an almost homogeneous bright luminescence radiation background and only a few dark spots are left. Finally, at $f = 14.7$ kHz all dark spots have disappeared and a homogeneous glow discharge covers the whole discharge area. This state stays stable up to the driving frequency of 20 kHz.

In a subsequent experiment the driving frequency has been decreased. The emerging patterns are presented in Fig. 6.8. In the beginning, the discharge exhibits a nearly homogeneous bright glow. At $f = 11.7$ kHz the first dark spots arise. Their appearance is shown in Fig. 6.8 in the upper row in four subsequent frames. The leftmost image belongs to the highest frequency. In contrast to the bright spots in the first run, the dark spot arise close to each other forming a connected island that grows with decreasing driving frequency. As the driving frequency becomes even samller, the same patterns as in the first run occur in the reverse order (Fig. 6.8, upper row). Finally, at about $f = 5.7$ kHz, the last bright spot disappears.

In Fig. 6.9 the amplitude difference A_{exp} of the pattern in dependence of the driving frequency is depicted. We define this amplitude difference A_{exp} as the difference of luminescence intensities between the brightest and the darkest area in the pattern. To reduce the influence of noise, the local brightness in a camera frame is computed as the average over a small area. The size of that area is chosen to be much smaller than the lateral structures in the pattern and large enough to reduce the noise sufficiently. To avoid effects of the boundary only the inner circle with a diameter of 32 mm of the discharge area is considered.

In Fig. 6.9 (a) below approximately $f = 8$ kHz the amplitude is determined only by the noise in the slightly glowing discharge. At approximately $f = 8$ kHz the first bright filaments occur and the amplitude jumps to a higher value. Within the jump some frames with intermediate amplitude occur. This is due to switching on and off of filaments during the exposure time. With increasing driving frequency the pattern varies from bright spots over stripes to dark spots and, the pattern amplitude develops continuously. At approximately $f = 14.5$ kHz, when the last dark spot disappears, the amplitude falls back to the low level of a homogeneous discharge.

In Fig. 6.9 (b) the amplitude difference A_{exp} corresponding to Fig. 6.8 is to be seen. The onset of the pattern at $f = 11.7$ kHz occurs abruptly. In the course of further increase of f the experiment the amplitude difference A_{exp} develops continuously. At $f = 5.7$ kHz the pattern has disappeared and A_{exp} falls back to the noise level. There is a single frame with an intermediate amplitude of 23 a.u. corresponding to a switching filament.

As inferred from Fig. 6.9 bifurcations from the homogeneous state to the patterned state is subcritical, and one might expect a hysteresis in the bifurcation point depending on the direction of the change of f. Although the overall phenomenon is reproducible, the precise positions of the discontinuities vary

from run to run over a wide range, i.e. from 4 to 8 kHz and from 9 to more than 20 kHz for the lower and upper transitions respectively. Therefore so far it is impossible to investigate the hysteresis of the reported bifurcations in detail.

Let us now classify these phenomenons in the framework of dissipative solitons and Turing structures.To begin with, spots behaving like dissipative solitons shall be identified. In both measurements, Fig. 6.7 (increasing driving frequency) and Fig. 6.8 (decreasing driving frequency), they appear at the end of the recording shortly before the pattern vanishes in favour of a homogeneous system state. For increasing driving frequency (Fig. 6.7) examples can be found in the images at $f = 13.6$ kHz and $f = 14.5$ kHz. For decreasing driving frequency (Fig. 6.8) they are exemplarily shown at $f = 7.7$ kHz. They can be identified as dissipative solitons by means of their arbitrary distances (i.e. their lack of long distance order) and their independent movement.

There are also a couple of spots in the beginning of both recordings, when the homogeneous state becomes unstable in favour of a structured discharge. For decreasing driving frequency (Fig. 6.8, $f = 11.7$ kHz) these spots appear as a domain of agglomeration with distinct short range order. As the driving frequency decreases, the domain of spots grows but spots existing independently do not appear. In combination with the amplitude behavior (Fig.6.9(b)) indicating a subcritical bifurcation, this pattern can safely be identified as a subcritical Turing bifurcation limited by a globally acting inhibition. For increasing driving frequency (Fig. 6.7) the situation is more complicated. For low frequencies ($f = 8$ kHz) single bright spots with arbitrary distances appear. But in contrast to the dark spots at the end of this recording and also in contrast to the bright spots at the end of the recording with decreasing driving frequency (Fig. 6.8, $f = 7.7$ kHz), these spots do not move at all. Moreover, they appear at rather low frequencies (at $f = 8$ kHz), long before the discharge area becomes covered with further spots (at $f = 9$ kHz). This leads us to the conclusion, that these early bright spots are induced by inhomogeneities or defects on the dielectric surfaces and are not truly self-organized. The following occupation of the discharge area with further bright spots (Fig. 6.7, $f = 9$ kHz) as the driving frequency increases is very similar to the appearance of dark spots with decreasing driving frequency (Fig. 6.8, $f = 11.7$ kHz – $f = 9.3$ kHz). The filaments occur in an agglomerated manner with a rather well defined short range order. Again, with respect to the behavior of the amplitude A_{exp} (Fig.6.9(a)) indicating a subcritical bifurcation, this bifurcation can be classified as a Turing bifurcation limited by a globally acting inhibition. Also in the present experimental situation, the physical nature of the global inhibition might be some serial resistance, i.e. the internal resistance of the driving.

Finally, we want to compare the experimental findings to the results ob-

tained from the RD-model. As the model is not specific for our gas-discharge system, we cannot draw any quantitative comparisons. However, a qualitative comparison is possible.

First, consider the development of the amplitude A_{exp} of the patterns as they are shown in Fig. 6.9 for the experiment and as A_{th} in Fig 6.5 for the RD-model. In both cases, all bifurcations – from a homogeneous state to a structured one and vice versa – are subcritical. In the course of the high-amplitude patterns the amplitude changes continuously.

For the pattern amplitude behavior A_{th} in the model (Fig. 6.5) the region of hysteresis of the subcritical bifurcations can be identified in the intervals $[\kappa_{1c}, \kappa_0]$ and $[-\kappa_0, -\kappa_{1c}]$ respectively. In the experiment, the range and the position of the high-amplitude patterns varies over a wide area between single runs making it impossible to compare the bifurcation points obtained from different runs. Nevertheless, as argued above, a region of hysteresis cannot be determined though is very probable.

Furthermore the comparison of the observed patterns in the experiment (Fig. 6.7, Fig. 6.8) and the RD-model (Fig. 6.3) leads to remarkable agreement. The appearance of spots, bright ones as well as dark ones, occurs via a subcritical Turing bifurcation limited by integral inhibition. The intermediate patterns in both cases consist of branched stripe-like patterns with a spatial scale inherited from the spots.

6.4 Turing instability in three-component system

6.4.1 Raus-Gurwitz Stability Criterion

As mentioned above, the concept of Turing instability was originally introduced for two-component RD systems. On the other hand spatially periodic structures are also emerged in systems described with RD equations with three and more components (see, e.g., [Sagués and Epstein, 2003, Maini et al., 1997]). The question is, how the idea of Turing can be adapt to such kind of multicomponent RD systems. To this end, let us turn back to RD system in general form (6.1) assuming now the case of n components $\mathbf{u} = (u_1, u_2, \dots u_n)^T$. As in Sec. 6.1 we suppose that the homogeneous solution $\mathbf{u_0}$ of (6.1) is stable in absence of diffusion, namely, all eigenvalues of the Jacobi matrix

$$\mathbf{A} = (\partial \mathbf{R}/\partial \mathbf{u})_{\mathbf{u}=\mathbf{u_0}}$$

should be less than zero. However, for the matrices of rather large size calculation of eigenvalues, as a rule, involves difficulties. On the other hand, one need to know not the eigenvalues, but their position on the complex plane. To do this, let us write down the characteristic equation for the matrix $\mathbf{A}$

$$\det(\mathbf{A} - \lambda E) = 0, \tag{6.21}$$

which always can be represented in the following polynomial form:

$$\lambda^n + p_1\lambda^{n-1} + \ldots + p_{n-1}\lambda + p_n = 0. \tag{6.22}$$

Using the coefficients $p_0 = 1,\ p_1,\ p_2,\ \ldots,\ p_n$ of the Eq. (6.22) one can construct so-called *Gurwitz matrix* :

$$G_n = \begin{pmatrix} p_1 & p_0 & 0 & 0 & \ldots & 0 \\ p_3 & p_2 & p_1 & p_0 & \ldots & 0 \\ p_5 & p_4 & p_3 & p_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & p_n \end{pmatrix}. \tag{6.23}$$

As can be seen, the coefficients $p_1,\ p_2,\ \ldots,\ p_n$ fills the principal diagonal of the matrix G_n. Indices of coefficients p_j decreases in each row, beginning from the index of the first column coefficient and are consistent with the index of the coefficient staying on the principal diagonal. If the index j becomes more than n or less than zero, the corresponding element of the matrix equals to zero.

Statement (Raus-Gurwitz Criterion)**.** *All roots of the characteristic equation (6.22) have negative real parts if and only if all principal minors of the matrix G_n*

$$\Delta_1 = p_1,\quad \Delta_2 = det\begin{vmatrix} p_1 & 1 \\ p_3 & p_2 \end{vmatrix},\quad \Delta_3 = det\begin{vmatrix} p_1 & 1 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix},\quad \ldots,\ \Delta_n \ = det(G_n) \tag{6.24}$$

are positive.

Remark: If only one inequality (6.24) changes the sign the characteristic equation (6.22) has roots with positive real parts.

For example, for the 2×2 matrix $\mathbf{A}$ the characteristic polynomial is

$$\lambda^2 - \mathrm{Sp}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

and the Gurwitz matrix G_2 takes a form

$$G_2 = \begin{pmatrix} -\mathrm{Sp}(\mathbf{A}) & 1 \\ 0 & \det(\mathbf{A}) \end{pmatrix}.$$

Then the Raus-Gurwitz criterion results in well-known stability conditions for trace and determinant of the matrix $\mathbf{A}$

$$\begin{array}{l} \Delta_1 = p_1 = -\mathrm{Sp}(\mathbf{A}) > 0 \\ \Delta_2 = p_1p_2 = -\mathrm{Sp}(A)\det(\mathbf{A}) > 0 \end{array} \Leftrightarrow \begin{array}{c} \mathrm{Sp}(\mathbf{A}) < 0 \\ \det(\mathbf{A}) > 0 \end{array}$$

In the case of the 3×3 matrix $\mathbf{A}$ the Gurwitz matrix can be written as

$$G_3 = \begin{pmatrix} p_1 & 1 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{pmatrix},$$

where

$$\begin{aligned} p_1 &= -\mathrm{Sp}(\mathbf{A}), \\ p_2 &= M_{11} + M_{22} + M_{33}, \\ p_3 &= -\det(\mathbf{A}). \end{aligned}$$

Here M_{ij} denote different minors of second order of the matrix $\mathbf{A}$, obtained by eliminating the i'th row and j'th column correspondingly. Then the Raus-Gurwitz criterion for the 3×3 matrix can be written as

$$\begin{aligned} \Delta_1 &= p_1 > 0, & \mathrm{Sp}(\mathbf{A}) &< 0, \\ \Delta_2 &= p_1 p_2 - p_3 > 0, \Leftrightarrow & \det(\mathbf{A}) - \mathrm{Sp}(\mathbf{A}) p_2 &> 0, \\ \Delta_3 &= p_3 \Delta_2 > 0 & \det(\mathbf{A}) &< 0 \end{aligned}$$

With this result, let us try to obtain conditions for Turing instability for three-component RD system.

6.4.2 Instability criterion for three-component RD system

Let us now consider a three-component reaction-diffusion system in the form

$$\begin{aligned} \partial_t u =& D_u \Delta u + \lambda u - u^3 - \kappa_3 v - \kappa_4 w + \kappa_1, \\ \tau \partial_t v =& D_v \Delta v + u - v, \\ \theta \partial_t w =& D_w \Delta w + u - w. \end{aligned} \tag{6.25}$$

The uniform solution $\mathbf{u_0} = (u_0, v_0, w_0)^T$ can be found solving

$$\begin{aligned} (\lambda - \kappa_3 - \kappa_4) u_0 - u_0^3 + \kappa_1 &= 0, \\ v_0 = w_0 &= u_0. \end{aligned} \tag{6.26}$$

First let us obtain conditions for the stability of the homogeneous solution $\mathbf{u_0}$ without diffusion. For the system (7.1) the Jacobi matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} f'(u_0) & -\kappa_3 & -\kappa_4 \\ 1/\tau & -1/\tau & 0 \\ 1/\theta & 0 & -1/\theta \end{pmatrix}$$

The coefficients $p_1,\ p_2,\ p_3$ of the Gurwitz matrix G_3 for the matrix $\mathbf{A}$ can be written as

$$\begin{aligned} p_1 &= \frac{\tau + \theta - f'(u_0)}{\tau\theta}, \\ p_2 &= \frac{1 + \theta\kappa_3 + \tau\kappa_4 - f'(u_0)(\tau + \theta)}{\tau\theta}, \\ p_3 &= \frac{f'(u_0) - \kappa_3 - \kappa_4}{\tau\theta} \end{aligned}$$

Then the conditions for local stability of homogeneous solution $(u_0, v_0, w_0)^T$ take a form:

$$\begin{aligned}
&f'(u_0) < \frac{1}{\tau} + \frac{1}{\theta},\\
&\tau\theta(\tau+\theta)f'(u_0)^2 - ((\tau+\theta)^2 + \tau\theta(\theta\kappa_3 + \tau\kappa_4))f'(u_0) + (\tau+\theta) + \theta^2\kappa_3 + \tau^2\kappa_4 > 0,\\
&f'(u_0) < \kappa_3 + \kappa_4
\end{aligned} \tag{6.27}$$

In some limit cases the latter can be represented in more simple form, e.g., if $\tau = 1$ and $\theta = 1$ the stability conditions (6.27) are

$$f'(u_0) \in \big(-\infty,\ 1\big) \ \cup\ \big(1 + \frac{\kappa_3 + \kappa_4}{2},\ \kappa_3 + \kappa_4\big) \tag{6.28}$$

As before, let us suppose that the homogeneous solution of the Eq. (7.1) is stable against uniform perturbations, i.e., the conditions (6.27) are fulfilled. We pursue now a goal to determine the stability of the homogeneous solution in presence of diffusion. With this objection in mind we consider a stability of the system with a matrix $\mathbf{B}(k) = \mathbf{A} - k^2\mathbf{D}$, where $\mathbf{D}$ again is the diffusion matrix,

$$\mathbf{D} = \begin{pmatrix} D_u & 0 & 0 \\ 0 & D_v/\tau & 0 \\ 0 & 0 & D_w/\theta \end{pmatrix}.$$

In the same manner as for the matrix $\mathbf{A}$ one can write down the characteristic polynomial for the matrix $\mathbf{B}$, using its coefficients $p_1,\ p_2,\ p_3$, construct the Gurwitz matrix and finally obtain Raus-Gurwitz stability criterion in the following form:

$$\begin{aligned}
&\Delta_1(k) = -\mathrm{Sp}(\mathbf{A}) + k^2\mathrm{Sp}(\mathbf{D}) > 0 \quad \forall k,\\
&\Delta_2(k) = p_1p_2 - p_3 > 0 \quad \forall k,\\
&\Delta_3(k) = \det(\mathbf{B}(k)) < 0 \quad \forall k.
\end{aligned} \tag{6.29}$$

It is easily seen that the first condition (6.29) is fulfilled identically for all k. The other two conditions are represented by a bicubic polynomial with coefficients, depending on the system parameters. It is clear that violation one of these two inequalities leads to instability to finite wave number perturbation, i.e., to Turing instability. Unfortunately, the coefficients staying in Δ_2 and Δ_3 have complicated form, what make the evaluation of the critical wave vector k as well as instability conditions in closed form impossible. On the other hand it can be done rather simple numerically for given parameters set.

6.4.3 Numerical results

We have solved system (7.1) numerically on a unit square $\Omega = [0,1] \times [0,1]$ with no-flux boundary conditions. As in Sec. 6.3 we have chosen the coefficients λ and κ_1 as control parameters; λ defines super- or subcritical regime