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Fell bundles and imprimitivity theorems

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Abstract. Our goal in this paper and two sequels is to apply the Yamagami–Muhly–Williams equivalence theorem for Fell bundles over groupoids to recover and extend all known imprimitivity theorems involving groups. Here we extend Raeburn's symmetric imprimitivity theorem, and also, in an appendix, we develop a number of tools for the theory of Fell bundles that have not previously appeared in the literature.

1. Introduction

In this paper, we take up in earnest the program that was suggested in [11] to use Fell bundles to unify and extend a broad range of imprimitivity theorems and equivalence theorems for C^*-dynamical systems, especially in settings that involve nonabelian duality. In a sense, it has long been understood that Fell bundles provide an important mechanism for illuminating the structure of C^*-dynamical systems and their associated crossed products. Fell invented Fell bundles precisely to understand better and extend the theory of induced representations that had been built up around Mackey's program [3, 4]. What is novel about our contribution is the use of Fell bundles over *groupoids.* Indeed, Fell bundles over groupoids appear to be essential in important situations involving groups acting and coacting on C^*-dynamical systems. For example, as we showed in [11], if one has a Fell bundle over a locally compact *group*, $\mathscr{A} \to G$, then there is a natural coaction δ of G on the C^*-algebra of the bundle, $C^*(G, \mathscr{A})$. Further, the cocrossed product, $C^*(G, \mathscr{A}) \rtimes_\delta G$, is naturally isomorphic to a Fell bundle over the transformation groupoid determined by the action of G on G through left translation [11, Thm. 5.1]. This fact turns out to be crucial for proving that the natural surjection from $C^*(G, \mathscr{A}) \rtimes_\delta G \rtimes_{\widehat{\delta}} G$ to $C^*(G, \mathscr{A}) \otimes \mathcal{K}(L^2(G))$, is an isomorphism [11, Thm. 8.1]—in other words, that δ is a maximal coaction. (Here $\widehat{\delta}$ denotes the natural action of G on $C^*(G, \mathscr{A}) \rtimes_\delta G$ that is dual to δ, and $\mathcal{K}(L^2(G))$ denotes the compact operators on $L^2(G)$.) This same technique—using Fell bundles over groupoids to prove that dual coactions on full cross-sectional C^*-algebras of Fell bundles over groups are maximal—was used in [2] for discrete groups.

This is the first of three papers that are dedicated to showing how all known imprimitivity theorems can be unified and extended under the umbrella of Fell bundles over groupoids. The notion of a system of imprimitivity, and the first imprimitivity theorem, appeared very early in the theory of groups. As discussed in [7, p. 64ff.], if a group G acts on a set X, then a system of imprimitivity is simply a partition $\mathcal{P}$ of X that is invariant under the action of G, in the sense that for each $P \in \mathcal{P}$ and for each $x \in G$, Px is another element of $\mathcal{P}$. The system is called transitive if G permutes the elements of $\mathcal{P}$ transitively. Suppose, in addition, that each $P \in \mathcal{P}$ is a vector space and that the action of G is linear in the sense that each $x \in G$ induces a linear map from $P \in \mathcal{P}$ to Px. Then in a natural way, G acts linearly on the direct sum $\bigoplus_{P\in\mathcal{P}} P$, yielding a representation of G, $U = \{U_x\}_{x\in G}$. Further, for each $P \in \mathcal{P}$, the restriction V of U to the isotropy group H of P is a representation of H on P and U is *induced* from V [7, Thm. 16.7.1]. And conversely, if H is the isotropy group of any $P \in \mathcal{P}$ and if V is a representation of H on P, then in a natural way V induces a representation of G on $\bigoplus_{P\in\mathcal{P}} P$. These two statements, taken together, are known as the imprimitivity theorem. Thus, even in the setting of finite groups, one begins to see the players that enter into our analysis: the elements of $\mathcal{P}$ may be viewed as a bundle over G/H, where H is the subgroup of G that fixes a particular $P \in \mathcal{P}$. The representation space $\bigoplus_{P\in\mathcal{P}} P$ is the space of all cross sections of this bundle. The representation U of G is induced by an action of H. We may also think of $\bigoplus_{P\in\mathcal{P}} P$ as arising from a coaction of G, or better of G/H.

Inspired by problems in quantum mechanics, Mackey discovered a generalization of the imprimitivity theorem that is valid for *unitary* representations of locally compact groups in [12]. His theorem may be formulated as follows. Suppose a second countable locally compact group G acts transitively and measurably on a standard Borel space X, and suppose H is the isotropy group of a point in X. Then a unitary representation U of G on a Hilbert space $\mathcal{H}$ is induced from a unitary representation of H if and only if there is a spectral measure E defined on the Borel sets of X with values in the projections on $\mathcal{H}$ such that for every Borel set M in X and every $g \in G$,

$$E(Mg) = U_g^{-1} E(M) U_g. \tag{1.1}$$

The connection with the imprimitivity theorem for finite groups becomes clear once one uses direct integral theory (applied to the spectral measure E) to decompose $\mathcal{H}$ as the direct integral of a bundle $\mathfrak{H} = \{H_x\}_{x\in X}$ of Hilbert spaces over X. The representation U permutes the fibers H_x transitively and is induced in the fashion indicated above from a representation of the isotropy group of any point in X. Of course, there are many technical difficulties to surmount when one works at this level of generality, but the idea is clear.

The contemporary view of Mackey's theorem is due to Rieffel [22]. He was motivated not only by Mackey's theorem, but also by other generalizations of it. In particular, he received considerable inspiration from Takesaki's paper [24] and from the work of Fell [4].

To understand Rieffel's perspective, observe first that if a second countable locally compact group acts measurably and transitively on a standard Borel space X and if H is the isotropy group of some point in X, then X is Borel isomorphic to the coset space G/H. Further, specifying a spectral measure on X is tantamount to specifying a C^*-representation, say π, of $C_0(G/H)$. The spectral measure satisfies the covariance equation (1.1) relative to some unitary representation U of G if and only if U and π, satisfy the following covariance equation

$$\pi(f_g) = U_g^{-1}\pi(f)U_g \tag{1.2}$$

for all $f \in C_0(G/H)$ and all $g \in G$. This equation, in turn, means that the pair (π, U) can be *integrated* to a C^*-representation of the crossed product C^*-algebra $C_0(G/H) \rtimes G$. Further, the space $C_c(G)$ may be endowed with (pre-) C^*-algebra-valued inner products in such a way that the completion $\mathcal{X}$ becomes a $(C_0(G/H) \rtimes G) - C^*(H)$ *imprimitivity bimodule*, also called a *Morita equivalence bimodule*, that links $C_0(G/H) \rtimes G$ and $C^*(H)$. The group C^*-algebra, $C^*(G)$, sits inside the multiplier algebra of $C_0(G/H) \rtimes G$ and one concludes that a representation ρ of $C^*(G)$ is induced from a representation σ, say, of $C^*(H)$ via $\mathcal{X}$ if and only if ρ can be extended in a natural way to a C^*-representation of $C_0(G/H) \rtimes G$.

Thus, Rieffel observed that Mackey's imprimitivity theorem is a special case of the following theorem:

Theorem 1.1 ([22, Thm. 6.29]). *Suppose π is a representation of a C^*-algebra A on a Hilbert space H_π. Suppose also that there is a representation of A in the space, $\mathcal{L}(\mathcal{X}_B)$, of bounded, adjointable operators on the right Hilbert C^*-module $\mathcal{X}_B$ over the C^*-algebra B. Then π is induced from a representation σ of B on a Hilbert space H_σ in the sense of Rieffel* [22, Def. 5.2] *if and only if π can be extended to a representation of the compact operators, $\mathcal{K}(\mathcal{X}_B)$, on $\mathcal{X}_B$, in such a way that*

$$\pi(ak) = \pi(a)\pi(k)$$

for all $a \in A$ and $k \in \mathcal{K}(\mathcal{X}_B)$.

Rieffel's theorem opens a whole new dimension to representation theory, not only of groups but of C^*-algebras, generally. One is led ineluctably to look for situations when a given C^*-algebra A may be represented in $\mathcal{L}(\mathcal{X}_B)$ for suitable C^*-algebras B and Hilbert C^*-modules $\mathcal{X}_B$. Such searches are really searches for Morita contexts $(\mathcal{K}(\mathcal{X}_B), \mathcal{X}_B, B)$ that reflect properties of A. Thus, an imprimitivity theorem arises whenever one finds an interesting Morita context. This point was made initially by Rieffel in [22] and was reinforced in his note [23]. Subsequently, in [6], Green parlayed Rieffel's *Imprimitivity-cum-Morita-equivalence*-perspective into a recovery of Takesaki's theorem in the form of a Morita equivalence between $(A \otimes C_0(G/H)) \rtimes G$ and $A \rtimes H$, where A is a C^*-algebra on which a locally compact group G acts continuously via $*$-automorphisms. In the same paper, Green also proved another imprimitivity theorem for induced actions. (We will have more to say about induced actions below.)

With the advent of the important theory of nonabelian crossed product duality, involving coactions as well as actions of locally compact groups, a dual version of Green's theorem became exigent. This was supplied by Mansfield [13]. It took the form of a Morita equivalence between $A \rtimes_\delta G \rtimes_{\widehat{\delta}} N$ and $A \rtimes_{\delta|} G/N$ when N is a normal subgroup of G. (Here, δ is a coaction of the locally compact group G on the C^*-algebra A, $\widehat{\delta}$ is the dual action of G determined by δ, and $\delta|$ is the natural coaction of G/N.) Mansfield's result was subsequently generalized to nonnormal N by an Huef and Raeburn [8]. Their study, in turn, led to the notion of a cocrossed product by a coaction of a homogeneous space.

A few other imprimitivity theorems involving crossed products by group actions have appeared: Combes [1] showed that an equivariant Morita equivalence gives rise to a Morita equivalence between the crossed products. Raeburn's symmetric imprimitivity theorem [19] recovers, among other results, Green's theorem for induced actions, the Green–Takesaki theorem for induced representations of C^*-dynamical systems, and several of the examples developed in [23].

Another general imprimitivity theorem that unifies various results, starting with Mackey's, was given by Fell (see, for example, [5]), using what are now known as Fell bundles over locally compact groups. Introducing groupoids to the realm of imprimitivity theorems, Muhly, Renault, and Williams [16] showed how a certain type of equivalence between locally compact groupoids gives rise to Morita equivalence between their C^*-algebras. In [21], Renault combined groupoid equivalence with groupoid crossed products, generalizing both Raeburn's symmetric imprimitivity theorem and the Muhly–Renault–Williams groupoid algebra imprimitivity theorem.

In an unpublished preprint [25], Yamagami stated a very general imprimitivity theorem for Fell bundles over locally compact groupoids (see also [26], [14]), and the complete details (in slightly greater generality) were worked out by Muhly and Williams in [17]. This imprimitivity theorem is central to our considerations, as we indicated at the outset, and so we will refer to this as the MWY theorem.

It is the MWY theorem that we will use to unify all known imprimitivity theorems involving groups. While an outline of what is necessary is fairly clear, the details are formidable and require a large amount of work. Due to their length, we will split this project over several papers. In this first paper, we will show how the MWY theorem can be used to deduce Raeburn's symmetric imprimitivity theorem, and thereby unify many imprimitivity theorems involving crossed products by actions of locally compact groups. To prepare the way for Raeburn's theorem, we first prove a general imprimitivity theorem, which we call the "Symmetric Action theorem", for commuting group actions by automorphisms on a Fell bundle. The Symmetric Action theorem will quickly imply not only Raeburn's theorem, but also Mansfield's theorem (which we postpone to a subsequent paper), thus giving a unified approach to the standard imprimitivity theorems for actions *and* for coactions. (We could use the Symmetric Action theorem to quickly derive the Green and the Green–Takesaki imprimitivity theorems, but we leave details aside since Raeburn has already shown how his symmetric imprimitivity theorem quickly implies the Green and Green–Takesaki theorems.) In subsequent papers, in addition to Mansfield's imprimitivity theorem, we will illuminate a curious connection between a one-sided version of the Symmetric Action theorem and Rieffel's imprimitivity theorem for his generalized-fixed-point algebras. And we shall deduce Fell's original imprimitivity theorem from the MWY theorem.

To help with some of the technicalities that arise in our constructions of Fell bundles, we have included in an appendix a "toolkit for Fell bundle constructions". In addition to playing an important role in our present analysis, we believe it will prove useful elsewhere. The constructions we develop in the appendix include semidirect products of Fell bundles (over groupoids) by actions of locally compact groups, quotients of Fell bundles by free and proper group actions, and a combination of these two that involve commuting actions of two groups. In addition, we give a structure theorem that characterizes all free and proper actions by automorphisms of a group on a Fell bundle. For this purpose, we employ a result, which we believe is due to Palais, that shows that such actions all arise from transformation Fell bundles (the theory of which we developed in [11]).

One final remark before getting down to business. One may wonder if all of our results have full groupoid variants. That is, one may wonder if we may replace all groups and group actions that we will be discussing by groupoids and groupoid actions. That may be possible and the more optimistic ones among us believe that it is. However, the technical details appear to be more formidable than those we must develop in this paper and its sequels. We feel, therefore, that it will be best to put them aside until the theory of group actions, along the lines we conceive, are more fully exposed.

2. Preliminaries

We adopt the conventions of [17, 11] for Fell bundles over locally compact groupoids. Whenever we refer to a space (in particular, to a groupoid or a group), we tacitly assume that it is locally compact, Hausdorff, and second countable. Whenever we refer to a Banach bundle over a space (in particular, to a Fell bundle over a groupoid or a group), we assume that it is upper semicontinuous and separable—as in [17, 11]. We say that a Fell bundle is *separable* if the base groupoid is second countable and the Banach space of continuous sections vanishing at infinity is separable. All groupoids will be assumed to be equipped with a left Haar system.

Whenever we say a groupoid $\mathcal{X}$ acts on the left of a space X, we tacitly assume that the associated fibring map

$$\rho : X \to \mathcal{X}^{(0)}$$

is continuous and open, and that the action is continuous in the appropriate sense.

If $p : \mathscr{A} \to \mathcal{X}$ is a Fell bundle over a locally compact groupoid, we define $s, r : \mathscr{A} \to \mathcal{X}^{(0)}$ by

$$s(a) = s(p(a)) \quad \text{and} \quad r(a) = r(p(a)).$$

Similarly, if Fell bundles $\mathscr{A} \to \mathcal{X}$ and $\mathscr{B} \to \mathcal{Y}$ act on the left and right, respectively, of a Banach bundle $q : \mathscr{E} \to \Omega$, with respective fibring maps

$$\mathcal{X}^{(0)} \xleftarrow{\rho} \Omega \xrightarrow{\sigma} \mathcal{Y}^{(0)},$$

we define maps

$$\mathcal{X}^{(0)} \xleftarrow{\rho} \mathscr{E} \xrightarrow{\sigma} \mathcal{Y}^{(0)}$$

by

$$\rho(e) = \rho(q(e)) \quad \text{and} \quad \sigma(e) = \sigma(q(e)).$$

3. The Symmetric Action theorem

In this section we derive from the Yamagami–Muhly–Williams equivalence theorem the following general imprimitivity theorem that involves commuting actions of groups on a Fell bundle. This theorem will be used to unify most (but not quite all) of the known imprimitivity theorems we will derive. For the background on actions of groups on Fell bundles and the associated groupoid constructions, see Section A.1.

Theorem 3.1. *If locally compact groups G and H act freely and properly on the left and right, respectively, of a Fell bundle $p : \mathscr{A} \to \mathcal{X}$ over a locally compact groupoid, and if the actions commute, then $\mathscr{A}$ becomes an $(\mathscr{A}/H \rtimes G) - (H \ltimes G\backslash\mathscr{A})$ equivalence in the following way:*

(i) $\mathscr{A}/H \rtimes G$ acts on the left of $\mathscr{A}$ by

$$(a \cdot H, t) \cdot b = a(t \cdot b) \quad \textit{if} \quad s(a) = r(t \cdot b);$$

(ii) the left inner product is given by

$${}_L\langle a, b\rangle = \big(a(t \cdot b^*) \cdot H, t\big) \quad \textit{if} \quad G \cdot s(a) = G \cdot s(b),$$

where t is the unique element of G such that $s(a) = t \cdot s(b)$;

(iii) $H \ltimes G\backslash\mathscr{A}$ acts on the right of $\mathscr{A}$ by

$$a \cdot (h, G \cdot b) = (a \cdot h)b \quad \textit{if} \quad s(a \cdot h) = r(b);$$

(iv) the right inner product is given by

$$\langle a, b\rangle_R = \big(h, G \cdot (a^* \cdot h)b\big) \quad \textit{if} \quad r(a) \cdot H = r(b) \cdot H,$$

where h is the unique element of H such that $r(a) \cdot h = r(b)$.

As we explain in Corollary A.12, by "free and proper action on $\mathscr{A}$" we mean that the corresponding action on $\mathcal{X}$ has these properties. By Proposition A.22 the action of G on $\mathscr{A}$ descends to an action on $\mathscr{A}/H$, and then the semidirect-product Fell Bundle $\mathscr{A}/H \rtimes G \to \mathcal{X}/H \rtimes G$ acts on the left of the Banach bundle $\mathscr{A}$. Since the hypotheses are symmetric in G and H, with the "sides" reversed, we immediately conclude that

- H acts on the right of the orbit bundle $G\backslash\mathscr{A}$ by $(G\cdot a)\cdot h = G\cdot(a\cdot h)$;
- the semidirect-product Fell bundle $H \ltimes G\backslash\mathscr{A}$ acts on the right of the Banach bundle $\mathscr{A}$ by $a\cdot(h, G\cdot b) = (a\cdot h)b$ if $s(a\cdot h) = r(b)$.

We list, for convenient reference, the formulas for the equivalence of the base groupoids: by Lemma A.21 the action of G on $\mathcal{X}$ descends to an action

$$t\cdot(x\cdot H) = (t\cdot x)\cdot H$$

on $\mathcal{X}/H$, and the semidirect-product groupoid $\mathcal{X}/H \rtimes G$ acts on the left of the space $\mathcal{X}$ by

$$(x\cdot H, t)\cdot y = x(t\cdot y) \quad \text{if} \quad s(x) = r(t\cdot y).$$

Again, by symmetry the action of H on $\mathcal{X}$ descends to an action

$$(G\cdot x)\cdot h = G\cdot(x\cdot h)$$

on the orbit groupoid $G\backslash\mathcal{X}$, and the semidirect-product groupoid $H \ltimes G\backslash\mathcal{X}$ acts on the right of the space $\mathcal{X}$ by

$$x\cdot(h, G\cdot y) = (x\cdot h)y \quad \text{if} \quad s(x\cdot h) = r(y).$$

We begin by observing that these formulas give a groupoid equivalence:

Lemma 3.2. *If locally compact groups G and H act freely and properly on the left and right, respectively, of a locally compact groupoid $\mathcal{X}$ by automorphisms and if the actions commute, then, with the actions indicated above, the space $\mathcal{X}$ becomes a $(\mathcal{X}/H \rtimes G) - (H \ltimes G\backslash\mathcal{X})$ equivalence.*

Remark. This lemma is a straightforward generalization of the well-known equivalence for transformation groupoids when $\mathcal{X}$ is just a space.

Proof. By [16, Def. 2.1] we must verify the following:

(i) $\mathcal{X}/H \rtimes G$ acts freely and properly;
(ii) $H \ltimes G\backslash\mathcal{X}$ acts freely and properly;
(iii) the actions of $\mathcal{X}/H \rtimes G$ and $H \ltimes G\backslash\mathcal{X}$ commute;
(iv) the associated fiber map $\rho : \mathcal{X} \to \mathcal{X}^{(0)}/H \times \{e\}$ factors through a homeomorphism of $\mathcal{X}/(H \ltimes G\backslash\mathcal{X})$ onto $\mathcal{X}^{(0)}/H \times \{e\}$;
(v) the associated fiber map $\sigma : \mathcal{X} \to \{e\} \times G\backslash\mathcal{X}^{(0)}$ factors through a homeomorphism of $(\mathcal{X}/H \rtimes G)\backslash\mathcal{X}$ onto $\{e\} \times G\backslash\mathcal{X}^{(0)}$.

Because our hypotheses are symmetric in G and H, it will suffice to verify (i), (iii), and (iv). Moreover, we already know (i) from Corollary A.19, and (iii) is clear.

So, it remains to verify (iv). Recall that ρ is defined by

$$\rho(x) = (r(x)\cdot H, e).$$

Since the range map $r : \mathcal{X} \to \mathcal{X}^{(0)}$ is open, so is ρ. Thus it suffices to show that ρ factors through a bijection of $\mathcal{X}/(H \ltimes G\backslash\mathcal{X})$ onto $\mathcal{X}^{(0)}/H \times \{e\}$. Clearly ρ is a surjection of $\mathcal{X}$ onto $\mathcal{X}^{(0)}/H \times \{e\}$, and it is invariant under the right $(H \ltimes G\backslash\mathcal{X})$-action:

$$\begin{aligned}\rho\big(x \cdot (h, G \cdot y)\big) &= \rho\big((x \cdot h)y\big) = \big(r\big((x \cdot h)y\big) \cdot H, e\big)\\ &= \big(r(x \cdot h) \cdot H, e\big) = \big(r(x) \cdot H, e\big)\\ &= \rho(x).\end{aligned}$$

Finally, suppose $\rho(x) = \rho(y)$. We must show that $y \in x \cdot (H \ltimes G\backslash\mathcal{X})$. We have $r(x) \cdot H = r(y) \cdot H$, so there exists $h \in H$ such that

$$r(y) = r(x) \cdot h = r(x \cdot h).$$

Put

$$z = (x \cdot h)^{-1} y.$$

Then

$$y = (x \cdot h)z = x \cdot (h, G \cdot z). \qquad \square$$

It will be convenient to record a few formulas associated with the above groupoid equivalence. Letting

$$\mathcal{X} *_\sigma \mathcal{X} = \big\{(x, y) \in \mathcal{X} \times \mathcal{X} \mid \sigma(x) = \sigma(y)\big\},$$

we have a pairing

$${}_L[\cdot,\cdot] : \mathcal{X} *_\sigma \mathcal{X} \to \mathcal{X}/H \rtimes G$$

characterized by the following property: ${}_L[x, y]$ is the unique element $(z \cdot H, t) \in \mathcal{X}/H \rtimes G$ such that

$$x = {}_L[x, y] \cdot y = (z \cdot H, t) \cdot y = z \cdot (t \cdot y),$$

so we have

$$(3.1)\quad {}_L[x, y] = \big(x(t \cdot y^{-1}) \cdot H, t\big), \quad \text{where } t \in G \text{ is unique with } s(x) = t \cdot s(y).$$

Similarly, if $(x, y) \in \mathcal{X} *_\rho \mathcal{X}$, so that $\rho(x) = \rho(y)$, then

(3.2)

$$[x, y]_R = \big(h, G \cdot (x^{-1} \cdot h)y\big), \quad \text{where } h \in H \text{ is unique with } r(y) = r(x) \cdot h,$$

is the unique element of $H \ltimes G\backslash\mathcal{X}$ such that $y = x \cdot [x, y]_R$.

Remark 3.3. Note that in the statement of Theorem 3.1 the formulas (i) and (iii) are expressed in "cleaned-up" form. For example, in (i) either a or b has been adjusted within its H-orbit to force $s(a) = r(b)$. In practice, we might not always have the luxury of making such adjustments, so we must be careful in computing the left action of $\mathscr{A}/H \rtimes G$ on $\mathscr{A}$: for $(a \cdot H, t) \in \mathscr{A}/H \rtimes G$ and $b \in \mathscr{A}$, the left module product $(a \cdot H, t) \cdot b$ is defined if and only if

$$s(a \cdot H, t) = \rho(b),$$

equivalently

$$(t^{-1} \cdot s(a) \cdot H, e) = (r(b) \cdot H, e),$$

which reduces to

$$t^{-1} \cdot s(a) \cdot H = r(b) \cdot H,$$

and then we have

$$(a \cdot H, t) \cdot b = (a \cdot h)(t \cdot b),$$

where $h \in H$ is unique with $s(a) \cdot h = t \cdot r(b)$. Similarly for the right action of $H \ltimes G \backslash \mathscr{A}$ on $\mathscr{A}$.

Proof of Theorem 3.1. We have seen in Lemma 3.2 that the space $\mathcal{X}$ is an $(\mathcal{X}/H \rtimes G) - (H \ltimes G \backslash \mathcal{X})$ equivalence. By Proposition A.22, the Fell bundle $\mathscr{A}/H \rtimes G$ acts on the left of the Banach bundle $\mathscr{A}$ as indicated in (i), and we have discussed at the beginning of this section how the right action of $H \ltimes G \backslash \mathscr{A}$ on $\mathscr{A}$ arises from symmetry.

We must verify the axioms in [17, Def. 6.1], which has three main items (a)–(c), with item (b) itself having four parts. To improve readability we break the verification into steps.

Step 1. Item (a) of [17, Def. 6.1] is that the actions of $\mathscr{A}/H \rtimes G$ and $H \ltimes G \backslash \mathscr{A}$ on $\mathscr{A}$ commute: let $(a \cdot H, t) \in \mathscr{A}/H \rtimes G$, $b \in \mathscr{A}$, and $(h, G \cdot c) \in H \ltimes G \backslash \mathscr{A}$, with

$$s(a) \cdot H = r(t \cdot b) \cdot H \quad \text{and} \quad G \cdot s(b \cdot h) = G \cdot r(c)$$

so that both

$$(a \cdot H, t) \cdot b \quad \text{and} \quad b \cdot (G \cdot c, h)$$

are defined. We must show that both

$$\big((a \cdot H, t) \cdot b\big) \cdot (G \cdot c, h) \quad \text{and} \quad (a \cdot H, t) \cdot \big(b \cdot (G \cdot c, h)\big)$$

are defined, and coincide in $\mathscr{A}$.

Adjust a within the orbit $a \cdot H$ so that $s(a) = r(t \cdot b)$. Then

$$(a \cdot H, t) \cdot b = a(t \cdot b).$$

Similarly, adjust c within $G \cdot c$ so that $s(b \cdot h) = r(c)$, and then

$$b \cdot (h, G \cdot c) = (b \cdot h)c.$$

We have

$$\begin{aligned} G \cdot s\big(\big(a(t \cdot b)\big) \cdot h\big) &= G \cdot s\big((a \cdot h)(t \cdot b \cdot h)\big) = G \cdot s(t \cdot b \cdot h) = G \cdot s(b \cdot h) \\ &= G \cdot r(c), \end{aligned}$$

so

$$\big(a(t \cdot b)\big) \cdot (h, G \cdot c)$$

is defined, and then we have

$$\big((a \cdot H, t) \cdot b\big) \cdot (h, G \cdot c) = \big(\big(a(t \cdot b)\big) \cdot h\big)(t' \cdot c),$$

where $t' \in G$ is unique with

$$t' \cdot r(c) = s\big(\big(a(t \cdot b)\big) \cdot h\big) = s\big((a \cdot h)(t \cdot b \cdot h)\big) = s(t \cdot b \cdot h) = t \cdot s(b \cdot h).$$

But we have arranged for $r(c) = s(b \cdot h)$, so in fact we have $t' = t$, and hence

$$\big((a \cdot H, t) \cdot b\big) \cdot (h, G \cdot c) = (a \cdot h)(t \cdot b \cdot h)(t \cdot c).$$

Similarly, we have
$$r\big((t\cdot((b\cdot h)c)\big)\cdot H = s(a)\cdot H,$$
so
$$\begin{aligned}(a\cdot H,t)\cdot\big(b\cdot(h,G\cdot c)\big) &= (a\cdot H,t)\cdot\big((b\cdot h)c\big) = (a\cdot h)\big(t\cdot\big((b\cdot h)c\big)\big)\\ &= (a\cdot h)(t\cdot b\cdot h)(t\cdot c),\end{aligned}$$
and we have shown that the actions of $(\mathscr{A}/H)\rtimes G$ and $H\ltimes(G\backslash\mathscr{A})$ commute.

Step 2. Item (b) of [17, Def. 6.1] has four parts, concerning the left and right inner products. Before we begin, we first check that the left and right inner products in Theorem 3.1 are well-defined, and by symmetry it suffices to check the left-hand inner product. Let $(a,b)\in\mathscr{A}*_\sigma\mathscr{A}$. Then $\sigma(a)=\sigma(b)$, and $\big(e,G\cdot s(a)\big)=\big(e,G\cdot s(b)\big)$, so that there is a unique $t\in G$ such that $s(a)=t\cdot s(b)$. Thus
$$s(a)=r\big((t\cdot b)^*\big),$$
and therefore the inner product is well-defined.

We proceed to the first part of [17, Def. 6.1(b)], namely
$$p_{\mathcal{X}/H\rtimes G}\left({}_L\langle a,b\rangle\right) = {}_{\mathcal{X}/H\rtimes G}\big[p(a),p(b)\big] \quad \text{for } (a,b)\in\mathscr{A}*_\sigma\mathscr{A},$$
where
$$p_{\mathcal{X}/H\rtimes G}:\mathscr{A}/H\rtimes G\to\mathcal{X}/H\rtimes G$$
is the bundle projection. According to the definition of the left inner product, we have
$$\begin{aligned}p\big({}_L\langle a,b\rangle\big) &= p\big(\big(a(t\cdot b^*)\cdot H,t\big)\big) = \big(p\big(a(t\cdot b^*)\big)\cdot H,t\big)\\ &= \big(p(a)\big(t\cdot p(b)^{-1}\big)\cdot H,t\big) = {}_L[p(a),p(b)],\end{aligned}$$
by (3.1).

Step 3. The second part of [17, Def. 6.1(b)] is
$${}_L\langle a,b\rangle^* = {}_L\langle b,a\rangle \quad\text{and}\quad \langle a,b\rangle_R^* = \langle b,a\rangle_R,$$
and again by symmetry it suffices to prove the first. Let $a,b\in\mathscr{A}$ with $\sigma(a)=\sigma(b)$, and let $t\in G$ be unique such that $s(a)=t\cdot s(b)$. Then
$$\begin{aligned}{}_L\langle a,b\rangle^* &= \big(a(t\cdot b^*)\cdot H,t\big)^* = \big(t^{-1}\cdot\big(a(t\cdot b^*)\big)^*\cdot H,t\big)\\ &= \big(t^{-1}\cdot\big((t\cdot b)a^*\big)^*\cdot H,t\big) = \big(b(t^{-1}\cdot a^*)\cdot H,t^{-1}\big) = {}_L\langle b,a\rangle.\end{aligned}$$

Step 4. The third part of [17, Def. 6.1(b)] is
$${}_L\big\langle(a\cdot H,t)\cdot b,c\big\rangle = (a\cdot H,t)\,{}_L\langle b,c\rangle,$$
and a similar equality involving $\langle\cdot,\cdot\rangle_R$, but by symmetry it suffices to show it for the left inner product: let
$$s\big(p_{\mathscr{A}/H\rtimes G}(a\cdot H,t)\big)=\rho(b) \quad\text{and}\quad \sigma(b)=\sigma(c),$$
so that
$$s(a)\cdot H = t\cdot r(b)\cdot H \quad\text{and}\quad G\cdot s(b) = G\cdot s(c).$$

Adjust a within its orbit $a \cdot H$ so that $s(a) = t \cdot r(b)$, and choose the unique $t' \in G$ such that $s(b) = t' \cdot s(c)$. Then

$$\begin{aligned}(a \cdot H, t)_L\langle b, c\rangle &= (a \cdot H, t)\big(b(t' \cdot c^*) \cdot H, t'\big) = \big(a\big(t \cdot \big(b(t' \cdot c^*)\big)\big) \cdot H, tt'\big) \\ &= \big(a(t \cdot b)(tt' \cdot c) \cdot H, tt'\big) = {}_L\big\langle a(t \cdot b), c\big\rangle = {}_L\big\langle (a \cdot H, t) \cdot b, c\big\rangle\end{aligned}$$

Step 5. The fourth part of [17, Def. 6.1(b)] is

$${}_L\langle a, b\rangle \cdot c = a \cdot \langle b, c\rangle_R.$$

More precisely, we need to show that for $a, b, c \in \mathscr{A}$, if both ${}_L\langle a, b\rangle$ and $\langle b, c\rangle_R$ are defined, then so are

$${}_L\langle a, b\rangle \cdot c \quad \text{and} \quad a \cdot \langle b, c\rangle_R,$$

and they are equal. Thus, we are assuming that

$$\sigma(a) = \sigma(b) \quad \text{and} \quad \rho(b) = \rho(c),$$

which entails that

$$G \cdot s(a) = G \cdot s(b) \quad \text{and} \quad r(b) \cdot H = r(c) \cdot H.$$

Choose the unique $t \in G$ and $h \in H$ such that $s(a) = t \cdot s(b)$ and $r(b) \cdot h = r(c)$. Then

$$\begin{aligned}{}_L\langle a, b\rangle \cdot c &= \big(a(t \cdot b^*) \cdot H, t\big) \cdot c = \big(\big(a(t \cdot b^*)\big) \cdot h\big)(t \cdot c) = (a \cdot h)(t \cdot b^* \cdot h)(t \cdot c) \\ &= (a \cdot h)\big(t \cdot \big((b^* \cdot h)c\big)\big) = a \cdot \big(h, G \cdot (b^* \cdot h)c\big) = a \cdot \langle b, c\rangle_R,\end{aligned}$$

where the second equality is justified by

$$\begin{aligned}s\big(\big(a(t \cdot b^*)\big) \cdot h\big) &= s\big((a \cdot h)(t \cdot b^* \cdot h)\big) = s(t \cdot b^* \cdot h) = t \cdot s(b^* \cdot h) \\ &= t \cdot r(b \cdot h) = t \cdot r(c),\end{aligned}$$

and a similar computation justifies the fifth equality.

Step 6. Finally, item (c) of [17, Def. 6.1(c)] is that that the operations (i)–(iv) make each fiber $A(x)$ of the Banach bundle $\mathscr{A}$ into an imprimitivity bimodule between the corresponding fibers

$$A(r(x)) \cdot H \times \{e\} \quad \text{and} \quad \{e\} \times G \cdot A(s(x))$$

of the Fell bundles

$$\mathscr{A}/H \rtimes G \quad \text{and} \quad H \ltimes G\backslash\mathscr{A},$$

respectively. For this we just have to observe that, in view of the obvious isomorphisms

$$A(r(x)) \cdot H \times \{e\} \cong A(r(x)) \quad \text{and} \quad \{e\} \times G \cdot A(s(x)) \cong A(s(x)),$$

our inner products and actions *coincide* with those on the $A(r(x)) - A(s(x))$ imprimitivity bimodule $A(x)$. □

We will need the special case of Theorem 3.1 where one group is trivial, and by symmetry it suffices to consider the case where the group H is trivial:

Corollary 3.4. *If a locally compact group G acts freely and properly on the left of a Fell bundle $p : \mathscr{A} \to \mathcal{X}$ over a locally compact groupoid, then $\mathscr{A}$ becomes an $\mathscr{A} \rtimes G - G\backslash\mathscr{A}$ equivalence in the following way:*

(i) $\mathscr{A} \rtimes G$ acts on the left of $\mathscr{A}$ by

$$(a, t) \cdot b = a(t \cdot b) \quad \text{if} \quad s(a) = t \cdot r(b);$$

(ii) the left inner product is given by

$$_L\langle a, b\rangle = \big(a(t \cdot b^*), t\big) \quad \text{if} \quad G \cdot s(a) = G \cdot s(b),$$

where t is the unique element of G such that $s(a) = t \cdot s(b)$;

(iii) $G\backslash\mathscr{A}$ acts on the right of $\mathscr{A}$ by

$$a \cdot (G \cdot b) = ab \quad \text{if} \quad s(a) = r(b);$$

(iv) the right inner product is given by

$$\langle a, b\rangle_R = G \cdot a^* b \quad \text{if} \quad r(a) = r(b).$$

Remark 3.5. Of course, if we have an action of a group H on the right of $\mathscr{A}$ instead of G acting on the left, the corresponding equivalence will be

$$\mathscr{A}/H \sim H \ltimes \mathscr{A}.$$

Remark 3.6. Note that (iii) has been expressed in "cleaned-up" form; in general, if $a, b \in \mathscr{A}$ then the right-module product $a \cdot (G \cdot b)$ is defined if and only if

$$G \cdot s(a) = G \cdot r(b),$$

and then we have

$$a \cdot (G \cdot b) = a(t \cdot b)$$

where t is the unique element of G such that $s(a) = t \cdot r(b)$.

It will be convenient to have another version of Corollary 3.4, recorded as Corollary 3.7 below, with the Fell bundle $\mathscr{A} \to \mathcal{X}$ replaced with the isomorphic transformation bundle.

Corollary 3.7. *Let $p : \mathscr{B} \to \mathcal{Y}$ be a Fell bundle over a locally compact groupoid, let $\mathcal{Y}$ act on the left of a locally compact Hausdorff space Ω, and let the associated fiber map $\rho : \Omega \to \mathcal{Y}^{(0)}$ be a principal G-bundle, with the locally compact group G acting on the left of Ω. Further assume that the actions of $\mathcal{Y}$ and G on Ω commute. Then the transformation bundle $\mathscr{B} * \Omega \to \mathcal{Y} * \Omega$ becomes a $(\mathscr{B} * \Omega) \rtimes G - \mathscr{B}$ equivalence in the following way:*

(i) $(\mathscr{B} * \Omega) \rtimes G$ acts on the left of $\mathscr{B} * \Omega$ by

$$(b, p(c) \cdot t \cdot u, t) \cdot (c, u) = (bc, t \cdot u) \quad \text{if} \quad s(b) = r(c);$$

(ii) the left inner product is given by

$$_L\big\langle (b, t \cdot u), (c, u)\big\rangle = \big(bc^*, t \cdot p(c) \cdot u, t\big) \quad \text{if} \quad s(b) = s(c);$$

(iii) $\mathscr{B}$ acts on the right of $\mathscr{B} * \Omega$ by

$$(b, u) \cdot c = (bc, p(c)^{-1} \cdot u) \quad \text{if} \quad s(b) = r(c);$$

(iv) *the right inner product is given by*

$$\big\langle (b, p(b^*c)\cdot u), (c,u)\big\rangle_{\mathscr{B}} = b^*c \quad \textit{if} \quad r(b)=r(c).$$

Note that the fibring maps

$$\rho : \mathcal{Y} * \Omega \to (\mathcal{Y}^{(0)} * \Omega) \times \{e\} \quad \text{and} \quad \sigma : \mathcal{Y} * \Omega \to \mathcal{Y}^{(0)}$$

associated to the actions of $(\mathcal{Y} * \Omega) \rtimes G$ and $\mathcal{Y}$ are given by

$$\rho(y,u) = (r(y), y\cdot u, e) \quad \text{and} \quad \sigma(y,u) = s(y).$$

Also note that every section $f \in \Gamma_c((\mathscr{B} * \Omega)\rtimes G)$ is uniquely of the form

$$f(y,u,s) = \big(f_1(y,u,s), u, s\big),$$

where $f_1 : (\mathcal{Y} * \Omega) \rtimes G \to \mathscr{B}$ is continuous with compact support and satisfies

$$f_1(y,u,s) \in B(y).$$

Once we have the Fell-bundle equivalence Theorem 3.1, by [17, Thm. 6.4] we have a Morita equivalence:

Corollary 3.8. *With the hypotheses of Theorem 3.1 we have a Morita equivalence*

$$C^*(\mathscr{A}/H) \rtimes_\alpha G \sim C^*(G\backslash\mathscr{A}) \rtimes_\beta H,$$

where α and β are the associated actions of G and H on the respective Fell-bundle C^*-algebras.

Proof. This follows from Theorem 3.1, [17, Thm. 6.4], and [11, Thm. 7.1]. □

Similarly, we have a one-sided Morita equivalence:

Corollary 3.9. *With the hypotheses of Corollary 3.4 we have a Morita equivalence*

$$C^*(\mathscr{A}) \rtimes_\alpha G \sim C^*(G\backslash\mathscr{A}),$$

where α is as in Corollary 3.8.

And here is the version for transformation bundles:

Corollary 3.10. *With the hypotheses of Corollary 3.7 we have a Morita equivalence*

$$C^*(\mathscr{B} * \Omega) \rtimes_\alpha G \sim C^*(\mathscr{B}),$$

where α is as in Corollary 3.9.

Special case: C^*-bundles. Here we specialize to the case where $\mathcal{X} = X$ is a *space* and $\mathscr{A}$ is just a C^*-bundle over X, so that $C^*(\mathscr{A}) = \Gamma_0(\mathscr{A})$. Then the orbit bundle is the C^*-bundle $G\backslash\mathscr{A} \to G\backslash X$.

Proposition 3.11. *If a locally compact group G acts freely and properly on a C^*-bundle $\mathscr{A} \to X$ over a locally compact Hausdorff space X, then we have a Morita equivalence*

$$\Gamma_0(\mathscr{A}) \rtimes_\alpha G \sim \Gamma_0(G\backslash\mathscr{A}),$$

where $\alpha : G \to \operatorname{Aut}\Gamma_0(\mathscr{A})$ is the associated action.

Special case: coaction-crossed products. We start with a Fell bundle $\mathscr{B}$ over a locally compact group G, then form the transformation Fell bundle $\mathscr{A} = \mathscr{B} \times G$ over the transformation groupoid $G \times_{\mathrm{lt}} G$, where G acts on itself by left translation. Then G acts freely and properly on $\mathscr{B} \times G$ by right translation in the second coordinate:

$$t \cdot (b, s) = (b, st^{-1}).$$

We identify the orbit bundle $G \backslash (\mathscr{B} \times G)$ with $\mathscr{B}$, and the orbit groupoid $G \backslash (G \times_{\mathrm{lt}} G)$ with the group G.

By [11, Thm. 5.1] there are a maximal coaction δ of G on $C^*(\mathscr{B})$ and an $\alpha - \hat{\delta}$ equivariant isomorphism

$$C^*(\mathscr{B} \times G) \cong C^*(\mathscr{B}) \rtimes_\delta G.$$

Thus Corollary 3.10 reduces in this case to crossed product duality for maximal coactions:

$$C^*(\mathscr{B}) \rtimes_\delta G \rtimes_{\hat{\delta}} G \sim C^*(\mathscr{B}).$$

In the even more special case where $\mathscr{B} = \mathbf{C} \times G$ is the trivial line bundle, we recover the well-known Morita equivalence

$$\mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho} G \sim C^*(G),$$

where ρ denotes the right regular representation of G.

4. Raeburn's Symmetric Imprimitivity Theorem

We recover Raeburn's symmetric imprimitivity theorem [19, Thm. 1.1] as a corollary to Corollary 3.8:

Corollary 4.1. *Suppose that locally compact groups G and H act freely and properly on the left and right, respectively, of a locally compact Hausdorff space X, and suppose the actions commute. Suppose, also, that σ and τ are commuting actions of G and H, respectively, on a C^*-algebra B, then we have a Morita equivalence*

$$\mathrm{Ind}_H^X B \rtimes_{\mathrm{Ind}\,\tau} G \sim \mathrm{Ind}_G^X B \rtimes_{\mathrm{Ind}\,\sigma} H.$$

Proof. Recall from [19] that we have induced actions $\mathrm{Ind}\,\tau$ and $\mathrm{Ind}\,\sigma$ of G and H on the induced C^*-algebras $\mathrm{Ind}_H^X B$ and $\mathrm{Ind}_G^X B$, respectively. We aim to apply Theorem 3.8 with $\mathscr{A} = B \times X$, $\mathcal{X} = X$, and G and H acting diagonally on $B \times X$ (with the right H-action given by $(b, x) \cdot h = (\tau_h^{-1}(b), x \cdot h))$. The orbit Fell bundle $\mathscr{A}/H \to \mathcal{X}/H$ is the C^*-bundle $(B \times X)/H \to X/H$, and similarly for $G \backslash (B \times X) \to G \backslash X$. Let α and β denote the associated actions of G and H on $\Gamma_0((B \times X)/H)$ and $\Gamma_0(G \backslash (B \times X))$, respectively. By Theorem 3.8 we can finish by recalling from the standard theory of induced algebras that there are equivariant isomorphims:

$$\big(\Gamma_0\big((B \times X)/H\big), \alpha\big) \cong (\mathrm{Ind}_H^X B, \mathrm{Ind}\,\tau)$$
$$\big(\Gamma_0\big(\big(G \backslash (B \times X)\big), \beta\big) \cong (\mathrm{Ind}_G^X B, \mathrm{Ind}\,\sigma).$$

For example, a suitable isomorphism $\theta : \operatorname{Ind}_H^X B \to \Gamma_0((B \times X)/H)$ is given by $\theta(f)(x \cdot H) = (f(x), x) \cdot H$. $\square$

Appendix A. Bundle constructions

In this appendix, we will review several constructions involving Fell bundles over groupoids, some from [11] and some new, that we need in the body of the paper.

A.1. **Transformation groupoids and bundles.** Suppose a locally compact groupoid $\mathcal{X}$ acts on the left of a locally compact Hausdorff space Ω. Let $\rho : \Omega \to \mathcal{X}^{(0)}$ be the associated fiber map, and let

$$\mathcal{X} * \Omega = \{(x, u) \in \mathcal{X} \times \Omega \mid s(x) = \rho(u)\}$$

be the fiber product. Then $\mathcal{X} * \Omega$ is locally compact Hausdorff, and becomes the *transformation groupoid* with multiplication

$$(x, y \cdot u)(y, u) = (xy, u).$$

Thus

$$(x, u)^{-1} = (x^{-1}, x \cdot u), \quad s(x, u) = (s(x), u) \quad \text{and} \quad r(x, u) = (r(x), x \cdot u).$$

Remark. We've made a choice of convention here for the transformation groupoid—another fairly common choice involves writing $\mathcal{X}$ and Ω in the opposite order. But we will always use the above convention.

Note that the coordinate projection $\pi_1 : \mathcal{X} * \Omega \to \mathcal{X}$ defined by

$$\pi_1(x, u) = x$$

is a groupoid homomorphism. If now $p : \mathscr{A} \to \mathcal{X}$ is a Fell bundle, then the associated *transformation Fell bundle* has total space

$$\mathscr{A} * \Omega := \{(a, u) \in \mathscr{A} \times \Omega \mid s(a) = \rho(u)\},$$

base groupoid $\mathcal{X} * \Omega$, bundle projection

$$p(a, u) = (p(a), u),$$

multiplication

$$(a, p(b) \cdot u)(b, u) = (ab, u),$$

and involution

$$(a, u)^* = (a^*, p(a) \cdot u).$$

The easiest way to see that this is a Fell bundle is to note that it is isomorphic to the pullback [11, Lemma 1.1] $\pi_1^* \mathscr{A}$ via the map

$$(a, u) \mapsto (a, p(a) \cdot u).$$

It is not hard to check that we get a left Haar system on $\mathcal{X} * \Omega$ via[1]

$$\int_{\mathcal{X} * \Omega} f(y, v)\, d\lambda^{r(x,u)}(y, v) = \int_{\mathcal{X}} f(y, y^{-1}x \cdot u)\, d\lambda^{r(x)}(y).$$

[1]Note that the formula in [15] for the Haar system on the transformation groupoid is incorrect.

Thus $C_c(\mathcal{X} * \Omega)$ has convolution and involution given by

$$(f * g)(x,u) = \int_{\mathcal{X}} f(y, y^{-1}x \cdot u) g(y^{-1}x, u)\, d\lambda^{r(x)}(y)$$

and

$$f^*(x,u) = \overline{f(x^{-1}, x \cdot u)}.$$

Every section $f \in \Gamma_c(\mathscr{A} * \Omega)$ is of the form

$$f(x,u) = (f_1(x,u), u)$$

for some function $f_1 \in C_c(\mathcal{X} * \Omega, \mathscr{A})$ satisfying

$$f_1(x,u) \in A(x) \quad \text{for } (x,u) \in \mathcal{X} * \Omega.$$

Convolution and involution in the section algebra $\Gamma_c(\mathscr{A} * \Omega)$ are determined by

$$(f * g)_1(x,u) = \int_{\mathcal{X}} f_1(y, y^{-1}x \cdot u) g_1(y^{-1}x, u)\, d\lambda^{r(x)}(y)$$

and

$$(f^*)_1(x,u) = f_1(x^{-1}, x \cdot u)^*,$$

and we will simplify the notation by writing f_1^* for $(f^*)_1$.

A.2. **Semidirect products.** Let G be a locally compact group and $\mathcal{X}$ be a locally compact groupoid. Recall from [11, Sec. 6] that an *action by automorphisms* of G on (the left of) $\mathcal{X}$ is a continuous map

$$(s,x) \mapsto s \cdot x : G \times \mathcal{X} \to \mathcal{X}$$

such that

- for each $s \in G$, the map $x \mapsto s \cdot x$ is an automorphism of the groupoid $\mathcal{X}$, and
- $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in G$ and $x \in \mathcal{X}$,

and that the associated *semidirect-product* groupoid $\mathcal{X} \rtimes G$ comprises the Cartesian product $\mathcal{X} \times G$ with multiplication

$$(x,s)(y,t) = \big(x(s \cdot y), st\big) \quad \text{if } x \text{ and } s \cdot y \text{ are composable.}$$

Thus the range and source maps are given by

$$r(x,t) = (r(x), e) \quad \text{and} \quad s(x,t) = \big(t^{-1} \cdot s(x), e\big),$$

and the inverse is given by

$$(x,s)^{-1} = (s^{-1} \cdot x^{-1}, s^{-1}).$$

Warning. Frequently we will have a group (or a groupoid) acting on a groupoid $\mathcal{X}$ *as a space*—so then the action on $\mathcal{X}$ is just by homeomorphisms, not automorphisms—and to avoid confusion we will usually emphasize the particular type of action on $\mathcal{X}$: G either *acts on $\mathcal{X}$ by automorphisms* or *acts on the space $\mathcal{X}$*.

Recall from [11, Sec. 6] that, when G acts on $\mathcal{X}$ by automorphisms, in order to get a Haar system on the semidirect-product groupoid $\mathcal{X} \rtimes G$ we need to assume that the action is *invariant* in the sense that

$$\int_{\mathcal{X}} f(s \cdot x)\, d\lambda^u(x) = \int_{\mathcal{X}} f(x)\, d\lambda^{s \cdot u}(x),$$

and then

$$d\lambda^{(u,e)}(x,s) = d\lambda^u(x)\, ds$$

is a Haar system on $\mathcal{X} \rtimes G$.

Also recall from [11] that if $p : \mathscr{A} \to \mathcal{X}$ is a Fell bundle, then an *action* of G on (the left of) $\mathscr{A}$ (by automorphisms) consists of an action of G on $\mathcal{X}$ and a continuous map $G \times \mathscr{A} \to \mathscr{A}$ such that

- for each $s \in G$, the map $a \mapsto s \cdot a$ is an automorphism of the Fell bundle $\mathscr{A}$ (which entails $p(s \cdot a) = s \cdot p(a)$), and
- $s \cdot (t \cdot a) = (st) \cdot a$ for all $s, t \in G$ and $a \in \mathscr{A}$,

and that the associated *semidirect-product* Fell bundle $\mathscr{A} \rtimes G$ comprises the Cartesian product $\mathscr{A} \times G$ with bundle projection

$$p(a,s) = (p(a), s),$$

multiplication

$$(a,s)(b,t) = \big(a(s \cdot b), st\big) \quad \text{if } a \text{ and } s \cdot b \text{ are composable,}$$

and involution

$$(a,s)^* = (s^{-1} \cdot a^*, s^{-1}).$$

Thus, convolution and involution on $C_c(\mathcal{X} \rtimes G)$ are given by

$$(\phi * \psi)(x,s) = \int_{\mathcal{X}} \int_G \phi(y,t) \psi\big(t^{-1} \cdot (y^{-1}x), t^{-1}s\big)\, dt\, d\lambda^{r(x)}(y)$$

and

$$\phi^*(x,s) = \overline{\phi(s^{-1} \cdot x^{-1}, s^{-1})}.$$

If G acts by automorphisms on a Fell bundle $p : \mathscr{A} \to \mathcal{X}$ then every section $f \in \Gamma_c(\mathscr{A} \rtimes G)$ is of the form

$$f(x,s) = (f_1(x,s), s)$$

for some function $f_1 \in C_c(\mathcal{X} \times G, \mathscr{A})$ satisfying

$$f_1(x,s) \in A(x) \quad \text{for } (x,s) \in \mathcal{X} \times G.$$

Convolution and involution in the section algebra $\Gamma_c(\mathscr{A} \rtimes G)$ are determined by

$$(f * f')_1(x,s) = \int_{\mathcal{X}} \int_G f_1(y,t) f_1'\big(t^{-1} \cdot (y^{-1}x), t^{-1}s\big)\, dt\, d\lambda^{r(x)}(y)$$

and

$$f_1^*(x,s) = f_1(s^{-1} \cdot x^{-1}, s^{-1})^*.$$

Similarly, if G acts on the right of $\mathcal{X}$ rather than the left, the semidirect product $G \ltimes \mathcal{X}$ has multiplication

$$(s,x)(t,y) = \big(st, (x \cdot t)y\big),$$

range and source maps

$$r(t,x) = (e, r(x)\cdot t^{-1}) \quad \text{and} \quad s(t,x) = (e, s(x)),$$

and inverse

$$(s,x)^{-1} = (s^{-1}, x^{-1}\cdot s^{-1}).$$

A.3. **Semidirect-product actions.**

Definition A.4. Let G be a locally compact group, $\mathcal{X}$ be a locally compact groupoid, and Ω be a locally compact Hausdorff space. Suppose G acts on $\mathcal{X}$ by groupoid automorphisms, and that both G and $\mathcal{X}$ act on Ω, with all actions being on the left. We say that the actions of G and $\mathcal{X}$ on Ω are *covariant* if

$$s\cdot(x\cdot u) = (s\cdot x)\cdot(s\cdot u) \quad \text{for all } (x,u)\in \mathcal{X} * \Omega.$$

Similarly for covariant right actions.

Lemma A.5. *Let G be a group, $\mathcal{X}$ be a groupoid, and Ω be a space. Suppose G acts on $\mathcal{X}$ by automorphisms, and that both G and $\mathcal{X}$ act on Ω, with all actions being on the left. If the actions of G and $\mathcal{X}$ on Ω are covariant, then the semidirect-product groupoid $\mathcal{X}\rtimes G$ acts on Ω by*

$$(x,t)\cdot u = x\cdot(t\cdot u) \quad \textit{if} \quad s(x) = \rho(t\cdot u), \tag{A.1}$$

where $\rho : \Omega \to \mathcal{X}^{(0)}$ is the associated fiber map.

Proof. Define

$$\rho' : \Omega \to (\mathcal{X}\rtimes G)^{(0)} = \mathcal{X}^{(0)} \times \{e\}$$

by

$$\rho'(u) = (\rho(u), e).$$

Then ρ' is clearly continuous and open. Moreover

$$s(x,t) = \rho'(u) \quad \text{if and only if} \quad s(x) = \rho(t\cdot u),$$

so the definition (A.1) of $(x,t)\cdot u$ is well-defined. It is routine to check the axioms for an action, even the continuity: if

$$\big((x_i,t_i),u_i\big) \to \big((x,t),u\big) \quad \text{in } (\mathcal{X}\rtimes) * \Omega,$$

then

$$(x_i,t_i)\cdot u_i = x_i\cdot(t_i\cdot u_i) \to x\cdot(t\cdot u) = (x,t)\cdot u$$

because $x_i \to x$ and $t_i\cdot u_i \to t\cdot u$. $\square$

For convenient reference, we record the corresponding result for right actions:

Corollary A.6. *If a group H acts on a groupoid $\mathcal{X}$, and H and $\mathcal{X}$ act covariantly on a space Ω, all actions being on the right, then $H \ltimes \mathcal{X}$ acts on the right of Ω by*

$$u\cdot(h,x) = (u\cdot h)\cdot x \quad \textit{if} \quad \sigma(u) = r(x\cdot h^{-1}),$$

where $\sigma : \Omega \to \mathcal{X}^{(0)}$ is the associated fiber map.

Definition A.7. Let G be a group, let $p : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a groupoid, and let $q : \mathscr{E} \to \Omega$ be a Banach bundle. Suppose G acts on $\mathscr{A}$ by Fell-bundle automorphisms, and that both G and $\mathscr{A}$ act on $\mathscr{E}$ (with G acting by isometric isomorphisms, and $\mathscr{A}$ acting as in [17]), with all actions being on the left. We say that the actions of G and $\mathscr{A}$ on $\mathscr{E}$ are *covariant* if

$$s \cdot (a \cdot e) = (s \cdot a) \cdot (s \cdot e) \quad \text{if} \quad s(a) = \rho(q(e)).$$

Corollary A.8. *Let G be a group, let $p : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a groupoid, and let $q : \mathscr{E} \to \Omega$ be a Banach bundle. Suppose G acts on $\mathscr{A}$ by Fell-bundle automorphisms, and that both G and $\mathscr{A}$ act on $\mathscr{E}$ (G acting by isometric isomorphisms and $\mathscr{A}$ acting as in* [17]*), with all actions being on the left. If the actions of G and $\mathscr{A}$ on $\mathscr{E}$ are covariant, then the semidirect-product Fell bundle $\mathscr{A} \rtimes G$ acts on $\mathscr{E}$ by*

$$(a, t) \cdot e = a \cdot (t \cdot e) \quad \textit{if} \quad s(a) = \rho(q(t \cdot e)). \tag{A.2}$$

Proof. Recall from Lemma A.5 that $\mathcal{X} \rtimes G$ acts on the space Ω, and that the associated fiber map $\rho' : \Omega \to \{e\} \times \mathcal{X}^{(0)}$ is

$$\rho'(u) = (\rho(u), e),$$

where $\rho : \Omega \to \mathcal{X}^{(0)}$ is the fiber map associated to the action of $\mathcal{X}$ on Ω. We have

$$s(p(a,t)) = \rho'(q(e))$$

if and only if

$$s(a) = \rho(q(t \cdot e)).$$

It is routine to check the axioms for an action of a Fell bundle on a Banach bundle. $\square$

A.9. **Quotients.** We want to know that if a group H acts freely by automorphisms on a groupoid $\mathcal{X}$, then the orbit space $\mathcal{X}/H$ is a groupoid, and similarly if H acts on a Fell bundle $\mathscr{A}$ over $\mathcal{X}$.

Of course we want all this to be topological. Thus we take H and $\mathcal{X}$ to be locally compact Hausdorff. To ensure that $\mathcal{X}/H$ also has these properties, we require that the action be free and proper.

Proposition A.10. *Let H be a locally compact group and $\mathcal{X}$ a locally compact Hausdorff groupoid. Suppose that H acts freely and properly on the right of $\mathcal{X}$ by automorphisms. Then the orbit space $\mathcal{X}/H$ becomes a locally compact Hausdorff groupoid, called an* orbit groupoid *or a* quotient groupoid*, with multiplication*

$$(x \cdot H)(y \cdot H) = (xy) \cdot H \quad \textit{whenever } s(x) = r(y). \tag{A.3}$$

Moreover, if the action of H on $\mathcal{X}$ is invariant in the sense of [11, Def. 6.3]*, then there is a Haar system $\dot\lambda$ on $\mathcal{X}/H$ given by*

$$\int_{\mathcal{X}/H} f(x \cdot H)\, d\dot\lambda^{u \cdot H}(x \cdot H) = \int_{\mathcal{X}} f(x \cdot H)\, d\lambda^u(x). \tag{A.4}$$

Proof. Because H acts freely, if $s(x)\cdot H = r(y)\cdot H$ then there is a unique $h \in H$ such that $s(x\cdot h) = r(y)$. But a moment's thought reveals that somewhat more is true: the set

$$\{zw \mid z \in x \cdot H,\ w \in y \cdot H,\ s(z) = r(w)\}$$

comprises a single orbit, represented by any such product zw. In particular, if we choose x within its orbit $x \cdot H$ so that $s(x) = r(y)$, then the above set of products coincides with the orbit $(xy)\cdot H$. Thus, replacing x by $x\cdot h$ we see that the multiplication (A.3) is well-defined. It is routine to check that the formula (A.3) does make $\mathcal{X}/H$ into a groupoid, and it is locally compact and Hausdorff because H acts properly. It remains to verify continuity of multiplication and inversion in $\mathcal{X}/H$.

Let

$$x_i \cdot H \to x \cdot H \quad \text{and} \quad y_i \cdot H \to y \cdot H \quad \text{in } \mathcal{X}/H,$$

and assume that

$$s(x_i \cdot H) = r(y_i \cdot H) \quad \text{for all } i.$$

We will show that $s(x \cdot H) = r(y \cdot H)$ and

$$(x_i \cdot H)(y_i \cdot H) \to (x \cdot H)(y \cdot H). \tag{A.5}$$

It suffices to show that every subnet has a subnet satisfying (A.5), and since any subnet will continue satisfy our hypotheses, it suffices to show that some subnet satisfies (A.5).

Note that the range and source maps on $\mathcal{X}/H$ are continuous and open, since the range and source maps on $\mathcal{X}$ are continuous and open, as is the quotient map $\mathcal{X} \to \mathcal{X}/H$. Thus we have $s(x \cdot H) = r(y \cdot H)$. If necessary, replace x within its orbit $x\cdot H$ so that $s(x) = r(y)$. However, we will *not* make corresponding choices at this time for the x_i's; rather, this will arise from other considerations in our argument.

Since the quotient map $\mathcal{X} \to \mathcal{X}/H$ is open, after passing to subnets and relabeling we may, if necessary, replace each x_i by another element of its orbit $x_i\cdot H$ so that we have $x_i \to x$ in $\mathcal{X}$, and similarly we can assume that $y_i \to y$. For each i there is a unique $h_i \in H$ such that $s(x_i)\cdot h_i = r(y_i)$. Then

$$s(x_i)\cdot h_i \to r(y) = s(x),$$

and of course by continuity we also have $s(x_i) \to s(x)$. By the elementary Lemma A.11 below we conclude that $h_i \to e$ in H. Thus, replacing x_i by $x_i \cdot h_i$, we now have $s(x_i) = r(y_i)$ for all i, and we still have $x_i \to x$, and hence

$$(x_i \cdot H)(y_i \cdot H) = (x_i y_i)\cdot H \to (xy)\cdot H = (x \cdot H)(y \cdot H).$$

Continuity of inversion is now easy: if $x_i \cdot H \to x \cdot H$ in $\mathcal{X}/H$, then as above we can pass to a subnet (which suffices, as before) so that $x_i \to x$, and then

$$(x_i \cdot H)^{-1} = x_i^{-1}\cdot H \to x^{-1}\cdot H = (x \cdot H)^{-1}.$$

Finally, assume that the action of H on $\mathcal{X}$ is invariant; since H acts on the right instead of the left here, we note explicitly that invariance in this case means

$$\int_{\mathcal{X}} f(x \cdot h)\, d\lambda^u(x) = \int_{\mathcal{X}} f(x)\, d\lambda^{u \cdot h}(x). \tag{A.6}$$

To show that (A.4) gives a left Haar system, note that the range map $r : \mathcal{X} \to \mathcal{X}^{(0)}$ is a continuous and open surjection that is H-equivariant, the left Haar system λ on $\mathcal{X}$ is an r-system, and by (A.6) it is H-equivariant in Renault's sense [21]. Thus the existence of a Haar system on $\mathcal{X}/H$ is guaranteed by [21, Lemme 1.3]. □

In the above proof we appealed to the following general lemma:

Lemma A.11. *Let a locally compact group H act freely and properly on the right of a locally compact Hausdorff space Ω. Suppose that we have $u \in \Omega$ and nets $\{u_i\}$ in Ω and $\{h_i\}$ in H such that $u_i \to u$ and $u_i \cdot h_i \to u$. Then $h_i \to e$.*

Proof. It suffices to show that some subnet of $\{h_i\}$ converges to e. Choose a compact neighborhood K of u. After passing to subnets and relabeling, without loss of generality we may assume that

$$u_i \in K \quad \text{and} \quad u_i \cdot h_i \in K \quad \text{for all } i.$$

Since H acts properly, the set

$$\{(v, h) \in \Omega \times H \mid v \in K \text{ and } v \cdot h \in K\}$$

is compact, and hence the set

$$\{h \in H \mid v \cdot h \in K \text{ for some } v \in K\}$$

is also compact. Thus all the h_i's lie in a compact set, so again by passing to subnets and relabeling we may assume that the net $\{h_i\}$ converges, say $h_i \to h$ in H. Then we have

$$u \cdot h = \lim u_i \cdot h_i = u,$$

so we must have $h = e$, again by freeness. □

Corollary A.12. *Let $p : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a locally compact Hausdorff groupoid, and let H be a locally compact group. Suppose H acts freely and properly by automorphisms on the right of $\mathscr{A}$ (by which we mean that the associated action of H on the groupoid $\mathcal{X}$ has these properties). Then the orbit space $\mathscr{A}/H$ becomes a Fell bundle over the orbit groupoid $\mathcal{X}/H$, called an* orbit Fell bundle *or a* quotient Fell bundle*, with operations*

(i) $p(a \cdot H) = p(a) \cdot H$;
(ii) $(a \cdot H)(b \cdot H) = (ab) \cdot H$ *if* $s(a) = r(b)$;
(iii) $(a \cdot H)^* = a^* \cdot H$.

Proof. It might be useful to explicitly record what a typical fiber of $\mathscr{A}/H$ is: if $x \in \mathcal{X}$, so that $x \cdot H$ is a typical element of $\mathcal{X}/H$, then the associated fiber of $\mathscr{A}/H$ is

$$p^{-1}(x \cdot H) = p^{-1}(x) \cdot H.$$

It might also be helpful to explicitly record the norm in $\mathscr{A}/H$:

$$\|a \cdot H\| = \|a\|,$$

which is well-defined since $\|a \cdot h\| = \|a\|$ for all $h \in H$. The map p defined in (i) is continuous and open because both the bundle projection $p : \mathscr{A} \to \mathcal{X}$ and the quotient map $\mathcal{X} \to \mathcal{X}/H$ are. Most of the axioms of a Fell bundle are routine to check. The crucial property is freeness of the H-action; for example, freeness is the reason that the sum of two elements in the same fiber of $\mathscr{A}/H$ is well-defined.

The only properties that are not quite routine are continuity of multiplication and of involution. But as in similar situations above, continuity can be established by lifting convergent nets from $\mathscr{A}/H$ to $\mathscr{A}$. □

A.13. **Principal bundles.** We now show that Fell bundles with free and proper group actions by automorphisms are really just special kinds of transformation bundles. First we must establish the analogous result for groupoids.

To motivate what follows, consider a locally compact groupoid $\mathcal{Y}$ acting on the left of a locally compact Hausdorff space Ω, and let $q : \Omega \to \mathcal{Y}^{(0)}$ be the associated fiber map, so that for each $y \in \mathcal{Y}$ the map $u \mapsto y \cdot u$ is a bijection of $q^{-1}(s(y))$ to $q^{-1}(r(y))$. We can form the transformation groupoid $\mathcal{Y} * \Omega = \{(y, u) \in \mathcal{Y} \times \Omega \mid s(y) = q(u)\}$, with multiplication as in Section A.1. Note that

$$(\mathcal{Y} * \Omega)^{(0)} = \mathcal{Y}^{(0)} * \Omega = \{(u, u) \in \mathcal{Y}^{(0)} \times \Omega \mid u = q(u)\} \cong \Omega.$$

Now suppose that the above map $q : \Omega \to \mathcal{Y}^{(0)}$ is a principal H-bundle[2] for some locally compact group H, acting on the right of Ω, and assume that the actions of H and $\mathcal{Y}$ on Ω commute.

Lemma A.14. *With the above setup, H acts freely and properly by automorphisms on the transformation groupoid $\mathcal{Y} * \Omega$ by*

$$(y, u) \cdot h = (y, u \cdot h).$$

In particular, $\pi_1 : \mathcal{Y} * \Omega \to \mathcal{Y}$ is a principal H-bundle.

Proof. We must show that for all $h, k \in H$,

(i) $(y, u) \mapsto (y, u \cdot h)$ is an automorphism of $\mathcal{Y} * \Omega$,
(ii) $((y, u) \cdot h) \cdot k = (y, u) \cdot (hk)$, and
(iii) $(y, u) \cdot e = (y, u)$.

[2]By which we mean that H acts freely and properly on Ω and that $q(u) = q(v)$ if and only if $u \cdot H = v \cdot H$—note that we do *not* assume the bundle is locally trivial!

Since (ii) and (iii) are obvious, we only prove (i). Let $s(y,u) = r(z,v)$ in $\mathcal{Y} * \Omega$, so that $s(y) = r(z)$ and $u = z \cdot v$. Then for $h \in H$ we have

$$u \cdot h = (z \cdot v) \cdot h = z \cdot (v \cdot h),$$

so $s((y,u) \cdot h) = r((z,v) \cdot h)$. Thus we can multiply:

$$\begin{aligned}\big((y,u) \cdot h\big)\big((z,v) \cdot h\big) &= (y, u \cdot h)(z, v \cdot h)\\ &= (yz, v \cdot h)\\ &= (yz, v) \cdot h)\\ &= \big((y,u)(z,v)\big) \cdot h.\end{aligned}$$

Finally, it is routine to verify that freeness and properness of the action of H on Ω transfers to the action on $\mathcal{Y} * \Omega$, and that $\pi_1 : \mathcal{Y} * \Omega \to \mathcal{Y}$ is the associated principal H-bundle. $\square$

Observe that in the setup of Lemma A.14, we have a commutative diagram

$$\begin{tikzcd} \mathcal{Y} * \Omega \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \Omega \arrow[d, "q"] \\ \mathcal{Y} \arrow[r, "s"'] & \mathcal{Y}^{(0)}, \end{tikzcd}$$

where the vertical maps are principal H-bundles, that π_1 is a surjective groupoid homomorphism, and that the map

$$(y,u) \cdot H \mapsto y$$

is an H-equivariant isomorphism of the quotient groupoid $(\mathcal{Y} * \Omega)/H$ onto $\mathcal{Y}$ making the diagram

$$\begin{tikzcd} \mathcal{Y} * \Omega \arrow[rr, "\pi_2"] \arrow[dr, "Q"] \arrow[dd, "\pi_1"'] & & \Omega \arrow[dd, "q"] \\ & (\mathcal{Y} * \Omega)/H \arrow[dl, "\cong"] \arrow[dr, "s"] & \\ \mathcal{Y} \arrow[rr, "s"'] & & \mathcal{Y}^{(0)} \end{tikzcd}$$

commute, where Q is the quotient map.

Lemma A.14 can be regarded as a source of free and proper actions by automorphisms of groups on groupoids. We now show that, remarkably, every such action arises in this manner. The underlying reason is a structure theorem for principal bundles, perhaps due to Palais (see the discussion preceding [18, Prop. 1.3.4]), although we have found it more convenient to quote the version recorded in [9, Prop. 1.3.4]; we feel that this structure theorem should be better known.

Theorem A.15. *Let H be a locally compact group acting freely and properly by automorphisms on the right of a locally compact Hausdorff groupoid $\mathcal{X}$, and let $\mathcal{Y} = \mathcal{X}/H$ be the quotient groupoid. Then there is an action of $\mathcal{Y}$ on $\mathcal{X}^{(0)}$ such that $\mathcal{X}$ is H-equivariantly isomorphic to the transformation groupoid $\mathcal{Y} * \mathcal{X}^{(0)}$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{X}^{(0)} \arrow[d, "q"'] & \mathcal{X} \arrow[l, "\tilde{r}"'] \arrow[r, "\tilde{s}"] \arrow[d, "q"] & \mathcal{X}^{(0)} \arrow[d, "q"] \\
\mathcal{Y}^{(0)} & \mathcal{Y} \arrow[l, "r"] \arrow[r, "s"'] & \mathcal{Y}^{(0)}
\end{tikzcd}$$

(where we denote the range and source maps for the groupoid $\mathcal{X}$ by $\tilde{r}$ and $\tilde{s}$, respectively), in which the vertical maps are principal H-bundles and the horizontal maps are continuous. We have a pullback diagram

$$\begin{tikzcd}
s^*(\mathcal{X}^{(0)}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{X}^{(0)} \arrow[d, "q"] \\
\mathcal{Y} \arrow[r, "s"'] & \mathcal{Y}^{(0)},
\end{tikzcd}$$

where $s^*(\mathcal{X}^{(0)}) = \{(y, u) \in \mathcal{Y} \times \mathcal{X}^{(0)} \mid s(y) = q(u)\}$. By the universal property of pullbacks, we have a unique continuous map θ_s making the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\tilde{s}"] \arrow[dr, dashed, "\theta_s", "!"'] \arrow[dd, "q"'] & & \mathcal{X}^{(0)} \arrow[dd, "q"] \\
& s^*(\mathcal{X}^{(0)}) \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
\mathcal{Y} \arrow[rr, "s"'] & & \mathcal{Y}^{(0)}
\end{tikzcd}$$

commute, namely $\theta_s(x) = \big(q(x), \tilde{s}(x)\big)$, and, by [9, Thm. 4.4.2], θ_s is in fact a principal H-bundle isomorphism. Similarly for $\theta_r : \mathcal{X} \cong r^*(\mathcal{X}^{(0)})$, and we get a big commuting diagram

$$\begin{tikzcd}
\mathcal{X}^{(0)} \arrow[dd, "q"'] & & \mathcal{X} \arrow[ll, "\tilde{r}"'] \arrow[rr, "\tilde{s}"] \arrow[dl, "\theta_r"', "\cong"] \arrow[dr, "\theta_s", "\cong"'] \arrow[dd, "q"] & & \mathcal{X}^{(0)} \arrow[dd, "q"] \\
& r^*(\mathcal{X}^{(0)}) \arrow[ul, "\pi_2"] \arrow[dr, "\pi_1"'] & & s^*(\mathcal{X}^{(0)}) \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
\mathcal{Y}^{(0)} & & \mathcal{Y} \arrow[ll, "r"'] \arrow[rr, "s"'] & & \mathcal{Y}^{(0)}.
\end{tikzcd} \tag{A.7}$$

We use the homeomorphisms θ_r and θ_s to (temporarily) *define* groupoid structures on the pullbacks $r^*(\mathcal{X}^{(0)})$ and $s^*(\mathcal{X}^{(0)})$, and this gives rise to a groupoid isomorphism θ making the diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\theta_r"'] \arrow[dr, "\theta_s"] & \\ r^*(\mathcal{X}^{(0)}) & & s^*(\mathcal{X}^{(0)}) \arrow[ll, "\theta", "\cong"'] \end{tikzcd}$$

commute. Commutativity of (A.7) tells us that θ has the form $\theta(y,u) = (y, y\cdot u)$, for some map $(y,u)\mapsto y\cdot u$ from $s^*(\mathcal{X}^{(0)})$ to $\mathcal{X}^{(0)}$.

We claim that this gives an action of the groupoid $\mathcal{Y}$ on the space $\mathcal{X}^{(0)}$. Continuity of the map $(y,u)\mapsto y\cdot u$ is clear. The associated fiber map will be $q:\mathcal{X}^{(0)}\to\mathcal{Y}^{(0)}$, which is continuous and open, and by construction the map $u\mapsto y\cdot u$ is a bijection of $q^{-1}(s(y))$ onto $q^{-1}(r(y))$. To help us see how to show that

$$y\cdot(z\cdot u) = (yz)\cdot u \tag{A.8}$$

whenever $s(y)=r(z)$ and $s(z)=q(u)$, we first observe a few properties of the groupoid $s^*(\mathcal{X}^{(0)})$: we have

- $s(z,u) = (s(z),u)$;
- $r(z,u) = (r(z), z\cdot u)$;
- $(y,u')(z,u)$ is defined if and only if $u' = z\cdot u$, and then $(y,z\cdot u)(z,u) = (yz,u)$.

Now take $w\in\mathcal{Y}$ with $s(w)=r(y)$. Then $s(w) = q(y\cdot(z\cdot u))$, so $(w, y\cdot(z\cdot u))\in s^*(\mathcal{X}^{(0)})$, and

$$s(w, y\cdot(z\cdot u)) = r(y, z\cdot u) = r\big((y,z\cdot u)(z,u)\big) = r(yz,u),$$

and (A.8) follows.

We now have an action of $\mathcal{Y}$ on $\mathcal{X}^{(0)}$, and hence we can form the transformation groupoid $\mathcal{Y}*\mathcal{X}^{(0)}$, which has the same underlying topological space as the pullback $s^*(\mathcal{X}^{(0)})$, and in fact the above reasoning now shows that the groupoid operation we temporarily gave to $s^*(\mathcal{X}^{(0)})$, by insisting that the bijection $\theta_s:\mathcal{X}\to s^*(\mathcal{X}^{(0)})$ be a groupoid isomorphism, coincides with the operation on the transformation groupoid $\mathcal{Y}*\mathcal{X}^{(0)}$. Thus θ_s is an isomorphism of $\mathcal{X}$ onto $\mathcal{Y}*\mathcal{X}^{(0)}$.

Finally, for the H-equivariance, we have

$$\begin{aligned}\theta_s(x)\cdot h &= (q(x),\tilde{s}(x))\cdot h = (q(x),\tilde{s}(x)\cdot h) = (q(x),\tilde{s}(x\cdot h))\\ &= (q(x\cdot h),\tilde{s}(x\cdot h)) = \theta_s(x\cdot h).\end{aligned}$$ □

We employ a by now familiar technique to promote the above result from groupoids to Fell bundles.

Let $p:\mathscr{B}\to\mathcal{Y}$ be a Fell bundle over a locally compact Hausdorff groupoid, let $\mathcal{Y}$ act on a locally compact Hausdorff space Ω, and let the map $q:\Omega\to\mathcal{Y}^{(0)}$ associated to the $\mathcal{Y}$-action be a principal H-bundle for some locally compact group H acting on the right of Ω. Keep the assumptions and notation from the preceding section, so that in particular the actions of $\mathcal{Y}$ and H on Ω commute.

We form the transformation Fell bundle

$$\mathscr{B} * \Omega = \{(b,u) \in \mathscr{B} \times \Omega \mid (p(b),u) \in \mathcal{Y} * \Omega\}$$

over $\mathcal{Y} * \Omega$.

Then H acts by automorphisms on $\mathscr{B} * \Omega$ by

$$(b,u) \cdot h = (b, u \cdot h),$$

and moreover this action is free and proper in the sense that, by Lemma A.14, the associated action on $\mathcal{Y} * \Omega$ has these properties.

It follows that the map

$$(b,u) \cdot H \mapsto b$$

is an isomorphism of the quotient Fell bundle $(\mathscr{B} * \Omega)/H$ onto $\mathscr{B}$ making the diagram

$$\begin{tikzcd}
\mathscr{B} * \Omega \arrow[rr, "p*\mathrm{id}"] \arrow[dr, "Q"] \arrow[dd, "\pi_1"'] & & \mathcal{Y} * \Omega \arrow[dd, "\pi_1"] \\
& (\mathscr{B} * \Omega)/H \arrow[dl, "\cong"'] \arrow[dr, "p"] & \\
\mathscr{B} \arrow[rr, "p"'] & & \mathcal{Y}
\end{tikzcd}$$

commute, where Q is the quotient map.

The above construction gives a source of free and proper actions (by automorphisms) of groups on Fell bundles; again, the structure theorem of Palais and Husemuller implies that all such actions arise in this manner:

Theorem A.16. *Let $\tilde{p} : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a locally compact Hausdorff groupoid, and let H be a locally compact group. Suppose H acts freely and properly by automorphisms on the right of $\mathscr{A}$, and let*

$$p : \mathscr{B} \to \mathcal{Y}$$

be the quotient Fell bundle (so that $\mathscr{B} = \mathscr{A}/H$ and $\mathcal{Y} = \mathcal{X}/H$). Then there is an action of $\mathcal{Y}$ on $\mathcal{X}^{(0)}$ such that $\mathscr{A}$ is H-equivariantly isomorphic to the transformation Fell bundle $\mathscr{B} * \mathcal{X}^{(0)}$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\tilde{p}"] \arrow[d, "q"'] & \mathcal{X} \arrow[r, "\tilde{s}"] \arrow[d, "q"'] & \mathcal{X}^{(0)} \arrow[d, "q"] \\
\mathscr{B} \arrow[r, "p"'] & \mathcal{Y} \arrow[r, "s"'] & \mathcal{Y}^{(0)},
\end{tikzcd}$$

where the vertical maps are principal H-bundles, and so it again follows from [9, Thm. 4.4.2] that the map

$$\tau(a) = \big(q(a), \tilde{s}(\tilde{p}(a)\big)$$

is a principal H-bundle isomorphism making the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "\tilde{s}\circ\tilde{p}"] \arrow[dr, "\tau", "\cong"'] \arrow[dd, "q"'] & & \mathcal{X}^{(0)} \arrow[dd, "q"] \\
& \mathscr{B} * \mathcal{X}^{(0)} \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
\mathscr{B} \arrow[rr, "s\circ p"'] & & \mathcal{Y}^{(0)}
\end{tikzcd}$$

commute. It is important to note that at this point the notation $\mathscr{B} * \mathcal{X}^{(0)}$ only stands for the pullback principal H-bundle. By Theorem A.15 we have an action of $\mathcal{Y}$ on $\mathcal{X}^{(0)}$ such that $\mathcal{X} \cong \mathcal{Y} * \mathcal{X}^{(0)}$. Thus we can form the transformation Fell bundle $\mathscr{B} * \mathcal{X}^{(0)}$.

We finish by showing that the principal H-bundle isomorphism $\tau : \mathscr{A} \to \mathscr{B} * \mathcal{X}^{(0)}$ is a homomorphism (and hence an isomorphism) of Fell bundles. If $a, b \in \mathscr{A}$ with $\tilde{s} \circ \tilde{p}(a) = \tilde{r} \circ \tilde{p}(b)$, then

$$\tilde{s}(\tilde{p}(a)) = \tilde{r}(\tilde{p}(b)) = q(\tilde{p}(b)) \cdot \tilde{s}(\tilde{p}(b)) = p(q(b)) \cdot \tilde{s}(\tilde{p}(b)),$$

so that $\tau(a)$ and $\tau(b)$ are composable in $\mathscr{B} * \mathcal{X}^{(0)}$, and we have

$$\begin{aligned}
\tau(a)\tau(b) &= \big(q(a), \tilde{s} \circ \tilde{p}(a)\big)\big(q(b), \tilde{s} \circ \tilde{p}(b)\big) = (q(a)q(b), \tilde{s} \circ \tilde{p}(b)) \\
&= (q(ab), \tilde{s} \circ \tilde{p}(ab)) = \tau(ab).
\end{aligned}$$

For the involution, we have

$$\begin{aligned}
\tau(a)^* &= \big(q(a)^*, p(q(a)) \cdot \tilde{s}(\tilde{p}(a))\big) = (q(a^*), \tilde{r}(\tilde{p}(a))) \\
&= (q(a^*), \tilde{s}(\tilde{p}(a)^{-1})) = (q(a^*), \tilde{s}(\tilde{p}(a^*))) = \tau(a^*).
\end{aligned}$$

□

A.17. **Orbit action.**

Lemma A.18. *Let $\mathcal{X}$ be a locally compact groupoid and H a locally compact group. Suppose H acts freely and properly by automorphisms on the right of $\mathcal{X}$. Then the orbit groupoid $\mathcal{X}/H$ acts on the left of the space $\mathcal{X}$ by*

$$(x \cdot H) \cdot y = xy \quad \textit{whenever } s(x) = r(y). \tag{A.9}$$

Remark. Note that $\mathcal{X}$ is being used in two different ways in the statement of the above lemma: first as a groupoid, and second as just a space.

Proof. This will be easier if we replace $\mathcal{X}$ by the isomorphic transformation groupoid $\mathcal{X}/H * \mathcal{X}^{(0)}$, using Theorem A.15. Then the formula (A.9) becomes

$$z \cdot (w, u) = (zw, u)$$

for $z, w \in \mathcal{X}/H$ with $s(z) = r(w)$ and $s(w) \cdot H = u \cdot H$. It is now clear that the action is well-defined and continuous, and that the associated map

$$\rho : \mathcal{X}/H * \mathcal{X}^{(0)} \to (\mathcal{X}/H)^{(0)} = \mathcal{X}^{(0)}/H$$

given by

$$\rho(x \cdot H, u) = r(x) \cdot H$$

is continuous and open (because both the range map $r : \mathcal{X} \to \mathcal{X}^{(0)}$ and the quotient map $\mathcal{X} \to \mathcal{X}/H$ are). $\square$

Corollary A.19. *Let $p : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a locally compact groupoid, and let H be a locally compact group. Suppose H acts freely and properly on the right of $\mathscr{A}$ by Fell-bundle automorphisms. Then the orbit Fell bundle $\mathscr{A}/H$ acts on the Banach bundle $\mathscr{A}$ by*

$$(a \cdot H) \cdot b = ab \quad \text{whenever } s(a) = r(b). \tag{A.10}$$

Remark. Again, note that $\mathscr{A}$ is being used in two different ways in the statement of the above corollary: first as a Fell bundle, and second as just a Banach bundle.

Proof. As in Lemma A.18, it is easier if we work with the isomorphic transformation Fell bundle $\mathscr{A}/H * \mathcal{X}^{(0)}$, using Theorem A.16. Then the formula (A.10) becomes

$$c \cdot (d, u) = (cd, u)$$

for $c, d \in \mathscr{A}$ with $s(c) = r(d)$ and $s(d) \cdot H = u \cdot H$. Again it is now clear that this action is well-defined and continuous, and is compatible with the action of the orbit groupoid $\mathcal{X}/H$ on $\mathcal{X}$ from Lemma A.18. $\square$

A.20. **Semidirect product orbit action.**

Proposition A.21. *Suppose that we are given commuting free and proper actions of locally compact groups G and H on the left and right, respectively, by automorphisms on a locally compact groupoid $\mathcal{X}$. Then:*

(i) G acts on the left of the orbit groupoid $\mathcal{X}/H$ by

$$s \cdot (x \cdot H) = (s \cdot x) \cdot H;$$

(ii) the semidirect-product groupoid $\mathcal{X}/H \rtimes G$ acts freely and properly on the left of the space $\mathcal{X}$ by

$$(x \cdot H, t) \cdot y = x(t \cdot y) \quad \text{whenever } s(x) = r(t \cdot y).$$

Proof. By Lemma A.18, the orbit groupoid $\mathcal{X}/H$ acts on the left of the space $\mathcal{X}$. It is routine to check (i), and also that the actions of G and $\mathcal{X}/H$ on $\mathcal{X}$ are covariant. Then it follows from Lemma A.5 that $\mathcal{X}/H \rtimes G$ acts on $\mathcal{X}$ as indicated.

To verify that the action of $\mathcal{X}/H \rtimes G$ is free and proper, it is easier if we replace $\mathcal{X}$ by the homeomorphic space $\mathcal{X}/H * \mathcal{X}^{(0)}$, using Theorem A.15. Then the action becomes

$$(z, t) \cdot (w, u) = \big(z(t \cdot w), t \cdot u\big) \quad \text{if} \quad s(z) = r(t \cdot w).$$

For the freeness, if $(z, t) \cdot (w, u) = (w, u)$, then $t \cdot u = u$, so $t = e$ since G acts freely, and then $zw = w$, so $z = r(w)$. Thus $(z, t) \in (\mathcal{X}/H \rtimes G)^{(0)}$.

For the properness, if $K \subset \mathcal{X}$ is compact, we can find compact sets

$$K_1 \subset \mathcal{X}/H \quad \text{and} \quad K_2 \subset \mathcal{X}^{(0)}$$

such that $K \subset K_1 \times K_2$. It suffices to find a compact set containing any $(z,t) \in \mathcal{X}/H \rtimes G$ for which $(z,t) \cdot K \cap K \neq \varnothing$. If $(w,u) \in K$ and $(z,t) \cdot (w,u) \in K$, then $t \cdot u \in K_2$, so because G acts properly there is a compact set $L \subset G$ containing any such t, and then using $z(t \cdot w) \in K_1$ we get

$$z \in K_1(t \cdot w^{-1}) \subset K_1(L \cdot K_1^{-1}),$$

which is compact in $\mathcal{X}/H$. Therefore we conclude that

$$(z,t) \in K_1(L \cdot K_1^{-1}) \times L,$$

which is compact in $\mathcal{X}/H \rtimes G$. □

Proposition A.22. *Let $p : \mathscr{A} \to \mathcal{X}$ be a Fell bundle over a locally compact groupoid, and let G and H be locally compact groups. Suppose that G and H act freely and properly on the left and right, respectively, of $\mathscr{A}$. Then*

(i) G acts on the left of the orbit bundle $\mathscr{A}/H$ by

$$s \cdot (a \cdot H) = (s \cdot a) \cdot H;$$

(ii) the semidirect-product Fell bundle $\mathscr{A}/H \rtimes G$ acts on the left of the Banach bundle $\mathscr{A}$ by

$$(a \cdot H, t) \cdot b = a(t \cdot b) \quad \textit{whenever } s(a) = r(p(t \cdot b)). \tag{A.11}$$

Proof. By Corollary A.19, the orbit Fell bundle $\mathscr{A}/H$ acts on the Banach bundle $\mathscr{A}$. By Corollary A.21, G acts on the orbit groupoid $\mathcal{X}/H$, and it is routine to check (i), and also that the actions of G and $\mathscr{A}/H$ on $\mathscr{A}$ are covariant. Then (ii) follows from Corollary A.8. □

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