item of Lemma 4.28 must hold and hence each of the four irreducible subrepresentations of $\pi_{G(2^3)}$ restricted to $Q(2^3)$ must be an irreducible subrepresentation of $\pi_{Q(2^3)}$. Thus, if the decomposition of $\pi_{G(2^3)}$ into irreducible representations is given by

$$\pi_{G(2^3)} = \overline{\pi}_1 \oplus \overline{\pi}_2 \oplus \overline{\pi}_3 \oplus \overline{\pi}_4 \tag{4.331}$$

then the decomposition of $\pi_{Q(2^3)}$ into irreducible representations is given by

$$\pi_{Q(2^3)} = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4 \tag{4.332}$$

where $\overline{\pi}_i$ and π_i are irreducible representations of $G(q)$ and $Q(q)$, respectively such that

$$\overline{\pi}_i \downarrow Q(2^3) = \pi_i, \quad i = 1, 2, 3, 4. \tag{4.333}$$

On the other hand, according to results in section 4.4, $\pi_{G(2^e)}$ with $e \ge 5$ contains the subrepresentation $\pi_{G(2^3),\overline{X}(2^e)} \cong \pi_{G(2^3)}$. Form this and since by Corollary 4.1 $\pi_{G(2^e)}$ is multiplicity-free, we can assume that the decomposition of $\pi_{G(2^e)}$ for $e \geq 5$ is given by

$$\pi_{G(2^e)} \cong \overline{\pi_1} \oplus \overline{\pi_2} \oplus \overline{\pi_3} \oplus \overline{\pi_4} \oplus \overline{\pi_5} \oplus \ldots \oplus \overline{\pi}_{\overline{N}} \tag{4.334}$$

where $\overline{\pi}_i$, $1 \leq i \leq 4$ are given in (4.331) and $\overline{N}$ denotes the number of non-isomorphic irreducible subrepresentations of $\pi_{G(2^e)}$ (or that of $U_{\overline{\Gamma}_0(2^e)}$) which according to Lemma 4.26 is given by

$$\overline{N} = \begin{cases} 7, & e = 5, \\ 2(e-2), & e \geq 6. \end{cases} \tag{4.335}$$

Then for the representation $\pi_{Q(2^e)}$ for $e \geq 5$ we have

$$\begin{aligned} &\pi_{Q(2^e)} = \pi_{G(2^e)} \downarrow Q(q) \cong \\ &\overline{\pi_1} \downarrow Q(q) \oplus \overline{\pi_2} \downarrow Q(q) \oplus \overline{\pi_3} \downarrow Q(q) \oplus \overline{\pi_4} \downarrow Q(q) \oplus \\ &\oplus \overline{\pi_5} \downarrow Q(q) \oplus \ldots \oplus \overline{\pi}_{\overline{N}} \downarrow Q(q) \end{aligned} \tag{4.336}$$

or, according to 4.333

$$\begin{aligned} &\pi_{Q(2^e)} = \pi_{G(2^e)} \downarrow Q(q) \cong \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4 \oplus \\ &\oplus \overline{\pi_5} \downarrow Q(q) \oplus \ldots \oplus \overline{\pi}_{\overline{N}} \downarrow Q(q) \end{aligned} \tag{4.337}$$

On the other hand, according to Lemma 4.28, for $4 \leq \alpha \leq \overline{N}$ we can assume that

$$\overline{\pi}_i \downarrow Q(q) = \pi_i, \quad 1 \leq i \leq \alpha \tag{4.338}$$

and

$$\overline{\pi}_i \downarrow Q(q) = \pi_i \oplus \pi_i', \quad \alpha < i \leq \overline{N} \tag{4.339}$$

where π_i and π_i' are non-isomorphic irreducible representations of $Q(2^e)$. Since all subrepresentations $\overline{\pi}_i$ of $\pi_{G(2^e)}$ are non-isomorphic, from the last part of Lemma 4.28 it follows that the irreducible representations π_i and π_i' of $Q(2^e)$ for $\alpha \leq i \leq \overline{N}$ are non-isomorphic. But for $5 \leq i \leq \alpha$ it can happen that some of π_i's are isomorphic. Hence, from (4.337) for the number N of non-isomorphic irreducible subrepresentations of $\pi_{Q(2^e)}$ it follows that

$$N \leq \alpha + 2(\overline{N} - \alpha). \tag{4.340}$$

Comparing this with the number of non-isomorphic irreducible subrepresentations of $\pi_{Q(2^e)}$ in Lemma 4.23 and (4.335) it follows that (4.340) holds if and only if $\alpha = 4$ and this yields equality in (4.340). That is, the decomposition of $\pi_{Q(2^e)}$ for $e \geq 5$ can be written as

$$\begin{aligned} &\pi_{Q(2^e)} \cong \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4 \oplus \\ &\oplus \pi_5 \oplus \pi_5' \oplus \ldots \oplus \pi_{\overline{N}} \oplus \pi_{\overline{N}}' \end{aligned} \tag{4.341}$$

Since all representations in the right hand side of this equation are non-isomorphic irreducible representations of $Q(2^e)$, $\pi_{Q(2^e)}$ is multiplicity-free. $\square$

Now we can prove the following:

LEMMA 4.30. *The representations* $U_{\Gamma_0(n)}$ *and* $U_{\overline{\Gamma}_0(n)}$ *with* $n \in \mathbb{N}$ *are multiplicity-free.*

PROOF. First we note that if (G_1, H_1) and (G_2, H_2) are Gelfand pairs then $(G_1 \times G_2, H_1 \times H_2)$ is a Gelfand pair, where $\times$ denotes the direct product of groups ([**20**], page 279, Lemma 2.4). We proved that $U_{\Gamma_0(q)}$ for $q = p^e$ and hence $\pi_{Q(q)} = ind_{K(q)}^{Q(q)}(1)$ is multiplicity-free. Thus $(Q(q), K(q))$ is a Gelfand pair. But as shown in section 4.1, for coprimes q and q' the group $Q(qq')$ is the direct product of the groups $Q(q)$ and $Q(q')$ and $K(qq')$ is the direct product of the groups $K(q)$ and $K(q')$. Therefore for any $n = \prod_i p_i^{e_i}$ $(Q(n), K(n))$ is a Gelfand pair. Thus $\pi_{Q(n)} = ind_{K(n)}^{Q(n)}(1)$ and hence $U_{\Gamma_0(n)}$ is multiplicity-free. Since the representation $U_{\Gamma_0(n)}$ is obtained by the restriction of $U_{\overline{\Gamma}_0(n)}$ to $\mathrm{PSL}(2, \mathbb{Z})$, it follows that $U_{\overline{\Gamma}_0(n)}$ must be multiplicity-free too. $\square$

CHAPTER 5

Atkin-Lehner type theory of old and new eigenfunctions of Mayer's transfer operator

In this chapter we consider the decomposition of Mayer's transfer operator and the space of its eigenfunctions. We use the Mathematica software for the calculations including matrix multiplication of vectors of high dimensions.

5.1. Decomposition of Mayer's transfer operator and Atkin-Lehner theory

First, we recall from [**23**] the notion of old and new Maass cusp forms for the Hecke congruence subgroups with trivial character. This is analogous to Atkin-Lehner's notion of old and new for the classical holomorphic automorphic functions [**3**].

For simplicity reasons, whenever χ is the trivial character, we put $\mathcal{S}(s;\Gamma;\chi)=\mathcal{S}(s;\Gamma)$ and $\mathcal{S}(\Gamma;\chi)=\mathcal{S}(\Gamma)$. Consider the matrix

$$\omega_d := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

By simple calculation, one can show that for $m \mid n$ and for any $dm \mid n$,

$$\omega_d \Gamma_0(n) \omega_d^{-1} \subseteq \Gamma_0(m). \tag{5.2}$$

Hence, if $v \in \mathcal{S}(\Gamma_0(m))$ we have for any $\gamma \in \Gamma_0(n)$

$$v(d\gamma z) = v(\omega_d \gamma z) = v(\omega_d \gamma \omega_d^{-1} \omega_d z). \tag{5.3}$$

But v is $\Gamma_0(m)$ invariant and according to (5.2) $\omega_d\gamma\omega_d^{-1} \in \Gamma_0(m)$, hence, for any $\gamma \in \Gamma_0(n)$ we have

$$v(d\gamma z) = v(dz), \tag{5.4}$$

that is, $v(dz) \in \mathcal{S}(\Gamma_0(n))$.

Definition 5.1. *For $m \mid n$ let $v(z) \in \mathcal{S}(\Gamma_0(m))$. Then for any d with $dm \mid n$, $u(z) := v(dz) \in \mathcal{S}(\Gamma_0(n))$ is called an old Maass cusp form of $\Gamma_0(n)$ coming from $\Gamma_0(m)$. We denote by $\mathcal{S}^{old}(\Gamma_0(n))$ the space of all old Maass cusp forms of $\Gamma_0(n)$. The space of old Maass cusp forms of $\Gamma_0(n)$ with spectral parameter s is defined to be*

$$\mathcal{S}^{old}(s;\Gamma_0(n)) := \mathcal{S}^{old}(\Gamma_0(n)) \cap \mathcal{S}(s;\Gamma_0(n)). \tag{5.5}$$

Definition 5.2. *The space of new Maass cusp forms of* $\Gamma_0(n)$ *is defined to be the orthogonal complement of* $\mathcal{S}^{old}(\Gamma_0(n))$ *in* $\mathcal{S}(\Gamma_0(n))$*. The space of new Maass cusp forms of* $\Gamma_0(n)$ *with spectral parameter* s *is defined to be*

$$\mathcal{S}^{new}(s;\Gamma_0(n)) := \mathcal{S}^{new}(\Gamma_0(n)) \cap \mathcal{S}(s;\Gamma_0(n)). \tag{5.6}$$

Thus, by definition we have

$$\mathcal{S}(\Gamma_0(n)) = \mathcal{S}^{old}(\Gamma_0(n)) \oplus \mathcal{S}^{new}(\Gamma_0(n)) \tag{5.7}$$

and

$$\mathcal{S}(s;\Gamma_0(n)) = \mathcal{S}^{old}(s;\Gamma_0(n)) \oplus \mathcal{S}^{new}(s;\Gamma_0(n)). \tag{5.8}$$

As we mentioned in section 3.2, the space $\mathcal{S}(s;\Gamma_0(n))$ is isomorphic to the space $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ and this isomorphism is described by the map B given in (3.26). Hence, we can distinguish the new and old subspaces of $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ as follows

$$\begin{aligned}\mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) &= B\mathcal{S}^{old}(s;\Gamma_0(n)),\\ \mathcal{S}^{new}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) &= B\mathcal{S}^{new}(s;\Gamma_0(n)).\end{aligned} \tag{5.9}$$

Indeed, for $u \in \mathcal{S}^{old}(s;\Gamma_0(n))$ and $v \in \mathcal{S}^{new}(s;\Gamma_0(n))$ with notations as in Lemma 3.2, we have

(5.10)

$$\int_F \langle Bu(z), Bv(z)\rangle_V \, d\mu(z) = \sum_{i=1}^{\mu_{\Gamma_0(n)}} \int_F \langle u(r_i z), v(r_i z)\rangle_W \, d\mu(z) =$$
$$\sum_{i=1}^{\mu_{\Gamma_0(n)}} \int_{r_i F} \langle u(z), v(z)\rangle_W \, d\mu(r_i^{-1} z) = \int_{\cup_{i=1}^{\mu_{\Gamma_0(n)}} r_i F} \langle u(z), v(z)\rangle_W \, d\mu(z) =$$
$$\int_{F_{\Gamma_0(n)}} \langle u(z), v(z)\rangle_W \, d\mu(z) = 0$$

where in the third equality we used invariance of the measure $d\mu(z)$ under the $\mathrm{PSL}(2,\mathbb{Z})$ action, in the fourth equality we used the fact that $F_{\Gamma_0(n)} = \cup_{i=1}^{\mu_{\Gamma_0(n)}} r_i F$ with F the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$, and the last equality follows from the orthogonality of the old Maass cusp form u and the new Maass cusp form v. Therefore the space of new Maass cusp forms $\mathcal{S}^{new}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ is the orthogonal complement of $\mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ in $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ and we have

$$\begin{aligned}&\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) =\\ &\mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) \oplus \mathcal{S}^{new}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}).\end{aligned} \tag{5.11}$$

Besides this decomposition of $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ into the subspaces of old and new forms, we can also decompose this space using a decomposition of the representation $U_{\Gamma_0(n)}$. Evidently, for a representation χ of Γ with $\chi = \chi_1 \oplus \chi_2$ we have

$$\mathcal{S}(s;\Gamma,\chi) = \mathcal{S}(s;\Gamma,\chi_1) \oplus \mathcal{S}(s;\Gamma,\chi_2). \tag{5.12}$$

Thus a decomposition of $U_{\Gamma_0(n)}$ provides us with a decomposition of $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ and hence of $\mathcal{S}(s;\Gamma_0(n))$. As we see in the next section for some examples, in general this decomposition does not coincide with the Atkin-Lehner decomposition into old and new Maass cusp forms. In particular this decomposition of $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ via a decomposition of $U_{\Gamma_0(n)}$ does not allow in general to characterize the space of new forms. But we can distinguish this way part of $\mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$. For instance, as mentioned in Section 4.4, the representation $U_{\Gamma_0(d)}$ with $d \mid n$ occurs in a decomposition of $U_{\Gamma_0(n)}$ and hence

$$\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(d)}) \subset \mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) \tag{5.13}$$

or equivalently $\mathcal{S}(s;\Gamma_0(d)) \subset \mathcal{S}^{old}(s;\Gamma_0(n))$ as expected from Atkin-Lehner's theory.

Recall from Theorem 3.10 that $F(s;\Gamma;\chi)$ denotes the space of eigenfunctions of the transfer operator $P\mathcal{L}_s^{\Gamma,\chi,+}$ in $\oplus_{i=1}^{\mu_{\Gamma_0(n)}\dim\chi} B(D)$ with eigenvalue $\lambda = \pm 1$. For χ the trivial character we put $F(s;\Gamma;\chi) = F(s;\Gamma)$ and $P\mathcal{L}_s^{\Gamma,\chi,+} = P\mathcal{L}_s^{\Gamma,+}$. According to Theorem 3.10 the space $F(s;\Gamma_0(n))$ and the space $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)})$ are in bijection via the transformation $\mathcal{P}\circ\mathcal{I}$. Hence we can distinguish the old and new subspace of $F(s;\Gamma_0(n))$ as follows

$$\begin{aligned}&F^{new}(s;\Gamma_0(n)) := \mathcal{P}\circ\mathcal{I}\mathcal{S}^{new}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) = \\ &\mathcal{P}\circ\mathcal{I}\circ B\mathcal{S}^{new}(s;\Gamma_0(n))\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}&F^{old}(s;\Gamma_0(n)) := \mathcal{P}\circ\mathcal{I}\mathcal{S}^{old}(s;\mathrm{PSL}(2,\mathbb{Z}),U_{\Gamma_0(n)}) = \\ &\mathcal{P}\circ\mathcal{I}\circ B\mathcal{S}^{old}(s;\Gamma_0(n)).\end{aligned} \tag{5.15}$$

Now let $\mathcal{M}$ be a matrix which decomposes the representation $U_{\Gamma_0(n)}$ into its irreducible components, that is,

$$\mathcal{M}U_{\Gamma_0(n)}\mathcal{M}^{-1} = \oplus_{i=1}^{N}\theta_i, \tag{5.16}$$

and decomposes the symmetry P as

$$\mathcal{M}P\mathcal{M}^{-1} = \oplus_{i=1}^{N}P_i, \tag{5.17}$$

where N is the number of non-isomorphic irreducible subrepresentations of $U_{\Gamma_0(n)}$, θ_i denotes an irreducible subrepresentation of $U_{\Gamma_0(n)}$, and $\dim P_i = \dim\theta_i$. That such a decomposition is possible will be shown later. Then the transfer operator $P\mathcal{L}_s^{\Gamma_0(n),+}$ is decomposed via conjugation by $\mathcal{M}$ as follows:

$$\mathcal{M}P\mathcal{L}_s^{\Gamma_0(n),+}\mathcal{M}^{-1} = \oplus_{i=1}^{N}P_i\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),\theta_i,+}. \tag{5.18}$$

Obviously, the spectra of $\mathcal{M}P\mathcal{L}_s^{\Gamma_0(n),+}\mathcal{M}^{-1}$ and $P\mathcal{L}_s^{\Gamma_0(n),+}$ are the same and we have

$$\det(1 \pm P\mathcal{L}_s^{\Gamma_0(n),+}) = \prod_{i=1}^{N} \det(1 \pm P_i\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),\theta_i,+}). \tag{5.19}$$

This and (3.23) lead to a factorization of Selberg's zeta function of the form

$$Z(s;\Gamma_0(n)) = \prod_{i=1}^{N} Z(s;\mathrm{PSL}(2,\mathbb{Z});\theta_i). \tag{5.20}$$

Moreover, the space $F(s;\Gamma_0(n))$ is decomposed as

$$F(s;\Gamma_0(n)) = \oplus_{i=1}^{N} F(s;\mathrm{PSL}(2,\mathbb{Z});\theta_i). \tag{5.21}$$

In the same way we have

$$\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_{\Gamma_0(n)}) = \oplus_{i=1}^{N} \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});\theta_i). \tag{5.22}$$

As we illustrate in the next section by an example, the decomposition of $U_{\Gamma_0(n)}$ into irreducibles leads in general to a subspace containing part of $F^{old}(s;\Gamma_0(n))$ and a mixed subspace of $F(s;\Gamma_0(n))$ which contains both $F^{new}(s;\Gamma_0(n))$ and another part of $F^{old}(s;\Gamma_0(n))$. Hence, if (5.16) is a decomposition of $U_{\Gamma_0(n)}$ into irreducibles then (5.21) can be considered as an Atkin-Lehner type decomposition of the space $F(s;\Gamma_0(n))$ of eigenfunctions of Mayer's transfer operator.

5.2. Decomposition of Mayer's transfer operator for $\Gamma_0(6)$

In this section we discuss in detail the decomposition of the transfer operator $P\mathcal{L}_s^{\Gamma_0(6),+}$ for $\Gamma_0(6)$ with trivial character and the space of its eigenfunctions $F(s;\Gamma_0(6))$. We fix the set of representatives for this group as follows

$$\begin{aligned} &R(\Gamma_0(6)\backslash\mathrm{PSL}(2,\mathbb{Z})) = R(\overline{\Gamma}_0(6)\backslash\mathrm{PGL}(2,\mathbb{Z})) = \\ &\{id, S, ST, ST^2, STS, ST^3, ST^2S, ST^4, ST^3S, ST^2ST, \\ &ST^3ST, ST^2ST^2\}. \end{aligned} \tag{5.23}$$

Then the induced representations $U_{\Gamma_0(6)}$ and $U_{\overline{\Gamma}(6)}$ for the generators of $\mathrm{PSL}(2,\mathbb{Z})$ and $\mathrm{PGL}(2,\mathbb{Z})$, respectively, are given by
(5.24)

$$U_{\overline{\Gamma}_0(6)}(T) = U_{\Gamma_0(6)}(T) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0 \end{pmatrix},$$

(5.25)

$$U_{\overline{\Gamma}_0(6)}(S) = U_{\Gamma_0(6)}(S) = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix},$$

respectively

$$U_{\overline{\Gamma}_0(6)}(M) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}. \tag{5.26}$$

As mentioned in Lemma 4.7,

$$U_{\Gamma_0(6)} = U_{\Gamma_0(2)} \otimes U_{\Gamma_0(3)}, \quad U_{\overline{\Gamma}_0(6)} = U_{\overline{\Gamma}_0(2)} \otimes U_{\overline{\Gamma}_0(3)}. \tag{5.27}$$

From this together with the decompositions of $U_{\Gamma_0(2)}$ and $U_{\Gamma_0(3)}$ into irreducible representations given in Lemma 4.12 and the decompositions of $U_{\overline{\Gamma}_0(2)}$ and $U_{\overline{\Gamma}_0(3)}$ into irreducible representations given in Lemma 4.13, we obtain the decompositions of $U_{\Gamma_0(6)}$ and $U_{\overline{\Gamma}(6)}$ into irreducible representations, namely

$$M_{\Gamma_0(6)}U_{\Gamma_0(6)}M^{-1}_{\Gamma_0(6)} = U_t \oplus U_2 \oplus U_3 \oplus (U_2 \otimes U_3) \tag{5.28}$$

and

$$M_{\overline{\Gamma}_0(6)}U_{\overline{\Gamma}_0(6)}M^{-1}_{\overline{\Gamma}_0(6)} = \overline{U}_t \oplus \overline{U}_2 \oplus \overline{U}_3 \oplus (\overline{U}_2 \otimes \overline{U}_3) \tag{5.29}$$

where the representations in the right hand sides are given in the Examples 4.1 and 4.2. Having the right hand sides of these decompositions we can calculate the matrices $M_{\Gamma_0(6)}$ and $M_{\overline{\Gamma}_0(6)}$ as follows

$$M_{\Gamma_0(6)} = M_{\overline{\Gamma}_0(6)} =$$
$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & 1 \\
1 & \omega^2 & \omega & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & 1 \\
1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 \\
1 & \omega & \omega^2 & -3\omega & -3\omega^2 & \omega^2 & 1 & \omega & \omega^2 & -3 & \omega & 1 \\
1 & \omega^2 & \omega & -3\omega^2 & -3\omega & \omega & 1 & \omega^2 & \omega & -3 & \omega^2 & 1 \\
1 & \omega & -3\omega^2 & \omega & \omega^2 & \omega^2 & -3 & -3\omega & \omega^2 & 1 & \omega & 1 \\
1 & \omega^2 & -3\omega & \omega^2 & \omega & \omega & -3 & -3\omega^2 & \omega & 1 & \omega^2 & 1 \\
1 & -3\omega & \omega^2 & \omega & \omega^2 & -3\omega^2 & 1 & \omega & \omega^2 & 1 & \omega & -3 \\
1 & -3\omega^2 & \omega & \omega^2 & \omega & -3\omega & 1 & \omega^2 & \omega & 1 & \omega^2 & -3
\end{pmatrix} \tag{5.30}$$

where $\omega = \exp(\frac{2\pi i}{3})$. This way we get the following decomposition of the space of Maass cusp forms

$$\begin{aligned}
&\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_{\Gamma_0(6)}) \cong \\
&\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z})) \oplus \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2) \oplus \\
&\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_3) \oplus \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2 \otimes U_3).
\end{aligned} \tag{5.31}$$

As mentioned in Section 3.2, $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_{\Gamma_0(6)})$ is isomorphic to $\mathcal{S}(s;\Gamma_0(6))$ via the isomorphism B given in (3.26). The map B^{-1} induces isomorphisms between the subspaces in the right hand side of (5.31) and certain subspaces of $S(s;\Gamma_0(6))$ which we are going to determine next. For this we define first the following spaces

(5.32)

$$\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6)) := \left\{ v(z) \in \mathcal{S}(s;\Gamma_0(2)) \mid \sum_{r \in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} v(rz) = 0 \right\},$$

(5.33)

$$\mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6)) := \left\{ v(z) \in \mathcal{S}(s;\Gamma_0(3)) \mid \sum_{r \in R(\Gamma_0(3)\backslash \mathrm{PSL}(2,\mathbb{Z}))} v(rz) = 0 \right\},$$

and

(5.34)
$\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6)) :=$

$$\left\{ v(z) \in \mathcal{S}(s;\Gamma_0(6)) \mid \sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(2))} v(rz) = \sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(3))} v(rz) = 0 \right\}.$$

Let $u \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))$. Then obviously we have

$$\sum_{r\in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} u(rz) = 3u(z) \neq 0 \tag{5.35}$$

and hence $u \notin \mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$. Thus $\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$ is the space of old Maass cusp forms of $\Gamma_0(6)$ which are $\Gamma_0(2)$ Maass cusp forms and not $\mathrm{PSL}(2,\mathbb{Z})$ Maass cusp forms. In particular, $\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$ contains the old form

$$v(z) = u(2z) - \frac{1}{3} \sum_{r\in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} u(2rz). \tag{5.36}$$

and the new forms of $\Gamma_0(2)$. In fact, by definition for $v \in \mathcal{S}^{new}(s;\Gamma_0(2))$ and $u \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))$ we have

$$\int_{F_{\Gamma_0(2)}} \langle v(z), u(z)\rangle \, d\mu(z) = 0 \tag{5.37}$$

where $F_{\Gamma_0(2)}$ denotes the fundamental domain of $\Gamma_0(2)$ and $\langle a,b\rangle = a\overline{b}$ with $\overline{b}$ the complex conjugate of b. On the other hand, we have

$$F_{\Gamma_0(2)} = \sqcup_{r\in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} rF \tag{5.38}$$

where F denotes the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$. Inserting this into (5.37), using the $\mathrm{PSL}(2,\mathbb{R})$-invariance of the measure and observing that $u \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))$ we get

$$\int_F \left\langle \sum_{r\in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} v(rz), u(z) \right\rangle d\mu(z) = 0 \tag{5.39}$$

and hence

$$\sum_{r\in R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z}))} v(rz) = 0, \tag{5.40}$$

because otherwise for $u(z)$ coinciding with the first argument in the Hermitian form we get a contradiction. That means all new forms of $\Gamma_0(2)$ fulfil the desired condition to be an element of $\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$. Similarly, $\mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6))$ is the space old Maass cusp forms of $\Gamma_0(6)$ which are $\Gamma_0(3)$ Maass cusp forms and not $\mathrm{PSL}(2,\mathbb{Z})$ Maass cusp forms.

In particular, $\mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6))$ contains the new forms of $\Gamma_0(3)$ and the old form

$$v(z) = u(3z) - \frac{1}{4}\sum_{r\in R(\Gamma_0(3)\backslash \mathrm{PSL}(2,\mathbb{Z}))} u(3rz) \tag{5.41}$$

where $u \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))$. Moreover, $\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$ is the space of Maass cusp forms of $\Gamma_0(6)$ which are neither $\Gamma_0(3)$ nor $\Gamma_0(2)$ Maass cusp forms.

Lemma 5.1. *The following isomorphisms of spaces hold:*

$$\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6)) \cong \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2), \tag{5.42}$$

$$\mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6)) \cong \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_3), \tag{5.43}$$

and

$$\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6)) \cong \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3). \tag{5.44}$$

Proof. We denote by r_i, the i-th element of $R(\Gamma_0(6)\backslash \mathrm{PSL}(2,\mathbb{Z}))$ given in (5.23). We fix

$$R(\Gamma_0(2)\backslash \mathrm{PSL}(2,\mathbb{Z})) = \{r_1, r_2, r_3\}. \tag{5.45}$$

Recall that the isomorphism

$$B : \mathcal{S}(s;\Gamma_0(6)) \to \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_{\Gamma_0(6)}) \tag{5.46}$$

is given by

$$B\mathfrak{f}(z) = (\mathfrak{f}(r_1z), \mathfrak{f}(r_2z), \ldots, \mathfrak{f}(r_{12}z))^t. \tag{5.47}$$

Consider then the restriction of B to $\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$. For an element v in $\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))$, Bv has the components

$$(Bv)_i = v(r_iz), \quad 1 \le i \le 12. \tag{5.48}$$

It is easy to verify that

$$\begin{aligned} r_1 \equiv r_7 \equiv r_{10} \equiv r_{12} \mod \Gamma_0(2) \\ r_2 \equiv r_4 \equiv r_8 \equiv r_{11} \mod \Gamma_0(2) \\ r_3 \equiv r_6 \equiv r_5 \equiv r_9 \mod \Gamma_0(2) \end{aligned} \tag{5.49}$$

where $r_i \equiv r_j \mod \Gamma_0(2)$ means that $\Gamma_0(2)r_i = \Gamma_0(2)r_j$. Hence, observing that $v(z)$ is $\Gamma_0(2)$-invariant we get

$$\begin{aligned} (Bv)_1 = (Bv)_7 = (Bv)_{10} = (Bv)_{12} \\ (Bv)_2 = (Bv)_4 = (Bv)_8 = (Bv)_{11} \\ (Bv)_3 = (Bv)_6 = (Bv)_5 = (Bv)_9. \end{aligned} \tag{5.50}$$

From these identities it follows that for $M_{\Gamma_0(6)}$ given in (5.30) one finds

$$M_{\Gamma_0(6)}Bv = (0, (M_{\Gamma_0(6)}Bv)_2, (M_{\Gamma_0(6)}Bv)_3, 0,0,0,0,0,0,0,0,0)^t \tag{5.51}$$

with

$$(M_{\Gamma_0(6)}Bv)_2(z) = 4[v(z) + \omega^2 v(r_2z) - \omega v(r_3z)] \tag{5.52}$$

respectively

$$(M_{\Gamma_0(6)}Bv)_3(z) = 4[v(z) - \omega v(r_2z) + \omega^2 v(r_3z)]. \tag{5.53}$$

Thus, $M_{\Gamma_0(6)}Bv \in \mathcal{S}(s; \mathrm{PSL}(2,\mathbb{Z}); U_2)$.

On the other hand, let

$$(w_1, w_2)^t \in \mathcal{S}(s; \mathrm{PSL}(2,\mathbb{Z}), U_2). \tag{5.54}$$

Then from the transformation property of the elements of this space under the representation U_2 of $\mathrm{PSL}(2,\mathbb{Z})$ we get (see (4.134))

$$\begin{aligned} (w_1(Tz), w_2(Tz))^t &= (w_2(z), w_1(z))^t, \\ (w_1(Sz), w_2(Sz))^t &= (\omega w_2(z), \omega^2 w_1(z))^t. \end{aligned} \tag{5.55}$$

For

$$w = (0, w_1, w_2, 0, 0, 0, 0, 0, 0, 0, 0, 0)^t \tag{5.56}$$

we can calculate $M^{-1}_{\Gamma_0(6)}w$ and get

$$v := (M^{-1}_{\Gamma_0(6)}w)_1 = \frac{1}{12}(w_1 + w_2). \tag{5.57}$$

Now we are going to prove that $v \in \mathcal{S}_{\Gamma_0(2)}(s; \Gamma_0(6))$. From the identities in (5.55) it follows that $v = \frac{1}{12}(w_1+w_2)$ is invariant under the generators of $\Gamma_0(2)$, namely T and ST^2S (see for example [**42**]) and that $v(r_1z) + v(r_2z) + v(r_3z) = 0$. This completes the proof of the first assertion.

For the second assertion the proof follows the same line of arguments. We fix

$$R(\Gamma_0(3)\backslash \mathrm{PSL}(2,\mathbb{Z})) = R(\overline{\Gamma}_0(3)\backslash \mathrm{PGL}(2,\mathbb{Z})) = \{r_1, r_2, r_3, r_4\} \tag{5.58}$$

and consider the restriction of B to $\mathcal{S}_{\Gamma_0(3)}(s; \Gamma_0(6))$

$$(Bv)_i = v(r_iz), \quad 1 \le i \le 12, \quad v \in \mathcal{S}_{\Gamma_0(3)}(s; \Gamma_0(6)). \tag{5.59}$$

For the representatives of $\Gamma_0(6)\backslash \mathrm{PSL}(2,\mathbb{Z})$ we have

$$\begin{aligned} r_1 \equiv r_9 \equiv r_{11} &\mod \Gamma_0(3) \\ r_2 \equiv r_6 \equiv r_{12} &\mod \Gamma_0(3) \\ r_3 \equiv r_7 \equiv r_8 &\mod \Gamma_0(3) \\ r_4 \equiv r_5 \equiv r_{10} &\mod \Gamma_0(3), \end{aligned} \tag{5.60}$$

By using this and the fact that v is $\Gamma_0(3)$-invariant we get

$$\begin{aligned} &M_{\Gamma_0(6)}Bv = \\ &(0, 0, 0, (M_{\Gamma_0(6)}Bv)_4, (M_{\Gamma_0(6)}Bv)_5, (M_{\Gamma_0(6)}Bv)_6, 0, 0, 0, 0, 0, 0)^t \end{aligned} \tag{5.61}$$

with

$$(M_{\Gamma_0(6)}Bv)_4(z) = 3[v(z) + v(r_2z) + v(r_3z) - 3v(r_4z)], \tag{5.62}$$

$$(M_{\Gamma_0(6)}Bv)_5(z) = 3[v(z) + v(r_2z) - 3v(r_3z) + v(r_4z)], \tag{5.63}$$

and

$$(M_{\Gamma_0(6)}Bv)_6(z) = 3[v(z) - 3v(r_2 z) + v(r_3 z) + v(r_4 z)]. \tag{5.64}$$

Hence, $M_{\Gamma_0(6)}Bv \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_3)$.

On the other hand, let

$$(w_1, w_2, w_3)^t \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_3). \tag{5.65}$$

Then from the transformation property of $(w_1,w_2,w_3)^t$ under the representation U_3 of $\mathrm{PSL}(2,\mathbb{Z})$ we get from (4.141)

$$(w_1(Tz), w_2(Tz), w_3(Tz))^t = (w_3(z), w_1(z), w_2(z))^t \tag{5.66}$$

and

$$(w_1(Sz), w_2(Sz), w_3(Sz))^t = (w_2(z), w_1(z), -w_1(z) - w_2(z) - w_3(z))^t. \tag{5.67}$$

Now for

$$w = (0,0,0,w_1,w_2,w_3,0,0,0,0,0,0)^t \tag{5.68}$$

we get by a simple calculation

$$v := (M_{\Gamma_0(6)}^{-1} w)_1 = \frac{1}{12}(w_1 + w_2 + w_3). \tag{5.69}$$

But from the identities (5.66) and (5.67), it follows that v is invariant under the generators of $\Gamma_0(3)$, namely T and ST^3S (see for example [**42**]) and also $v(r_1z) + v(r_2z) + v(r_3z) + v(r_4z) = 0$. Thus $v \in \mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6))$ and this completes the proof of the second assertion.

To prove the third assertion we fix

$$R(\Gamma_0(6)\backslash\Gamma_0(2)) = \{r_1, r_7, r_{10}, r_{12}\} \tag{5.70}$$

and

$$R(\Gamma_0(6)\backslash\Gamma_0(3)) = \{r_1, r_9, r_{11}\}\,. \tag{5.71}$$

Then for $v \in \mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$ we have by definition

$$\sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(2))} v(rz) = v(r_1z) + v(r_7z) + v(r_{10}z) + v(r_{12}z) = 0 \tag{5.72}$$

and

$$\sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(3))} v(rz) = v(r_1z) + v(r_9z) + v(r_{11}z) = 0. \tag{5.73}$$

Hence we get

$$\begin{aligned} &v(r_1Sz) + v(r_7Sz) + v(r_{10}Sz) + v(r_{12}Sz) = 0,\\ &v(r_1STz) + v(r_7STz) + v(r_{10}STz) + v(r_{12}STz) = 0,\\ &v(r_1Sz) + v(r_9Sz) + v(r_{11}Sz) = 0,\\ &v(r_1STz) + v(r_9STz) + v(r_{11}STz) = 0,\\ &v(r_1ST^2z) + v(r_9ST^2z) + v(r_{11}ST^2z) = 0. \end{aligned} \tag{5.74}$$

But for the representatives $r_i \in R(\Gamma_0(6)\backslash \mathrm{PSL}(2,\mathbb{Z}))$ we have

$$\begin{aligned}
&r_1 S \equiv r_2 \mod \Gamma_0(6),\\
&r_7 S \equiv r_4 \mod \Gamma_0(6),\\
&r_{10} S \equiv r_8 \mod \Gamma_0(6),\\
&r_{12} S \equiv r_{11} \mod \Gamma_0(6),\\
&r_1 ST \equiv r_3 \mod \Gamma_0(6),\\
&r_7 ST \equiv r_6 \mod \Gamma_0(6),\\
&r_{10} ST \equiv r_5 \mod \Gamma_0(6),\\
&r_{12} ST \equiv r_9 \mod \Gamma_0(6)
\end{aligned} \tag{5.75}$$

respectively

$$\begin{aligned}
&r_1 S \equiv r_2 \mod \Gamma_0(6),\\
&r_9 S \equiv r_6 \mod \Gamma_0(6),\\
&r_{11} S \equiv r_{12} \mod \Gamma_0(6),\\
&r_1 ST \equiv r_3 \mod \Gamma_0(6),\\
&r_9 ST \equiv r_7 \mod \Gamma_0(6),\\
&r_{11} ST \equiv r_8 \mod \Gamma_0(6),\\
&r_1 ST^2 \equiv r_4 \mod \Gamma_0(6),\\
&r_9 ST^2 \equiv r_5 \mod \Gamma_0(6),\\
&r_{11} ST^2 \equiv r_{10} \mod \Gamma_0(6).
\end{aligned} \tag{5.76}$$

Therefore, (5.72), (5.73), and (5.74) can be written in terms of $(Bv)_i = v(r_i z)$ as

$$\begin{aligned}
&(Bv)_1 + (Bv)_7 + (Bv)_{10} + (Bv)_{12} = 0,\\
&(Bv)_2 + (Bv)_4 + (Bv)_8 + (Bv)_{11} = 0,\\
&(Bv)_3 + (Bv)_6 + (Bv)_5 + (Bv)_9 = 0,\\
&(Bv)_1 + (Bv)_9 + (Bv)_{11} = 0,\\
&(Bv)_2 + (Bv)_6 + (Bv)_{12} = 0,\\
&(Bv)_3 + (Bv)_7 + (Bv)_8 = 0,\\
&(Bv)_4 + (Bv)_5 + (Bv)_{10} = 0.
\end{aligned} \tag{5.77}$$

By using these identities, we get for $v \in \mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$

$$M_{\Gamma_0(6)} Bv = (0,0,0,0,0,0,w_1,w_2,w_3,w_4,w_5,w_6)^t, \tag{5.78}$$

where the components w_i can again be written explicitly in terms of the components of Bv. Conversely, let

$$(w_1,w_2,w_3,w_4,w_5,w_6)^t \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2 \otimes U_3). \tag{5.79}$$

Then we have

$$
\begin{aligned}
&(w_1(Tz), w_2(Tz), w_3(Tz), w_4(Tz), w_5(Tz), w_6(Tz))^t = \\
&(U_2 \otimes U_3)(T)(w_1(z), w_2(z), w_3(z), w_4(z), w_5(z), w_6(z))^t
\end{aligned} \tag{5.80}
$$

and

$$
\begin{aligned}
&(w_1(Sz), w_2(Sz), w_3(Sz), w_4(Sz), w_5(Sz), w_6(Sz))^t = \\
&(U_2 \otimes U_3)(S)(w_1(z), w_2(z), w_3(z), w_4(z), w_5(z), w_6(z))^t
\end{aligned} \tag{5.81}
$$

where

$$
(U_2 \otimes U_3)(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.82}
$$

and

$$
(U_2 \otimes U_3)(S) = \begin{pmatrix} 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & -\omega & 0 & -\omega \\ -\omega^2 & 0 & -\omega^2 & 0 & -\omega^2 & 0 \end{pmatrix}. \tag{5.83}
$$

Inserting (5.82) respectively (5.83) into (5.80) respectively (5.81) gives

$$
\begin{aligned}
w_1(Tz) &= w_6(z) \\
w_2(Tz) &= w_5(z) \\
w_3(Tz) &= w_2(z) \\
w_4(Tz) &= w_1(z) \\
w_5(Tz) &= w_4(z) \\
w_6(Tz) &= w_3(z)
\end{aligned} \tag{5.84}
$$

and

$$
\begin{gathered}
w_1(Sz) = \omega w_4(z) \\
w_2(Sz) = \omega^2 w_3(z) \\
w_3(Sz) = \omega w_2(z) \\
w_4(Sz) = \omega^2 w_1(z) \\
w_5(Sz) = -\omega w_2(z) - \omega w_4(z) - \omega w_6(z) \\
w_6(Sz) = -\omega^2 w_1(z) - \omega^2 w_3(z) - \omega^2 w_5(z).
\end{gathered} \tag{5.85}
$$

For

$$
w = (0, 0, 0, 0, 0, 0, w_1, w_2, w_3, w_4, w_5, w_6)^t \tag{5.86}
$$

we get by a simple calculation

$$
v := (M_{\Gamma_0(6)}^{-1} w)_1 = \frac{1}{12}(w_1 + w_2 + w_3 + w_4 + w_5 + w_6). \tag{5.87}
$$

Using identities (5.84) and (5.85), we can show that $(M^{-1}_{\Gamma_0(6)}w)_1$ is invariant under the generators of $\Gamma_0(6)$, namely the elements T, ST^6S and $ST^{-2}ST^3ST^2ST$ ([**42**], page 69), and that the identities

$$\sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(2))} v(rz) = \sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(3))} v(rz) = 0 \tag{5.88}$$

are fulfilled. Thus the proof is complete. □

According to this Lemma, we can characterize the old subspaces of $\mathcal{S}(s;\Gamma_0(6))$ coming from the over groups $\mathrm{PSL}(2,\mathbb{Z})$, $\Gamma_0(2)$, and $\Gamma_0(3)$ via the irreducible representations U_t, U_2, and U_3, respectively. Furtheremore, the representation $U_2\otimes U_3$ characterizes $\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$. This space contains evidently the new subspace $\mathcal{S}^{new}(s;\Gamma_0(6))$. But for $u\in\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))$ one can check that

$$\begin{aligned} v(z) := u(6z) &- \frac{1}{4}\sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(2))} u(6rz) \\ &- \frac{1}{3}\sum_{r\in R(\Gamma_0(6)\backslash\Gamma_0(3))} u(6rz) + \frac{1}{12}\sum_{r\in R(\Gamma_0(6)\backslash\mathrm{PSL}(2,\mathbb{Z}))} u(6rz). \end{aligned} \tag{5.89}$$

also belongs to $\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$. Now the question arises, if this Maass cusp form v is really nontrivial. Unfortunately, this is a difficult question which one can not answer easily. Nevertheless the transfer operator approach supports the nontriviality of v. Indeed, according to (4.134) and (4.141), the cusp at infinity is open for both representations U_2 and U_3 and hence they are regular representations. Therefore the tensor product $U_2\otimes U_3$ is also regular and the cusp of $\Gamma_0(6)$ at infinity is open. From this it follows that the transfer operator $P\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3;+}$ has a meromorphic continuation to the left of the line $\Re(s)=\frac{1}{2}$ and therefore its Fredholm determinant and hence Selberg's zeta function $Z(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$ is a meromorphic function in the complex s-plane. If $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$ would be isomorphic to $\mathcal{S}^{new}(s;\Gamma_0(6))$ and hence in bijection with $F^{new}(s;\Gamma_0(6))$, then according to the Jacquet-Langlands correspondence $Z(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$ must coincide with Selberg's zeta function for a cocompact group and hence it must be a holomorphic function. But this is in contradiction with the aforementioned meromorphy of $Z(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$, which supports the nontriviality of (5.89).

Thus $\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))$ must be a mixture of old and new Maass cusp forms. Therefore, the decomposition of $U_{\Gamma_0(6)}$ into irreducible representations does not characterize the space of new forms.

Since

$$MT = T^{-1}M, \quad MS = SM, \quad M^2 = id_{2\times 2} \tag{5.90}$$

and $U_{\overline{\Gamma}_0(6)}$ is a representation of $\mathrm{PGL}(2,\mathbb{Z})$ obviously we have

$$U_{\overline{\Gamma}_0(6)}(M)U_{\overline{\Gamma}_0(6)}(T) = U_{\overline{\Gamma}_0(6)}(T)^{-1}U_{\overline{\Gamma}_0(6)}(M), \tag{5.91}$$

$$U_{\overline{\Gamma}_0(6)}(M)U_{\overline{\Gamma}_0(6)}(S) = U_{\overline{\Gamma}_0(6)}(S)U_{\overline{\Gamma}_0(6)}(M), \tag{5.92}$$

and

$$U_{\overline{\Gamma}_0(6)}(M)^2 = id_{12\times 12}. \tag{5.93}$$

Let $P = U_{\overline{\Gamma}_0(6)}(M)$. Then observing that the restriction of $U_{\overline{\Gamma}_0(6)}$ to $\mathrm{PSL}(2,\mathbb{Z})$ coincides with $U_{\Gamma_0(6)}$, from the identities (5.91), (5.92), and (5.93) we get

$$PU_{\Gamma_0(6)}(T) = U_{\Gamma_0(6)}(T)^{-1}P, \tag{5.94}$$

$$PU_{\Gamma_0(6)}(S) = U_{\Gamma_0(6)}(S)P, \tag{5.95}$$

and

$$P^2 = id_{12\times 12}. \tag{5.96}$$

Thus P is a symmetry of Mayer's transfer operator $\mathcal{L}_s^{\Gamma_0(6)}$ (see Definition 3.3).

We consider now the transfer operator $P\mathcal{L}_s^{\Gamma_0(6),+}$ with the symmetry operator $P = U_{\overline{\Gamma}_0(6)}(M)$. Evidently, we have

$$\begin{aligned} &M_{\Gamma_0(6)}P\mathcal{L}_s^{\Gamma_0(6),+}M_{\Gamma_0(6)}^{-1} = \\ &P_1\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),+} \oplus P_2\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2,+} \oplus P_3\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_3,+} \oplus \\ &P_{2\times 3}\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2\otimes U_3,+} \end{aligned} \tag{5.97}$$

where

$$M_{\Gamma_0(6)}PM_{\Gamma_0(6)}^{-1} = P_1 \oplus P_2 \oplus P_3 \oplus P_{2\times 3} \tag{5.98}$$

and

$$P_1 = \overline{U}_t(M), \quad P_2 = \overline{U}_2(M), \quad P_3 = \overline{U}_3(M), \quad P_{2\times 3} = \overline{U}_2 \otimes \overline{U}_3(M). \tag{5.99}$$

The representations $\overline{U}_t$, $\overline{U}_2$, and $\overline{U}_3$ are given in Example 4.1 and 4.2. Then the space of eigenfunctions $F(s;\Gamma_0(6))$ of $P\mathcal{L}_s^{\Gamma_0(6),+}$ is decomposed as the following

$$\begin{aligned} &F(s;\Gamma_0(6)) \cong M_{\Gamma_0(6)}F(s;\Gamma_0(6)) = F(s;\mathrm{PSL}(2,\mathbb{Z}))\oplus \\ &F(s;\mathrm{PSL}(2,\mathbb{Z});U_2) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_3) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3). \end{aligned} \tag{5.100}$$

According to Theorem 3.11 and Lemma 5.1 we get the following bijections

(5.101)
$$\begin{aligned}
&\mathcal{P}\circ\mathcal{I}\circ M_{\Gamma_0(6)}\circ B:\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z}))\to F(s;\mathrm{PSL}(2,\mathbb{Z}))\\
&\mathcal{P}\circ\mathcal{I}\circ M_{\Gamma_0(6)}\circ B:\mathcal{S}_{\Gamma_0(2)}(s;\Gamma_0(6))\to F(s;\mathrm{PSL}(2,\mathbb{Z});U_2)\\
&\mathcal{P}\circ\mathcal{I}\circ M_{\Gamma_0(6)}\circ B:\mathcal{S}_{\Gamma_0(3)}(s;\Gamma_0(6))\to F(s;\mathrm{PSL}(2,\mathbb{Z});U_3)\\
&\mathcal{P}\circ\mathcal{I}\circ M_{\Gamma_0(6)}\circ B:\mathcal{S}_{\Gamma_0(6)}(s;\Gamma_0(6))\to F(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)
\end{aligned}$$

where $M_{\Gamma_0(6)}$ acts on the vector Bv as a matrix.

5.3. The space of new eigenfunctions of Mayer's transfer operator for $\Gamma(2)$

As we mentioned in the previous section, by the decomposition of $U_{\Gamma_0(6)}$ into irreducibles we cannot distinguish the subspace of new eigenfunctions of Mayer's transfer operator. Indeed, we obtained in the case of the group $\Gamma_0(6)$ the mixed subspace $F(s;\mathrm{PSL}(2,\mathbb{Z});U_2\otimes U_3)$, which contains the subspace of the new eigenfunctions and part of the old eigenfunctions. But there are examples where we can distinguish the subspace consisting only of new eigenfunctions via an irreducible representation of $\mathrm{PSL}(2,\mathbb{Z})$. Here we recall the example of Maass cusp forms for the principal congruence group $\Gamma(2)$ from Balslev and Venkov [**5**] and Mayer's transfer operator for $\Gamma(2)$ with trivial character as already discussed by Chang [**13**].

First, we need to determine the representation $U_{\Gamma(2)}$ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial character on $\Gamma(2)$. For this we fix the following ordered set of representatives of the right cosets of $\Gamma(2)$ in $\mathrm{PSL}(2,\mathbb{Z})$

$$R(\Gamma(2)\backslash\mathrm{PSL}(2,\mathbb{Z}))=\{id,S,T,ST,TS,STS\}\,. \tag{5.102}$$

With this choice of representatives the induced representations $U_{\Gamma(2)}$ of the generators S and T of $\mathrm{PSL}(2,\mathbb{Z})$ are given by

$$\begin{aligned}
U_{\Gamma(2)}(S)&=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{pmatrix},\\
U_{\Gamma(2)}(T)&=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}.
\end{aligned} \tag{5.103}$$

Since $\Gamma(2)$ is a normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, the set of right and left cosets of $\Gamma(2)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is the same and has a group structure.

We denote this group by $\mathcal{G}(2)$, that is (see for example [**40**], page 90, Proposition 2.2.),

$$\begin{aligned} \mathcal{G}(2) &:= \mathrm{PSL}(2,\mathbb{Z})\backslash\Gamma(2) = \\ &\{\Gamma(2), S\,\Gamma(2), T\,\Gamma(2), ST\,\Gamma(2), TS\,\Gamma(2), STS\,\Gamma(2)\}\,. \end{aligned} \tag{5.104}$$

Now we are going to relate the representation $U_{\Gamma(2)}$ to the regular representation of $\mathcal{G}(2)$. Let V_6 be a 6-dimensional Hermitian vector space with an orthonormal basis $\{e_{g\Gamma(2)}\}$ indexed by the elements $g\Gamma(2) \in \mathcal{G}(2)$. Then the regular representation $\rho : \mathcal{G}(2) \to V_6$ is defined by

$$\rho(\gamma\Gamma(2))e_{g\Gamma(2)} = e_{\gamma g\Gamma(2)}. \tag{5.105}$$

LEMMA 5.2. *For the representation $U_{\Gamma(2)}$ of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial character on $\Gamma(2)$ and the regular representation ρ of the group $\mathcal{G}(2)$ defined in* (5.105) *we have, up to a conjugation,*

$$U_{\Gamma(2)}(\gamma) = \rho(\gamma\Gamma(2)), \quad \gamma \in \mathrm{PSL}(2,\mathbb{Z}). \tag{5.106}$$

PROOF. In the basis $\{e_{g\Gamma(2)}\}$, $\rho(\gamma\Gamma(2))$ is a 6×6 permutation matrix with the entries given by

$$[\rho(\gamma\Gamma(2))]_{i,j} = \langle e_{g_i\Gamma(2)}, \rho(\gamma\Gamma(2))e_{g_j\Gamma(2)}\rangle = \langle e_{g_i\Gamma(2)}, e_{\gamma g_j\Gamma(2)}\rangle \tag{5.107}$$

where $g_i\Gamma(2)$'s, $1 \le i \le 6$, are the elements of $\mathcal{G}(2)$, and $\langle , \rangle$ denotes the scalar product in V_6. Then we have

$$[\rho(\gamma\Gamma(2))]_{i,j} = \begin{cases} 1, & e_{g_i\Gamma(2)} = e_{\gamma g_j\Gamma(2)} \Leftrightarrow g_i\Gamma(2) = \gamma g_j\Gamma(2), \\ 0, & e_{g_i\Gamma(2)} \ne e_{\gamma g_j\Gamma(2)} \Leftrightarrow g_i\Gamma(2) \ne \gamma g_j\Gamma(2). \end{cases} \tag{5.108}$$

Or,

$$[\rho(\gamma\Gamma(2))]_{i,j} = \begin{cases} 1, & g_i^{-1}\gamma g_j \in \Gamma(2), \\ 0, & g_i^{-1}\gamma g_j \notin \Gamma(2) \end{cases} = \delta_{\Gamma(2)}(g_i^{-1}\gamma g_j) \tag{5.109}$$

where

$$\delta_{\Gamma(2)}(\gamma) = \begin{cases} 1, & \gamma \in \Gamma(2), \\ 0, & \gamma \notin \Gamma(2). \end{cases} \tag{5.110}$$

But a set of representatives $\{r_1, r_2, \ldots, r_6\}$ of the set of right cosets $\Gamma(2)\backslash\mathrm{PSL}(2,\mathbb{Z})$ can be chosen such that $r_i = g_i^{-1}$. Then we have

$$[\rho(\gamma\Gamma(2))]_{i,j} = \delta_{\Gamma(2)}(g_i^{-1}\gamma g_j) = \delta_{\Gamma(2)}(r_i\gamma r_j^{-1}) = \left[U_{\Gamma(2)}(\gamma)\right]_{ij} \tag{5.111}$$

which completes the proof. ☐

Therefore, the study of the decomposition of $U_{\Gamma(2)}$ into irreducibles is reduced to that of ρ. To decompose the representation ρ into irreducible representations we use the fact that in the regular representation of a given group all the irreducible representations of the group appear with multiplicity equal to their dimensions ([**52**], page

18). Hence, we determine first the irreducible representations of $\mathcal{G}(2)$. To begin with, we note that $\mathcal{G}(2)$ contains the subgroups

$$A = \{a_1 := \Gamma(2), a_2 := ST\,\Gamma(2), a_3 := TS\,\Gamma(2)\} \tag{5.112}$$

is a cyclic group of order 3 generated the the element a_2 and

$$H = \{h_1 := \Gamma(2), h_2 := T\,\Gamma(2)\} \tag{5.113}$$

which is a cyclic group of order 2 generated the the element h_2. Furtheremore, one can easily check that $A \lhd \mathcal{G}(2)$, $\mathcal{G}(2) = AH$, and $A \cap H = id$. Hence,

$$\mathcal{G}(2) = A \rtimes H \tag{5.114}$$

where $\rtimes$ denotes the semidirect product of groups.

Remark 5.1. *The group $\mathcal{G}(2)$ is called the modulary group of level 2 ([**43**], page 21). This group is generated by the elements $S\Gamma(2)$ and $T\Gamma(2)$ (see for example [**33**], page 285) and it is isomorphic to the symmetric group S_3. The alternating group A_3 of order 3 is the unique normal subgroup of S_3 (see for example [**33**], page 286). The group A, which is a cyclic group of order 3 generated by the element $ST\Gamma(2)$, is isomorphic to A_3 and hence it is the unique normal subgroup of $\mathcal{G}(2)$.*

As mentioned in (5.114), $\mathcal{G}(2)$ has a semidirect product structure with H an abelian group. Thus we can apply Wigner's and Mackey's little group method to obtain all irreducible representations of $\mathcal{G}(2)$ (see for example [**52**], page 62). We illustrate this method for the group $\mathcal{G}(2)$.

Step 1. In the first step we determine the irreducible representations of A. Since A is abelian, all its irreducible representations are of degree one coinciding with the corresponding characters of A. The characters of A form a group,

$$X = Hom(A, \mathbb{C}^*), \qquad \mathbb{C}^* = \mathbb{C} - \{0\}\,. \tag{5.115}$$

As mentioned already A is a cyclic group of order 3 generated by the element $ST\Gamma(2)$. We denote its elements by a_i, $1 \le i \le 3$. Thus the character table of A with $\omega = \exp(\frac{2\pi i}{3})$ is given by

X\A	a_1	a_2	a_3
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

Table 1. Character table of A.

Step 2. In the next step we determine the orbits of H in X. The action of $\mathcal{G}(2)$ on X is defined by

$$(s\chi)(a) = \chi(s^{-1}as), \quad s \in \mathcal{G}(2), \quad \chi \in X, \quad a \in A. \tag{5.116}$$

We note that A is normal in $\mathcal{G}(2)$ and hence $s^{-1}as \in A$ for all $s \in \mathcal{G}(2)$ and for all $a \in A$. Next we determine the orbits of H in X. For the trivial representation $\chi_1 \in X$ obviously we have

$$(h\chi_1)(a) = \chi_1(h^{-1}ah) = \chi_1(a), \quad \forall h \in H, \quad \forall a \in A. \tag{5.117}$$

That means $\{\chi_1\}$ is an orbit of H in X. Then according to Table 1, we calculate the action of H on χ_2. Since $h_1 \in H$ acts trivially we have to consider only $h_2 \in H$:

$$\begin{aligned} (h_2\chi_2)(a_1) &= \chi_2(a_1) = \chi_3(a_1), \\ (h_2\chi_2)(a_2) &= \chi_2(h_2^{-1}a_2h_2) = \chi_2(a_3) = \chi_3(a_2), \\ (h_2\chi_2)(a_3) &= \chi_2(h_2^{-1}a_3h_2) = \chi_2(a_2) = \chi_3(a_3), \end{aligned} \tag{5.118}$$

and hence $h_2\chi_2 = \chi_3$. Thus $\{\chi_2, \chi_3\}$ is another orbit of H in X and evidently there is no other orbit. Therefore the set of orbits of H in X is given by

$$X/H = \{\{\chi_1\}, \{\chi_2, \chi_3\}\}. \tag{5.119}$$

We choose the following system of representatives of X/H:

$$R(X/H) = \{\chi_1, \chi_2\}. \tag{5.120}$$

Step 3. We determine the subgroups of H which are the stabilizer groups of χ_1 and χ_2. Let H_i be the subgroup of H consisting of the elements h such that for $\chi_i \in R(X/H)$, $h\chi_i = \chi_i$. Then evidently we have

$$H_1 = H, \quad H_2 = \{h_1\}. \tag{5.121}$$

Step 4. We define the groups G_1 and G_2 as follows:

$$G_1 := AH_1 = AH = \mathcal{G}(2), \quad G_2 := AH_2 = A. \tag{5.122}$$

Step 5. In this step we extend the representations χ_1 and χ_2 of the abelian group A to the groups G_1 and G_2. For $i = 1, 2$ the extension $\widetilde{\chi}_i$ of $\chi_i \in R(X/H)$ to G_i is given by

$$\widetilde{\chi}_i(ah) = \chi_i(a), \quad \forall a \in A, \quad \forall h \in H_i. \tag{5.123}$$

Evidently, $\widetilde{\chi}_i$ is an irreducible representation of G_i (see also [**52**], page 62). The representation $\widetilde{\chi}_1$ is the trivial representation of $G_1 = \mathcal{G}(2)$ and the representation $\widetilde{\chi}_2$ of $G_2 = A$ is identical with the representation χ_2 of A.

Step 6. In this step, we define certain irreducible representations of G_1 and G_2 by tensor products of representations. For $i = 1, 2$ let ρ_i be an irreducible representation of H_i. Then by composing ρ_i with the canonical projection $G_i \to H_i$, we obtain an irreducible representation $\widetilde{\rho}_i$ of G_i. Finally by taking the tensor product of $\widetilde{\chi}_i$ and $\widetilde{\rho}_i$ we obtain another irreducible representation $\widetilde{\chi}_i \otimes \widetilde{\rho}_i$ of G_i.

Since the group $H_1 = H$ is a cyclic group of order two, it has only two irreducible representations as given in the following table:

Rep.\H	h_1	h_2
$\rho_{1,t}$	1	1
$\rho_{1,s}$	1	-1

TABLE 2. Irreducible representations of $H_1 = H$.

The extension of the representations of $H_1 = H$ to $G_1 = \mathcal{G}(2)$ leads to a trivial representation and a sign representation of $G_1 = \mathcal{G}(2)$ as given in the following table:

Rep.\ G_1	g_1	g_2	g_3	g_4	g_5	g_6
$\widetilde{\rho}_{1,t}$	1	1	1	1	1	1
$\widetilde{\rho}_{1,s}$	1	-1	-1	1	1	-1

TABLE 3. The extension of the representations of $H_1 = H$ to $G_1 = \mathcal{G}(2)$ where g_i denotes the ith element of the set of elements of $\mathcal{G}(2)$ given in (5.104) and $g_1 = a_1h_1$, $g_2 = a_2h_2$, $g_3 = a_1h_2$, $g_4 = a_2h_1$, $g_5 = a_3h_1$, $g_6 = a_3h_2$.

On the other hand, the representation $\widetilde{\chi}_1$ is the trivial representation of $G_1 = \mathcal{G}(2)$. Hence we obtain the tensor product representations $\widetilde{\chi}_1 \otimes \widetilde{\rho}_{1,t}$ and $\widetilde{\chi}_1 \otimes \widetilde{\rho}_{1,s}$ of $G_1 = \mathcal{G}(2)$ as given in the following Table 4.

Rep.\ G_1	g_1	g_2	g_3	g_4	g_5	g_6
$\widetilde{\chi}_1 \otimes \widetilde{\rho}_{1,t}$	1	1	1	1	1	1
$\widetilde{\chi}_1 \otimes \widetilde{\rho}_{1,s}$	1	-1	-1	1	1	-1

TABLE 4. The tensor product representations of $G_1 = \mathcal{G}(2)$ where g_i denotes the ith element of the set of elements of $\mathcal{G}(2)$ given in (5.104).

The group H_2 is trivial and hence has only the trivial representation. Its extention to $G_2 = A$ is simply the trivial representation $\widetilde{\rho}_2$ of $G_2 = A$. Using the representation $\widetilde{\chi}_2 = \chi_2$ of $G_2 = A$ as given in Table 1, the representation $\widetilde{\chi}_2 \otimes \widetilde{\rho}_2$ of $G_2 = A$ is then given in the following Table 5.

Rep.\ G_2	a_1	a_2	a_3
$\widetilde{\chi}_2 \otimes \widetilde{\rho}_2$	1	ω	ω^2

TABLE 5. The tensor product representation $\widetilde{\chi}_2 \otimes \widetilde{\rho}_2$ of $G_2 = A$.

Step 7. All the irreducible representations of $\mathcal{G}(2)$ are obtained via induction from the representations of G_1 and G_2 given in Tables 4 and 5, respectively (see for example [**52**], page 62, Proposition 25). First, we derive the representations of $\mathcal{G}(2)$ induced from the representations of G_1 given in Table 4. Since $G_1 = \mathcal{G}(2)$, the representations in Table 4 are already irreducible representations of $\mathcal{G}(2)$. Next, we derive the representation of $\mathcal{G}(2)$, induced from the representation $\widetilde{\chi}_2 \otimes \widetilde{\rho}_2$ of G_2 given in Table 5. Let us now for $\gamma \in \mathcal{G}(2)$ define:

$$\delta_{G_2,\widetilde{\chi}_2\otimes\widetilde{\rho}_2}(\gamma) := \begin{cases} \widetilde{\chi}_2 \otimes \widetilde{\rho}_2(\gamma), & \gamma \in G_2, \\ 0, & \gamma \notin G_2. \end{cases} \tag{5.124}$$

Then

$$[\rho_2(\gamma)]_{ij} := \left[ind_{G_2}^{\mathcal{G}(2)}(\widetilde{\chi}_2 \otimes \widetilde{\rho}_2)(\gamma)\right]_{i,j} = \delta_{G_2,\widetilde{\chi}_2\otimes\widetilde{\rho}_2}(h_i\gamma h_j^{-1}), \quad 1 \le i,j \le 2 \tag{5.125}$$

where $H = \{\Gamma(2), T\Gamma(2)\} = \{h_1, h_2\}$ can be taken as the set of representatives of the right cosets of $G_2 = A$ in $\mathcal{G}(2)$. Since $\mathcal{G}(2)$ is generated by $g_2 := S\Gamma(2)$ and $g_3 := T\Gamma(2)$, it is enough to calculate the induced representation of these generators. By a simple calculation we have

$$\begin{aligned} h_1g_3h_1^{-1} &= T\Gamma(2) \notin A, \\ h_1g_3h_2^{-1} &= \Gamma(2) = a_1 \in A, \\ h_2g_3h_1^{-1} &= \Gamma(2) = a_1 \in A, \\ h_2g_3h_2^{-1} &= T\Gamma(2) \notin A. \end{aligned} \tag{5.126}$$

Thus according to Table 5 we get

$$ind_{G_2}^{\mathcal{G}(2)}(\widetilde{\chi}_2 \otimes \widetilde{\rho}_2)(g_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.127}$$

For the element $g_2 = S\Gamma(2)$ we have

$$\begin{aligned} h_1g_2h_1^{-1} &= S\Gamma(2) \notin A, \\ h_1g_2h_2^{-1} &= ST\Gamma(2) = a_2 \in A, \\ h_2g_2h_1^{-1} &= TS\Gamma(2) = a_3 \in A, \\ h_2g_2h_2^{-1} &= STS\Gamma(2) \notin A. \end{aligned} \tag{5.128}$$

Hence according to Table 5

$$ind_{G_2}^{\mathcal{G}(2)}(\widetilde{\chi}_2 \otimes \widetilde{\rho}_2)(g_2) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}. \tag{5.129}$$

Thus we obtained all irreducible representations of $\mathcal{G}(2)$ as listed in the following table:

Rep.\ $\mathcal{G}(2)$	$S\Gamma(2)$	$T\Gamma(2)$
ρ_t	1	1
ρ_s	-1	-1
ρ_2	$\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

TABLE 6. The irreducible representations of $\mathcal{G}(2)$.

Recall that $\rho : \mathcal{G}(2) \to V_6$ is the regular representation of $\mathcal{G}(2)$ given in (5.105). Hence, as mentioned, all irreducible representations of $\mathcal{G}(2)$ occur in the irreducible decomposition of ρ with multiplicity equal to their dimensions. Therefore, there is a matrix $M_{\Gamma(2)}$ such that

$$M_{\Gamma(2)}\rho M_{\Gamma(2)} = \rho_t \oplus \rho_s \oplus \rho_2 \oplus \rho_2 \tag{5.130}$$

where the irreducible representations in the right hand side are given in Table 6. The matrix $M_{\Gamma(2)}$ can be calculated explicitly to be

$$M_{\Gamma(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 0 & \omega^2 & \omega & 0 & 0 & 1 \\ \omega & 0 & 0 & \omega^2 & 1 & 0 \\ \omega^2 & 0 & 0 & \omega & 1 & 0 \\ 0 & \omega & \omega^2 & 0 & 0 & 1 \end{pmatrix}. \tag{5.131}$$

Hence, Lemma 5.2 yields the decomposition of $U_{\Gamma(2)}$ into its irreducible subrepresentations which has been given already in [**13**]:

LEMMA 5.3. *Let ρ be the regular representation of $\mathcal{G}(2)$, given in 5.105, and $U_{\Gamma(2)}$ be the representation of* $\mathrm{PSL}(2,\mathbb{Z})$ *induced from the trivial character on* $\Gamma(2)$. *Then the decomposition of $U_{\Gamma(2)}$ into its irreducible components is given by*

$$M_{\Gamma(2)}U_{\Gamma(2)}M_{\Gamma(2)}^{-1} = U_t \oplus U_s \oplus U_2 \oplus U_2 \tag{5.132}$$

where $M_{\Gamma(2)}$ is given in (5.131) *and for $\gamma \in$* $\mathrm{PSL}(2,\mathbb{Z})$ *we have*

$$\begin{aligned} U_t(\gamma) &= \rho_t(\gamma\Gamma(2)), \\ U_s(\gamma) &= \rho_s(\gamma\Gamma(2)), \\ U_2(\gamma) &= \rho_2(\gamma\Gamma(2)). \end{aligned} \tag{5.133}$$

We recall the notion of old and new Maass forms for $S(s;\Gamma(2))$ from [**5**]. Let

$$\Gamma^0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \mid b = 0 \mod 2 \right\}. \tag{5.134}$$

Then the old forms in $\mathcal{S}(s;\Gamma^0(2))$ are given by ([**5**], page 446, formula 14)

$$\mathcal{S}^{old}(s;\Gamma^0(2)) := \left\{ f_1(z) + f_2(\frac{z}{2}) \mid f_1, f_2 \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z})) \right\}. \tag{5.135}$$

The space of new forms in $\mathcal{S}(s;\Gamma^0(2))$ is defined as the orthogonal complement of $\mathcal{S}^{old}(s;\Gamma^0(2))$ in $\mathcal{S}(s;\Gamma^0(2))$, that is,

$$\mathcal{S}^{new}(s;\Gamma^0(2)) := \mathcal{S}(s;\Gamma^0(2)) \ominus \mathcal{S}^{old}(s;\Gamma^0(2)). \tag{5.136}$$

The old forms in $\mathcal{S}(s;\Gamma(2))$ are given by ([**5**], page 446, Formula 16)

$$\begin{aligned} \mathcal{S}^{old}(s;\Gamma(2)) &:= \{f_1(z) + f_2(\frac{z}{2}) + f_3(2z) + g_1(z) + g_2(2z) \mid \\ & f_1, f_2, f_3 \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z})), g_1, g_2 \in \mathcal{S}^{new}(s;\Gamma^0(2))\}. \end{aligned} \tag{5.137}$$

That $f_1(z)$, $f_2(\frac{z}{2})$, $f_3(2z)$, $g_1(z)$, and $g_2(2z)$ are indeed $\Gamma(2)$ Maass cusp forms was proved in ([**5**], page 445, Lemma. 5.1.). Then by definition, the space of new forms in $\mathcal{S}(s;\Gamma(2))$ is defined as the orthogonal complement of $\mathcal{S}^{old}(s;\Gamma(2))$ in $\mathcal{S}(s;\Gamma(2))$, that is,

$$\mathcal{S}^{new}(s;\Gamma(2)) := \mathcal{S}(s;\Gamma(2)) \ominus \mathcal{S}^{old}(s;\Gamma(2)). \tag{5.138}$$

Further, one shows that ([**5**], page 446, Theorem 1)

$$\begin{aligned} &\mathcal{S}^{new}(s;\Gamma(2)) = \\ &\left\{ h \in \mathcal{S}(s;\Gamma(2)) \mid \sum_{r \in R(\Gamma(2)\backslash\Gamma_0(2))} h(rz) = 0, \quad \sum_{r \in R(\Gamma(2)\backslash\Gamma^0(2)} h(rz) = 0 \right\}. \end{aligned} \tag{5.139}$$

Now we can characterize the new forms in the space $\mathcal{S}(s;\Gamma(2))$ via the representation U_s.

Lemma 5.4. *The spaces $\mathcal{S}^{new}(s;\Gamma(2))$ and $\mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_s)$ are isomorphic. Indeed, the map*

$$M_{\Gamma(2)}B : \mathcal{S}^{new}(s;\Gamma(2)) \to \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_s) \tag{5.140}$$

is an isomorphism where B is defined in (3.26) *and $M_{\Gamma(2)}$ is given in* (5.131).

Proof. Let $u \in \mathcal{S}^{new}(s;\Gamma(2))$ and let

$$Bu = \begin{pmatrix} u(z) \\ u(Sz) \\ u(Tz) \\ u(STz) \\ u(TSz) \\ u(STSz) \end{pmatrix} = \begin{pmatrix} u_1(z) \\ u_2(z) \\ u_3(z) \\ u_4(z) \\ u_5(z) \\ u_6(z) \end{pmatrix}. \tag{5.141}$$

Then we have

$$M_{\Gamma(2)}Bu = \begin{pmatrix} u_1+u_2+u_3+u_4+u_5+u_6 \\ -u_1+u_2+u_3-u_4-u_5+u_6 \\ \omega^2u_2+\omega u_3+u_6 \\ \omega u_1+\omega^2u_4+u_5 \\ \omega^2u_1+\omega u_4+u_5 \\ \omega u_2+\omega^2u_3+u_6 \end{pmatrix}. \tag{5.142}$$

We fix the following sets of representatives of $\Gamma(2)$ in $\Gamma_0(2)$.

$$R(\Gamma(2)\backslash\Gamma_0(2)) = \{id, T\} \tag{5.143}$$

respectively in $\Gamma^0(2)$

$$R(\Gamma(2)\backslash\Gamma^0(2)) = \{id, STS\}\,. \tag{5.144}$$

By definition, for $u \in \mathcal{S}^{new}(s;\Gamma(2))$ the following identities hold:

$$\sum_{r\in R(\Gamma(2)\backslash\Gamma_0(2))} u(rz) = u_1+u_3 = 0 \tag{5.145}$$

and

$$\sum_{r\in R(\Gamma(2)\backslash\Gamma^0(2))} u(rz) = u_1+u_6 = 0. \tag{5.146}$$

From (5.145) we have

$$u_1 = -u_3 \quad \text{or} \quad u(z) = -u(Tz). \tag{5.147}$$

By replacing z by Sz and z by STz in this identity, we get

$$u(Sz) = -u(TSz), \quad \text{or} \quad u_2(z) = -u_5(z) \tag{5.148}$$

respectively

$$u(STz) = -u(TSTz), \quad \text{or} \quad u_4(z) = -u_6(z). \tag{5.149}$$

Similarly, from (5.146), we get the identities

$$u_1(z) = -u_6(z), \quad u_2(z) = -u_4(z), \quad u_3(z) = -u_5(z). \tag{5.150}$$

Summarizing all these identities, we have

$$u_1 = u_5 = u_4 = -u_2 = -u_3 = -u_6. \tag{5.151}$$

From these identities together with the fact that $1+\omega+\omega^2 = 0$, we get

$$M_{\Gamma(2)}Bu = \begin{pmatrix} 0 \\ -6u(z) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{S}(s;\mathrm{PSL}(2,\mathbb{Z});U_s). \tag{5.152}$$

Conversely, let

$$\mathfrak{v} = \begin{pmatrix} 0 \\ v(z) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{S}(s; \mathrm{PSL}(2,\mathbb{Z}); U_s) \tag{5.153}$$

Then we find

$$M_{\Gamma(2)}^{-1}\mathfrak{v} = \frac{1}{6}\begin{pmatrix} -v \\ v \\ v \\ -v \\ -v \\ v \end{pmatrix}. \tag{5.154}$$

Now we must show that $u := (M_{\Gamma(2)}^{-1}\mathfrak{v})_1 = -v \in \mathcal{S}^{new}(s;\Gamma(2))$. For this we note first, that

$$u(\gamma z) = U_s(\gamma)u(z), \quad \gamma \in \mathrm{PSL}(2,\mathbb{Z}). \tag{5.155}$$

From this one can show that u is invariant under the generators of $\Gamma(2)$, namely T^2 and ST^2S and that

$$Bu := \begin{pmatrix} u_1(z) \\ u_2(z) \\ u_3(z) \\ u_4(z) \\ u_5(z) \\ u_6(z) \end{pmatrix} = \begin{pmatrix} u(r_1 z) \\ u(r_2 z) \\ u(r_3 z) \\ u(r_4 z) \\ u(r_5 z) \\ u(r_6 z) \end{pmatrix} = \begin{pmatrix} u(z) \\ -u(z) \\ -u(z) \\ u(z) \\ u(z) \\ -u(z) \end{pmatrix} = \begin{pmatrix} -v(z) \\ v(z) \\ v(z) \\ -v(z) \\ -v(z) \\ v(z) \end{pmatrix}. \tag{5.156}$$

with r_i the i-th element of the set of representatives of $\Gamma(2)$ in $\mathrm{PSL}(2,\mathbb{Z})$ (see for example [**40**], page 90, Proposition 2.2.)

$$R(\Gamma(2)\backslash \mathrm{PSL}(2,\mathbb{Z})) = \{id, S, T, ST, TS, STS\}.$$

Moreover, from (5.156) it follows that

$$\sum_{r\in R(\Gamma(2)\backslash\Gamma_0(2))} u(rz) = u_1 + u_3 = 0, \tag{5.157}$$

and

$$\sum_{r\in R(\Gamma(2)\backslash\Gamma^0(2))} u(rz) = u_1 + u_6 = 0. \tag{5.158}$$

Therefore $u := (M_{\Gamma(2)}\mathfrak{v})_1 \in \mathcal{S}^{new}(s;\Gamma(2))$ and this completes the proof. □

Now we consider Mayer's transfer operator $\mathcal{L}_s^{\Gamma(2)}$ for $\Gamma(2)$ with trivial character. A symmetry operator for this transfer operator is given by $P = U_{\overline{\Gamma}(2)}(M) = id_{6\times 6}$ and hence we have $\mathcal{L}_s^{\Gamma(2)} = \mathcal{L}_s^{\Gamma(2),+} = \mathcal{L}_s^{\Gamma(2),-}$. Evidently, Lemma 5.3 yields the following decomposition of $\mathcal{L}_s^{\Gamma(2)}$:

$$\begin{aligned} &M_{\Gamma(2)}\mathcal{L}_s^{\Gamma(2)}M_{\Gamma(2)}^{-1} = \\ &\mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z})} \oplus \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_s} \oplus \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2} \oplus \mathcal{L}_s^{\mathrm{PSL}(2,\mathbb{Z}),U_2}. \end{aligned} \tag{5.159}$$

Moreover, we find the space $F(s;\Gamma(2))$ of eigenfunctions of $\mathcal{L}_s^{\Gamma(2)}$ is decomposed as

(5.160)

$$\begin{aligned} &M_{\Gamma(2)}F(s;\Gamma(2)) = F(s;\mathrm{PSL}(2,\mathbb{Z}))\oplus \\ &F(s;\mathrm{PSL}(2,\mathbb{Z});U_s) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_2) \oplus F(s;\mathrm{PSL}(2,\mathbb{Z});U_2). \end{aligned}$$

According to Theorem 3.11 there is a bijection

$$\mathcal{P}\circ\mathcal{I}\circ B : \mathcal{S}(s;\Gamma(2)) \to F(s;\Gamma(2)). \tag{5.161}$$

We define the space of new periodic functions in the space $F(s;\Gamma(2))$ by

$$F^{new}(s;\Gamma(2)) := \mathcal{P}\circ\mathcal{I}\circ B(\mathcal{S}^{new}(s;\Gamma(2))). \tag{5.162}$$

Then Lemma 5.4 and linearity of $\mathcal{P}\circ\mathcal{I}$, lead to the following bijection

$$M_{\Gamma(2)}F^{new}(s;\Gamma(2)) \cong F(s;\mathrm{PSL}(2,\mathbb{Z});U_s) \tag{5.163}$$

which characterizes $F^{new}(s;\Gamma(2))$ by the representation U_s.

CHAPTER 6

Congruence properties of the induced representation

This chapter reviews my joint work with Dieter Mayer and Alexei Venkov [**31**]. We show first that a unitary representation χ of a group $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ is congruence, that is, $\ker\chi$ contains a principal congruence group, if and only if the induced representation ρ_χ is congruence. This provides us with a new method to determine if a representation is congruence or not. We illustrate this method in the example of Selberg's character on $\Gamma_0(4)$ and we compare it with Zograf's geometric approach.

6.1. Congruence representations, Selberg's character and its induction to $\mathrm{PSL}(2,\mathbb{Z})$

Let Γ be a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ and χ be a unitary representation of Γ on the Hermitian vector space W.

Definition 6.1. *A unitary representation χ of $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ is called a congruence representation if $\ker\chi$ is a congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, that is, if $\ker\chi$ contains a principal congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$.*

The motivation for this definition comes from the belief that the spectrum of the automorphic Laplacian $\Delta(\Gamma;\chi)$ and hence the position of the zeros of Selberg's zeta function $Z(s;\Gamma;\chi)$ strongly depends on the representation χ being congruence or not (see for example [**40**]). We illustrate this by an example from [**5**]. Consider the principal congruence subgroup $\Gamma(2)$ with two generators given by

$$A := T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B := ST^2S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{6.1}$$

Each element $\gamma \in \Gamma(2)$ can be presented uniquely as a product

$$\gamma = A^{n_1}B^{m_1}A^{n_2}B^{m_2}\dots A^{n_k}B^{m_k}, \quad n_j, m_j \in \mathbb{Z}, \quad 1 \leq j \leq k. \tag{6.2}$$

Then an integer valued function on elements of $\Gamma(2)$ is defined by

$$P_A(\gamma) = n_1 + \dots n_k. \tag{6.3}$$

Using this a character on $\Gamma(2)$ is defined by

$$\widehat{\chi}_\alpha(\gamma) = \exp(2\pi i\alpha P_A(\gamma)). \tag{6.4}$$

In [**41**], it is shown under a multiplicity assumption on the eigenvalues of the automorphic Laplacian, that for at most countably many $\alpha \in (0,1)$ the pair $(\Gamma(2), \alpha)$ is essentially cuspidal, namely the eigenvalues of $\Delta(\Gamma(2); \widehat{\chi}_\alpha)$ fulfil a Weyl law. As an example where a special case of the congruence property of a representation plays a role we mention the Phillips-Sarnak conjecture saying that $(\Gamma(2), \alpha)$ is essentially cuspidal only for α-values for which $\widehat{\chi}_\alpha$ is a congruence character.

Consequently, the problem arises of finding the α-values for which $\widehat{\chi}_\alpha$ is congruence. For α-values with

$$\alpha = \frac{n}{d}, \quad n, d \in \mathbb{Z}_+, \quad (n,d) = 1, \tag{6.5}$$

$\ker \widehat{\chi}_\alpha$ is given by the group Γ_d defined by Balslev and Venkov in [**5**]:

$$\Gamma_d := \{\gamma \in \Gamma(2) \mid P_A(\gamma) = 0 \mod d\}, \tag{6.6}$$

which is a normal subgroup of $\Gamma(2)$ of genus zero. The groups Γ_d, as defined by Balslev and Venkov, coincide with the groups Γ_{6d} of G. Sansone [**48**] and studied in [**37**] by M. Newman. He solved indeed the congruence problem for these subgroups of $\Gamma(2)$ by showing Γ_{6d} to be congruent only for $d = 1, 2, 4, 8$ with $\Gamma(2d) \le \Gamma_{6d}$. In terms of the α-values in $[0, 1)$ this means that $\alpha = \frac{j}{8}$ with $j = 0, 1, \ldots, 7$ ([**5**], page 439). This was also reproved in [**40**].

Recall the Hecke congruence subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ of level 4 defined by

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \mid c = 0 \mod 4 \right\}. \tag{6.7}$$

This group is freely generated by

$$T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad ST^4S = \pm \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}. \tag{6.8}$$

Selberg's character

$$\chi_\alpha : \Gamma_0(4) \to \mathrm{Aut}\mathbb{C}, \quad 0 \le \alpha \le 1, \tag{6.9}$$

on the group $\Gamma_0(4)$ is defined by the following assignments:

$$\chi_\alpha(T) = \exp(2\pi i \alpha), \quad \chi_\alpha(ST^4S) = 1. \tag{6.10}$$

Indeed, Selberg defined originally a character $\widetilde{\chi}_\alpha$ by the assignments $\widetilde{\chi}_\alpha(T) = 1$ and $\widetilde{\chi}_\alpha(ST^4S) = \exp(i\alpha)$ where $-\pi \le \alpha \le \pi$ and then he proved that each point on the critical line $\Re(s) = \frac{1}{2}$ is a limit point of zeros of $Z(s; \Gamma_0(4); \widetilde{\chi}_\alpha)$ as $\alpha \to 0$ ([**51**], page 15).

In Mayer's transfer operator approach to Selberg's zeta function for the group $\Gamma_0(4)$ with Selberg's character χ_α this character is replaced by the representation U_α of $\mathrm{PSL}(2, \mathbb{Z})$ induced from χ_α [**19**]. Selberg's zeta function $Z(\Gamma_0(4); \chi_\alpha; s)$ thereby gets expressed as

$$Z(s; \Gamma_0(4); \chi_\alpha) = \det(1 - P\mathcal{L}_s^{\Gamma, \chi_\alpha, +}) \det(1 + P\mathcal{L}_s^{\Gamma, \chi_\alpha, +}) \tag{6.11}$$

where P is a symmetry of Mayer's transfer operator and

$$\mathcal{L}_s^{\Gamma,\chi_\alpha,+} f(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2s} U_\alpha(ST^n) f(\frac{1}{z+n}), \quad f \in B(D). \tag{6.12}$$

We fix the ordered set of representatives of $\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})$ to be

$$R(\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})) = \left\{id, S, ST, ST^2, ST^3, ST^2S\right\}. \tag{6.13}$$

With this choice of representatives, the induced representations U_α for the generators S and T of $\mathrm{PSL}(2,\mathbb{Z})$ are given by

$$U_\alpha(S) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp(-2\pi i\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \exp(2\pi i\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{6.14}$$

and

$$U_\alpha(T) = \begin{pmatrix} \exp(2\pi i\alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp(-2\pi i\alpha) \end{pmatrix}. \tag{6.15}$$

The eigenvalues and resonances of the automorphic Laplace operator $A(\Gamma_0(4),\chi_\alpha)$ determine the nontrivial zeros of $Z(\Gamma_0(4);\chi_\alpha;s)$ and hence, because of (6.11) depend on the induced representation U_α being congruence or not. Indeed, this is the case as one knows from the work of Selberg and confirmed also by numerical calculations of Fraczek [**18**]. Indeed we show U_α is congruence iff χ_α is congruence, that is, iff $\alpha = \frac{j}{8}$, $0 \le j \le 7$ for $\alpha \in [0,1)$. Using identity (6.11), Fraczek [**18**] was able to calculate the zeros of $Z(s;\Gamma_0(4);\chi_\alpha)$ and trace them as α varies. His numerical calculations confirm that $\alpha = \frac{j}{8}$, $0 \le j \le 7$ are the only congruence values for U_α respectively for χ_α. Based on these numerical calculations Mayer et al studied the zeros of Selberg's zeta function $Z(s;\Gamma_0(4);\chi_\alpha)$ in detail and give in particular a detailed description of Selberg's phenomenon mentioned above [**8**]. Motivated by this works of Mayer and Fraczek, we can prove the more general result:

Theorem 6.1. *Let Γ be a subgroup of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$*, χ be a unitary representation of Γ on the Hermitian vector space W and ρ_χ be the induced representation of* $\mathrm{PSL}(2,\mathbb{Z})$ *given in Definition 3.1. Then χ is congruence if and only if ρ_χ is congruence.*

Proof. According to the definition of ρ_χ an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ is in the kernel $\ker \rho_\chi$ of the representation iff for all elements r in

R, the set of representatives of the right cosets $\Gamma\backslash\mathrm{PSL}(2,\mathbb{Z})$, one has $\delta_{\Gamma,\chi}(rgr^{-1}) = id_{\dim\chi}$. But according to (3.9) this is equivalent to $rgr^{-1} \in \ker\chi$ for all $r \in R$. Therefore we have

$$\ker\rho_\chi = \left\{g \in \mathrm{PSL}(2,\mathbb{Z}) \mid rgr^{-1} \in \ker\chi, \forall r \in R\right\}. \tag{6.16}$$

From this and the fact that $id_{2\times 2} \in R$ it follows that $\ker\rho_\chi$ is a subgroup of $\ker\chi$,

$$\ker\rho_\chi \leq \ker\chi. \tag{6.17}$$

Thus, if $\ker\rho_\chi$ is a congruence subgroup then also $\ker\chi$ is a congruence subgroup and hence if ρ_χ is a congruence representation then χ is a congruence representation too. To prove the converse, we note that the kernel $\ker\rho_\chi$ in (6.16) can be written as the intersection of the sets given by

$$\ker\rho_\chi = r_1\ker\chi r_1^{-1} \cap r_2\ker\chi r_2^{-1} \cap \ldots \cap r_6\ker\chi r_6^{-1}, \quad r_i \in R. \tag{6.18}$$

If χ is a congruence representation and hence $\ker\chi$ is a congruence group, then $\Gamma(n) \leq \ker\chi$ for some $n \in \mathbb{N}$. Since $\Gamma(n)$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$ one gets $\Gamma(n) \subset r\ker\chi r^{-1}$ for all $r \in R$. Therefore, according to (6.18), $\Gamma(n) \leq \ker\rho_\chi$. Hence $\ker\rho_\chi$ is also a congruence subgroup and therefore ρ_χ is a congruence representation. $\square$

Hence the problem to determine if a representation χ is congruence or not, is reduced to determine if ρ_χ is congruence or not. This provides us with an alternative method to study the congruence properties of χ via that of ρ_χ. In the following sections, we apply this method to study the congruence properties of Selberg's character.

6.2. The kernel of the induced representation U_α

In this section we study the group $\ker U_\alpha \lhd \mathrm{PSL}(2,\mathbb{Z})$. For this we need some auxiliary results for the U_α-image of $\mathrm{PSL}(2,\mathbb{Z})$, namely the group

$$G_\alpha := U_\alpha(\mathrm{PSL}(2,\mathbb{Z})) = \{U_\alpha(g) \mid g \in \mathrm{PSL}(2,\mathbb{Z})\}. \tag{6.19}$$

Since $\mathrm{PSL}(2,\mathbb{Z})$ is generated by S and T, the group G_α is generated by $U_\alpha(S)$ and $U_\alpha(T)$.

Before proceeding further we recall some definitions and results. The general linear group $\mathrm{GL}(\mathrm{n},\mathbb{C})$ is the group of all invertible $n \times n$ matrices with entries in $\mathbb{C}$. A matrix is called monomial if each row and column has exactly one nonzero element ([**1**], page 48). We denote by $M(6,\mathbb{C})$ the group of all monomial matrices in $\mathrm{GL}(6,\mathbb{C})$. We further denote by $\Delta(6,\mathbb{C})$ the group of all diagonal matrices in $\mathrm{GL}(6,\mathbb{C})$. The group $M(6,\mathbb{C})$ is the normalizer of $\Delta(6,\mathbb{C})$ in $\mathrm{GL}(6,\mathbb{C})$ (see [**1**], page 48, Exercise 7). Hence, $\Delta(6,\mathbb{C})$ is normal in $M(6,\mathbb{C})$. A permutation matrix is a monomial matrix in which all nonzero elements are equal to one. We denote by W the set of all permutation matrices in $\mathrm{GL}(6,\mathbb{C})$. W is a subgroup of $\mathrm{GL}(6,\mathbb{C})$ which is called the Weyl group (see [**1**], page 42, Proposition 4). The group W is isomorphic to S_6, the symmetric group of degree 6. The group $M(6,\mathbb{C})$ has the following semidirect product structure (see [**1**], page 48, Exercise 7)

$$M(6,\mathbb{C}) = \Delta(6,\mathbb{C}) \rtimes W. \tag{6.20}$$

Hence, each element m in $\in M(6,\mathbb{C})$ has a unique expression as $m = \delta w$ where $\delta \in \Delta(6,\mathbb{C})$ and $w \in W$ (see [**1**], page 21, second item).

The generators of G_α, namely $U_\alpha(S)$ and $U_\alpha(T)$ belong to $M(6,\mathbb{C})$. Hence, G_α is a subgroup of $M(6,\mathbb{C})$:

$$G_\alpha \leq M(6,\mathbb{C}). \tag{6.21}$$

Lemma 6.2. *Let U_0 be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial one- dimensional representation of $\Gamma_0(4)$. Then each element $U_\alpha(g) \in G_\alpha$ has a unique representation as*

$$U_\alpha(g) = D_\alpha(g) U_0(g), \tag{6.22}$$

where $D_\alpha(g) \in \Delta(6,\mathbb{C})$.

Proof. To $g \in \mathrm{PSL}(2,\mathbb{Z})$ we assign the diagonal matrix $D_\alpha(g) \in \Delta(6,\mathbb{C})$ with entries

$$[D_\alpha(g)]_{ik} = \delta_{ik}\chi_\alpha(r_i g r(i)^{-1}), \quad 1 \leq i,k \leq 6. \tag{6.23}$$

Here, r_i and $r(i)$ are elements of the set $R(\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z}))$ with $r(i)$ uniquely determined by the condition $r_i g r(i)^{-1} \in \Gamma_0(4)$. Then we have

$$[D_\alpha(g)U_0(g)]_{ij} = \sum_{k=1}^{6} [D_\alpha(g)]_{ik} \, [U_0(g)]_{kj} \, . \tag{6.24}$$

Inserting (6.23) into this identity we get

$$[D_\alpha(g)U_0(g)]_{ij} = \chi_\alpha(r_i g r(i)^{-1}) \, [U_0(g)]_{ij} \, . \tag{6.25}$$

But according to the definition of the induced representation (Definition 3.1) U_0 we get

$$[D_\alpha(g)U_0(g)]_{ij} = \chi_\alpha(r_i g r(i)^{-1}) \delta_{\Gamma_0(4)}(r_i g r_j^{-1}). \tag{6.26}$$

Hence

$$[D_\alpha(g)U_0(g)]_{ij} = \begin{cases} \chi_\alpha(r_i g r_j^{-1}) & \text{if } r_i g r_j^{-1} \in \Gamma_0(4), \\ 0 & \text{if } r_i g r_j^{-1} \notin \Gamma_0(4) \end{cases} \tag{6.27}$$

or

$$[D_\alpha(g)U_0(g)]_{ij} = \delta_{\Gamma_0(4),\chi_\alpha}(r_i g r_j^{-1}). \tag{6.28}$$

That is, we have

$$D_\alpha(g)U_0(g) = U_\alpha(g). \tag{6.29}$$

Since $U_0(g)$ is a permutation matrix in W and G_α is a subgroup of $M(6,\mathbb{C})$, this decomposition according to (6.20) is unique. $\square$

As already mentioned, the group $\Delta(6,\mathbb{C})$ is normal in the group of monomial matrices $M(6,\mathbb{C})$ and G_α is a subgroup of $M(6,\mathbb{C})$. Hence $A_\alpha := G_\alpha \cap \Delta(6,\mathbb{C})$ is normal in G_α. By definition A_α is the group of all diagonal matrices in G_α. Hence, according to lemma 6.2, A_α is the image of the kernel of the representation U_0 under the map U_α, that means

$$A_\alpha = \{U_\alpha(\gamma) | \gamma \in \ker U_0\} . \tag{6.30}$$

But from Lemma 4.4 we have

$$\ker U_0 = \Gamma(4). \tag{6.31}$$

Hence the normal subgroup A_α of G_α in (6.30) is given by

$$A_\alpha = \{U_\alpha(\gamma) | \gamma \in \Gamma(4)\} . \tag{6.32}$$

According to this, generators of A_α can be calculated explicitly from generators of $\Gamma(4)$. A set of generators of $\Gamma(4)$ is given for instance by (see for example [**26**], page 104)

$$\begin{aligned}
g_1 = T^4 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\
g_2 = ST^{-4}S &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \\
g_3 = T^{-1}ST^4ST &= \begin{pmatrix} -5 & -4 \\ 4 & 3 \end{pmatrix}, \\
g_4 = T^{-2}ST^{-4}ST^{-2} &= \begin{pmatrix} 7 & -12 \\ -4 & 7 \end{pmatrix}, \\
g_5 = TST^{-4}ST^{-1} &= \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix}.
\end{aligned} \tag{6.33}$$

The corresponding generators of A_α are obtained by calculating the induced representation U_α for each generator of $\Gamma(4)$:

$$A_1(\alpha) := U_\alpha(g_1) = \mathrm{diag}(\exp(8\pi i\alpha), 1, 1, 1, 1, \exp(-8\pi i\alpha)), \tag{6.34}$$

$$A_2(\alpha) := U_\alpha(g_2) = \mathrm{diag}(1, \exp(-8\pi i\alpha), 1, \exp(8\pi i\alpha), 1, 1), \tag{6.35}$$

$$A_3(\alpha) := U_\alpha(g_3) = \mathrm{diag}(1, 1, \exp(8\pi i\alpha), 1, \exp(-8\pi i\alpha), 1) \tag{6.36}$$

where $\mathrm{diag}(a_1, \ldots, a_6)$ denotes a 6 dimensional diagonal matrix with entries a_i. The generators g_4 and g_5 do not lead to new generators for A_α. In fact, one finds

$$U_\alpha(g_5) = U_\alpha(g_3), \quad U_\alpha(g_4) = [U_\alpha(g_1)U_\alpha(g_2)]^{-1} . \tag{6.37}$$

Thus, A_α is an abelian group, generated by three elements:

$$A_\alpha = \langle A_1(\alpha), A_2(\alpha), A_3(\alpha)\rangle . \tag{6.38}$$

Next we consider the factor group G_α/A_α.

LEMMA 6.3. *The factor group G_α/A_α is isomorphic to the modulary group $G(4) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4)$.*

PROOF. First, we consider the group homomorphisms

$$\begin{aligned} h_1 &: \mathrm{PSL}(2,\mathbb{Z}) \to G_\alpha \\ h_1(g) &= U_\alpha(g) \end{aligned} \tag{6.39}$$

and

$$\begin{aligned} h_2 &: \Gamma(4) \to A_\alpha \\ h_2(g) &= U_\alpha(g). \end{aligned} \tag{6.40}$$

According to lemma 6.2, $\ker h_1 \leq \ker U_0 = \Gamma(4)$. Thus $\ker h_1 \leq \Gamma(4)$. Moreover, h_2 is the restriction of h_1 to $\Gamma(4)$ and hence $\ker h_1 = \ker h_2 \leq \Gamma(4)$. From the definitions of h_1 and h_2 it is also clear that

$$\ker U_\alpha = \ker h_1 = \ker h_2. \tag{6.41}$$

Since $G_\alpha = h_1(\mathrm{PSL}(2,\mathbb{Z}))$ and $A_\alpha = h_2(\Gamma(4))$, the following isomorphisms hold

$$\mathrm{PSL}(2,\mathbb{Z})/\ker U_\alpha \cong G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z})) \tag{6.42}$$

and

$$\Gamma(4)/\ker U_\alpha \cong A_\alpha. \tag{6.43}$$

Then the "second isomorphism theorem" (see [**1**], page 12) yields the desired result,

$$G_\alpha/A_\alpha \cong \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4) = \mathrm{G}(4). \tag{6.44}$$

□

We denote by $N = N(\alpha)$ the order of a generator of the group A_α defined in (6.38). Obviously all generators of this group have the same order, which for α irrational for example is given by $N = \infty$. We recall that (see (6.43))

$$\Gamma(4)/\ker U_\alpha \cong A_\alpha. \tag{6.45}$$

Hence, the index $\mu(\alpha) = [\mathrm{PSL}(2,\mathbb{Z}) : \ker U_\alpha]$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ is equal to the number of elements of A_α times $[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(4)] = 24$, the index of $\Gamma(4)$ in $\mathrm{PSL}(2,\mathbb{Z})$. Thus we have

$$\mu(\alpha) = 24N^3 = 24N(\alpha)^3. \tag{6.46}$$

From this formula it is clear that for irrational α the subgroup $\ker U_\alpha$ is of infinite index in $\mathrm{PSL}(2,\mathbb{Z})$. In the following let α be a rational number with $N(\alpha) = N$, $N \in \mathbb{N}$.

Based on the Gauss-Bonnet formula, we can determine the number of generators of $\ker U_\alpha$. As can be seen from (6.29) and (6.31) $\ker U_\alpha \leq \Gamma(4)$ and hence $\ker U_\alpha$ has no elliptic elements (see for example [**23**], page 44). The Gauss-Bonnet formula for a group Γ without elliptic elements shows (see for example [**57**], page 15), that

$$|F| = 2\pi(2g - 2 + h), \tag{6.47}$$

where $|F|$ is the area of the fundamental domain, g its genus, and h is the number of cusps of Γ. It is also known that the number of generators of Γ is $2g + h$ (see [**57**], page 14). For the group $\ker U_\alpha$ we have

$$|F| = \mu(\alpha)\frac{\pi}{3}, \tag{6.48}$$

where $\pi/3$ is the area of the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ and $\mu(\alpha)$ is the index of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ as given in (6.46). Hence, the number of generators of $\ker U_\alpha$ is given by

$$2g + h = 4N^3 + 2. \tag{6.49}$$

The number of free generators on the other hand is given by (see [**57**], page 14)

$$\mathcal{N}(\alpha) = 2g + h - 1 = 4N^3 + 1. \tag{6.50}$$

Before continuing further we recall the concept of the width of a cusp (see [**58**], page 529).

DEFINITION 6.2. *For $x \in \mathbb{Q} \cup \{\infty\}$ a cusp of the group $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ and $\sigma \in \mathrm{PSL}(2,\mathbb{Z})$ with $\sigma\infty = x$, let $P \in \Gamma$ be a primitive parabolic element with $Px = x$. If*

$$\sigma P \sigma^{-1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}). \tag{6.51}$$

then $|m|$ is called the width of the cusp x of Γ.

Next we recall Wohlfahrt's definition of the level of a group (see [**58**], page 530)

DEFINITION 6.3. *Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ and $C(\Gamma) \subset \mathbb{N}$ be the set of widths of the cusps of Γ. If $C(\Gamma)$ is nonempty and bounded in $\mathbb{N}$, the level $n(\Gamma)$ of Γ is defined to be the least common multiple of the elements of $C(\Gamma)$. Otherwise the level is defined to be zero.*

For congruence subgroups $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ F. Klein on the other hand defined the level as follows (see [**58**] and the references there)

DEFINITION 6.4. *The level of a congruence subgroup is defined to be the smallest integer n such that $\Gamma(n) \subset \Gamma$.*

The following theorem, which is crucial for our purpose, shows that for congruence groups Wohlfahrt's and F. Klein's definition of the level coincide (see [**58**] and the references there).

THEOREM 6.4. *If Γ is a congruence subgroup of level n in the sense of Wohlfahrt then n is the smallest integer with $\Gamma(n) \leq \Gamma$.*

Next we determine the level of $\ker U_\alpha$ in the sense of Wohlfahrt. To this end, we note that $\ker U_\alpha$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$. Hence, all cusps of $\ker U_\alpha$ have the same width (see [**57**], page 160 third paragraph). Thus it is enough to find the width just for one cusp. According to (6.34), for α with $N(\alpha) = N \in \mathbb{N}$, we have $U_\alpha(g_1)^N = Id_{6\times 6}$ where

$$g_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \tag{6.52}$$

is given in (6.33). Hence

$$g_1^N = \begin{pmatrix} 1 & 4N \\ 0 & 1 \end{pmatrix} \tag{6.53}$$

belongs to $\ker U_\alpha$ and it is obviously primitive. Thus Wohlfahrt's level $n(\alpha)$ of $\ker U_\alpha$ is given for α with $N(\alpha) = N$ by

$$n(\alpha) = 4N. \tag{6.54}$$

Next, based on a formula due to Morris Newman [**38**], we derive a formula for the genus of $\ker U_\alpha$. Let Γ be a normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ with index μ, genus g, and h the number of cusps, respectively with Wohlfahrt's level n. Put $t := \dfrac{\mu}{n}$. Then according to Newman (see [**38**], page 268 and also [**57**] page 160) one has

$$g = 1 + \frac{\mu}{12} - \frac{t}{2}. \tag{6.55}$$

For the group $\ker U_\alpha$ according to (6.46) and (6.54) we get $t = 6N^2$. Inserting this and (6.46) into (6.55) we get for the genus of $\ker U_\alpha$

$$g(\alpha) = 1 + 2N^3 - 3N^2. \tag{6.56}$$

From this formula and (6.49) we obtain the number $h(\alpha)$ of cusps of $\ker U_\alpha$ as follows

$$h(\alpha) = 6N^2. \tag{6.57}$$

Let us summarize the information obtained for $\ker U_\alpha$ in the following theorem

THEOREM 6.5. *For α with $N(\alpha) = N \in \mathbb{N}$ the order of the generators of A_α, let $\mu(\alpha)$ be the index of the group $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, $g(\alpha)$ its genus, $h(\alpha)$ the number of its cusps, $n(\alpha)$ its Wohlfahrt's level, and $\mathcal{N}(\alpha)$ the number of its free generators. Then we have*

- $\mu(\alpha) = 24N^3$
- $g(\alpha) = 1 + 2N^3 - 3N^2$
- $h(\alpha) = 6N^2$
- $n(\alpha) = 4N$
- $\mathcal{N}(\alpha) = 4N^3 + 1.$

6.3. Determination of the α-values for U_α to be congruent

In this section we determine the α-values for which the representation U_α is congruence. As mentioned in the previous section, for irrational α the index $\mu(\alpha)$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ is infinite and hence $\ker U_\alpha$ can not be congruence. Thus we restrict α to be rational with $N = N(\alpha) \in \mathbb{N}$. Since the Wohlfahrt level of $\ker U_\alpha$ is given by $n = 4N$, one finds in the case where $\ker U_\alpha$ is a congruence group

$$\Gamma(4N) \leq \ker U_\alpha. \tag{6.58}$$

From this one determines easily those values of α for which $\ker U_\alpha$ is indeed a congruence subgroup. Since the index of $\Gamma(4N)$ in $\mathrm{PSL}(2,\mathbb{Z})$, given by

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(4N)] = \frac{1}{2}(4N)^3 \prod_{p|4N}(1 - \frac{1}{p^2}), \tag{6.59}$$

must then be larger or equal to the index $\mu(\alpha)$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, one finds

$$\frac{1}{2}(4N)^3 \prod_{p|4N}(1 - \frac{1}{p^2}) \geq 24N^3 \tag{6.60}$$

or

$$\frac{4}{3}\prod_{p|4N}(1 - \frac{1}{p^2}) \geq 1. \tag{6.61}$$

Obviously this inequality holds if and only if $N = 2^k$, $0 \leq k < \infty$.

Lemma 6.6. *If $N(\alpha) = 2^k$ and $\ker U_\alpha$ is a congruence group then $\ker U_\alpha = \Gamma(2^{k+2})$ and hence $A_\alpha \cong \Gamma(4)/\Gamma(2^{k+2})$.*

Proof. For α with $N(\alpha) = 2^k$ the group A_α has order 2^{3k}. If $\ker U_\alpha$ is a congruence subgroup then $\Gamma(2^{k+2}) \leq \ker U_\alpha$. On the other hand one finds for the index $\left[\Gamma(4) : \Gamma(2^{k+2})\right]$ by a simple calculation $\left[\Gamma(4) : \Gamma(2^{k+2})\right] = 2^{3k}$. But $\Gamma(4)/\Gamma(2^{k+2}) \geq \Gamma(4)/\ker U_\alpha \cong A_\alpha$ and hence $2^{3k} = |A_\alpha| \leq |\Gamma(4)/\Gamma(2^{k+2})| = 2^{3k}$. Therefore $\ker U_\alpha = \Gamma(2^{k+2})$. $\square$

Next we show, that only for $k = 0, 1, 2$ the principal congruence subgroup $\Gamma(2^{k+2})$, that means, only $\Gamma(4), \Gamma(8)$ and $\Gamma(16)$ can coincide with the group $\ker U_\alpha$. This follows immediately from the following lemma

Lemma 6.7. *The group $\Gamma(4)/\Gamma(2^{k+2})$ is abelian iff k=0,1,2.*

Since we did not find this result, which is presumably well known, in the literature, we give a simple proof.

PROOF. For $h_i = \begin{pmatrix} 1+4a_i & 4b_i \\ 4c_i & 1+4d_i \end{pmatrix} \in \Gamma(4)$, $i = 1, 2$, one finds for $h_{i,j} := h_i h_j$, $i, j = 1, 2$:

$$h_{1,2} = h_{2,1} = \begin{pmatrix} 1+4(a_1+a_2) & 4(b_1+b_2) \\ 4(c_1+c_2) & 1+4(d_1+d_2) \end{pmatrix} \mod 16$$

respectively

$$h_{1,2}^{-1} h_{2,1} = \begin{pmatrix} 1+4(a_1+a_2+d_1+d_2) & 0 \\ 0 & 1+4(a_1+a_2+d_1+d_2) \end{pmatrix} \mod 16.$$

But $4|(a_i+d_i), i = 1, 2$, and therefore $h_{1,2} = h_{2,1} \mod \Gamma(16)$. Hence $\Gamma(4)/\Gamma(2^{k+2})$ is abelian for $k = 0, 1, 2$. To show that this group is not abelian for $k \geq 3$ take the two elements $h_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ respectively $h_2 = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \in \Gamma(4)$. Then one finds $h_{1,2}^{-1} h_{2,1} = \begin{pmatrix} 17 & 64 \\ 64 & 241 \end{pmatrix}$ which does not belong to $\Gamma(2^{k+2)})$ for $k \geq 3$. $\square$

Next we show, that $\Gamma(16)$ cannot be a subgroup of $\ker U_\alpha$. Assume this is the case. By lemma 6.6 $\ker U_\alpha = \Gamma(16)$. But according to (6.37) $U_\alpha(g_5) = U_\alpha(g_3)$ for the generators g_3 and g_5 of $\Gamma(4)$ in (6.33) and hence $g_3^{-1} g_5 \in \ker U_\alpha$. But $g_3^{-1} g_5 = \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix}$ mod 16 which does not belong to $\Gamma(16)$. Hence $\ker U_\alpha > \Gamma(16)$, which is a contradiction. This proves

COROLLARY 6.1. *The group* $\ker U_\alpha$ *can be a congruence group only for* $N(\alpha) = 1, 2$.

Let then α_1 and α_2 denote the α-values for which $N(\alpha_1) = 1$ and $N(\alpha_2) = 2$, respectively. We are going to prove $\ker U_{\alpha_1}$ and $\ker U_{\alpha_2}$ are indeed congruence groups. To this end recall that $\Gamma(4)/\ker U_\alpha \cong A_\alpha$. Since A_{α_1} is the trivial group, $\ker U_{\alpha_1} = \Gamma(4)$ and hence is a congruence group.
It remains to prove the congruence property of $\ker U_{\alpha_2}$. Since $N(\alpha_2) = 2$ and $U_{\alpha_2}(\Gamma(4)) = A_{\alpha_2}$, it follows that $(U_{\alpha_2}(g))^2 = id$ for all $g \in \Gamma(4)$ and hence $g^2 \in \ker U_{\alpha_2}$ for all $g \in \Gamma(4)$. But $g^2 \in \Gamma(8)$ for $g \in \Gamma(4)$. Therefore also the group $\langle g^2 : g \in \Gamma(4) \rangle$ generated by $\Gamma(4)^2$ belongs to $\ker U_{\alpha_2}$ and hence $\ker U_{\alpha_2} \cap \Gamma(8) \neq \emptyset$. Next we will show that the groups $\Gamma(8)$ and $\ker U_{\alpha_2}$ coincide. To this end we note that $A_{\alpha_2} \cong C_2 \times C_2 \times C_2$ where C_2 is the cyclic group of order 2. But $A_{\alpha_2} \cong \Gamma(4)/\ker U_{\alpha_2}$ under the following well known natural group isomorphism $\imath_1 : \Gamma(4)/\ker U_{\alpha_2} \to A_{\alpha_2}$:

$$\imath_1(g \ker U_{\alpha_2}) = U_{\alpha_2}(g). \tag{6.62}$$

Thereby the generators $A_i(\alpha_2), 1 \leq i \leq 3$ of the group A_{α_2} in (6.34)-(6.36) are mapped to the generators $g_i \ker U_{\alpha_2}, 1 \leq i \leq 3$ of the group $\Gamma(4)/\ker U_{\alpha_2}$ with the $\{g_i\}$ as given in (6.33). Indeed, from equation (6.33) it follows that $g_3 = g_5 \mod \ker U_{\alpha_2}$ and $g_4 = g_2^{-1} g_1^{-1} \mod \ker U_{\alpha_2}$. On the other hand, it is known [**33**], that $\Gamma(4)/\Gamma(8)$ is also isomorphic to $C_2 \times C_2 \times C_2$. Indeed, the elements $g_i\Gamma(8), 1 \leq i \leq 3$ with $\{g_i, 1 \leq i \leq 5\}$ defined in (6.33), are generators of the group $\Gamma(4)/\Gamma(8)$: we know that the five elements g_i, $1 \leq i \leq 5$, generate the group $\Gamma(4)$ and fulfill $g_i^2 = id \mod \Gamma(8)$. Furthermore one checks easily that $g_3 = g_5 \mod \Gamma(8)$ and $g_4 = g_2^{-1} g_1^{-1} \mod \Gamma(8)$. Therefore the following map of their generators defines an isomorphism $\imath$ of the two groups $\Gamma(4)/\ker U_{\alpha_2}$ and $\Gamma(4)/\Gamma(8)$

$$\imath : \Gamma(4)/\ker U_{\alpha_2} \to \Gamma(4)/\Gamma(8) \tag{6.63}$$

defined by

$$\imath(g_i \ker U_{\alpha_2}) = g_i\Gamma(8) \tag{6.64}$$

Indeed we have

$$\imath(g_i \ker U_{\alpha_2} g_j \ker U_{\alpha_2}) = \imath(g_1 g_2 \ker U_{\alpha_2}) = g_i g_j \Gamma(8) = g_i\Gamma(8) g_j\Gamma(8).$$

Since any $g \in \Gamma(4)$ can be expressed both modulo $\ker U_{\alpha_2}$ and modulo $\Gamma(8)$ in terms of the generators $g_i, 1 \leq i \leq 3$, this implies for all $g \in \Gamma(4)$ $\imath(g \ker U_{\alpha_2}) = g\Gamma(8)$. For $g \in \ker U_{\alpha_2}$ this implies necessarily $g \in \Gamma(8)$, that means $\Gamma(8) \geq \ker U_{\alpha_2}$. Similar arguments as in lemma 6.6 then imply, that the two groups must coincide. This shows

COROLLARY 6.2. *The kernel* $\ker U_{\alpha_2}$ *is given by* $\Gamma(8)$ *and hence* U_{α_2} *is a congruence representation.*

From the definition of the generators of A_α in (6.34), (6.35), and (6.36) it is clear that $N(\alpha_1) = 1$ iff $8\pi i\alpha_1 = 2\pi ik$ iff $\alpha_1 = (1/4)k$ with $k \in \mathbb{Z}$. Moreover, $N(\alpha_2) = 2$ iff $8\pi i\alpha_2 = \pi ik$ iff $\alpha_2 = (1/8)k$ with $k \in \mathbb{Z}$ and $(k, 2) = 1$.

Summarizing our discussion of the congruence properties of the kernels $\ker U_\alpha$ we have

THEOREM 6.8. *The representation* $U_\alpha, 0 \leq \alpha \leq 1/2$ *is congruence only for the* α*-values* $0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. *Moreover we have*

$$\ker U_0 = \ker U_{\frac{2}{8}} = \ker U_{\frac{4}{8}} = \Gamma(4), \tag{6.65}$$

respectively

$$\ker U_{\frac{1}{8}} = \ker U_{\frac{3}{8}} = \Gamma(8). \tag{6.66}$$

This obviously implies the well known result of Newman et al. on the congruence properties of the character χ_α. Contrary to the latter case, where the principal congruence groups $\Gamma(2d)$, $d = 1, 2, 4, 8$ appear as subgroups for the congruence character χ_α, for the induced representation U_α only the two groups $\Gamma(4)$ and $\Gamma(8)$ are related to the congruence properties of its representations.

6.4. Zograf's criterion

As mentioned in Section 6.1, for $\alpha = \frac{n}{d}$, $n, d \in \mathbb{Z}_+$ with $(n,d) = 1$ we have $\Gamma_d = \ker \widehat{\chi}_\alpha$ and Γ_d is congruence only for $d = 1, 2, 4, 8$. In terms of α-values in $[0,1)$, this means that $\ker \widehat{\chi}_\alpha$ is congruence for $\alpha = \frac{j}{8}$ with $j = 0, 1, \dots, 7$.

The fact that Γ_d for large d cannot be a congruence subgroup follows already from a remarkable geometric result of Zograf (see [**61**], [**62**]), based on previous results of Yang and Yau [**59**] respectively Hersh [**22**], together with Selberg's famous theorem on small eigenvalues (see [**50**]). We recall here these two Theorems of Zograf [**60**] and Selberg [**50**]:

Theorem 6.9 (Zograf). *Let Γ be a discrete cofinite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of signature $(g; m_1, m_2, \dots, m_k; h)$ and $a(F_\Gamma)$ be the hyperbolic area of its fundamental domain F_Γ. Assume $a(F_\Gamma) \geq 32\pi(g+1)$. Then the set of eigenvalues of the automorphic Laplacian $\Delta(\Gamma)$ in $(0, 1/4)$ is not empty and*

$$\lambda_1 < \frac{8\pi(g+1)}{a(F_\Gamma)} \tag{6.67}$$

where λ_1, $0 < \lambda_1 \leq \lambda_2, \dots$ is the first non zero eigenvalue of $\Delta(\Gamma)$.

Theorem 6.10 (Selberg). *Let Γ be a congruence subgroup of the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$. Then*

$$\frac{3}{16} \leq \lambda_1 \tag{6.68}$$

where the notations are the same as in Theorem 6.9.

Selberg's eigenvalue conjecture for congruence subgroups is indeed $\lambda_1 \geq 1/4$ (see [**50**]). Notice, that the interval $[0, 1/4)$ is free from the continuous spectrum of the automorphic Laplacian $\Delta(\Gamma)$ which is real and given by $[1/4, \infty)$. If we assume now Γ to be congruence and that $a(F_\Gamma) \geq 32\pi(g+1)$ then the following inequality holds

$$3/16 < \frac{8\pi(g(\Gamma)+1)}{a(F_\Gamma)}. \tag{6.69}$$

If we assume that for a given d the group Γ_d, which has vanishing genus g, is a congruence subgroup, we get from (6.69) that $3/16 < 8\pi/2\pi d$ or $d < 64/3$ and hence there are only finitely many d with Γ_d a congruence subgroup.

The groups $\Gamma_0(4)$ and $\Gamma(2)$ are conjugate, that is,

$$\Gamma_0(4) = V_2 \Gamma(2) V_2^{-1}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{6.70}$$

Also for the characters χ_α and $\widehat{\chi}_\alpha$ we have

$$\chi_\alpha(\gamma) = \widehat{\chi}_\alpha(V_2^{-1}\gamma V_2), \quad \gamma \in \Gamma_0(4). \tag{6.71}$$

Hence $\ker \chi_\alpha = V_2 \ker \widehat{\chi}_\alpha V_2^{-1}$. Thus for $\alpha = \frac{n}{d}$ with $(n,d)=1$, $n,d \in \mathbb{Z}_+$ we have

$$\ker \chi_\alpha = V_2 \Gamma_d V_2^{-1}. \tag{6.72}$$

From this it follows that $\ker \chi_\alpha$ has the same genus and area of its fundamental domain as Γ_d. Then from Zograf's criterion it follows that $\ker \chi_\alpha$ is congruent only for finitely many α in the interval $[0,1)$.

Contrary to χ_α, the congruence property of U_α can not be detected by Zograf's criterion. Indeed, according to Theorem 6.5 we have

$$a(F_{\ker U_\alpha}) = \frac{\pi}{3}(24N^3), \quad 32\pi(g(\ker U_\alpha)+1) = 32\pi(2+2N^3-3N^2). \tag{6.73}$$

Then one can easily check that the group $\ker U_\alpha$ does not satisfy Zograf's assumption.

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