On linear differential-algebraic equations with properly stated leading term.[1]

II: Critical points

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Abstract

This paper addresses critical points of linear differential-algebraic equations (DAEs) of the form $A(t)(D(t)x(t))' + B(t)x(t) = q(t)$ within a projector-based framework. We present a taxonomy of critical points which reflects the phenomenon from which the singularity stems; this taxonomy is proved independent of projectors and also invariant under linear time-varying coordinate changes and refactorizations. Under certain working assumptions, the analysis of such critical problems can be carried out through a *scalarly implicit* decoupling, and certain harmless problems in which such decoupling can be rewritten in explicit form are characterized. A linear, time-varying analogue of Chua's circuit is discussed with illustrative purposes.

Keywords: differential-algebraic equation, index, critical point, singular ODE, Chua's circuit.

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1 Introduction

The present paper extends the results of [12], focused on linear differential-algebraic equations (DAEs) of the form

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{J}, \tag{1}$$

by addressing so-called singular or *critical* points.

Roughly speaking, critical points of linear DAEs are often defined in terms of the non-existence or non-uniqueness of solutions at those points. This is the case for instance in [6], where a definition of critical point is provided for analytic problems relying upon the impossibility to continue a given solution. In the present paper we face critical points from a different point of view, described below.

Most approaches to the analysis of DAEs [1, 2, 5, 7, 8, 9, 13, 14] are based on an iterative or recursive definition of an index, and end up with some kind of related (underlying/inherent/reduced etc.) ODE. The index definition usually implies that this ODE is uniquely solvable. But, sometimes, although the analysis procedure can be completed, it ends up with a non-solvable (or a non-uniquely solvable) ODE. In linear cases, non-solvable continuous problems are related to *singularities* and, in the analytic setting, can be tackled via Fuchs-Frobenius theory. These singularities arise typically in the last step of the analysis procedure, and drive the problem to the setting of singular ODEs.

But we look at somehow pathological behavior not from the ODE perspective but *from the DAE viewpoint*. Hence, broadly speaking, critical points will be those where the DAE analysis procedure meets difficulties; formally, a point $t_* \in \mathcal{J}$ will be called critical if no neighborhood of t_* admits an index or, equivalently, there is no regularity interval including t_*. Instances of these phenomena, beyond the above-mentioned last-step singularities, are rank-changing points of an identically singular, non-analytic matrix function $A(t)$ in (1). If the analysis procedure can be adapted in order to handle these critical points, then it would typically end up with a singular ODE, although some (informally called *harmless*) cases may result in a non-singular (hence solvable) linear ODE. In contrast, in the most involved cases there is simply no way to obtain such an ODE.

From this point of view, we show in this paper how to adapt the projector-based analysis of linear DAEs introduced in [9, 10, 11] in order to accommodate critical points. Our approach extends the results of [15], proved for index-1 DAEs in standard form, and is directly based on [12]. Critical points will be classified according to a taxonomy which reflects the phenomenon from which the singularity stems. To emphasize that critical points arise at a well-defined step we use from [12] the notions of a *nice* at level k DAE, and *admissible* up to level k projector sequences. The different types of critical points will be proved independent of projectors and invariant with respect to rescaling and linear, time-dependent coordinate changes. This discussion is carried out in Section 2.

We then discuss in Section 3 working assumptions which, allowing to relax constant rank conditions, still make it possible to construct a chain of continuous matrix functions which particularizes to a tractability chain at regular points. Several algebraic properties of this modified or critical matrix chain will be analyzed. As shown in Section 4, under the same working assumptions the dynamical behavior of the DAE can be unveiled through a decoupling based on a scalarly implicit inherent ODE. Our working scenario allows for a *uniform over singularities* treatment, in problems which include non-isolated critical cases beyond the analytic setting of [6, 13]. Finally, this framework will be applied in Section 5 to the analysis of critical points of the linear, time-varying analogue of Chua's circuit introduced in [12].

2 Critical points

The reader is here referred to [12, Section 2] for the concept of a DAE with properly stated leading term, and to Section 3 of that paper for the notions of (algebraically) nice DAEs, (pre)admissible projector sequences, and regular DAEs with index μ on a given subinterval $\mathcal{I} \subseteq \mathcal{J}$. All these concepts are based on the construction of a tractability chain $\{G_i(t)\}$ on $\mathcal{I}$ satisfying certain

a) constant rank;

b) transversality;

c) smoothness

assumptions at every step k. Section 4 of the same work discusses the corresponding notions at a given point $t_* \in \mathcal{J}$.

Definition 1 *Assume that the DAE (1) has continuous coefficients $A(t), D(t), B(t)$. A point t_* is said to be* critical *if there is no regularity interval comprising it.*

As indicated in the Introduction, critical points may well arise in the last step of the tractability chain construction (see [12, Section 3]), leading to a singular inherent ODE. This is the case in example 1 below.

Example 1 Consider the DAE

$$\begin{bmatrix}1\\1\end{bmatrix}([1\ \ -t]x(t))' + \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}x(t) = q(t),\ t \in \mathcal{J} = (-\infty, \infty) \tag{2}$$

which has a properly stated leading term with $m = 2,\ n = 1,\ \ker A = 0,\ \operatorname{im} D = I\!R^n,\ R = 1$. The chain construction can be performed up to level $k = 1$:

$$G_0 = \begin{bmatrix}1 & -t\\1 & -t\end{bmatrix},\ P_0 = \begin{bmatrix}1 & -t\\0 & 0\end{bmatrix},\ Q_0 = \begin{bmatrix}0 & t\\0 & 1\end{bmatrix},\ D^- = \begin{bmatrix}1\\0\end{bmatrix},\ G_1 = \begin{bmatrix}1 & 1\\1 & 2-t\end{bmatrix},$$

with $\det G_1 = 1 - t,\ G_1^{-1} = \frac{1}{1-t}\begin{bmatrix}2-t & -1\\-1 & 1\end{bmatrix}$. While all points being not equal to 1 are regular (with index one), $t_* = 1$ is a critical point, since G_1 undergoes a rank deficiency there. For $t < 1$ and $t > 1$ the solutions of the DAE (2) are given by the expression

$$x(t) = \frac{1}{1-t}\begin{bmatrix}1\\1\end{bmatrix}u(t) - \frac{1}{1-t}\begin{bmatrix}t(q_1(t) - q_2(t))\\q_1(t) - q_2(t)\end{bmatrix},$$

where u solves the singular ODE (see [12])

$$u'(t) + \frac{2}{1-t}u(t) = \frac{1}{1-t}(2q_1(t) - (t+1)q_2(t)). \tag{3}$$

Observe that here, for $t > 0$ and $t < 0$,

$$S_{can_1}(t) = S_0(t) := \{z \in I\!R^2 : B(t)z \in imG_0(t)\} = \{z \in I\!R^2 : z_1 = z_2\},$$

$$N_0(t) \cap S_0(t) = \{z \in I\!R^2 : z_1 = z_2,\ z_1 = tz_2\} = 0.$$

A geometrical picture of the critical phenomenon occurring at t_* is the loss of transversality of these subspaces, since $N_0(t_*) \cap S_0(t_*)$ has dimension one. The canonical projector $\Pi_{can_1} = (I - \mathcal{K}_0)P_0$ onto S_0 along N_0

$$\Pi_{can_1}(t) = \frac{1}{1-t}\begin{bmatrix}1 & -t\\1 & -t\end{bmatrix}$$

does not exist for $t \to t_* = 1$.

The homogeneous inherent ODE (3) with $q_1 = 0,\ q_2 = 0$ has the solutions $u(t) = (t-1)^2u(0)$. The solutions of the homogeneous DAE (2) with $q = 0$ are then

$$x(t) = (1-t)u(0)\begin{bmatrix}1\\1\end{bmatrix},\ u(0) \in I\!R,$$

which shows that the space of the solution values at $t_* = 1$ consists of the origin only while, for $t \neq t_*$, the space $S_0(t)$ is filled by solution values. Note that uniqueness of solutions is broken at the critical point.

Nevertheless, critical points do not only follow from rank deficiencies occurring in the last step of the chain construction. In more involved cases, the chain construction cannot be carried out up to the last step in the whole interval $\mathcal{J}$. In this regard, we will not be interested in problems that arise if a given continuous component should be continuously differentiable but is not, that is, we do not consider critical points that arise just from the fact that pre-admissible projector sequences may be not admissible. We suppose the DAE coefficients to be smooth enough to avoid those cases; sufficient conditions for this will be given in Proposition 3. Our interest is directed to critical points corresponding to failures of the used algebraic constant-rank and transversality conditions, as acknowledged in Definition 3. Proposition 1 will prove that these notions are actually independent of the (admissible up to level $k-1$) choice of projectors.

Definition 2 *A continuous matrix function $G : \mathcal{I} \to L(\mathbb{R}^k)$, $\mathcal{I} \subseteq \mathbb{R}$ an interval, has a rank drop at $t_* \in \mathcal{I}$, if each neighborhood of t_* contains points where the rank is different from $\operatorname{rk} G(t_*)$. Then, t_* is called a rank-change or rank-drop point of G.*

Definition 3 *Given the DAE (1) with continuous coefficients, $t_* \in \mathcal{J}$ will be said to be a critical point of*

(i) type 0 if G_0 has a rank drop at t_*;

(ii) type k-A, $k \geq 1$, if there exists a neighborhood $\mathcal{I} \subseteq \mathcal{J}$ of t_* where the DAE is nice up to level $k-1$, but G_k has a rank drop at t_* for some (hence any) admissible sequence $Q_0, \ldots, Q_{k-1}$;

(iii) type k-B, $k \geq 1$, if there exists a neighborhood $\mathcal{I} \subseteq \mathcal{J}$ of t_* where the DAE is nice up to level $k-1$ and G_k has constant rank for some (hence any) admissible sequence $Q_0, \ldots, Q_{k-1}$, but the intersection $N_k(t_*) \cap \{N_0(t_*) \oplus \cdots \oplus N_{k-1}(t_*)\}$ is nontrivial, for these (hence any other) projectors and G_k.

We will often speak of level-k critical points to refer either to type k-A or type k-B; and we will say that a critical point is of type A or B if it has type k-A or k-B with arbitrary k.

Definition 4 *A critical point $t_* \in \mathcal{J}$ of the DAE (1) is isolated if there is a neighborhood of t_* such that all points that are different from t_* are regular.*

By definition, an isolated critical point t_* of type $(k+1)$ is the border of two regularity intervals, say $\mathcal{I}^-$ and $\mathcal{I}^+$, $\mathcal{I}^-, \mathcal{I}^+ \subset \mathcal{I}$. The characteristic values $r_0, \ldots, r_k$ apply to both intervals since there is a sequence $Q_0, \ldots, Q_k$ that is admissible on $\mathcal{I}$. In case of a type $(k+1)$-B critical point, G_{k+1} has uniform rank r_{k+1} on $\mathcal{I}^-$ and $\mathcal{I}^+$. However, there may be different further characteristics $\mu^-, r^-_{k+2}, \ldots, r_{\mu^- -1}$, and $\mu^+, r^+_{k+2}, \ldots, r_{\mu^+ -1}$ on $\mathcal{I}^-$ resp. $\mathcal{I}^+$. In case of a type $(k+1)$-A critical point, there are possibly different characteristic values r^-_{k+1}, r^+_{k+1}, too.

With this terminology, $t_* = 1$ in Example 1 is an isolated critical point of type 1-A. More subtle phenomena are defined by the critical points of types 1-B and 0, respectively, in Examples 2 and 3 below.

Example 2 Consider the circuit displayed in Figure 1, defined by a parallel connection of an independent voltage source, a capacitor and an inductor which are linear time-invariant, and a time-varying current-controlled current source (CCCS).

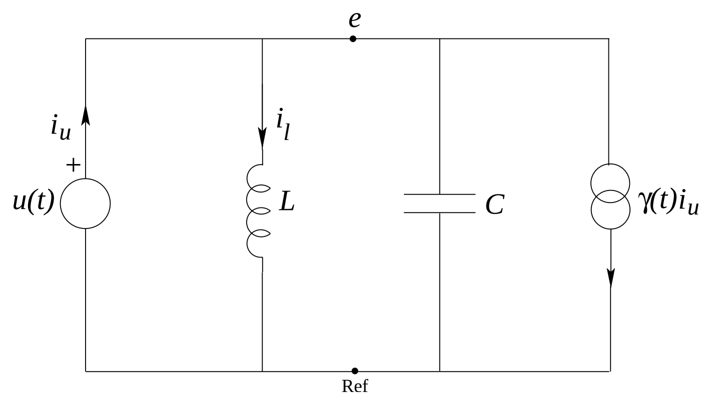

Figure 1: A linear time-varying circuit with a current attenuator/amplifier.

Modified Node Analysis (MNA) equations [4] read for this circuit

$$(Ce)' + i_l + (\gamma(t) - 1)i_u = 0, \tag{4a}$$
$$(Li_l)' - e = 0, \tag{4b}$$
$$e = u(t), \tag{4c}$$

and this system can be written as a DAE with properly stated leading term letting

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} C & 0 & 0 \\ 0 & L & 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} C & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & \gamma(t) - 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The function $\gamma(t)$ is continuous, and satisfies $\gamma(t) < 1$ if $t < 0$ and $\gamma(t) > 1$ if $t > 0$. At the point $t = 0$ it is $\gamma(0) = 1$ and the behavior of the CCCS switches from that of an attenuator to one of an amplifier. From an electrical point of view, it is worth emphasizing that the network includes a C-V loop, and that the controlling current of the CCCS is the one of a voltage source within a C-V loop, hence falling out of the scope of [4] (see specifically item 4 of Table V there).

Write in the sequel $\alpha(t) = \gamma(t) - 1$, so that $\alpha(t) = 0$ iff $t = 0$. Choose $D^- = \begin{bmatrix} 1/C & 0 \\ 0 & 1/L \\ 0 & 0 \end{bmatrix}$, and $Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that $G_1 = \begin{bmatrix} C & 0 & \alpha(t) \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}$ results. $G_1(t)$ has constant rank $r_1 = 2$, and the nullspace $N_1 = \ker G_1$ is continuous. Compute $N_0(t) \cap N_1(t) = \{z \in I\!R^3 : z_1 = 0, z_2 = 0, \alpha(t)z_3 = 0\}$, hence $N_0(t) \cap N_1(t) = \{0\}$, for $t \neq 0$, $N_0(0) \cap N_1(0) = N_0(0)$. Therefore, $t_* = 0$ is a critical point of type 1-B.

On $\mathbb{R}^-$ and $\mathbb{R}^+$ we may choose $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\alpha} & 0 & 0 \end{bmatrix}$, yielding $Q_1 = Q_1 P_0,\ \ P_0 Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $DP_0Q_1D^- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B_1 = BP_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} C & 0 & \alpha \\ -1 & L & 0 \\ 1 & 0 & 0 \end{bmatrix}$. It results that $G_2(t)$ is nonsingular except at the critical point $t_* = 0$, the problem hence being regular with index 2 in $\mathbb{R}^-$ and $\mathbb{R}^+$.

The last example in this Section, discussed below, attempts to illustrate that critical points of type A do not necessarily yield a singular inherent ODE. This *harmless* phenomenon will be discussed in more detail in Section 3.

Example 3 Consider the DAE

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} ([0\ \ \alpha]\ x)' + x = q, \tag{5}$$

where α is a continuous scalar function, $\mathcal{J} = (-\infty, \infty), n = 1, m = 2, D = [0 \quad \alpha], A = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, $B = I, G_0 = \begin{bmatrix} 0 & \alpha^2 \\ 0 & 0 \end{bmatrix}$. All points t_* with $\alpha(t_*) \neq 0$ are regular ones. Namely, G_0 has constant rank $r_0 = 1$ in a neighborhood of t_*. There we may choose $D^- = \begin{bmatrix} 0 \\ \frac{1}{\alpha} \end{bmatrix}, Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & \alpha^2 \\ 0 & 0 \end{bmatrix}$ has constant rank $r_1 = 1$, it holds that $N_0 \cap N_1 = 0, Q_0$ and $Q_1 := \begin{bmatrix} 0 & -\alpha^2 \\ 0 & 1 \end{bmatrix}$ form an admissible projector sequence, $DP_0P_1D^- = 0, G_2 = \begin{bmatrix} 1 & \alpha^2 \\ 0 & 1 \end{bmatrix}$, hence $r_2 = 2, \mu = 2$. On the regularity interval the solution of the DAE is given by the expression

$$x = (P_0Q_1 + Q_0P_1)G_2^{-1}q + Q_0Q_1D^-(DP_0Q_1G_2^{-1}q)' = q - \begin{bmatrix} \alpha \\ 0 \end{bmatrix} (\alpha q_2)', \tag{6}$$

where the coefficients are

$$Q_0Q_1D^- = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad DP_0Q_1G_2^{-1} = (\alpha\ \ 0), \quad (P_0Q_1 + Q_0P_1)G_2^{-1} = I. \tag{7}$$

On intervals where $\alpha(t)$ vanishes identically, we have simply $A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $D = [0\ 0]$, $D^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $G_0 = 0,\ Q_0 = I,\ r_0 = 0,\ G_1 = I,\ r_1 = 2,\ \mu = 1$, that is, the DAE (5) is there regular with index one. Letting $Q_1 = 0,\ P_1 = I,\ G_2 = G_1$, we find $DP_0P_1D^- = 0$ and (6), (7) keeps its value also in this case.

If $t_* \in (-\infty, \infty)$ is such that $\alpha(t_*) = 0$ but in any neighborhood of t_* there are points t with $\alpha(t) \neq 0$, then t_* is no longer regular but a type 0 critical point. In the particular case

$$\alpha(t) = t \quad \text{if} \quad t \geq 0, \qquad \alpha(t) = 0 \quad \text{if} \quad t \leq 0, \tag{8}$$

the DAE (5) is regular with index two on $(0, \infty)$ but regular with index one on $(-\infty, 0)$. The point $t_* = 0$ is no longer regular. Here, at t_*, the characteristic numbers r_0, r_1 and μ change their value. Nevertheless, $DP_0P_1D^- = 0$ and (7) hold true on both intervals $(-\infty, 0)$ and $(0, \infty)$ so that all these terms have continuous extensions on $(-\infty, \infty)$, and the solution expression (6) holds true on $(-\infty, \infty)$.

If we consider (5) with $\alpha(t) = t^k$ or with $\alpha(t) = t^{1/3}$ instead of (8), formulas (6), (7) result again, and, furthermore, now the characteristics $r_0 = 1, r_1 = 1, r_2 = 2, \mu = 2$ are equal for both $(-\infty, 0)$ and $(0, \infty)$. Observe that, in all cases, it holds on regularity intervals that $DP_0P_1D^- = 0$, so that there is a trivial smooth extension on the whole interval $\mathcal{J}$. Moreover, the function G_2 can be continuously extended on $\mathcal{J}$ and the extension remains nonsingular.

Critical points of type 0, that is, rank drops in G_0, may be caused by rank drops in A or D, or in both, but also by failures of the transversality condition for $\ker A$ and $\operatorname{im} D$. At those points, D^-, R and P_0 are no longer continuous, however, they may have continuous extensions through the critical point. We will focus our interest on cases in which P_0 has a continuous extension.

Propositions 1 and 2 below are an easy consequence of [12, Proposition 2].

Proposition 1 *The definitions of critical points of types k-A and k-B are independent of the (admissible) choice of projectors.*

Proposition 2 *With every critical point of type k we may associate a* characteristic critical value *in an invariant manner, namely:*

(i) rk $G_k(t_*)$ *for those of type A;*

(ii) dim $(N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k(t_*)$ *for those of type B.*

Furthermore, if t_* is an isolated k-A critical point, and $r_k = \text{rk } G_k(t)$ is constant in some punctured neighborhood $\mathcal{I}^{t_*} - \{t_*\}$, then we may speak properly of the *rank deficiency* at t_*, which is also independent of projectors.

In sufficiently smooth cases, the only critical points are those of types A and B, as proved below.

Proposition 3 *Assume that the coefficients $A(t)$, $D(t)$, $B(t)$ in the DAE (1) are C^{m-1}. Then every critical point is of type k-A or k-B, with $1 \leq k \leq m$, or of type 0.*

Proof: Note that, if the DAE is algebraically nice at a given level $k \leq m-1$, then the smoothness requirement (iv) in [12, Definition 5] can be satisfied. This is due to the fact that, in the matrix chain construction, supposed the DAE to be algebraically nice at level 0, we can take Q_0 in the class C^{m-1}, so that $G_1 = G_0 + BQ_0$ is also C^{m-1}. If neither type 1-A nor type 1-B singularities are met, then we may choose a preadmissible Q_1 in C^{m-1}, so that Q_1 will actually be admissible. Then B_1 and so G_2 will be in the class C^{m-2}.

If no critical points are displayed in subsequent levels nor an invertible G_i is met, we can continue the sequence in an admissible manner up to G_{m-1}, Q_{m-1} in C^1, so that continuous B_{m-1} and G_m can be constructed. Now, if G_m is singular and has constant rank (so that regular points with index m and critical points of type m-A are also ruled out), then $(N_0 \oplus \ldots \oplus N_{m-1}) \cap N_m = \mathbb{R}^m \cap N_m = N_m \neq \{0\}$ and a critical point of type m-B is met.

□

This means that, for sufficiently smooth (meaning in this context C^{m-1}) DAEs, there will be no difficulties in the particular smoothness requirement (iv). In this situation, a DAE will be nice at a given level if and only if it is algebraically nice at the same level, and if a preadmissible sequence up to a given level exists, then an admissible one exists at the same level. This is implicit in the classification of critical points depicted in Figure 4 at the end of the paper.

We finish this Section by showing that the taxonomy of critical points presented in Definition 3 is invariant under linear time-varying coordinate changes $x(t) = K(t)y(t)$ and premultiplication by a continuous matrix function $L(t)$.

Proposition 4 *Let t_* be a critical point of type k-A or k-B, $1 \leq k \leq m$, or of type 0, for the DAE (1). Then t_* is a critical point of the same type for the rescaled, transformed DAE*

$$\tilde{A}(t)(\tilde{D}(t)y(t))' + \tilde{B}(t)y(t) = L(t)q(t), \quad t \in \mathcal{J}, \tag{9}$$

with nonsingular $L(t),\ K(t) \in C(\mathcal{J}, L(\mathbb{R}^m))$, $\tilde{A}(t) = L(t)A(t)$, $\tilde{D}(t) = D(t)K(t)$, $\tilde{B}(t) = L(t)B(t)K(t)$.

Proof: The result follows from the construction of the projectors $\tilde{Q}_i = K^{-1}Q_iK$ [9] for (9), which results in the identities $\tilde{G}_i = LG_iK$. The rank of G_i is therefore transferred to $\tilde{G}_i$ and type 0 and type-A singularities are hence invariant. Additionally, $\tilde{N}_i = \ker \tilde{G}_i = K^{-1}N_i$, so that the loss of transversality in the N_i spaces defining type-B singularities is also transferred to $\tilde{N}_i$.

□

3 Algebraic aspects of critical chains

Generally speaking, the existence of critical points in $\mathcal{J}$ precludes the construction of a tractability chain defined on the whole $\mathcal{J}$. We figure out in this Section working assumptions which make it possible a "uniform over singularities" treatment, that is, the construction of a globally defined matrix chain which yields a tractability chain at regular points. Several algebraic properties of such *critical* chains are considered in this Section, whereas the fact that they allow to unravel the behavior through a scalarly implicit inherent ODE is addressed in Section 4.

Assumption 1. *The set $\mathcal{I}_{\rm reg}$ of regular points is dense in $\mathcal{J}$.*

We restrict further the attention to problems with *almost uniform characteristic values*, defined by the following working hypothesis.

Assumption 2. *There exist projector functions $Q_0, \ldots, Q_{m-1}$ on $\mathcal{J}$ such that, for $0 \leq i \leq m-1$:*

(i) Q_i *is continuous in the whole* $\mathcal{J}$*;*

(ii) Q_i *is onto* $\ker G_i$ *for all* $t \in \mathcal{I}_{\rm reg}$*;*

(iii) $Q_iQ_j = 0$ *for all* $t \in \mathcal{J}$, $0 \leq j < i$*;*

(iv) $DP_0 \cdots P_iD^-$ *is continuously differentiable in* $\mathcal{I}_{reg}$*, and* $(DP_0 \cdots P_iD^-)'$, $D^-(DP_0 \cdots P_iD^-)'D$ *have continuous extensions on* $\mathcal{J}$*.*

These working assumptions make it possible to define a matrix chain as in the regular setting. But now it accommodates rank deficiencies in some of the G_i matrices at critical points. This *critical* chain has two important properties, described in Propositions 5 and 6 below. In particular, Proposition 5 follows immediately from the constant rank condition on G_i in $\mathcal{I}_{\rm reg}$ implied by (i) and (ii) in assumption 2.

Proposition 5 *Under assumptions 1-2, the DAE has the same characteristic values and, in particular, the same index μ in the whole $\mathcal{I}_{\rm reg}$.*

Note that $Q_\mu = ... = Q_{m-1} = 0$ and then $G_\mu = ... = G_m$ in the whole $\mathcal{J}$. Since the projectors Q_i realize the "regular" matrix chain on $\mathcal{I}_{\rm reg}$ and are well-defined and continuous on the whole interval, we will say that the DAE is *almost uniformly regular with index μ* on $\mathcal{J}$.

Proposition 6 *Assumptions 1-2 rule out type B critical points on $\mathcal{J}$.*

Proof: Fix $t_* \in \mathcal{J}$. Assume that the DAE is nice up to level $k-1$, for some $k \geq 1$, and that $\operatorname{rk} G_k$ is constant in some neighborhood of t_*, so that t_* is not a type k-A critical point, and $N_k = \ker G_k$ has constant dimension around t_*. We need to show that t_* cannot be a type k-B critical point, namely, that $N_k(t_*)$ and $N_0(t_*) \oplus \ldots \oplus N_{k-1}(t_*)$ intersect trivially. From assumption 2-(iii), it follows that $\operatorname{im} Q_k(t)$ is transversal to $N_0(t) \oplus \ldots \oplus N_{k-1}(t)$ for all t in some neighborhood of t_*; it then suffices to show that $\operatorname{im} Q_k(t_*) = N_k(t_*)$. On the one hand, both spaces have the same dimension due to the continuity of Q_k and the constant dimension of N_k; on the other hand, from the vanishing of the continuous product G_kQ_k in the dense set $\mathcal{I}_{\rm reg}$, it follows that $G_k(t_*)Q_k(t_*) = 0$, so that $\operatorname{im} Q_k(t_*) \subseteq \ker G_k(t_*) = N_k(t_*)$. This proves that type k-B critical points are precluded by assumptions 1-2, for $1 \leq k \leq m$.

$\square$

We will hence call the matrix chain G_i constructed under assumptions 1-2 an A-critical chain. This provides a setting beyond just μ-A critical points; note that these last step critical points do not put in question the matrix chain construction (and the regular framework could be essentially used), in contrast to the ones in previous levels.

On harmless A-critical chains

We will show in Section 4 that the behavior of critical DAEs, under assumptions 1-2, can be unveiled through a scalarly implicit decoupling. The leading coefficient will actually vanish iff G_μ is singular. Therefore we might speak of "harmless" critical points t_* in A-critical chains if $G_\mu(t_*)$ is non-singular. Obviously, μ-A critical points are never harmless. But this is not always the case for lower level A critical points. Consider the DAE

$$\begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \right)' + x = q, \tag{10}$$

with the type 0 critical point $t_* = 0$, and which is regular for $t \neq 0$,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ D^- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ G_0 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix},\ Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ G_1 = \begin{bmatrix} 0 & 0 \\ t & 1 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 & 0 \\ -t & 0 \end{bmatrix},$$

$$G_1 + B_0 P_0 Q_1 = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

It therefore seems to be an interesting problem to check when type k-A critical points are lifted to the $(k+1)$-level or, more generally, how rank deficiencies *overlap* or *accumulate* at different levels, for every critical point t_*. A result in this direction follows:

Proposition 7 *Under assumptions 1-2, a type $(\mu-1)$-A critical point t_* leads to a singular $G_\mu(t_*)$.*

Proof: Note that the identity $G_{\mu-1}Q_{\mu-1} = 0$ holds in the whole $\mathcal{J}$ by continuity and due to the fact that it holds in the dense subset $\mathcal{I}_{\text{reg}}$. Since $G_{\mu-1}$ undergoes a rank drop at t_*, and $Q_{\mu-1}$ has constant rank by assumption, it follows that $\operatorname{im} Q_{\mu-1}(t_*)$ is strictly contained in $\ker G_{\mu-1}(t_*)$. Equivalently, there exists a nontrivial vector $z \in \ker G_{\mu-1}(t_*) - \operatorname{im} Q_{\mu-1}(t_*)$. Because of this, it is

$$P_{\mu-1}(t_*)z = (I - Q_{\mu-1}(t_*))z \neq 0,$$

and

$$G_\mu(t_*)P_{\mu-1}(t_*)z = G_{\mu-1}(t_*)P_{\mu-1}(t_*)z + B_{\mu-1}(t_*)Q_{\mu-1}(t_*)P_{\mu-1}(t_*)z = 0,$$

since $G_\mu(t_*)P_{\mu-1}(t_*)z = G_\mu(t_*)z = 0$ and $Q_{\mu-1}(t_*)P_{\mu-1}(t_*) = 0$. This proves that $\ker G_\mu(t_*)$ includes a non-vanishing vector.

$\square$

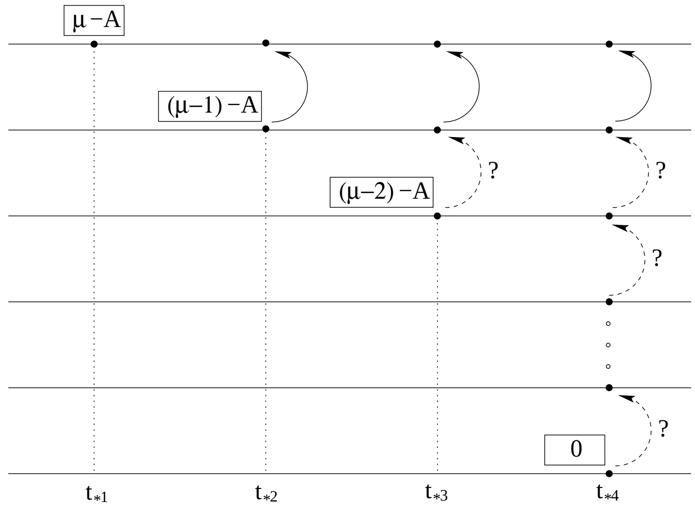

Figure 2: Accumulation or overlapping of rank deficiencies in an A-chain.

Corollary 1 *A necessary condition for t_* to be harmless is that $G_{\mu-1}$ has constant rank in some neighborhood of t_*.*

4 Dynamical aspects

We generalize [15, Th. 3], holding for standard-form index-1 problems, along the lines defined by [9, 10], making use of the fact that any continuous function which vanishes on a dense subset of $\mathcal{I}$ actually vanishes on the whole $\mathcal{I}$.

Proposition 8 *Let assumptions 1-2 hold. Then, $x \in C^1_D(\mathcal{I}, \mathbb{R}^m)$, $\mathcal{I} \subseteq \mathcal{J}$, is a solution of the DAE (1) if and only if it can be written as*

$$x = D^- u + v_0 + \cdots + v_{\mu-1}, \tag{11}$$

where $u \in C^1(\mathcal{I}, \mathbb{R}^n)$ is a solution of the scalarly implicit ODE

$$\omega_\mu u' - \omega_\mu (DP_0 \cdots P_{\mu-1} D^-)' u + DP_0 \cdots P_{\mu-1} G_\mu^{adj} BD^- u = DP_0 \cdots P_{\mu-1} G_\mu^{adj} q, \tag{12}$$

on the locally invariant space $\operatorname{im} DP_0 \cdots P_{\mu-1}$*, and $v_i \in C^1_D(\mathcal{I}, \mathbb{R}^m)$, $i = 1, \ldots, \mu-1$, $v_0 \in C(\mathcal{I}, \mathbb{R}^m)$ satisfy*

$$\omega_\mu v_{\mu-1} = -\mathcal{K}^{adj}_{\mu-1} D^- u + \mathcal{L}^{adj}_{\mu-1} q, \tag{13a}$$

$$\omega_\mu v_k = -\mathcal{K}^{adj}_k D^- u + \mathcal{L}^{adj}_k q + \omega_\mu \sum_{j=k+1}^{\mu-1} \mathcal{N}_{kj} (Dv_j)' + \omega_\mu \sum_{j=k+2}^{\mu-1} \mathcal{M}_{kj} v_j, \ \ k = \mu-2, \ldots, 1, 0. \tag{13b}$$

Here, ω_μ stands for $\det G_\mu$, and G_μ^{adj} is the transposed matrix of cofactors of G_μ. The coefficients $\mathcal{K}^{adj}_k$, $\mathcal{L}^{adj}_k$, $\mathcal{N}_{kj}$, $\mathcal{M}_{kj}$ are given in the Appendix.

Proof: Let $x \in C^1_D(\mathcal{I}, I\!R^m)$ be a solution of (1) on some subinterval $\mathcal{I} \subseteq \mathcal{J}$. Since the identity $AR = A$ holds in the dense subset $\mathcal{I} \cap \mathcal{I}_{\text{reg}}$, it remains true in the whole $\mathcal{I}$. We may then write $A = ADD^-$ in (1). Premultiplying by G_μ^{adj} and using

$$G_\mu^{adj} G_k \quad = \quad G_\mu^{adj} G_\mu P_{\mu-1} \cdots P_k = \omega_\mu P_{\mu-1} \cdots P_k, \tag{14}$$

in the particular case $k = 0$, we transform (1) into

$$\omega_\mu P_{\mu-1} \cdots P_0 D^-(Dx)' + G_\mu^{adj} Bx = G_\mu^{adj} q. \tag{15}$$

Now, writing

$$B = BP_0 \cdots P_{\mu-1} + BP_0 \cdots P_{\mu-2} Q_{\mu-1} + \ldots + BP_0 Q_1 + BQ_0, \tag{16}$$

taking into account the definition of B_i as well as the relations

$$G_\mu^{adj} B_k Q_k \quad = \quad G_\mu^{adj} G_{k+1} Q_k = \omega_\mu P_{\mu-1} \cdots P_{k+1} Q_k = \omega_\mu Q_k, \ 0 \le k \le \mu - 1, \tag{17}$$

and premultiplying by $DP_0 \cdots P_{\mu-1}$, we arrive at

$$\begin{aligned} \omega_\mu (DP_0 \cdots P_{\mu-1} x)' - \omega_\mu (DP_0 \cdots P_{\mu-1} D^-)' DP_0 \cdots P_{\mu-1} x + \\ + DP_0 \cdots P_{\mu-1} G_\mu^{adj} BD^- DP_0 \cdots P_{\mu-1} x = \quad DP_0 \cdots P_{\mu-1} G_\mu^{adj} q, \end{aligned}$$

which is the scalarly implicit inherent ODE (12) with $u = DP_0 \cdots P_{\mu-1} x$.

Proceeding as in [10], if we multiply (15) by $Q_{\mu-1}$, then by $P_0 \cdots P_{\mu-2}$ if $\mu \ge 2$, and in turn by $Q_0 P_1 \cdots P_{\mu-1}$ and $P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{\mu-1}$, we get the relations depicted in (13) for $v_0 = Q_0 x, \ v_i = P_0 \cdots P_{i-1} Q_i x, \ \ i = 1, \ldots, \mu - 1$.

The local invariance of the space $\operatorname{im} DP_0 \cdots P_{\mu-1}$ owes to the fact that $\alpha = (I - DP_0 \cdots P_{\mu-1} D^-) u$ satisfies the homogeneous equation $\omega_\mu[\alpha' + (DP_0 \cdots P_{\mu-1} D^-)' \alpha] = 0$ on $\mathcal{I}$. Since $\omega_\mu \neq 0$ on a dense set, it follows that $\alpha' + (DP_0 \cdots P_{\mu-1} D^-)' \alpha = 0$ on $\mathcal{I}$, and therefore a vanishing initial condition for α (in virtue of $u = DP_0 \cdots P_{\mu-1} D^- u$) yields a trivial solution in the whole $\mathcal{I}$.

Conversely, we need to show that, provided $u, \ v_0, \ldots, v_{\mu-1}$ verify (12)-(13) in some subinterval $\mathcal{I}$, then $x = D^- u + v_0 + \cdots + v_{\mu-1}$ solves the DAE (1) on $\mathcal{I}$. But this is easier, since from the smoothness properties of $u, \ v_0, \ldots, v_{\mu-1}$ it follows that $x \in C^1_D(\mathcal{I}, \mathbb{R}^m)$, and additionally we know the relation $A(Dx)' + Bx - q = 0$ to be satisfied on $\mathcal{I} \cap \mathcal{I}_{\text{reg}}$. Now, $A(Dx)' + Bx - q$ is continuous and, again by density of $\mathcal{I} \cap \mathcal{I}_{\text{reg}}$, the identity $A(Dx)' + Bx - q = 0$ must hold in the whole $\mathcal{I}$, showing that x actually solves (1).

□

Under assumptions 1-2, the behavior of a DAE with critical points is therefore addressed in terms of the scalarly implicit decoupling (12)-(13). This way, the analysis of a critical DAE is driven to the singular ODE setting; this is analogous to the approach carried out in the reduction approach of Rabier and Rheinboldt [13] and also in the framework of the strangeness index by Ilchmann and Mehrmann [6]. Note that in those works the results hold only for analytic problems.

Note, however, that not necessarily ω_μ vanishes at all critical points. This reflects that critical points, defined in terms of the failing of the regularity assumptions in the chain construction, do not exclude cases with unique solvability properties. A well-known example is (10), for which $x_1 = q_1(t),\ x_2 = q_2(t) - tq_1'(t)$ is a well-defined (actually unique) solution. The same phenomenon is acknowledged in the above-mentioned approaches [6, 13]. A complete formal characterization of these "harmless" critical points is currently an open problem. Proposition 7 provides a result in this direction. An important consequence of this proposition is that this type of harmless critical points cannot follow from rank deficiencies in the leading matrix in almost uniformly index-1 problems such as those of [15]. Obviously, a type μ-A critical point will always yield a zero in ω_μ.

5 Critical points of a linear time-varying Chua's circuit

Following [12], we consider the linear time varying analogue of Chua's circuit [3] with current-controlled resistors depicted in Figure 3. The framework presented in previous sections will make it possible to classify the critical points arising in the DAE model of this circuit; in particular, the harmless nature of certain type-0 critical points owing to the vanishing of the values of reactances will be discussed in an index-2 context.

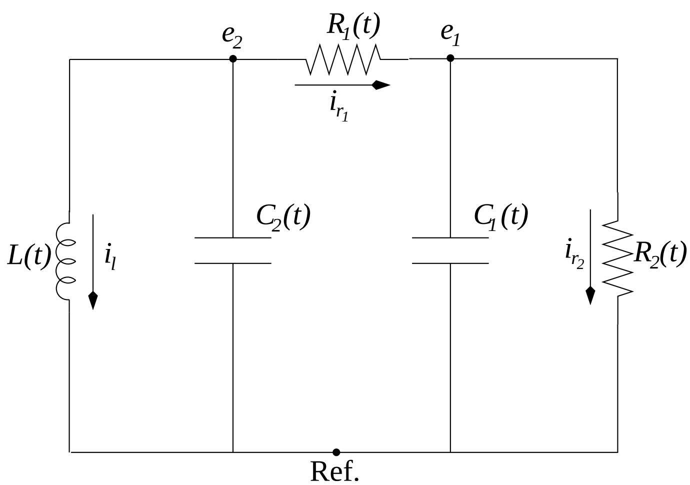

Figure 3: Linear time-varying Chua's circuit with current-controlled resistors.

Due to the current-control assumption and the eventual vanishing of $R_1(t)$ and $R_2(t)$, resistors' currents appear as variables in the Modified Nodal Analysis (MNA) model

$$(C_1(t)e_1)' - i_{r_1} + i_{r_2} = 0 \tag{18a}$$

$$(C_2(t)e_2)' + i_l + i_{r_1} = 0 \tag{18b}$$

$$(L(t)i_l)' - e_2 = 0 \tag{18c}$$

$$e_2 - e_1 - R_1(t)i_{r_1} = 0 \tag{18d}$$

$$e_1 - R_2(t)i_{r_2} = 0. \tag{18e}$$

This setting precludes the standard state reduction to Chua's equation [3] and drives the problem to the DAE context.

We assume in the sequel that the resistor R_1 verifies $R_1(t) > 0$ for $t < 0$, and $R_1(t) = 0$ for $t \geq 0$. This models a persistent short-circuit in the interval $[0, \infty)$; note that the model (18) is valid in the whole real line. Below we consider critical points due to the vanishing of R_2, C_1, C_2 or L in both subintervals.

$\boldsymbol{R_1(t) > 0.}$ In [12], it is shown that the simultaneous non-vanishing at a given t of R_1, R_2, C_1, C_2 and L defines t as a regular index-1 point. Assume that the non-vanishing of these values holds true in some open dense subset of $\mathbb{R}^-$ (where $R_1 > 0$), and let us analyze the effect of the vanishing of R_2, C_1, C_2 or L. To this end, write

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 \end{bmatrix}, \; B_0 = B = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -R_1 & 0 \\ 1 & 0 & 0 & 0 & -R_2 \end{bmatrix}, \tag{19}$$

and

$$G_0 = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

The vanishing of C_1, C_2 or L yields a rank-deficiency in D and G_0, and therefore a critical point of type 0. Nevertheless, under the assumed density of the regular set $\mathcal{I}_{\mathrm{reg}} \cap \mathbb{R}^-$, we may define there

$$D^- = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 1/C_2 & 0 \\ 0 & 0 & 1/L \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{21}$$

yielding continuous projectors P_0, Q_0 in the whole $\mathbb{R}^-$, namely

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

From this projector, we get

$$G_1 = G_0 + B_0 Q_0 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & C_2 & 0 & 1 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & -R_1 & 0 \\ 0 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{23}$$

At points where C_1, C_2 and L do not vanish, but $R_2 = 0$, we have a critical point of type 1-A, due to the rank deficiency on G_1. In contrast, at points where all C_1, C_2, L and R_2 are nonzero, the problem is indeed regular with index-1, so that the problem is almost uniformly regular with index-1 in $\mathbb{R}^-$.

Due to the asserted exclusion of harmless critical points in these a.u. index-1 problems, the vanishing at a given t of any of the values C_1, C_2, L or R_2 is expected to yield a singularity in the scalarly implicit decoupling of the DAE. This is indeed the case, since the scalarly implicit inherent ODE (12) can be checked to read

$$LC_1C_2R_1R_2u' + \begin{bmatrix} LC_2(R_1+R_2) & -LC_1R_2 & 0 \\ -LC_2R_2 & LC_1R_2 & C_1C_2R_1R_2 \\ 0 & -LC_1R_1R_2 & 0 \end{bmatrix} u = 0. \tag{24}$$

$\boldsymbol{R_1(t) = 0.}$ Let us now consider the behavior in $\mathbb{R}^+ \cup \{0\}$, where R_1 does vanish. Assume that the conditions $R_2 \neq 0$, $0 \neq C_1 \neq -C_2 \neq 0$, $L \neq 0$ hold in some dense subset of $\mathbb{R}^+$, so that the DAE has index-2 there, according to [12]. Let us again consider the effect of the vanishing of some of these values at certain points. We assume that no more than one of the values of R_2, C_1, C_2 or L vanishes at a given t.

Looking at (19), the vanishing of C_1, C_2 or L yields again a critical point of type 0. Take P_0 and Q_0 as in (22), so that

$$G_1 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & C_2 & 0 & 1 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{25}$$

The vanishing of R_2 yields a rank drop in G_1 and therefore the zeros of R_2 define critical points of type 1-A. But in the light of G_1 we may get additional conclusions regarding critical points of type 0. When L does vanish, G_1 undergoes a rank deficiency and, according to Corollary 1, critical points owing to the vanishing of L cannot be harmless. In contrast, the vanishing of C_1 or C_2 alone does not change rank in G_1, so that these cases might well yield harmless critical points.

Assuming $C_1 + C_2 \neq 0$, we may take

$$Q_1 = \begin{bmatrix} \frac{C_2}{C_1+C_2} & \frac{-C_2}{C_1+C_2} & 0 & 0 & 0 \\ \frac{-C_1}{C_1+C_2} & \frac{C_1}{C_1+C_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{C_1C_2}{C_1+C_2} & \frac{-C_1C_2}{C_1+C_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

which yields, after some computations,

$$G_2 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & C_2 & 0 & 1 & 0 \\ \frac{C_1}{C_1+C_2} & \frac{-C_1}{C_1+C_2} & L & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \frac{C_2}{C_1+C_2} & \frac{-C_2}{C_1+C_2} & 0 & 0 & -R_2 \end{bmatrix}. \tag{27}$$

It can be checked that neither type-B nor type 2-A critical points are displayed under the condition $C_1 + C_2 \neq 0$. In this setting, the matrix G_2 is singular if and only if $L = 0$ or $R_2 = 0$, so that the vanishing of C_1 or C_2 alone indeed defines a harmless critical point. The latter cases define non-singular points of the scalarly implicit inherent ODE

$$LR_2(C_1+C_2)u' - \begin{bmatrix} \frac{LC_1}{C_1+C_2} - LR_2(C_1+C_2)\left(\frac{C_1}{C_1+C_2}\right)' & \frac{LC_1}{C_1+C_2} - LR_2(C_1+C_2)\left(\frac{C_1}{C_1+C_2}\right)' & R_2C_1 \\ \frac{LC_2}{C_1+C_2} - LR_2(C_1+C_2)\left(\frac{C_2}{C_1+C_2}\right)' & \frac{LC_2}{C_1+C_2} - LR_2(C_1+C_2)\left(\frac{C_2}{C_1+C_2}\right)' & R_2C_2 \\ -LR_2 & -LR_2 & 0 \end{bmatrix} u = 0.$$

Appendix: The coefficients of (13)

For $k = 1, \ldots, \mu-1,\ j = k+2, \ldots, \mu-1$:

$$\begin{aligned} \mathcal{L}_k^{adj} &= P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{\mu-1} G_\mu^{adj}, \\ \mathcal{K}_k^{adj} &= -P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{\mu-1} G_\mu^{adj} B P_0 \cdots P_{\mu-1} \\ &\quad -\omega_\mu P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{\mu-1} P_k D^- (D P_0 \cdots P_{\mu-1} D^-)' D P_0 \cdots P_{\mu-1}, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_0^{adj} &= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{adj}, \\ \mathcal{K}_0^{adj} &= -Q_0 P_1 \cdots P_{\mu-1} G_\mu^{adj} B P_0 \cdots P_{\mu-1} - \omega_\mu Q_0 P_1 \cdots P_{\mu-1} P_0 D^- (D P_0 \cdots P_{\mu-1} D^-)' D P_0 \cdots P_{\mu-1}, \end{aligned}$$

$$\begin{aligned} \mathcal{N}_{k\,k+1} &= P_0 \cdots P_{k-1} Q_k Q_{k+1} D^-, \\ \mathcal{N}_{kj} &= P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{j-1} Q_j D^-, \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{kj} &= -P_0 \cdots P_{k-1} Q_k \{ Q_{k+1} D^- (D P_0 \cdots P_k Q_{k+1} D^-)' + P_{k+1} Q_{k+2} D^- (D P_0 \cdots P_{k+1} Q_{k+2} D^-)' \\ &\quad + \cdots + P_{k+1} \cdots P_{\mu-2} Q_{\mu-1} D^- (D P_0 \cdots P_{\mu-2} Q_{\mu-1} D^-)' \} D P_0 \cdots P_{j-1} Q_j \\ &\quad - \sum_{l=1}^{i} P_0 \cdots P_{k-1} Q_k P_{k+1} \cdots P_{\mu-1} \cdots P_l D^- (D P_0 \cdots P_l D^-)' D P_0 \cdots P_{j-1} Q_j. \end{aligned}$$

For $k = 0$, in the top of these expressions, $P_0 \cdots P_{k-1} Q_k$ has to be replaced by Q_0.

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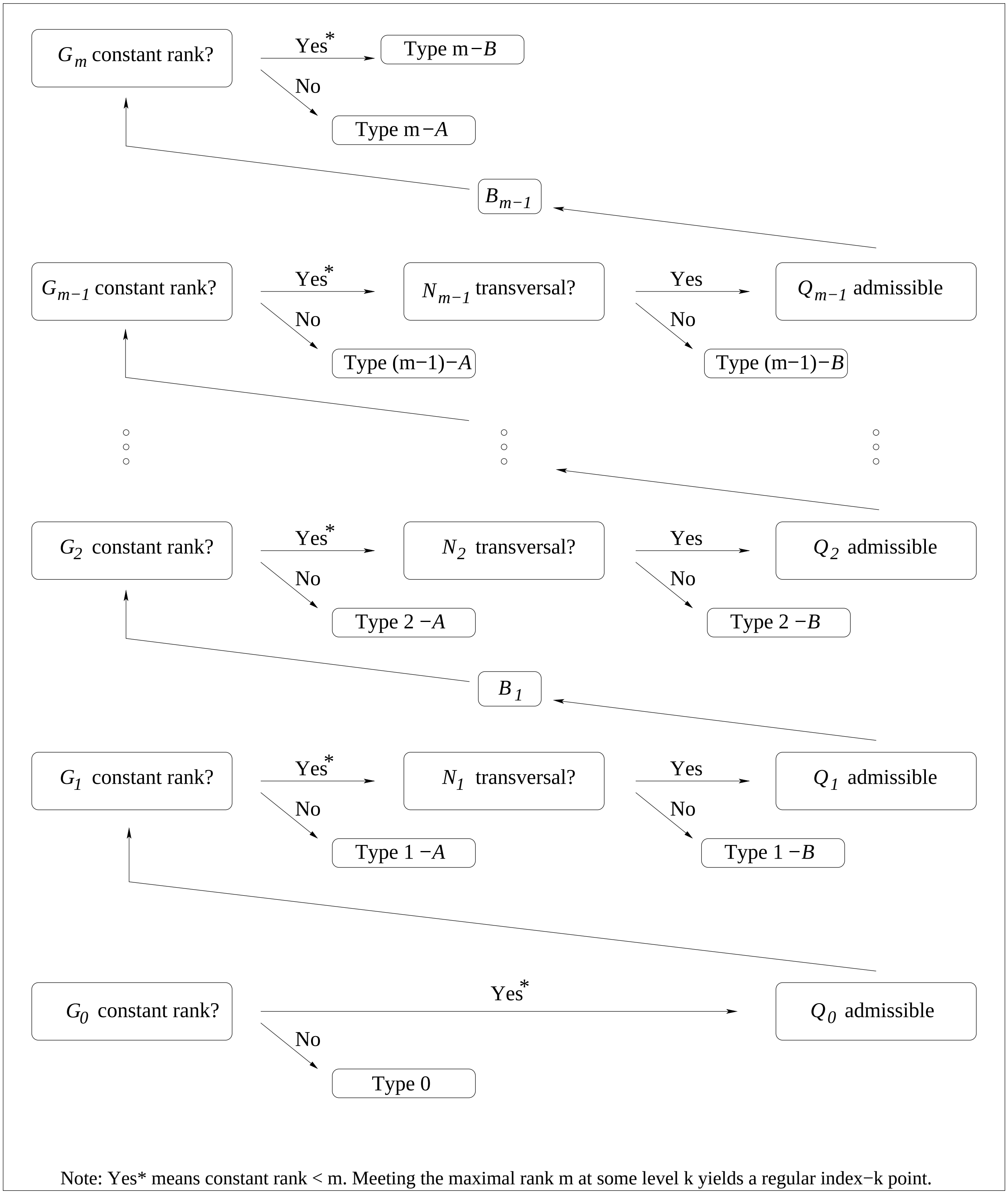

Figure 4: The smooth G-building and critical points in $\mathbb{R}^m$.