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André Casajus, HHL Leipzig Graduate School of Management

Frank Huettner, ESMT Berlin

The Coleman-Shapley-Index: Being Decisive Within the Coalition of the Interested☆

André Casajus[a,*], Frank Huettner[b]

[a]HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany
[b]ESMT European School of Management and Technology, Schlossplatz 1, 10178 Berlin, Germany

Abstract

The Coleman Power of the Collectivity to Act (CPCA) is a popular statistic that reflects the ability of a committee to pass a proposal. Applying the Shapley value to this measure, we derive a new power index that indicates each voter's contribution to the CPCA. This index is characterized by four axioms: anonymity, the null voter property, transfer property, and a property that stipulates that sum of the voters' power equals the CPCA. Similar to the Shapley-Shubik index (SSI) and the Penrose-Banzhaf index (PBI), our new index emerges as the expectation of being a pivotal voter. Here, the coalitional formation model underlying the CPCA and the PBI is combined with the ordering approach underlying the SSI. In contrast to the SSI, the voters are not ordered according to their agreement with a potential bill but according to their vested interest in it. Among the most interested voters, the power is then measured in a similar way as with the PBI. Althoug we advocate the CSI against the PBI to capture a voter's influence on whether a proposal passes, the CSI gives new meaning to the PBI. The CSI is the decomposer of the PBI, splitting it into a voter's power as such and a her impact on the power of the other voters by threatening to block any proposal. We apply the index to the EU Council and the UN Security Council.

Keywords: Decomposition, Shapley value, Shapley-Shubik index, power index, Coleman Power of the Collectivity to Act, Penrose-Banzhaf index, EU Council, UN Security Council
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1. Introduction

The Coleman Power of the Collectivity to Act (CPCA) is a popular measure for the ease with which individual members' attitutes to a proposal can be translated into whether this

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*Corresponding author.
Email addresses: `mail@casajus.de` (André Casajus), `mail@frankhuettner.de` (Frank Huettner)
URL: `www.casajus.de` (André Casajus), `www.frankhuettner.de` (Frank Huettner)

proposal is actually going to pass, accepted and applied across disciplines (Coleman, 1971). It is a simple and transparent committee-level statistic that counts the share of winning coalitions among all possible coalitions that may form in a committee. The Penrose-Banzhaf index (PBI) is an individual-level statistic that measures each voter's power in a committee (Penrose, 1946; Banzhaf, 1965). It is the depreciation of the CPCA of a committee caused by a voter's abstention or a voter's rejection of proposals. As a consequence, the PBI of the voters sums up to an expression that itself seems hard to interpret and that is not applicable for comparisons across committees. Shapley and Shubik (1954) propose another index measuring the power of an individual in a committee. This Shapley-Shubik index (SSI) differs from the PBI in that it always sums up to 1. Conceptually, the SSI indicates the influence of a committee member on *which* bill is going to pass, i.e., it is assumed that something will be decided and the question is in whose favor. The PBI in contrast refers to the influence on *whether* a bill is going to pass (cf. Dubey and Shapley, 1979).

In this paper, we propose a new power index—henceforth called the Coleman-Shapley-index (CSI)—which resembles the PBI as it transpires from the CPCA and which resembles the SSI in that the voter's power sums up to a meaningful term. For non-contradictory committees, it is reasonable to normalize the CPCA since at maximum only half of the coalitions can be winning, i.e., it is reasonable to consider twice the CPCA which we denote by 2CPCA. It is exactly this entity that is distributed among the voters by the CSI:

$$\text{Sum of CSI over all committee members} = 2 \times \text{Coleman Power of the Collectivity to Act}$$

Postulating this as a property, together with the standard axioms of transfer, symmetry, and null voter, is characteristic of the CSI. Having an index that sums up to a interpretable term is not only of practical advantage (e.g., for cross committee comparison) but supports the soundness of the concept, in particular given a clear understanding of the overall power of a committee in terms of 2CPCA. We therefore advocate the CSI against the PBI as a measure of the influence on whether a proposal passes – even though the CSI gives new meaning to the PBI.

The CSI is the decomposer of the PBI, a concept studied by Casajus and Huettner (2017). Consequently, the CSI splits PBI into a direct part (reflecting a voter's influence on whether a proposal passes) and an indirect part that reflects the impact of her threats to no longer support any proposal on the other voters' direct power:

$$\begin{aligned}\text{A voter's power according to PBI} &= \text{her power according to CSI}\\ &\quad + \text{what the others gain or lose according to CSI}\\ &\qquad \text{when she no longer supports any proposal}\end{aligned}$$

In this sense, the PBI not only captures a voter's power but also her impact on the power of the other voters arising from threats. This induces a "double count" if the PBI is summed up across all voters and explains the problematic behaviour of the sum the PBI.

Both SSI and PBI can be obtained as the expectation of being a pivotal voter. The CSI also emerges from such a model. More concretely, consider the binomial model of coalition formation underlying both the CPCA and the PBI. Behind the veil of ignorance, any of the voters is in favor of a proposal or against it with probability 1/2. This implies that the probability of any coalition $S \subseteq N$ being in favor of the proposal, whilst the counter coalition $N \setminus S$ is against it, is $1/2^n$. The CPCA is then just the chance that a winning coalition forms and the proposal passes. Based on the same model of coalition formation, the PBI of voter i measures the expectation of voter i being pivotal, i.e., the expectation that voter i faces a situation where the voters in favor of a proposal will only win if she joins them.

The SSI in contrast rests on a different model of coalition formation. Given a proposal, the voters are sorted starting with the most supportive voter and ending with the voter least in favor of the proposal. Such an ordering defines a pivotal voter without whom the voters that are more supportive than this voter cannot win, while if this voter agrees to the proposal, then the coalition wins. Being the voter who actually determines whether the proposal passes, the pivotal voter is the one most contested by both sides. Hence, she might be most influential regarding the details of the bill. Behind the veil of ignorance, all orderings are equally likely and the SSI emerges as the expectation of being pivotal.

The CSI combines both approaches, yet with a twist concerning the interpretation of the rank ordering underlying the SSI. We remark that voters might have a limited capacity to participate in all of the ballots that they are entitled to. In particular, we assume that the voters may be more or less interested in the topic to be decided. Let us rank the voters according to their interest, starting with the voter who will most definitely show up first when this topic is on the agenda and ending with the voter who is least interested. Assume all orderings regarding interest to be equally likely. In expectation, a voter thus faces other voters of the committee that are more interested than her and who already deal with the topic. Within these most interested voters, she may now be pivotal in the sense that her joining the subcoalition comprising those voters in favor of the proposal and more interested than her makes this subcoalition a winning coalition. Given that being interested in a topic and being in favor of a proposal is independent behind the veil of ignorance, the CSI emerges as the expectation of being pivotal,

$$\text{CSI} = \text{expectation of being pivotal among the most interested}$$

The three upper equations constitute the cuneate contributions of our paper: we suggest a power index that (i) distributes a meaningful measure of committee agility – the Coleman power of the collectivity to act – among the voters, (ii) splits the Penrose-Banzhaf index into a constructive power component and a veto power component, and (iii) is the expected value of being a pivotal voter among the voters most interested in the topic at stake. Consequently, our index facilitates the comparison of (individual) power across committees, provides a better understanding of the Penrose-Banzhaf index, and integrates the idea of voters having priorities on topics they may engage in.

The remainder of this paper is organized as follows. Section 2 surveys related literature. Section 3 introduces and characterizes the CSI. The relationship between the CSI, the Shapley value, and the PBI is studied in Section 4. In Section 5, we consider examples. Some remarks conclude the paper. The appendix supports the computation of the index and contains data for various examples.

2. Related literature

For a survey of further power indices we refer to Bertini et al. (2013). A rich literature compares PBI and SSI. Many contributions are of axiomatic character. Employing standard axioms – transfer, symmetry, and null player – the difference of both indices boils down to the sum of the power held by the voters (Dubey and Shapley (1979), Laruelle and Valenciano (2001)). Einy and Haimanko (2011) characterize the SSI by weakening the assumption that power sums up to one to the assumption that whenever a voter gains in power if the voting rule of a committee changes, another voter needs to lose power. Lehrer (1988), Casajus (2012, 2014), and Haimanko (2017) employ equivalences of payoffs referring to the amalgamation of two voters to a singleton. While this amalgamation of two voters is clearly of mathematical interest, it is rather counterintuitive for an index measuring the impact of voters on whether a proposal passes (joining forces typically increases this impact). However, in light of the decomposition of the PBI by the CSI, this property gains plausibility.

Beyond the axiomatic approach, Felsenthal and Machover (1998) stress paradoxes to illicit (mis)understandings of indices. They further distinguish two notions of voting power, I-power and P-power. While I-power stands for the degree to which a player determines whether a coalition is winning or losing, P-power indicates a player's expected share in a fixed prize gained by a winning coalition containing her. We present a cooperative game that assigns to every coalition an I-power score and we use the Shapley value to divide this I-power among the voters. Therefore, the CSI can be considered an I-power index. Wiese (2009) extends the scope of notions of power by discussing the idea of power over others as power that emerges from threats to no longer support any proposal and connects cooperative game theory to the sociology literature. We connect to this notion of power over others when decomposing the PBI into a direct power component and a voter's impact from threats, where the latter can be understood as power over others.

Some empirical analyses favor the PBI, e.g., Leech (1988) and Renneboog and Trojanowski (2011), while others rest on the SSI, e.g., Köke and Renneboog (2005), Basu et al. (2016), and Bena and Xu (2017). Finally, there are non-cooperative game theoretical studies of voting systems that may support one or the other power index. To this end, we refer to Kurz et al. (2017b,a) who derive the SSI in the context of a two-tier voting procedure.

Power indices are applied for various purposes and are often extended to serve a particular purpose. A typical extension manipulates the possibility of particular coalitions forming, e.g., based on judgement about the preferences of the committee members. Amaral and Tsay (2009) use such a modification of the PBI to measure influence in an outsourcing game. Owen (1977) assumes prior unions of voters and imposes that a union can only vote for a proposal with all its members supporting. This invokes an intermediate committee with unions as voters, which is used to measure power – first of the unions and second of their members. Karos and Peters (2015) study mutual control structures, i.e., systems of committees where a committee might also take the role of a voter in another committee (e.g., the shareholders of a company are the voters on this company's general meeting which in turn might act as a shareholder of another company, cf. Gorton and Schmid (2000)). We remark that these extensions are also possible for the CSI.

Efficient computation of power indices is studied by, among others, Leech (2003) who suggest an approximative algorithm based on the multilinear extension introduced by Owen (1972). Kurz (2016) highlights the efficiency of dynamic programming to count number of coalitions of a particular size containing a particular player for the computation of power indices in weighted voting games. We provide the means to pursue computation of the CSI along those lines in the appendix. A survey of various algorithms is provided by Matsui and Matsui (2000).

3. The Coleman-Shapley index

A committee for the assembly $N = \{1, \dots, n\}$ is a collection $\mathcal{W}$ of subsets of N that satisfies the following properties:[1]

(i) $\emptyset \notin \mathcal{W}$.

(ii) If $S \subseteq T \subseteq N$ and $S \in \mathcal{W}$, then $T \in \mathcal{W}$.

Members $i \in N$ are called voters. Coalitions $S \in \mathcal{W}$ are called winning coalitions. The set of all assemblies is denoted by $\mathbb{W}$. Unless specified otherwise, we further assume that a voting game is non-contradictory, i.e., a coalition and its complement cannot both be winning:

(iii) If $S \in \mathcal{W}$, then $N \setminus S \notin \mathcal{W}$.

Coleman (1971, 1990) introduces a measure for the **Power of the Collectivity to Act** (**CPCA**) that is given by

$$\mathrm{CPCA}\,(\mathcal{W}) = \frac{1}{2^n} \sum_{S \subseteq N} [S \in \mathcal{W}]\,, \tag{1}$$

where we use the square brackets are understood as logical operators,

$$[\text{statement}] = \begin{cases} 0, & \text{statement is false,} \\ 1, & \text{statement is true.} \end{cases}$$

In other words, the CPCA counts the number of winning coalitions and divides it by the number of all coalitions (including the empty coalition). One interpretation is as follows. With a probability of $1/2$, any of the voters is in favor of a given proposal or against it. This implies that the probability of coalition $S \subseteq N$ being in favor of the proposal whilst the coalition $N \setminus S$ is against it, is $1/2^n$. Thus, the CPCA equals the chance that a proposal meets a winning coalition and passes – followed by some corresponding action, hence the name. For non-contradictory games, it is reasonable to normalize this measure since at maximum only half of the coalitions can be winning. We will use **2CPCA** to denote this normalization,

$$2\mathrm{CPCA}\,(\mathcal{W}) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} [S \in \mathcal{W}]\,.$$

[1] We do without the assumption $N \in \mathcal{W}$ frequently made in the literature. This extends the domain of our axioms. However, our results would also hold true if the axioms were restricted to the smaller domain.

While the power indices of the collectivity to act refer to the whole assembly N, the Penrose-Banzhaf index measures the voting power of each individual voter $i \in N$. It is given as the probability to be a pivotal voter. Voter i is a pivot in coalition $T \subseteq N,\ i \in T$ if leaving this coalition turns it from a winning coalition into a losing coalition, i.e., $\{i\} \in T \in \mathcal{W}$ and $T \setminus \{i\} \notin \mathcal{W}$. Suppose that every other voter is in favor (or against) a proposal with probability 1/2. Then, the probability that exactly the voters of coalition $T \setminus \{i\} \subseteq N \setminus \{i\}$ are in favor of the proposal is 2^{n-1}. The **Penrose-Banzhaf index** (**PBI**) equals the expectation of being a pivot voter,

$$\mathrm{PBI}_i\left(\mathcal{W}\right) = \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} \left[T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}\right]$$

for all $\mathcal{W} \in \mathbb{W}$ and $i \in N$, where $[T \in \mathcal{W}$ and $T \setminus \{i\} \notin \mathcal{W}]$ takes the value 1 if i is pivotal in T and the value 0 if i is not pivot in T.

The **Shapley-Shubik index** (**SSI**) is based on a different notion of when being pivotal matters with the consequence that the probability distribution over the set of coalitions that are in favor of the proposal and can be joined by voter i is different. The probability model behind the SSI supposes that the voters have a different degree of support for a bill that is to be decided on. Ordering the voters beginning with the most supportive voter and ending with the most dismissive opponent, there is a decisive voter along the line that makes the bill pass (or fail). It appears that this voter is targeted most by the others that are trying to convince her into their camp. Consequently, this voter has crucial influence on the details of the bill. Assuming that all orderings are a priori equal probable, then the SSI is the probability of being the decisive voter. Formally, a rank order for N is a bijection $\sigma : N \to \{1, \ldots, n\}$; R denotes the set of all rank orders. For N, $\sigma \in R$, and $i \in N$, we denote the coalition of voter i together with the voters that appear before i in σ by $B_i\left(\sigma\right) := \{j \in N \mid \sigma\left(j\right) \leq \sigma\left(i\right)\}$. The **Shapley-Shubik index** (**SSI**) is given by

$$\mathrm{SSI}_i\left(\mathcal{W}\right) = \frac{1}{n!} \sum_{\sigma \in R} \left[B_i\left(\sigma\right) \in \mathcal{W} \text{ and } \left(B_i\left(\sigma\right) \setminus \{i\}\right) \notin \mathcal{W}\right].$$

As is discussed above, we capture power in terms of the influence on both whether a bill can be passed (similar to PBI) and which coalition is essentially designing the bill (similar to SSI). To this end, suppose that the voters might have varying interest to invest time and resources into a topic and this determines whether they are present in the room when a bill is worked on. This invokes an ordering of the voters according to their enthusiasm on a topic, starting with the voter that is most interested and eager to work on a bill, and ending with the voter who is least concerned about the topic. We assume that all such orderings are a equal probable. Within this coalition of the most interested voters $B_i\left(\sigma\right)$, it is now crucial to make a difference on whether a bill can be passed. Assuming that preferences are for or against with equal probabilities (as for the CPCA or the PBI), this is measured by $1/2^{|B_i(\sigma)|-1} \times \left|\{S \subseteq B_i\left(\sigma\right) \mid i \in S \in \mathcal{W} \text{ and } S \setminus \{i\} \notin \mathcal{W}\}\right|$. The *expectation of being pivotal* *in a coalition of most interested* voters is then given by

$$\mathrm{CSI}_i(\mathcal{W}) = \frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}[T\in\mathcal{W} \text{ and } T\setminus\{i\}\notin\mathcal{W}].$$

In what follows, we discuss this new index. We start with a characterization. The property that distinguishes the CSI from SSI and PBI is the sum across the voters.

2CPCA-Efficiency, 2CE. For all $\mathcal{W}\in\mathbb{W}$, we have $\sum_{i\in N}\varphi_i(\mathcal{W}) = 2\mathrm{CPCA}(\mathcal{W})$.

The motivation for this property arises from the idea that the overall power of a committee is measured by the model of Coleman. However, $\mathrm{CPCA}(\mathcal{W}) = 1/2$ if $\mathcal{W}$ is a non-contradictory committee with maximal number of winning coalitions, e.g., if there is a dictator. We therefore suggest the normalized CPCA, i.e., twice the CPCA. The following properties are standard.

Anonymity, A. For all permutations σ of N and all $\mathcal{W}\in\mathbb{W}$, we have $\varphi_{\sigma(i)}(\mathcal{W}) = \varphi_i(\sigma^{-1}(\mathcal{W}))$.

Null voter, N. For all $\mathcal{W}\in\mathbb{W}$ such that $i\in N$ is a null voter in $\mathcal{W}$ (i.e., $S\in\mathcal{W}$ implies $(S\setminus\{i\})\in\mathcal{W}$), we have $\varphi_i(v) = 0$.

Transfer, T. For all $\mathcal{V}$, $\mathcal{W}\in\mathbb{W}$, and $i\in N$, we have $\varphi_i(\mathcal{V}\cap\mathcal{W}) + \varphi_i(\mathcal{V}\cup\mathcal{W}) = \varphi_i(\mathcal{V}) + \varphi_i(\mathcal{W})$.

Anonymity just requires that the index does not depend on the labeling of the voters. The null voter property normalizes the index such that a voter without influence receives zero and a dictator receives 1. Transfer property is less innocuous. Laruelle and Valenciano (2001) offer an equivalent version. Consider a coalition that is minimal winning in both $\mathcal{V}$ and $\mathcal{W}$, i.e., $S\in\mathcal{V}$ and $S\in\mathcal{W}$ but for every strict subset $T\subsetneq S$ we have $T\notin\mathcal{V}$ and $T\notin\mathcal{W}$. Then, the transfer property is tantamount to the requirement that $\varphi_i(\mathcal{V})-\varphi_i(\mathcal{W}) = \varphi_i(\mathcal{V}\setminus\{S\}) - \varphi_i(\mathcal{W}\setminus\{S\})$ for every voter $i\in N$, i.e., the difference of the power of a voter in two committees is preserved if we make the same minimal winning coalition in both committees losing. If we keep in mind that a committee is specified by its collection of minimal winning coalitions, the transfer property implies that differences in the power of a voter in two committees originate from differences in the committees, i.e., their different minimal winning sets. We can now present a characterization of the CSI.

Theorem 1. *The CSI is the unique power index that satisfies 2CPCA-Efficiency* ***(2CE)****, Anonymity* ***(A)****, Null voter* ***(N)****, and Transfer* ***(T)****.*

Proof. We leave it to the reader to verify that the CSI satisfies **2CE**, **A**, **N**, and **T**. To show uniqueness, assume that φ satisfies the four properties. For $T\subseteq N$, $T\neq\emptyset$, let $\mathcal{W}_T = \{S \mid T\subseteq S\}$ denote the unanimity committee. By Lemma 2.3 in Einy (1987), φ is already determined by the numbers assigned to $\mathcal{W}_T$. Remains to show that these are unique given the properties of φ. Indeed, **2CE** of φ implies $\sum_{i\in N}\varphi_i(\mathcal{W}_T) = 2^{n-t}/2^{n-1}$. Now, **N** implies $\varphi_i(\mathcal{W}_T) = 0$ for $i\notin T$ and consequently $\sum_{i\in T}\varphi_i(\mathcal{W}_T) = 2^{1-t}$. Finally, **A** implies $\varphi_i(\mathcal{W}_T) = 2^{1-t}/t$ for $i\in T$. □

The present characterization allows us to nail down the difference between CSI, SSI and, PBI, to just one property. Replacing 2CPCA-Efficiency in the upper theorem by efficiency, i.e., the requirement that the voters' power sums up to one, $\sum_{i\in N}\varphi_i(\mathcal{W})=1$, gives a characterization of the SSI. Replacing 2CPCA-Efficiency in the upper theorem by PBI-efficiency, i.e., the requirement that the voters' power sums up to the PBI, $\sum_{i\in N}\varphi_i(\mathcal{W})=\sum_{i\in N}\mathrm{PBI}_i(\mathcal{W})$, gives a characterization of the PBI.

4. The CSI, the Shapley value, and the PBI

To gain a better understanding of the CSI, it is helpful to study a particular auxiliary TU game. A TU game with player set N is a function v that assigns to every coalition $S\subseteq N$ a number $v(S)$ with the interpretation that $v(S)$ is the worth that can be created by this coalition. By convention, $v(\emptyset)=0$. Note that a so-called simple TU game can be understood as a representation of a committee. A game is simple if $v(S)\in\{0,1\}$ for all $S\subseteq N$, $v(S)\leq v(T)$ whenever $S\subseteq T$. Coalition S is then said to be winning if $v(S)=1$ and losing if $v(S)=0$. Next, we want to investigate in a different TU game associated to a committee, a game that is not to be confused with the representation of a committee.

For every committee $\mathcal{W}$, define the (non-simple) TU game $v^{\mathcal{W}}$ by

$$v^{\mathcal{W}}(S)=\frac{1}{2^{s-1}}\sum_{T\subseteq S}[T\in\mathcal{W}]. \tag{2}$$

The game $v^{\mathcal{W}}$ assigns to every coalition S its share of possible winning subcoalitions. Analogously to the 2CPCA, $v^{\mathcal{W}}(S)$ measures *the power of coalition S to act* (if the other voters $N\setminus S$ reject any proposal). Assuming that every voter is equally likely in favor or against a proposal, $v^{\mathcal{W}}(S)/2$ is the probability that a proposal is supported by a winning subcoalition $T\subseteq S$ and passes in the committee $\mathcal{W}$. Consider for example the committee with $N=\{1,2,3\}$ with simple majority rule. One of the four subcoalitions of $\{1,2\}$ is winning such that $v^{\mathcal{W}}(\{1,2\})=0.5$ reflects the power of $\{1,2\}$ to act. Moreover, $v^{\mathcal{W}}(\{1\})=0$ and $v^{\mathcal{W}}(\{1,2,3\})=1=\mathrm{2CPCA}(\mathcal{W})$.

Next we show, that using the auxiliary TU game $v^{\mathcal{W}}$, we can understand the PBI as the a voter's contribution to the the power of the others voters to act. In other words, PBI_i quantifies the reduction of the power of the collectivity to act if voter i refuses to support any proposal.

Proposition 2. *For every committee $\mathcal{W}$ and all $i\in\mathcal{W}$, we have*

$$\mathrm{PBI}_i(\mathcal{W})=v^{\mathcal{W}}(N)-v^{\mathcal{W}}(N\setminus\{i\}),$$

where $v^{\mathcal{W}}$ is given in (2).

Proof. Straightforward computation yields

$$\begin{aligned} v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus i) &= \frac{1}{2^{n-1}} \sum_{T \subseteq N} [T \in \mathcal{W}] - \frac{1}{2^{n-2}} \sum_{T \subseteq N : i \in T} [T \setminus \{i\} \in \mathcal{W}] \\ &= \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \in \mathcal{W}] - \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \setminus \{i\} \in \mathcal{W}] \\ &= \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}] \end{aligned}$$

which completes the proof. □

There are two basic arguments against measuring an individual's performance in a TU game by looking at the contribution to the others. First, summing up over all individuals yields a problematic term. In particular, the sum $\sum_i \mathrm{PBI}_i(\mathcal{W}) = \sum_i \left(v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus \{i\})\right)$ is difficult to interpret. Second – and conceptually more important – the last marginal contribution $v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus i)$ cannot simply be attributed to an individual but also requires participation of other voters (contrast this with $v^{\mathcal{W}}(i) - v^{\mathcal{W}}(\emptyset) = [i \text{ is a dictator}]$, which can be seen as an indicator of voter i's individual performance). Concretely, claiming that voter i's share of the 2CPCA is reflected by PBI_i is flawed because it gives some credit to i for the other voters providing their power to act.

Consequently, we are interested in a concept that disentangles this interrelationship inherent to a cooperatively generated power to act. The most prominent solution to this end is the Shapley value (Shapley, 1953), given by

$$\mathrm{Sh}_i(v) = \frac{1}{n!} \sum_{\sigma \in R} \left(v(B_i(\sigma)) - v(B_i(\sigma) \setminus \{i\})\right).$$

As is shown in the next result, it turns out that this yields precisely the CSI.

Theorem 3. *For every committee $\mathcal{W}$ and all $i \in \mathcal{W}$, we have*

$$\mathrm{CSI}_i(\mathcal{W}) = \mathrm{Sh}_i\left(v^{\mathcal{W}}\right),$$

where $v^{\mathcal{W}}$ is given by (2).

Proof. Apllying the definitions gives

$$\mathrm{Sh}_i\left(v^{\mathcal{W}}\right) = \frac{1}{n!} \sum_{\sigma \in R} \left(\frac{1}{2^{|B_i(\sigma)|-1}} \sum_{T \subseteq B_i(\sigma)} [T \in \mathcal{W}] - \frac{1}{2^{|B_i(\sigma) \setminus \{i\}|-1}} \sum_{T \subseteq B_i(\sigma) \setminus \{i\}} [T \in \mathcal{W}] \right).$$

This can be rearranged as follows:

$$\begin{aligned}\mathrm{Sh}_i\left(v^{\mathcal{W}}\right) &= \frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left([T\in\mathcal{W}]+[T\setminus\{i\}\in\mathcal{W}]\right)\\ &-\frac{1}{n!}\sum_{\sigma\in R}\frac{2}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}[T\setminus\{i\}\in\mathcal{W}]\\ &=\frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left([T\in\mathcal{W}]-[T\setminus\{i\}\in\mathcal{W}]\right),\end{aligned}$$

which establishes $\mathrm{Sh}\left(v^{\mathcal{W}}\right)=\mathrm{CSI}\left(\mathcal{W}\right)$. □

In light of Theorem 3, we can say that CSI measures a voter's contribution to the power of the collectivity to act. Moreover, the CSI relates to the PBI as the Shapley value relates to the naïve approach, which assigns to any player the difference between the worth of the grand coalition and its worth after this player left the game.

This relationship is studied by Casajus and Huettner (2017), who introduce the notion of a decomposer. Starting from the understanding that $v^{\mathcal{W}}(N)-v^{\mathcal{W}}(N\setminus\{i\})$ is not solely due to i, they are interested in a solution that distributes $v^{\mathcal{W}}(N)-v^{\mathcal{W}}(N\setminus\{i\})$ among all players. It turns out that the natural decomposition is the Shapley value, since the Shapley value is the only decomposition that itself can be decomposed. In the present context, this has the consequence that the CSI is the decomposer of the PBI, with the following implication.[2]

Corollary 4. *For every committee $\mathcal{W}$ and all $i\in\mathcal{W}$, we have.*

$$\mathrm{PBI}_i\left(\mathcal{W}\right)=\mathrm{CSI}_i\left(\mathcal{W}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{CSI}_j\left(\mathcal{W}\right)-\mathrm{CSI}_j\left(\mathcal{W}-i\right)\right) \tag{3}$$

where $\mathcal{W}-i=\{S\in\mathcal{W}\mid i\notin S\}$ denotes the committee in which voter j is never supportive of any proposal.

Proof. We apply Theorem 3 to the RHS of (3), giving

$$\begin{aligned}\mathrm{CSI}_i\left(\mathcal{W}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{CSI}_j\left(\mathcal{W}\right)-\mathrm{CSI}_j\left(\mathcal{W}-i\right)\right) &= \mathrm{Sh}_i\left(v^{\mathcal{W}}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{Sh}_j\left(v^{\mathcal{W}}\right)-\mathrm{Sh}_j\left(v^{\mathcal{W}-i}\right)\right)\\ &=\sum_{j\in N}\mathrm{Sh}_j\left(v^{\mathcal{W}}\right)-\sum_{j\in N\setminus\{i\}}\mathrm{Sh}_j\left(v^{\mathcal{W}-i}\right)\\ &=v^{\mathcal{W}}(N)-v^{\mathcal{W}-i}(N)\end{aligned}$$

The claim now follows with $v^{\mathcal{W}-i}(N)=v^{\mathcal{W}}(N\setminus\{i\})$ and Proposition 2. □

[2]Equation (3) is not only satisfied by the CSI. Yet, CSI is the only index that itself is decomposable. Indeed, decomposability (i.e., requiring that there exists an index that decomposer) and 2CPCA-Efficiency are characteristic of the CSI.

The term $\mathrm{CSI}_j(\mathcal{W}) - \mathrm{CSI}_j(\mathcal{W} - j)$ reflects the impact of voter i's threats on voters j due to the fact that j's power (measured by the CSI) changes if i refuses to support any proposal. In this sense, the RHS of (3) consists of the power of voter i and the impact of voter i's threats on all other voters. This expression might be helpful to identify voters who are critical for keeping the committee agile even though their influence on whether a proposal passes is low, i.e., voters with high $\mathrm{CSI}_i(\mathcal{W}) / (\mathrm{PBI}_i(\mathcal{W}) - \mathrm{CSI}_i(\mathcal{W}))$ ratio.

Moreover, (3) clarifies once more why summing up the PBI across all players results in a "double count" of power and yields a sum that is cumbersome in its interpretation. It further gives plausibility to a the following property that is characteristic of the PBI.

2Efficiency. For all committees $\mathcal{W} \in \mathbb{W}$ and voters i, j, we have $\varphi_i(\mathcal{W}) + \varphi_j(\mathcal{W}) = \varphi_{\widehat{ij}}(\mathcal{W}_{\widehat{ij}})$, where $\mathcal{W}_{\widehat{ij}} = \{S \in \mathcal{W} \mid i \notin S \text{ and } j \notin S\} \cup \{(S \setminus \{i,j\}) \cup \{\widehat{ij}\} \in \mathcal{W} \mid i \in S \text{ or } j \in S\}$ represents the committee with voters $(N \setminus \{i,j\}) \cup \{\widehat{ij}\}$ in which i and j merged into one voter $\widehat{ij}$.

Although of mathematical and philosophical interest, this property appears rather counter-intuitive as one would expect the total power of two voters to change if they join forces.[3] In light of (3) however, 2Efficiency merely means that a change in power is counterweighted by a change of the impact on the power of other voters. This becomes more clear when looking at an example in the next section.

5. Examples

In this section, we consider some committees to exemplify the usability of the new index. In particular, we use the fact that the CSI constitutes the constructive power part of the PBI while the remainder measures the impact of threats.

5.1. Dictator

Let $d \in N$ be a dictator, i.e., $\mathcal{W}_1 = \{S \mid d \in S\}$. Then,

$$\mathrm{CSI}_i(\mathcal{W}_1) = \mathrm{PBI}_i(\mathcal{W}_1) = \mathrm{SSI}_i(\mathcal{W}_1) = \begin{cases} 1, & \text{if } i = d \\ 0, & \text{if } i \neq d. \end{cases}$$

The fact that CSI and PBI are equal in this example indicates that threats have no impact and all power is constructive. Indeed, if the dictator blocks every proposal, a voter without power loses no power. Likewise, the power of the dictator is unaffected if another voter "blocks" every proposal.

5.2. One big, two small

Consider a committee where voter 1 needs either 2 or 3 to win, while 2 and 3 together lose, $N = \{1,2,3\}$ and $\mathcal{W}_{1b2s} = \{\{1,2\}, \{1,3\}, \{1,2,3\}\}$. The following tableau contains

[3]Note that although 2Efficiency requires invariance of the merging voters' power, it is silent about the other players' power such that the proportions of power might change.

the resulting power scores:

	CSI	PBI	CSI/(PBI – CSI)	SSI
Voter 1	$\frac{5}{12}$	$\frac{9}{12}$	$\frac{5}{4}$	$\frac{8}{12}$
Voter 2	$\frac{2}{12}$	$\frac{3}{12}$	2	$\frac{8}{12}$
Overall	$\frac{9}{12}$	$\frac{15}{12}$	$\frac{7}{4}$	1

The previous two examples clarify that the sum of PBI is not an intuitive measure of overall constructive power present in a committee: A dictator is very decisive, while coming to a decision is more difficult in the second example, but $\sum_i \mathrm{PBI}_i(\mathcal{W}_1) = 1 < \frac{15}{12} = \sum_i \mathrm{PBI}_i(\mathcal{W}_{1b2s})$. However, if we measure power in terms of CSI and take into account that the threats in $\mathcal{W}_{1b2s}$ are severe, an overall increase of the PBI becomes plausible. If for example voter 1 blocks any decision, then voters 2 and 3 become null voters such that the impact of her threats equals $4/12$ – almost as much as her power.

As mentioned in the previous section, a particularity of the PBI is its property 2Efficiency: if two voters merge, then their power sums up as well. If for instance voters 1 and 2 merge, we obtain a committee with the two voters $\widehat{12}$ and 3 where now $\widehat{12}$ is a dictator. Clearly, this alliance changes the impact of 1 and 2 on whether a proposal passes. While a coalition between 1 and 2 or 3 was necessary before, $\widehat{12}$ can now easily decide on a proposal. The PBI therefore is a doubtful measure for the impact of whether a proposal passes. Understanding the PBI as a sum of power and impact of threats gives plausibility to 2Efficiency: while constructive power of 1 and 2 increases from $\mathrm{CSI}_1(\mathcal{W}_{1b2s}) + \mathrm{CSI}_2(\mathcal{W}_{1b2s}) = 7/12$ to $\mathrm{CSI}_{\widehat{12}}(\mathcal{W}_1) = 1$, this is balanced by a loss of threats.

5.3. Unanimity committees

Consider a committee where proposals require unanimous approval, $\mathcal{W}_n = \{N\}$. Then, reaching an agreement is more difficult the larger is N. This is captured by all well-known indices, e.g.

$$\mathrm{CSI}_i(\mathcal{W}_n) = \frac{1}{n2^{2-n}}, \quad \mathrm{PBI}_i(\mathcal{W}_n) = \frac{1}{2^{2-n}}, \quad \mathrm{SSI}_i(\mathcal{W}_n) = \frac{1}{n}.$$

The relation of power to impact from threats also decreases, $\mathrm{CSI}_i(\mathcal{W}_n)/(\mathrm{PBI}_i(\mathcal{W}_n) - \mathrm{CSI}_i(\mathcal{W}_n)) = 1/(n-1)$. While every individual player has less impact on whether an agreement is reached, the threats become relatively stronger since blocking all proposals means that all players lose their constructive power. This trend – a decreasing ratio of power to the impact of threats for increasing committee size – prevails in other committees as well. Note however that adding null voters does not affect the outcomes, e.g., for any $T \subseteq N$ we have $\mathrm{CSI}_i(\mathcal{W}_T) = \frac{1}{|T|2^{2-|T|}}$ if $\mathcal{W}_T = \{S \mid T \subseteq S\}$.

5.4. UN Security Council.

Let $N = \{1, \ldots, 15\}$ and $\mathcal{W}_{UN} = \{S \mid \{1, \ldots, 5\} \subseteq S \text{ and } |S| > 9\}$

	CSI	PBI	CSI/(PBI – CSI)	SSI
Permanent member	0.00809	0.05176	0.18516	0.19627
Nonpermanent member	0.00113	0.00513	0.28348	0.00186
Overall	0.05176	0.31006	0.25071	1

The fact that, on average, power is about a quarter of the impact of threats reveals once more that the threat from blocking a proposal is rather convincing. Not surprisingly, this is more pronounced for the powerful "veto powers" than for the nonpermanent members.

5.5. European Council

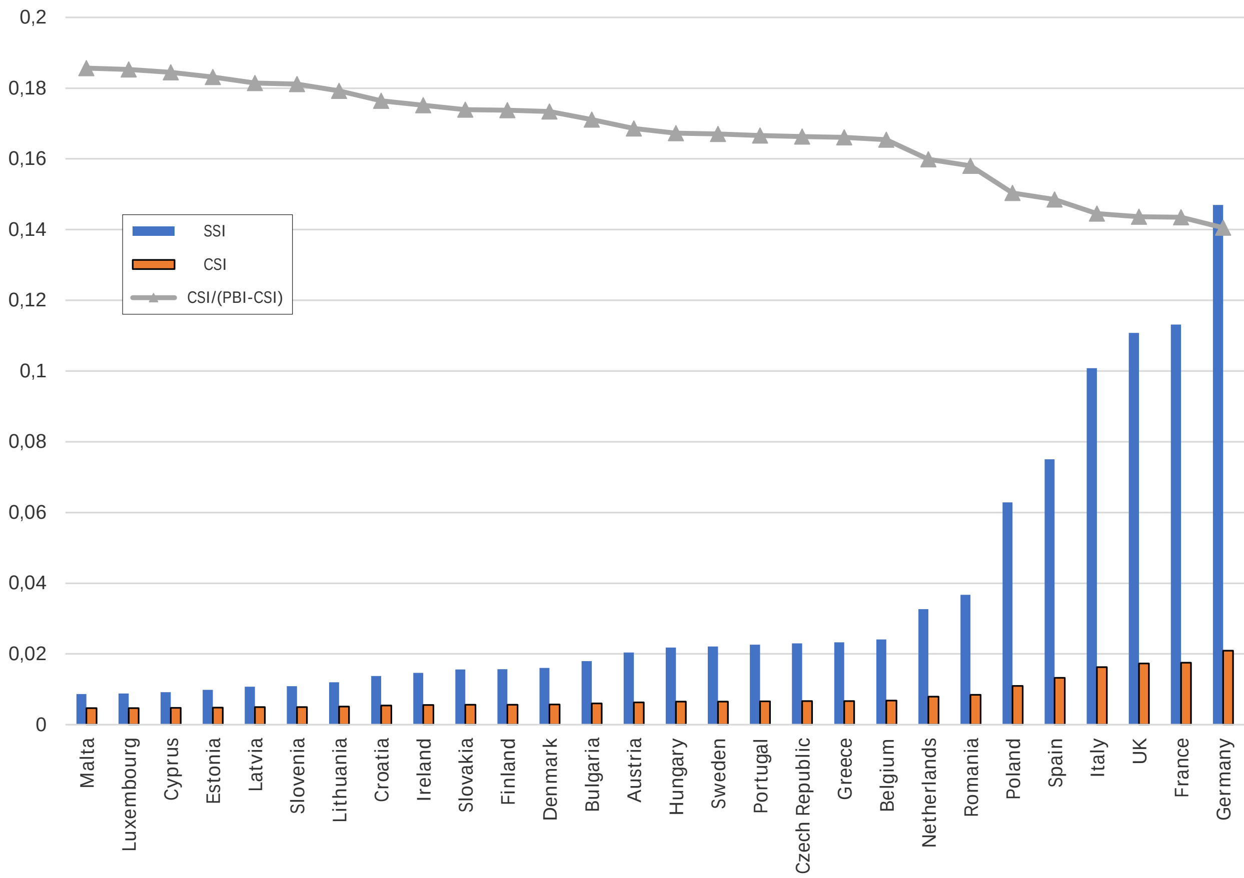

Figure 1: SSI, CSI, and CSI/(PBI-CSI), for the EU Council based on population sizes on 01.01.2017.

Let N contain the 28 countries of the EU. A coalition is winning if it represents 65% of the population and 55% of the countries, i.e., $\mathcal{W}_{EU} = \{S \subseteq N \mid \sum_{i \in S} w_i \geq 332,489,700$ and

$|S| \geq 16\}$, where w_i is the number of inhabitants in country i.[4] Figure 1 contains the SSI and CSI values as well as the CSI/(PBI − CSI)-ratio. It becomes clear that the differences between powerful and less powerful countries according to the CSI (and the PBI) are smaller than according to the SSI, e.g., $\text{CSI}_{Germany}/\text{CSI}_{Malta} \approx 4.5$ while $\text{SSI}_{Germany}/\text{SSI}_{Malta} \approx 17$. An interpretation for this is that in the EU Council, the impact on the design of proposals is more concentrated than the impact on whether a proposal passes. Further note that the CSI/(PBI − CSI)-ratio is 1/6 on average and that this ratio is more balanced in the EU than in the UN Security Council.

6. Concluding remarks

We introduce a new power index, the Coleman-Shapley index (CSI), with the purpose of distributing the Coleman Power of the Collectivity to Act (more precisely its normalized, i.e., doubled, version) among the voters by help of the Shapley value. Imposing structural similarities to the mostly used indices, we single out the CSI as the only index that serves this purpose. The index supports the idea that voters may have different interests in participating in a vote. The most interested voters would then show up before less interested voters, such that the CSI emerges as the expectation of being pivotal in a coalition of most interested.

The CSI is the decomposer of the Penrose-Banzhaf index (PBI), i.e., it relates to the PBI in the same way as the Shapley value relates to the naïve solution, which looks at the difference between the ability to act of all voters and the ability to act of all but this particular voter. This is problematic because all voters are necessary to create this difference and a mere attribution to one voter is flawed. Moreover, the CSI sums up to a well-understood entity and allows for cross-committee comparison. This is a requirement when analyzing the consequences of changing majorities (e.g., the consequences of a population growth driven change of the set of winning coalitions in the EU council).

Although we advocate the CSI against the PBI for measuring the influence on whether a proposal can pass (I-power in the terminology of Felsenthal and Machover (1998)), using the CSI to measure power reveals another interpretation of the PBI in the sense that it equals the sum of power and the impact of threats. The latter captures the change of power of other voters if a voter no longer supports any proposal. Apparently, the threats tend become stronger when a committee grows in size.

Because of structural similarities to the Shapley-Shubik index and to the PBI, the CSI serves well as a building block for various extensions. In particular, two-tier voting systems and mutual control structures can be analysed well based on the CSI. Finally, insights concerning the efficient computation carry over.

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[4] We use the EU population sizes on 01.01.2017 rounded up to 100 (http://ec.europa.eu/eurostat). Detailed numbers can be found in Appendix B.

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Appendix A. Computation

We first present an effective way to compute the CSI for weighted voting games. Thereafter, we provide an algorithm to recursively compute the CSI via the potential of the Shapley value. Finally, we develop multilinear extension of the CSI that can be used to approximate the CSI for large games along the lines of Leech (2003).

Computation for weighted voting games

Let $\left[q, (w_i)_{i\in N}, k\right]$ denote a weighted voting game with minimal winning coalition size k where coalition S is winning if (i) $\sum_{i\in S} w_i \geq q$ and (ii) $|S| \geq k$. Building upon the study of Kurz (2016) who describes an effective way to count the number of coalitions of size t that include i and has weight x

$$c_i(x,t) = \left| \left\{ T \subseteq N \mid i \in T, \ \ |T| = t, \text{ and } \sum_{j\in T} w_j = x \right\} \right|, \tag{A.1}$$

we obtain the CSI as follows:

Proposition 5. *Let $\mathcal{W}$ represent a weighted voting game with quota q, weights $(w_i)_{i\in N}$, and minimal winning coalition size k. Then,*

$$\begin{aligned} \mathrm{CSI}_i(\mathcal{W}) &= \frac{1}{n!} \sum_{t=k}^{n} (n-t)! \sum_{s=0}^{n-t} \frac{(s+t-1)!}{s!} \frac{1}{2^{s+t-1}} \sum_{x=q}^{q+w_i-1} c_i(x,t) \\ &+ \frac{1}{n!}(n-k)! \sum_{s=0}^{n-k} \frac{(s+k-1)!}{s!} \frac{1}{2^{s+k-1}} \sum_{x=q+w_i}^{\sum_j w_j} c_i(x,k). \end{aligned}$$

For $k = 1$ the above simplyfies to $\frac{1}{n!} \sum_{t=1}^{n} (n-t)! \sum_{s=0}^{n-t} \dfrac{(s+t-1)!}{(s)!} \frac{1}{2^{s+t-1}} \sum_{x=q}^{q+w_i-1} c_i(x,t)$.

Proof. Denote set of coalitions passing the quota by $\mathcal{W}_1 = \left\{S \mid \sum_{i\in S} w_i \geq q\right\}$, and set of coalitions of size greater k by $\mathcal{W}_2 = \{S \mid |S| \geq k\}$. We have:

$$
\begin{aligned}
&\mathrm{CSI}_i\left(\mathcal{W}\right)\\
&= \frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left[T\in\mathcal{W}\text{ and }T\setminus\{i\}\notin\mathcal{W}\right]\\
&= \frac{1}{n!}\sum_{B\subseteq N:B\ni i}\left(|B|-1\right)!\left(n-|B|\right)!\frac{1}{2^{|B|-1}}\sum_{T\subseteq B:i\in T}\left[T\in\mathcal{W}\text{ and }T\setminus\{i\}\notin\mathcal{W}\right]\\
&= \frac{1}{n!}\sum_{T\subseteq N:T\ni i}\sum_{S\subseteq N\setminus T}\frac{\left(s+t-1\right)!\left(n-s-t\right)!}{2^{s+t-1}}\left[T\in\mathcal{W}\text{ and }T\setminus\{i\}\notin\mathcal{W}\right]\\
&= \frac{1}{n!}\sum_{T\subseteq N:T\ni i}\sum_{s=0}^{n-t}\frac{\left(n-t\right)!}{s!\left(n-t-s\right)!}\left(s+t-1\right)!\left(n-s-t\right)!\frac{1}{2^{s+t-1}}\left[T\in\mathcal{W}\text{ and }T\setminus\{i\}\notin\mathcal{W}\right]\\
&= \frac{1}{n!}\sum_{t=1}^{n}\sum_{T\subseteq N:T\ni i,|T|=t}\left(n-t\right)!\sum_{s=0}^{n-t}\frac{\left(s+t-1\right)!}{s!2^{s+t-1}}\left[T\in\mathcal{W}\text{ and }T\setminus\{i\}\notin\mathcal{W}\right]\\
&= \frac{1}{n!}\sum_{t=1}^{n}\sum_{T\subseteq N:T\ni i,|T|=t}\left(n-t\right)!\sum_{s=0}^{n-t}\frac{\left(s+t-1\right)!}{s!}\frac{1}{2^{s+t-1}}\left[T\in\mathcal{W}_1,\, T\setminus\{i\}\notin\mathcal{W}_1\text{ and }t\geq k\right]\\
&+ \frac{1}{n!}\sum_{t=1}^{n}\sum_{T\subseteq N:T\ni i,|T|=t}\left(n-t\right)!\sum_{s=0}^{n-t}\frac{\left(s+t-1\right)!}{s!}\frac{1}{2^{s+t-1}}\left[T\in\mathcal{W}_1,\, T\setminus\{i\}\in\mathcal{W}_1,\text{ and }t=k\right]\\
&\overset{(A.1)}{=} \frac{1}{n!}\sum_{t=k}^{n}\left(n-t\right)!\sum_{s=0}^{n-t}\frac{\left(s+t-1\right)!}{s!2^{s+t-1}}\sum_{x=q}^{q+w_i-1}c_i\left(x,t\right)+\frac{\left(n-k\right)!}{n!}\sum_{s=0}^{n-k}\frac{\left(s+k-1\right)!}{s!2^{s+k-1}}\sum_{x=q+w_i}^{\sum_j w_j}c_i\left(x,k\right)
\end{aligned}
$$

which completes the proof. □

Computation via Potential of the Shapley value

In order to compute the Shapley value exactly for any TU game, one can use the potential function, recursively defined by $P\left(\emptyset\right) = 0$ and $P\left(S\right) = v\left(S\right) + \sum_{i\in S} P\left(S\setminus\{i\}\right)$, since $\mathrm{Sh}_i\left(v\right) = P\left(N\right) - P\left(N\setminus\{i\}\right)$. In order to compute the CSI, we need to keep track of $v^{\mathcal{W}}\left(S\right) = \frac{1}{2^{|S|-1}}\sum_{T\subseteq S}\left[T\in\mathcal{W}\right]$ as well. To this end, we need to know about the number of winning subcoalitions of S. This can be done recursively as well. Since $\sum_{i\in S}\sum_{T\subseteq S\setminus\{i\}} v(T) = \sum_{T\subsetneq S} v(T)(s-t)$, we can count the winning coalitions recursively but need to take care of coalition sizes. For every $T\subseteq N$, let $\omega^T\in\mathbb{R}^{|T|}$ denote the vector that contains for each coalition size $k = 1,\dots,|T|$ the number of winning coalitions. Then, $\sum_{T\subseteq S:|T|=|S|-1}\frac{\omega_k^T}{|T|-k+1} = \omega_k^S$ for $k = 1,\dots,|T|$ and $v^{\mathcal{W}}\left(S\right) = \frac{1}{2^{|S|-1}}\sum_k \omega_k^S$. This motivates the following pseudo code for the iterative computation of CSI.

Input: set of voters $N = \{1,\dots,n\}$, set of winning coalitions $\mathcal{W}$

Output: vector CSI containing $\mathrm{CSI}_i\left(\mathcal{W}\right)$

$P(\emptyset) \longleftarrow 0$
for every $t = 0, ..., n$ **do**:
  for every $S \subseteq N$ such that $|S| = t + 1$ **do**:
    if $S \in \mathcal{W}$:
      then $\omega_s^S \longleftarrow 1$
      else $\omega_s^S \longleftarrow 0$
  end
  for every $T \subseteq N$ such that $|T| = t$ **do**:
    for every $i \in N \setminus T$ **do**:
      for every $k = 1 \ldots t$ **do**:
        $\omega_k^{T\cup\{i\}} \longleftarrow \omega_k^{T\cup\{i\}} + \omega_k^T / (t - k + 1)$
      end
    end
  end
  for every $S \subseteq N$ such that $|S| = t + 1$ **do**:
    $P(S) \longleftarrow \frac{1}{2^{|S|-1}} \sum_{k=1}^{|S|} \omega_k^S$
  end
  for every $T \subseteq N$ such that $|T| = t$ **do**:
    for every $i \in N \setminus T$ **do**:
      $P(T \cup \{i\}) \longleftarrow P(T \cup \{i\}) + \frac{P(T)}{t+1}$
    end
  end
end
for every $i = 1, ..., n$ **do**:
  $\mathrm{CSI}_i(\mathcal{W}) = P(N) - P(N \setminus \{i\})$
end

Clearly, $P(S) = 0$ if $v(T) = 0$ for all T with $|T| \leq |S|$, which allows to exploit knowledge about the minimal winning coalition size. Likewise, one may reduce the tracking of ω_k^S to those k that are greater than the minimal winning coalition size.

Multilinear extension

Fix the player set N and denote the set of all games on N by $\mathbb{V}$. The **multi-linear extension** Owen (1972) $\bar{v} : [0,1]^N \to \mathbb{R}$ of a TU game $v \in \mathbb{V}$ is given by

$$\bar{v}(x) = \sum_{S\subseteq N: S\neq\emptyset} v(S) \cdot \prod_{i\in S} x_i \prod_{i\in N\setminus S} (1 - x_i) \qquad \text{for all } x \in [0,1]^N . \tag{A.2}$$

Owen (1972) shows that the Shapley value can be calculated using the *partial* derivatives of the multi-linear extension as follows: For all v and $i \in N$, we have

$$\mathrm{Sh}_i(v) = \int_0^1 \frac{\partial \bar{v}}{\partial x_i} (\theta, \theta, \ldots, \theta)\, d\theta. \tag{A.3}$$

That is, the Shapley payoffs are the players' expected marginal productivities along the diagonal of the standard cube representing the players' probabilities with the uniform distribution on the diagonal. For the CSI we find the following expression. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. As pointed out in Shapley (1953), these unanimity games form a basis of the vector space[5] $\mathbb{V}$, i.e., any $v \in \mathbb{V}$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{A.4}$$

where the **Harsanyi dividends** $\lambda_T(v)$ can be determined recursively via $\lambda_T(v) = v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v)$ for all $T \subseteq N$, $T \neq \emptyset$ (see Harsanyi, 1959).

The CSI on generalizes naturally to a solution for TU games as follows:

$$\mathrm{CSI}_i(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{2^{t-1}t}$$

for all $v \in \mathbb{V}$ and $i \in N$. Note that $\mathrm{CSI}_i(w) = \mathrm{CSI}_i(\mathcal{W})$ if w is the game representing $\mathcal{W}$, i.e., $w(S) = 1$ if $S \in \mathcal{W}$ and $w(S) = 0$ if $S \notin \mathcal{W}$. The following proposition gives a formula of CSI via the multilinear extension that can be useful for computing the index for large games.

Proposition 6. *For all $v \in \mathbb{V}$ and $i \in N$, we have $\mathrm{CSI}_i(v) = 2\int_0^{\frac{1}{2}} \frac{\partial \bar{v}}{\partial x_i}(\theta, \theta, \ldots, \theta)\, d\theta$.*

Proof. Fix $T \subseteq N$, $T \neq \emptyset$. By (A.2), we have $\bar{u}_T(x) = \prod_{\ell \in T} x_\ell$ and therefore

$$\frac{\partial \bar{u}_T(x)}{\partial x_i} = \begin{cases} \prod_{\ell \in T \setminus \{i\}} x_\ell, & i \in T, \\ 0, & i \in N \setminus T \end{cases} \qquad \text{for all } x \in [0,1]^N. \tag{A.5}$$

With $\int_0^{\frac{1}{2}} x^{t-1} dx = \frac{1}{t} x^t \Big|_0^{\frac{1}{2}} = \frac{1}{t}\frac{1}{2^t}$ we get $\mathrm{CSI}_i(\bar{u}) = 2\int_0^{\frac{1}{2}} \frac{\partial \bar{u}_T(x)}{\partial x_i}$. The claim now follows from linearity of the multi-linear extension, of its partial derivatives, and of CSI . $\square$

[5] For $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v+w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.

Appendix B. EU Council data

Country	Population	SSI	CSI	PBI	CSI/(PBI -CSI)
Malta	46030	0.00864677	0.00467474	0.0298542	0.185656881
Luxembourg	59067	0.00882527	0.00470043	0.03007219	0.185262276
Cyprus	85480	0.00919977	0.00475264	0.03051504	0.184479707
Estonia	131564	0.00982827	0.00484335	0.03128485	0.183172286
Latvia	195012	0.01069484	0.0049682	0.0323446	0.181477477
Slovenia	206590	0.0108584	0.00499101	0.03253836	0.181179315
Lithuania	284790	0.01196332	0.00514449	0.03384352	0.179256581
Croatia	415421	0.01379073	0.00539998	0.03601063	0.176408538
Ireland	478438	0.0146639	0.00552318	0.03705747	0.175148386
Slovakia	543534	0.01558604	0.00565048	0.03813914	0.173921608
Finland	550330	0.01567687	0.00566378	0.03825193	0.173798758
Denmark	574877	0.0160319	0.00571161	0.03865863	0.173357408
Bulgaria	710186	0.01794073	0.00597544	0.04090185	0.171086579
Austria	877286	0.02037591	0.00630017	0.04366372	0.168618078
Hungary	979756	0.02183706	0.00649936	0.04535731	0.167259467
Sweden	999515	0.02213365	0.00653773	0.0456836	0.167009444
Portugal	1030957	0.02257792	0.00659883	0.04620332	0.166618229
Czech Republic	1057882	0.02297514	0.00665123	0.0466492	0.166289189
Greece	1076819	0.02324958	0.00668782	0.04696088	0.166061879
Belgium	1135173	0.02409938	0.00680144	0.04792701	0.165382267
Netherlands	1708151	0.03266343	0.00791782	0.05743203	0.159910054
Romania	1964435	0.03673456	0.00842906	0.06176846	0.158026899
Poland	3797296	0.06283548	0.01094856	0.0837624	0.150363722
Spain	4652802	0.0750305	0.01322536	0.10228021	0.148508026
Italy	6058944	0.10084021	0.01623156	0.12854309	0.144522651
UK	6580857	0.11080917	0.01727328	0.13753348	0.143632557
France	6698908	0.11316	0.01753157	0.13973897	0.143457516
Germany	8252165	0.1469712	0.02094077	0.16995624	0.14052749
Overall	51152265	1	0.22657389	1.65293233	0.167156902

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